1. Potential theory for general Markov processes

1.1. **Functional framework.** Let (E, d) be a Polish space endowed with its Borel σ-algebra and a reference probability measure π. Denote by $L^2 = L^2(\pi)$ the space of square-integrable, real-valued functions defined in E. The norm of L^2 is represented by $\| \cdot \|$. Consider a Markov semigroup with generator $L \colon D(L) \to L^2(\pi)$, with domain $D(L) \subset L^2(\pi)$, see [1, Definition 1.8].

Denote by $\mathcal{C}$ a core for the generator L and assume that $\mathcal{C}$ is closed by multiplication. Denote by L^* the adjoint of L, and assume that $\mathcal{C}$ is also a core for L^*.

Denote by $C(E, D(L))$ the space of continuous functions $\xi \colon E \to D(L)$. For a function $\xi \in C(E, D(L))$, we represent $\xi(x) \in D(L)$, $x \in E$, by ξ_x and $\xi_x(y) \in E$, $y \in E$, by $\xi(x, y)$.

We assume some sort of sector condition: For each function $f \in D(L)$, there exists a finite constant C_f such that for all $\varphi \in C(E, D(L))$,

$$\Big(\int \pi(dx)\, f(x)\, (L\, \varphi_x)(x) \Big)^2 \;\le\; C_f \int \pi(dx)\, (L\, \varphi_x^2)(x)\;. \tag{1.1}$$ 03

If $\varphi(x,y) = g(y) - g(x)$, then this condition is nothing but the sector condition with $C_f = c_K D(f,f)$.

If $L = \Delta$ is the Laplacian on $\mathbb{R}^d$, then define $V(x) = (\nabla_y \varphi)(x,x)$. Then

$$
\begin{gathered}
V_i(x) = (\partial_{y_i}\varphi)(x,x) \\
\partial_{x_i} V_i(x) = (\partial_{x_i}\partial_{y_i}\varphi)(x,x) + (\partial_{y_i}\partial_{y_i}\varphi)(x,x) \\
\operatorname{div}(V)(x) = \sum_i \partial_{x_i} V_i(x) = (\Delta_y \varphi)(x,x) + \sum_i (\partial_{x_i}\partial_{y_i}\varphi)(x,x) = (L\varphi_x)(x) \\
(L\,\varphi_x^2)(x) = 2V(x)\cdot V(x)
\end{gathered}
$$

Then the sector-like condition is

$$\Big(\int dx\, f(x)\operatorname{div}(V(x)) \Big)^2 \;\le\; C_f \int dx\, 2V(x)\cdot V(x)$$

which holds since integrating by parts this is equivalent to

$$\Big(\int dx\, (\nabla f)(x)\, V(x) \Big)^2 \;\le\; 2C_f \int dx\, V(x)\cdot V(x)$$

and therefore it is enough to take $C_f = D(f,f)$.

$$
\begin{gathered}
\Gamma(f,g) = L(fg) - fLg - gLf \\
D^s(f,g) = \int d\pi(x)\,\Gamma(g,f)
\end{gathered}
$$

$$\mathbf{\Gamma}(\varphi,\psi)(x) = L(\varphi_x\psi_x)(x)$$

Notice that $\mathbf{\Gamma}(df,dg) = \Gamma(f,g)$ If $L = \Delta$, then $\mathbf{\Gamma}(\varphi,\psi) = 2V_\varphi\, V_\psi$

If $Lf = b\cdot df$, then $\mathbf{\Gamma}(f,g) = 0$. Then the condition

$$\Big(\int \pi(dx)\, f(x)\,(L\,\varphi_x)(x) \Big)^2 \;\le\; C_f \int \pi(dx)\,(L\,\varphi_x^2)(x)$$

writes as

$$\Big(\int \pi(dx)\, f(x)\,(\mathbf{L}\varphi)(x) \Big)^2 \;\le\; C_f \int \pi(dx)\,\mathbf{\Gamma}(\varphi,\varphi)(x)$$

If we restrict to the case $\varphi = dg$ then the last condition becomes

$$\Big(\int \pi(dx)\, f(x)\,(Lg)(x) \Big)^2 \;\le\; C_f \int \pi(dx)\,\Gamma(g,g)(x) = 2C_f D(g,g)$$

which is implied by the sector condition.

Let us try to prove

$$\Big(\int \pi(dx)\, f(x)\, (\mathbf{L}\varphi)(x) \Big)^2 \;\le\; C_f \int \pi(dx)\, \mathbf{\Gamma}(\varphi,\varphi)(x)$$

assuming L is self-adjoint in $L^2(d\pi)$. By self-adjointness

$$2\int \pi(dx)\, f(x)\, (\mathbf{L}\varphi)(x) = -\int d\pi(x) \mathbf{\Gamma}(df,\varphi) \le \sqrt{\int d\pi(x)\mathbf{\Gamma}(df,df)}\sqrt{\int d\pi(x)\mathbf{\Gamma}(\varphi,\varphi)}$$

The first equality should follow as

$$\int \pi(dx)\, f(x)\, (\mathbf{L}\varphi)(x) = \int \pi(dx)\, \lim_{t\to 0} \mathbb{E}_x \varphi(X_t, X_0) f(X_0)/t = \lim_{t\to 0} \mathbb{E}_\pi \varphi(X_t, X_0) f(X_0)/t$$

By self-adjointness the last quantity also equals

$$\lim_{t\to 0} \mathbb{E}_\pi \varphi(X_0, X_t) f(X_t)/t = -\lim_{t\to 0} \mathbb{E}_\pi \varphi(X_t, X_0) f(X_t)/t = -\lim_{t\to 0} \tfrac{1}{2} \mathbb{E}_\pi \varphi(X_t, X_0)(f(X_t) - f(X_0)/t$$
$$= -\tfrac{1}{2}\int \pi(dx)\, \mathbf{\Gamma}(df,\varphi)(x)$$

Probably it is enough to do it for pure jump processes. Namely, is it true that

$$\Big(\int \pi(dx) c(x,dy) f(x) \varphi(x,y) \Big)^2 \le C_f \int \pi(dx)\, c(x,dy) \varphi(x,y)^2$$

if we know

$$\Big(\int \pi(dx) c(x,dy) f(x) (g(y) - g(x)) \Big)^2 \le c\, D(f,f) D(g,g)$$

?

Given a Markov process, for each $t \geq 0$ it is defined a measurable $p_t \colon E \to \mathcal{P}(E)$, which we denote $p_t(x, dy)$, given by

$$\int p_t(x, dy) f(y) = \mathbb{E}_x[f(X_t)]$$

Moreover

$$\int_{y \in E} p_t(x, dy) p_s(y, dz) = p_{t+s}(x, dz)$$

Given a Markov process, with transition probability p_t, one can weakly approximate it, with a pure jump process with generator $L_\varepsilon f = \varepsilon^{-1} \int_y p_\varepsilon(x, dy)(f(y) - f(x))$. Notice that if the original Markov process admits a generator L, then

$$Lf(x) = \lim_{t \to 0} \mathbb{E}_x[f(X_t) - f(X_0)]/t = \lim_{t \to 0} \int p_t(x, dy)(f(y) - f(x))/t = \lim_{\varepsilon \to 0} L_\varepsilon f(x)$$

Notice $c_\varepsilon(x, dy) = \varepsilon^{-1} p_\varepsilon(x, dy)$ has TV ε^{-1}

Puro salto: siano $k(dx, dy) = \pi(dx) c(x, dy)$ e $k^\dagger(dx, dy) = k(dy, dx)$, $k^s(dx, dy) = (k(dx, dy) + k^\dagger(dx, dy))/2$. In particolare $k \leq 2\,k^s$. Sia

$$q(x, y) = \frac{k - k^\dagger}{k^s}(x, y) \in [-2, 2]$$

$$\begin{aligned}
(\int \pi(dx) c(x, dy) f(x) \varphi(x, y))^2 &= (\int k(dx, dy) f(x) \varphi(x, y))^2 \\
&= (\tfrac{1}{2} \int k(dx, dy) f(x) \varphi(x, y) - k(dy, dx) f(y) \varphi(x, y))^2 \\
= (\tfrac{1}{2} \int k^s(dx, dy)(f(x) - f(y)) \varphi(x, y) &- \tfrac{1}{2} \int (k(dx, dy) - k(dy, dx)) f(x) \varphi(x, y))^2 \\
&\leq 2D(f, f) \mathbb{D}(\varphi, \varphi) + 4(\int k^s(dx, dy) q(x, y) f(x) \varphi(x, y))^2
\end{aligned}$$

Esempio: $E = \mathbb{T}_N$, $k^s(dx, dy) = 1/(2N)(\delta_{y-1}(dy) + \delta_{y+1}(dy))$, δ

Let $\mathcal{U}_0$, $\mathcal{A}$ be the set given by

$$\mathcal{U}_0 \;=\; \big\{\, \xi \in C(E, D(L)) : \xi_x(x) = 0 \;\; \forall\, x \in E \,\big\}\;.$$

$$\mathcal{A} \;=\; \big\{\, \xi \in \mathcal{U}_0 : \xi_x(y) = -\,\xi_y(x) \;\; \forall\, x \in E,\, \xi_x \in \mathcal{C} \,\big\}\;.$$

Define the operator $\mathbb{L} : \mathcal{U}_0 \to L^2(\pi)$ by

$$(\mathbb{L}\,\xi)(x) \;=\; (L\,\xi_x)(x)\;, \quad x \in E\;.$$

Denote by $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ the scalar product on $\mathcal{U}_0$ defined by

$$\langle\!\langle\, \xi\,,\, \xi' \,\rangle\!\rangle \;:=\; \frac{1}{2}\int \pi(dx)\,(\mathbb{L}\,\xi\,\xi') \;=\; \frac{1}{2}\int \pi(dx)\,(L\,\xi_x\,\xi'_x)(x)\;.$$

Note that $\langle\!\langle\, \xi\,,\, \xi \,\rangle\!\rangle \ge 0$ because, since ξ vanish on the diagonal,

$$\begin{aligned}\langle\!\langle\, \xi\,,\, \xi \,\rangle\!\rangle \;&=\; \lim_{t\downarrow 0}\frac{1}{2}\int \pi(dx)\,\frac{(P_t\,\xi_x^2)(x) - \xi_x^2(x)}{t}\\ &=\; \lim_{t\downarrow 0}\frac{1}{2}\int \pi(dx)\,\frac{(P_t\,\xi_x^2)(x)}{t} \;\ge\; 0\;.\end{aligned}$$

Denote by $\|\cdot\|$ the pre-norm associated to this scalar product and by $\sim$ the equivalence relation in $\mathcal{U}_0$ given by $\xi \sim \xi'$ whenever $\|\,\xi \,-\, \xi'\,\| = 0$. Let $\mathcal{U}$, $\mathcal{H}$ be the completions of $\mathcal{U}_0$, $\mathcal{A}$, respectively, with respect to the scalar product $\langle\cdot\,,\,\cdot\rangle$: $\mathcal{U} = \overline{\mathcal{U}_0\big|_\sim}$, $\mathcal{H} = \overline{\mathcal{A}\big|_\sim}$.

For every function $f \in L^2(\pi)$, denote by $df : E \times E \to \mathbb{R}$ the function defined by $[(df)(x)](y) = f(y) - f(x)$. Clearly, if $f \in D(L)$, then df belongs to $\mathcal{A}$. Note that $\mathbb{L}(df) \;=\; Lf$ and that for every g in $D(L)$,

$$\langle\!\langle\, df\,,\, dg \,\rangle\!\rangle \;=\; \frac{1}{2}\int \pi(dx)\,\big\{L\,f\,g \;-\; f\,Lg \;-\; g\,Lf\,\big\}(x)\;.$$

As π is the stationary state, $\int \pi(dx)\,(L\,f\,g)(x) \;=\; 0$ so that

$$\langle\!\langle\, df\,,\, dg \,\rangle\!\rangle \;=\; \frac{1}{2}\,\mathcal{D}(f,g) \;+\; \frac{1}{2}\,\mathcal{D}(g,f)\;. \tag{1.2}$$

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In the particular case where $f = g$,

$$\|\,df\,\|_{\mathcal{H}}^2 \;=\; -\int \pi(dx)\,f(x)\,(Lf)(x) \;=\; \mathcal{D}(f,f)\;. \tag{1.3}$$

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1.2. **Markov Flows.** Denote by $\mathcal{F}$ the dual of $\mathcal{H}$. We refer to $\mathcal{F}$ as the space of flows.

For a function $f : E \to \mathbb{R}$, let $\Phi_f : \mathcal{H} \to \mathbb{R}$ be the linear functional defined by

$$\Phi_f(\chi) \;=\; -\int \pi(dx)\,f(x)\,(\mathbb{L}\,\chi)(x) \;=\; -\int \pi(dx)\,f(x)\,(L\,\chi_x)(x)\;.$$

By (1.1), for each $f \in D(L)$, Φ_f is a bounded functional and therefore belongs to $\mathcal{F}$.

Notice that for every functions f, g in $D(L)$,

$$\Phi_g(df) \;=\; -\int \pi(dx)\,g(x)\,(\mathbb{L}\,df)(x) \;=\; -\int \pi(dx)\,g(x)\,(L\,f)(x) \;=\; \mathcal{D}(f,g)\;. \tag{1.4}$$

phig

For a function f in $D(L)$. let Ψ_f the element of $\mathcal{F}$ defined by

$$\Psi_f(\chi) \;=\; \langle\!\langle\, df\,,\, \chi \,\rangle\!\rangle\;, \quad \chi\,\in\,\mathcal{H}\;. \tag{1.5}$$ 06

As df belongs to $\mathcal{H}$, Ψ_f is an element of $\mathcal{F}$ and

$$\|\Psi_f\|^2_{\mathcal{F}} \;=\; \|df\|^2_{\mathcal{H}} \;=\; \mathcal{D}(f,f)\;, \tag{1.6}$$ 07

where the last identity follows from (1.3). Indeed, by definition,

$$\|\Psi_f\|^2_{\mathcal{F}} \;:=\; \sup_{\chi}\big\{\, 2\,\Psi_f(\chi) \,-\, \|\chi\|^2_{\mathcal{H}}\,\big\} \;=\; \sup_{\chi\in\mathcal{H}}\big\{\, 2\,\langle\!\langle\, df\,,\,\chi\,\rangle\!\rangle \,-\, \|\chi\|^2_{\mathcal{H}}\,\big\}\;,$$

and this expression is equal to $\|df\|^2_{\mathcal{H}}$. Moreover, for every function g in $D(L)$,

$$\Psi_f(dg) \;=\; \langle\!\langle\, df\,,\, dg\,\rangle\!\rangle \tag{1.7}$$ 09

Define the codifferential d^* on $\mathcal{F}$ as the dual operator of d, namely

$$(d^*\Phi)(f) \;:=\; -\,\Phi(df)\;.$$

Fix two disjoint, non-empty subsets A, B of E: $A\neq\varnothing$, $B\neq\varnothing$, $A\cap B=\varnothing$. Let $C_{\alpha,\beta}$, α, $\beta\in\mathbb{R}$, be the subspace of real continuous functions given by

$$C_{\alpha,\beta} \;:=\; \big\{\, f\in C(E) \,:\, f(x) \;=\; \alpha \text{ for } x\in A \text{ and } f(y) \;=\; \beta \text{ for } y\in B,\; f\in D(\mathcal{D}^s)\big\}\;.$$

Let $\mathcal{F}_\gamma$, $\gamma\in\mathbb{R}$, be given by

$$\mathcal{F}_\gamma \;:=\; \big\{\, \Phi\in\mathcal{F} \,:\, d^*\Phi(f) \;=\; -\,\gamma \text{ for all } f\in C_{1,0}\,\big\}\;. \tag{1.8}$$ fgamma

2. Capacity

A closed set $A\subset E$ is recurrent if for all $x\in E$, $\mathbb{P}_x(\tau_A<\infty)=1$.

def1 **Definition 2.1.** *Two closed recurrent sets A, $B\subset E$ have* finite capacity *if the function $h(x)\equiv h_{A,B}(x)=\mathbb{P}_x(\tau_A<\tau_B)$ is in the domain of $\mathcal{D}(\cdot)$ or equivalently $dh\in\mathcal{H}$. In such a case we define*

$$\mathrm{cap}(A,B)=\mathcal{D}(h,h) \tag{2.1}$$ 08

102 **Lemma 2.2.** *Fix two disjoint, non-empty subsets A, B of E.*

$$\mathrm{cap}(A\,,\,B) \;=\; \mathrm{cap}^\dagger(A\,,\,B)\;.$$

101 **Lemma 2.3.** *Fix two disjoint, non-empty subsets A, B of E. For each α, $\gamma\in\mathbb{R}$, $f\in C_{\alpha,0}$ and $\Phi\in\mathcal{F}_\gamma$,*

$$\big[\,\Phi_f \,-\, \Phi\,\big]\,(dh_{A,B}) \;=\; \alpha\,\mathrm{cap}(A,B) \,-\, \gamma\;. \tag{2.2}$$ 04

Proof. From (1.4), we have that

$$\Phi_f(dh) \;=\; -\int \pi(dx)\,f(x)\,(L\,h)(x)\;.$$

Since LH vanishes on $\Omega=(A\cup B)^c$ and $f=\alpha, h$ 0 on $A\cup B$, we may replace in the previous formula, f by $\alpha\,h$ to get that $\Phi_f(dh)\,=\,\alpha\,\mathcal{D}(h,h)\,=\,\alpha\,\mathrm{cap}(A,B)$.

On the other hand, by the definition (1.8) of $\mathcal{F}_\gamma$ and since h belongs to $C_{1,0}$,

$$\Phi(dh) \;=\; \gamma\;.$$

This proves the lemma. □

prop1 **Proposition 2.4** (Dirichlet Principle)**.** *Fix two disjoint, non-empty subsets A, B of E. We have that*

$$\mathrm{cap}(A.B) \;=\; \inf_{f\in C_{1,0}} \inf_{\Phi\in\mathcal{F}_0} \|\Phi_f - \Phi\|^2_{\mathcal{F}}\;.$$

Proof. Fix f in $C_{1,0}$ and Φ in $\mathcal{F}_0$. By the previous lemma and by Schwarz inequality,

$$\mathrm{cap}(A,B)^2 \;=\; \Big\{ \big[\, \Phi_f \,-\, \Phi \,\big]\, (dh_{A,B}) \Big\}^2 \;\le\; \big\| \, \Phi_f - \Phi \, \big\|^2_{\mathcal{F}} \, \big\| \, dh \, \big\|^2_{\mathcal{H}}\;.$$

By (1.3) and (2.1), the last term is equal to $\mathrm{cap}(A,B)$, so that $\mathrm{cap}(A,B) \le \| \, \Phi_f - \Phi \, \|^2_{\mathcal{F}}$ for every f in $C_{1,0}$ and Φ in $\mathcal{F}_0$:

$$\mathrm{cap}(A,B) \;\le\; \inf_{f\in C_{1,0}} \inf_{\Phi\in\mathcal{F}_0} \|\Phi_f - \Phi\|^2_{\mathcal{F}}\;.$$

Let $f_\star = (h+h^\dagger)/2$ and $\Phi_\star = \Phi_{f_\star} - \Psi_h$ so that $\Phi_{f_\star} - \Phi_\star = \Psi_h$. By (2.1) and (1.6), $\mathrm{cap}(A,B) = \mathcal{D}(h,h) = \| \, \Psi_h \, \|^2_{\mathcal{F}} \;=\; \| \, \Phi_{f_\star} - \Phi_\star \, \|^2_{\mathcal{F}}$. Hence, to prove that

$$\inf_{f\in C_{1,0}} \inf_{\Phi\in\mathcal{F}_0} \|\Phi_f - \Phi\|^2_{\mathcal{F}} \;\le\; \mathrm{cap}(A,B)\;,$$

it remains to check that $f_\star$ belongs to $C_{1,0}$, and $\Phi_\star$ to $\mathcal{F}_0$.

Clearly, $f_\star = (h+h^\dagger)/2 \in C_{1,0}$. To shows that $\Phi_\star \in \mathcal{F}_0$, fix g in $C_{1,0}$. By definition, $\Phi_\star(dg) = \Phi_{f_\star}(dg) - \Psi_h(dg)$. By definition of $f_\star$, (1.4), (1.2) and (1.7), this expression is equal to

$$\frac{1}{2} \Big\{ \mathcal{D}(h\,,\,g) \;+\; \mathcal{D}(h^\dagger\,,\,g) \;-\; \mathcal{D}(h\,,\,g) \;-\; \mathcal{D}(g\,,\,h) \Big\} \;=\; \frac{1}{2} \big\{ \mathcal{D}(h^\dagger\,,\,g) \;-\; \mathcal{D}(g\,,\,h) \big\}\;.$$

As g belongs to $C_{1,0}$, g coincides with h amd $h^\dagger$ on $A \cup B$. Hence, as $Lh = L^\dagger h^\dagger = 0$ on Ω, the previous expression is equal to

$$\frac{1}{2} \big\{ -\, \langle L^\dagger h^\dagger\,,\, h^\dagger \rangle \;+\; \langle Lh\,,\,h \rangle \big\} \;=\; \frac{1}{2} \big\{ \mathcal{D}(h^\dagger\,,\,h^\dagger) \;-\; \mathcal{D}(h\,,\,h) \big\} \;=\; 0$$

because both quantities are equal to $\mathrm{cap}(A,B)$. □

prop2 **Proposition 2.5** (Thomson Principle)**.** *Fix two disjoint, non-empty subsets A, B of E. We have that*

$$\frac{1}{\mathrm{cap}(A.B)} \;=\; \inf_{\Phi\in\mathcal{F}_1} \inf_{f\in C_{0,0}} \|\Phi_f - \Phi\|^2_{\mathcal{F}}\;.$$

Proof. Fix f in $C_{0,0}$ and Φ in $\mathcal{F}_1$. By the previous lemma and by Schwarz inequality,

$$1 \;=\; \Big\{ \big[\, \Phi_f \,-\, \Phi \,\big]\, (dh_{A,B}) \Big\}^2 \;\le\; \big\| \, \Phi_f - \Phi \, \big\|^2_{\mathcal{F}} \, \big\| \, dh \, \big\|^2_{\mathcal{H}}\;.$$

By (1.3) and (2.1), the last term is equal to $\mathrm{cap}(A,B)$, so that $1/\mathrm{cap}(A,B) \le \| \, \Phi_f - \Phi \, \|^2_{\mathcal{F}}$ for every f in $C_{0,0}$ and Φ in $\mathcal{F}_1$:

$$\frac{1}{\mathrm{cap}(A.B)} \;\le\; \inf_{\Phi\in\mathcal{F}_1} \inf_{f\in C_{0,0}} \|\Phi_f - \Phi\|^2_{\mathcal{F}}\;.$$

To complete the proof of the theorem, it remains to find g in $C_{0,0}$ and Φ in $\mathcal{F}_1$ such that $\| \, \Phi_g \;-\; \Phi \, \|^2 \;=\; 1/\mathrm{cap}(A,B)$. Let $g_\star \;=\; (h^\dagger - h)/2\,\mathrm{cap}(A,B)$ and $\Psi_\star = \Phi_{g_\star} + \Psi_{h_0}$, where $h_0 = h/\mathrm{cap}(A,B)$, so that $\Phi_{g_\star} - \Psi_\star = -\,\Psi_{h_0}$. By (2.1) and (1.6), $1/\mathrm{cap}(A,B) = \mathcal{D}(h_0,h_0) = \| \, \Psi_{h_0} \, \|^2_{\mathcal{F}} \;=\; \| \, \Phi_{g_\star} - \Psi_\star \, \|^2_{\{}$. Hence, to prove that

$$\inf_{f\in C_{1,0}} \inf_{\Phi\in\mathcal{F}_0} \|\Phi_f - \Phi\|^2_{\mathcal{F}} \;\le\; \mathrm{cap}(A.B)\;,$$

it remains to check that $g_\star$ belongs to $C_{0,0}$, and $\Psi_\star$ to $\mathcal{F}_1$.

Clearly, $g_\star = (h^\dagger - h)/2\,\mathrm{cap}(A,B) \in C_{0,0}$. To shows that $\Psi_\star \in \mathcal{F}_1$, fix f in $C_{1,0}$. By definition, $\Psi_\star(df) = \Phi_{g_\star}(df) + \Psi_{h_0}(df)$. By definition of $g_\star$, (1.4), (1.7) and (1.2), this expression is equal to

$$\begin{aligned}&\frac{1}{2\,\mathrm{cap}(A,B)}\Big\{\mathcal{D}(f\,,\,h^\dagger)\;-\;\mathcal{D}(f,h)\;+\;\mathcal{D}(h\,,\,f)\;+\;\mathcal{D}(f\,,\,h)\Big\}\\ &\quad=\;\frac{1}{2\,\mathrm{cap}(A,B)}\,\big\{\,\mathcal{D}(f\,,\,h^\dagger)\;+\;\mathcal{D}(h\,,\,f)\,\big\}\;.\end{aligned}$$

As f belongs to $C_{1,0}$, f coincides with h and $h^\dagger$ on $A\cup B$. Hence, as $Lh = L^\dagger h^\dagger = 0$ on Ω, the previous expression is equal to

$$\begin{aligned}&\frac{1}{2\,\mathrm{cap}(A,B)}\,\big\{\,-\,\langle L^\dagger h^\dagger\,,\,h^\dagger\rangle\;-\;\langle Lh\,,\,h\rangle\,\big\}\\ &\quad=\;\frac{1}{2\,\mathrm{cap}(A,B)}\,\big\{\,\mathcal{D}(h^\dagger\,,\,h^\dagger)\;+\;\mathcal{D}(h\,,\,h)\,\big\}\;=\;1\end{aligned}$$

because both quantities are equal to $\mathrm{cap}(A,B)$. □

3. Examples

Discrete spaces.

The operator $\mathbb{L} : \mathcal{A} \to E$ is given by

$$\begin{aligned}(\mathbb{L}\,\xi)(x)\;&=\;(L\,\xi_x)(x)\;,\quad x\in E\\ &=\;\sum_{y\in E} r(x,y)\,[\xi_x(y)-\xi_x(x)]\\ &=\;\sum_{y\in E} r(x,y)\,\xi(x,y)\;.\end{aligned}$$

The operator $L^\eta : D(L^\eta) \to B$ given by

$$(L^\eta\, f)(x)\;=\;[\,\mathbb{L}\,(e^\eta\,df)\,](x);=\;\sum_{y\in E} r(x,y)\,e^{\eta(x,y)}\,[f(y)-f(x)]\;.$$

$$\langle\!\langle\,\xi\,,\,\xi'\,\rangle\!\rangle\;:=\;\frac{1}{2}\int_{E\times E}\pi(dx)\,R(x,dy)\,\xi(x,y)\,\xi'(x,y)\;=\;\frac{1}{2}\int\pi(dx)\,(\mathbb{L}\xi\,\xi')\;.$$

$$\langle\,\xi\,,\,\xi'\,\rangle\;=\;\frac{1}{2}\sum_{x,y}\pi(x)r_s(x,y)\,\xi(x,y)\,\xi'(x,y)\;.$$

$$\Phi_f(\chi)\;=\;\int\pi(dx)\,f(x)\,(\mathbb{L}\,\chi)(x)\;.$$

3.1. **Diffusions.** Fix $d \ge 1$, and denote by $\mathbb{T}^d = [0,1)^d$ the d-dimensional torus of length 1. Denote by $a(x)$ a uniformly positive-definite matrix whose entries $a_{i,j}$ are smooth functions: There exist $c_0 > 0$ such that for all $x \in \mathbb{T}^d$, $y \in \mathbb{R}^d$,

$$y\cdot a(x)\,y\;\ge\;c_0\,\|y\|^2\;, \tag{3.1}$$ 19

where $y \cdot z$ represents the scalar product in $\mathbb{R}^d$.

Generator. Denote by $\mathcal{L}$ the generator given by

$$\mathcal{L}f\;=\;\nabla\cdot(a\,\nabla f)\;+\;b\cdot\nabla f\;, \tag{3.2}$$ 2-12

where $b : \mathbb{T}^d \to \mathbb{R}^d$ is a smooth vector field. By modifying the drift b we could assume the matrix a to be symmetric. We will not assume this condition for reasons which will become below. There exists a unique Borel probability measure μ on $\mathbb{T}^d$ such that $\mu\mathcal{L} = 0$. This measure is absolutely continuous, $\mu(dx) = m(x)dx$, where m is the unique solution to

$$\nabla \cdot (a^\dagger \nabla m) \;-\; \nabla \cdot (b\, m) \;=\; 0\;, \tag{3.3}$$ 2-16

where $a^\dagger$ stands for the transpose of a. For existence, uniqueness and regularity conditions of solutions of elliptic equations, we refer to [?]. Let $V(x) = -\log m(x)$, so that $m(x) = e^{-V(x)}$.

We may rewrite the generator $\mathcal{L}$ introduced in (3.2) as

$$\mathcal{L} f \;=\; e^V \nabla \cdot \big(e^{-V} a \nabla f \big) \;+\; c \cdot \nabla f\;,$$

where $c = b + a^\dagger \nabla V$. It follows from (3.3) that

$$\nabla \cdot (e^{-V} c) \;=\; 0\;. \tag{3.4}$$ 2-07

This implies that the operator $c \cdot \nabla$ is skew-adjoint in $L_2(\mu)$: for any smooth functions f, $g : \mathbb{T}^d \to \mathbb{R}$,

$$\int f\, c \cdot \nabla g \, d\mu \;=\; -\int g\, c \cdot \nabla f \, d\mu\;. \tag{3.5}$$ 2-01

In view of (3.5), the adjoint of $\mathcal{L}$ in $L_2(\mu)$, represented by $\mathcal{L}^*$, is given by

$$\mathcal{L}^* f \;=\; e^V\, \nabla \cdot \big(e^{-V} a^\dagger \nabla f \big) \;-\; c \cdot \nabla f\;,$$

while the symmetric part, denoted by $\mathcal{L}^s$, $\mathcal{L}^s = (1/2)(\mathcal{L} + \mathcal{L}^*)$, takes the form

$$\mathcal{L}^s f \;=\; e^V\, \nabla \cdot \big(e^{-V} a_s \nabla f \big)\;. \tag{3.6}$$ 2-8

where a_s stands for the symmetrization of the matrix a: $a_s = (1/2)[a + a^\dagger]$.

Recall from (**??**), (**??**), the definition of the spaces $C(\mathbb{T}^d, D(L))$, $\mathcal{A}$ and the operator $\mathbb{L}$. In this context, for any smooth element φ of $C(\mathbb{T}^d, D(L))$,

$$\langle\!\langle\, \varphi\,,\, \varphi\,\rangle\!\rangle \;=\; \int_{\mathbb{T}^d} \mu(dx)\, (\nabla_y \varphi)(x,x) \cdot a(x)\, (\nabla_y \varphi)(x,x)\;. \tag{3.7}$$ b02

Hence, as the matrix a is strictly elliptic, $\|\varphi\| = 0$ if and only if $(\nabla_y \varphi)(x,x) = 0$ for all $x \in \mathbb{T}^d$.

Consider a smooth, conservative vector field $\mathfrak{v} : \mathbb{T}^d \to \mathbb{R}^d$. Recall that conservative means that its line integral over closed paths vanishes or, equivalently, that its line integral is path independent. Denote by $\varphi_{\mathfrak{v}} : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ the function defined by

$$\varphi_{\mathfrak{v}}(x,y) \;=\; \int_\gamma \mathfrak{v}(\ell)\, d\ell\;,$$

where γ is a path from x to y. The function $\varphi_{\mathfrak{v}}$ is well defined because $\mathfrak{v}$ is conservative. The same property yields that $\varphi_{\mathfrak{v}}(x,y) \;=\; -\,\varphi_{\mathfrak{v}}(y,x)$ so that $\varphi_{\mathfrak{v}} \in \mathcal{A}$. Furthermore, $(\nabla_y \varphi_{\mathfrak{v}})(x,x) = \mathfrak{v}(x)$ so that, by (3.7),

$$\langle\!\langle\, \varphi_{\mathfrak{v}}\,,\, \varphi_{\mathfrak{v}}\,\rangle\!\rangle \;=\; \int_{\mathbb{T}^d} \mu(dx)\, \mathfrak{v}(x) \cdot a(x)\, \mathfrak{v}(x)\;. \tag{3.8}$$ b01

Denote by $\mathcal{V}$ the closure of this set in $\mathcal{A}$.

We claim that $\mathcal{H} = \mathcal{V}$. Indeed, since $\mathcal{V}$ is contained in $\mathcal{H}$, it is enough to show that for any smooth φ in $\mathcal{A}$ there exists a conservative vector field $\mathfrak{v}$ such that $(\nabla_y \varphi)(x,x) \;=\; (\nabla_y \varphi_{\mathfrak{v}})(x,x)$ for all $x \in \mathbb{T}^d$. Fix such a smooth function φ in $\mathcal{A}$, define the vector field $\mathfrak{v}$ by $\mathfrak{v}(x) = (\nabla_y \varphi)(x,x)$. It is clear that $(\nabla_y \varphi)(x,x) = (\nabla_y \varphi_{\mathfrak{v}})(x,x)$, which proves the claim. Thus, in the context of diffusions, the space $\mathcal{H}$ can be identified with the space of vector fields.

Jump processes.

Consider a family of jump rates $r_\epsilon(x, dy)$ on $\mathbb{R}^d$. Denote by $\pi_\epsilon(dx)$ the invariant measure. Let $\lambda_\epsilon(x) = r_\epsilon(x, \mathbb{R}^d) \in [0, \infty]$.

$$\langle\!\langle \xi \, , \, \xi' \rangle\!\rangle \;=\; \frac{1}{2} \int \pi(dx)\, (\mathbb{L}\xi\, \xi') \;:=\; \frac{1}{2} \int_{E\times E} \pi(dx)\, R(x, dy)\, \xi(x,y)\, \xi'(x,y)\ .$$

$$\mathrm{cap}(A, B) \;=\; \inf$$

$$\begin{aligned}
\Phi_f(\xi) \;&=\; \int \pi(dx)\, f(x)\, (\mathbb{L}\,\xi)(x) \\
&=\; - \int \pi(dx)\, r(x, dy)\, f(x)\, \xi(x,y) \\
&=\; \int \pi(dx)\, r_s(x, dy) \Big(\frac{f(y) - f(x)}{2} \Big)\, \xi(x,y) \\
&\quad -\; \int \pi(dx)\, r_a(x, dy) \Big(\frac{f(y) + f(x)}{2} \Big)\, \xi(x,y)
\end{aligned}$$

Now

$$r_a(x, dy) \;=\; \rho(x,y)\, r_s(x, dy)$$

last line

$$- \int \pi(dx)\, r_s(x, dy)\, \rho(x,y) \Big(\frac{f(y) + f(x)}{2} \Big)\, \xi(x,y)$$

$$(L_{\epsilon,\beta} f)(x) \;=\; \int \mu_\epsilon(x, dy)\, \frac{1}{2} [\, f(y) - f(x) \,]\, e^{-\beta q(x,y)}\ ,$$

where $q(x,y) = V(y) - V(x) + c(x,y)$ and c is symmetric and $\mu_\epsilon(x, dy) = [\mu(x + \epsilon dy) + \mu(x - \epsilon dy)]/2$.

4. REMAINS

Let $\mathcal{B}$ be the set given by

$$\mathcal{B} \;=\; \left\{ \eta \in C(E, D(L)) : e^{\eta} - 1 \in \mathcal{A} \right\} .$$

For $\eta \in \mathcal{B}$, define the operator $L^{\eta} : D(L^{\eta}) \to B$ given by

$$(L^{\eta} f)(x) \;=\; [\,\mathbb{L}\,(e^{\eta}\, df)\,](x) \;.$$

$$\eta \sim \eta' \quad if \quad L^{\eta} = L^{\eta'}$$

References

MR92 [1] Zhi Ming Ma and Michael Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms.* Universitext. Springer-Verlag, Berlin, 1992.