

Differential one-forms on Dirichlet spaces and Bakry-Émery estimates on metric graphs

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Abstract

We develop a general framework on Dirichlet spaces to prove a weak form of the Bakry-Émery estimate and study its consequences. This estimate may be satisfied in situations, like metric graphs, where generalized notions of Ricci curvature lower bounds are not available.

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1 Introduction

In the last few years, there has been much work toward defining curvature bounds for general metric measure spaces (see [AGS14a, LV09, Stu06a, Stu06b]). In this work, we are interested in mostly one-dimensional Dirichlet spaces, like metric graphs, which are spaces for which good notions of curvature have been elusive so far. Despite the lack of good *curvature bounds* on those spaces, we prove that metric graphs with finite number of edges satisfy a weak form of the Bakry-Émery estimate:

$$\sqrt{\Gamma(e^{t\Delta}f)} \leq C_1 e^{t\Delta} \sqrt{\Gamma(f)}, \quad 0 \leq t \leq 1, \quad (1)$$

where Δ is the generator of a Dirichlet form and Γ the associated carré du champ defined by

$$\Gamma(f, g) = \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f).$$

The equality defining Γ is usually understood in the weak sense of [BH91, Proposition 4.1.3]. As with all bilinear operators we will denote $\Gamma(f) = \Gamma(f, f)$.

We show that for metric graphs the optimal C_1 in the inequality (1) is bounded from below by $(\max \deg v - 1)$, where the maximum is taken over the set of vertices of the graph. Therefore, for non trivial graphs, $C_1 > 1$. Weak Bakry-Émery estimates of the type (1) with $C_1 > 1$, have already been met in the literature and are known to be satisfied in the H-type Heisenberg groups (see [Li06, BBBC08, Eld10]). It turns out that weak Bakry-Émery estimates are actually sufficient to recover several key consequences of the classical one (which corresponds to $C_1 = 1$), see [BBBC08] for heat kernel functional inequalities, [KM07] for the Hardy-Littlewood-Sobolev theory and [Kuw10] for Wasserstein spaces Lipschitz continuity properties. Therefore, it is interesting to develop a general framework to understand them.

In classical situations, the Bakry-Émery estimate with $C_1 = 1$ is proved as a consequence of a lower bound on the Bakry's Γ_2 operator

$$\Gamma_2(f, g) = \frac{1}{2}(\Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f)).$$

However, in a general framework, the Γ_2 operator may not even be defined in a strong sense, since $\Gamma(f, g)$ fails to be in the domain of Δ for a reasonable class \mathcal{C} of $f, g \in \mathcal{C}$.

More recently, it has been proved in [AGS14b] and [AGS15], that under mild conditions, the Bakry-Émery estimate with $C_1 = 1$ is actually equivalent to the underlying metric space satisfying a Riemannian Ricci curvature lower bound in the sense of [AGS14a].

In singular spaces, where no generalized Ricci curvature lower bounds are satisfied and no Γ_2 -calculus is available, to prove the weak Bakry-Émery estimate, it seems fruitful to take advantage of and further develop the theory of measurable one-forms on Dirichlet spaces that was originally devised in [CS03, Hin10, IRT12, HRT13, HKT15]. A main observation is the intertwining property

$$\partial e^{t\Delta} = e^{t\bar{\Delta}} \partial, \quad (2)$$

where ∂ is the exterior derivative and $e^{t\bar{\Delta}}$ a semigroup on *one-forms*. Proving the semigroup domination

$$\|e^{t\bar{\Delta}} \eta\| \leq C_1 e^{t\Delta} \|\eta\|, \quad 0 \leq t \leq 1. \quad (3)$$

therefore implies (1).

The paper is organized as follows. In Section 2, we present the necessary preliminaries about the space of measurable one-forms on a Dirichlet space and prove the intertwining (2). As a consequence, we prove that the weak Bakry-Émery estimate is satisfied in large times, with exponential decay, in a general class of compact Dirichlet spaces. Admittedly, this is mainly a spectral effect. *Curvature* is what controls the estimate in small times.

In Section 3, we explore consequences of (1) that go beyond the usual applications. We are mostly interested in the space of bounded variation functions and isoperimetric type inequalities. Bounded variation functions in Dirichlet spaces may be defined by adapting the original ideas of De Giorgi [DG54]. More precisely, one defines

$$\text{Var } f = \sup \{ \langle f, \partial^* \eta \rangle_2 \mid \eta \in \text{Dom } \partial^*, \|\eta\| \leq 1 \}$$

where ∂^* is the adjoint of the exterior derivative (one may also think of it as a generalized divergence in a distributional sense, see [HRT13]). If $f \in \text{Dom } \mathcal{E}$, then $\text{Var } f$ is comparable to $\int \sqrt{\Gamma(f)} d\mu$, but in geometric measure theory, one is typically interested in situations where $f \notin \text{Dom } \mathcal{E}$. With this definition in hand and the semigroup domination (3), we obtain Sobolev embeddings of the type

$$\|f\|_p \leq C_Q (\text{Var } f + \|f\|_1),$$

where $p = \frac{Q}{Q-1}$ and Q is the semigroup dimension (in the sense of Varopoulos). The corresponding isoperimetric inequality writes

$$\mu(E)^{\frac{Q-1}{Q}} \leq C_{iso} P(E).$$

where $P(E) := \text{Var } 1_E$ is the perimeter of a Caccioppoli set. We also prove generalizations of the Buser's and Ledoux's isoperimetric inequalities. The main work is to adapt to our framework some ideas originally due to Ledoux [BGL14, Led94, Led03].

In Section 4, we specialize our study to the case of Hino index one Dirichlet spaces. In those spaces, one forms may be identified with functions. As a consequence the semigroup domination (3) is equivalent to a semigroup domination on functions:

$$|e^{t\Delta^\perp} f| \leq C e^{t\Delta} |f|$$

where Δ^\perp is a self-adjoint operator (in general non-Markovian), that we call Poincaré dual of Δ . For instance, if Δ is the Laplace operator on an interval with Neumann boundary conditions, then Δ^\perp is the Laplace operator with Dirichlet boundary conditions. We end the section with the study of the situation where the reference measure comes from a harmonic form. In this special situation, the Dirichlet space associated to Δ^\perp is an extension of the original Dirichlet space and the Bakry-Émery estimate with constant $C_1 = 1$ is satisfied for a suitable class of functions f .

In the last Section 5, we study in detail a class of examples to which the previous results apply. We first show that on the Walsh spider with N legs, one has

$$\sqrt{\Gamma(e^{t\Delta} f)} \leq (N - 1) e^{t\Delta} \sqrt{\Gamma(f)}, \quad t \geq 0.$$

where the constant $N - 1$ is optimal. The weak Bakry-Émery estimate is then generalized to any metric graph (with standard boundary conditions at the vertices) that has a finite number of edges. Some consequences are then explored.

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2 Differential one-forms on Dirichlet spaces

2.1 Preliminaries

This section mostly establishes definitions and collects known results from the theory of differential forms on Dirichlet spaces as can be found in [IRT12, HRT13, HKT15]. These works built on ideas found in [CS03].

Let $(\mathcal{E}, \text{Dom } \mathcal{E})$ be a symmetric strictly local regular Dirichlet form on $L^2(X, \mu)$ with self-adjoint generator Δ , where (X, d) is a locally compact separable metric space, and μ is a locally finite Borel measure such that $\mu(U) > 0$ when U is a non empty open set. Throughout the paper it is always assumed that \mathcal{E} admits a carré du champ (see [BH91]) which shall be denoted by Γ . We adopt the convention that Δ is a negative operator in the sense that for $u \in \text{Dom } \Delta$,

$$\mathcal{E}(u, u) = - \int u \Delta u d\mu$$

Example 2.1. If \mathbb{M} is a smooth complete Riemannian manifold and

$$\mathcal{E}(f, g) = \int_{\mathbb{M}} \langle df, dg \rangle_{T^*\mathbb{M}} d\mu, \quad f, g \in W^{1,2}(\mathbb{M})$$

where μ is the Riemannian volume measure, then $\Gamma(f, g) = \langle df, dg \rangle_{T^*\mathbb{M}}$ and the restriction of Δ to smooth functions coincides with the Laplace-Beltrami operator.

Let $\mathcal{B}_b(X)$ be defined to be the set of bounded Borel measurable functions on X , and $\mathcal{C} := C_b(X) \cap \text{Dom } \mathcal{E}$. By regularity, \mathcal{C} is dense in $\text{Dom } \mathcal{E}$. For simple tensors in the vector space $\mathcal{C} \otimes \mathcal{B}_b(X)$, define the scalar product

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{\mathcal{H}} = \int g_1 g_2 \Gamma(f_1, f_2) d\mu.$$

The above product is non-negative definite, and thus by factoring out the 0-seminorm elements and completing defines a Hilbert space which will be denoted by \mathcal{H} . We think of \mathcal{H} as the space of L^2 differential one-forms \mathcal{H} on X (as in [CS03]).

Example 2.2. For the Dirichlet form in Example 2.1, one has

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{\mathcal{H}} = \int_{\mathbb{M}} g_1 g_2 \langle df_1, df_2 \rangle_{T^*\mathbb{M}} d\mu.$$

and, up to isomorphism, \mathcal{H} is the Hilbert space of square integrable one-forms (see [HRT13]).

The space \mathcal{H} is a \mathcal{C} -left module structure and $\mathcal{B}_b(X)$ -right module with multiplication defined to be

$$a \cdot f \otimes g = (af) \otimes g - a \otimes (fg) \quad \text{and} \quad f \otimes g \cdot a = f \otimes ag$$

respectively. Since \mathcal{E} is strictly local, left and right multiplications actually coincide (see [FOT11, Lemma 3.2.5]). We define an exterior derivative $\partial : \mathcal{C} \rightarrow \mathcal{H}$ by

$$\partial f = f \otimes \mathbf{1},$$

where $\mathbf{1}$ is the constant function equal to 1. From the definition

$$\langle \partial f, \partial g \rangle_{\mathcal{H}} = \int \Gamma(f, g) d\mu = \mathcal{E}(f, g),$$

and hence ∂ is a closable operator because \mathcal{E} is a closed Dirichlet form. Therefore ∂ extends to a densely defined closed linear operator $L^2(X, \mu) \rightarrow \mathcal{H}$ with domain

$$\text{Dom } \partial = \text{Dom } \mathcal{E}.$$

Example 2.3. In the case of Example 2.1, the restriction of ∂ to smooth functions coincides with the usual derivative d .

Note that since \mathcal{E} is assumed to be strictly local ∂ admits a chain rule (see [FOT11, Lemma 3.2.5]): If F is continuously differentiable and $f^1, f^2, \dots, f^n \in \text{Dom } \mathcal{E}$, then

$$\partial F(f^1, f^2, \dots, f^n) = \sum_{i=1}^n \frac{\partial F}{\partial x^i} \partial f^i.$$

The co-differential is defined as the adjoint of the exterior differential ∂ . More precisely, the operator ∂^* is the densely defined operator from $\mathcal{H} \rightarrow L^2(X, \mu)$ with domain

$$\text{Dom } \partial^* := \{ \eta \in \mathcal{H} : \exists f \in L^2(X, \mu), \text{ with } \langle \eta, \partial \phi \rangle_{\mathcal{H}} = \langle f, \phi \rangle_2 \ \forall \phi \in \text{Dom } \mathcal{E} \},$$

and we have $\partial^* \eta = f$. The operator ∂^* may also be interpreted in a distributional sense (see [HRT13]).

Example 2.4. In the case of Example 2.1, the restriction of ∂^* to smooth forms coincides with the usual divergence δ .

Observe that $u \in \text{Dom } \mathcal{E}$ is in $\text{Dom } \Delta$ if and only if there exists $\Delta u \in L^2(X, \mu)$ such that $\mathcal{E}(u, \phi) = -\langle \Delta u, \phi \rangle$ for all $\phi \in \text{Dom } \mathcal{E}$. Hence

$$\text{Dom } \Delta = \{ u \in \text{Dom } \mathcal{E} \mid \partial u \in \text{Dom } \partial^* \}.$$

and for $u \in \text{Dom } \Delta$ we have $\partial^* \partial u = -\Delta u$.

2.2 Laplacian on one-forms

Define the 1-form Laplacian, as in [HKT15] by $\vec{\Delta} = -\partial \partial^*$ with the domain

$$\text{Dom } \vec{\Delta} = \{ \eta \in \mathcal{H} \mid \partial^* \eta \in \text{Dom } \partial \}.$$

Since ∂^* is densely defined and closed, from a Von Neumann's theorem (confer [Tay96, theorem 8.4] or the proof of theorem VIII.32 in [RS72]), the operator $\vec{\Delta} = -(\partial^*)^* \partial^*$ is self-adjoint. Alternatively, since ∂^* is closed, we may also see $\vec{\Delta}$ as the self-adjoint generator of the closed symmetric bilinear form on \mathcal{H}

$$\vec{\mathcal{E}}(\omega, \eta) = \langle \partial^* \omega, \partial^* \eta \rangle_{L^2}.$$

Remark 2.1. In the vein of Example 2.1, if X is a one-dimensional Riemannian manifold, the space of 2-forms is trivial, therefore the restriction of $\vec{\Delta}$ to smooth forms coincides with the Hodge–de Rham Laplacian $dd^* + d^*d$. In [HT14], it is shown more generally that for topologically one dimensional spaces, there are no associated differential 2-forms, and hence $\vec{\Delta}$ is analogous to the Hodge–de Rham Laplacian. This definition of form Laplacian is tailor made for 1-dimensional situations.

Our first result is the following:

Theorem 2.1. *For $f \in \text{Dom } \mathcal{E}$,*

$$\partial e^{t\Delta} f = e^{t\vec{\Delta}} \partial f, \quad t \geq 0.$$

Proof. For every $f \in \text{Dom}(\Delta)$, $\omega \in \text{Dom } \partial^*$, one has

$$\langle \Delta f, \partial^* \omega \rangle_2 = -\vec{\mathcal{E}}(\partial f, \omega).$$

The result is then a consequence of Theorem 3.1 in Shigekawa [Shi00]. \square

We can describe more precisely $\vec{\Delta}$ and its domain in the special case where Δ has pure point spectrum, i.e. there exists an increasing sequence $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ of eigenvalues of $-\Delta$, with finite multiplicity, and a complete orthonormal basis $(\phi_j)_{j \geq 1}$ of corresponding eigenfunctions such that

$$-\Delta f = \sum_{j=1}^{+\infty} \lambda_j \langle f, \phi_j \rangle \phi_j, \quad f \in \text{Dom } \Delta.$$

Lemma 2.2. *Assume that Δ has a pure point spectrum. Then,*

$$\text{Dom } \partial^* = \left\{ \eta \in \mathcal{H}, \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}}^2 < +\infty \right\},$$

and for $\eta \in \text{Dom } \partial^*$,

$$\partial^* \eta = \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}} \phi_j.$$

Furthermore

$$\text{Dom } \vec{\Delta} = \left\{ \eta \in \mathcal{H}, \sum_{j=1}^{+\infty} \lambda_j \langle \eta, \partial \phi_j \rangle_{\mathcal{H}}^2 < +\infty \right\}$$

and for every $\eta \in \text{Dom } \vec{\Delta}$,

$$-\vec{\Delta} \eta = \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}} \partial \phi_j.$$

Proof. We observe first that

$$\begin{aligned} \text{Dom } \mathcal{E} &= \left\{ f \in L^2(X, \mu), \lim_{t \rightarrow 0} \left\langle \frac{f - e^{t\Delta} f}{t}, f \right\rangle_{L^2(X, \mu)} \text{ exists} \right\} \\ &= \left\{ f \in L^2(X, \mu), \lim_{t \rightarrow 0} \frac{1}{t} \sum_{j=1}^{+\infty} (1 - e^{-\lambda_j t}) \langle f, \phi_j \rangle^2 \text{ exists} \right\} \\ &= \left\{ f \in L^2(X, \mu), \sum_{j=1}^{+\infty} \lambda_j \langle f, \phi_j \rangle^2 < +\infty \right\}, \end{aligned}$$

and moreover that for $f \in \text{Dom } \partial = \text{Dom } \mathcal{E}$,

$$\partial f = \sum_{j=1}^{+\infty} \langle f, \phi_j \rangle \partial \phi_j. \tag{4}$$

As a consequence,

$$\text{Dom } \partial^* = \left\{ \eta \in \mathcal{H}, \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}}^2 < +\infty \right\},$$

and for $\eta \in \text{Dom } \partial^*$,

$$\partial^* \eta = \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}} \phi_j.$$

From the definition of $\vec{\Delta}$, this immediately yields

$$\text{Dom } \vec{\Delta} = \left\{ \eta \in \mathcal{H}, \sum_{j=1}^{+\infty} \lambda_j \langle \eta, \partial \phi_j \rangle_{\mathcal{H}}^2 < +\infty \right\}$$

and for every $\eta \in \text{Dom } \vec{\Delta}$ we have,

$$-\vec{\Delta} \eta = \sum_{j=1}^{+\infty} \langle \eta, \partial \phi_j \rangle_{\mathcal{H}} \partial \phi_j.$$

□

2.3 Bakry-Émery estimates

The intertwining property in Theorem 2.1 may be used to establish Bakry-Émery type estimates for $e^{t\Delta}$.

It is possible to think of \mathcal{H} as measurable sections of a vector bundle over X . Our presentation follows [HRT13], but follows from [Ebe99]. Let $\{f_n\}_{n=1}^{\infty}$ be a countable set of functions which is \mathcal{E} -dense in \mathcal{C} . Define $\mathcal{A} := \text{span } \{f_n\}_{n=1}^{\infty}$. Define a positive bilinear form on simple tensors of $\mathcal{A} \otimes \mathcal{B}_b(X)$ by

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{\mathcal{H}_x} = g_1(x)g_2(x)\Gamma(f_1, f_2)(x).$$

The fibre of \mathcal{H} at x is defined to be the space $\mathcal{H}_x := \mathcal{A} / \ker \langle \cdot, \cdot \rangle_{\mathcal{H}_x}$, where

$$\ker \langle \cdot, \cdot \rangle_{\mathcal{H}_x} := \left\{ \eta \in \mathcal{A} \otimes \mathcal{B}_b(X) : \langle \eta, \eta \rangle_{\mathcal{H}_x} = 0 \right\}.$$

Example 2.5. In the case of Example 2.1, \mathcal{H}_x can be identified with $T_x^* \mathbb{M}$.

Theorem 2.3 ([HRT13], Theorem 2.1 or [Ebe99], Theorem 3.9). *The fibres \mathcal{H}_x are a measurable field over X , and \mathcal{H} is isometrically isomorphic to $\int_X^{\oplus} \mathcal{H}_x d\mu(x)$. In particular, for any $\eta_1, \eta_2 \in \mathcal{H}$,*

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{H}} = \int_X \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_x} d\mu(x).$$

$\|\cdot\|_{\mathcal{H}_x}$ shall be used to denote the fiberwise norm associated to $\langle \cdot, \cdot \rangle_{\mathcal{H}_x}$. Note that, for any $a_1, a_2 \in \mathcal{B}_b(X)$ and $\eta_1, \eta_2 \in \mathcal{H}$ then, $\langle a_1 \eta_1, a_2 \eta_2 \rangle_{\mathcal{H}_x} = a_1(x)a_2(x) \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_x}$.

The following result is then easy to establish.

Theorem 2.4. *Let $C_1 \geq 1$. Assume that for every $\eta \in \mathcal{H}$, we have μ -almost everywhere*

$$\|e^{t\vec{\Delta}}\eta\|_{\mathcal{H}_x} \leq C_1(e^{t\Delta}\|\eta\|_{\mathcal{H}})(x), \quad 0 \leq t \leq 1.$$

Then, the semigroup $e^{t\Delta}$ satisfies the Bakry-Émery estimate

$$\sqrt{\Gamma(e^{t\Delta}f)} \leq C_1 e^{C_2 t} e^{t\Delta} \sqrt{\Gamma(f)}, \quad f \in \text{Dom } \mathcal{E}, \quad t \geq 0,$$

for some $C_2 \in \mathbb{R}$.

Proof. From Theorem 2.1, we have for $f \in \text{Dom } \mathcal{E}$,

$$\partial e^{t\Delta}f = e^{t\vec{\Delta}}\partial f.$$

Since $\|\partial e^{t\Delta}f\|_{\mathcal{H}_x} = \sqrt{\Gamma(e^{t\Delta}f)(x)}$, we deduce that for $0 \leq t \leq 1$, we have

$$\sqrt{\Gamma(e^{t\Delta}f)} \leq C_1 e^{t\Delta} \sqrt{\Gamma(f)}.$$

Applying this inequality with $e^{t\Delta}f$ instead of f , we deduce that for $0 \leq t \leq 1$, we have

$$\sqrt{\Gamma(e^{2t\Delta}f)} \leq C_1^2 e^{2t\Delta} \sqrt{\Gamma(f)}.$$

By induction, we then easily deduce that for $n \in \mathbb{N}$, $n \geq 1$, and $t \in [n-1, n]$, we have

$$\sqrt{\Gamma(e^{t\Delta}f)} \leq C_1^n e^{t\Delta} \sqrt{\Gamma(f)}.$$

We conclude then that

$$\sqrt{\Gamma(e^{t\Delta}f)} \leq C_1 C_1^{n-1} e^{t\Delta} \sqrt{\Gamma(f)} \leq C_1 e^{C_2 t} e^{t\Delta} \sqrt{\Gamma(f)},$$

with $C_2 = \ln C_1$. □

Remark 2.2. One can deduce from [Shi97] several statements equivalent to the semigroup domination

$$\|e^{t\vec{\Delta}}\eta\|_{\mathcal{H}_x} \leq (e^{t\Delta}\|\eta\|_{\mathcal{H}})(x), \quad 0 \leq t \leq 1.$$

However, in this work, we shall be more interested in situations for which the optimal C_1 is strictly larger than 1. Such situations include the metric graphs studied in Section 5 of the present paper.

Example 2.6. In the case of Example 2.1, the semigroup domination

$$\|e^{t\vec{\Delta}}\eta\|_{\mathcal{H}_x} \leq (e^{t\Delta}\|\eta\|_{\mathcal{H}})(x), \quad 0 \leq t \leq 1.$$

is equivalent to the non negativity of the Ricci curvature tensor.

In large times, the Bakry-Émery estimate may be obtained under some *compactness* assumptions.

Theorem 2.5. Assume that Δ has a pure point spectrum, $1 \in \text{Dom } \Delta$ and that the Dirichlet space $(\mathcal{E}, \text{Dom } \mathcal{E})$ satisfies the Poincaré inequality:

$$\int_X \left(f - \int_X f d\mu \right)^2 d\mu \leq \frac{1}{\lambda_1} \mathcal{E}(f, f), \quad f \in \text{Dom } \mathcal{E}.$$

Assume moreover that the heat kernel $p_t(x, y)$ of $e^{t\Delta}$ satisfies the estimates: For some $t_0 > 0$, there exists a $M > 0$ such that for μ -almost every $x, y \in X$

$$p_{t_0}(x, y) \leq M, \quad |\Gamma(p_{t_0}(\cdot, y))(x)| \leq M.$$

Then, there exist constants $C > 0$ and $t_1 > t_0$ depending only on M , t_0 and the spectrum of Δ such that for every $t > t_1$ and $f \in \text{Dom } \mathcal{E}$,

$$\sqrt{\Gamma(e^{t\Delta} f)} \leq C e^{-\lambda_1 t} e^{t\Delta} \sqrt{\Gamma(f)}.$$

Proof. From the assumptions $\mu(X) < +\infty$. For convenience, we assume that $\mu(X) = 1$. From the spectral decomposition

$$-\Delta f = \sum_{j=1}^{+\infty} \lambda_j \langle f, \phi_j \rangle \phi_j, \quad f \in \text{Dom } \Delta,$$

where the λ_i 's are the non-zero eigenvalues, and the Poincaré inequality we deduce that

$$p_t(x, y) = 1 + \sum_{j=1}^{+\infty} e^{-\lambda_j t} \phi_j(x) \phi_j(y).$$

Since $e^{t_0 \Delta} \phi_j = e^{-\lambda_j t_0} \phi_j$, we deduce that μ almost everywhere

$$\begin{aligned} |\phi_j(x)| &= e^{\lambda_j t_0} \left| \int_X p_{t_0}(x, y) \phi_j(y) d\mu(y) \right| \\ &\leq e^{\lambda_j t_0} \left(\int_X p_{t_0}(x, y)^2 d\mu(y) \right)^{1/2} \\ &\leq M e^{\lambda_j t_0}. \end{aligned}$$

Similarly, from the bound $|\Gamma(p_{t_0}(\cdot, y))(x)| \leq M$, one obtains

$$\|\partial \phi_j\|_{\mathcal{H}_x} \leq M e^{\lambda_j t_0}.$$

As a first consequence, for $t > 2t_0$,

$$\begin{aligned} |p_t(x, y) - 1| &\leq \sum_{j=1}^{+\infty} e^{-\lambda_j t} |\phi_j(x) \phi_j(y)| \\ &\leq M^2 \sum_{j=1}^{+\infty} e^{-\lambda_j (t-2t_0)}, \end{aligned}$$

and thus

$$p_t(x, y) \geq 1 - M^2 \sum_{j=1}^{+\infty} e^{-\lambda_j(t-2t_0)}.$$

Now from (4), one has for $f \in \text{Dom } \mathcal{E}$,

$$\begin{aligned} \partial e^{t\Delta} f &= \sum_{j=1}^{+\infty} \langle e^{t\Delta} f, \phi_j \rangle_{\mathcal{H}} \partial \phi_j \\ &= \sum_{j=1}^{+\infty} e^{-\lambda_j t} \langle f, \phi_j \rangle_{\mathcal{H}} \partial \phi_j \\ &= - \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} e^{-\lambda_j t} \langle \partial f, \partial \phi_j \rangle_{\mathcal{H}} \partial \phi_j \end{aligned}$$

This implies that for $t > 2t_0$, μ almost everywhere

$$\|\partial e^{t\Delta} f\|_{\mathcal{H}_x} \leq M^2 \int_X \|\partial f\|_{\mathcal{H}_y} d\mu(y) \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} e^{-\lambda_j(t-2t_0)}.$$

We conclude that for t large enough

$$\|\partial e^{t\Delta} f\|_{\mathcal{H}_x} \leq C(t) (e^{t\Delta} \|\partial f\|_{\mathcal{H}})(x)$$

where

$$C(t) = \frac{M^2 \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} e^{-\lambda_j(t-2t_0)}}{1 - M^2 \sum_{j=1}^{+\infty} e^{-\lambda_j(t-2t_0)}}.$$

The conclusion easily follows. \square

3 Sobolev and isoperimetric inequalities on Dirichlet spaces

3.1 Setting

In this section, we work under the assumptions of Section 2.1. We shall moreover assume that (X, d) is a compact metric space and that $\mu(X) = 1$. Let $P_t = e^{t\Delta}$ be the heat semigroup generated by Δ . We shall assume that the semigroup on forms $\vec{P}_t = e^{t\vec{\Delta}}$ satisfies μ -almost everywhere the semigroup domination:

$$\|\vec{P}_t \eta\|_{\mathcal{H}_x} \leq C_1 (P_t \|\eta\|_{\mathcal{H}})(x), \quad \eta \in \mathcal{H}, \quad 0 \leq t \leq 1, \quad (5)$$

where $C \geq 1$ is a constant. From the results in Section 5, this assumption is for instance satisfied in any metric graph which has a finite number of edges. It follows from Theorem 2.4 that a Bakry-Émery estimate

$$\sqrt{\Gamma(P_t f)} \leq C_1 e^{C_2 t} P_t \sqrt{\Gamma(f)}, \quad f \in \text{Dom } \mathcal{E}, \quad t \geq 0, \quad (6)$$

is satisfied for some $C_2 \geq 0$. We note that, as in Section 6, Remark 6.6, of [BBBC08], this inequality alone automatically implies a large number of functional inequalities for the heat semigroup. We also note that (6) is equivalent to a Lipschitz continuity property of P_t in the Wasserstein distance (see [Kuw10]). In this section, we are interested in the space of bounded variation functions and take full advantage of the semigroup domination (5) to prove several Sobolev and isoperimetric type inequalities.

In the sequel, we will say that \mathcal{E} satisfies a spectral gap inequality if there exists a positive λ_1 , such that for every $f \in \text{Dom } \mathcal{E}$,

$$\int \left(f - \int f d\mu \right)^2 d\mu \leq \frac{1}{\lambda_1} \mathcal{E}(f, f)$$

The best constant λ_1 in this inequality is then called the spectral gap.

We will say that \mathcal{E} satisfies a log-Sobolev inequality if there exists a positive ρ_0 , such that for every $f \in \text{Dom } \mathcal{E}$,

$$\int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \leq \frac{1}{\rho_0} \int \Gamma(f) d\mu, \quad (7)$$

The best constant ρ_0 in this inequality is then called the log-Sobolev constant.

Criteria to ensure that a spectral gap and a log-Sobolev inequality are well known. Define $p_t(x, y) : \mathbb{R}^+ \times X \times X \rightarrow \mathbb{R}$ be the heat kernel of Δ if it is the integral kernel of P_t , that is $P_t f(x) = \int_X f(y) p_t(x, y) dy$. We will assume that such kernel exists, is jointly continuous in (t, x, y) and that for some $t > 0$, $\inf_{x, y} p_t(x, y) > 0$.

A first consequence of those assumptions is the regularization property of P_t . If $f \in L^2(X)$, then the integral $\int p_t(x, y) f(y) d\mu(y)$, $t > 0$, is convergent for every $x \in X$ since

$$\begin{aligned} \int p_t(x, y) |f(y)| d\mu(y) &\leq \sqrt{\int p_t(x, y)^2 d\mu(y)} \|f\|_2 \\ &\leq \sqrt{p_{2t}(x, x)} \|f\|_2. \end{aligned}$$

As a consequence, one can define $P_t f(x)$ for every $x \in X$. Observe also that $P_t f$ is then actually a bounded continuous function since

$$\begin{aligned} |P_t f(x) - P_t f(y)| &\leq \int |p_t(x, z) - p_t(y, z)| |f(z)| d\mu(z) \\ &\leq \sqrt{\int |p_t(x, z) - p_t(y, z)|^2 d\mu(z)} \|f\|_2 \\ &\leq \sqrt{p_{2t}(x, x) - p_{2t}(x, y) + p_{2t}(y, y)} \|f\|_2 \end{aligned}$$

and similarly

$$|P_t f(x)| \leq \sqrt{p_{2t}(x, x)} \|f\|_2 \leq \sup_{x \in X} \sqrt{p_{2t}(x, x)} \|f\|_2.$$

Under our assumptions, both the spectral gap and the log-Sobolev inequality are actually satisfied. Indeed, the previous computation also shows that P_t is supercontractive, i.e for every $t > 0$, $\|P_t\|_{2 \rightarrow 4} < \infty$. Therefore from Gross' theorem (e.g. [BGL14, theorem 5.2.3] and [Dav89, theorem 2.2.3]), a defective logarithmic Sobolev inequality is satisfied, that is there exist two constants $A, B > 0$ such that

$$\int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \leq A \int \Gamma(f) d\mu + B \int f^2 d\mu.$$

Since the heat kernel is positive and the invariant measure a probability, we deduce from the uniform positivity improving property (see [Aid98], Theorem 2.11) that Δ admits a spectral gap. That is, a Poincaré inequality is satisfied. It is then classical (see [ABC⁺00]), that the conjunction of a spectral gap and a defective logarithmic Sobolev inequality implies the log-Sobolev inequality (i.e. we may actually take $B = 0$ in the above inequality).

As a consequence, for instance, the compact metric graphs considered in Section 5 satisfy a spectral gap and a log-Sobolev inequality.

3.2 Bounded variation functions and Sobolev embeddings

In this section, as a preliminary, we prove some results about the theory of bounded variation functions and Cacciopoli sets in our framework. Define

$$\text{Var } f = \sup \{ \langle f, \partial^* \eta \rangle_2 \mid \eta \in \text{Dom } \partial^*, \|\eta\|_{\mathcal{H}_x} \leq 1, \mu\text{-almost everywhere} \}$$

Define the set of functions of finite variation

$$BV(X) := \{ f \in \mathcal{B}_b(X) \mid \text{Var}(f) < \infty \}.$$

This allows one to define the perimeter of a Borel set $E \subset X$ with $\mathbf{1}_E \in BV(X)$ by

$$P(E) = \text{Var } \mathbf{1}_E.$$

Call any set E with $P(E) < \infty$ a Caccioppoli set.

Lemma 3.1. *If $f \in \text{Dom } \mathcal{E}$, one has*

$$\frac{1}{C_1} \int \sqrt{\Gamma(f)} d\mu \leq \text{Var } f \leq \int \sqrt{\Gamma(f)} d\mu.$$

Proof.

$$\begin{aligned} \text{Var } f &= \sup \{ \langle f, \partial^* \eta \rangle_2 \mid \eta \in \text{Dom } \partial^*, \|\eta\|_{\mathcal{H}_x} \leq 1, \mu\text{-almost everywhere} \} \\ &\leq \int \|\partial f\|_{\mathcal{H}_x} d\mu(x) = \int \sqrt{\Gamma(f)} d\mu. \end{aligned}$$

On the other side, let

$$\eta = \frac{1}{C_1} e^{s\vec{\Delta}} \left(\frac{\partial f}{\|\partial f\|_{\mathcal{H}} + \varepsilon} \right)$$

where $s, \varepsilon > 0$. We note that $\eta \in \text{Dom } \partial^*$. Indeed $\frac{\partial f}{\|\partial f\|_{\mathcal{H}} + \varepsilon} \in \mathcal{H}$ and thus by spectral theory $\eta \in \text{Dom } \vec{\Delta} \subset \text{Dom } \partial^*$. Also, from our assumption (5), $\|\eta\|_{\mathcal{H}_x} \leq 1$. As a consequence,

$$\text{Var } f \geq \langle f, \partial^* \eta \rangle_2 = \langle \partial f, \eta \rangle_{\mathcal{H}} = \frac{1}{C_1} \left\langle \partial f, e^{s\vec{\Delta}} \left(\frac{\partial f}{\|\partial f\|_{\mathcal{H}} + \varepsilon} \right) \right\rangle_{\mathcal{H}}.$$

Letting $s \rightarrow 0$ and then $\varepsilon \rightarrow 0$ finishes the proof. \square

Theorem 3.2. *Let $f \in BV(X)$.*

$$\text{Var } f \leq \liminf_{t \rightarrow 0} \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu \leq \limsup_{t \rightarrow 0} \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu \leq C_1^2 \text{Var } f$$

Proof. Let $f \in BV(X)$ and $\eta \in \text{Dom } \partial^*$, $\|\eta\|_{\mathcal{H}_x} \leq 1$ almost everywhere. We have

$$\langle e^{t\Delta} f, \partial^* \eta \rangle_2 = \langle \partial e^{t\Delta} f, \eta \rangle_{\mathcal{H}} \leq \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu.$$

Since we have in L^2 , $\lim_{t \rightarrow 0} e^{t\Delta} f = f$, we deduce that

$$\langle f, \partial^* \eta \rangle_2 \leq \liminf_{t \rightarrow 0} \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu,$$

and therefore

$$\text{Var } f \leq \liminf_{t \rightarrow 0} \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu.$$

By symmetry of $e^{t\Delta}$,

$$\langle e^{t\Delta} f, \partial^* \eta \rangle_2 = \langle f, e^{t\Delta} \partial^* \eta \rangle_2 = \left\langle f, \partial^* e^{t\vec{\Delta}} \eta \right\rangle_2.$$

Where the last equality is because $\partial^* e^{t\vec{\Delta}} \eta = e^{t\Delta} \partial^* \eta$ from the same logic as the proof of theorem 2.1. (i.e. [Shi00, Theorem 3.1]).

We now observe that $e^{t\vec{\Delta}} \eta \in \text{Dom } \vec{\Delta} \subset \text{Dom } \partial^*$, and $\|e^{t\vec{\Delta}} \eta\|_{\mathcal{H}_x} \leq C_1$ almost everywhere. Therefore

$$\langle e^{t\Delta} f, \partial^* \eta \rangle_2 \leq C_1 \text{Var } f,$$

which implies

$$\int \sqrt{\Gamma(e^{t\Delta} f)} d\mu \leq C_1 \text{Var}(e^{t\Delta} f) \leq C_1^2 \text{Var } f,$$

and of course

$$\limsup_{t \rightarrow 0} \int \sqrt{\Gamma(e^{t\Delta} f)} d\mu \leq C_1^2 \text{Var } f.$$

\square

The following inequality is the cornerstone of the section.

Theorem 3.3. *Assume that (6) is satisfied with $C_2 = 0$. Let $f \in BV(X)$. For $t \geq 0$,*

$$\|P_t f - f\|_1 \leq C_1^3 \sqrt{2t} \text{Var } f.$$

Proof. We adapt an argument due to Ledoux (see [Led94, p. 953]). Let $f \in \mathcal{B}_b(X) \cap \text{Dom } \mathcal{E}$. From the Bakry-Émery estimate, we have

$$P_t(f^2) - (P_t f)^2 = 2 \int_0^t P_s(\Gamma(P_{t-s} f)) ds \geq \frac{2}{C_1^2} t \Gamma(P_t f).$$

The first equality follows from the fact that

$$\frac{d}{ds} (P_s(P_{t-s} f)^2) = P_s \Gamma(P_{t-s} f).$$

To prove this, let ϕ be a positive continuous uniformly bounded function. Then, for any given $0 < s \leq t$ $P_s \phi$ and $P_{t-s} f$ are bounded from the previous section. Further, $\Gamma(P_{t-s} f)$ exists for $0 \leq s \leq t$ because $P_{t-s} f \in \mathcal{B}_b(X) \cap \text{Dom } \mathcal{E}$. Thus we have

$$\begin{aligned} \frac{d}{ds} \int \phi P_s(P_{t-s} f)^2 d\mu &= \frac{d}{ds} \int P_s \phi (P_{t-s} f)^2 d\mu \\ &= \int (\Delta P_s \phi) (P_{t-s} f)^2 - 2(P_s \phi) (\Delta P_{t-s} f) (P_{t-s} f) d\mu \\ &= -\mathcal{E}(P_s \phi, (P_{t-s} f)^2) + 2\mathcal{E}(P_s \phi P_{t-s} f, P_{t-s} f) \\ &= 2 \int (P_s \phi) \Gamma(P_{t-s} f) d\mu. \end{aligned}$$

Since this is true for all such ϕ , this implies that $P_s \Gamma(P_{t-s} f)$ exists and is equal to $\frac{d}{ds} P_s(P_{t-s} f)^2$ where the derivative is taken in L^2 .

We have thus deduced that

$$\Gamma(P_t f) \leq \frac{C_1^2}{2t} (P_t(f^2) - (P_t f)^2).$$

This implies

$$\|\sqrt{\Gamma(P_t f)}\|_\infty \leq \frac{C_1}{\sqrt{2t}} \|f\|_\infty.$$

Let now $g \in \text{Dom } \mathcal{E}$, with $g \geq 0$ and $\|g\|_\infty \leq 1$, and $f \in \mathcal{B}_b(X) \cap \text{Dom } \mathcal{E}$. We have

$$\begin{aligned} \int g(f - P_t f) d\mu &= \int_0^t \int g \frac{\partial P_s f}{\partial s} d\mu ds = \int_0^t \int g \Delta P_s f d\mu ds = - \int_0^t \int \Gamma(P_s g, f) d\mu ds \\ &\leq \int_0^t \|\sqrt{\Gamma(P_s g)}\|_\infty \int \sqrt{\Gamma(f)} d\mu ds \leq C_1 \sqrt{2t} \int \sqrt{\Gamma(f)} d\mu. \end{aligned}$$

By the regularity of \mathcal{E} and pointwise approximation, this is true for every bounded Borel g with $\|g\|_\infty \leq 1$, and thus

$$\|P_t f - f\|_1 \leq C_1 \sqrt{2t} \int \sqrt{\Gamma(f)} d\mu.$$

Let now $f \in BV(X)$. For $s > 0$, we have $P_s f \in \mathcal{B}_b(X) \cap \text{Dom } \mathcal{E}$. Thus we deduce

$$\|P_{s+t} f - P_s f\|_1 \leq C_1 \sqrt{2t} \int \sqrt{\Gamma(P_s f)} d\mu.$$

Taking the limit when $s \rightarrow 0$ finishes the proof. \square

Remark 3.1. If we do not assume $C_2 = 0$, then the inequality

$$\|P_t f - f\|_1 \leq C_1^3 \sqrt{2t} \text{Var } f.$$

only holds for $0 \leq t \leq 1$.

The previous theorem has many implications in terms of Sobolev embedding theorems. Ledoux proves in Section 2 of [Led03] that a L^1 -bound of the type

$$\|P_t f - f\|_1 \leq C \sqrt{t} \text{Var } f,$$

implies, in very general frameworks, improved Sobolev embeddings involving Besov norms. In particular we obtain the following result:

Corollary 3.4. *Assume that (6) is satisfied with $C_2 = 0$ and that $e^{t\Delta}$ has a heat kernel $p_t(x, y)$ that satisfies for some constant $Q > 1$,*

$$\sup_{x, y \in X, t \in (0, 1]} t^{Q/2} p_t(x, y) < +\infty. \quad (8)$$

Then, there exists a constant $C_Q > 0$, such that for every $f \in BV(X)$,

$$\|f\|_p \leq C_Q (\text{Var } f + \|f\|_1), \quad (9)$$

where $p = \frac{Q}{Q-1}$.

Remark 3.2. It is consequence of the celebrated Varopoulos' theorem that the heat kernel bound (8) alone implies the Sobolev inequality

$$\|f\|_p \leq C_Q (\mathcal{E}(f, f) + \|f\|_2),$$

where $p = \frac{2Q}{Q-1}$. The assumptions $C_1 < +\infty, C_2 = 0$ are therefore used to improve this inequality into (9).

3.3 Isoperimetric inequality

The inequality (9) obviously has an isoperimetric flavor when applied to $f = 1_E$, where E is a Caccioppoli set. Actually, adapting some beautiful ideas of Varopoulos (see [Var89], pp.256-258), Ledoux (see pp. 22 in [Led93], see also Theorem 8.4 in [Led96]) and [BB16] yields:

Theorem 3.5 (Isoperimetric inequality). *Assume that $e^{t\Delta}$ has a heat kernel $p_t(x, y)$ that satisfies for some constant $Q > 1$,*

$$\sup_{x, y \in X, t \in (0, 1]} t^{Q/2} p_t(x, y) < +\infty. \quad (10)$$

There exist constants $C_{\text{iso}}, \mu_{\text{max}} > 0$, such that for every Caccioppoli set $E \subset X$ with $\mu(E) \leq \mu_{\text{max}}$

$$\mu(E)^{\frac{Q-1}{Q}} \leq C_{\text{iso}} P(E).$$

Proof. Let $f \in BV(X)$. From the proof of Theorem 3.3, one has for $0 \leq t \leq 1$,

$$\|P_t f - f\|_1 \leq C_1^3 \sqrt{2t} \text{Var } f.$$

Therefore, if E is a Caccioppoli set

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_1 \leq C_1^3 \sqrt{2t} \text{Var}(\mathbf{1}_E) = C_1^3 \sqrt{2t} P(E),$$

Observe now that, because $P_E \mathbf{1}_E \leq 1$ on E and we have

$$\begin{aligned} \|P_t \mathbf{1}_E - \mathbf{1}_E\|_1 &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_{E^c} P_t(\mathbf{1}_E) d\mu \\ &= \int_E (1 - P_t \mathbf{1}_E) d\mu + \int_E (P_t \mathbf{1}_{E^c}) d\mu \\ &= 2 \left(\mu(E) - \int_E P_t(\mathbf{1}_E) d\mu \right) \end{aligned}$$

On the other hand, we have

$$\int_E P_t \mathbf{1}_E d\mu = \int (P_{t/2} \mathbf{1}_E)^2 d\mu.$$

We thus obtain

$$\|P_t \mathbf{1}_E - \mathbf{1}_E\|_1 = 2 \left(\mu(E) - \int (P_{t/2} \mathbf{1}_E)^2 d\mu \right). \quad (11)$$

We now note that

$$\begin{aligned} \int (P_{t/2} \mathbf{1}_E)^2 d\mu &\leq \left(\int_E \left(\int p_{t/2}(x, y)^2 d\mu(y) \right)^{\frac{1}{2}} d\mu(x) \right)^2 \\ &= \left(\int_E p_t(x, x)^{\frac{1}{2}} d\mu(x) \right)^2 \leq \frac{A}{t^{Q/2}} \mu(E)^2. \end{aligned}$$

for some constant $A > 0$. Combining these equations yields

$$\mu(E) \leq B\sqrt{t} P(E) + \frac{C}{t^{Q/2}} \mu(E)^2, \quad 0 < t \leq 1,$$

for some positive constants B, C . Applying the inequality at $t = D\mu(E)^{2/Q}$ where D is large enough concludes the proof. \square

Remark 3.3. We note that we do not assume $C_2 = 0$ in the theorem.

3.4 Buser's isoperimetric inequality

In the context of a smooth compact Riemannian manifold with Riemannian measure μ , Cheeger [Che70] introduced the following isoperimetric constant

$$h = \inf \frac{\mu(\partial A)}{\mu(A)},$$

where the infimum runs over all open subsets A with smooth boundary ∂A such that $\mu(A) \leq \frac{1}{2}$. Cheeger's constant can be used to bound from below the first non zero eigenvalue of the manifold. Indeed, it is proved in [Che70] that

$$\lambda_1 \geq \frac{h^2}{4}.$$

Buser [Bus82] then proved that if the Riemannian Ricci curvature of the manifold is non-negative, then we actually have

$$\lambda_1 \leq Ch^2$$

where C is a universal constant depending only on the dimension. Buser's inequality was reproved by Ledoux [Led94] using heat semigroup techniques. Under proper assumptions, by using the tools we introduced, Ledoux' technique can essentially reproduced in our general framework of Dirichlet spaces.

In this section, we assume that \mathcal{E} satisfies a spectral gap inequality and that (6) is satisfied with $C_2 = 0$. We define the Cheeger's constant of X by

$$h = \inf \frac{P(E)}{\mu(E)}$$

where the infimum runs over all Caccioppoli sets E such that $\mu(E) \leq \frac{1}{2}$. We denote by λ_1 the spectral gap of Δ .

Theorem 3.6.

$$\lambda_1 \leq C_{buser} h^2,$$

where C_{buser} is a constant depending on C_1 only.

Proof. Let A be a Caccioppoli set with finite perimeter. By symmetry and stochastic completeness of the semigroup, we have from equation (11)

$$\|1_A - P_t 1_A\|_1 = 2 \left(\mu(A) - \|P_{\frac{t}{2}}(1_A)\|_2^2 \right).$$

By Theorem 3.3, we have

$$\|P_t 1_A - 1_A\|_1 \leq C_1^3 \sqrt{2t} P(A).$$

We deduce that

$$\mu(A) \leq C_1^3 \sqrt{\frac{t}{2}} P(A) + \|P_{\frac{t}{2}}(1_A)\|_2^2.$$

Now, by spectral theorem,

$$\|P_{\frac{t}{2}}(1_A)\|_2^2 = \mu(A)^2 + \|P_{\frac{t}{2}}(1_A - \mu(A))\|_2^2 \leq \mu(A)^2 + e^{-\lambda_1 t} \|1_A - \mu(A)\|_2^2$$

This yields

$$\mu(A) \leq C_1^3 \sqrt{\frac{t}{2}} P(A) + \mu(A)^2 + e^{-\lambda_1 t} \|1_A - \mu(A)\|_2^2.$$

Equivalently, one obtains

$$C_1^3 \sqrt{\frac{t}{2}} P(A) \geq \mu(A)(1 - \mu(A))(1 - e^{-\lambda_1 t}).$$

Therefore,

$$h \geq \frac{1}{C_1^3 \sqrt{2}} \sup_{t>0} \left(\frac{1 - e^{-\lambda_1 t}}{\sqrt{t}} \right).$$

We conclude

$$h^2 \geq \frac{1}{2C_1^6} (1 - e^{-1})^2 \lambda_1.$$

□

Let us observe that it is known that the Cheeger lower bound on λ_1 may be obtained under further assumptions on the Dirichlet space (X, d, \mathcal{E}) . Indeed, assume that Lipschitz functions are in the domain of \mathcal{E} and that $\sqrt{\Gamma(f)}$ is an upper gradient in the sense that for any Lipschitz function f ,

$$\sqrt{\Gamma(f)}(x) = \limsup_{d(x,y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In that case, if A is a closed set of X , one defines its Minkowski exterior boundary measure by

$$\mu_+(A) = \liminf_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mu(A_\varepsilon) - \mu(A)),$$

where $A_\varepsilon = \{x \in X, d(x, X) < \varepsilon\}$. We can then define the second Cheeger's constant of X by

$$h_+ = \inf \frac{\mu_+(E)}{\mu(E)}$$

where the infimum runs over all closed sets E such that $\mu(E) \leq \frac{1}{2}$. Then, according to Theorem 8.5.2 in [BGL14], one has

$$\lambda_1 \geq \frac{h_+^2}{4}.$$

3.5 Ledoux isoperimetric inequality

In this section, we assume that \mathcal{E} satisfies a log-Sobolev inequality and that (6) is satisfied with $C_2 = 0$. We define the Gaussian isoperimetric constant of X by

$$k = \inf \frac{P(E)}{\mu(E) \sqrt{-\ln \mu(E)}}$$

where the infimum runs over all Caccioppoli sets E such that $\mu(E) \leq \frac{1}{2}$. We denote by ρ_0 the log-Sobolev constant of X , that is the best constant in the inequality (7).

Theorem 3.7.

$$\rho_0 \leq C_{\text{ledoux}} k^2$$

where C_{ledoux} is a constant depending on C_1 only.

Proof. Let A be a Caccioppoli with finite perimeter. From the proof of Theorem 3.6, we have

$$\mu(A) \leq C_1^3 \sqrt{\frac{t}{2}} P(A) + \|P_{\frac{t}{2}}(1_A)\|_2^2.$$

Now we can use the hypercontractivity constant to bound $\|P_{\frac{t}{2}}(1_A)\|_2^2$. Indeed, from Gross' theorem it is well known that the logarithmic Sobolev inequality

$$\int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \leq \frac{1}{\rho_0} \int \Gamma(f) d\mu,$$

is equivalent to hypercontractivity property

$$\|P_t f\|_q \leq \|f\|_p$$

for all f in L^p whenever $1 < p < q < \infty$ and $e^{\rho_0 t} \geq \sqrt{\frac{q-1}{p-1}}$.

Therefore, with $p(t) = 1 + e^{-2\rho_0 t} < 2$, we get,

$$\begin{aligned} C_1^3 \sqrt{2t} P(A) &\geq 2 \left(\mu(A) - \mu(A)^{\frac{2}{p(t)}} \right) \\ &\geq 2\mu(A) \left(1 - \mu(A)^{\frac{1-e^{-2\rho_0 t}}{1+e^{-2\rho_0 t}}} \right). \end{aligned}$$

By using then the computation page 956 in [Led94], one deduces that if A is a set which has a finite perimeter $P(A)$ and such that $0 \leq \mu(A) \leq \frac{1}{2}$, then

$$P(A) \geq \tilde{C} \sqrt{\rho_0} \mu(A) \left(\ln \frac{1}{\mu(A)} \right)^{\frac{1}{2}},$$

where \tilde{C} is a constant depending on C_1 only. □

4 Poincaré Duality on Hino index-1 spaces

In this section, we come back to the general framework of Section 2.1. Our goal is to construct a scalarization of the closed symmetric form

$$\vec{\mathcal{E}}(\omega, \eta) = \langle \partial^* \omega, \partial^* \eta \rangle_2.$$

This can be achieved on Hino index-1 spaces where one-forms may be identified with functions. In such spaces, we will prove that the semigroup domination

$$\|\vec{P}_t \eta\|_{\mathcal{H}_x} \leq C_1 (P_t \|\eta\|_{\mathcal{H}})(x), \quad \eta \in \mathcal{H}, \quad 0 \leq t \leq 1,$$

is then equivalent to a semigroup domination

$$|e^{t\Delta^\perp} f|(x) \leq C_1 e^{t\Delta} |f|(x), \quad 0 \leq t \leq 1.$$

where Δ^\perp is a self adjoint operator on $L^2(X)$ that we call Poincaré dual of Δ . We stress that Δ^\perp , in general, is not Markovian, that is the semigroup $e^{t\Delta^\perp}$ is not positivity preserving.

4.1 Poincaré duality

We first recall the following definition.

Definition 4.1 (Definition 2.9 in [Hin10]). The pointwise Hino index $p(x)$ of $(\mathcal{E}, \text{Dom } \mathcal{E})$ is the function such that

- (a) For any $N \in \mathbb{N}$ and $f_1, \dots, f_N \in \text{Dom } \mathcal{E}$ the rank of the $N \times N$ matrix with entries $Z_{ij} := \Gamma(f_i, f_j)$ is less than $p(x)$ almost everywhere.
- (b) If $p'(x)$ is another function which satisfies (a), then $p(x) \leq p'(x)$ almost everywhere.

The essential supremum of $p(x)$ with respect to μ is referred to as the Hino index of $(\mathcal{E}, \text{Dom } \mathcal{E})$.

Proposition 4.2 (Lemma 3.2 in [Hin13]). *If $p(x)$ is the pointwise Hino index of $(\mathcal{E}, \text{Dom } \mathcal{E})$ then $p(x) = \dim \mathcal{H}_x$ almost everywhere.*

For $\omega \in \mathcal{H}$, define ν_ω to be the measure on X such that for $\phi \in C_b(X)$

$$\int_X \phi \, d\nu_\omega = \langle \omega \cdot \phi, \omega \rangle_{\mathcal{H}}.$$

The following lemma is then trivial.

Lemma 4.3. *There exists $\omega \in \mathcal{H}$ such that $\mu = \nu_\omega$ if and only if there exists $\omega \in \mathcal{H}$ such that $\|\omega\|_{\mathcal{H}_x} = 1$, μ -a.e.*

We now set the following definition of the Hodge star operator on Hino index-1 spaces.

Definition 4.4. Assume that $(\mathcal{E}, \text{Dom } \mathcal{E})$ has Hino index 1 and that $\mu = \nu_\omega$ for some $\omega \in \mathcal{H}$. For μ -almost every $x \in X$, we define the Hodge star operator $\star : L^2(X, \mu) \rightarrow \mathcal{H} = \int^\oplus \mathcal{H}_x \, d\mu$ by $\star f$ which is defined to be $(\star f)_x := f(x)\omega_x$ on almost every fiber \mathcal{H}_x of \mathcal{H} . \star shall also be used to denote the inverse of this map $\star(\omega \cdot f) = f$.

Classically, for n -dimensional Riemannian manifolds, Poincaré duality states the differential p forms are isometric to $n-p$ forms, and this isometry is given by the Hodge star. For 1-dimensional spaces (i.e. the line or the circle), the classical Hodge star provides an isometry between 0 forms (functions) and 1 forms. Hence, the following proposition is a measurable version of Poincaré duality for 1 dimensional spaces.

Proposition 4.5. *Assume that $(\mathcal{E}, \text{Dom } \mathcal{E})$ has Hino index 1 and that $\mu = \nu_\omega$ for some $\omega \in \mathcal{H}$. The operator \star is an isometry both fibre-wise and globally. i.e. $\|\star f\|_{\mathcal{H}_x} = |f(x)|$ almost everywhere, and $\|\star f\|_{\mathcal{H}} = \|f\|_2$. Thus \mathcal{H} is isometric to $L^2(X, \mu)$.*

Proof. This holds because

$$\langle \star f, \star g \rangle_{\mathcal{H}, x} = \langle f(x) \cdot \omega_x, g(x) \cdot \omega_x \rangle_{\mathcal{H}_x} = f(x)g(x)$$

almost everywhere and

$$\langle \star f, \star g \rangle_{\mathcal{H}} = \int \langle f\omega, g\omega \rangle_{\mathcal{H}_x}^2 \, d\mu(x) = \int f(x)g(x) \, d\mu(x).$$

□

Definition 4.6. Assume that $(\mathcal{E}, \text{Dom } \mathcal{E})$ has Hino index 1 and that $\mu = \nu_\omega$ for some $\omega \in \mathcal{H}$. The self-adjoint operator

$$\Delta^\perp = \star \vec{\Delta} \star$$

will be called the (Poincaré) dual operator of Δ . It is the self-adjoint generator of the closed symmetric form on $L^2(X, \mu)$

$$\mathcal{E}^\perp(f, g) = \langle \partial^* \star f, \partial^* \star g \rangle_2.$$

Examples 4.1. 1. Let $X = \mathbb{R}$ or $X = \mathbb{S}^1$. Consider the standard Dirichlet form on X which is the closure of

$$\mathcal{E}(f, g) = \int_X f'(x)g'(x)dx, \quad f, g \in C_0^\infty(X).$$

Then, $(\mathcal{E}^\perp, \text{Dom } \mathcal{E}^\perp) = (\mathcal{E}, \text{Dom } \mathcal{E})$ and $\Delta^\perp = \Delta$.

2. Let $X = I$, where I is an interval of \mathbb{R} . Denote $(\mathcal{E}_D, \text{Dom } \mathcal{E}_D)$ the standard Dirichlet form $\int_X f'(x)g'(x)dx$ with Dirichlet boundary condition, and denote $(\mathcal{E}_N, \text{Dom } \mathcal{E}_N)$ the one with Neumann boundary condition. Then,

$$(\mathcal{E}_N^\perp, \text{Dom } \mathcal{E}_N^\perp) = (\mathcal{E}_D, \text{Dom } \mathcal{E}_D).$$

Therefore,

$$\Delta_N^\perp = \Delta_D.$$

This duality between the Dirichlet and the Neumann boundary conditions is exceptional — In general $(\mathcal{E}^\perp, \text{Dom } \mathcal{E}^\perp)$ is not a Dirichlet form, since it may fail to satisfy the Markovian property, as is the case with the metric graphs in Section 5.1, 5.2 (see in particular the Walsh spider, Example 5.1).

We are interested in $(\mathcal{E}^\perp, \text{Dom } \mathcal{E}^\perp)$ because of the following intertwining property:

Theorem 4.7. *Assume that $(\mathcal{E}, \text{Dom } \mathcal{E})$ has Hino index 1 and that $\mu = \nu_\omega$ for some $\omega \in \mathcal{H}$. For $f \in \text{Dom } \mathcal{E}$,*

$$\star \partial e^{t\Delta} f = e^{t\Delta^\perp} \star \partial f, \quad t \geq 0.$$

Proof. From Theorem 2.1, one has

$$\partial e^{t\Delta} = e^{t\tilde{\Delta}} \partial.$$

Thus,

$$\star \partial e^{t\Delta} = \star e^{t\tilde{\Delta}} \partial.$$

Since \star is an isometry one has

$$\star e^{t\tilde{\Delta}} \star = e^{t\Delta^\perp},$$

and the conclusion easily follows. □

The following corollary is then obvious:

Corollary 4.8. *Let $C_1 \geq 1$. Assume that for every $f \in L^2$, we have μ -almost everywhere*

$$|e^{t\Delta^\perp} f| \leq C_1 e^{t\Delta} |f|, \quad 0 \leq t \leq 1.$$

Then, the semigroup $e^{t\Delta}$ satisfies the Bakry-Émery estimate

$$\sqrt{\Gamma(e^{t\Delta} f)} \leq C_1 e^{C_2 t} e^{t\Delta} \sqrt{\Gamma(f)}, \quad f \in \text{Dom } \mathcal{E}, \quad t \geq 0,$$

for some $C_2 \geq 0$.

4.2 Harmonic forms

A form ω in \mathcal{H} is called harmonic if $\partial^*\omega = 0$. In this subsection we assume that \mathcal{E} has Hino index 1 and we consider the Hodge star \star with respect to a harmonic form.

Lemma 4.9. *Assume that $(\mathcal{E}, \text{Dom } \mathcal{E})$ has Hino index 1 and that $\mu = \nu_\omega$ for some harmonic form $\omega \in \mathcal{H}$. Then, for every $f, g \in \text{Dom } \mathcal{E}$,*

$$\langle f, \star \partial g \rangle_2 = -\langle \star \partial f, g \rangle_2.$$

Therefore, $(\mathcal{E}^\perp, \text{Dom } \mathcal{E}^\perp)$ is an extension of $(\mathcal{E}, \text{Dom } \mathcal{E})$.

Proof. If $f, g \in \mathcal{C}$,

$$\langle f, \star \partial g \rangle_2 = \langle \star f, \partial g \rangle_{\mathcal{H}} = \langle f\omega, \partial g \rangle_{\mathcal{H}} = \langle \omega, f\partial g \rangle_{\mathcal{H}} = -\langle \omega, g\partial f \rangle_{\mathcal{H}} = -\langle \star \partial f, g \rangle_2,$$

because $\partial(fg) = f\partial g + g\partial f$ and $\partial^*\omega = 0$. The identity extends to every $f, g \in \text{Dom } \mathcal{E}$ by regularity of \mathcal{E} as follows: for $f \in \text{Dom } \mathcal{E}$ we can find a sequence of f_i in \mathcal{C} with $\lim \mathcal{E}(f_i - f) + \|f_i - f\|_2 = 0$. For $g \in \text{Dom } \mathcal{E}$, $\lim_{i \rightarrow \infty} \langle f_i, \star \partial g \rangle_2 = \langle f, \star \partial g \rangle_2$ because f_i converges to f in $L^2(X)$. On the other hand $\|\partial(f_i - f)\|_{\mathcal{H}} = \sqrt{\mathcal{E}(f_i - f)} \rightarrow 0$. This implies, $\lim_{i \rightarrow \infty} \partial f_i = \partial f$ strongly in \mathcal{H} , and thus $\lim_{i \rightarrow \infty} \langle \partial f_i, \star g \rangle_{\mathcal{H}} = \langle \partial f, \star g \rangle_{\mathcal{H}}$. \square

From the previous proposition we have $\star \partial \subset -\partial^* \star$. However, in general it is not true that $\star \partial = -\partial^* \star$ (see the following discussion on the union of circles for an example). In the case, where $\star \partial = -\partial^* \star$, then $\mathcal{E} = \mathcal{E}^\perp$ and therefore $\Delta = \Delta^\perp$, which implies from Corollary 4.8 that the Bakry-Émery estimate is satisfied with a constant 1. In general, one can prove the Bakry-Émery estimate with constant 1 only on a subspace of $\text{Dom } \mathcal{E}$.

Theorem 4.10. *Let*

$$\mathfrak{L} = \left\{ f \in L^2(X, \mu), \text{ for every } t \geq 0, e^{t\Delta} f = e^{t\Delta^\perp} f \right\}.$$

Then \mathfrak{L} is an L^2 -closed subspace \mathfrak{L} of $L^2(X, \mu)$ such that $\star \partial(\mathfrak{L} \cap \text{Dom } \mathcal{E}) \subset \mathfrak{L}$ and for every $f \in \mathfrak{L} \cap \text{Dom } \mathcal{E}$ and $t \geq 0$,

$$\sqrt{\Gamma(e^{t\Delta} f)} \leq e^{t\Delta} \sqrt{\Gamma(f)}.$$

Proof. The fact that \mathfrak{L} is an L^2 -closed subspace \mathfrak{L} of $L^2(X, \mu)$ is obvious. Let now $f \in \mathfrak{L} \cap \text{Dom } \mathcal{E}$. We have $e^{t\Delta} f = e^{t\Delta^\perp} f$. Therefore $\star \partial e^{t\Delta} f = \star \partial e^{t\Delta^\perp} f$. Now, from Theorem 4.7, $\star \partial e^{t\Delta} f = e^{t\Delta^\perp} \star \partial f$. On the other hand, from the previous lemma $\star \partial e^{t\Delta^\perp} f = -\partial^* \star e^{t\Delta^\perp} f = -\partial^* e^{t\Delta} \star f = -e^{t\Delta} \partial^* \star f = e^{t\Delta} \star \partial f$. We conclude $e^{t\Delta^\perp} \star \partial f = e^{t\Delta} \star \partial f$ and thus $\star \partial f \in \mathfrak{L}$. Finally, if $f \in \mathfrak{L} \cap \text{Dom } \mathcal{E}$, then

$$\star \partial e^{t\Delta} f = e^{t\Delta} \star \partial f,$$

which immediately implies the Bakry-Émery estimate. \square

We conclude the section with a detailed example that satisfies the assumptions of this section. Assume that X is a union of n circles connected at one point. One can represent a function $f : X \rightarrow \mathbb{R}$ as a function

$$f = (f_1, \dots, f_n)$$

where the $f_i : [0, 1] \rightarrow \mathbb{R}$ are subject to the boundary conditions

$$f_i(1) = f_i(0) = f_j(0) = f_j(1) \quad \text{for all } i \neq j.$$

One considers then the Dirichlet form

$$\mathcal{E}(f, g) = \sum_{i=1}^n \int_0^1 f'_i(x) g'_i(x) dx$$

with domain the \mathcal{E} -closure of

$$\{f \in C^\infty([0, 1], \mathbb{R}^n), f_i(1) = f_i(0) = f_j(0) = f_j(1), \forall i \neq j\}.$$

For every $f \in \text{Dom } \mathcal{E}$, one has

$$\sum_{i=1}^n \int_0^1 f'_i(x) dx = 0,$$

where the derivatives are understood in the distribution sense. Therefore the reference measure dx is the the energy measure of a harmonic form (namely, the energy measure of the differential form $\mathbf{1}$ in the isometry described in proposition 5.1). For every $f \in \text{Dom } \mathcal{E}$, one has

$$\star \partial f = (f'_1, \dots, f'_n).$$

One deduces that $\text{Dom } \partial^* \star$ is the \mathcal{E} -closure of

$$\{f \in C^\infty([0, 1], \mathbb{R}^n), \sum_{i=1}^n f_i(0) = \sum_{i=1}^n f_i(1)\}$$

and that for $f \in \text{Dom } \partial^* \star$,

$$\partial^* \star f = -(f'_1, \dots, f'_n),$$

where, once again, the derivatives are understood in the distribution sense.

We denote as before by Δ the generator of \mathcal{E} and $P_t = e^{t\Delta}$. Denote now P_t^S the standard heat semigroup on $[0, 1]$ with periodic boundary condition and by P_t^D the Dirichlet heat semigroup (zero boundary condition) on $[0, 1]$. By extension for $f = (f_1, \dots, f_n) \in L^2([0, 1], dx)^n$, we denote

$$P_t^S f = (P_t^S f_1, \dots, P_t^S f_n),$$

and we adopt a similar convention for P_t^D . The generators of P_t^S and P_t^D are respectively denoted by Δ^S and Δ^D and the corresponding Dirichlet form by \mathcal{E}^S and \mathcal{E}^D . If we denote

$$\mathfrak{L} = \{f \in L^2(X), f_1 = \cdots = f_n\},$$

any function f can uniquely be decomposed as $f = f_{\mathfrak{L}} + f_{\mathfrak{L}^\perp}$ where $f_{\mathfrak{L}} \in \mathfrak{L}$ and $f_{\mathfrak{L}^\perp} \in \mathfrak{L}^\perp$. We have then the following proposition:

Proposition 4.11.

1. Let $f \in L^2(X, dx)$, then $f \in \text{Dom } \Delta$ if and only if $f_{\mathfrak{L}} \in \text{Dom } \Delta^S$ and $f_{\mathfrak{L}^\perp}$ is in $\text{Dom } \Delta^D$. In that case,

$$\Delta f = \Delta^S f_{\mathfrak{L}} + \Delta^D f_{\mathfrak{L}^\perp}.$$

2. Let $f \in L^2(X, dx)$, then $f \in \text{Dom } \Delta^\perp$ if and only if $f_{\mathfrak{L}} \in \text{Dom } \Delta^S$ and $f_{\mathfrak{L}^\perp} \in \text{Dom } \Delta^N$. In that case,

$$\Delta^\perp f = \Delta^S f_{\mathfrak{L}} + \Delta^N f_{\mathfrak{L}^\perp}.$$

Proof. If $f, g \in C^\infty([0, 1], \mathbb{R}^n)$, $f_i(1) = f_i(0) = f_j(0) = f_j(1)$, $\forall i \neq j$. Then $f_{\mathfrak{L}}, g_{\mathfrak{L}} \in \text{Dom } \mathcal{E}^S$, $f_{\mathfrak{L}^\perp}, g_{\mathfrak{L}^\perp} \in \text{Dom } \mathcal{E}^D$ and

$$\mathcal{E}(f, g) = \mathcal{E}^S(f_{\mathfrak{L}}, g_{\mathfrak{L}}) + \mathcal{E}^D(f_{\mathfrak{L}^\perp}, g_{\mathfrak{L}^\perp}).$$

Thus

$$\begin{aligned} \text{Dom } \mathcal{E} &\rightarrow \text{Dom } \mathcal{E}^S \otimes \text{Dom } \mathcal{E}^D \\ f &\rightarrow (f_{\mathfrak{L}}, f_{\mathfrak{L}^\perp}) \end{aligned}$$

is seen to be a Dirichlet space isomorphism and Part 1 follows. Part 2 follows from the fact that $(\Delta^S)^\perp = \Delta^S$ and $(\Delta^D)^\perp = \Delta^N$. \square

The next corollary easily follows and illustrates Theorem 4.10.

Corollary 4.12.

1. Let $f \in L^2(X, dx)$. Then for every $t \geq 0$,

$$P_t f = P_t^S f_{\mathfrak{L}} + P_t^D f_{\mathfrak{L}^\perp}.$$

2. Let $f \in L^2(X, dx)$. Then for every $t \geq 0$,

$$P_t^\perp f = P_t^S f_{\mathfrak{L}} + P_t^N f_{\mathfrak{L}^\perp}.$$

As a consequence

$$\mathfrak{L} = \left\{ f \in L^2(X, \mu), \text{ for every } t \geq 0, e^{t\Delta} f = e^{t\Delta^\perp} f \right\}.$$

5 Bakry-Émery estimate on metric graphs

In this section we prove the validity of the Bakry-Émery estimate on metric graphs with finite number of edges and rays. The results of Section 3 may therefore be applied in that class of examples.

5.1 Function spaces and differential one-forms on metric graphs

In Section 4 we developed a Poincaré duality based on a Hodge star operator when the reference measure is an energy form ν_ω for some $\omega \in \mathcal{H}$. This requires the total measure of the space to be finite, ruling therefore out non-compact metric graphs. Our first task will therefore be to find an isomorphism between one-forms and functions that works for any metric graph. This will be made possible by the existence of the derivative operator.

For a reference on the general theory of metric graphs we refer to [Pos12]. We start off with notations concerning (discrete) weighted graphs. We use G to denote a graph, which is composed of vertexes V , (internal) edges E and rays R . For each edge $e \in E$ there is two endpoints e^- and e^+ in V as well as a length $r(e) > 0$. Rays have one associated endpoint e^- in V and the length is infinite. For $v \in V$ define the set of adjacent edges $E_v = \{e \in E \cup R \mid v = e^- \text{ or } v = e^+\}$. We assume that E and R are finite.

Define G^{met} to be the metric graph associated with G : For $e \in E$ let $I_e = [0, r(e)]$ and if $e \in R$ then $I_e = [0, \infty)$. In this case G^{met} is the set $\sqcup_{e \in E \cup R} I_e$ modulo the equivalence relation which identifies endpoints of I_{e_1} and I_{e_2} if associated endpoints of e_1 and e_2 are the same vertex. Define $\Phi_e : I_e \rightarrow G^{\text{met}}$ to be the projection onto the equivalence classes. For example $\Phi_{e_1}(0) = \Phi_{e_2}(r(e_2))$ if $e_1^- = e_2^+$. We may think of I_e as subsets of G^{met} and refer to $0 \in I_e$ as e^- and $r(e) \in I_e$ as e^+ .

Now, we shall define some notations concerning the function spaces on G^{met} . Define the reference measure μ on G^{met} to be that which is Lebesgue measure when restricted to each I_e . Functions $f \in L^2(G^{\text{met}}) = L^2(G^{\text{met}}, \mu)$ will be denoted as vectors $f = (f_e)_{e \in E \cup R}$ where $f_e \in L^2(I_e)$, i.e. $L^2(G^{\text{met}}) = \oplus_e L^2(I_e)$. Other function spaces have similar vector decompositions, perhaps with boundary conditions. For example, we shall think of continuous functions $C(G^{\text{met}})$ to be the vectors with entries in $C(I_e)$ where, if $v \in I_{e_1}$ and I_{e_2} then $f_{e_1}(v) = f_{e_2}(v)$. Define the Sobolev space $H_0^1(G^{\text{met}})$ to be the functions f such that $f_e \in H^1(I_e)$, i.e. both f_e and f'_e are in $L^2(I_e)$, with the boundary conditions ensuring that f is continuous at vertexes.

When it is well defined, we consider $f(v)$ to be the vector $(f_e(v))_{e \in E_v}$ of values of f (or traces of f) at the associated endpoint of e . We shall need to also denote the multiplication (diagonal) operator $U_v(e) = 1$ if $v = e^-$ and $U_v(e) = -1$ if $v = e^+$ for each $v \in V$. In this way, the inward facing normal derivative of f at $v \in V$ along an edge e is $U_v(e)f'_e(v)$. Here and later, $f'_e(0)$ or $f'_e(r(e))$ is understood to be the trace of f'_e onto the

boundary of I_e .

One defines first a derivative operator $d : H_0^1(\mathbf{G}^{\text{met}}) \rightarrow L^2(\mathbf{G}^{\text{met}})$ by $(df)_e(x) = f'_e(x)$, which we will concisely denote by $df = f'$. Note that, up to a sign, d depends on the orientation of the graph. However, the Dirichlet form defined by

$$\mathcal{E}(f, g) = \int f'g' d\mu = \langle f', g' \rangle$$

nor its generator $\Delta f = -f''$ depend on this orientation. The domain of \mathcal{E} is $H_0^1(\mathbf{G}^{\text{met}})$ and the domain of Δ is

$$\text{Dom } \Delta = \left\{ f \in H_0^1(\mathbf{G}^{\text{met}}) \mid \forall e, f'_e \in H^1(I_e), \forall v \in V, \sum_{e \in \mathbf{E}_v} U_v(e) f'_e(v) = 0 \right\}.$$

These boundary conditions are called standard or Kirchhoff boundary conditions. The carré du champ associated to \mathcal{E} or Δ is $\Gamma(f, g) = f'g'$ for $f, g \in H_0^1(\mathbf{G}^{\text{met}})$.

We also define the codifferential $d^*f := -f'$ to be the adjoint of d . Using the integration by parts formula

$$\langle f', g \rangle = -\langle f, g' \rangle + \sum_{v \in V} \sum_{e \in \mathbf{E}_v} U_v(e) f_e(v) g_e(v),$$

one sees that

$$\text{Dom } d^* = H_1^1(\mathbf{G}^{\text{met}}) := \left\{ f \in L^2(\mathbf{G}^{\text{met}}) \mid f_e \in H^1(I_e), \forall v \in V, \sum_{e \in \mathbf{E}_v} U_v(e) f_e(v) = 0 \right\}.$$

The following result shows that we can identify the space of one-forms in the sense of Section 2 with $L^2(\mathbf{G}^{\text{met}})$.

Proposition 5.1. *Let \mathbf{G}^{met} be a metric graph with a finite number of edges and rays, and \mathcal{E} be the Dirichlet form defined above with $\text{Dom } \mathcal{E} = H_0^1(\mathbf{G}^{\text{met}})$. If \mathcal{H} be the space of differential 1-forms, then $\mathcal{H} \cong L^2(\mathbf{G}^{\text{met}})$ via an isometry which sends $\partial f \mapsto f'$ for all $f \in \text{Dom } \mathcal{E}$. Under this isometry $\partial^* = d^*$, $\text{Dom } \partial^* = H_1^1(\mathbf{G}^{\text{met}})$, and $\vec{\Delta} = dd^*$.*

Proof. This is an expansion of comments made in [IRT12, Section 5], we include a quick argument for the sake of completeness. It is straightforward to see that,

$$\|f \otimes g\|_{\mathcal{H}}^2 = \int g^2(f')^2 d\mu = \|gf'\|_{L^2(\mathbf{G}^{\text{met}})}^2$$

and thus the function which maps $f \otimes g \rightarrow gf'$ is an isomorphism of simple tensors and thus extends to an isomorphism. Under this isomorphism, $\partial = d$ and hence $\partial^* = d^*$. \square

In view of the previous isomorphism, we will simply denote $\vec{\Delta}f = dd^*f = -f''$. The domain is

$$\text{Dom } \vec{\Delta} = \{f \in H_1^1(\mathbf{G}^{\text{met}}) \mid f' \in H_0^1(\mathbf{G}^{\text{met}})\}$$

i.e. for each $v \in V$ $\sum_{e \in \mathbf{E}_v} U_v(e) f_e(v) = 0$ and for any pair of $e_1, e_2 \in \mathbf{E}_v$ then $f'_{e_1}(v) = f'_{e_2}(v)$. These are sometimes referred to as anti-Kirchhoff boundary conditions.

Remark 5.1. A metric graph \mathbf{G}^{met} admits a Poincaré duality in the sense of Section 4 if there is a function h in $H_1^1(\mathbf{G}^{\text{met}})$ with $|h| = 1$ almost everywhere. i.e. $h = \pm 1$ on each edge where the \pm depends on the orientation of the edge. Alternatively, such a form exists, if there is an orientation such that $\sum_e \int_0^{r_e} f' dx = \langle f', h \rangle = -\langle f, h' \rangle = 0$ for all $f \in H_0^1(\mathbf{G}^{\text{met}})$. In the case that \mathbf{G}^{met} admits a Poincaré duality, $\vec{\Delta} = \Delta^\perp$.

5.2 Heat Kernels and Bakry-Émery Estimates on Metric Graphs

In this section, we present a formula for the kernel of the semigroups generated by Δ and $\vec{\Delta}$ as a sum over (combinatorial) paths. We assume, as before, that \mathbf{G}^{met} is a metric graph with a finite number of edges and rays, and that \mathbf{G}^{met} has no tadpoles — that is edges e such that $e^+ = e^-$. This assumption does not limit the metric spaces which the following discussion applies to: one can introduce a vertex at the midpoint of any tadpole, producing a metric graph which is isometric (as a metric space) to the original space.

A combinatorial path c from $x \in e_0$ to $y \in e_{n+1}$ is the $2n + 1$ -tuple

$$(e_0, v_0, e_1, v_1, \dots, v_n, e_{n+1}),$$

where for $k = 0, 1, 2, \dots, n$, v_k and v_{k+1} are distinct endpoints of e_k . Without loss of generality we assume that $v_0 = e_0^+ = \Phi_{e_0}(r(e_0))$ and $v_n = e_n^- = \Phi_{e_n}(0)$.

We can define two distinct notions of the length of a path c : the combinatorial length, which will be denoted $|c|$ and is $n + 1$ (the number of vertices c passes through) and the metric length (or simply length)

$$d_c(x, y) := |r(e) - x| + |y| + \sum_{k=1}^n r(e_k).$$

This is the length of the shortest path which follows the combinatorial path from x to y , and hence depends on the endpoints as well as c .

Using the work of [FOT11, Sto10], we observe that the natural distance

$$\rho(x, y) = \sup \{ |f(x) - f(y)| : \Gamma(f, f) = |f'|^2 \leq 1 \}$$

coincides with the natural length metric on the space.

Define $C(x, y)$ to be the set of the combinatorial paths connecting x to y , including, if x and y are in e_0 , then the trivial path (e_0) , defining $S((e_0)) = 1$ and $d_{(e_0)}(x, y) = |x - y|$. Define the scattering amplitude associated to a combinatorial path

$$S(c) = \prod_{k=0}^n \left(\frac{2}{\deg_{v_k}} - \delta_{e_k, e_{k+1}} \right) \quad (12)$$

where \deg_v is the vertex degree of v and $\delta_{e_k, e_{k+1}}$ is the Dirac Delta (i.e. 1 if $e_k = e_{k+1}$ and 0 otherwise).

Letting $g_t(u) := (4\pi t)^{-1/2} e^{-u^2/4t}$, according to the formula in [KPS07, Corollary 3.4], the heat kernel of Δ has the form

$$p_t(x, y) = \sum_{c \in C(x, y)} S(c) g_t(d_c(x, y)). \quad (13)$$

Proposition 5.2. *The integral kernel of the semigroup associated to the anti-Kirchhoff Laplacian $\vec{\Delta}$ is*

$$\vec{p}_t(x, y) = \sum_{c \in C(x, y)} \vec{S}(c) g_t(d_c(x, y)),$$

where $\vec{S}(c)$ is the anti-Kirchhoff scattering amplitude defined

$$\vec{S}(c) = \prod_{k=0}^n U_v(e_k) U_v(e_{k+1}) \left(\frac{2}{\deg_{v_k}} - \delta_{e_k, e_{k+1}} \right) = \pm S(c). \quad (14)$$

Proof. To apply [KPS07, Corollary 3.4], we need to verify the technical condition that anti-Kirchhoff boundary conditions correspond to a maximal isotropic subspace in the sense of [KPS07]. Following example 2.8 of [KPS07], the anti-Kirchoff vertex space at each vertex is that of Kirchhoff vertex space multiplied by the diagonal matrix $-U_v$, thus the conclusion. Alternatively, from remark 5.8 in [Pos09], the maximal isotropic condition is equivalent to the associated Laplacian being self-adjoint and we know that $\vec{\Delta} = dd^*$ is self-adjoint. \square

Example 5.1 (Walsh spider). One can illustrate the previous formulas in the case of the Walsh spider. The Walsh spider with N legs is the graph consisting on N copies of $[0, +\infty)$ which we shall call $\{I_j\}_{j=1}^N$ identified at the respective 0. Calculating from the formula (13) or using [BPY89], one sees that the heat kernel has the form

$$p_t(x_j, y_k) = \begin{cases} \frac{2}{N} \frac{e^{-|x_j + y_k|^2/4t}}{\sqrt{4\pi t}} & \text{if } j \neq k \\ \frac{1}{\sqrt{4\pi t}} \left(e^{-|x_j - y_k|^2/4t} - \left(1 - \frac{2}{N}\right) e^{-|x_j + y_k|^2/4t} \right) & \text{if } j = k. \end{cases}$$

where $x_i \in I_i$ and $y_k \in I_k$. It follows that, if \vec{p}_t is the integral kernel of $\vec{\Delta}$, then

$$\vec{p}_t(x_j, y_k) = \begin{cases} -\frac{2}{N} \frac{e^{-|x_j + y_k|^2/4t}}{\sqrt{4\pi t}} & \text{if } j \neq k \\ \frac{1}{\sqrt{4\pi t}} \left(e^{-|x_j - y_k|^2/4t} + \left(1 - \frac{2}{N}\right) e^{-|x_j + y_k|^2/4t} \right) & \text{if } j = k. \end{cases}$$

Observe that this kernel takes values which are both positive and negative. From this one sees that the ratio

$$\frac{p_t(x_j, y_k)}{|\vec{p}_t^\perp(x_j, y_k)|} = \begin{cases} 1 & \text{if } j \neq k \\ \frac{1 - K e^{-x_j y_k/2t}}{1 + K e^{-x_j y_k/2t}} & \text{if } j = k. \end{cases}$$

where $K = (1 - 2/N)$. It is easy to see that the above ratio is bounded between 1 and $(1 - K)/(1 + K) = 1/(N - 1)$. Integrating we get the inequality $|e^{t\Delta^\perp} f|(x) \leq (N - 1)e^{t\Delta} |f|(x)$, which implies that the following Bakry-Émery estimate holds on the Walsh spider

$$\sqrt{\Gamma(e^{t\Delta} f)}(x) \leq (N - 1)e^{t\Delta} \sqrt{\Gamma(f)}(x).$$

Observe that the constant $N - 1$ is optimal in the previous estimate. Indeed, in the Walsh spider, the range of d is dense in L^2 , as a consequence the inequality

$$\sqrt{\Gamma(e^{t\Delta} f)}(x) \leq Ce^{t\Delta} \sqrt{\Gamma(f)}(x), \quad f \in \text{Dom } \mathcal{E}.$$

is equivalent to the inequality

$$|e^{t\tilde{\Delta}} f|(x) \leq Ce^{t\Delta} |f|(x), \quad f \in L^2(X),$$

which is equivalent to the bound $|\vec{p}_t(x, y)| \leq Cp_t(x, y)$.

With this example in mind, we now return to the study of general graphs.

Lemma 5.3. *Assume that \mathbf{G}^{met} has a finite number of edges and rays. For $T > 0$, there exists a constant $C_1 > 0$ that depends only on T and the graph \mathbf{G}^{met} , such that for $0 < t \leq T$ and μ almost every x, y*

$$|\vec{p}_t(x, y)| \leq C_1 g_t(\rho(x, y)), \quad p_t(x, y) \leq C_1 g_t(\rho(x, y))$$

Further, there exists a T_0 such that for all $0 < t < T_0$, and μ almost every x, y

$$p_t(x, y) \geq C_0 g_t(\rho(x, y)).$$

Here C_0 and T_0 only depend on the geometry of \mathbf{G}^{met} — on the maximum vertex degree, the minimum edge length and the number of internal edges.

Remark 5.2. The absolute values around \vec{p}_t are important because it may be negative, as is the case in the case of the Walsh spider studied in the previous example.

Proof. Upper bound. First, since $\rho(x, y) = \inf_{c \in C(x, y)} d_c(x, y)$, for a fixed $a > 0$ and any x, y there is a bounded number of paths $c \in C(x, y)$ such that $d_c(x, y) \leq \rho(x, y) + a$. To see this we may assume that x, y are both in finite length edges. This is because a combinatorial path to/from a point on a ray is determined by the path taken until the last time it crosses the 0 of that ray. So either x and y are in internal edges, or we can replace them with the endpoints of the ray they are in.

For any x in an internal edge, the number of paths starting from x and of length bounded by $M > 0$ is less than $(\deg_{max} + 1)^{M/r_{min}}$ where \deg_{max} is the maximum vertex degree and r_{min} is the minimum edge length. Thus there is an upper bound independent of our choice of x . The claim follows because the interior of the graph is compact. Further, if we take

$$\text{diam} = \sup \{ \rho(x, y) \mid \exists e_1, e_2 \in \mathbf{E}, x \in I_{e_1}, y \in I_{e_2} \}$$

to be the farthest apart two points on finite length edges of G can be, then the number of paths from x to any point y of length less than $\rho(x, y) + a$ is bounded by $(\deg_{\max} + 1)^{(\text{diam} + a)/r_{\min}}$.

Because

$$g_t(d_c(x, y))/g_t(\rho(x, y)) = \exp\left(-\frac{d_c(x, y)^2 - \rho(x, y)^2}{4t}\right),$$

both $|\vec{p}_t|$ and $|p_t| = p_t$ are bounded above by

$$g_t(\rho(x, y)) \sum_{c \in C(x, y)} |S(c)| \exp\left(-\frac{d_c(x, y)^2 - \rho(x, y)^2}{4t}\right)$$

Factoring out the $g_t(\rho(x, y))$, we break the sum

$$\sum_{c \in C(x, y)} |S(c)| \exp\left(-\frac{d_c(x, y)^2 - \rho(x, y)^2}{4t}\right) = A_I + A_{II}$$

into parts A_I and A_{II} . Here A_I is the sum over $c \in C(x, y)$ with $d_c(x, y) \leq \rho(x, y) + a$, and A_{II} is the sum over $c \in C(x, y)$ with $d_c(x, y) > \rho(x, y) + a$. Then

$$A_I \leq \sum_{c \in C(x, y), d_c(x, y) \leq \rho(x, y) + a} |S(c)| \leq (\deg_{\max} + 1)^{\frac{\text{diam} + a}{r_{\min}}},$$

and, using an argument similar to the proof of Lemma 3.2 in [KPS07], one sees that

$$\begin{aligned} A_{II} &= \sum_{c \in C(x, y), d_c(x, y) > \rho(x, y) + a} |S(c)| \exp\left(-\frac{(d_c(x, y) + \rho(x, y))(d_c(x, y) - \rho(x, y))}{4t}\right) \\ &\leq \sum_{c \in C(x, y), d_c(x, y) > \rho(x, y) + a} \exp\left(-\frac{ar_{\min}|c|}{4t}\right) \\ &= \sum_{n=1}^{\infty} \sum_{|c|=n} \exp\left(-\frac{ar_{\min}|c|}{4t}\right) \\ &\leq \sum_{n=1}^{\infty} |\mathbf{E}|^n \exp\left(-\frac{ar_{\min}n}{4t}\right) \end{aligned}$$

because $d_c(x, y) + \rho(x, y) > d_c(x, y) \geq r_{\min}|c|$, $d_c(x, y) - \rho(x, y) > a$, $|S(c)| \leq 1$ and the number of paths $c \in C(x, y)$ of combinatorial length $|c| = n$ is less than $|\mathbf{E}|^n$. Here $|\mathbf{E}|$ is the number of finite length edges of G .

For t small enough

$$\sum_{n=1}^{\infty} |\mathbf{E}|^n \exp\left(-\frac{ar_{\min}n}{4t}\right) = \frac{|\mathbf{E}| e^{-ar_{\min}/4t}}{1 - |\mathbf{E}| e^{-ar_{\min}/4t}},$$

and choosing a large, we can show this is bounded in the interval $(0, T)$ for any $T > 0$.

Lower Bound. Note that if c_0 is such that $d_{c_0}(x, y) = \rho(x, y)$, then $0 < S(c_0)$ because, with Kirchhoff conditions any negative terms in the product that make up $S(c_0)$ would come from a combinatorial path which has two consecutive edges which are the same, in which case, a shorter combinatorial path c' could be constructed by removing this sequence of two edges.

If x, y are not in the same edge, then the combinatorial path c must visit 2 vertices for any c with $S(c) < 0$, and in this case, this implies that, using the notation from the previous paragraph that $d_{c_0}(x, y) + r_{\min} \leq d_c(x, y)$. Thus, if x, y are not on the same edge, and $d_c(x, y) - \rho(x, y) < r_{\min}$ then $S(c) \geq 0$. Hence, setting the a above to be r_{\min} ,

$$p_t(x, y) \geq g_t(d_{c_0}(x, y)) (A_I - A_{II}).$$

Since there is at least one path from x, y with $d_{c_0}(x, y)$,

$$A_I \geq S(c_0) > \left(\frac{2}{\deg_{\max}} \right)^{\text{diam}/r_{\min}}$$

and since $A_{II} \rightarrow 0$ at $t \rightarrow 0$, then we can find T_0 such that the lower bound holds.

If x and y are in the same edge e , and e has vertices v_- and v_+ then, choosing $a < r_{\min}$ implies that the sum becomes, if x and y are in an internal edge

$$\begin{aligned} p_t(x, y) &= g_t(|x - y|) + \frac{2 - d_{v_-}}{d_{v_-}} g_t(x + y) + \frac{2 - d_{v_+}}{d_{v_+}} g_t(2r_e - x - y) + \sum_{c: d_c(x, y) > \rho(x, y) + a} S(c) g_t(d_c(x, y)) \\ &\geq g_t(|x - y|) \left(\frac{1}{3} - A_{II} \right). \end{aligned}$$

If x, y are in the same external edge, a slight modification above shows that $p_t(x, y) \geq g_t(|x - y|) \left(\frac{2}{3} - A_{II} \right)$. \square

We are now ready to prove the main result of the section.

Theorem 5.4. *Assume that \mathbf{G}^{met} has a finite number of edges.*

1. *If \mathbf{G}^{met} is compact, then there exist a constant $C > 1$ and a constant $K > 0$ such that for every $f \in \text{Dom } \mathcal{E}$ and $t \geq 0$,*

$$\sqrt{\Gamma(P_t f)} \leq C e^{-Kt} P_t \sqrt{\Gamma(f)}.$$

2. *If \mathbf{G}^{met} is not compact, then there exist a constant $C > 1$ and a constant $K \geq 0$ such that for every $f \in \text{Dom } \mathcal{E}$ and $t \geq 0$*

$$\sqrt{\Gamma(P_t f)} \leq C e^{Kt} P_t \sqrt{\Gamma(f)}.$$

Proof. Since \mathbf{G}^{met} has a finite number of edges, as a consequence of Lemma 5.3, we deduce that there exists a constant $C > 1$ such that for $0 < t \leq 1$,

$$\frac{|\vec{p}_t(x, y)|}{p_t(x, y)} \leq C.$$

From Theorem 2.1, for $f \in \text{Dom } \mathcal{E}$,

$$\partial e^{t\Delta} f = e^{t\tilde{\Delta}} \partial f, \quad t \geq 0.$$

Thus, for $0 \leq t \leq 1$, we have

$$\sqrt{\Gamma(P_t f)} \leq C P_t \sqrt{\Gamma(f)}. \quad (15)$$

We now discuss the two cases:

1. **G is compact.** In that case Δ has a pure point spectrum, $1 \in \text{Dom } \Delta$ and the Dirichlet space $(\mathcal{E}, \text{Dom } \mathcal{E})$ satisfies a Poincaré inequality:

$$\int_X \left(f - \int_X f d\mu \right)^2 d\mu \leq \frac{1}{\lambda_1} \mathcal{E}(f, f), \quad f \in \text{Dom } \mathcal{E}.$$

Moreover, it is easy to check that there exists a $M > 0$ such that for μ -almost every $x, y \in X$

$$p_1(x, y) \leq M, \quad |\Gamma(p_1(\cdot, y))(x)| \leq M.$$

See (16) for the bound on $\Gamma(p_1(\cdot, y))(x)$. We conclude then as a consequence of Theorem 2.5.

2. **G is not compact.** One can use Theorem 2.4.

□

We can give a lower bound estimate on the optimal constant in the inequality

$$\sqrt{\Gamma(P_t f)} \leq C e^{Kt} P_t \sqrt{\Gamma(f)}.$$

Theorem 5.5. *Assume that \mathbf{G} has a finite number of edges. Let $\tau > 0$. The optimal constant C in the inequality*

$$\sqrt{\Gamma(P_t f)} \leq C P_t \sqrt{\Gamma(f)}, \quad f \in \text{Dom } \mathcal{E}, 0 \leq t \leq \tau$$

satisfies

$$C \geq \max(\deg v - 1)$$

where the maximum is taken over the set of vertices of \mathbf{G} .

Proof. The idea is to use a local comparison to the Walsh spider around vertexes and a scaling argument. Let v be a vertex in \mathbf{G} . For $c > 0$, we denote by \mathbf{G}^c the metric graph obtained from \mathbf{G} by multiplying all distances by c . Denote by $\delta_c : \mathbf{G} \rightarrow \mathbf{G}^c$ the dilation that fixes v . Let now X be the Walsh spider with N legs where $N = \deg v$. A function $f = (f_1, \dots, f_N) \in L^2(X)$ defines a function \tilde{f} on the graph \mathbf{G}^c by identifying v with the center of the Walsh spider, numbering the edges adjacent to v and defining

$\tilde{f}(x_i) = f_i(d(v, x_i))$ when x_i is in the edge numbered i and $\tilde{f} = 0$ on edges which are not adjacent to v . When $c \rightarrow +\infty$, one has

$$P_{t/c^2}(\tilde{f} \circ \delta_c)(\delta_c^{-1}x_i) \rightarrow (P_t^X f)(x_i^*),$$

where $x_i^* \in X$ is the point on the leg i such that $x_i^* = d(v, x_i)$. Rescaling then the inequality

$$\sqrt{\Gamma(P_t \tilde{f})} \leq C P_t \sqrt{\Gamma(\tilde{f})}, \quad 0 \leq t \leq \tau$$

and taking the limit when $c \rightarrow +\infty$ yields

$$\sqrt{\Gamma^X(P_t^X f)} \leq C P_t^X \sqrt{\Gamma^X(f)}, \quad t \geq 0.$$

Since it is true for every f , one must have $C \geq \deg v - 1$. \square

5.3 Local Riesz transform on non-compact metric graphs

For this subsection we assume that G^{met} is a non-compact metric graph with a finite number of edges and rays. We prove the following theorem. We wish to use results from [ACDH04], so we first need to establish that the current setting matches that in the article. In particular, we follow the checklist indicated on page 922 in the local form. For all $t \geq 0$,

$$\frac{t}{2} \leq \mu(B_t(x)) \leq Ct$$

where C is bounded by the number of edges. This is stronger than volume doubling. Doubling is important in the proofs, because it allows us to use the Hardy–Littlewood maximal operators on G as indicated in [Hei01]. Also, the lower bound above does not hold for all times if G is compact.

It is established in [Hae11] that these metric graphs satisfy Gaussian Heat Kernel estimates and Poincaré inequality (alternatively, earlier in this section we established local upper Gaussian estimates, which are sufficient for our situation). It is also well established that P_t is conservative, i.e. $\int p_t(x, y) d\mu(x) = 1$. Further, from the standard theory of Dirichlet forms $\|(-\Delta)^{1/2} f\|^2 = \mathcal{E}(f) = \|df\|^2$, and thus the Riesz transform is L^2 bounded. The Laplacian operator is elliptic, by virtue of the fact that it looks like the 1-dimensional Laplacian almost everywhere.

Theorem 5.6. *There is $\alpha > 0$ such that for all $a \geq \alpha$, the local Riesz transform $d(-\Delta + a)^{-1/2}$ is bounded in L^p for all p with $1 < p < \infty$.*

Proof. The proof leverages [ACDH04, Theorem 1.8] which states that, for a metric measure space with local volume doubling (in our case, implied because there are a finite number of edges) and local upper estimates on the diagonal of the heat kernel (in our case, implied by lemma 5.3), if there is $\beta > 0$ such that

$$|d_x p_t(x, y)| \leq \frac{C e^{\beta t}}{\sqrt{t} \mu(B_{\sqrt{t}}(x))} \quad (16)$$

then the local Riesz transform is bounded. This heat kernel gradient estimate is the condition referred to as G_{loc} in [ACDH04].

Calculating from above

$$\begin{aligned} |d_x p_t(x, y)| &= \left| \sum_{c \in C(x, y)} S(c) \frac{\pm d_c(x, y)}{2t} g_t(d_c(x, y)) \right| \\ &\leq \frac{1}{t\sqrt{4\pi}} \sum_{n=0}^{\infty} |\mathbb{E}|^n \frac{n r_{min}}{t^{1/2}} e^{-r_{min}^2 n^2 / (4t)} \\ &\leq \frac{1}{t\sqrt{4\pi}} \sum_{n=0}^{\infty} |\mathbb{E}|^n e^{-r_{min}^2 n^2 / 8t}. \end{aligned}$$

Where r_{min} is the minimum length of an edge, the second inequality is by the same argument from Lemma 3.2 in [KPS07] (as we used before), and the last inequality is because $x e^{-x^2} \leq e^{-x^2/2}$. Letting $L = \log(|\mathbb{E}|)$,

$$|\mathbb{E}|^n e^{-n^2 r_{min}^2 / 8t} = e^{Ln - n^2 r_{min}^2 / 8t} \leq e^{-2Ln + 8Lr_{min}^{-2}t}$$

by taking the Taylor expansion of $xL - \frac{x^2 r_{min}^2}{8t}$ around $x = 8tLr_{min}^{-2}$.

The above sum is thus bounded, and we get

$$|d_x p_t(x, y)| \leq \frac{1}{t\sqrt{4\pi}} \frac{e^{8Lr_{min}^{-2}t}}{1 - e^{-L}} \leq \frac{e^{8Lr_{min}^{-2}t}}{\sqrt{t}V(x, \sqrt{t})} \frac{1}{\sqrt{\pi}(1 + |\mathbb{E}|)}.$$

□

5.4 Invalidity of Ricci Curvature lower bounds

In this section we point out that no metric graph with standard boundary conditions and a vertex with degree more than two can satisfy the Ricci Curvature lower bounds of Sturm–Lott–Villani, which shall be denoted $CD(K, \infty)$ for any K . This is obviously not surprising since, from recent works (see [AGS14b] and [AGS15]), under suitable assumptions a generalized Ricci curvature lower bound is actually equivalent to a classical Bakry–Émery estimate:

$$\Gamma(e^{t\Delta} f) \leq e^{2Kt} e^{t\Delta} \Gamma(f).$$

Let the set

$$[A, B]_t = \{z \in A \mid \exists x \in A, y \in B \text{ such that } d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y)\}$$

for $t \in [0, 1]$ is the set of points which are convex combinations of A and B in that the lie on a geodesic connecting a point x in A to a point y in B at the portion t along the curve. The idea is to prove the invalidity of the Brunn–Minkowski inequality.

Let W denote the Wasserstein distance function on probability measures on a geodesic metric measure space (X, d, μ) . We shall need no properties of the Wasserstein distance

other than the fact that it is a metric on probability measures on a metric space, and hence is positive for two different measures.

The Brunn-Minkowski inequality refers to the following convexity condition

$$\log(\mu([A, B]_t)) \geq t \log(\mu(A)) + (1 - t) \log(\mu(B)) + \frac{1}{2} K t(1 - t) W \left(\frac{1_A}{\mu(A)} \mu, \frac{1_B}{\mu(B)} \mu \right)^2.$$

It is proven in [Stu06b, Proposition 2,1] that if a metric measures space (X, d, μ) which satisfies $CD(K, \infty)$, then for all sets A, B and times $t \in [0, 1]$, the above inequality holds. Showing that this inequality doesn't hold was used in [Kaj13, Section 8.2] to prove that $CD(K, \infty)$ does not hold for any K on the harmonic Sierpinski gasket.

The intuitive reasoning why this inequality does not hold on metric graphs is that at each vertex with degree at least 3, geodesics branch off from one another.

Theorem 5.7. *Let \mathbf{G} be a metric graph with standard boundary conditions and d is the intrinsic (geodesic) distance function on \mathbf{G} , and let \mathbf{G} has at least one vertex with degree greater than 2. There are sets A and B in \mathbf{G} for which the Brunn-Minkowski inequality does not hold. Hence it is not possible for \mathbf{G} to satisfy $CD(K, \infty)$ for any K .*

Proof. We shall prove the inequality is not satisfied for the Walsh spider with three legs, $E_i = [0, \infty)$ for $i = 0, 1, 2$. This can be generalized to any metric graph by considering a small neighborhood of a vertex with degree at least 3. Let $A = (a_1, a_2) \subset E_0$ with $a_2 - a_1 = \ell$. Let B consist of two intervals (b_1, b_2) contained in E_1 and E_2 with $b_2 - b_1 = \ell$. Then, for t close enough to 1,

$$[A, B]_t = (ta_1 - (1 - t)b_2, ta_2 - (1 - t)b_1) \subset E_0$$

and hence

$$\mu([A, B]_t) = t(a_2 - a_1) + (1 - t)(b_2 - b_1) = \ell.$$

On the other hand

$$t \log(\mu(A)) + (1 - t) \log(\mu(B)) = \log(\ell) + (1 - t) \ln(2) \geq \log(\ell) = \log(\mu([A, B]_t)).$$

Thus it is impossible to satisfy the Brunn-Minkowski inequality. \square

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