1 Setting

We fix a smooth bounded connected set $\Omega \subset \mathbb{R}^d$, a function $V \in C^2(\overline{\Omega}; \mathbb{R})$ and consider the differential operator

$$L_0 \;:=\; \frac{1}{2}\Delta \;-\; \nabla V \cdot \nabla \;.$$

More generally one may take

$$L_0 \;:=\; \frac{1}{2}\nabla \cdot a\nabla \;+\; b \cdot \nabla \;,$$

or even work with general Markov processes...

We denote by $u : [0, \infty) \times \overline{\Omega} \to \mathbb{R}$ the unique solution of

$$\begin{cases} \partial_t u\,(t,x) \;=\; L_0 u\,(t,x) & \text{for } (t,x) \in (0,\infty) \times \Omega \\ u(0,x) \;=\; 1 & \text{for } x \in \Omega \\ u(t,x) \;=\; 0 & \text{for } (t,x) \in (0,\infty) \times \partial\Omega \;. \end{cases}$$

Probabilistic interpretation: $u(t,x) \;=\; \mathbb{P}_x(\tau_\Omega > t)$.

Note that $u > 0$ and that $u(t,\cdot) \in C^2(\Omega; \mathbb{R})$. For each $0 \leq t \leq T$ let $U_{t,T} : \Omega \to \mathbb{R}$ be given by

$$U_{t,T}(x) \;:=\; \log u(T-t, x) \;,$$

and consider the differential operator

$$L_t^T \;:=\; L_0 \;+\; \nabla U_{t,T} \cdot \nabla \;.$$

For each fixed $T > 0$, we denote by $\{\mathbb{P}_x^T\}_{x\in\Omega}$ the family of Markov distributions induced by the time-dependent generator $L^T := (L_t^T)_{t\in[0,T]}$ on the path space $C([0,T]; \Omega)$.

Probabilistic interpretation: condition to not be absorbed until time T...

We define for each path $X \in C([0,T];\Omega)$ the empirical measure

$$\mu_T(X) \ := \ \frac{1}{T}\int_0^T \delta_{X_t}\,dt \ .$$

The goal is to study the asymptotic behaviour when $T \to \infty$ of the probability distribution P_x^T on $\mathcal{P}(\Omega)$ (the set of probability distributions on Ω) given by

$$P_x^T \ := \ \mathbb{P}_x^T \circ \mu_T^{-1} \ .$$

More precisely, we would like to show:

"Theorem"
$\{P_x^T\}_{T>0}$ satisfies for $T \to \infty$ a large deviation principle (uniformly for x in a compact ?) with speed T and rate function

$$I(\mu) \ = \ -\inf_{f \in C_c^2(\mathbb{R};(0,\infty))} \int \frac{Lf}{f}\,d\mu \ \ ,$$

where

$$L \ := \ L_0 \ + \ \nabla \log\varphi \cdot \nabla \ ,$$

and φ is characterized by $\varphi > 0$ and existence of $\lambda > 0$ such that

$$\begin{cases} -L_0\varphi\,(x) \ = \ \lambda\varphi & \text{for } x \in \Omega \\ \varphi(x) \ = \ 0 & \text{for } x \in \partial\Omega \ . \end{cases}$$

We denote in the sequel by $\{\mathbb{Q}_x^T\}_{x\in\Omega}$ the family of Markov distributions induced by the generator L on the path space $C([0,T];\Omega)$.

Probabilistic interpretation: process conditioned to never be absorbed, "Q-process".

Remark 1.1. *It follows from the usual Donsker-Varadhan principle that $Q_x^T := \mathbb{Q}_x^T \circ \mu_T^{-1}$ satisfies an LDP with rate I (**reference?**).*

2 proof

Upper bound

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do usual perturbation with gradient fields, the problem is: time dependent drift, how should the perturbing drift depend on t, T?

mi sembra ne avevamo discusso a luglio, dovrei riuscire a ricostruirlo.

Lower bound ....

Instead of doing upper and lower bound, maybe it is easier to show that Q_x^T and P_x^T are exponentially equivalent? Something like: let ν, η random elements in $\mathcal{P}(\Omega)$ define on the same probability space and having distribution respectively P_x and Q_x. We want to show that for every $\delta > 0$

$$\limsup_{T \to \infty} T^{-1} \log \mathscr{P}(|\nu - \eta| > \delta) \;=\; -\infty \;.$$

References