Quasi-stationarity and Quasi-ergodicity of General Markov Processes

Junfei Zhang, Shoumei Li*and Renming Song

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Abstract

In this paper we give some general, but easy-to-check, conditions guaranteeing the quasi-stationarity and quasi-ergodicity of Markov processes. We also present several classes of Markov processes satisfying our conditions.

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1 Introduction

Suppose that E is a topological space with Borel σ-field $\mathcal{B}(E)$ and that $X = \{X_t : t \geq 0\}$ is a Markov process on E. For any $x \in E$, we use $\mathbb{P}_x$ to denote the law of the process X with initial position x. For any distribution ν on E, we define $\mathbb{P}_\nu(\cdot) := \int_E \mathbb{P}_x(\cdot)\nu(dx)$. Expectation with respect to $\mathbb{P}_\nu$ will be denoted by $\mathbb{E}_\nu$. Suppose that ζ is the lifetime of X. A distribution ν on E is called a quasi-stationary distribution of X if

$$\nu(A) = \mathbb{P}_\nu(X_t \in A|\zeta > t), \qquad \text{for all } A \in \mathcal{B}(E).$$

A distribution ν on E is called a quasi-ergodic distribution if for any distribution μ on E

$$\lim_{t\to\infty} \mathbb{E}_\mu \left(\frac{1}{t} \int_0^t 1_A(X_s)ds|\zeta > t \right) = \nu(A), \qquad \text{for all } A \in \mathcal{B}(E).$$

A quasi-ergodic distribution is also called a mean-ratio quasi-stationary distribution in some literature. Quasi-stationary and quasi-ergodic distributions are very important concepts in the investigation of asymptotic behavior of Markov processes. They have been studied intensively for a long time. There are many results on the existence and uniqueness of quasi-stationary and quasi-egodic distributions. When the state space E of the Markov process is countable, quasi-stationarity and

*Corresponding Author

quasi-ergodicity have been thoroughly studied, see, for instance, [7, 8, 10, 23]. For Markov processes on general state spaces, Breyer and Roberts [2] established the existence and uniqueness of quasi-ergodic distributions under the assumption that the Markov process is positive λ-recurrent for some constant $\lambda \leq 0$ (see also the recent paper [3, 18]). For a survey on quasi-stationary distributions, see [24]. However, for general state spaces, checking a Markov process is positive λ-recurrent is not an easy thing to do. In the literature there does not seem to be any general but easy-to-check sufficient conditions for positive λ-recurrence. The purpose of this paper is to give some general, but easy-to-check, conditions which guarantee positive λ-recurrence, the existence and uniqueness of quasi-stationarity and quasi-ergodic distributions. We then give a quite a few examples satisfying our conditions. The main results of [4] on quasi-stationary distributions of killed Brownian motions follow as consequences of our main results.

The organization of this paper is as follows. In Section 2, we give the setup of this paper and prove a key lemma which plays a crucial role in this paper. In Section 3, we use the key lemma to study quasi-stationarity and quasi-ergodicity. In the last section we give some examples satisfying our conditions.

2 The setup and a key lemma

In this paper, we always assume that E is a locally compact separable metric space with Borel σ-field $\mathcal{B}(E)$ and that m is a positive σ-finite measure on $(E, \mathcal{B}(E))$ such that $\text{Supp}[m] = E$. Suppose that $X = \{X_t : t \geq 0\}$ is a standard Markov process on E with lifetime ζ. For any $x \in E$, we use $\mathbb{P}_x$ to denote the law of the process X with initial position x. For any distribution ν on E, we define $\mathbb{P}_\nu(\cdot) := \int_E \mathbb{P}_x(\cdot)\nu(dx)$. Expectation with respect to $\mathbb{P}_\nu$ will be denoted by $\mathbb{E}_\nu$. We will use $\{P_t\}$ to denote the semigroup of X.

In this paper we will assume that X has a dual with respect to m, that is, there is a strong Markov process $\widehat{X} = \{\widehat{X}_t : t \geq 0\}$ on E with semigroup $\{\widehat{P}_t\}$ such that any $t > 0$ and nonnegative functions f and g on E,

$$\int_E f(x)P_t g(x)m(dx) = \int_E g(x)\widehat{P}_t f(x)m(dx).$$

We further assume that there exists a family of continuous and strictly positive functions $\{p(t, \cdot, \cdot)\}$ on $E \times E$ such that for any $(t, x) \in (0, \infty) \times E$ and any nonnegative function f on E,

$$P_t f(x) = \int_E p(t, x, y)f(y)m(dy), \qquad \widehat{P}_t f(x) = \int_E p(t, y, x)f(y)m(dy).$$

Since X is a standard process, for any $f \in C_0(E)$, we have by the dominated convergence theorem

$$\lim_{t\downarrow 0} P_t f(x) = f(x) \qquad x \in E.$$

Since $C_0(E)$ is dense in $L^2(E, m)$, it follows from [19, Proposition II.4.3] that $\{P_t\}$ is a strongly continuous semigroup in $L^2(E, m)$. Now the strong continuity of $\{\widehat{P}_t\}$ in $L^2(E, m)$ follows from general theory, see, for instance, [20, Corollary 1.10.6].

Under the above assumptions, the semigroups $\{P_t\}$ and $\{\widehat{P}_t\}$ are both strongly continuous contraction semigroups in $L^2(E, m)$. In fact, for any $f \in L^2(E, m)$,

$$\left(\int_E p(t,x,y)f(y)m(dy)\right)^2 \le \int_E p(t,x,y)f^2(y)m(dy),$$

thus by Fubini's theorem

$$\begin{aligned}\int_E (P_t f(x))^2 m(dx) &\le \int_E \int_E p(t,x,y)f^2(y)m(dy)m(dx) \\ &= \int_E f^2(y)\int_E p(t,x,y)m(dx)m(dy) \\ &\le \int_E f^2(y)m(dy)\end{aligned}$$

which implies that $\{P_t\}$ is a contraction semigroup on $L^2(E, m)$. Similarly, we can show that $\{\widehat{P}_t\}$ is also a contraction semigroup in $L^2(E, m)$.

The standing assumptions in this section are the following:

(**A1**) For any $t > 0$,

$$\int_E \int_E p^2(t,x,y)m(dx)m(dy) < \infty; \tag{2.1}$$

(**A2**) For any $t > 0$,

$$\sup_{x\in E}\int_E p^2(t,x,y)m(dy) < \infty, \qquad \sup_{x\in E}\int_E p^2(t,y,x)m(dy) < \infty. \tag{2.2}$$

The assumption (**A1**) implies that, for any $t > 0$, P_t and $\widehat{P}_t$ are Hilbert-Schmidt operators.

Let L and $\widehat{L}$ be the infinitesimal generators of the semigroups $\{P_t\}$ and $\{\widehat{P}_t\}$ in $L^2(E, m)$ respectively. Under the above assumptions, it follows from Jentzsch's theorem (Theorem V.6.6 on page 337 of [22]) that the common value $\lambda_0 := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(\widehat{L}))$ is non-positive and an eigenvalue of multiplicity 1 for both L and $\widehat{L}$, and that an eigenfunction ϕ_0 of L associated with λ_0 can be chosen to be strictly positive a.e. with $\|\phi_0\|_2 = 1$ and an eigenfunction ψ_0 of $\widehat{L}$ associated with λ_0 can be chosen to be strictly positive a.e. with $\|\psi_0\|_2 = 1$. Thus for all $t > 0$ and a.e. $x \in E$,

$$e^{\lambda_0 t}\phi_0(x) = \int_E p(t,x,y)\phi_0(y)m(dy), \tag{2.3}$$

$$e^{\lambda_0 t}\psi_0(x) = \int_E p(t,y,x)\psi_0(y)m(dy). \tag{2.4}$$

It follows from (**A2**) and the two displays above that ϕ_0 and ψ_0 are bounded. Using (**A2**) we easily see that

$$\sup_{x\in E}\int_E p(t,x,y)\phi_0(y)m(dy) < \infty, \qquad \sup_{x\in E}\int_E p(t,y,x)\psi_0(y)m(dy) < \infty.$$

Consequently the families $\{p(t,x,\cdot)\phi_0(\cdot) : x \in E\}$ and $\{p(t,\cdot,x)\psi_0(\cdot) : x \in E\}$ are uniformly integrable with respect to m.

Put $M := \int_E \phi_0(x)\psi_0(x)m(dx)$. It is easy to see that $M \in (0, 1]$. We could have normalized ϕ_0 and ψ_0 by $\|\phi_0\|_2 = 1$ and $\int_E \phi_0(x)\psi_0(x)m(dx) = 1$. We chose the normalization of this paper to emphasize the fact that ϕ_0 and ψ_0 are equally important.

The following lemma will play a crucial role in this paper. The idea of the proof comes from the proof of [21, Theorem 3] and the proof of [13, Theorem 2.7].

Lemma 2.1 *There exist constants $c, \gamma > 0$ such that for large t, we have*

$$|Me^{-\lambda_0 t}p(t, x, y) - \phi_0(x)\psi_0(y)| \leq ce^{-\gamma t}, \qquad x, y \in E.$$

Proof. Put $N = \{c\psi_0 : c \in \mathbb{R}\}$. It follows from the Riesz-Schauder theory of compact operators (see Section 6.6 of [1]) that $L^2(E, m) = N \oplus R$, $\hat{P}_t$ leaves N and R invariant, and that $\sigma(\hat{P}_t|_R) = \sigma(\hat{P}_t) \setminus \{e^{\lambda_0 t}\}$. Since the nonzero eigenvalues of a compact operator are isolated, it follows that there exist positive constants c_1 and γ_1 such that for large t

$$e^{-\lambda_0 t}||\hat{P}_t|_R||_2 \leq c_1 e^{-\gamma_1 t}.$$

It follows from the decomposition $L^2(E, m) = N \oplus R$ that any $f \in L^2(E, m)$ can be written as $f = c_f\psi_0 + \psi_f$, where $\psi_f \in R$. Thus

$$||e^{-\lambda_0 t}\hat{P}_t f - c_f\psi_0||_2 = e^{-\lambda_0 t}||\hat{P}_t\psi_f||_2 \leq c_1 e^{-\gamma_1 t}||\psi_f||_2. \tag{2.5}$$

Now we identify c_f. By the dominated convergence theorem, we have

$$\begin{aligned} 0 &= \lim_{t\to\infty} \int_E (e^{-\lambda_0 t}\hat{P}_t f(x) - c_f\psi_0(x))\phi_0(x)m(dx) \\ &= \lim_{t\to\infty} \int_E (e^{-\lambda_0 t}f(x)P_t\phi_0(x)m(dx) - c_f M \\ &= \int_E f(x)\phi_0(x)m(dx) - c_f M. \end{aligned}$$

Thus $c_f = \frac{1}{M}\int_E f(x)\phi_0(x)m(dx)$ and $|c_f| \leq \frac{1}{M}||f||_2||\phi_0||_2 = \frac{1}{M}||f||_2$. Consequently

$$||\psi_f||_2 \leq ||f||_2 + |c_f|||\phi_0||_2 \leq (1 + \frac{1}{M})||f||_2.$$

Therefore, it follows from (2.5) that for large t

$$||e^{-\lambda_0 t}\hat{P}_t f - c_f\psi_0||_2 \leq c_1(1 + \frac{1}{M})e^{-\gamma_1 t}||f||_2. \tag{2.6}$$

For $t > 1$ we have

$$p(t, x, y) = \int_E p(1, x, z)p(t - 1, z, y)m(dz) = \hat{P}_{t-1}f_x(y), \tag{2.7}$$

where $f_x(z) = p(1, x, z)$. It follows from (**A2**) that $f_x(\cdot) \in L^2(E, m)$, thus

$$c_{f_x} = \frac{1}{M}\int_E p(1, x, z)\phi_0(z)m(dz) = \frac{1}{M}e^{\lambda_0}\phi_0(x) \tag{2.8}$$

Put $c_2^2 := \sup_{x\in E}\int_E p^2(1,x,z)m(dz)$. Then by (2.6), (2.7) and (2.8) we have for large t

$$\begin{aligned}
& \sup_{x\in E}\int_E |Me^{-\lambda_0 t}p(t,x,y)-\phi_0(x)\psi_0(y)|^2 m(dy)\\
= & \sup_{x\in E}\int_E |Me^{-\lambda_0 t}\hat{P}_{t-1}f_x(y)-Me^{-\lambda_0}c_{f_x}\psi_0(y)|^2 m(dy)\\
\le & \sup_{||f||_2\le c_2}\int_E |Me^{-\lambda_0 t}\hat{P}_{t-1}f(y)-Me^{-\lambda_0}c_f\psi_0(y)|^2 m(dy)\\
\le & (Me^{-\lambda_0})^2 \sup_{||f||_2\le c_2}\int_E |e^{-\lambda_0(t-1)}\hat{P}_{t-1}f(y)-c_f\psi_0(y)|^2 m(dy)\\
\le & (Me^{-\lambda_0}c_1c_2(1+\frac{1}{M})e^{-\gamma_1(t-1)})^2.
\end{aligned}$$

Thus we have shown that there exist $c_3, \gamma_2 > 0$ such that for large t,

$$\sup_{x\in E}\int_E |Me^{-\lambda_0 t}p(t,x,z)-\phi_0(x)\psi_0(z)|^2 m(dz) \le c_3 e^{-\gamma_2 t}.$$

Repeating the above argument with $\{P_t\}$ and ϕ_0, we get that there exist $c_4, \gamma_3 > 0$ such that for large t,

$$\sup_{y\in E}\int_E |Me^{-\lambda_0 t}p(t,z,y)-\phi_0(z)\psi_0(y)|^2 m(dz) \le c_4 e^{-\gamma_3 t}.$$

By the semigroup property of P_t, (2.3)–(2.4) and the definition of M, we have

$$\begin{aligned}
& Me^{-\lambda_0 t}p(t,x,y)-\phi_0(x)\psi_0(y)\\
= & \frac{1}{M}\{M^2e^{-\lambda_0 t}\int_E p(\frac{t}{2},x,z)p(\frac{t}{2},z,y)m(dz)-M\phi_0(x)\psi_0(y)\}\\
= & \frac{1}{M}\{\int_E Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},x,z)Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},z,y)m(dz)\\
& -\int_E Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},x,z)\phi_0(z)\psi_0(y)m(dz)\\
& -\int_E Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},z,y)\phi_0(x)\psi_0(z)m(dz)\\
& +\int_E \phi_0(x)\psi_0(y)\phi_0(z)\psi_0(z)m(dz)\}\\
= & \frac{1}{M}\int_E (Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},x,z)-\phi_0(x)\psi_0(z))(Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},z,y)-\phi_0(z)\psi_0(y))m(dz).
\end{aligned}$$

Therefore, for large t we have,

$$\begin{aligned}
& \sup_{(x,y)\in E\times E}|Me^{-\lambda_0 t}p(t,x,y)-\phi_0(x)\psi_0(y)|^2\\
\le & \frac{1}{M^2}(\sup_{x\in E}\int_E (Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},x,z)-\phi_0(x)\psi_0(z))^2 m(dz))\\
& \cdot(\sup_{y\in E}\int_E (Me^{-\lambda_0\frac{t}{2}}p(\frac{t}{2},z,y)-\phi_0(z)\psi_0(y))^2 m(dz))
\end{aligned}$$

$$\leq \quad (\frac{1}{M}c_3 e^{-\gamma_2 \frac{t}{2}})(\frac{1}{M}c_4 e^{-\gamma_3 \frac{t}{2}}).$$

The proof is now complete. □

3 Quasi-stationarity and quasi-ergodicity

In this section we assume that m is a finite measure on E with $\mathrm{Supp}[m] = E$ and that $X = \{X_t : t \geq 0\}$ is a Markov process on E satisfying all the assumptions of the previous section up to (**A1**) and (**A2**). Instead of (**A1**) and (**A2**), we assume that, for any $t > 0$, there exists $c_t > 0$ such that

$$p(t, x, y) \leq c_t, \qquad (x, y) \in E \times E. \tag{3.1}$$

It is clear that the finiteness of m and the condition above imply (**A1**) and (**A2**).

Since m is a finite measure, we have

$$\int_E \phi_0(x)m(dx) < \infty, \qquad \int_E \psi_0(x)m(dx) < \infty.$$

Define

$$\nu_0(A) := \frac{\int_A \psi_0(y)m(dy)}{\int_E \psi_0(y)m(dy)}, \qquad A \in \mathcal{B}(E).$$

Then ν_0 is a distribution on E. The following result says that ν_0 is the unique quasi-stationary distribution of X.

Theorem 3.1 *(i) For any $f \in L^1(E, m)$ and any distribution ν on E, we have*

$$\lim_{t\to\infty} \mathbb{E}_\nu(f(X_t)|\zeta > t) = \int_E f(y)\nu_0(dy).$$

(ii) ν_0 is the unique quasi-stationary distribution of X.

Proof. (i) It suffices to prove (i) for $\nu = \delta_x, x \in E$. It follows from Lemma 2.1 that there exists $\gamma > 0$ such that, for all $x, y \in E$, we have

$$p(t, x, y) = \frac{1}{M} e^{\lambda_0 t}\phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)t}), \qquad t \to \infty.$$

For any $f \in L^1(E, m)$,

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_x(f(X_t)|\zeta > t) &= \lim_{t\to\infty} \frac{\int_E p(t, x, y)f(y)m(dy)}{\int_E p(t, x, y)m(dy)} \\
&= \lim_{t\to\infty} \frac{\int_E (\frac{1}{M} e^{\lambda_0 t}\phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)t}))f(y)m(dy)}{\int_E (\frac{1}{M} e^{\lambda_0 t}\phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)t}))m(dy)} \\
&= \frac{\int_E \psi_0(y)f(y)m(dy)}{\int_E \psi_0(y)m(dy)} = \int_E f(y)\nu_0(dy).
\end{aligned}$$

(ii) It suffices to show that for any $f \in C_0(E)$ and $t > 0$,

$$\mathbb{E}_{\nu_0}(f(X_t)|\zeta > t) = \int_E f(y)\nu_0(dy).$$

It follows from (i) that for any $x \in E$,

$$\begin{aligned}
\mathbb{E}_{\nu_0}(f(X_t)|\zeta > t) &= \frac{\int_E P_t f(x)\nu_0(dx)}{\mathbb{P}_{\nu_0}(\zeta > t)} \\
&= \frac{1}{\mathbb{P}_{\nu_0}(\zeta > t)} \lim_{s\to\infty} \frac{P_s(P_t f)(x)}{\mathbb{P}_x(\zeta > s)} \\
&= \frac{1}{\mathbb{P}_{\nu_0}(\zeta > t)} \lim_{s\to\infty} \frac{P_s(P_t f)(x)}{\mathbb{P}_x(\zeta > s+t)} \frac{\mathbb{P}_x(\zeta > s+t)}{\mathbb{P}_x(\zeta > s)} \\
&= \frac{1}{\mathbb{P}_{\nu_0}(\zeta > t)} \lim_{s\to\infty} \mathbb{E}_x(f(X_{t+s})|\zeta > s+t)\mathbb{E}_x(P_t 1(X_s)|\zeta > s) \\
&= \frac{1}{\mathbb{P}_{\nu_0}(\zeta > t)} \int_E f(y)\nu_0(dy) \int_E P_t 1(y)\nu_0(dy) \\
&= \int_E f(y)\nu_0(dy).
\end{aligned}$$

That is, ν_0 is a quasi-stationary distribution of X. Now we prove uniqueness. Suppose that $\widetilde{\nu_0}$ is any quasi-stationary distribution of X, that is, for any $f \in L^1(E,m)$, $\mathbb{E}_{\widetilde{\nu_0}}(f(X_t)|\zeta > t) = \int f(y)\widetilde{\nu_0}(dy)$. Hence, we have the following from (i)

$$\int_E f(y)\widetilde{\nu_0}(dy) = \lim_{t\to\infty} \mathbb{E}_{\widetilde{\nu_0}}(f(X_t)|\zeta > t) = \int_E f(y)\nu_0(dy).$$

That is $\widetilde{\nu_0} = \nu_0$. $\square$

Define

$$\nu_1(A) := \frac{1}{M}\int_A \phi_0(x)\psi_0(x)m(dx), \qquad A \in \mathcal{B}(E).$$

Then by the definition of M, ν_1 is a distribution on E.

Theorem 3.2 *For any $0 < a < b < 1$, any distribution ν on E and any $f, g \in L^1(E,m)$, we have*

$$\lim_{t\to\infty} \mathbb{E}_\nu(f(X_{at})g(X_t)|\zeta > t) = \int_E f(x)\nu_1(dx)\int_E g(x)\nu_0(dx), \tag{3.2}$$

$$\lim_{t\to\infty} \mathbb{E}_\nu(f(X_{at})g(X_{bt})|\zeta > t) = \int_E f(x)\nu_1(dx)\int_E g(x)\nu_1(dx). \tag{3.3}$$

In particular,

$$\lim_{t\to\infty} \mathbb{E}_\nu(f(X_{at})|\zeta > t) = \int_E f(x)\nu_1(dx).$$

Proof. By the semigroup property, we have for any $(t, x, y) \in (0, \infty) \times E \times E$,

$$p(t, x, y) = \int_E p(at, x, z)p(t - at, z, y)m(dz).$$

It follows from Lemma 2.1 that there exists $\gamma > 0$ such that for any $0 < a < 1$ and $x, y \in E$,

$$p(t, x, y) = \frac{1}{M} e^{\lambda_0 t} \phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)t}), \qquad t \to \infty,$$

$$p(at, x, y) = \frac{1}{M} e^{\lambda_0 at} \phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)at}), \qquad t \to \infty$$

and

$$p((1-a)t, x, y) = \frac{1}{M} e^{\lambda_0 (1-a)t} \phi_0(x)\psi_0(y) + o(e^{(-\gamma+\lambda_0)(1-a)t}), \qquad t \to \infty.$$

It suffices to prove for $\nu = \delta_x, x \in E$. Then, it now follows from the dominated convergence theorem that for any $0 < a < 1$ and $f \in L^1(E, m)$,

$$\begin{aligned}
& \lim_{t\to\infty} \mathbb{E}_x(f(X_{at})g(X_t)|\zeta > t) \\
= & \lim_{t\to\infty} \frac{\mathbb{E}_x(f(X_{at}g(X_t), \zeta > t))}{\int_E p(t, x, y)m(dy)} \\
= & \lim_{t\to\infty} \frac{\int_E \int_E p(at, x, z)f(z)p(t - at, z, y)g(y)m(dz)m(dy)}{\int_E p(t, x, y)m(dy)} \\
= & \frac{\int_E \phi_0(z)\psi_0(z)f(z)m((dz) \int_E \psi_0(y)g(y)m(dy)}{M \int_E \psi_0(y)m(dy)} \\
= & \int_E f(x)\nu_1(dx) \int_E g(x)\nu_0(dx),
\end{aligned}$$

that is, (3.2) holds. The proof of (3.3) is similar. □

We also have the following result.

Theorem 3.3 *For any distribution ν on E and any bounded Borel function f on E,*

$$\lim_{t\to\infty} \lim_{T\to\infty} \mathbb{E}_\nu(f(X_t)|\zeta > T) = \int_E f(x)\nu_1(dx).$$

Proof. It suffice to prove the theorem for $\nu = \delta_x, x \in E$. By Lemma 2.1 and the semigroup property,

$$\begin{aligned}
& \lim_{t\to\infty} \lim_{T\to\infty} \mathbb{E}_x(f(X_t)|\zeta > T) \\
= & \lim_{t\to\infty} \lim_{T\to\infty} \frac{\mathbb{E}_x(f(X_t), \zeta > T)}{\mathbb{P}_x(\zeta > T)} \\
= & \lim_{t\to\infty} \lim_{T\to\infty} \frac{\int_E \int_E p(t, x, z)f(z)p(T - t, z, y)m(dz)m(dy)}{\int_E p(T, x, y)m(dy)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_E \int_E \phi_0(x)\psi_0(z)f(z)\phi_0(z)\psi_0(y)m(dz)m(dy)}{M\int_E \phi_0(x)\psi_0(y)m(dy)} \\
&= \int_E f(y)\nu_1(dy).
\end{aligned}
$$

The proof is now complete. $\square$

The following result implies that ν_1 is the unique quasi-ergodic distribution of X.

Theorem 3.4 *For any distribution ν on E and any bounded Borel function f on E,*

$$
\lim_{t\to\infty} \mathbb{E}_\nu(\frac{1}{t}\int_0^t f(X_s)ds|\zeta > t) = \int_E f(x)\nu_1(dx).
$$

Proof. It suffice to prove the theorem for $\nu = \delta_x, x \in E$. Combining Fubini's theorem, the semigroup property and Lemma 2.1, we have

$$
\begin{aligned}
& \lim_{t\to\infty} \mathbb{E}_x(\frac{1}{t}\int_0^t f(X_s)ds|\zeta > t) \\
&= \lim_{t\to\infty} \frac{\mathbb{E}_x(\frac{1}{t}\int_0^t f(X_s)ds, \zeta > t)}{\mathbb{P}_x(\zeta > t)} \\
&= \lim_{t\to\infty} \frac{\frac{1}{t}\int_0^t \mathbb{E}_x(f(X_s), \zeta > t)ds}{\mathbb{P}_x(\zeta > t)} \\
&= \lim_{t\to\infty} \frac{\frac{1}{t}\int_0^t \int_E \int_E p(s,x,z)f(z)p(t-s,z,y)m(dz)m(dy)ds}{\int_E p(t,x,y)m(dy)} \\
&= \lim_{t\to\infty} \frac{\frac{1}{tM}\int_0^t \int_E \phi_0(z)\psi_0(z)f(z)m(dz)m(dy)\int_E \psi_0(y)m(dy)ds}{\int_E \psi_0(y)m(dy)} \\
&= \int_E f(x)\nu_1(dx).
\end{aligned}
$$

$\square$

Similarly, we also have the following

Lemma 3.5 *For any distribution ν on E and any bounded Borel function f on E,*

$$
\lim_{t\to\infty} \mathbb{E}_\nu((\frac{1}{t}\int_0^t f(X_s)ds)^2|\zeta > t) = (\int_E f(x)\nu_1(dx))^2.
$$

Proof. It suffice to prove the lemma for $\nu = \delta_x, x \in E$. The proof of this lemma is similar to that of Theorem 3.4 or Theorem 3.2. We only need to note that

$$
\begin{aligned}
& \mathbb{E}_x((\frac{1}{t}\int_0^t f(X_s)ds)^2|\zeta > t) \\
&= \mathbb{E}_x(\frac{1}{t}\int_0^t f(X_s)ds\frac{1}{t}\int_0^t f(X_r)dr|\zeta > t)
\end{aligned}
$$

$$\begin{aligned} &= 2\mathbb{E}_x(\frac{1}{t^2}\int_0^t\int_0^t 1_{\{s<r\}}f(X_s)f(X_r)dsdr|\zeta > t) \\ &= \frac{2}{t^2}\int_0^t\int_0^t 1_{\{s<r\}}\mathbb{E}_x(f(X_s)f(X_r)|\zeta > t)dsdr. \end{aligned}$$

We omit the details. □

Using an argument similar to the proof of Lemma 3.5, we can also find the limits of higher order conditional moments of $\int_0^t f(X_s)ds$.

As a consequence of Lemma 3.5, we can get the following result

Theorem 3.6 *For any distribution ν on E, any bounded Borel function f on E and any $\varepsilon > 0$,*

$$\lim_{t\to\infty}\mathbb{P}_\nu(|\frac{1}{t}\int_0^t f(X_s)ds - \int_E f(x)\nu_0(dx)| > \varepsilon|\zeta > t) = 0.$$

Furthermore, for any positive integer p,

$$\lim_{t\to\infty}\mathbb{E}_\nu[|\frac{1}{t}\int_0^t f(X_s)ds - \int_E f(x)\nu_0(dx)|^p|\zeta > t] = 0.$$

Proof. The first assertion follows immediately from Lemma 3.5 and Chebyshev's inequality. So we only give the proof of the second assertion. It suffice to prove the assertion for $\nu = \delta_x, x \in E$. Put $Y(t) := |\frac{1}{t}\int_0^t f(X_s)ds - \int_E f(x)\nu_0(dx)|$, for any given $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{E}_x[|\frac{1}{t}\int_0^t f(X_s)ds - \int_E f(x)\nu_0(dx)|^p|\zeta > t] \\ = \; &\mathbb{E}_x[Y(t)^p 1_{\{Y(t)>\varepsilon\}}|\zeta > t] + \mathbb{E}_x[Y(t)^p 1_{\{Y(t)\le\varepsilon\}}|\zeta > t] \\ \le \; &(2\|f\|_\infty)^p\mathbb{P}_x(Y(t) > \varepsilon|\zeta > t) + \varepsilon^p. \end{aligned}$$

Letting $\varepsilon \to 0$, we arrive at the second assertion. □

4 Examples

In this section we give several classes of processes that satisfy all the conditions of Section 3. The purpose of these examples is to show that our conditions are easy to check. We will not try to give the most general examples possible.

The first few examples are symmetric Markov processes.

Example 4.1 Suppose that for $A(x) = (a_{ij}(x))$ is a symmetric $(d \times d)$-matrix-valued function on $\mathbb{R}^d$ such that there exist positive $\eta_1 < \eta_2$ such that

$$\eta_1\sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le \eta_2\sum_{i=1}^d \xi_i^2, \qquad x, \xi \in \mathbb{R}^d.$$

Let $Y = \{Y_t : t \geq 0\}$ be a diffusion process associated with the Dirichlet form $(\mathcal{E}, H^1(\mathbb{R}^d))$, where

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} (\nabla u(x), A\nabla v(x))dx, \qquad u, v \in H^1(\mathbb{R}^d).$$

It is well-known, see for instance, [9], that Y is a symmetric Hunt process with respect to the Lebesgue measure and the process Y has a strictly positive continuous transition density $q(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and that there exists $c > 0$ such that

$$q(t, x, y) \leq ct^{-d/2}, \qquad t > 0, x, y \in \mathbb{R}^d.$$

Suppose that D is an open connected subset D of $\mathbb{R}^d$ with $|D| < \infty$. Put $E = D$ and let m be the Lebesgue measure on D, let $X = \{X_t : t \geq 0\}$ be the process on D obtained by killing Y upon exiting D. Then X is a symmetric Hunt process on E and X has a strictly positive continuous transition density $p(t, x, y)$ with respect to m. It is easy to see that X satisfies all the conditions of Section 3.

Example 4.2 Let $Y = \{Y_t : t \geq 0\}$ be a symmetric Lévy process on $\mathbb{R}^d$ with a Gaussian component. Y has a smooth density $q(t, x, y)$ with respect to the Lebesgue measure. The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is given by

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^d} (\nabla u(x), A\nabla v(x))dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 J(dy - x)dx$$

and $D(\mathcal{E}) = \overline{C_0(\mathbb{R}^d)}^{\mathcal{E}_1}$, where A is a symmetric positive definite $d \times d$ matrix, J is the Lévy measure of Y and $\mathcal{E}_1(u, u) = \mathcal{E}(u, u) + (u, u)$. So we have the following Nash's inequality

$$\|f\|_2^{2+4/d} \leq c_1 \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \cdot \|f\|_1^{4/d} \leq c_2 \mathcal{E}(u, u)\|f\|_1^{4/d}, \qquad u \in D(\mathcal{E}).$$

It follows that there exists $c > 0$ such that

$$q(t, x, y) \leq ct^{-d/2}, \qquad t > 0, x, y \in \mathbb{R}^d.$$

Suppose that D is an open connected subset D of $\mathbb{R}^d$ with $|D| < \infty$. Put $E = D$ and let m be the Lebesgue measure on D, let $X = \{X_t : t \geq 0\}$ be the process on D obtained by killing Y upon exiting D. Then X is a symmetric Hunt process on E and X has a strictly positive continuous transition density $p(t, x, y)$ with respect to m. It is easy to see that X satisfies all the conditions of Section 3.

Example 4.3 Consider the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ given by

$$\begin{aligned} \mathcal{E}(f, f) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y) dx dy \\ D(\mathcal{E}) &= \{f \in L^2(\mathbb{R}^d) : \mathcal{E}(f, f) < \infty\} \end{aligned}$$

where $J(x,y)$ is a symmetric kernel given by

$$J(x,y)=\int_{[\alpha_1,\alpha_2]}\frac{c(\alpha,x,y)}{|x-y|^{d+\alpha}\Phi(|x-y|)}\xi(d\alpha),$$

where ξ is a probability measure on $[\alpha_1,\alpha_2]\subset(0,2)$, Φ is an increasing function on $[0,\infty)$ with $c_1e^{c_2r^\beta}\le\Phi(r)\le c_3e^{c_4r^\beta}$ for some positive constants c_1,c_2,c_3,c_4 and $\beta\in(0,1]$, and $c(\alpha,x,y)$ is a function that is symmetric in (x,y) and bounded between two positive constants. Let $Y=\{Y_t:t\ge 0\}$ be a symmetric Markov process associated with the Dirichlet form $(\mathcal{E},D(\mathcal{E}))$. Then it follows from [5] that Y is a Hunt process and it admits a strictly positive continuous density $q(t,x,y)$ with respect to the Lebesgue measure such that

$$q(t,x,y)\le c_t,\qquad t>0,x,y\in\mathbb{R}^d.$$

for some positive constant $c_t>0$. Suppose that D is an open subset D of $\mathbb{R}^d$ with $|D|<\infty$. Put $E=D$ and let m be the Lebesgue measure on D, let $X=\{X_t:t\ge 0\}$ be the process on D obtained by killing Y upon exiting D. Then X is a symmetric Hunt process on E and X has a strictly positive continuous transition density $p(t,x,y)$ with respect to m. It is easy to see that X satisfies all the conditions of Section 3.

Example 4.4 Suppose that $A(x)=(a_{ij}(x))$ is a symmetric $(d\times d)$-matrix-valued function on $\mathbb{R}^d$ such that there exist positive $\eta_1<\eta_2$ such that

$$\eta_1\sum_{i=1}^d\xi_i^2\le\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j\le\eta_2\sum_{i=1}^d\xi_i^2,\qquad x,\xi\in\mathbb{R}^d.$$

Suppose that $J(x,y)$ is a symmetric kernel such that

$$\frac{c_1}{|x-y|^d\Phi(|x-y|)}\le J(x,y)\le\frac{c_2}{|x-y|^d\Phi(|x-y|)},$$

for some positive constants c_1,c_2 and a strictly increasing function Φ on $[0,\infty)$ satisfying the following: there exist $c_3,c_4>1$ and $0<\beta_1\le\beta_2<2$ such that

$$c_3^{-1}r^{\beta_1}\le\Phi(r)\le c_3r^{\beta_2},\qquad r\ge 1$$

and

$$c_4^{-1}r^{\beta_2}\le\Phi(r)\le c_4r^{\beta_1},\qquad 0\le r\le 1.$$

Let $Y=\{Y_t:t\ge 0\}$ be a symmetric Hunt process associated with the Dirichlet form $(\mathcal{E},D(\mathcal{E}))$, where

$$\mathcal{E}(u,v)=\int_{\mathbb{R}^d}(\nabla u(x),A\nabla v(x))dx+\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x)-u(y))^2J(dy-x)dx$$

and $D(\mathcal{E}) = \overline{C_0(\mathbb{R}^d)}^{\mathcal{E}_1}$, where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)$. Then it follows from [6] that Y is a Hunt process and it admits a strictly positive continuous density $q(t,x,y)$ with respect to the Lebesgue measure such that

$$q(t,x,y) \le c_t, \qquad t > 0, x, y \in \mathbb{R}^d.$$

for some positive constant $c_t > 0$. Suppose that D is an open subset D of $\mathbb{R}^d$ with $|D| < \infty$. Put $E = D$ and let m be the Lebesgue measure on D, let $X = \{X_t : t \ge 0\}$ be the process on D obtained by killing Y upon exiting D. Then X is a symmetric Hunt process on E and X has a strictly positive continuous transition density $p(t,x,y)$ with respect to m. It is easy to see that X satisfies all the conditions of Section 3.

The next few examples are non-symmetric Markov processes.

Example 4.5 Suppose that $\alpha \in (0,2)$ and $Y = \{Y_t : t \ge 0\}$ is a strictly α-stable process in $\mathbb{R}^d$. Suppose that, in the case $d \ge 2$, the the spherical part η of the Lévy measure of Y satisfying the following assumption: there exist $\Phi : S \to (0,\infty)$ and $\kappa > 1$ such that

$$\Phi = \frac{d\eta}{d\sigma} \quad \text{and} \quad \kappa^{-1} \le \Phi(z) \le \kappa,$$

where σ is the surface measure on the unit sphere S in $\mathbb{R}^d$. In the case $d = 1$, we assume that the Lv́ey measure is given by

$$J(dx) = c_1 x^{-1-\alpha} 1_{\{x>0\}} dx + c_2 |x|^{-1-\alpha} 1_{\{x<0\}} dx$$

with $c_1, c_2 > 0$. The dual of Y with respect to the Lebesgue measure is the process $\{-Y_t : t \ge 0\}$. Y has a smooth density $q(t,x,y)$ with respect to the Lebesgue measure and that there exists $c > 0$ such that

$$q(t,x,y) \le c t^{-d/\alpha}, \qquad t > 0,\ x, y \in \mathbb{R}^d.$$

For these basic facts, see [25]. Suppose that D is an open subset D of $\mathbb{R}^d$ with $|D| < \infty$. Put $E = D$ and let m be the Lebesgue measure on D, let $X = \{X_t : t \ge 0\}$ be the process on D obtained by killing Y upon exiting D. Then X is a Hunt process on E and X has a strictly positive continuous transition density $p(t,x,y)$ with respect to m. For these facts, see [15, Example 4.1]. It is easy to see that X satisfies all the conditions of Section 3.

Example 4.6 Suppose that $d \ge 3$ and that $\mu = (\mu^1, \cdots, \mu^d)$, where each μ^j is a signed measure on $\mathbb{R}^d$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j|(dy)}{|x-y|^{d-1}} = 0.$$

Let $Y = \{Y_t : t \ge 0\}$ is a Brownian motion with drift μ in $\mathbb{R}^d$, see [11]. It is known (see, again, [11]) that Y admits a continuous strictly positive transition density $q(t,x,y)$ with respect to the Lebesgue measure and that there exist $c_1, c_2 > 0$ such that

$$q(t,x,y) \le c_1 e^{c_2 t} t^{-d/2}, \qquad t > 0, x, y \in \mathbb{R}^d.$$

Suppose that D is a bounded connected open subset of $\mathbb{R}^d$ and suppose $K > 0$ is such that $D \subset B(0, K/2)$. Put $B = B(0, K)$. Let G_B be the Green function of Y in B and define

$$H(x) = \int_B G_B(y, x)dy.$$

Then H is a strictly positive continuous function on B. Take $E = D$ and let m be the measure defined by $m(dx) = H(x)dx$ on E. Let $X = \{X_t : t \geq 0\}$ be the process obtained by killing Y upon exiting E. X is a Hunt process on E. Let $q_E(t, x, y)$ be the transition density of X with respect to the Lebesgue measure. $q_E(t, x, y)$ is strictly positive and continuous. It follows from [12, 14] that X has a dual with respect to the measure m and the transition density of X with respect to m is given by

$$p(t, x, y) = \frac{q_E(t, x, y)}{H(y)}, \qquad (t, x, y) \in (0, \infty) \times E \times E.$$

Thus there exists $c_3, c_4 > 0$ such that

$$p(t, x, y) \leq c_3 e^{c_4 t} t^{-d/2}, \qquad (t, x, y) \in (0, \infty) \times E \times E.$$

It is easy to see that X satisfies all the conditions of Section 3.

Example 4.7 Suppose that $d \geq 2$, $\alpha \in (1, 2)$ and that $\mu = (\mu^1, \cdots, \mu^d)$, where each μ^j is a signed measure on $\mathbb{R}^d$ such that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j|(dy)}{|x - y|^{d-\alpha+1}} = 0.$$

Let $Y = \{Y_t : t \geq 0\}$ is an α-stable with drift μ in $\mathbb{R}^d$, see [16]. It is known (see, again, [16]) that Y admits a continuous transition density $q(t, x, y)$ with respect to the Lebesgue measure and that there exist $c_1, c_2 > 0$ such that

$$q(t, x, y) \leq c_1 e^{c_2 t} t^{-d/\alpha}, \qquad t > 0, x, y \in \mathbb{R}^d.$$

Suppose that D is a bounded open subset of $\mathbb{R}^d$ and suppose $K > 0$ is such that $D \subset B(0, K/2)$. Put $B = B(0, K)$. Let G_B be the Green function of Y in B and define

$$H(x) = \int_B G_B(y, x)dy.$$

Then H is a strictly positive continuous function on B. Take $E = D$ and let m be the measure defined by $m(dx) = H(x)dx$ on E. Let $X = \{X_t : t \geq 0\}$ be the process obtained by killing Y upon exiting E. X is a Hunt process on E. Let $q_E(t, x, y)$ be the transition density of X with respect to the Lebesgue measure. $q_E(t, x, y)$ is strictly positive and continuous. It follows from [17] that X has a dual with respect to the measure m and the transition density of X with respect to m is given by

$$p(t, x, y) = \frac{q_E(t, x, y)}{H(y)}, \qquad (t, x, y) \in (0, \infty) \times E \times E.$$

Thus there exists $c_3, c_4 > 0$ such that

$$p(t,x,y) \leq c_3 e^{c_4 t} t^{-d/\alpha}, \qquad (t,x,y) \in (0,\infty) \times E \times E.$$

It is easy to see that X satisfies all the conditions of Section 3.

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Junfei Zhang
College of Applied Sciences, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing, 100124, P.R. China
Email: `junfeizhang@emails.bjut.edu.cn`

Shoumei Li
College of Applied Sciences, Beijing University of Technology, 100 Pingleyuan, Chaoyang District, Beijing, 100124, P.R. China
Email: `lisma@bjut.edu.cn`

Renming Song
Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
E-mail: `rsong@math.uiuc.edu`