Final Project No. 2: Uncertainty Quantification in Stationary Heat Diffusion Using Finite Element Methods

Introduction to Scientific Computing - Academic Year 2023/2024

Many natural and industrial phenomena can be modeled using differential problems. However, due to various reasons (incomplete knowledge of the phenomenon, measurement errors, etc.), it is often not possible to know the data of the problem (domain, coefficients, known term, boundary conditions, etc.) with certainty. A possible approach to incorporate such uncertainty into the mathematical model is to assume that the data are random variables distributed according to a certain probability law. This randomness is inherited by the solutions, which in turn become random variables. In this context, a typical problem is to determine, starting from information on the distribution of the data of a differential problem (complete distribution, mean, variance, etc.), information on the distribution of the solution or quantities of interest (QoI) that depend on it. The problem just described is an example of an uncertainty quantification (UQ) problem.

As an illustrative example, consider the boundary value problem

$$-\left(a(x,\omega)u'(x,\omega)\right)' = 1, \quad x \in (0,1),$$

$$u(0) = 0,$$

$$u(1) = 0,$$

where

$$a(x,\omega) = \mu + \sum_{p=1}^{P} \sigma \frac{\cos(\pi p x)\xi_p(\omega)}{p^2\pi^2}.$$

In the differential equation, the derivatives are taken with respect to the spatial variable $x \in (0,1)$. The coefficient $a = a(x,\omega)$, and consequently the solution $u = u(x,\omega)$, depends not only on the spatial variable $x \in (0,1)$, but also on the realization of a sample $\omega \in \Omega$ (where Ω denotes the sample space of a probability space). In the equation, $P \in \mathbb{N}$, $\mu, \sigma \in \mathbb{R}_{>0}$ are such that $\sigma < 6\mu$, while $\{\xi_p\}_{p=1,\ldots,P}$ is a family of P independent and identically distributed random variables such that, for each $p = 1,\ldots,P$, ξ_p is a random variable taking values in the interval $[-1, 1]$ with uniform distribution ($\xi_p \sim U(-1, 1)$). The problem is a simplified model of stationary heat diffusion in a unit-length rod with uncertain and non-constant thermal conductivity. Suppose we are interested in the integral mean of the solution

$$Q(u) = \int_0^1 u(x)\, dx.$$

(i) For each fixed sample $\omega \in \Omega$, derive the variational formulation of the problem solved by the weak solution $u(\cdot, \omega) \in V = H_0^1(0, 1)$. Prove that, for every $\omega \in \Omega$, the weak solution $u(\cdot, \omega)$ exists and is unique.

(ii) Consider the case $P = 50$, $\mu = 1$, and $\sigma = 4$. Starting from the variational formulation obtained in the previous point, numerically approximate the weak solution $u(\cdot, \omega)$ using the linear finite element method ($u_h(\cdot, \omega) \in V_h = S_0^1(T_h)$, where T_h is a uniform grid of the interval $(0, 1)$ with constant spacing h). Generate 20 random samples $\omega_1, \ldots, \omega_{20} \in \Omega$ and graphically compare the approximate solutions $u_h(\cdot, \omega_m)$ (for $m = 1, \ldots, 20$).

(iii) An approximation of the expected value of $Q(u)$ can be obtained using the Monte Carlo method:

$$Q_{M,h}^{\text{MC-FEM}}(u) = \frac{1}{M} \sum_{m=1}^{M} Q(u_h(\cdot, \omega_m)).$$

The quantity $Q_{M,h}^{\text{MC-FEM}}(u)$, known as the Monte Carlo finite element estimator (MC-FEM) of $\mathbb{E}[Q(u)]$, can be calculated by performing the following steps:

(a) Generate M independent samples $\omega_1, \ldots, \omega_M \in \Omega$.

(b) Compute the M approximate solutions $u_h(\cdot, \omega_1), \ldots, u_h(\cdot, \omega_M)$ using the finite element method.

(c) Evaluate the QoI for each of the computed approximate solutions.

(d) Define the Monte Carlo finite element estimator as the empirical mean of the M QoI evaluations.

Extend the code from the previous point to implement this procedure. Investigate numerically the convergence of $Q_{M,h}^{\text{MC-FEM}}(u)$ as $h \to 0$ and $M \to \infty$.