Final Project No. 1: The Helmholtz Equation and Acoustic Pressure

Introduction to Scientific Computing – Academic Year 2023/24

Sound, the sensation generated by the vibration of particles (atoms or molecules) in a physical propagation medium, can be quantitatively described by acoustic pressure $P = P(x,t)$, a quantity that locally describes (at point x and time t) the pressure fluctuations relative to a reference pressure caused by the passage of sound waves. In the one-dimensional case, if certain conditions are met (if the oscillations have low amplitude and if the propagation medium is homogeneous and isotropic), an adequate model to describe the dynamics of acoustic pressure is the wave equation:

$$\frac{\partial^2 P(x,t)}{\partial t^2} - c^2 \frac{\partial^2 P(x,t)}{\partial x^2} = 0, \tag{1}$$

where $c > 0$ is the speed of sound in the propagation medium.

(i) Assume that the acoustic pressure has the form

$$P(x,t) = \mathrm{Re}\{u(x)e^{-i\omega t}\}, \tag{2}$$

for a certain function $u = u(x)$, independent of time and complex-valued, and angular frequency $\omega > 0$ (the so-called time-harmonic case). Show that (2) solves (1) if the stationary part u of the acoustic pressure solves the Helmholtz equation

$$-u''(x) - k^2 u(x) = 0, \tag{3}$$

where $k = \omega/c > 0$ is the wave number (the corresponding wavelength is $\lambda = 2\pi/k$). Determine the expression of the general solution of the Helmholtz equation and deduce from it (using (2)) the general expression of the solutions of (1) in the time-harmonic case.

(ii) Consider the following boundary value problem for the Helmholtz equation:

$$\begin{aligned} -u''(x) - k^2 u(x) &= f(x), \quad x \in (0,1), \\ u(0) &= 0, \\ u'(1) - iku(1) &= \beta \end{aligned}$$

($\beta \in \mathbb{C}$). Derive the variational formulation of the problem solved by the weak solution $u \in V = H^1_{\{0\}}(0,1) = \{u \in H^1(0,1) : u(0) = 0\}$. Do the hypotheses of the Lax-Milgram theorem hold?

(iii) Starting from the variational formulation obtained in the previous point, numerically approximate the weak solution u with the linear finite element method ($u_h \in V_h = S^1(T_h) \cap H^1_{\{0\}}(0,1)$, where T_h is a uniform grid of the interval $(0,1)$ with constant width h). Consider the case $f(x) = 0$ and $\beta = ke^{-ik}$, for which the expression of the exact solution is $u(x) = \sin(kx)$. Fix $k = 2\pi$ and graphically compare the exact solution with the approximate solution (representing their graphs in the same figure, distinguishing between real and imaginary parts). Then numerically study the convergence of the method, representing the behavior of the errors $|u - u_h|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ as h decreases. Finally, study how the behavior of the method changes as the wave number k increases.