SCALING LIMIT OF SMALL RANDOM PERTURBATION OF DYNAMICAL SYSTEMS

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Abstract. In this article, we prove that a small random perturbation of dynamical system with multiple stable equilibria converges to a Markov chain whose states are neighborhoods of the deepest stable equilibria, under a suitable time-rescaling, provided that the perturbed dynamics is reversible in time. Such a result has been anticipated from 1970s, when the foundation of mathematical treatment for this problem has been established by Freidlin and Wentzell. We solve this long-standing problem by reducing the entire analysis to an investigation of the solution of an associated Poisson equation, and furthermore provide a method to carry out this analysis by using well-known test functions in a novel manner.

1. Introduction

Dynamical systems that are perturbed by small random noises are known to exhibit *metastable* behavior. There have been numerous progresses in the last two decades on the rigorous verification of metastability for a class of models that are collectively known as *Small Random Perturbation of Dynamical System (SRPDS)*. In this introductory section, we briefly review some of the existing results on SRPDS, and describe the main contribution of this article. We refer to a classical monograph [12] and a recent monograph [9] for the comprehensive discussion on SRPDS.

1.1. **Small random perturbation of dynamical systems: historical review.**

Consider a dynamical system given by the ordinary differential equation in $\mathbb{R}^d$

$$d\boldsymbol{x}(t) = b(\boldsymbol{x}(t))dt \ , \tag{1.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field. Suppose that this dynamical system owns multiple stable equilibria as illustrated in Figure 1.1, and consider the random dynamical system obtained by perturbing (1.1) with a small Brownian noise. Such a random dynamical system is defined by a stochastic differential equation of the form

$$d\boldsymbol{x}_\epsilon(t) = b(\boldsymbol{x}_\epsilon(t))dt + \sqrt{2\epsilon}\, d\boldsymbol{w}_t \ ; \ t \ge 0 \ , \tag{1.2}$$

where $(\boldsymbol{w}_t : t \ge 0)$ is the standard d-dimensional Brownian motion, and $\epsilon > 0$ is a small positive parameter representing the magnitude of the noise. Suppose now that the diffusion

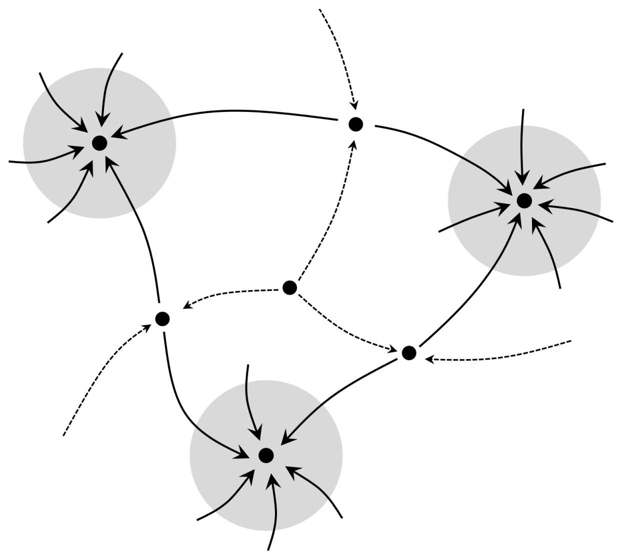

FIGURE 1.1. The flow chart of the dynamical systems $d\boldsymbol{x}(t) = -b(\boldsymbol{x}(t))dt$ with three stable equilibria. There are four unstable equilibria as well.

process $\boldsymbol{x}_\epsilon(t)$ starts from a neighborhood of a stable equilibrium of the unperturbed dynamics (1.1). Then, because of the small random noise, one can expect that the perturbed dynamics (1.2) exhibits a rare transition from this starting neighborhood to another one around different stable equilibrium. This is a typical metastable or tunneling transition and its quantitative analysis was originated from Freidlin and Wentzell [12, 13, 14]. However, beyond the large-deviation type estimate that was obtained by Freidlin and Wentzell (explained below), not much is known about the precise nature of the metastable behavior of the model (1.2), unless the drift b is a gradient vector field. For instance, we do not know of any sharp asymptotic for the expectation of the metastable transition time.

1.2. **Small random perturbation of dynamical systems: gradient model.** Suppose that the vector field b in (1.2) can be expressed as $b = -\nabla U$, for a smooth *potential* function $U : \mathbb{R}^d \to \mathbb{R}$. In other words, the stochastic differential equation (1.2) is of the form

$$d\boldsymbol{x}_\epsilon(t) = -\nabla U(\boldsymbol{x}_\epsilon(t))dt + \sqrt{2\epsilon}\, d\boldsymbol{w}_t \ ; \ t \ge 0 \ . \tag{1.3}$$

In particular, if the function $U(\cdot)$ has several local minima as illustrated in Figure 1.1, then the dynamical system associated with the unperturbed equation $d\boldsymbol{x}(t) = -\nabla U(\boldsymbol{x}(t))dt$, has multiple stable equilibria, and hence the diffusion process $(\boldsymbol{x}_\epsilon(t) : t \ge 0)$ is destined to exhibit a metastable behavior.

In order to explain some of the classical results obtained in [12, 13] by Freidlin and Wentzell in its simplest form, let us assume that U is a double-well potential. That is, the function U has exactly two local minima $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, and a saddle point $\boldsymbol{\sigma}$ between them, as illustrated

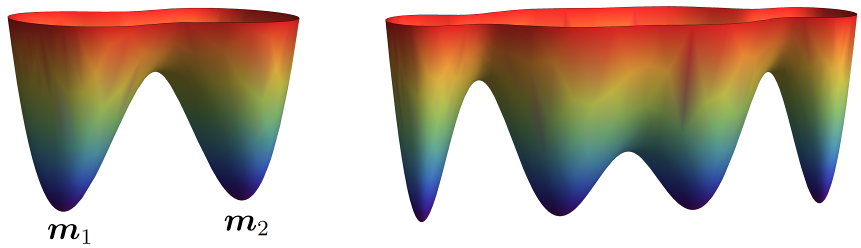

FIGURE 1.2. Potential U with two global minima $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ (left) and multiple global minima (right).

in Figure 1.2-(left). For such a choice of U, the diffusion $\boldsymbol{x}_\epsilon$, wonders mostly in one of the two *potential wells* surrounding $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$, and occasionally makes transitions from one well to the other. To understand the metastable nature of $\boldsymbol{x}_\epsilon$ qualitatively, we analyze the asymptotic behavior of the transition time of $\boldsymbol{x}_\epsilon$ between the two potential wells. Writing τ_ϵ for the time that it takes for $\boldsymbol{x}_\epsilon(t)$ to reach a small ball around $\boldsymbol{m}_2$, we wish to estimate the mean transition time $\mathbb{E}^\epsilon_{\boldsymbol{m}_1}[\tau_\epsilon]$, where $\mathbb{E}^\epsilon_{\boldsymbol{m}_1}$ denotes the expectation with respect to the law of $\boldsymbol{x}_\epsilon(t)$ starting from $\boldsymbol{m}_1$. Freidlin and Wentzell in [12, 13] establishes a large-deviation type estimate of the form

$$\log \mathbb{E}^\epsilon_{\boldsymbol{m}_1}[\tau_\epsilon] \;\simeq\; \frac{U(\boldsymbol{\sigma}) - U(\boldsymbol{m}_1)}{\epsilon} \quad \text{as } \epsilon \to 0 \; . \tag{1.4}$$

For the precise metastable behavior of $\boldsymbol{x}_\epsilon$, we need to go beyond (1.4) and evaluate the low ϵ limit of

$$\mathbb{E}^\epsilon_{\boldsymbol{m}_1}[\tau_\epsilon] \;\exp\left\{-\frac{U(\boldsymbol{\sigma}) - U(\boldsymbol{m}_1)}{\epsilon}\right\} \; .$$

This was achieved by Bovier et. al. in [8] by verifying a classical conjecture of Eyring [11] and Kramers [18]. By developing a robust methodology which is now known as *the potential theoretic approach*, Bovier et. al. derive an Eyring-Kramers type formula in the form

$$\mathbb{E}^\epsilon_{\boldsymbol{m}_1}[\tau_\epsilon] \simeq \frac{2\pi}{\lambda_\sigma}\sqrt{\frac{-\det(\nabla^2 U)(\boldsymbol{\sigma})}{\det(\nabla^2 U)(\boldsymbol{m}_1)}}\;\exp\left\{\frac{U(\boldsymbol{\sigma}) - U(\boldsymbol{m}_1)}{\epsilon}\right\} \quad \text{as } \epsilon \to 0 \; , \tag{1.5}$$

provided that the Hessians of U at $\boldsymbol{m}_1$, $\boldsymbol{m}_2$, and $\boldsymbol{\sigma}$ are non-degenerate, $(\nabla^2 U)(\boldsymbol{\sigma})$ has a unique negative eigenvalue $-\lambda_{\boldsymbol{\sigma}}$, and some additional technical assumptions on U (corresponding to (2.1) and (2.2) of the current paper) are valid. It is also verified in the same work that $\tau_\epsilon / \mathbb{E}^\epsilon_{\boldsymbol{m}_1}[\tau_\epsilon]$ converges to the mean-one exponential random variable. Similar formulas can be derived when U has multiple local minima as in Figure 1.2 (right).

1.3. **Main result.** We starts with an informal explanation of our main result when U is a double-well potential with $U(\boldsymbol{m}_1) = U(\boldsymbol{m}_2)$. Heuristically speaking, the process starting from a neighborhood of $\boldsymbol{m}_1$ makes a transition to that of $\boldsymbol{m}_2$ after an exponentially long time, as suggested by (1.4). After spending another exponentially long time, the process makes a transition back to the neighborhood of $\boldsymbol{m}_1$. These tunneling-type transitions take place repeatedly and may be explained in terms of a Markov chain among two valleys around $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. More generally, if U has several global minima as in Figure 1.2 (right), then the successive inter-valley dynamics seems to be approximated by a Markov chain whose states are the deepest valleys of U. In spite of the appeal of the above heuristic description, and its consistency with (1.4), its rigorous verification for our process (1.3) was not known before. In the main result of the current paper (Theorem 2.3), we show that after a rescaling of time, a finite state Markov chain governs the inner-valley dynamics of $\boldsymbol{x}_\epsilon$.

1.4. **Scaling limit of metastable random processes.** The most natural way to describe the inter-valley dynamics of metastable random processes is to demonstrate that their scaling limits are governed by finite state Markov chains whose jump rates are evaluated with the aid of Eyring-Kramers type formulas. Recently, there have been numerous active researches toward this direction, especially when the underlying metastable process lives in a discrete space. Beltran and Landim in [2, 3] provide a general framework, known as the *martingale approach* to obtain the scaling limit of metastable Markov chains. This method is quite robust and has been applied to a wide scope of metastable processes including the condensing zero-range processes [1, 4, 20, 32], the condensing simple inclusion processes [5, 17], the random walks in potential fields [24, 25], and the Potts models [26, 29].

The method of Beltran and Landim relies on a careful analysis of the so-called trace process. A trace process is obtained from the original process by turning off the clock when the process is not in a suitable neighborhood of a stable equilibrium. However, as Landim pointed out in [21], it is not clear how to apply this methodology when the underlying metastable process is a diffusion. In this paper, instead of modifying the approach outlined in [2, 3], we appeal to an entirely new method that is a refinement of a scheme that was utilized in [10, 31]. We establish the metastable behavior of our diffusion $\boldsymbol{x}_\epsilon$ by analyzing the solutions of certain classes of Poisson equations related to its infinitesimal generator. Theorem 4.1 is the main step of our approach and will play an essential role in the proof of our main result Theorem 2.3. The proof of Theorem 4.1 is to some extent model-dependent, though the deduction of the main result from this Theorem is robust and applicable to many other examples. Hence, we hope that our work reveals the importance of studying the Poisson equation of type (4.2) below in the study of scaling limit of metastable random processes.

1.5. **Non-gradient model.** As we mentioned earlier, except for the exponential estimate similar to (1.4), the analog of (1.5) is not known for the general case (1.2). Even for (1.4), the term $U(\boldsymbol{\sigma}) - U(\boldsymbol{m}_1)$ on the right-hand side is replaced with the so-called *quasi-potential* $V(\boldsymbol{\sigma}; \boldsymbol{m}_1)$. For the sake of comparison, let us describe three simplifying features of the diffusion (1.3) that play essential roles in our work:

- The quasi-potential function governing the rare behaviors of the process (1.3) is given by U. In general, the quasi-potential V is given by a variational principle in a suitable function space. For the metastability questions, we need to study the regularity of this quasi-potential that in general is a very delicate issue.
- The diffusion $\boldsymbol{x}_\epsilon$ of the equation (1.3) admits an invariant measure with a density of the form $Z_\epsilon^{-1} \exp\{-U/\epsilon\}$. For the general case, no explicit formula for the invariant measure is expected. The invariant measure density is specified as the unique solution of an elliptic PDE associated with the adjoint of the generator of (1.2).
- The diffusion $\boldsymbol{x}_\epsilon$ of the equation (1.3) is reversible with respect to its invariant measure. This is no longer the case for non-gradient models.

The main tool for proving the Eyring-Kramers formula for the gradient model (1.2) in [8] is the potential theory associated with reversible processes. Of course the special form of the invariant measure is also critically used, and hence its extension to general case requires non-trivial additional work. Recently, in [23] a potential theory for non-reversible processes is obtained, and accordingly the Eyring-Kramers formula is extended to a class of non-reversible diffusions with Gibbsian invariant measures. This result offers a meaningful advance to the general case.

The current work can be regarded as an entirely new alternative approach to the general case. Comparing to previous approaches, the main difference of ours is the fact that we do not rely on potential theory, especially the estimation of the capacity. Hence our approach does not rely on the reversibility of the process $\boldsymbol{x}_\epsilon$. Keeping in mind that one of main challenge of the non-reversible case is the estimation of the capacity between valleys, the methodology adopted in the current paper appears to be well-suited for treating non-reversible models. This possibility is partially verified in [27] by Landim and an author of the current paper. In this work, the scaling limit for the diffusion $\boldsymbol{x}_\epsilon$ of the equation (1.2) on a circle is obtained. It is worth mentioning that in the case of a circle, many simplifications and explicit computations are available. Nonetheless, the results of [27] demonstrates that the Eyring-Kramers' formula as well as the limiting Markov chain are very different from the reversible case, and many peculiar features are observed.

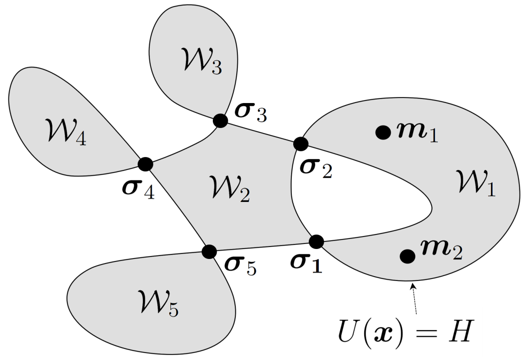

FIGURE 2.1. Shadow area represents Ω. For this case $S = \{1, 2, 3, 4, 5\}$, $\mathcal{S} = \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_4, \boldsymbol{\sigma}_5\}$, and $\mathcal{M}_1 = \{\boldsymbol{m}_1, \boldsymbol{m}_2\}$.

2. Model and Main result

Our main interest in this paper is the metastable behavior of the diffusion process (1.3) when the potential function U has multiple global minima. In Section 2.1, we explain basic assumptions on U and the geometric structure of its graph related to the metastable valleys and saddle points between them. In Section 2.2 some elementary results about the invariant measure of the process (1.3) is recalled. Finally, in Section 2.3 we describe the main result of the paper, which is a convergence theorem for the metastable process (1.3). We remark that the presentation and the result in the current section are similar to a discrete counterpart model considered in [24], though our proof of the main result is entirely different from the one that is presented therein.

2.1. **Potential function and its landscape.** We shall consider the potential function $U : \mathbb{R}^d \to \mathbb{R}$ that belongs to $C^2(\mathbb{R}^d)$, satisfying the growth condition

$$\lim_{|\boldsymbol{x}|\to\infty} \frac{U(\boldsymbol{x})}{|\boldsymbol{x}|} = \infty \;, \tag{2.1}$$

and the tightness condition

$$\int_{\{\boldsymbol{x}:U(\boldsymbol{x})\geq a\}} e^{-U(\boldsymbol{x})/\epsilon} dx \leq C_a e^{-a/\epsilon} \;\text{ for all } a \in \mathbb{R} \text{ and } \epsilon \in (0,\, 1] \;, \tag{2.2}$$

where C_a, $a \in \mathbb{R}$, is a constant that depends on a, but not on ϵ. These two conditions are required to confine the process $\boldsymbol{x}_\epsilon(t)$ in a compact region with high probability.

The metastable behavior of our model critically depends on the graphical structures of the level sets of the potential function U. To guarantee the occurrence of a metastable behavior of the type we have described in Section 1, we need to make some standard assumptions on U. We refer to Figure 2.1 for the visualization of some the notations that appear in the rest of the current section.

2.1.1. *Structure of the metastable wells.* Fix $H \in \mathbb{R}$ and let $\mathcal{S} = \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \cdots, \boldsymbol{\sigma}_L\}$ be the set of saddle points of U with height H, i.e.,

$$U(\boldsymbol{\sigma}_1) = U(\boldsymbol{\sigma}_2) = \cdots = U(\boldsymbol{\sigma}_L) = H \ .$$

Denote by $\mathcal{W}_1, \cdots, \mathcal{W}_K$ the connected components of the set

$$\Omega = \{\boldsymbol{x} : U(\boldsymbol{x}) < H\} \ . \tag{2.3}$$

Let us write $S = \{1, 2, \cdots, K\}$. By the growth condition (2.1), all the sets $\mathcal{W}_i$, $i \in S$, are bounded. We assume that $\overline{\Omega} = \cup_{i\in S}\overline{\mathcal{W}}_i$ is a connected set, where $\overline{\mathcal{A}}$ represents the topological closure of the set $\mathcal{A} \subset \mathbb{R}^d$.

Let h_i, $i \in S$, be the minimum of the function U in the well $\mathcal{W}_i$. We regard $H - h_i$ as the depth of the well $\mathcal{W}_i$. Define

$$h = \min_{i\in S} h_i \tag{2.4}$$

and let

$$S_\star = \{i \in S : h_i = h\} \subset S \ . \tag{2.5}$$

Note that the collection $\{\mathcal{W}_i : i \in S_\star\}$ represents the set of *deepest* wells. The purpose of the current article is to describe the metastable behavior of the diffusion process $\boldsymbol{x}_\epsilon(t)$ among these deepest wells. For a non-trivial result, we assume that $|S_\star| \ge 2$.

Remark 2.1. When the set $\overline{\Omega}$ is not connected, we can still apply our result to each connected component to get the metastability among the neighborhood of this component. In order to deduce the global result instead, one must find a larger H to unify the connected components. Because of this, our assumptions are quite general. For the details for such a multi-scale analysis, we refer to [24].

2.1.2. *Assumptions on the critical points of U.* For $i \in S$, define

$$\mathcal{M}_i = \{\boldsymbol{m} \in \mathcal{W}_i : U(\boldsymbol{m}) = h_i\}$$

which represents the set of minima of U in the set $\mathcal{W}_i$. We assume that $\mathcal{M}_i$ is a finite set for all $i \in S$. Define

$$\mathcal{M} = \bigcup_{i\in S} \mathcal{M}_i \ \text{ and } \ \mathcal{M}_\star = \bigcup_{i\in S_\star} \mathcal{M}_i \ , \tag{2.6}$$

so that the set $\mathcal{M}_\star$ denotes the set of global minima of U. We assume that those critical points of U that belong to $\mathcal{M}_\star \cup \mathcal{S}$ are non-degenerate, i.e., the Hessian of U is invertible at each point of $\mathcal{M}_\star \cup \mathcal{S}$. Furthermore, we assume that the Hessian $(\nabla^2 U)(\boldsymbol{\sigma})$ has one negative eigenvalue and $(d-1)$ positive eigenvalues for all $\boldsymbol{\sigma} \in \mathcal{S}$. These assumptions are standard in the study of metastability (cf. [8, 23, 24, 25]). In particular, they are satisfied if the function U is a Morse function .

2.1.3. *Metastable valleys.* Fix a small constant $a > 0$ such that there is no critical point $\boldsymbol{c}$ of U satisfying $U(\boldsymbol{c}) \in [H - a,\, H)$. For $i \in S$, denote by $\mathcal{W}_i^o$ the unique connected component of the level set $\{\boldsymbol{x} : U(\boldsymbol{x}) < H - a\}$ which is a subset of $\mathcal{W}_i$. We write $\mathcal{B}(\boldsymbol{x},\, r)$ for the ball of radius $r > 0$ centered at $\boldsymbol{x} \in \mathbb{R}^d$, i.e.,

$$\mathcal{B}(\boldsymbol{x},\, r) = \{\boldsymbol{y} \in \mathbb{R}^d : |\boldsymbol{x} - \boldsymbol{y}| < r\} \ . \tag{2.7}$$

Pick r_0' and r_0 with $0 < r_0 < r_0'$. Assume that r_0' is small enough so that the ball $\mathcal{B}(\boldsymbol{m},\, r_0')$ does not contain any critical points of U other than $\boldsymbol{m}$, and $\mathcal{B}(\boldsymbol{m},\, r_0') \subset \bigcup_{i \in S} \mathcal{W}_i^o$ for all $\boldsymbol{m} \in \mathcal{M}$. For $i \in S$, the metastable valley corresponding to the well $\mathcal{W}_i$ is defined by

$$\mathcal{V}_i = \bigcup_{\boldsymbol{m} \in \mathcal{M}_i} \mathcal{B}(\boldsymbol{m},\, r_0) \ . \tag{2.8}$$

For our purposes, we need to consider a larger valley

$$\mathcal{V}_i' = \bigcup_{\boldsymbol{m} \in \mathcal{M}_i} \mathcal{B}(\boldsymbol{m},\, r_0') \ . \tag{2.9}$$

Finally, we write

$$\mathcal{V}_\star = \bigcup_{i \in S_\star} \mathcal{V}_i \ , \ \ \text{and} \ \ \Delta = \mathbb{R}^d \setminus \mathcal{V}_\star \ . \tag{2.10}$$

2.2. **Invariant measure.** The generator corresponding to the diffusion process $\boldsymbol{x}_\epsilon(t)$ of the equation (1.3), can be written as

$$\mathscr{L}_\epsilon = \epsilon\, \Delta - \nabla U \cdot \nabla = \epsilon\, e^{U(\boldsymbol{x})/\epsilon}\, \nabla \cdot \left[e^{-U(\boldsymbol{x})/\epsilon} \nabla \right] \ .$$

From this, it is not hard to show that the invariant measure for the process $\boldsymbol{x}_\epsilon(\cdot)$ is given by

$$\mu_\epsilon(d\boldsymbol{x}) = Z_\epsilon^{-1} e^{-U(\boldsymbol{x})/\epsilon} d\boldsymbol{x} := \hat{\mu}_\epsilon(\boldsymbol{x})\ d\boldsymbol{x} \tag{2.11}$$

where Z_ϵ is the partition function defined by

$$Z_\epsilon = \int_{\mathbb{R}^d} e^{-U(\boldsymbol{x})/\epsilon} d\boldsymbol{x} < \infty \ .$$

Notice that Z_ϵ is finite because of (2.2). Define

$$\nu_i = \sum_{\boldsymbol{m} \in \mathcal{M}_i} \frac{1}{\sqrt{\det(\nabla^2 U)(\boldsymbol{m})}} \ \ \text{for } i \in S_\star \ \ \text{and} \ \ \nu_\star = \sum_{j \in S_\star} \nu_j \ . \tag{2.12}$$

We state some asymptotic results for the partition function Z_ϵ and the invariant measure $\mu_\epsilon(\cdot)$. We write $o_\epsilon(1)$ for a term that vanishes as $\epsilon \to 0$.

Proposition 2.2. *It holds that*

$$Z_\epsilon = (1+o_\epsilon(1))\,(2\pi\epsilon)^{d/2}\,e^{-h/\epsilon}\,\nu_\star\;, \tag{2.13}$$

$$\mu_\epsilon(\mathcal{V}_i) = (1+o_\epsilon(1))\,\frac{\nu_i}{\nu_\star}\;\; \textit{for } i\in S_\star\;, \tag{2.14}$$

$$\mu_\epsilon(\mathcal{V}_i') = (1+o_\epsilon(1))\,\frac{\nu_i}{\nu_\star}\;\; \textit{for } i\in S_\star\;, \tag{2.15}$$

$$\mu_\epsilon(\Delta) = o_\epsilon(1)\;. \tag{2.16}$$

Proof. By Laplace's method, we can deduce that, for $i\in S_\star$,

$$\mu_\epsilon(\mathcal{V}_i) = Z_\epsilon^{-1}\,(1+o_\epsilon(1))\,(2\pi\epsilon)^{d/2}\,e^{-h/\epsilon}\,\nu_i\;. \tag{2.17}$$

$$\mu_\epsilon(\mathcal{V}_i') = Z_\epsilon^{-1}\,(1+o_\epsilon(1))\,(2\pi\epsilon)^{d/2}\,e^{-h/\epsilon}\,\nu_i\;. \tag{2.18}$$

On the other hand, by (2.2), we have

$$\mu_\epsilon(\Delta) = Z_\epsilon^{-1}\,o_\epsilon(1)\,\epsilon^{d/2}\,e^{-h/\epsilon}\;. \tag{2.19}$$

Now, (2.13) follows from (2.17) and (2.19) because

$$1 = \mu_\epsilon(\Delta) + \sum_{i\in S_\star}\mu_\epsilon(\mathcal{V}_i)\;.$$

Moreoever, (2.14), (2.15) and (2.16) are obtained by inserting (2.13) into (2.17), (2.18), and (2.19), respectively. □

2.3. **Main result.** The metastable behavior of the process $\boldsymbol{x}_\epsilon(t)$ is a consequence of its convergence to a Markov chain $\mathbf{y}(t)$ on $S_\star$ in a proper sense, as is explained in Section 2.3.3 below. The Markov chain $\mathbf{y}(t)$ is defined in Section 2.3.2, based on an auxiliary Markov chain $\mathbf{x}(t)$ on S that is introduced below.

2.3.1. *Markov chain* $\mathbf{x}(t)$ *on* S. For a saddle point $\boldsymbol{\sigma}\in\mathcal{S}$, we write $-\lambda_{\boldsymbol{\sigma}}$ for the unique negative eigenvalue of the Hessian $(\nabla^2 U)(\boldsymbol{\sigma})$, and define

$$\omega_{\boldsymbol{\sigma}} = \frac{\lambda_{\boldsymbol{\sigma}}}{2\pi\sqrt{-\det(\nabla^2 U)(\boldsymbol{\sigma})}}\;.$$

For distinct $i,\,j\in S$, let $\mathcal{S}_{i,\,j}$ be the set of saddle points between wells $\mathcal{W}_i$ and $\mathcal{W}_j$ in the sense that

$$\mathcal{S}_{i,\,j} = \overline{\mathcal{W}}_i\cap\overline{\mathcal{W}}_j\subset\mathcal{S}\;.$$

Define

$$\omega_{i,\,j} = \sum_{\boldsymbol{\sigma}\in\mathcal{S}_{i,\,j}}\omega_{\boldsymbol{\sigma}}\;.$$

For convenience, we set $\omega_{i,\,i}=0$ for all $i\in S$. For $i\in S$, we define

$$\omega_i=\sum_{j\in S}\omega_{i,\,j}\ \text{ and }\ \mu(i)=\omega_i/(\sum_{j\in S}\omega_j)\ .$$

We have $\omega_i>0$ since the set $\overline{\Omega}$ is connected by our assumption. Denote by $\{\mathbf{x}(t):t\ge 0\}$ the continuous time Markov chain on S whose jump rate from $i\in S$ to $j\in S$ is given by $\omega_{i,\,j}/\mu(i)$. For $i\in S$, denote by $\mathbf{P}_i$ the law of the Markov chain $\mathbf{x}(t)$ starting from i. Notice that this Markov chain is reversible with respect to the probability measure $\mu(\cdot)$. The generator $L_{\mathbf{x}}$ corresponding to the chain $\mathbf{x}(t)$ can be written as,

$$(L_{\mathbf{x}}\mathbf{f})(i)=\sum_{j\in S}\frac{\omega_{i,\,j}}{\mu(i)}\,[\mathbf{f}(j)-\mathbf{f}(i)]\ \ ;\ i\in S\ ,$$

for $\mathbf{f}\in\mathbb{R}^S$. Define, for $\mathbf{f},\,\mathbf{g}\in\mathbb{R}^S$,

$$D_{\mathbf{x}}(\mathbf{f},\,\mathbf{g})=\sum_{i\in S}\mu(i)\,\mathbf{f}(i)(-L_{\mathbf{x}}\mathbf{g})(i)=\frac{1}{2}\sum_{i,\,j\in S}\omega_{i,\,j}\,[\mathbf{f}(j)-\mathbf{f}(i)]\,[\mathbf{g}(j)-\mathbf{g}(i)]\ . \tag{2.20}$$

Then, $D_{\mathbf{x}}(\mathbf{f},\,\mathbf{f})$ represents the Dirichlet form associated with the chain $\mathbf{x}(t)$.

Now we define the equilibrium potential and the capacity corresponding to the chain $\mathbf{x}(t)$. For $A\subseteq S$, denote by H_A the hitting time of the set A, i.e., $H_A=\inf\{t\ge 0:\mathbf{x}(t)\in A\}$. For two non-empty disjoint subsets A and B of S, define a function $\mathbf{h}_{A,\,B}:S\to[0,\,1]$ by

$$\mathbf{h}_{A,\,B}(i)=\mathbf{P}_i(H_A<H_B)\ . \tag{2.21}$$

The function $\mathbf{h}_{A,\,B}$ is called the equilibrium potential between two sets A and B with respect to the Markov chain $\mathbf{x}(t)$. One of the notable fact about the equilibrium potential is that, $\mathbf{h}_{A,\,B}$ can be characterized as the unique solution of the following equation:

$$\begin{cases}(L_{\mathbf{x}}\mathbf{h}_{A,\,B})(i)=0 & \text{for all } i\in(A\cup B)^c\ ,\\ \mathbf{h}_{A,\,B}(a)=1 & \text{for all } a\in A\ ,\\ \mathbf{h}_{A,\,B}(b)=0 & \text{for all } b\in B\ .\end{cases} \tag{2.22}$$

The capacity between these two sets A and B is now defined as

$$\mathrm{cap}_{\mathbf{x}}(A,\,B)=D_{\mathbf{x}}(\mathbf{h}_{A,\,B},\,\mathbf{h}_{A,\,B})\ .$$

2.3.2. *Markov chain $\mathbf{y}(t)$ on $S_\star$.* For distinct $i,\,j\in S_\star$, define

$$\beta_{i,\,j}=\frac{1}{2}\,[\mathrm{cap}_{\mathbf{x}}(\{i\},\,S_\star\setminus\{i\})+\mathrm{cap}_{\mathbf{x}}(\{j\},\,S_\star\setminus\{j\})-\mathrm{cap}_{\mathbf{x}}(\{i,\,j\},\,S_\star\setminus\{i,\,j\})] \tag{2.23}$$

and set $\beta_{i,\,i}=0$ for all $i\in S_\star$. Note that $\beta_{i,\,j}=\beta_{j,\,i}$ for all $i,\,j\in S$. Recall ν_i from (2.12) and let $\{\mathbf{y}(t):t\ge 0\}$ be a continuous time Markov chain on $S_\star$ whose jump rate from $i\in S_\star$ to $j\in S_\star$ is given by $\beta_{i,\,j}/\nu_i$. Denote by $\mathbf{Q}_i$, $i\in S_\star$, the law of Markov chain $\mathbf{y}(t)$ starting from

i. Notice that the probability measure $\mu_\star$ on $S_\star$, defined by

$$\mu_\star(i) = \frac{\nu_i}{\nu_\star} \quad \text{for } i \in S_\star \tag{2.24}$$

is the invariant measure for the Markov chain $\mathbf{y}(t)$. For $\mathbf{f} \in \mathbb{R}^{S_\star}$, the generator $L_\mathbf{y}$ corresponding to the Markov chain $\mathbf{y}(t)$ is given by

$$(L_\mathbf{y}\mathbf{f})(i) = \sum_{j\in S_\star} \frac{\beta_{i,j}}{\nu_i} [\mathbf{f}(j) - \mathbf{f}(i)] \ \ ; \ i \in S_\star \ .$$

Similar to (2.20), we define, for $\mathbf{f}$, $\mathbf{g} \in \mathbb{R}^{S_\star}$,

$$D_\mathbf{y}(\mathbf{f},\, \mathbf{g}) = \sum_{i\in S} \frac{\nu_i}{\nu_\star} \mathbf{f}(i)(-L_\mathbf{y}\mathbf{g})(i) = \frac{1}{2\nu_\star} \sum_{i,\, j\in S_\star} \beta_{i,j}\, [\mathbf{f}(j) - \mathbf{f}(i)]\, [\mathbf{g}(j) - \mathbf{g}(i)] \ .$$

We acknowledge here that a similar construction has been carried out in [30] at which a sharp asymptotics of the low-lying spectra of the metastable diffusions on σ-compact Riemannian manifold has been carried out for special form of the potential function U.

2.3.3. *Main result.* It is anticipated from (1.5) that the time scale corresponding to the metastable transition is given by

$$\theta_\epsilon = e^{(H-h)/\epsilon} \ . \tag{2.25}$$

Define the rescaled process $\{\widehat{\boldsymbol{x}}_\epsilon(t) : t \ge 0\}$ as of $\boldsymbol{x}_\epsilon(t)$

$$\widehat{\boldsymbol{x}}_\epsilon(t) = \boldsymbol{x}_\epsilon(\theta_\epsilon t) \ .$$

We now define the trace process $\boldsymbol{y}^\epsilon(t)$ of $\widehat{\boldsymbol{x}}^\epsilon(t)$ inside $\mathcal{V}_\star$. To this end, define the total time spent by $(\widehat{\boldsymbol{x}}_\epsilon(s) : \ s \in [0,t])$ in the valley $\mathcal{V}_\star$ as

$$T^\epsilon(t) = \int_0^t \chi_{\mathcal{V}_\star}(\widehat{\boldsymbol{x}}_\epsilon(s))ds \ \ ; \ t \ge 0 \ ,$$

where the function $\chi_\mathcal{A} : \mathbb{R}^d \to \{0,\, 1\}$ represents the characteristic function of $\mathcal{A} \subseteq \mathbb{R}^d$. Then, define

$$S^\epsilon(t) = \sup\{s \ge 0 : T^\epsilon(s) \le t\} \ \ ; \ t \ge 0 \ , \tag{2.26}$$

which is the generalized inverse of the increasing function $T^\epsilon(\cdot)$. Finally, the trace process of $\widehat{\boldsymbol{x}}_\epsilon(t)$ in the set $\mathcal{V}_\star$ is defined by

$$\boldsymbol{y}_\epsilon(t) = \widehat{\boldsymbol{x}}_\epsilon(S^\epsilon(t)) \ \ ; \ t \ge 0 \ . \tag{2.27}$$

One can readily verify that $\boldsymbol{y}_\epsilon(t) \in \mathcal{V}_\star$ for all $t \ge 0$. Define a projection function $\Psi : \mathcal{V}_\star \to S_\star$ by

$$\Psi(\boldsymbol{x}) = \sum_{i\in S_\star} i\, \chi_{\mathcal{V}_i}(\boldsymbol{x}) \ . \tag{2.28}$$

Since $\boldsymbol{y}_\epsilon(t)$ is always in the set $\mathcal{V}_\star$, the following process is well-defined:

$$\mathbf{y}_\epsilon(t) = \Psi(\boldsymbol{y}_\epsilon(t)) \ ; \ t \geq 0 \ . \tag{2.29}$$

The process $\mathbf{y}_\epsilon(t)$ represents the index of the valley in which the process $\boldsymbol{y}_\epsilon(t)$ is residing. Denote by $\mathbb{P}^\epsilon_{\boldsymbol{x}}$ and $\widehat{\mathbb{P}}^\epsilon_{\boldsymbol{x}}$ the law of processes $\boldsymbol{x}_\epsilon(\cdot)$ and $\widehat{\boldsymbol{x}}_\epsilon(\cdot)$ starting from $\boldsymbol{x} \in \mathbb{R}^d$, respectively, and denote by $\mathbb{E}^\epsilon_{\boldsymbol{x}}$ and $\widehat{\mathbb{E}}^\epsilon_{\boldsymbol{x}}$ the corresponding expectations. For $\boldsymbol{x} \in \mathcal{V}_\star$, denote by $\mathbf{Q}^\epsilon_{\boldsymbol{x}}$ the law of process $\mathbf{y}^\epsilon(\cdot)$ when the underlying diffusion process $\boldsymbol{x}_\epsilon(t)$ follows $\mathbb{P}^\epsilon_{\boldsymbol{x}}$, i.e.,

$$\mathbf{Q}^\epsilon_{\boldsymbol{x}} = \widehat{\mathbb{P}}^\epsilon_{\boldsymbol{x}} \circ \Psi^{-1} \ .$$

For any Borel probability measure π on $\mathcal{V}_\star$, we denote by $\mathbb{P}^\epsilon_\pi$ the law of process $\boldsymbol{x}_\epsilon(\cdot)$ with initial distribution π. Then, define $\widehat{\mathbb{P}}^\epsilon_\pi$, $\mathbb{E}^\epsilon_\pi$, $\widehat{\mathbb{E}}^\epsilon_\pi$, and $\mathbf{Q}^\epsilon_\pi$ similarly as above. We are now ready to state the main result of this article:

Theorem 2.3. *For all $i \in S_\star$ and for any sequence of Borel probability measures $(\pi_\epsilon)_{\epsilon>0}$ concentrated on $\mathcal{V}_i$, the sequence of probability laws $(\mathbf{Q}^\epsilon_{\pi_\epsilon})_{\epsilon>0}$ converges to $\mathbf{Q}_i$, the law of the Markov process $(\mathbf{y}(t))_{t\geq 0}$ starting from i, as ϵ tends to 0.*

We finish this section by explaining the organization of the rest of the paper. In Section 3, we construct a class of test functions which are useful in some of the computations we carry out in Section 4. In Section 4, we analyze a Poisson equation that will play a crucial role in the proof of both the *tightness* in Section 5, and the *uniqueness of the limit point* in Section 6. These two ingredients complete the proof of the convergence result stated in Theorem 2.3, as we will demonstrate in Section 6.

3. Test functions

The purpose of the current section is to construct some test functions. We acknowledge that these functions are not new; similar functions have already been used in [7] and [23] in order to obtain sharp estimates on the capacity associated with pairs of valleys. Hence we refer to those papers for some proofs. We also remark here that the way we utilize these test functions will be entirely different from how they are used in [7] and [23]. We use these functions to estimate the value of a solution of our *Poisson Problem* in each valley (see Theorem 4.1).

3.1. **Neighborhoods of saddle points.** We now introduce some subsets of $\mathbb{R}^d$ related to the inter-valley structure of U. For each saddle point $\boldsymbol{\sigma} \in \mathcal{S}$, denote by $-\lambda^{\boldsymbol{\sigma}}_1$ the unique negative eigenvalue of $(\nabla^2 U)(\boldsymbol{\sigma})$, and by $\lambda^{\boldsymbol{\sigma}}_2, \cdots \lambda^{\boldsymbol{\sigma}}_d$ the positive eigenvalues of $(\nabla^2 U)(\boldsymbol{\sigma})$. We choose unit eigenvectors $\boldsymbol{v}^{\boldsymbol{\sigma}}_1, \dots \boldsymbol{v}^{\boldsymbol{\sigma}}_d$ of $(\nabla^2 U)(\boldsymbol{\sigma})$ corresponding to the eigenvalues $-\lambda^{\boldsymbol{\sigma}}_1, \lambda^{\boldsymbol{\sigma}}_2, \dots, \lambda^{\boldsymbol{\sigma}}_d$.

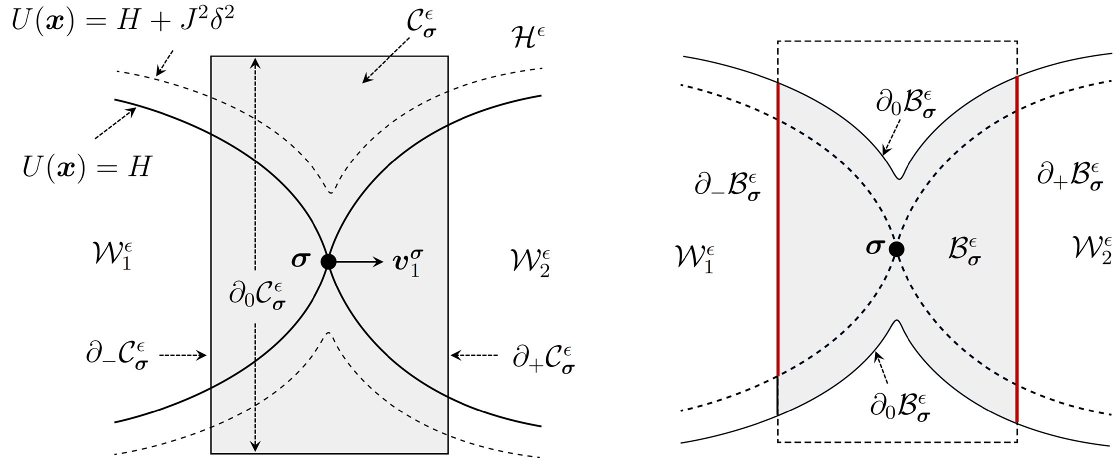

FIGURE 3.1. Visualization of a neighborhood of $\boldsymbol{\sigma} \in \mathcal{S}_{1,2}$.

Remark 3.1. Some care is needed as we select the direction of $\boldsymbol{v}_1^{\boldsymbol{\sigma}}$. If $\boldsymbol{\sigma} \in \mathcal{S}_{i,j}$ for some $i < j$, we choose $\boldsymbol{v}_1^{\boldsymbol{\sigma}}$ to be directed toward the valley $\mathcal{W}_j$. Formally stating, we assume that $\boldsymbol{\sigma} + \alpha \boldsymbol{v}_1^{\boldsymbol{\sigma}} \in \mathcal{W}_j$ for all sufficiently small $\alpha > 0$.

We define

$$\delta = \delta(\epsilon) = \sqrt{\epsilon \log(1/\epsilon)}\ . \tag{3.1}$$

A closed box $\mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon}$ around the saddle point $\boldsymbol{\sigma}$ is defined by

$$\mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} = \left\{ \boldsymbol{\sigma} + \sum_{i=1}^{d} \alpha_i \boldsymbol{v}_i^{\boldsymbol{\sigma}} : \alpha_1 \in \left[-\frac{J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}}, \frac{J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \right] \text{ and } \alpha_i \in \left[-\frac{2J\delta}{\sqrt{\lambda_i^{\boldsymbol{\sigma}}}}, \frac{2J\delta}{\sqrt{\lambda_i^{\boldsymbol{\sigma}}}} \right] \text{ for } 2 \le i \le d \right\} \ ,$$

where J is a constant which is larger than $\sqrt{12d}$. We refer to Figure 3.1 for the illustration of the sets defined in this subsection.

Notation 3.2. We summarize the notations used in the remaining of the paper. We regard J as a constant so that the terms like $o_\epsilon(1)$, $O(\delta^2)$ may depend on J as well. All the constants without subscript or superscript ϵ are independent of ϵ (and hence of δ) but may depend on J or the function U. Constants are usually denoted by c or C and different appearances may take different values.

Decompose the boundary $\partial \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon}$ into

$$\partial_+ \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} = \left\{ \boldsymbol{\sigma} + \sum_{i=1}^{d} \alpha_i \boldsymbol{v}_i^{\boldsymbol{\sigma}} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} : \alpha_1 = \frac{J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \right\} \ ,$$

$$\partial_- \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} = \left\{ \boldsymbol{\sigma} + \sum_{i=1}^{d} \alpha_i \boldsymbol{v}_i^{\boldsymbol{\sigma}} \in \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} : \alpha_1 = -\frac{J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \right\} \ , \ \text{ and } \ \partial_0 \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} = \partial \mathcal{C}_{\epsilon}^{\boldsymbol{\sigma}} \setminus (\partial_+ \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} \cup \partial_- \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon})\ .$$

The following is a direct consequence of a Taylor expansion of U around $\boldsymbol{\sigma}$, since $U(\boldsymbol{\sigma}) = H$.

Lemma 3.3. *For all $\boldsymbol{x} \in \partial_0 \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon}$, we have that*

$$U(\boldsymbol{x}) \geq H + (1 + o_\epsilon(1)) \, \frac{3J^2\delta^2}{2} \ .$$

Proof. This follows from the Taylor expansion of U at σ (see [23, Lemma 6.1]). □

Now we define

$$\mathcal{H}^\epsilon = \left\{ \boldsymbol{x} \in \mathbb{R}^d : U(\boldsymbol{x}) \leq H + J^2\delta^2 \right\} \ ,$$

and let $\mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} = \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} \cap \mathcal{H}^\epsilon$ for $\boldsymbol{\sigma} \in \mathcal{S}$. Decompose the boundary $\partial \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon}$ as

$$\partial_+ \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} = \partial \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \cap \partial_+ \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} \ , \ \ \partial_- \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} = \partial \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \cap \partial_- \mathcal{C}_{\boldsymbol{\sigma}}^{\epsilon} \ , \text{ and } \ \partial_0 \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} = \partial \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \setminus (\partial_+ \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \cup \partial_- \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon}) \ . \tag{3.2}$$

Then, by Lemma 3.3, for small enough ϵ, we have

$$U(\boldsymbol{x}) = H + J^2\delta^2 \text{ for all } \boldsymbol{x} \in \partial_0 \mathcal{B}_{\epsilon}^{\boldsymbol{\sigma}} \ . \tag{3.3}$$

Thus, the set $\mathcal{H}^\epsilon \setminus \bigcup_{\boldsymbol{\sigma} \in \mathcal{S}} \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon}$ consists of K connected components $\mathcal{W}_1^\epsilon, \cdots, \mathcal{W}_K^\epsilon$ such that $\mathcal{V}_i' \subset \mathcal{W}_i^\epsilon$ for all $i \in S$. Furthermore, if $\boldsymbol{\sigma} \in \mathcal{S}_{i,j}$ with $i < j$, then by Remark 3.1 we have that

$$\partial_- \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \subset \partial \mathcal{W}_i^\epsilon \ \text{ and } \ \partial_+ \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \subset \partial \mathcal{W}_j^\epsilon \ . \tag{3.4}$$

We shall assume from now on that $\epsilon > 0$ is small enough so that the construction above is in force.

3.2. **Test function and basic estimates.** For $\boldsymbol{\sigma} \in \mathcal{S}$, define a normalizing constant $c_\epsilon^{\boldsymbol{\sigma}}$ by

$$c_\epsilon^{\boldsymbol{\sigma}} = \int_{-J\delta/\sqrt{\lambda_1^{\boldsymbol{\sigma}}}}^{J\delta/\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \sqrt{\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\pi\epsilon}} \exp\left\{ -\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\epsilon} t^2 \right\} \, dt = 1 + o_\epsilon(1) \ , \tag{3.5}$$

and define a function $f_{\boldsymbol{\sigma}}^{\epsilon}(\cdot)$ on $\mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon}$ by,

$$f_\epsilon^{\boldsymbol{\sigma}}(\boldsymbol{x}) = (c_\epsilon^{\boldsymbol{\sigma}})^{-1} \int_{-J\delta/\sqrt{\lambda_1^{\boldsymbol{\sigma}}}}^{(\boldsymbol{x}-\boldsymbol{\sigma})\cdot \boldsymbol{v}_1^{\boldsymbol{\sigma}}} \sqrt{\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\pi\epsilon}} \exp\left\{ -\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\epsilon} t^2 \right\} dt \ \ ; \ \boldsymbol{x} \in \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \ . \tag{3.6}$$

By (3.5) we have

$$f_\epsilon^{\boldsymbol{\sigma}}(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \partial_- \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \\ 1 & \text{if } \boldsymbol{x} \in \partial_+ \mathcal{B}_{\boldsymbol{\sigma}}^{\epsilon} \end{cases} \ . \tag{3.7}$$

We next investigate two basic properties of $f_\epsilon^{\boldsymbol{\sigma}}$ in Lemmas 3.4 and 3.5 below. The statement and the proof of the first lemma is similar to those of [23, Lemma 8.7] (in terms of the notations of [23], our model corresponding to the special case $\mathbb{M} = \mathbb{I}$, where $\mathbb{I}$ denotes the identity matrix). Since the proof is much simpler for our specific case, and some of the computations carried out below will be useful later, we give the full proof of this lemma.

Lemma 3.4. *For all $\boldsymbol{\sigma} \in \mathcal{S}$, we have that*

$$\theta_\epsilon \int_{\mathcal{C}_{\boldsymbol{\sigma}}^\epsilon} |(\mathscr{L}_\epsilon f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x})| \, \hat{\mu}_\epsilon(\boldsymbol{x}) \, d\boldsymbol{x} = o_\epsilon(1) \, . \tag{3.8}$$

Proof. To ease the notation, we may assume that $\boldsymbol{\sigma} = 0$. For $\boldsymbol{x} \in \mathcal{C}_{\boldsymbol{\sigma}}^\epsilon$, write $\alpha_i := \alpha_i(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{v}_i^{\boldsymbol{\sigma}}$ so that $\boldsymbol{x} = \sum_{i=1}^d \alpha_i \boldsymbol{v}_i^{\boldsymbol{\sigma}}$. By elementary computations, we can write

$$(\mathscr{L}_\epsilon f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x}) = -\frac{1}{c_\epsilon^{\boldsymbol{\sigma}}} \sqrt{\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\pi\epsilon}} e^{-\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\epsilon}\alpha_1^2} \left[(\nabla U(\boldsymbol{x}) + \lambda_1^{\boldsymbol{\sigma}} \boldsymbol{x}) \cdot \boldsymbol{v}_1^{\boldsymbol{\sigma}} \right] \, . \tag{3.9}$$

By the Taylor expansion of ∇U around $\boldsymbol{\sigma}$, we have

$$\nabla U(\boldsymbol{x}) + \lambda_1^{\boldsymbol{\sigma}} \boldsymbol{x} = (\nabla^2 U)(\boldsymbol{\sigma}) \, \boldsymbol{x} + O(\delta^2) + \lambda_1^{\boldsymbol{\sigma}} \boldsymbol{x} = \sum_{i=2}^d (\alpha_i \lambda_i + \alpha_i \lambda_1)^{\boldsymbol{\sigma}} \boldsymbol{v}_i^{\boldsymbol{\sigma}} + O(\delta^2) \, . \tag{3.10}$$

Since $\boldsymbol{v}_1^{\boldsymbol{\sigma}} \cdot \boldsymbol{v}_i^{\boldsymbol{\sigma}} = 0$ for $2 \le i \le d$, we conclude from (3.9) and (3.10) that

$$(\mathscr{L}_\epsilon f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x}) = O(\delta^2) \, \epsilon^{-\frac{1}{2}} \, \exp\left\{ -\frac{\lambda_1^{\boldsymbol{\sigma}}}{2\epsilon} \alpha_1^2 \right\} \, . \tag{3.11}$$

Therefore, the left-hand side of (3.8) is bounded above by

$$\begin{aligned} & O(\delta^2) \, \theta_\epsilon \, \epsilon^{-\frac{1}{2}} Z_\epsilon^{-1} \int_{\mathcal{C}_{\boldsymbol{\sigma}}^\epsilon} \exp\left\{ -\frac{U(\boldsymbol{x}) + (1/2)\lambda_1^{\boldsymbol{\sigma}} \alpha_1^2}{\epsilon} \right\} d\boldsymbol{x} \\ & \qquad = O(\delta^2) \, \theta_\epsilon \, \epsilon^{-\frac{1}{2}} Z_\epsilon^{-1} e^{-\frac{H}{\epsilon}} \int_{\mathcal{C}_{\boldsymbol{\sigma}}^\epsilon} \exp\left\{ -\frac{1}{2\epsilon} \sum_{i=2}^d \lambda_i^{\boldsymbol{\sigma}} \alpha_i^2 \right\} d\boldsymbol{x} \, , \end{aligned} \tag{3.12}$$

where the identity follows from the second-order Taylor expansion of U around $\boldsymbol{\sigma}$ and the fact that $O(\delta^3/\epsilon) = o_\epsilon(1)$. By the change of variables, the last integral can be bounded as

$$\begin{aligned} & \frac{2J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \int_{-2J\delta/\sqrt{\lambda_2^{\boldsymbol{\sigma}}}}^{2J\delta/\sqrt{\lambda_2^{\boldsymbol{\sigma}}}} \cdots \int_{-2J\delta/\sqrt{\lambda_d^{\boldsymbol{\sigma}}}}^{2J\delta/\sqrt{\lambda_d^{\boldsymbol{\sigma}}}} \exp\left\{ -\frac{1}{2\epsilon} \sum_{i=2}^d \lambda_i^{\boldsymbol{\sigma}} \alpha_i^2 \right\} d\alpha_2 \cdots d\alpha_d \\ & \qquad \le \epsilon^{\frac{d-1}{2}} \frac{2J\delta}{\sqrt{\lambda_1^{\boldsymbol{\sigma}}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} \sum_{i=2}^d \lambda_i^{\boldsymbol{\sigma}} y_i^2 \right\} dy_2 \cdots dy_d = C \epsilon^{\frac{d-1}{2}} \delta \, . \end{aligned}$$

Inserting this into (3.12) finishes the proof. □

Lemma 3.5. *For all $\boldsymbol{\sigma} \in \mathcal{S}$, we have*

$$\theta_\epsilon \, \epsilon \int_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon} |\nabla f_\epsilon^{\boldsymbol{\sigma}}(\boldsymbol{x})|^2 \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = (1 + o_\epsilon(1)) \, \nu_\star^{-1} \, \omega_{\boldsymbol{\sigma}} \, .$$

Proof. See [23, Lemma 8.4]. □

For $\mathbf{q} = (\mathbf{q}(i) : i \in S) \in \mathbb{R}^S$, we now define a test function $F_\epsilon^{\mathbf{q}} : \mathbb{R}^d \to \mathbb{R}$. This test function is used in Sections 4 and 5. In particular, in Section 4, the vector $\mathbf{q}$ may depend on ϵ. For this reason, we will keep track of the dependence of the constants on $\mathbf{q}$ in the inequalities that appear in this section.

We start by defining a real-valued function $\widehat{F}_\epsilon^{\mathbf{q}}$ on $\mathcal{H}^\epsilon$. This function is defined by

$$\widehat{F}_\epsilon^{\mathbf{q}}(\boldsymbol{x}) = \begin{cases} \mathbf{q}(i) & \text{if } \boldsymbol{x} \in \mathcal{W}_i^\epsilon \ , \ i \in S \ , \\ \mathbf{q}(i) + (\mathbf{q}(j) - \mathbf{q}(i)) f_\epsilon^{\boldsymbol{\sigma}}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \mathcal{B}_{\boldsymbol{\sigma}}^\epsilon \ , \ \boldsymbol{\sigma} \in \mathcal{S}_{i,j} \text{ with } i < j \ . \end{cases} \tag{3.13}$$

By (3.7), the function $\widehat{F}_\epsilon^{\mathbf{q}}$ is continuous on $\mathcal{H}^\epsilon$. Evidently,

$$\|\widehat{F}_\epsilon^{\mathbf{q}}\|_{L^\infty(\mathcal{H}^\epsilon)} \le \|\mathbf{q}\|_\infty := \max\{|\mathbf{q}(i)| : i \in S\} \ . \tag{3.14}$$

Furthermore, since $\|\nabla f_\epsilon^{\boldsymbol{\sigma}}\| \le C\,\epsilon^{-1/2}$, we deduce that the function $\widehat{F}_\epsilon^{\mathbf{q}}$ satisfies

$$\|\nabla \widehat{F}_\epsilon^{\mathbf{q}}\|_{L^\infty(\mathcal{H}^\epsilon)} \le C\,\epsilon^{-1/2} \max\{|\mathbf{q}(i) - \mathbf{q}(j)| : i,\, j \in S\} \le C\,\epsilon^{-1/2} [D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q})]^{1/2} \ . \tag{3.15}$$

Here we stress that the constant C is independent of $\mathbf{q}$.

Let $\mathcal{K}$ be a compact set containing $\mathcal{H}^\epsilon$ for all $\epsilon \in (0,\, 2]$. For instance, one can select $\mathcal{K} = \mathcal{H}^a$ for any $a > 2$. Then, for $\epsilon \in (0,\, 1]$, by (3.14) and (3.15), there exists a continuous extension $F_\epsilon^{\mathbf{q}} : \mathbb{R}^d \to \mathbb{R}$ of $\widehat{F}_\epsilon^{\mathbf{q}}$ satisfying

$$\operatorname{supp}\, F_\epsilon^{\mathbf{q}} \subset \mathcal{K} \ , \quad \|F_\epsilon^{\mathbf{q}}\|_{L^\infty(\mathbb{R}^d)} \le \|\mathbf{q}\|_\infty \ , \text{and } \ \|\nabla F_\epsilon^{\mathbf{q}}\|_{L^\infty(\mathbb{R}^d)} \le C\,\epsilon^{-1/2} [D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q})]^{1/2} \ . \tag{3.16}$$

Suppose from now on that ϵ is not larger than 1 so that we can define $F_\epsilon^{\mathbf{q}}$ satisfying (3.16).

Note that the Dirichlet form $\mathscr{D}_\epsilon(\cdot)$ corresponding to the process $\boldsymbol{x}_\epsilon(t)$ is given by

$$\mathscr{D}_\epsilon(f) = \epsilon \int_{\mathbb{R}^d} |\nabla f(\boldsymbol{x})|^2 \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} \ \ ; \ f \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \ . \tag{3.17}$$

Lemma 3.6. *For all $\mathbf{q} = (\mathbf{q}(i) : i \in S) \in \mathbb{R}^K$, we have that*

$$\theta_\epsilon\, \mathscr{D}_\epsilon(F_\epsilon^{\mathbf{q}}) = (1 + o_\epsilon(1))\, \nu_\star^{-1} D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q}) \ .$$

Proof. It is immediate from Lemma 3.5 that

$$\theta_\epsilon\, \epsilon \int_{\mathcal{H}^\epsilon} |\nabla F_\epsilon^{\mathbf{q}}(\boldsymbol{x})|^2 \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = (1 + o_\epsilon(1))\, \nu_\star^{-1}\, D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q}) \ .$$

Thus, it suffices to show that

$$\theta_\epsilon\, \epsilon \int_{(\mathcal{H}^\epsilon)^c} |\nabla F_\epsilon^{\mathbf{q}}(\boldsymbol{x})|^2 \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = o_\epsilon(1)\, D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q}) \ . \tag{3.18}$$

Since $\nabla F_\epsilon^{\mathbf{q}} \equiv 0$ on $\mathcal{K}^c$ we can replace the domain of integration in (3.18) with $\mathcal{K} \setminus \mathcal{H}_\epsilon$. Then, by (3.16), (2.13), and by the fact that $U(\boldsymbol{x}) \ge H + J^2\delta^2$ for $\boldsymbol{x} \notin \mathcal{H}_\epsilon$,

$$\theta_\epsilon\, \epsilon \int_{\mathcal{K} \setminus \mathcal{H}^\epsilon} |\nabla F_\epsilon^{\mathbf{q}}(\boldsymbol{x})|^2 \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} \le\ m_d(\mathcal{K})\, D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q})\, \theta_\epsilon\, Z_\epsilon^{-1}\, e^{-H/\epsilon} \epsilon^{J^2} \ \le C\, D_{\mathbf{x}}(\mathbf{q},\, \mathbf{q})\, \epsilon^{J^2 - (d/2)} \ ,$$

where $m_d(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$. This completes the proof since $J > \sqrt{12d}$. $\square$

4. A Poisson equation

For each $i \in S_\star$, we pick a smooth function $\zeta^i : \mathbb{R}^d \to \mathbb{R}$ such that $0 \le \zeta^i \le 1$, $\zeta^i = 1$ on $\mathcal{V}_i$, and $\zeta^i = 0$ on the complement of $\mathcal{V}_i'$. We also set

$$\bar{\zeta}^i = \int \zeta^i(\boldsymbol{x}) \, d\boldsymbol{x}.$$

Define $\mathbf{a}_\epsilon = (\mathbf{a}_\epsilon(i) : i \in S_\star) \in \mathbb{R}^{S_\star}$ by

$$\mathbf{a}_\epsilon(i) = \frac{Z_\epsilon^{-1} \, (2\pi\epsilon)^{d/2} \, e^{-h/\epsilon} \, \nu_i}{\bar{\zeta}_i} \ \ ; \ i \in S_\star \ .$$

From $\mu_\epsilon(\mathcal{V}_i) \le \bar{\zeta}^i \le \mu_\epsilon(\mathcal{V}_i')$, and (2.17), we learn

$$\mathbf{a}_\epsilon(i) = 1 + o_\epsilon(1) \text{ for all } i \in S_\star \ . \tag{4.1}$$

The main result of the current section is stated in the following theorem.

Theorem 4.1. *For all $\mathbf{f} : S_\star \to \mathbb{R}$, there exists a bounded function $\phi_\epsilon = \phi_\epsilon^{\mathbf{f}} : \mathbb{R}^d \to \mathbb{R}$ satisfying all the following properties:*

(1) $\phi_\epsilon \in C^2$.

(2) *ϕ_ϵ satisfies the equation*

$$\theta_\epsilon \mathscr{L}_\epsilon \phi_\epsilon = \sum_{i \in S_\star} \mathbf{a}_\epsilon(i) \, (L_{\mathbf{y}} \mathbf{f})(i) \, \zeta^i \ . \tag{4.2}$$

(3) *For all $i \in S_\star$, it holds that*

$$\lim_{\epsilon \to \infty} \sup_{\boldsymbol{x} \in \mathcal{V}_i} |\phi_\epsilon(\boldsymbol{x}) - \mathbf{f}(i)| = 0 \ . \tag{4.3}$$

Remark 4.2. In [31, Theorem 5.3], a similar analysis has been carried out for a slightly different situation. In [31], we treat a Poisson equation of the form (4.2) for a different right-hand side. The form of the right-hand we have chosen in (4.2) enables us to use the Poincare's inequality (see subsection 4.3 below). Furthermore, the proof therein relies on the capacity estimates between metastable valleys. Our proof though does not use any capacity estimates and has a chance to be applicable to the non-reversible variant of our model. This fact deserves to be highlighted here once more.

Note that the function $L_{\mathbf{y}}\mathbf{f} : S_\star \to \mathbb{R}$ satisfies

$$\sum_{i \in S_\star} (L_{\mathbf{y}} \mathbf{f})(i) \, \mu_\star(i) = 0 \tag{4.4}$$

since $\mu_\star(\cdot)$ defined in (2.24) is the invariant measure for the Markov chain $\mathbf{y}(t)$. Let $\mathbf{e}_i \in \mathbb{R}^{S_\star}$, $i \in S_\star$, be the ith unit vector defined by

$$\mathbf{e}_i(j) = \mathbf{1}\{i = j\} \ \ ; \ j \in S_\star \ . \tag{4.5}$$

For $i,\, j\in S_\star$, let $\mathbf{S}_{i,j}$ be the collection of $\mathbf{f}\in\mathbb{R}^{S_\star}$ satisfying

$$L_{\mathbf{y}}\mathbf{f}=\frac{1}{\mu_\star(i)}\mathbf{e}_i-\frac{1}{\mu_\star(j)}\mathbf{e}_j=\frac{\nu_\star}{\nu_i}\mathbf{e}_i-\frac{\nu_\star}{\nu_j}\mathbf{e}_j\ .$$

Remark that the selection $\mathbf{S}_{i,j}$ is consistent with the condition (4.4) for $L_{\mathbf{y}}\mathbf{f}$. It is immediate from the irreducibility of the Markov chain $\mathbf{y}(t)$ that $\bigcup_{i,\,j\in S_\star}\mathbf{S}_{i,j}$ spans whole space $\mathbb{R}^{S_\star}$. Note that for $\mathbf{f}\equiv 0$, it suffices to select $\phi_\epsilon\equiv 0$ and thus it suffices to consider non-zero $\mathbf{f}$. Therefore, by the linearity of the statement of Theorem 4.1 with respect to $\mathbf{f}$, it suffices to prove the theorem for $\mathbf{f}\in\mathbf{S}_{i,j}$ only. To simplify notations, let us assume that $1,\,2\in S_\star$, and assume that $\mathbf{f}\in\mathbf{S}_{1,2}$, i.e,

$$L_{\mathbf{y}}\mathbf{f}=\frac{\nu_\star}{\nu_1}\mathbf{e}_1-\frac{\nu_\star}{\nu_2}\mathbf{e}_2\ . \tag{4.6}$$

Now we fix such $\mathbf{f}$ throughout the remaining part of the current section. We note that $(L_{\mathbf{y}}\mathbf{f})(i)=0$ for all $i\neq 1,\,2$.

Our plan is to select the test function ϕ_ϵ that appeared in Theorem 4.1 as a minimizer of a functional $\mathscr{I}_\epsilon(\cdot)$ that will be defined in Section 4.1. More precisely, we first take a minimizer ψ_ϵ of that functional satisfies a certain symmetry condition (see (4.7) below) and analyze its property thoroughly in Sections 4.2-4.5. Then, we shall prove that a translation of ψ_ϵ, which is also a minimizer of $\mathscr{I}_\epsilon(\cdot)$, satisfies all the requirements of Theorem 4.1 in Section 4.6.

4.1. **A Variational principle.** Recall from (3.17) the functional $\mathscr{D}_\epsilon(\cdot)$ and define a functional $\mathscr{I}_\epsilon(\cdot)$ on $H^1(\mathbb{R}^d)$ as

$$\mathscr{I}_\epsilon(\phi)=\frac{1}{2}\theta_\epsilon\,\mathscr{D}_\epsilon(\phi)+\sum_{i=1,\,2}\mathbf{a}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\int\zeta^i(\boldsymbol{x})\phi(\boldsymbol{x})\,\hat{\mu}_\epsilon(\boldsymbol{x})d\boldsymbol{x}\ . \tag{4.7}$$

Denote by ψ_ϵ a minimizer of $\mathscr{I}_\epsilon(\cdot)$. Then, it is well-known that ψ_ϵ classically solves (4.2), i.e.,

$$\theta_\epsilon\mathscr{L}_\epsilon\psi_\epsilon=\sum_{i\in S_\star}\mathbf{a}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\,\zeta^i\ . \tag{4.8}$$

Our purpose in the remaining part is to find a constant c_ϵ such that $\phi_\epsilon=\psi_\epsilon+c_\epsilon$ satisfies (4.3). Note that this ϕ_ϵ also satisfies (4.2) and hence, this finishes the proof.

Write

$$\mathbf{p}_\epsilon(i)=\mathbf{a}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\int\zeta^i(\boldsymbol{x})\psi_\epsilon(\boldsymbol{x})\,\hat{\mu}_\epsilon(\boldsymbol{x})d\boldsymbol{x}\ \ ;\ i\in S_\star\ , \tag{4.9}$$

so that $\mathbf{p}_\epsilon(i)=0$ for all $i\neq 1,\,2$ because of (4.6). Note that if we add a constant a to ψ_ϵ, then the value of $\mathbf{p}_\epsilon(i)$ for $i=1,2$ changes to $\mathbf{p}'_\epsilon(i)$, with

$$\mathbf{p}'_\epsilon(1)=\mathbf{p}_\epsilon(1)+ab\ ,\quad \mathbf{p}'_\epsilon(2)=\mathbf{p}_\epsilon(2)-ab\ ,\quad \mathbf{p}_\epsilon(i)=0\ ,$$

for $i \neq 1, 2$, where $b = Z_\epsilon^{-1} (2\pi\epsilon)^{d/2} e^{-h/\epsilon} \nu_*$. Hence, by adding a constant a to ψ_ϵ if necessary, we can assume without loss of generality that $\mathbf{p}_\epsilon(1) = \mathbf{p}_\epsilon(2)$. Set

$$\lambda_\epsilon := -\mathbf{p}_\epsilon(1) = -\mathbf{p}_\epsilon(2) \ . \tag{4.10}$$

We now multiply both sides of the equation (4.8) by $-\psi_\epsilon$ and integrate with respect to the invariant measure μ_ϵ to deduce

$$\theta_\epsilon \, \mathscr{D}_\epsilon(\psi_\epsilon) = 2\lambda_\epsilon \ . \tag{4.11}$$

Consequently, $\lambda_\epsilon > 0$ and furthermore, by (4.7), (4.10), and (4.11) we obtain

$$\mathscr{I}_\epsilon(\psi_\epsilon) = -\lambda_\epsilon \ . \tag{4.12}$$

4.2. **Lower bound on λ_ϵ.** In this subsection, we prove a rough lower bound for λ_ϵ in Proposition 4.4.

We start by providing some relations between Dirichlet forms $D_{\mathbf{x}}(\cdot,\, \cdot)$ and $D_{\mathbf{y}}(\cdot,\, \cdot)$. For $\mathbf{u} : S_\star \to \mathbb{R}$ and $\mathbf{u}' : S \to \mathbb{R}$, we say that $\mathbf{u}'$ is an extension of $\mathbf{u}$ if $\mathbf{u}'(i) = \mathbf{u}(i)$ for all $i \in S_\star$. For $\mathbf{u} : S_\star \to \mathbb{R}$, we define the harmonic extension $\widetilde{\mathbf{u}} : S \to \mathbb{R}$ of $\mathbf{u}$ as the extension of $\mathbf{u}$ satisfying

$$(L_{\mathbf{x}} \widetilde{\mathbf{u}})(i) = 0 \ \text{ for all } i \in S \setminus S_\star \ . \tag{4.13}$$

The following lemma will be used in several instances in the remaining part of the article.

Lemma 4.3. *For all $\mathbf{u}$, $\mathbf{v} : S_\star \to \mathbb{R}$, the following properties hold.*

(1) *For harmonic extension $\widetilde{\mathbf{u}}$ and $\widetilde{\mathbf{v}}$ of $\mathbf{u}$ and $\mathbf{v}$, respectively, we have*

$$D_{\mathbf{x}}(\widetilde{\mathbf{u}},\, \widetilde{\mathbf{v}}) = \nu_\star D_{\mathbf{y}}(\mathbf{u},\, \mathbf{v}) \ . \tag{4.14}$$

(2) *For any extensions $\mathbf{v}_1$, $\mathbf{v}_2 : S \to \mathbb{R}$ of $\mathbf{v}$, we have*

$$D_{\mathbf{x}}(\widetilde{\mathbf{u}},\, \mathbf{v}_1) = D_{\mathbf{x}}(\widetilde{\mathbf{u}},\, \mathbf{v}_2) \ . \tag{4.15}$$

Proof. For part (1), recall the function $\mathbf{e}_i$, $i \in S_\star$, that was defined in (4.5). Since both $D_{\mathbf{x}}(\cdot,\, \cdot)$ and $D_{\mathbf{y}}(\cdot,\, \cdot)$ are bi-linear forms, it suffices to check (4.14) for $(\mathbf{u},\, \mathbf{v}) = (\mathbf{e}_i,\, \mathbf{e}_j)$ for $i \in S_\star$ and $j \in S_\star$. By (2.22), the harmonic extension of $\mathbf{e}_i$, namely $\widetilde{\mathbf{e}}_i : S \to \mathbb{R}$, is the equilibrium potential between $\{i\}$ and $S_\star \setminus \{i\}$, with respect to the process $\mathbf{x}(\cdot)$, and hence we have

$$D_{\mathbf{x}}(\widetilde{\mathbf{e}}_i,\, \widetilde{\mathbf{e}}_i) = \mathrm{cap}_{\mathbf{x}}(\{i\},\, S_\star \setminus \{i\}) \ . \tag{4.16}$$

Similarly, for $i \neq j \in S_\star$, the function $\widetilde{\mathbf{e}}_i + \widetilde{\mathbf{e}}_j : S \to \mathbb{R}$ is the equilibrium potential between $\{i,\, j\}$ and $S_\star \setminus \{i,\, j\}$, with respect to the process $\mathbf{x}(\cdot)$, and therefore it holds

$$D_{\mathbf{x}}(\widetilde{\mathbf{e}}_i + \widetilde{\mathbf{e}}_j,\, \widetilde{\mathbf{e}}_i + \widetilde{\mathbf{e}}_j) = \mathrm{cap}_{\mathbf{x}}(\{i, j\},\, S_\star \setminus (\{i\} \cap \{j\})) \ . \tag{4.17}$$

By (4.16), (4.17) and the bi-linearity of $D_{\mathbf{x}}$, we have

$$D_{\mathbf{x}}(\widetilde{\mathbf{e}}_i,\, \widetilde{\mathbf{e}}_j) = -\beta_{i,\,j} \;\; ;\; i \neq j \in S_\star \; . \tag{4.18}$$

It also follows from the definition

$$D_{\mathbf{y}}(\mathbf{e}_i,\, \mathbf{e}_j) = \frac{1}{2\nu_\star}\beta_{i,\,j}(0-1)(1-0) + \frac{1}{2\nu_\star}\beta_{j,\,i}(1-0)(0-1) = -\frac{\beta_{i,\,j}}{\nu_\star} \; . \tag{4.19}$$

From (4.18) and (4.19), we deduce (4.14) for $(\mathbf{u},\, \mathbf{v}) = (\mathbf{e}_i,\, \mathbf{e}_j)$ with $i \neq j$.

Now, we turn to the case $(\mathbf{u},\, \mathbf{v}) = (\mathbf{e}_i,\, \mathbf{e}_i)$ for some $i \in S_\star$. For this case, since $\sum_{j\in S_\star} \mathbf{e}_j = 1$ on $S_\star$, it is immediate that $\sum_{j\in S_\star} \widetilde{\mathbf{e}}_j = 1$ on S. Therefore,

$$D_{\mathbf{x}}\bigg(\widetilde{\mathbf{e}}_i,\, \sum_{j\in S_\star} \widetilde{\mathbf{e}}_j\bigg) = 0 \; .$$

By this equation, (4.18), and the bi-linearity of $D_{\mathbf{x}}$, we obtain

$$D_{\mathbf{x}}(\widetilde{\mathbf{e}}_i,\, \widetilde{\mathbf{e}}_i) = -\sum_{j\in S_\star : j\neq i} D_{\mathbf{x}}(\widetilde{\mathbf{e}}_i,\, \widetilde{\mathbf{e}}_j) = \sum_{j\in S_\star : j\neq i} \beta_{i,\,j} \; .$$

This finishes the proof for part (1) since by the direct computation we can verify that $D_{\mathbf{y}}(\mathbf{e}_i,\, \mathbf{e}_i) = \nu_\star^{-1} \sum_{j\in S_\star : j\neq i} \beta_{i,\,j}$.

For part (2), by the definition (2.20) of $D_{\mathbf{x}}$, we can write

$$D_{\mathbf{x}}(\widetilde{\mathbf{u}},\, \mathbf{v}_1 - \mathbf{v}_2) = \sum_{i\in S} \mu(i)\,(-L_{\mathbf{x}}\widetilde{\mathbf{u}})(i)\,(\mathbf{v}_1(i) - \mathbf{v}_2(i)) \; .$$

The last summation is 0 since $(L_{\mathbf{x}}\widetilde{\mathbf{u}})(i) = 0$ for $i \in S \setminus S_\star$ and $\mathbf{v}_1(i) - \mathbf{v}_2(i) = 0$ for $i \in S_\star$. This completes the proof. □

Now we are ready to establish an a priori lower bound on λ_ϵ. We remark that a sharp asymptotic of λ_ϵ will be given in Section 4.6. Recall that we have fixed $\mathbf{f}$ as in (4.6).

Proposition 4.4. *We have*

$$\lambda_\epsilon \geq (1/2) D_{\mathbf{y}}(\mathbf{f}, \mathbf{f}) + o_\epsilon(1) \; .$$

Proof. Recall from Section 3.2 the test function $F_\epsilon^{\widetilde{\mathbf{f}}} : \mathbb{R}^d \to \mathbb{R}$ where $\widetilde{\mathbf{f}}$ is the harmonic extension of $\mathbf{f}$ defined above. By (2.14), (2.15), (2.24) and Lemma 3.6,

$$\mathscr{I}_\epsilon(F_\epsilon^{\widetilde{\mathbf{f}}}) = \frac{1}{2}\nu_\star^{-1} D_{\mathbf{x}}(\widetilde{\mathbf{f}},\, \widetilde{\mathbf{f}}) + \sum_{i\in S_\star} \mu_\star(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\,\mathbf{f}(i) + o_\epsilon(1) \; .$$

By Lemma 4.3, we can conclude that the right-hand side of the previous display is equal to

$$\frac{1}{2} D_{\mathbf{y}}(\mathbf{f},\, \mathbf{f}) - \sum_{i\in S_\star} \mu_\star(i)\,(-L_{\mathbf{y}}\mathbf{f})(i)\,\mathbf{f}(i) + o_\epsilon(1) = -\frac{1}{2} D_{\mathbf{y}}(\mathbf{f},\, \mathbf{f}) + o_\epsilon(1) \; .$$

The proof is completed by recalling that $\mathscr{I}_\epsilon(F_\epsilon^{\widetilde{\mathbf{f}}}) \geq \mathscr{I}_\epsilon(\psi_\epsilon) = -\lambda_\epsilon$. □

4.3. **L^2-estimates based on Poincaré's inequality.** For small enough $\epsilon > 0$ the set

$$\left\{\boldsymbol{x} : U(\boldsymbol{x}) \le H - \frac{1}{4}J^2\delta^2\right\} \tag{4.20}$$

consists of K connected components. For such ϵ and $i \in S$, we write $\mathcal{V}_i^{(1)} := \mathcal{V}_{i,\epsilon}^{(1)}$ a connected component of (4.20) containing $\mathcal{V}_i'$. Similarly, define $\mathcal{V}_i^{(2)} := \mathcal{V}_{i,\epsilon}^{(2)}$ as a connected component of the set

$$\left\{\boldsymbol{x} : U(\boldsymbol{x}) \le H - \frac{1}{8}J^2\delta^2\right\} \tag{4.21}$$

containing $\mathcal{V}_i'$ for sufficiently small $\epsilon > 0$. We shall assume that $\epsilon > 0$ is small enough so that the descriptions above hold. Then, for $i \in S$, we have $\mathcal{V}_i \subset \mathcal{V}_i' \subset \mathcal{V}_i^{(1)} \subset \mathcal{V}_i^{(2)} \subset \mathcal{W}_i$. It is not hard to show

$$\mathrm{dist}(\partial\mathcal{V}_i^{(1)},\, \partial\mathcal{V}_i^{(2)}) = cJ\delta + o(\delta) \tag{4.22}$$

for some constant $c > 0$ where this distance is achieved around a saddle point $\boldsymbol{\sigma} \in \mathcal{S}$ that belongs to $\overline{\mathcal{W}}_i$. The constant c depends only on the Hessian of U at that saddle point.

For $i \in S$, define

$$\mathbf{q}_\epsilon(i) = \frac{1}{m_d(\mathcal{V}_i^{(1)})}\int_{\mathcal{V}_i^{(1)}} \psi_\epsilon(\boldsymbol{x})d\boldsymbol{x} \quad \text{and} \quad \widehat{\mathbf{q}}_\epsilon(i) = \frac{1}{m_d(\mathcal{V}_i^{(2)})}\int_{\mathcal{V}_i^{(2)}} \psi_\epsilon(\boldsymbol{x})d\boldsymbol{x}\ ,$$

where m_d denotes the Lebesgue measure of $\mathbb{R}^d$.

Proposition 4.5. *There exists a constant $C > 0$ such that the following estimate holds for all $i \in S$:*

$$\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^2(\mathcal{V}_i^{(2)})} \le C\,\epsilon^{J^2/16}\lambda_\epsilon\ .$$

Remark 4.6. Here and elsewhere in this paper, L^p norms are computed with respect to the Lebesgue measure of $\mathbb{R}^d$.

Proof. By Poincaré's inequality, the definition of $\mathcal{V}_i^{(2)}$, (4.11), and (2.13),

$$\begin{aligned}\int_{\mathcal{V}_i^{(2)}} |\psi_\epsilon(\boldsymbol{x}) - \widehat{\mathbf{q}}_\epsilon(i)|^2\, d\boldsymbol{x} &\le C\int_{\mathcal{V}_i^{(2)}} |\nabla\psi_\epsilon(\boldsymbol{x})|^2 d\boldsymbol{x} \le Ce^{(H-(1/8)J^2\delta^2)/\epsilon}\int_{\mathcal{V}_i^{(2)}} |\nabla\psi_\epsilon(\boldsymbol{x})|^2 e^{-U(\boldsymbol{x})/\epsilon}d\boldsymbol{x}\\ &\le Ce^{(H-(1/8)J^2\delta^2)/\epsilon}Z_\epsilon\epsilon^{-1}\mathscr{D}_\epsilon(\psi_\epsilon) \le C\epsilon^{J^2/8+d/2-1}\lambda_\epsilon \le C\epsilon^{J^2/8}\lambda_\epsilon\ ,\end{aligned}$$

From this and Cauchy-Schwarz's inequality we deduce,

$$|\mathbf{q}_\epsilon(i) - \widehat{\mathbf{q}}_\epsilon(i)| \le \frac{1}{m_d(\mathcal{V}_i^{(1)})}\int_{\mathcal{V}_i^{(1)}} |\psi_\epsilon(\boldsymbol{x}) - \widehat{\mathbf{q}}_\epsilon(i)|\, d\boldsymbol{x} \le C\int_{\mathcal{V}_i^{(2)}} |\psi_\epsilon(\boldsymbol{x}) - \widehat{\mathbf{q}}_\epsilon(i)|\, d\boldsymbol{x} \le C'\epsilon^{J^2/16}\lambda_\epsilon^{1/2}\ .$$

Combining the above two bounds yields

$$\int_{\mathcal{V}_i^{(2)}} |\psi_\epsilon(\boldsymbol{x}) - \mathbf{q}_\epsilon(i)|^2\, d\boldsymbol{x} \le C\epsilon^{J^2/8}\lambda_\epsilon\ .$$

Thus, the proposition follows immediately from this estimate and Proposition 4.4. $\square$

4.4. **L^∞-estimates on valleys.** In this subsection, we use the interior elliptic regularity techniques and a suitable bootstrapping argument to reinforce the L^2-estimate in $\mathcal{V}_i^{(2)}$ that was obtained in Proposition 4.5 to L^∞-estimate in the smaller set $\mathcal{V}_i^{(1)}$. This type of argument has been introduced originally in [10], and is suitably modified to yield a desired L^∞-estimate.

We start by a lemma. Let us write, for $\epsilon > 0$,

$$Z_\epsilon = (1+\eta_\epsilon)\,(2\pi\epsilon)^{d/2}\,e^{-h/\epsilon}\,\nu_\star\ . \tag{4.23}$$

Then, by (2.13), we have $\eta_\epsilon = o_\epsilon(1)$.

Lemma 4.7. *We have*

$$\begin{aligned} &|(1+\eta_\epsilon)\mathbf{p}_\epsilon(1) - \mathbf{q}_\epsilon(1)| \le \|\psi_\epsilon - \mathbf{q}_\epsilon(1)\|_{L^\infty(\mathcal{V}_1)} \ \text{ and} \\ &|(1+\eta_\epsilon)\mathbf{p}_\epsilon(2) + \mathbf{q}_\epsilon(2)| \le \|\psi_\epsilon - \mathbf{q}_\epsilon(2)\|_{L^\infty(\mathcal{V}_2)}\ . \end{aligned}$$

Proof. By (4.6), (4.9), and (4.23) we can write

$$\mathbf{p}_\epsilon(1) = \frac{1}{1+\eta_\epsilon}\,\frac{1}{\mu_\epsilon(\mathcal{V}_1)}\int_{\mathcal{V}_1}\psi_\epsilon(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})\ .$$

Therefore, we have

$$|(1+\eta_\epsilon)\mathbf{p}_\epsilon(1) - \mathbf{q}_\epsilon(1)| \le \frac{1}{\mu_\epsilon(\mathcal{V}_1)}\int_{\mathcal{V}_1}|\psi_\epsilon(\boldsymbol{x}) - \mathbf{q}_\epsilon(1)|\ \mu_\epsilon(d\boldsymbol{x})\ .$$

Thus, the estimate for $\mathbf{p}_\epsilon(1)$ follows. The proof for $\mathbf{p}_\epsilon(2)$ is identical. □

Proposition 4.8. *For all $i \in S$, we have*

$$\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^\infty(\mathcal{V}_i^{(1)})} = o_\epsilon(1)\lambda_\epsilon\ .$$

Proof. On $\mathcal{V}_i^{(2)}$, $i \in S$, the function ψ_ϵ satisfies the equation

$$\mathscr{L}_\epsilon\psi_\epsilon = \begin{cases} \theta_\epsilon^{-1}\mathbf{a}_\epsilon(i)\,\mathbf{g}(i)\,\zeta^i & \text{if } i \in S_\star \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbf{g} = L_{\mathbf{y}}\mathbf{f}$. For both cases, we can rewrite the equation as

$$\epsilon\Delta(\psi_\epsilon - \mathbf{q}_\epsilon(i)) = \nabla\cdot[(\psi_\epsilon - \mathbf{q}_\epsilon(i))\nabla U] - (\psi_\epsilon - \mathbf{q}_\epsilon(i))\Delta U + \frac{C}{\theta_\epsilon}\zeta^i\ ,$$

for some constant $C \ge 0$. Then, by the local interior elliptic estimate [15, Theorem 8.17] with $R = c\delta$ for small enough constant $c > 0$ (we are allowed to do this because of (4.22)), we obtain that, for any $p > d$ and for some constant $C_p > 0$,

$$\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^\infty(\mathcal{V}_i^{(1)})} \le \frac{C_p}{\epsilon\delta^{d/2}}\,\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^2(\mathcal{V}_i^{(2)})} + \frac{C_p}{\epsilon}\delta^{1-(d/p)}\,\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^p(\mathcal{V}_i^{(2)})} + o_\epsilon(1)\ .$$

Let us select $p = 2d$ for the sake of definiteness and let us write $\|\psi_\epsilon\|_\infty := \|\psi_\epsilon\|_{L^\infty(\mathbb{R}^d)}$ for the simplicity of notation. Then, by Propositions 4.4, 4.5, Hölder's inequality, and the trivial fact that $|\mathbf{q}_\epsilon(i)| \le \|\psi_\epsilon\|_\infty$, we obtain

$$\begin{aligned}\|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|_{L^\infty(\mathcal{V}_i^{(1)})} &\le o_\epsilon(1)\lambda_\epsilon + \frac{C}{\epsilon}\delta^{1/2}\, \|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|^{1/d}_{L^2(\mathcal{V}_i^{(2)})}\, \|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|^{1-(1/d)}_{L^\infty(\mathcal{V}_i^{(2)})} \\ &= o_\epsilon(1)\left[\lambda_\epsilon + \lambda_\epsilon^{1/d}\, \|\psi_\epsilon - \mathbf{q}_\epsilon(i)\|^{1-(1/d)}_{L^\infty(\mathcal{V}_i^{(2)})}\right] , \\ &\le o_\epsilon(1)\left[\lambda_\epsilon + \lambda_\epsilon^{1/d}\, \|\psi_\epsilon\|^{1-(1/d)}_\infty\right] . \\ &\le o_\epsilon(1)\left[\lambda_\epsilon + \|\psi_\epsilon\|_\infty\right] .\end{aligned} \tag{4.24}$$

Now we present a bootstrapping argument. Write

$$\mathbf{m}_\epsilon(i) = \|\psi_\epsilon\|_{L^\infty(\mathcal{V}_i^{(1)})} \ \text{ for } i \in S \quad \text{and} \quad \xi = \xi_\epsilon = \max\{\mathbf{m}_\epsilon(1),\, \mathbf{m}_\epsilon(2)\}$$

Then, it holds that $\|\psi_\epsilon\|_\infty = \xi$, since otherwise $\mathscr{I}(u_\epsilon \circ \psi_\epsilon) < \mathscr{I}(\psi_\epsilon)$ where

$$u_\epsilon(t) = \begin{cases} \|\psi_\epsilon\|_\infty & \text{if } t \ge \xi \ , \\ t & \text{if } |t| < \|\psi_\epsilon\|_\infty \, , \\ -\|\psi_\epsilon\|_\infty & \text{if } t \le -\xi \ . \end{cases}$$

Thus we can write $\|\psi_\epsilon\|_\infty = \mathbf{m}_\epsilon(k)$ where k is either 1 or 2. Then,

$$\|\psi_\epsilon\|_\infty = \mathbf{m}_\epsilon(k) = \|\psi_\epsilon\|_{L^\infty(\mathcal{V}_k^{(1)})} \le \|\psi_\epsilon - \mathbf{q}_\epsilon(k)\|_{L^\infty(\mathcal{V}_k^{(1)})} + |\mathbf{q}_\epsilon(k)| \ . \tag{4.25}$$

By Lemma 4.7 and (4.10), we have that

$$|\mathbf{q}_\epsilon(k)| \le (1 + o_\epsilon(1))\lambda_\epsilon + \|\psi_\epsilon - \mathbf{q}_\epsilon(k)\|_{L^\infty(\mathcal{V}_k^{(1)})} \ . \tag{4.26}$$

By combining (4.25) and (4.26), we obtain

$$\|\psi_\epsilon\|_\infty \le (1 + o_\epsilon(1))\lambda_\epsilon + 2\, \|\psi_\epsilon - \mathbf{q}_\epsilon(k)\|_{L^\infty(\mathcal{V}_k^{(1)})} \ . \tag{4.27}$$

Inserting (4.27) into (4.24) with $i = k$ yields

$$\|\psi_\epsilon - \mathbf{q}_\epsilon(k)\|_{L^\infty(\mathcal{V}_k^{(1)})} \le o_\epsilon(1)\lambda_\epsilon \ . \tag{4.28}$$

By (4.27) and (4.28), we have

$$\|\psi_\epsilon\|_\infty \le (1 + o_\epsilon(1))\lambda_\epsilon \ . \tag{4.29}$$

Finally, inserting this into (4.24) finishes the proof. □

4.5. **Characterization of $\mathbf{q}_\epsilon$ on deepest valleys.** In the previous subsection, we proved that if the constant λ_ϵ is bounded above, then for every $i \in S$, the function $\psi_\epsilon(x) - \mathbf{q}_\epsilon(i)$ is almost 0 in each valley $\mathcal{V}_i^{(1)}$. This boundedness of λ_ϵ will be established later in (4.55). In this subsection, we shall prove that, for each $i \in S_\star$, the value $\mathbf{q}_\epsilon(i)$ is close to $\mathbf{f}(i)$ up to a constant c_ϵ that does not depend on i. The following is a formulation of this result.

Proposition 4.9. *For all small enough $\epsilon > 0$, there exists a constant c_ϵ such that, for all $i \in S_\star$,*

$$|\mathbf{q}_\epsilon(i) - \mathbf{f}(i) - c_\epsilon| = o_\epsilon(1)\,\lambda_\epsilon\,.$$

Indeed, this characterization of $\mathbf{q}_\epsilon$ is the main innovation of the current work. We shall use the test function constructed in Section 3.2 in a novel manner to establish Proposition 4.9. For each $\epsilon > 0$, we consider a function $\mathbf{h}_\epsilon : S_\star \to \mathbb{R}$ and write $\widetilde{\mathbf{h}}_\epsilon : S \to \mathbb{R}$ for its harmonic extension as was introduced in Section 4.2. Our selection for $\mathbf{h}_\epsilon$ will be revealed at the last stage of the proof (cf. (4.51)). To simplify the notation, we write

$$F_\epsilon := F_\epsilon^{\widetilde{\mathbf{h}}_\epsilon}\,, \tag{4.30}$$

where the notation $F_\epsilon^{\widetilde{\mathbf{h}}_\epsilon}$ was introduced in Section 3.2. We denote by $\|\mathbf{h}_\epsilon\|_\infty$ and $\|\widetilde{\mathbf{h}}_\epsilon\|_\infty$ the maximum of $|\mathbf{h}_\epsilon|$ and $|\widetilde{\mathbf{h}}_\epsilon|$ on $S_\star$ and S, respectively. Using a discrete Maximum Principle, one can readily verify that $\|\mathbf{h}_\epsilon\|_\infty = \|\widetilde{\mathbf{h}}_\epsilon\|_\infty$.

Since ψ_ϵ satisfies the equation (4.2) and since $F_\epsilon \equiv \widetilde{\mathbf{h}}_\epsilon(i) = \mathbf{h}_\epsilon(i)$ on $\mathcal{V}_i'$, $i \in S_\star$, we have the identity

$$\theta_\epsilon \int_{\mathbb{R}^d} F_\epsilon(\boldsymbol{x})\,(\mathscr{L}_\epsilon \psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x}) = \sum_{i\in S_\star} \mathbf{h}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\,\mathbf{a}_\epsilon(i)\,\bar{\zeta}^i\,. \tag{4.31}$$

In order to prove Proposition 4.9, we compute two sides of (4.31) separately. From the comparison of these computations, we obtain the characterization described in Proposition 4.9.

The right-hand side of (4.31) is relatively easy to compute. By Proposition 2.2 and (4.1), we have

$$\mathbf{a}_\epsilon(i)\,\bar{\zeta}^i = (1 + o_\epsilon(1))(\nu_i/\nu_\star)$$

and thus we can rewrite the right-hand side of (4.31) as

$$\sum_{i\in S_\star} \mathbf{h}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\,\mathbf{a}_\epsilon(i)\,\bar{\zeta}^i = -D_{\mathbf{y}}(\mathbf{h}_\epsilon,\,\mathbf{f}) + o_\epsilon(1)\|\mathbf{h}_\epsilon\|_\infty\,. \tag{4.32}$$

The main difficulty of the proof lies on the computation of the left-hand side of (4.31). We carry out this computation in several lemmas below.

Lemma 4.10. *With the notations above, it holds that*

$$\begin{aligned}\theta_\epsilon &\int_{\mathbb{R}^d} F_\epsilon(\boldsymbol{x})\,(\mathscr{L}_\epsilon \psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x}) \\ &= -\theta_\epsilon\,\epsilon \sum_{\boldsymbol{\sigma}\in\mathcal{S}} \int_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon} (\nabla F_\epsilon \cdot \nabla \psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x}) + o_\epsilon(1)\,\lambda_\epsilon^{1/2}\,\|\mathbf{h}_\epsilon\|_\infty\,.\end{aligned} \tag{4.33}$$

Proof. By the divergence theorem, the left-hand side of (4.33) is equal to

$$-\,\theta_\epsilon\,\epsilon\int_{\mathbb{R}^d}(\nabla F_\epsilon\cdot\nabla\psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})\;. \tag{4.34}$$

By the definition of $F_\epsilon = F_\epsilon^{\widetilde{\mathbf{h}}_\epsilon}$, we have that

$$\nabla F_\epsilon\equiv 0\;\text{ in }\mathcal{W}_i^\epsilon\text{ for all }i\in S\;. \tag{4.35}$$

Since

$$\mathcal{H}^\epsilon\setminus\left(\bigcup_{i\in S}\mathcal{W}_i^\epsilon\right)=\bigcup_{\boldsymbol{\sigma}\in\mathcal{S}}\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon\;,$$

it suffices to show that

$$-\,\theta_\epsilon\,\epsilon\int_{(\mathcal{H}^\epsilon)^c}(\nabla F_\epsilon\cdot\nabla\psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})=o_\epsilon(1)\,\lambda_\epsilon^{1/2}\,\|\mathbf{h}_\epsilon\|_\infty\;. \tag{4.36}$$

By the Cauchy-Schwarz inequality, the square of the left-hand side of (4.36) is bounded above by

$$\theta_\epsilon\epsilon\left(\int_{(\mathcal{H}^\epsilon)^c}|\nabla F_\epsilon(\boldsymbol{x})|^2\mu_\epsilon(d\boldsymbol{x})\right)^{\frac12}\left(\int_{(\mathcal{H}^\epsilon)^c}|\nabla\psi_\epsilon(\boldsymbol{x})|^2\mu_\epsilon(d\boldsymbol{x})\right)^{\frac12}\;.$$

By (3.18) and (4.11), the last expression is $o_\epsilon(1)\,\lambda_\epsilon^{1/2}\,\|\mathbf{h}_\epsilon\|_\infty$. Thus, (4.36) follows. □

Recall the function $f_\epsilon^{\boldsymbol{\sigma}}$ from (3.6). The estimate below corresponds to that of each summand on the right-hand side of (4.33).

Lemma 4.11. *For $i,\,j\in S$ with $i<j$ and for $\boldsymbol{\sigma}\in\mathcal{W}_{i,\,j}$, it holds that*

$$\theta_\epsilon\,\epsilon\int_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}(\nabla f_\epsilon^{\boldsymbol{\sigma}}\cdot\nabla\psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})=\frac{\omega_{\boldsymbol{\sigma}}}{\nu_\star}\,[\mathbf{q}_\epsilon(j)-\mathbf{q}_\epsilon(i)]+o_\epsilon(1)\,\lambda_\epsilon\;. \tag{4.37}$$

Proof. Recall the decomposition of boundary of $\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon$ from (3.2). By applying the divergence theorem to the left-hand side of (4.37), we can write

$$\theta_\epsilon\,\epsilon\int_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}(\nabla f_\epsilon^{\boldsymbol{\sigma}}\cdot\nabla\psi_\epsilon)(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})=A_1+A_2+A_3+A_4\;, \tag{4.38}$$

where

$$\begin{aligned}
A_1&=-\theta_\epsilon\int_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}(\mathscr{L}_\epsilon f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x})\,\psi_\epsilon(\boldsymbol{x})\,\mu_\epsilon(d\boldsymbol{x})\;,\\
A_2&=\theta_\epsilon\,\epsilon\int_{\partial_0\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big[(\nabla f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x})\cdot\boldsymbol{n}_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big]\,\psi_\epsilon(\boldsymbol{x})\,\hat\mu_\epsilon(\boldsymbol{x})\,\sigma(d\boldsymbol{x})\;,\\
A_3&=\theta_\epsilon\,\epsilon\int_{\partial_+\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big[(\nabla f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x})\cdot\boldsymbol{n}_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big]\,\psi_\epsilon(\boldsymbol{x})\,\hat\mu_\epsilon(\boldsymbol{x})\,\sigma(d\boldsymbol{x})\;,\\
A_4&=\theta_\epsilon\,\epsilon\int_{\partial_-\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big[(\nabla f_\epsilon^{\boldsymbol{\sigma}})(\boldsymbol{x})\cdot\boldsymbol{n}_{\mathcal{B}_{\boldsymbol{\sigma}}^\epsilon}\Big]\,\psi_\epsilon(\boldsymbol{x})\,\hat\mu_\epsilon(\boldsymbol{x})\,\sigma(d\boldsymbol{x})\;,
\end{aligned}$$

where the vector $\boldsymbol{n}_{\mathcal{B}^{\boldsymbol{\sigma}}_\epsilon}$ denotes the outward unit normal vector to the domain $\mathcal{B}^{\epsilon}_{\boldsymbol{\sigma}}$, and $\sigma(d\boldsymbol{x})$ represents the surface integral. We now compute these four expressions.

Without loss of generality, we may assume that $\boldsymbol{\sigma} = 0$. First, we claim that A_1 and A_2 are negligible in the sense that

$$A_1 = o_\epsilon(1)\,\lambda_\epsilon \quad \text{and} \quad A_2 = o_\epsilon(1)\,\lambda_\epsilon\ . \tag{4.39}$$

The estimate for A_1 is immediate from Lemma 3.4 and (4.29). For A_2, notice first that by the definition (3.6) of $f^{\boldsymbol{\sigma}}_\epsilon$, we can write

$$(\nabla f^{\boldsymbol{\sigma}}_\epsilon)(\boldsymbol{x}) = \frac{1}{c^{\boldsymbol{\sigma}}_\epsilon}\sqrt{\frac{\lambda^{\boldsymbol{\sigma}}_1}{2\pi\epsilon}}\, e^{-\frac{\lambda^{\boldsymbol{\sigma}}_1}{2\epsilon}(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_1)^2}\boldsymbol{v}^{\boldsymbol{\sigma}}_1\ . \tag{4.40}$$

By inserting this into A_2, and applying (3.3), (3.5), and (4.29), we are able to deduce

$$|A_2| \le C\,\theta_\epsilon\,\epsilon^{1/2}\,\lambda_\epsilon\,Z_\epsilon^{-1}\,e^{-(H+J^2\delta^2)/\epsilon}\,\delta^{d-1} \;= o_\epsilon(1)\,\lambda_\epsilon\ . \tag{4.41}$$

Here we have used trivial facts such as $|\boldsymbol{v}^{\boldsymbol{\sigma}}_1\cdot\boldsymbol{n}_{\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}}| \le 1$, $e^{-\frac{\lambda^{\boldsymbol{\sigma}}_1}{2\epsilon}(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_1)^2} \le 1$, and that the σ-measure of $\partial_0\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}$ is of order δ^{d-1}.

Next, we shall prove that

$$A_3 = \frac{\omega_{\boldsymbol{\sigma}}}{\nu_\star}\,\mathbf{q}_\epsilon(j) + o_\epsilon(1)\,\lambda_\epsilon \quad \text{and} \quad A_4 = -\frac{\omega_{\boldsymbol{\sigma}}}{\nu_\star}\,\mathbf{q}_\epsilon(i) + o_\epsilon(1)\,\lambda_\epsilon\ . \tag{4.42}$$

Since the proofs for these two estimates are identical, we only focus on the former. Note that the surface $\partial_+\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}$ is flat, and hence the outward normal vector $\boldsymbol{n}_{\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}}$ is merely equal to $\boldsymbol{v}^{\boldsymbol{\sigma}}_1$. Hence, by (2.13), (3.5) and (4.40) we can rewrite A_3 as

$$A_3 = (1+o_\epsilon(1))\theta_\epsilon\,\epsilon\sqrt{\frac{\lambda^{\boldsymbol{\sigma}}_1}{2\pi\epsilon}}\,\frac{1}{(2\pi\epsilon)^{d/2}e^{-h/\epsilon}\nu_\star}\int_{\partial_+\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}} e^{-\frac{\lambda^{\boldsymbol{\sigma}}_1}{2\epsilon}(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_1)^2-\frac{U(\boldsymbol{x})}{\epsilon}}\psi_\epsilon(\boldsymbol{x})\,\sigma(d\boldsymbol{x})\ .$$

By the Taylor expansion, we have

$$U(\boldsymbol{x}) = H + \frac{1}{2}\left(-\lambda^{\boldsymbol{\sigma}}_1(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_1)^2 + \sum_{i=2}^d \lambda^{\boldsymbol{\sigma}}_i(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_i)^2\right) + o(\delta^2)\ .$$

Inserting this into the penultimate display, we can reorganize the right-hand side so that

$$A_3 = (1+o_\epsilon(1))\,\frac{\sqrt{\lambda^{\boldsymbol{\sigma}}_1}}{2\pi\nu_\star}\int_{\partial_+\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}}\frac{1}{(2\pi\epsilon)^{(d-1)/2}}e^{-\frac{1}{2\epsilon}\sum_{i=2}^d\lambda^{\boldsymbol{\sigma}}_i(\boldsymbol{x}\cdot\boldsymbol{v}^{\boldsymbol{\sigma}}_i)^2}\psi_\epsilon(\boldsymbol{x})\,\sigma(d\boldsymbol{x})\ . \tag{4.43}$$

Now we introduce a change of variable to estimate the last integral. Define a map $g^{\boldsymbol{\sigma}}_\epsilon : \mathbb{R}^{d-1}\to\mathbb{R}^d$ as, for $\boldsymbol{y} = (y_2,\,\cdots,\,y_d)\in\mathbb{R}^{d-1}$,

$$g^{\boldsymbol{\sigma}}_\epsilon(\boldsymbol{y}) = \frac{J\delta}{\sqrt{\lambda_1}}\boldsymbol{v}^{\boldsymbol{\sigma}}_1 + \sum_{k=2}^d\sqrt{\frac{\epsilon}{\lambda_k}}y_k\boldsymbol{v}^{\boldsymbol{\sigma}}_k\ , \tag{4.44}$$

(recall $\boldsymbol{\sigma} = 0$). Notice here that $\partial_+ \mathcal{B}^\epsilon_{\boldsymbol{\sigma}} \subset g^{\boldsymbol{\sigma}}_\epsilon(\mathbb{R}^{d-1})$. Write

$$\mathcal{D}^\epsilon_{\boldsymbol{\sigma}} = (g^{\boldsymbol{\sigma}}_\epsilon)^{-1}(\partial_+ \mathcal{B}^\epsilon_{\boldsymbol{\sigma}}) \subset \mathbb{R}^{d-1} \ .$$

Then, by a change of variable $\boldsymbol{x} = g^{\boldsymbol{\sigma}}_\epsilon(\boldsymbol{y})$, we can rewrite (4.43) as

$$A_3 = (1 + o_\epsilon(1)) \, \frac{1}{\nu_\star} \frac{\sqrt{\lambda^{\boldsymbol{\sigma}}_1}}{2\pi \sqrt{\prod_{k=2}^d \lambda^{\boldsymbol{\sigma}}_k}} \int_{\mathcal{D}^\epsilon_{\boldsymbol{\sigma}}} \frac{1}{(2\pi)^{(d-1)/2}} e^{-\frac{1}{2}|\boldsymbol{y}|^2} \psi_\epsilon(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) \, d\boldsymbol{y} \ . \tag{4.45}$$

Now we analyze $\mathcal{D}^\epsilon_{\boldsymbol{\sigma}}$. For $\boldsymbol{y} \in \mathcal{D}^\epsilon_{\boldsymbol{\sigma}}$, we note that $|g^{\boldsymbol{\sigma}}(\boldsymbol{y}) - \boldsymbol{\sigma}| = O(\delta)$ and thus by the Taylor expansion,

$$U(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) = H - \frac{1}{2} J^2 \delta^2 + \frac{\epsilon}{2} \sum_{k=2}^d y_k^2 + o(\delta^2) \ .$$

Denote by $\mathcal{Q}_{d-1}(r)$ the $(d-1)$-dimensional ball of radius $r > 0$, centered at origin. Then, for $\boldsymbol{y} \in \mathcal{Q}_{d-1}(\frac{J}{2}\sqrt{\log \frac{1}{\epsilon}})$, by the previous display we have that

$$U(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) \le H - \frac{1}{2} J^2 \delta^2 + \frac{1}{8} J^2 \delta^2 + o(\delta^2) < H - \frac{1}{4} J^2 \delta^2$$

for all sufficiently small $\epsilon > 0$. For such ϵ, we can conclude that $\boldsymbol{y} \in \partial_+ \mathcal{B}^\epsilon_{\boldsymbol{\sigma}} \cap \mathcal{V}^{(1)}_j$ by (4.20), and therefore by Proposition 4.8, we have that $\psi_\epsilon(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) = \mathbf{q}_\epsilon(j) + o_\epsilon(1)\lambda_\epsilon$. Consequently, we have

$$\int_{\mathcal{Q}_{d-1}(\frac{J}{2}\sqrt{\log \frac{1}{\epsilon}})} \frac{1}{(2\pi)^{(d-1)/2}} e^{-\frac{1}{2}|\boldsymbol{y}|^2} \psi_\epsilon(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) \, d\boldsymbol{y} = (1 + o_\epsilon(1)) \mathbf{q}_\epsilon(j) + o_\epsilon(1)\lambda_\epsilon \ ,$$

because the integral of the probability density function of the $(d-1)$-dimensional standard normal distribution on $\mathcal{Q}_{d-1}(\frac{J}{2}\sqrt{\log \frac{1}{\epsilon}})$ is $1 + o_\epsilon(1)$.

On the other hand, by (4.29),

$$\begin{aligned} &\left| \int_{\mathcal{D}^\epsilon_{\boldsymbol{\sigma}} \setminus \mathcal{Q}_{d-1}(\frac{J}{2}\sqrt{\log \frac{1}{\epsilon}})} \frac{1}{(2\pi)^{(d-1)/2}} e^{-\frac{1}{2}|\boldsymbol{y}|^2} \psi_\epsilon(g^{\boldsymbol{\sigma}}(\boldsymbol{y})) \, d\boldsymbol{y} \right| \\ &\qquad \le \|\psi_\epsilon\|_\infty \int_{\mathcal{Q}_{d-1}(\frac{J}{2}\sqrt{\log \frac{1}{\epsilon}})^c} \frac{1}{(2\pi)^{(d-1)/2}} e^{-\frac{1}{2}|\boldsymbol{y}|^2} \, d\boldsymbol{y} = o_\epsilon(1)\lambda_\epsilon \ . \end{aligned}$$

By the two last centered displays and by the definition of $\omega_{\boldsymbol{\sigma}}$, we can rewrite (4.45) as

$$A_3 = \frac{\omega_{\boldsymbol{\sigma}}}{\nu_\star} \left[(1 + o_\epsilon(1)) \mathbf{q}_\epsilon(j) + o_\epsilon(1)\lambda_\epsilon \right] \ .$$

The proof of (4.42) is completed by recalling that the fact that by (4.29)

$$|\mathbf{q}_\epsilon(i)| \le \|\psi_\epsilon\|_\infty \le (1 + o_\epsilon(1))\lambda_\epsilon.$$

By combining (4.38), (4.39), and (4.42), we complete the proof. $\square$

Lemma 4.12. *Assume that* $\mathbf{f} \neq 0$*. It then holds,*

$$\theta_\epsilon \int_{\mathbb{R}^d} F_\epsilon(\boldsymbol{x})\, (\mathscr{L}_\epsilon \psi_\epsilon)(\boldsymbol{x})\, \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} = -\frac{1}{\nu_\star} D_{\mathbf{x}}(\widetilde{\mathbf{h}}_\epsilon,\, \mathbf{q}_\epsilon) + o_\epsilon(1)\, \lambda_\epsilon\, \|\mathbf{h}_\epsilon\|_\infty \,. \tag{4.46}$$

Proof. By Lemma 4.10 and the definition (4.30) (cf. (3.13)) of F_ϵ we can rewrite the left-hand side as

$$-\,\theta_\epsilon\, \epsilon \sum_{1\le i<j<K} \left[(\widetilde{\mathbf{h}}_\epsilon(j) - \widetilde{\mathbf{h}}_\epsilon(i)) \sum_{\boldsymbol{\sigma}\in\mathcal{W}_{i,\,j}} \int_{\mathcal{B}^\epsilon_{\boldsymbol{\sigma}}} (\nabla f^{\boldsymbol{\sigma}}_\epsilon \cdot \nabla \psi_\epsilon)(\boldsymbol{x})\, \hat{\mu}_\epsilon(\boldsymbol{x}) d\boldsymbol{x} \right] + o_\epsilon(1)\, \lambda_\epsilon^{1/2}\, \|\mathbf{h}_\epsilon\|_\infty \,. \tag{4.47}$$

From this and Lemma 4.11, we deduce that the left-hand side of (4.46) equals to

$$-\frac{1}{\nu_\star} D_{\mathbf{x}}(\widetilde{\mathbf{h}}_\epsilon,\, \mathbf{q}_\epsilon) + o_\epsilon(1)\, \lambda_\epsilon^{1/2}\, \|\mathbf{h}_\epsilon\|_\infty + o_\epsilon(1)\, \lambda_\epsilon\, \|\widetilde{\mathbf{h}}_\epsilon\|_\infty.$$

Therefore, the proof is completed because by Maximum Principle $\|\widetilde{\mathbf{h}}_\epsilon\|_\infty = \|\mathbf{h}_\epsilon\|_\infty$, and λ_ϵ is uniformly positive whenever $\mathbf{f} \neq 0$ by Proposition 4.4. □

Now we are ready to prove Proposition 4.9.

Proof of Proposition 4.9. The proof of the Proposition is trivial when $\mathbf{f} = 0$, because we may choose $\psi_\epsilon = c_\epsilon = 0$. From now on, we assume that $\mathbf{f} \neq 0$. By (4.32), Proposition 4.4, and Lemma 4.12, we have

$$D_{\mathbf{y}}(\mathbf{h}_\epsilon,\, \mathbf{f}) = \frac{1}{\nu_\star} D_{\mathbf{x}}(\widetilde{\mathbf{h}}_\epsilon,\, \mathbf{q}_\epsilon) + o_\epsilon(1)\, \lambda_\epsilon\, \|\mathbf{h}_\epsilon\|_\infty \,. \tag{4.48}$$

Denote by $\mathbf{q}^\star_\epsilon \in \mathbb{R}^{S_\star}$ the restriction of $\mathbf{q}_\epsilon$ on $S_\star$, i.e., $\mathbf{q}^\star_\epsilon(i) = \mathbf{q}_\epsilon(i)$ for all $i \in S_\star$, and denote by $\widetilde{\mathbf{q}}^\star_\epsilon \in \mathbb{R}^S$ the harmonic extension of $\mathbf{q}^\star_\epsilon$ to S. Note that $\widetilde{\mathbf{q}}^\star_\epsilon$ and $\mathbf{q}_\epsilon$ are two different extensions of $\mathbf{q}^\star_\epsilon \in \mathbb{R}^{S_\star}$ to S. Thus, by Lemma 4.3 we have

$$D_{\mathbf{x}}(\widetilde{\mathbf{h}}_\epsilon,\, \mathbf{q}_\epsilon) = D_{\mathbf{x}}(\widetilde{\mathbf{h}}_\epsilon,\, \widetilde{\mathbf{q}}^\star_\epsilon) = \nu_\star D_{\mathbf{y}}(\mathbf{h}_\epsilon,\, \mathbf{q}^\star_\epsilon) \,. \tag{4.49}$$

Hence, by (4.48) and (4.49), we obtain

$$D_{\mathbf{y}}(\mathbf{h}_\epsilon,\, \mathbf{q}^\star_\epsilon - \mathbf{f}) = o_\epsilon(1)\, \lambda_\epsilon\, \|\mathbf{h}_\epsilon\|_\infty \,. \tag{4.50}$$

Finally, let us define the test function $\mathbf{h}_\epsilon \in \mathbb{R}^{S_\star}$ as

$$\mathbf{h}_\epsilon(i) := \mathbf{q}_\epsilon(i) - \mathbf{f}(i) - c_\epsilon \;\; \text{for all } i \in S_\star \,, \tag{4.51}$$

where

$$c_\epsilon = \frac{1}{|S_\star|} \sum_{i\in S_\star} [\mathbf{q}_\epsilon(i) - \mathbf{f}(i)] \,. \tag{4.52}$$

By inserting this test function $\mathbf{h}_\epsilon$ in (4.50), we obtain

$$D_{\mathbf{y}}(\mathbf{h}_\epsilon,\, \mathbf{h}_\epsilon) = o_\epsilon(1)\, \lambda_\epsilon\, \|\mathbf{h}_\epsilon\|_\infty \,. \tag{4.53}$$

Write

$$\beta_\star = \frac{1}{2\nu_\star} \min_{i\in S_\star,\, j\in S_\star,\, i\neq j} \beta_{i,\,j} > 0 \ .$$

Then, we have

$$D_{\mathbf{y}}(\mathbf{h}_\epsilon,\, \mathbf{h}_\epsilon) \geq \beta_\star \sum_{i,\, j\in S_\star} (\mathbf{h}_\epsilon(i) - \mathbf{h}_\epsilon(j))^2 = 2\beta_\star\, |S_\star| \sum_{i\in S_\star} \mathbf{h}_\epsilon^2 \geq 2\beta_\star\, |S_\star|^2 \|\mathbf{h}_\epsilon\|_\infty^2 \ , \tag{4.54}$$

where the identity follows from the fact that $\sum_{i\in S_\star} \mathbf{h}_\epsilon = 0$ thanks to our selection (4.51) and (4.53) of $\mathbf{h}_\epsilon$. By (4.53) and (4.54), we obtain

$$\|\mathbf{h}_\epsilon\|_\infty \leq o_\epsilon(1)\, \lambda_\epsilon \ .$$

This completes the proof since $\mathbf{h}_\epsilon(i) = \mathbf{q}_\epsilon(i) - \mathbf{f}(i) - c_\epsilon$ for $i \in S_\star$. □

4.6. **Proof of Theorem 4.1.** Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Define $\phi_\epsilon = \psi_\epsilon - c_\epsilon$ where c_ϵ is the constant appearing in the statement of Proposition 4.9. Then, by Propositions 4.8 and 4.9, we obtain

$$\|\phi_\epsilon - \mathbf{f}(i)\|_{L^\infty(\mathcal{V}_i^{(1)})} = o_\epsilon(1)\, \lambda_\epsilon \ \text{ for all } i \in S_\star.$$

Since it already has been shown that ϕ_ϵ satisfies (4.2), and $\phi_\epsilon \in W^{2,\,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $p \geq 1$, it only remains to show that λ_ϵ is bounded above. By Lemma 4.7, Proposition 4.8, and (4.10), we have that

$$\mathbf{q}_\epsilon(1) = -(1 + o_\epsilon(1))\lambda_\epsilon \ \text{ and } \ \mathbf{q}_\epsilon(2) = (1 + o_\epsilon(1))\lambda_\epsilon \ .$$

By combining these results with Proposition 4.9, we obtain

$$\mathbf{f}(1) = -c_\epsilon - (1 + o_\epsilon(1))\lambda_\epsilon \ \text{ and } \ \mathbf{f}(2) = -c_\epsilon + (1 + o_\epsilon(1))\lambda_\epsilon \ .$$

Therefore, we have

$$\lambda_\epsilon = \frac{1 + o_\epsilon(1)}{2} (\mathbf{f}(2) - \mathbf{f}(1)) \ . \tag{4.55}$$

This proves the boundedness of λ_ϵ. □

5. Tightness

The main result of the current section is the following theorem regarding the tightness of the family of processes $\{\mathbf{y}_\epsilon(\cdot) : \epsilon \in (0,\, 1]\}$.

Theorem 5.1. *For all $i \in S_\star$ and for any sequence of Borel probability measures $(\pi_\epsilon)_{\epsilon>0}$ concentrated on $\mathcal{V}_i$, the family $\{\mathbf{Q}^\epsilon_{\pi_\epsilon} : \epsilon \in (0,\, 1]\}$ is tight on $D([0,\, \infty),\, S_\star)$, and every limit point $\mathbf{Q}^*$, as $\epsilon \to 0$, of this sequence satisfies*

$$\mathbf{Q}^*(\mathbf{x}(0) = i) = 1 \ \textit{ and } \ \mathbf{Q}^*(\mathbf{x}(t) \neq \mathbf{x}(t-)) = 0 \ \textit{ for all } t > 0 \ .$$

We first introduce in Subsection 5.1 two main ingredients of the proof of the tightness. These technical estimates are the tight bound of the transition time from a valley to other valleys (Proposition 5.2), and the negligibility of the time spent by $\widehat{\boldsymbol{x}}_\epsilon(t)$ in Δ (Proposition 5.4). These are common technical steps in the proof of tightness in the metastable situation, and Beltran and Landim [2, 3] developed a robust methodology to verify these when the underlying dynamics are discrete Markov chain. In [27], the corresponding tightness when the underlying dynamics is a 1-dimensional diffusion is obtained. The common feature for these models which allows to prove the tightness is the coupling of two trajectories starting from different points in the same well. Since two diffusion processes living in $\mathbb{R}^d$, $d \ge 2$, cannot be exactly coupled, we have to developed another machinery. We shall use Theorem 4.1 to bound the inter-valleys transition times, and Freidlin-Wentzell theory [12] for the negligibility of the time spent outside valleys. Then, the proof of Theorem 5.1 is given in Subsection 5.2.

5.1. **Two preliminary estimates.** For $\mathcal{A} \subset \mathbb{R}^d$, we denote by $H_{\mathcal{A}}$ the hitting time of the set $\mathcal{A}$. Then the hitting time $H_{\mathcal{V}_\star \setminus \mathcal{V}_i}$ under the law $\mathbb{P}^\epsilon_{\boldsymbol{x}}$, $\boldsymbol{x} \in \mathcal{V}_i$, can be regarded as the transition time from valley $\mathcal{V}_i$ to other deepest valleys. We now verify that this inter-valley transition time cannot be too small.

Proposition 5.2. *For all $i \in S_\star$, it holds that,*

$$\lim_{a\to 0} \limsup_{\epsilon\to 0} \sup_{\boldsymbol{x}\in\mathcal{V}_i} \mathbb{P}^\epsilon_{\boldsymbol{x}} \left[H_{\mathcal{V}_\star\setminus\mathcal{V}_i} \le a\theta_\epsilon \right] = 0 \; . \tag{5.1}$$

Remark 5.3. The result of Freidlin and Wentzell [12] provides that, for all $i \in S_\star$,

$$\limsup_{\epsilon\to 0} \sup_{\boldsymbol{x}\in\mathcal{V}_i} \mathbb{P}^\epsilon_{\boldsymbol{x}} \left[H_{\mathcal{V}_\star\setminus\mathcal{V}_i} \le e^{-\eta/\epsilon}\theta_\epsilon \right] = 0 \quad \text{for all } \eta > 0 \; . \tag{5.2}$$

This estimate is definitely weaker than (5.1). On the other hand, Bovier et. al. [8] demonstrated that $\theta_\epsilon^{-1} H_{\mathcal{V}_\star\setminus\mathcal{V}_i}$ converges to an exponential random variable with constant mean, and this result does implies (5.1). However, in this paper, we provide another proof without using this result. Two main advantages of our proof of (5.1) is that it is short, and is has a good chance to be applicable to the non-reversible case (1.5); our proof of (5.1) relies only on our analysis on the elliptic equations carried out in the previous section. The reader can readily notice that this result is a direct consequence of Theorem 4.1.

Proof. We fix $i \in S_\star$ and $\boldsymbol{x} \in \mathcal{V}_i$. Consider a function $\mathbf{b}_i : S_\star \to \mathbb{R}$ given by

$$\mathbf{b}_i(j) = \begin{cases} 0 & \text{if } j = i \\ 1 & \text{if } j \in S_\star \setminus \{i\} \; . \end{cases}$$

Denote by $\phi_\epsilon = \phi_\epsilon^i$ the test function we obtain in Theorem 4.1 for $\mathbf{f} = \mathbf{b}_i$. Then, by Ito's formula and part (2) of Theorem 4.1, we get

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{x}}^\epsilon\left[\phi_\epsilon(\boldsymbol{x}_\epsilon(a\theta_\epsilon \wedge H_{\mathcal{V}_\star\setminus\mathcal{V}_i}))\right]\\&\quad = \phi_\epsilon(\boldsymbol{x}) + \sum_{i\in S_\star}\mathbb{E}_{\boldsymbol{x}}^\epsilon\left[\int_0^{a\theta_\epsilon\wedge H_{\mathcal{V}_\star\setminus\mathcal{V}_i}} \theta_\epsilon^{-1}\mathbf{a}_\epsilon(i)\,(L_{\mathbf{y}}\mathbf{f})(i)\,\zeta^i(\boldsymbol{x}_\epsilon(s))ds\right] .\end{aligned}$$

Note that the last integral is bounded by Ca for some constant $C>0$. Hence, by part (3) of Theorem 4.1, the right-hand side is bounded by $Ca + o_\epsilon(1)$.

Now we turn to the left-hand side. Again by part (3) of Theorem 4.1, we can add small constant $\alpha_\epsilon = o_\epsilon(1)$ so that $\widetilde{\phi}_\epsilon = \phi_\epsilon + \alpha_\epsilon \ge 0$ on $\mathcal{V}_\star$. Then, by the maximum principle, $\widetilde{\phi}_\epsilon \ge 0$ on $\mathbb{R}^d$, and furthermore, $\widetilde{\phi}_\epsilon \ge 1/2$ on $\mathcal{V}_\star \setminus \mathcal{V}_i$ provided that ϵ is sufficiently small. Hence,

$$\mathbb{E}_{\boldsymbol{x}}^\epsilon\left[\phi_\epsilon(\boldsymbol{x}_\epsilon(a\theta_\epsilon \wedge H_{\mathcal{V}_\star\setminus\mathcal{V}_i}))\right] \ge -\alpha_\epsilon + \frac{1}{2}\mathbb{P}_{\boldsymbol{x}}^\epsilon\left[H_{\mathcal{V}_\star\setminus\mathcal{V}_i} < a\theta_\epsilon\right] .$$

Summing up, there exists a constant $C>0$ such that

$$\mathbb{P}_{\boldsymbol{x}}^\epsilon\left[H_{\mathcal{V}_\star\setminus\mathcal{V}_i} < a\theta_\epsilon\right] \le Ca + o_\epsilon(1) ,$$

as desired. □

Now we show that the process $\boldsymbol{x}_\epsilon(t)$ does not spend too much time in Δ (cf. (2.10)). Define the amount of time the rescaled process $\widehat{\boldsymbol{x}}_\epsilon(\cdot)$ spends in the set Δ up to time t as

$$\widehat{\Delta}(t) = \widehat{\Delta}_\epsilon(t) = \int_0^t \chi_\Delta(\widehat{\boldsymbol{x}}_\epsilon(s))\,ds .$$

Proposition 5.4. *For any sequence of Borel probability measures $(\pi_\epsilon)_{\epsilon>0}$ concentrated on $\mathcal{V}_\star$, it holds that*

$$\lim_{\epsilon\to 0}\mathbb{E}_{\pi_\epsilon}^\epsilon\left[\widehat{\Delta}(t)\right] = 0 \ \text{ for all } t \ge 0.$$

The proof of this proposition can be deduced by combining several classical results of Freidlin and Wentzell [12] in a careful manner. Since we have to introduce numerous new notations and have to recall previous results that are not related to the other part of the current article, we postpone the full proof of this proposition to the appendix. Here, we only provide the proof of Proposition 5.4 when π_ϵ has a density function with respect to the equilibrium measure μ_ϵ (cf. (2.11)) for each $\epsilon>0$, and this density function belongs to $L^p(\mu_\epsilon)$ for some $p>1$, with a uniform L^p bound, i.e.,

$$\limsup_{\epsilon\to 0}\int_{\mathcal{V}_\star}\left(\frac{d\pi_\epsilon}{d\mu_\epsilon}\right)^p d\mu_\epsilon < \infty . \tag{5.3}$$

For this case, we can offer a simple proof.

Proof of Proposition 5.4 under the assumption (5.3). We fix $t \ge 0$. Write

$$u_\epsilon(\boldsymbol{x}) = \mathbb{E}^\epsilon_{\boldsymbol{x}} \left[\widehat{\Delta}(t) \right] .$$

Then, by Fubini's theorem we get

$$\int_{\mathbb{R}^d} u_\epsilon \, d\mu_\epsilon = \mathbb{E}^\epsilon_{\mu_\epsilon} \left[\int_0^t \chi_\Delta(\widehat{\boldsymbol{x}}_\epsilon(s)) \, ds \right] = \int_0^t \mathbb{P}^\epsilon_{\mu_\epsilon} \left[\widehat{\boldsymbol{x}}_\epsilon(s) \in \Delta \right] ds = t \mu_\epsilon(\Delta) \, . \tag{5.4}$$

Write $f_\epsilon = \frac{d\pi_\epsilon}{d\mu_\epsilon}$ so that we can write

$$\mathbb{E}^\epsilon_{\pi_\epsilon} \left[\widehat{\Delta}(t) \right] = \int_{\mathbb{R}^d} u_\epsilon f_\epsilon \, d\mu_\epsilon .$$

Now we apply H́older's inequality, the bound $u_\epsilon \le t$ and (5.4) to the right-hand side of the previous identity to deduce

$$\mathbb{E}^\epsilon_{\pi_\epsilon} \left[\widehat{\Delta}(t) \right] \le \left[\int_{\mathbb{R}^d} u_\epsilon \, d\mu_\epsilon \right]^{1/q} \left[\int_{\mathbb{R}^d} u_\epsilon f_\epsilon^p \, d\mu_\epsilon \right]^{1/p} \le t^{1/q} \mu_\epsilon(\Delta)^{1/q} \left[\int_{\mathbb{R}^d} f_\epsilon^p \, d\mu_\epsilon \right]^{1/p} ,$$

where q is the conjugate exponent of p satisfying $\frac{1}{p} + \frac{1}{q} = 1$. This, Proposition 2.2 and the condition (5.3) complete the proof. $\square$

5.2. **Proof of tightness.** For the completeness of the discussion, we start by summarizing well-known properties related to the current situation. For the full discussion of this material with the detailed proof, we refer to [27, Section 7]. Denote by $\{\mathscr{F}_t^0 : t \ge 0\}$ the natural filtration of $C([0, \infty), \mathbb{R}^d)$ with respect to $\widehat{\boldsymbol{x}}_\epsilon(\cdot)$, namely,

$$\mathscr{F}_t^0 = \sigma(\widehat{\boldsymbol{x}}_\epsilon(s) : s \in [0, t]) \, .$$

and define $\{\mathscr{F}_t : t \ge 0\}$ as the usual augmentation of $\{\mathscr{F}_t^0 : t \ge 0\}$ with respect to $\widehat{\mathbb{P}}^\epsilon_{\pi_\epsilon}$ where (π_ϵ) is a sequence of probability measures that appeared in Theorem 5.1. Define $\mathscr{G}_t = \mathscr{F}_{S^\epsilon(t)}$ for $t \ge 0$, where S^ϵ was defined in (2.26).

Lemma 5.5. *The following statements are true:*

1. *For each $u \ge 0$, the random time $S^\epsilon(u)$ is a stopping time with respect to the filtration $\{\mathscr{F}_t\}$.*
2. *Let τ be a stopping time with respect to the filtration $\{\mathscr{G}_t\}$. Then, $S^\epsilon(\tau)$ is a stopping time with respect to the filtration $\{\mathscr{F}_t\}$.*
3. *The process $\{\boldsymbol{y}_\epsilon(t) : t \ge 0\}$ defined in (2.27) is a continuous-time Markov chain on $\mathcal{V}_\star$ with respect to the filtration $\{\mathscr{G}_t\}$.*

Proof. See [27, Lemma 7.2 and the paragraph below]. $\square$

For $M > 0$, define $\mathscr{T}_M$ as the collection of stopping times with respect to the filtration $\{\mathscr{G}_t\}_{t\geq 0}$ which is bounded by M. The following lemma is required to apply the Aldous criterion to prove the tightness.

Lemma 5.6. *For any sequence of Borel probability measures $(\pi_\epsilon)_{\epsilon>0}$ concentrated on $\mathcal{V}_\star$ and for all $M > 0$, we have*

$$\lim_{a_0\to 0} \limsup_{\epsilon\to 0} \sup_{\tau\in\mathscr{T}_M} \sup_{a\in(0,\,a_0)} \mathbb{P}^\epsilon_{\pi_\epsilon} \left[S^\epsilon(\tau + a) - S^\epsilon(\tau) \geq 2a_0\right] = 0 \; .$$

Proof. Since $S^\epsilon(\cdot)$ is a generalized inverse of $T^\epsilon(\cdot)$, the set $\{S^\epsilon(\tau + a) - S^\epsilon(\tau) \geq 2a_0\}$ is a subset of

$$\{T^\epsilon(S^\epsilon(\tau) + 2a_0) - T^\epsilon(S^\epsilon(\tau)) < a\} \; . \tag{5.5}$$

Since $T^\epsilon(S^\epsilon(\tau) + 2a_0) - T^\epsilon(S^\epsilon(\tau))$ can be rewritten as

$$\int_{S^\epsilon(\tau)}^{S^\epsilon(\tau)+2a_0} \chi_{\mathcal{V}_\star}(\widehat{\boldsymbol{x}}_\epsilon(s))ds = 2a_0 - \int_{S^\epsilon(\tau)}^{S^\epsilon(\tau)+2a_0} \chi_\Delta(\widehat{\boldsymbol{x}}_\epsilon(s))ds \; ,$$

the set (5.5) is a subset of

$$\left\{\int_{S^\epsilon(\tau)}^{S^\epsilon(\tau)+2a_0} \chi_\Delta(\widehat{\boldsymbol{x}}_\epsilon(s))ds \geq 2a_0 - a\right\} \; .$$

Therefore, we can replace the probability appeared in the statement of the lemma with

$$\mathbb{P}^\epsilon_{\pi_\epsilon}\left[\int_{S^\epsilon(\tau)}^{S^\epsilon(\tau)+2a_0} \chi_\Delta(\widehat{\boldsymbol{x}}_\epsilon(s))ds \geq 2a_0 - a\right] \; .$$

This probability is bounded above by

$$\mathbb{P}^\epsilon_{\pi_\epsilon}\left[\widehat{\Delta}(2M + 2a_0) \geq 2a_0 - a\right] + \mathbb{P}^\epsilon_{\pi_\epsilon}\left[S^\epsilon(\tau) > 2M\right] \; . \tag{5.6}$$

By Chebyshev's inequality the first term is bounded from above by

$$\frac{\mathbb{E}^\epsilon_{\pi_\epsilon}\left[\widehat{\Delta}(2M + 2a_0)\right]}{2a_0 - a} \; ,$$

and therefore by Proposition 5.4 we have

$$\lim_{a_0\to 0} \limsup_{\epsilon\to 0} \sup_{\tau\in\mathscr{T}_M} \sup_{a\in(0,\,a_0)} \mathbb{P}^\epsilon_{\pi_\epsilon}\left[\widehat{\Delta}(2M + 2a_0) \geq 2a_0 - a\right] = 0 \; . \tag{5.7}$$

For the second term of (5.6), we observe that $S^\epsilon(\tau) > 2M$ and $\tau \leq M$ imply that $\widehat{\Delta}(2M) \geq M$. Hence again by Chebyshev's inequality this probability is bounded by $M^{-1}\mathbb{E}^\epsilon_{\pi_\epsilon}[\widehat{\Delta}(2M)]$, and therefore by Proposition 5.4 we have

$$\lim_{a_0\to 0} \limsup_{\epsilon\to 0} \sup_{\tau\in\mathscr{T}_M} \sup_{a\in(0,\,a_0)} \mathbb{P}^\epsilon_{\pi_\epsilon}\left[S^\epsilon(\tau) > 2M\right] = 0 \; .$$

This, (5.6), and (5.7) complete the proof. $\square$

Now we are ready to prove the main tightness result.

Proof of Theorem 5.1. By Aldous' criterion, it suffices to show that, for all $M > 0$,

$$\lim_{a_0 \to 0} \limsup_{\epsilon \to 0} \sup_{\tau \in \mathscr{T}_M} \sup_{a \in (0,\, a_0)} \mathbb{P}^{\epsilon}_{\pi_\epsilon} \left[\mathbf{y}_\epsilon(\tau + a) \neq \mathbf{y}_\epsilon(\tau) \right] = 0 \ . \tag{5.8}$$

By Lemma 5.6, it suffices to show that

$$\lim_{a_0 \to 0} \limsup_{\epsilon \to 0} \sup_{\tau \in \mathscr{T}_M} \sup_{a \in (0,\, a_0)} \mathbb{P}^{\epsilon}_{\pi_\epsilon} \left[\mathbf{y}_\epsilon(\tau + a) \neq \mathbf{y}_\epsilon(\tau),\, S^\epsilon(\tau + a) - S^\epsilon(\tau) \le 2a_0 \right] = 0 \ .$$

Since $\mathbf{y}_\epsilon(t) = \Psi(\widehat{\boldsymbol{x}}_\epsilon(S^\epsilon(t)))$, the last probability can be bounded above by

$$\mathbb{P}^{\epsilon}_{\pi_\epsilon} \left[\Psi(\widehat{\boldsymbol{x}}_\epsilon(S^\epsilon(\tau) + t) \neq \Psi(\widehat{\boldsymbol{x}}_\epsilon(S^\epsilon(\tau))) \text{ for some } t \in (0,\, 2a_0] \right] \ .$$

Since $S^\epsilon(\tau)$ is a stopping time with respect to the filtration $\{\mathscr{F}_t\}$ by Lemma 5.5, and since $\widehat{\boldsymbol{x}}_\epsilon(S^\epsilon(\tau)) \in \mathcal{V}_\star$, the last probability is bounded above by

$$\sup_{\boldsymbol{y} \in \mathcal{V}_\star} \mathbb{P}^{\epsilon}_{\boldsymbol{y}} \left[\Psi(\widehat{\boldsymbol{x}}_\epsilon(t)) \neq \Psi(\boldsymbol{y}) \text{ for some } t \in (0,\, 2a_0] \right] = \sup_{i \in S_\star} \sup_{\boldsymbol{y} \in \mathcal{V}_i} \mathbb{P}^{\epsilon}_{\boldsymbol{y}} \left[H_{\mathcal{V}_\star \setminus \mathcal{V}_i} \le 2a_0 \theta_\epsilon \right] \ .$$

Thus, the proof of (5.8) is completed by Proposition 5.2.

The assertion $\mathbf{Q}^*(\mathbf{x}(0) = i) = 1$ is trivial. For the last assertion of the proposition, it suffices to prove that

$$\lim_{a_0 \to 0} \limsup_{\epsilon \to 0} \mathbb{P}^{\epsilon}_{\pi_\epsilon} \left[\mathbf{y}_\epsilon(t - a) \neq \mathbf{y}_\epsilon(t) \text{ for some } a \in (0,\, a_0) \right] = 0 \ .$$

The proof of this estimate is almost identical to that of (5.8) and is omitted. □

6. Proof of Theorem 2.3

We are now ready to prove the main convergence theorem. In view of the tightness result obtained in Section 5, it is enough to demonstrate the uniqueness of limit point. The main ingredient is Theorem 4.1.

Proof of Theorem 2.3. Fix $\mathbf{f} \in \mathbb{R}^{S_\star}$, and let $\phi_\epsilon = \phi_\epsilon^{\mathbf{f}}$ be the function obtained in Theorem 4.1 for the function $\mathbf{f}$. Note that the distribution of $\boldsymbol{x}_\epsilon(0)$ is concentrated on a valley $\mathcal{V}_i$ for some $i \in S_\star$. We fix i in the proof.

We begin with the observation that

$$M_\epsilon(t) = \phi_\epsilon(\widehat{\boldsymbol{x}}_\epsilon(t)) - \theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\widehat{\boldsymbol{x}}_\epsilon(s)) ds$$

is a martingale with respect to the filtration $\{\mathscr{F}_t\}$ defined in Section 5. Write

$$M_\epsilon(t) = U_\epsilon(t) + N_\epsilon(t)$$

where

$$U_\epsilon(t) = \phi_\epsilon(\widehat{\boldsymbol{x}}_\epsilon(t)) - \theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\widehat{\boldsymbol{x}}_\epsilon(s)) \, \mathbf{1}\{\widehat{\boldsymbol{x}}_\epsilon(s) \in \mathcal{V}_\star\} ds \ ,$$

$$N_\epsilon(t) = -\theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\widehat{\boldsymbol{x}}_\epsilon(s)) \, \mathbf{1}\{\widehat{\boldsymbol{x}}_\epsilon(s) \in \Delta\} ds \ .$$

Since $\theta_\epsilon(\mathscr{L}_\epsilon \phi_\epsilon)(\cdot)$ is bounded function by its construction (cf. (4.2)) and hence there exists $c > 0$ such that

$$|N_\epsilon(t)| \le c\widehat{\Delta}(t) \ . \tag{6.1}$$

By definition of $\boldsymbol{y}_\epsilon(t)$, we can write

$$U_\epsilon(S_\epsilon(t)) = \phi_\epsilon(\boldsymbol{y}_\epsilon(t)) - \theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\boldsymbol{y}_\epsilon(s)) ds \ ,$$

and hence

$$\widetilde{M}_\epsilon(t) = M_\epsilon(S_\epsilon(t)) = \phi_\epsilon(\boldsymbol{y}_\epsilon(t)) - \theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\boldsymbol{y}_\epsilon(s)) ds + N_\epsilon(S_\epsilon(t)) \ . \tag{6.2}$$

By Lemma 5.5, $S_\epsilon(t)$ is a stopping time with respect to the filtration $\{\mathscr{F}_t\}$, and therefore $\widetilde{M}_\epsilon(t)$ is a martingale with respect to $\{\mathscr{G}_t\}$. We now investigate each terms in the expression (6.2) separately. Recall Ψ from (2.28). Then, by Theorem 4.1, we can write $\phi_\epsilon = \mathbf{f} \circ \Psi + o_\epsilon(1)$ on $\mathcal{V}_\star$. Since the process $\boldsymbol{y}_\epsilon(t)$ takes values in $\mathcal{V}_\star$, and by definition $\mathbf{y}_\epsilon = \Psi(\boldsymbol{y}_\epsilon)$, we have

$$\phi_\epsilon(\boldsymbol{y}_\epsilon(t)) = \mathbf{f}(\Psi(\boldsymbol{y}_\epsilon(t))) + o_\epsilon(1) = \mathbf{f}(\mathbf{y}_\epsilon(t)) + o_\epsilon(1) \ . \tag{6.3}$$

Next we consider the second term at the right-hand side of (6.2). Since $\theta_\epsilon \mathscr{L}_\epsilon \phi_\epsilon = (L_{\mathbf{y}} \mathbf{f}) \circ \Psi + o_\epsilon(1)$ on $\mathcal{V}_\star$ by Theorem 4.1 and (4.1), we can write

$$\theta_\epsilon \int_0^t (\mathscr{L}_\epsilon \phi_\epsilon)(\boldsymbol{y}_\epsilon(s)) ds = \int_0^t (L_{\mathbf{y}} \mathbf{f})(\mathbf{y}_\epsilon(s)) ds + o_\epsilon(1) \ . \tag{6.4}$$

Hence, by (6.2), (6.3), and (6.4), we can write $\widetilde{M}_\epsilon(t)$ as

$$\widetilde{M}_\epsilon(t) = \mathbf{f}(\mathbf{y}_\epsilon(t)) - \int_0^t (L_{\mathbf{y}} \mathbf{f})(\mathbf{y}_\epsilon(s)) ds + N_\epsilon(S_\epsilon(t)) + o_\epsilon(1) \ . \tag{6.5}$$

Recall that $\mathbf{Q}^\epsilon_{\pi_\epsilon}$ represents the law of the process $\mathbf{y}_\epsilon(\cdot)$ under $\mathbb{P}^\epsilon_{\pi_\epsilon}$ and let $\mathbf{Q}^*$ be a limit point of the family $\{\mathbf{Q}^\epsilon_{\pi_\epsilon}\}_{\epsilon \in (0,\,1]}$. Then, by (6.1), (6.5), and Proposition 5.4, we can conclude that the process

$$\widetilde{M}(t) = \mathbf{f}(\mathbf{x}(t)) - \int_0^t (L_{\mathbf{y}} \mathbf{f})(\mathbf{x}(s)) ds$$

is a martingale under $\mathbf{Q}^*$. Furthermore, by Theorem 5.1, we have that $\mathbf{Q}^*[\mathbf{x}(0) = i] = 1$ and $\mathbf{Q}^*(\mathbf{x}(t) \neq \mathbf{x}(t-)) = 0$ for all $t > 0$. The only probability measure on $D([0,\,\infty), \mathbb{R}^d)$ satisfying these properties is $\mathbf{Q}_i$, and thus we can conclude that $\mathbf{Q}^* = \mathbf{Q}_i$. This completes the characterization of the limit point of the family $\{\mathbf{Q}^\epsilon_{\pi_\epsilon}\}_{\epsilon \in (0,\,1]}$.

$\square$

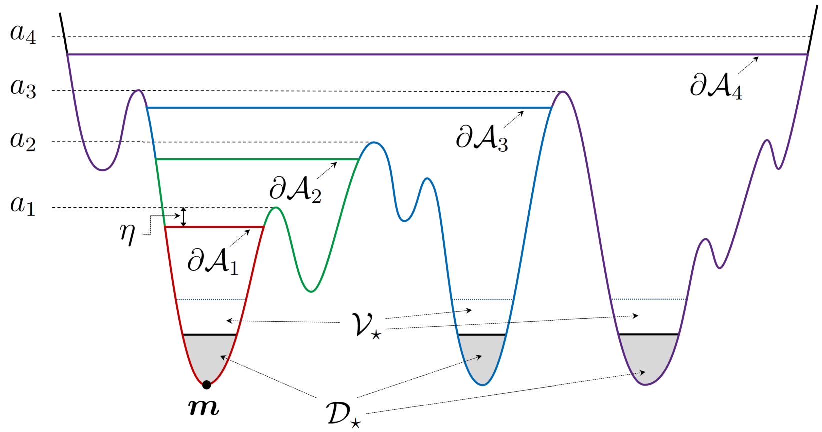

FIGURE A.1. Cycle structure associated to $\boldsymbol{m}$: in this example, $l = 3$ so that $a_3 = H$.

APPENDIX A. NEGLIGIBILITY OF $\widehat{\Delta}$

In this appendix, we prove Proposition 5.4. The proof relies solely on the Freidlin-Wentzell theory, and hence our result is not restricted to the reversible process (1.3), but also holds for the general dynamics (1.2) as well. The verification of this generality is immediate from a careful reading of our proof.

A.1. **Notations and idea of proof.** We introduce some additional notations to those in Section 2.1. Denote by $\mathcal{C}$ the set of critical points of U. Let η be any sufficiently small number such that

$$\eta < \frac{1}{5}\min\left\{|U(\boldsymbol{c}') - U(\boldsymbol{c})| : \boldsymbol{c},\, \boldsymbol{c}' \in \mathcal{C} \text{ and } U(\boldsymbol{c}') \neq U(\boldsymbol{c})\right\} . \tag{A.1}$$

In particular, there is no local minima $\boldsymbol{m}$ of U such that $U(\boldsymbol{m}) \in (h,\, h + 5\eta]$. We write the level set of U as

$$\mathcal{Q}_a := \{\boldsymbol{x} : U(\boldsymbol{x}) < a\} \ \ ;\ a \in \mathbb{R}\ . \tag{A.2}$$

For each $\boldsymbol{m} \in \mathcal{M}_\star$, define $\mathcal{D}_{\boldsymbol{m}}$ as a connected component of $\mathcal{Q}_{h+\eta}$ containing $\boldsymbol{m}$ and let

$$\mathcal{D}_\star := \bigcup_{\boldsymbol{m}\in\mathcal{M}_\star} \mathcal{D}_{\boldsymbol{m}}\ .$$

We take $\eta > 0$ small enough so that $\mathcal{D}_{\boldsymbol{m}} \subset \mathcal{B}(\boldsymbol{m},\, r_0)$ (cf. (2.8)) for all $\boldsymbol{m} \in \mathcal{M}_\star$. This implies that $\mathcal{D}_\star \subset \mathcal{V}_\star$. From now on we regard η as a constant.

Strategy of proof. Define the time spent in the set Δ (without time-rescaling) as

$$\Delta(t) = \Delta_\epsilon(t) := \int_0^t \chi_\Delta(\boldsymbol{x}_\epsilon(s))\, ds\ .$$

Then, by a change of variable, we get

$$\widehat{\Delta}(t) = \theta_\epsilon^{-1}\Delta(\theta_\epsilon t) \; . \tag{A.3}$$

Our main purpose is to estimate $\Delta(t)$ and verify that it is negligible in the sense of Proposition 5.4. To this end, define two sequences $(\tau_i)_{i\in\mathbb{N}}$, $(\sigma_i)_{i\in\mathbb{N}}$ of hitting times recursively according to the following rules: set $\tau_0 = 0$, and

$$\begin{aligned} \sigma_i &= \inf\{s > \tau_{i-1} : \boldsymbol{x}_\epsilon(s) \in \partial\mathcal{V}_\star\} \;\; ; \; i \geq 1 \; . \\ \tau_i &= \inf\{s > \sigma_i : \boldsymbol{x}_\epsilon(s) \in \partial\mathcal{D}_\star\} \;\; ; \; i \geq 1 \; , \end{aligned} \tag{A.4}$$

With these notations, we have the following bound on $\Delta(t)$:

$$\Delta(t) \leq \sum_{i=1}^{\nu(t)} (\tau_i - \sigma_i) \; , \tag{A.5}$$

where $\nu(t) = \sup\{n \in \mathbb{N} : \tau_n \leq t\}$. Hence, for the negligibility of $\Delta(t)$, it suffices to estimate the term $\tau_i - \sigma_i$, which measures the length of the ith excursion from $\partial\mathcal{V}_\star$ to $\partial\mathcal{D}_\star$. This length is typically short since the drift term $-\nabla U(\boldsymbol{x}_\epsilon(t))dt$ pushes the process toward the deeper part of the valley. However, because of the small random noise, some of these excursions are extraordinarily long, though such a long excursion is extremely rare. Therefore, in order to control the right-hand side of (A.5), one has to characterize these long excursions and control both the length and the frequency of them in a careful manner. This will be carried out in the remaining part of the appendix.

A.2. **Cyclic structure and Freidlin-Wentzell type estimates.** We introduce a hierarchy structure of the landscape associated to each global minimum of U. Let us fix $\boldsymbol{m} \in \mathcal{M}_\star$ throughout this subsection. The constructions below are illustrated in Figure A.1.

For each $a \in \mathbb{R}$, denote by $\mathcal{Q}_a(\boldsymbol{m})$ the connected component of the level set $\mathcal{Q}_a$ (cf. (A.2)) containing $\boldsymbol{m}$. For $\mathcal{A} \subset \mathbb{R}^d$, we denote by $\mathcal{M}(\mathcal{A})$ the set of local minima of U contained in $\mathcal{A}$. Then, define an increasing sequence $(a_i)_{i=0}^{l+1}$ recursively as follows: set $a_0 = h + 5\eta$ and

$$a_{k+1} = \inf\{a : \mathcal{M}(\mathcal{Q}_{a_k}(\boldsymbol{m})) \subsetneq \mathcal{M}(\mathcal{Q}_a(\boldsymbol{m}))\} \; ; \; k \geq 0 \; .$$

If $a_l = H$, we stop the recursion procedure and set $a_{l+1} = H + 3\eta$. Now we define

$$\mathcal{A}_k = \mathcal{Q}_{a_k - \eta}(\boldsymbol{m}) \;\; ; \; k \in [\![1, \, l+1]\!] \; .$$

By (A.1), one can notice that $\mathcal{A}_k$ is a connected set. The sequence of connected sets $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_{l+1}$ represents a growing landscape surrounding $\boldsymbol{m}$. According to the classical monograph [12, Chapter 6.6], the set $\mathcal{A}_k$ (or $\mathcal{M}(\mathcal{A}_k)$) corresponds to the rank-k cycle containing $\boldsymbol{m}$. We shall classify each excursions in (A.5) by the maximum k such that the corresponding trajectory hit $\partial\mathcal{A}_k$ before arriving at a point in $\partial\mathcal{D}_\star$. Hitting $\partial\mathcal{A}_k$ for large k means that we may have a long excursion.

We define a sequence $(J_k)_{k=0}^{l+1}$ as

$$J_k = a_k - h - 5\eta \ .$$

With the notations introduced above, we are ready to recall several classical results from [12].

Theorem A.1. *There exists ϵ_0 such that for all $\epsilon \in (0,\, \epsilon_0)$ and the followings hold.*

$$\sup_{\boldsymbol{x}\in\mathcal{A}_k\setminus\overline{\mathcal{D}_\star}} \mathbb{E}_{\boldsymbol{x}}^{\epsilon}\left[H_{\partial\mathcal{A}_k\cup\partial\mathcal{D}_\star}\right] < \exp\frac{J_{k-1}+\eta}{\epsilon} \ \text{ for all } k\in[\![1,\, l+1]\!]\ , \tag{A.6}$$

$$\sup_{\boldsymbol{x}\in\mathcal{D}_\star} \mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[H_{\partial\mathcal{A}_k} < \exp\frac{J_k+3\eta}{\epsilon}\right] < \frac{1}{64} \ \text{ for all } k\in[\![1,\, l+1]\!]\ , \text{ and} \tag{A.7}$$

$$\sup_{\boldsymbol{x}\in\partial\mathcal{Q}_{H+\eta}} \mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[H_{\partial\mathcal{A}_{l+1}} < H_{\partial\mathcal{D}_\star}\right] < \frac{1}{8}\ . \tag{A.8}$$

Remark A.2. Of course, we can replace constants $1/64$ and $1/8$ appeared in the statement of theorem with any small positive number. From now on, ϵ_0 always denotes the constant that appeared in this theorem.

Proof. All of these results are consequence of well-known Freidlin-Wentzell theory. The bound (A.6) follows from [12, Theorem 5.3 in Chapter 6] since the deepest possible depth of a valley in $\mathcal{A}_{k+1}$, which does not contain a global minimum of $\mathcal{M}$ is at most J_{k-1} by (A.1). The bound (A.7) is a consequence of [12, Theorem 6.2 in Chapter 6], since the depth of $\mathcal{A}_k$ is $(a_k-\eta)-h = J_k+4\eta$. Finally, (A.8) can be deduced from [12, Theorem 5.1 in Chapter 6]. □

We next present some exponential-type tail estimates that are consequences of Theorem A.1. We acknowledge that these estimates are inspired by [28, Lemmas B.1 and B.2]. For the simplicity of notation we write

$$\rho_k = \exp\left(-\frac{J_k+2\eta}{\epsilon}\right)\ ,\ \text{ for } k\in[\![0,\, \ell]\!]\ .$$

Lemma A.3. *There exists a constant $c_0>0$ such that for all $\epsilon\in(0,\,\epsilon_0)$, we have*

$$\sup_{\boldsymbol{x}\in\mathcal{A}_k\setminus\overline{\mathcal{D}_\star}} \mathbb{E}_{\boldsymbol{x}}^{\epsilon}\exp\left(c_0\rho_{k-1}H_{\partial\mathcal{A}_k\cup\partial\mathcal{D}_\star}\right) < 2 \ \ \forall k\in[\![1,\, l+1]\!] \text{ and} \tag{A.9}$$

$$\sup_{\boldsymbol{x}\in\partial\mathcal{A}_{l+1}} \mathbb{E}_{\boldsymbol{x}}^{\epsilon}\exp\left(c_0\rho_l H_{\partial\mathcal{D}_\star}\right) < 4\ . \tag{A.10}$$

Proof. For (A.9), it suffices to prove that there exists $c>0$ such that

$$\sup_{\boldsymbol{x}\in\mathcal{A}_k\setminus\overline{\mathcal{D}_\star}} \mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[\rho_{k-1}H_{\partial\mathcal{A}_k\cup\partial\mathcal{D}_\star}>t\right]<\exp\left(-\frac{c}{\epsilon}t\right) \tag{A.11}$$

for all $t>0$ and for all $\epsilon\in(0,\,\epsilon_0)$. Write the left-hand side of the previous inequality as $f(t)$. Then, by the strong Markov property, Chebyshev's inequality, and (A.6), one can deduce that, for $n\in\mathbb{N}$,

$$f(n)\le f(1)^n\le \sup_{\boldsymbol{x}\in\mathcal{A}_k\setminus\overline{\mathcal{D}_\star}}\left(\rho_{k-1}\mathbb{E}_{\boldsymbol{x}}^{\epsilon}H_{\partial\mathcal{A}_k\cup\partial\mathcal{D}_\star}\right)^n\le\exp\left(-\frac{\eta}{\epsilon}n\right)$$

provided that ϵ is sufficiently small. This completes the proof of (A.9).

For (A.10), we first claim that there exists $c>0$ such that

$$\sup_{\boldsymbol{x}\in\partial\mathcal{A}_{l+1}}\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[H_{\partial\mathcal{Q}_{H+\eta}}>t\right]<\exp\left(-\frac{ct}{\epsilon}\right)\;\;\text{for all }\epsilon\in(0,\,\epsilon_0)\;. \tag{A.12}$$

The proof is identical to [28, Proof of Lemma B.2] and we will omit the detail. The main ingredient of the proof therein is the fact that for any trajectory $\phi:[0,\,t]\to\mathbb{R}^d$ such that $\phi(s)\in\mathcal{Q}_{H+\eta}^c$ for all $s\in[0,\,t]$ must satisfy

$$\int_0^t|\dot{\phi}(s)+\nabla U(\phi(s))|^2ds\ge ct \tag{A.13}$$

for some $c>0$. This follows mainly because there is no critical point of U in $\mathcal{Q}_{H+\eta}^c$. Then, (A.12) is immediate from (A.13) through Schilder's classical large deviation theorem.

Now we define two sequences of hitting times $(\pi_i)_{i=0}^{\infty}$, $(\zeta_i)_{i=1}^{\infty}$ recursively as, $\pi_0=0$ and

$$\begin{aligned}\zeta_i&=\inf\left\{s>\pi_{i-1}:\boldsymbol{x}_\epsilon(s)\in\partial\mathcal{Q}_{H+\eta}\right\}\;\;;\,i\ge1\;,\\ \pi_i&=\inf\left\{s>\zeta_i:\boldsymbol{x}_\epsilon(s)\in\partial\mathcal{A}_{l+1}\text{ or }\partial\mathcal{D}_\star\right\}\;\;;\,i\ge1\;.\end{aligned}$$

Let $N=\inf\{n:\boldsymbol{x}_\epsilon(\pi_n)\in\partial\mathcal{D}_\star\}$. Then, we can write

$$H_{\partial\mathcal{D}_\star}=\pi_N=\sum_{i=1}^{N}(\zeta_i-\pi_{i-1})+\sum_{i=0}^{N}(\pi_i-\zeta_i)\;. \tag{A.14}$$

Then, by Hölder's inequality,

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{x}}^{\epsilon}\exp\left(c\rho_lH_{\partial\mathcal{D}_\star}\right)\\ &=\sum_{n=1}^{\infty}\mathbb{E}_{\boldsymbol{x}}^{\epsilon}\left[\exp\left\{c\rho_l\left(\sum_{i=1}^{N}(\zeta_i-\pi_{i-1})+\sum_{i=0}^{N}(\pi_i-\zeta_i)\right)\right\}\mathbf{1}\{N=n\}\right]\\ &\le\sum_{n=1}^{\infty}\left[\mathbb{E}_{\boldsymbol{x}}^{\epsilon}\exp\left\{3c\rho_l\sum_{i=0}^{n}(\pi_i-\zeta_i)\right\}\right]^{\frac13}\left[\mathbb{E}_{\boldsymbol{x}}^{\epsilon}\exp\left\{3c\rho_l\sum_{i=0}^{n}(\zeta_i-\pi_{i-1})\right\}\right]^{\frac13}\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[N=n\right]^{\frac13}\;.\end{aligned} \tag{A.15}$$

Now we consider the terms appeared in the last line separately. By the strong Markov property, (A.12) and the first part of the current lemma with $k = l + 1$, we get

$$\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \exp\left(3c\rho_l \sum_{i=0}^{n} (\zeta_i - \pi_{i-1})\right) \le \sup_{\boldsymbol{y} \in \partial\mathcal{A}_{l+1}} \left[\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \exp\left(3c\rho_l H_{\partial\mathcal{Q}_{H+\eta}}\right)\right]^n < 2^n \quad \text{and} \tag{A.16}$$

$$\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \exp\left(\frac{c\rho_l}{3} \sum_{i=0}^{n} (\pi_i - \zeta_i)\right) \le \sup_{\boldsymbol{y} \in \partial\mathcal{Q}_{H+\eta}} \left[\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \exp\left(\frac{c\rho_l}{3} H_{\partial\mathcal{A}_{l+1} \cup \partial\mathcal{D}_\star}\right)\right]^n < 2^n \ , \tag{A.17}$$

for all small enough c and $\epsilon \in (0,\, \epsilon_0)$. On the other hand, the strong Markov property and (A.8) implies that

$$\sup_{\boldsymbol{x} \in \partial\mathcal{A}_{l+1}} \mathbb{P}_{\boldsymbol{x}}^{\epsilon} \left[N = n\right] < \frac{1}{8^{n-1}} \ \ ; \ n \ge 1 \ . \tag{A.18}$$

Now applying (A.16), (A.17), and (A.18) to (A.15) finally yields

$$\sup_{\boldsymbol{x} \in \partial\mathcal{A}_{l+1}} \mathbb{E}_{\boldsymbol{x}}^{\epsilon} \exp\left(c\rho_l H_{\partial\mathcal{D}_\star}\right) < \sum_{n=1}^{\infty} 4^{\frac{n}{3}} \frac{1}{8^{\frac{n-1}{3}}} \le 4 \ .$$

□

A.3. **Proof of Proposition 5.4.** The main ingredient to prove Proposition 5.4 is the following exponential tail estimate for $\Delta(t)$.

Lemma A.4. *For any $\upsilon \in (0,\, 1)$, there exist constants $C_1,\, C_2,\, \epsilon_1(\upsilon) > 0$ such that,*

$$\sup_{\boldsymbol{x} \in \mathcal{V}_\star} \mathbb{P}_{\boldsymbol{x}}^{\epsilon} \left[\Delta(t) > \alpha t\right] \le C_1 \exp\left\{-C_2(\alpha - \upsilon)\rho_l t\right\} \ , \tag{A.19}$$

for all $\alpha \in (\upsilon,\, 1)$, $\epsilon \in (0,\, \epsilon_1(\upsilon))$, and $t > 0$.

Before proving this proposition, we show how it implies Proposition 5.4.

Proof of Proposition 5.4. Fix $\upsilon \in (0,\, 1)$. Then, by Lemma A.4, for all $\epsilon \in (0,\, \epsilon_1(\upsilon))$, $\boldsymbol{x} \in \mathcal{V}_\star$ and $t > 0$, we obtain

$$\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \left[\frac{\Delta(t)}{t}\right] = \int_0^1 \mathbb{P}_{\boldsymbol{x}}^{\epsilon} \left[\frac{\Delta(t)}{t} > \alpha\right] d\alpha \le \upsilon + \int_\upsilon^\infty C_1 \exp\left\{-C_2(\alpha - \upsilon)\rho_l t\right\} d\alpha = \upsilon + \frac{C}{t\rho_l} \ .$$

Therefore, by (A.3), we have

$$\mathbb{E}_{\boldsymbol{x}}^{\epsilon} \left[\widehat{\Delta}(t)\right] \le \upsilon t + \frac{C}{\theta_\epsilon \rho_l} = \upsilon t + C \exp\left(-\frac{H - h - 3\eta}{\epsilon}\right) \ .$$

Hence,

$$\limsup_{\epsilon \to 0} \sup_{\boldsymbol{x} \in \mathcal{V}_\star} \mathbb{E}_{\boldsymbol{x}}^{\epsilon} \left[\widehat{\Delta}(t)\right] \le \upsilon t \ .$$

The proof is now completed by letting $\upsilon \to 0$. □

Now we turn to the proof of Lemma A.4. Let us fix $\boldsymbol{x} \in \mathcal{B}(\boldsymbol{m}, r_0)$ for some $\boldsymbol{m} \in \mathcal{M}_\star$, and recall the cycle structure $\mathcal{A}_0 \subset \cdots \subset \mathcal{A}_{l+1}$ associated to $\boldsymbol{m}$. Recall the sequences of hitting times (σ_i) and (τ_i) from (A.4). For each i, we define a sequence of hitting times $\sigma_i = \tau_i^{(0)} \le \tau_i^{(1)} \le \cdots \le \tau_i^{(l+2)} = \tau_i$ recursively as,

$$\begin{aligned}\tau_i^{(k)} &= \inf\{s \ge \tau_i^{(k-1)} : \boldsymbol{x}_\epsilon(s) \in \partial\mathcal{D}_\star \cup \partial\mathcal{A}_k\} \ ; \ k \in [\![1,\, l+1]\!] \ ,\\ \tau_i^{(l+2)} &= \inf\{s \ge \tau_i^{(l+1)} : \boldsymbol{x}_\epsilon(s) \in \partial\mathcal{D}_\star\} \ .\end{aligned}$$

Now we write

$$\Delta^{(k)}(t) = \sum_{i=1}^{\nu(t)} (\tau_i^{(k+1)} - \tau_i^{(k)}) \ ; \ k \in [\![0,\, l+1]\!] \ . \tag{A.20}$$

With these notations, it suffices to prove the following lemma. For convenience, we set $\rho_{l+1} := \rho_l$.

Lemma A.5. *For all $k \in [\![0,\, l+1]\!]$ and $\upsilon \in (0,\, 1)$, there exist constants C_1, C_2 and $\epsilon_1 = \epsilon_1(\upsilon)$ such that,*

$$\mathbb{P}_{\boldsymbol{x}}^\epsilon \left[\Delta^{(k)}(t) > \alpha t\right] \le C_1 \exp\left\{-C_2(\alpha - \upsilon)\rho_k t\right\} \ ,$$

for all $\alpha \in (\upsilon,\, 1)$, $\epsilon \in (0,\, \epsilon_0)$, and $t > 0$.

Proof. We fix $k \in [\![0,\, l+1]\!]$. Observe first that $\tau_i^{(k+1)} - \tau_i^{(k)} \neq 0$ if and only if $\boldsymbol{x}_\epsilon(\tau_i^{(k)}) \in \partial\mathcal{A}_k$. Denote by $\{i_1,\, i_2,\, \cdots\}$ the (random) set of i such that $\boldsymbol{x}_\epsilon(\tau_i^{(k)}) \in \partial\mathcal{A}_k$, and write $\nu_k(t) = \sup\{i : \tau_i^{(k)} \le t\}$. With these notations, we can write

$$\Delta^{(k)}(t) = \sum_{m=1}^{\nu^{(k)}(t)} (\tau_{i_m}^{(k+1)} - \tau_{i_m}^{(k)}) \ .$$

Then, by Chebyshev's inequality and Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}^\epsilon \left[\Delta^{(k)}(t) > \alpha t\right] &\le e^{-\lambda\alpha\rho_k t} \sum_{n=0}^{\infty} \mathbb{E}_{\boldsymbol{x}}^\epsilon \left[\exp\left\{\lambda\rho_k \sum_{m=1}^{\nu^{(k)}(t)} (\tau_{i_m}^{(k+1)} - \tau_{i_m}^{(k)})\right\} \mathbf{1}\left\{\nu^{(k)}(t) = n\right\}\right]\\ &\le e^{-\lambda\alpha\rho_k t} \sum_{n=0}^{\infty} \mathbb{E}_{\boldsymbol{x}}^\epsilon \left[\exp\left\{2\lambda\rho_k \sum_{m=1}^{n} (\tau_{i_m}^{(k+1)} - \tau_{i_m}^{(k)})\right\}\right]^{\frac{1}{2}} \mathbb{P}_{\boldsymbol{x}}^\epsilon \left[\nu^{(k)}(t) = n\right]^{\frac{1}{2}} \ .\end{aligned}$$

Now let $\lambda = c_0/2$ be the half of the constant that appeared in Lemma A.3. By the strong Markov property and Lemma A.3 (we use (A.9) for $k \le l$ and (A.10) for $k = l+1$),

$$\mathbb{E}_{\boldsymbol{x}}^\epsilon \left[\exp\left\{2\lambda\rho_k \sum_{m=1}^{n} (\tau_{i_m}^{(k+1)} - \tau_{i_m}^{(k)})\right\}\right]^{\frac{1}{2}} \le \sup_{\boldsymbol{y} \in \partial\mathcal{A}_k} \mathbb{E}_{\boldsymbol{y}}^\epsilon \left[\exp\left\{2\lambda\rho_k H_{\partial\mathcal{A}_{k+1} \cup \partial\mathcal{D}_\star}\right\}\right]^{\frac{n}{2}} < 2^{\frac{n}{2}} \ .$$

Summing up, we get

$$\mathbb{P}_{\boldsymbol{x}}^\epsilon \left[\Delta^{(k)}(t) > \alpha t\right] \le e^{-\lambda\alpha\rho_k t} \sum_{n=0}^{\infty} 2^{\frac{n}{2}} \mathbb{P}_{\boldsymbol{x}}^\epsilon \left[\nu^{(k)}(t) = n\right]^{\frac{1}{2}} \ . \tag{A.21}$$

Now we estimate the probability $\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[\nu^{(k)}(t)=n\right]$. Fix $\upsilon>0$ and suppose that $n>\upsilon\rho_k t$. Conditioned on the event $\{\nu^{(k)}(t)=n\}$, consider $n-1$ disjoint sub-intervals of $[0,\,t]$:

$$[\sigma_{i_1},\,\tau_{i_1}^{(k)}],\,[\sigma_{i_2},\,\tau_{i_2}^{(k)}],\cdots,\,[\sigma_{i_{n-1}},\,\tau_{i_{n-1}}^{(k)}]\,. \tag{A.22}$$

Note that the last interval $[\sigma_{i_n},\,\tau_{i_n}^{(k)}]$ is excluded since it is possible that $\tau_{i_n}^{(k)}>t$. Then, since $n>\upsilon\rho_k t$, we can find $(n-1)/2$ intervals among (A.22) that have length at most $2/(\upsilon\rho_k)$. Hence, by the strong Markov property and (A.7), there exists $\epsilon_1(\upsilon)>0$ such that

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[\nu^{(k)}(t)=n\right]&\le\sum_{S\subset\{i_1,i_2,\cdots,i_n\},\,|S|=\frac{n-1}{2}}\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[\tau_i^{(k)}-\sigma_i\le\frac{2}{\upsilon\rho_{k+1}}\ \forall i\in S\right]\\&\le\binom{n}{(n-1)/2}\sup_{\boldsymbol{y}\in\mathcal{D}_\star}\mathbb{P}_{\boldsymbol{y}}^{\epsilon}\left[H_{\partial\mathcal{A}_k}\le\frac{2}{\upsilon\rho_{k+1}}\right]^{\frac{n-1}{2}}\le\binom{n}{(n-1)/2}\frac{1}{8^{n-1}}\le\frac{1}{4^{n-1}}\end{aligned}$$

for all $\epsilon\in(0,\,\epsilon_1(\upsilon))$ and $n>\upsilon\rho_{k+1}t$. Combining this computation with (A.21), we get

$$\mathbb{P}_{\boldsymbol{x}}^{\epsilon}\left[\Delta^{(k)}(t)>\alpha t\right]\le e^{-\alpha\beta\rho_{k+1}t}\left[\sum_{n=0}^{\lceil\upsilon\rho_{k+1}t}2^{\frac{n}{2}}+\sum_{n=\upsilon\rho_{k+1}t+1}^{\infty}2^{\frac{n}{2}}\frac{1}{4^{n-1}}\right]\le Ce^{-\alpha\beta\rho_{k+1}t}e^{\upsilon\rho_{k+1}t}\,.$$

This completes the proof. ☐

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