PROPAGATION OF CHAOS FOR MEAN-FIELD MODELS

ABSTRACT. Some notes about convergence and fluctuations of mean-field models.

1. INTRODUCTION AND RESULT

s:1

We are concerned with the limiting behavior of the empirical measure and current of mean-field interacting Brownian motions. We deal with the general framework of a smooth Riemmanian manifold (M, g), with smooth (possibly empty), reflecting boundary ∂M. A system of N mean-field interacting Brownian particles, is a stochastic differential equation of the form

$$\begin{aligned} &dX_i = \frac{1}{N}\sum_{j=1}^{N} K(X_i, X_j)\, dt + \sqrt{\nu}\, dW_i(t) \qquad i = 1, \ldots, N \\ &X_i(0) = x_{0,i} \end{aligned} \tag{1.1}$$

e:mf

where $\nu > 0$ is a constant viscosity coefficient and the W_i are independent Brownian motions on M, and reflection conditions are assumed at the boundary ∂M. Precise definitions and notations are given in the next section. Typical cases include $K(x, y) = F(x - y)$ on $\mathbb{R}^2$, and in particular the McKean vortexes models (converging to a Navier-Stokes equation) [5], and the Keller-Segel system [4].

One is generally interested in investigating the limiting behavior of the system (1.1) in the thermodynamic limit, namely when N is large. A typical observable in this context is the empirical measure $\theta_N(t) := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)}$. Assuming $\theta_N(0) \to u_0\, dx$, one expects that, as the number of particles N diverges to infinity, θ_N approximates a continuous density $u(t, x)$, that is a weak solution to the parabolic equation

$$\begin{aligned} &\partial_t u + \nabla \cdot (u\, K[u]) = \frac{\nu}{2}\Delta u \qquad \text{on } M \\ &\nabla u \cdot \hat{n} = 0 \qquad \text{on } \partial M \\ &u(0, x) = u_0(x) \end{aligned} \tag{1.2}$$

e:gng

where $K[\cdot]$ is the operator $K[u](x) := \int K(x, y) u(y) dy$, Δ is the Beltrami-Laplace operator, and $\hat{n}$ is the normal to ∂M.

More recently, the macroscopic fluctuation theory focused the interest on an observable other than (and indeed richer than) the empirical measure, namely the empirical current J_N defined weakly as

$$\langle dJ_N(t), \omega \rangle := \frac{1}{N}\sum_{i=1}^{N} \omega(X_i(s)) \circ dX_i(s)$$

e:curr1

The analogue of the convergence result to (1.2) in the context of currents, is the convergence of the couple (θ_N, J_N) to the solution (u, J) of the system

$$\begin{aligned} &\partial_t u + \nabla \cdot J = 0 \\ &J = u\, K[u] - \frac{\nu}{2} \nabla u \end{aligned} \tag{1.3}$$ e:gng2

with boundary and initial data given similarly to (1.2). Note that this is a stronger statement, since from the convergence of θ_N to (1.2) one can only recover the limiting current J up to a divergence-free (and time dependent) additive current.

Convergence results for θ_N, or even for J_N are not hard to prove rigorously, if one assumes that $K[\cdot]$ is a continuous operator in a suitable topology. However, the most interesting models do feature interaction kernels $K(\cdot, \cdot)$ with singularities. Convergence of the empirical measure for such models has been the object of intense mathematical investigation in the last decades. Here we cite the early results by Osada [7] for the vortexes model, the much more recent paper [3] that finally proved the convergence in this case, and [2] which proves convergence for sub-critical Keller-Segel models.

In this paper we aim to build over the technique introduced in [1] to prove this type of convergence results, and we apply it to two classes of systems, including?????? the aforementioned vortexes and Keller-Segel models, where $M \subset \mathbb{R}^n$ or $M = \mathbb{T}^n$:

t:conv1 **Theorem 1.1.** *Assume that ??????. Then (θ_N, J_N) converges in law to the solution to* (1.3).

t:conv2 **Theorem 1.2.** *Assume that ??????. Then (θ_N, J_N) satisfies a large deviations principle with speed N, and good rate function I given by* (2.2) *(see also* (2.3)*).*

ss:1.1

1.1. Mean-field models and variational techniques.

ss:1.2

1.2. Main result.

s:2

2. The variational approach

In this section we first introduce some precise notation, and next describe a variational technique to prove propagation of chaos.

ss:2.1

2.1. The smooth case. Hereafter (M, g) is a smooth Riemmanian manifold without boundary, and we fix $T > 0$ to be interpreted as a final time. Elements of M are denoted with latin letters x, y etc. Fixed $N \in \mathbb{N}^+$, elements of the product manifold $M^{\otimes N}$ are denoted $\underline{x} \equiv (x_1, \dots, x_N)$. We denote by dx and $d\underline{x}$ the volume measure on M and $M^{\otimes N}$ respectively. $\mathcal{P}(M)$, $\mathcal{V}(M)$ and $\mathcal{J}(M)$ are respectively the spaces of Borel probability measures, and distributional vector fields and currents on M. Then we regard the kernel K as a map $K\colon M \to \mathcal{V}(M)$, meaning that K maps $y \in M$ in the vector field $(K(x,y))_{x\in M}$.

We assume that

- (A1) $\nu = 2$ (with no loss of generality).
- (A2) (M, g) is compact, and without boundary.
- (A3) The solution (in law) to (1.1) is a Feller process on $M^{\otimes N}$ with generator

$$(L_N f)(\underline{x}) := \tfrac{1}{2}(\Delta_N f)(\underline{x}) + \frac{1}{N} \sum_{i,j=1}^{N} K(x_i, x_j)(\nabla_i f)(\underline{x})$$ e:gen

(A4) There exists $U \in C^1(M \times M)$ such that the σ-finite measure

$$d\pi_N(\underline{x}) = d\underline{x} \exp\left(-\tfrac{1}{N}\sum_{i,j=1}^{N} U(x_i, x_j)\right)$$ e:pin

is invariant, namely $\pi_N L_N = 0$. With no loss of generality, we assume $U(x,y) = U(y,x)$, and we denote $(\nabla U)(x,y) := (\nabla_1 U)(x,y) = (\nabla_2 U)(y,x)$.

We introduce the space

$$\mathcal{X}_N := \left\{(\mu_\cdot, J_\cdot) \in C\left([0,T]; \mathcal{P}(M^{\otimes N}) \times \mathcal{J}(M^{\otimes N})\right) \,:\, \partial_t \mu + \nabla \cdot J = 0 \,\text{weakly}\right\}$$ e:meascurr

and $\mathcal{X} \equiv \mathcal{X}_1$.

Given $\mu \in \mathcal{P}(M)$, define the current $\overline{J}^\mu$ as

$$\langle \overline{J}^\mu, \omega\rangle := \int_M d\mu(x)\left(\langle K[\mu], \omega\rangle_x + (\nabla \cdot \omega^\sharp)(x)\right)$$ e:jmu1

d:mec **Definition 2.1.** *A solution to the* mean-field equation for currents*, with initial data $\mu_0 \in \mathcal{P}(M)$, is a curve $(\mu, J) \in \mathcal{X}$ such that $\mu_{t=0} = \mu_0$ and for a.e. $t \in [0,T]$*

$$J_t = \overline{J}^{\mu_t} \tag{2.1}$$ e:mec

namely (1.3) *holds weakly.*

r:mec1 **Remark 2.2.** *If (μ, J) is a solution to the mean-field equation for currents, then (μ_t) is a weak solution to* (1.2).

Conversely, if μ_t solves (1.2) *weakly and $(\mu, J) \in \mathcal{X}_N$, then $J_t - \overline{J}^{\mu_t}$ is divergence-free.*

Let $I \colon \mathcal{X} \to [0, +\infty]$ be defined as

$$I(\mu, J) := \sup_{\omega \in C([0,T];\Omega)} \int_{[0,T]} dt\left[\langle J_t - \overline{J}^{\mu_t}, \omega_t\rangle - \tfrac{1}{2}\int_M d\mu_t(x)\langle \omega_t^\sharp, \omega_t\rangle_x\right] \tag{2.2}$$ e:ldfunc1

r:ldfunc1 **Remark 2.3.** *I is convex, has compact sub-level sets, and $I(\mu, J) = 0$ iff (μ, J) solves the mean-field equation for currents* (2.1).

r:ldfunc2 **Proposition 2.4.** *Suppose that $I(\mu, J) < +\infty$. Then*

$$I(\mu, J) = S(\mu_T) - S(\mu_0) + \tfrac{1}{2}\int_{[0,T]} dt\left[\mathcal{E}(\mu_t) + \left\|J_t - \overline{J}^{a,\mu_t}\right\|_{\mu_t}^2\right] \tag{2.3}$$ e:ldfunc2

Proof.

$$I(\mu, J) = \tfrac{1}{2}\int_{[0,T]} dt \left\|J_t - \overline{J}^{a,\mu_t}\right\|_{\mu_t}^2 + \left\|\overline{J}^{s,\mu_t}\right\|_{\mu_t}^2 + \left\langle \overline{J}^{a,\mu_t} - J_t, \overline{J}^{s,\mu_t}\right\rangle_{\mu_t}$$ e:decmec4

Moreover □

Let $S, \mathcal{E} \colon \mathcal{P}(M) \to [0, +\infty]$ be defined as

$$S(\mu) := \begin{cases} \int d\mu(x) d\mu(y)\, U(x,y) + \int d\mu(x) \,\log \frac{d\mu}{dx} & \text{if } \mu \ll dx \\ +\infty & \text{otherwise} \end{cases}$$ e:qp1

$$\mathcal{E}(\mu) := \begin{cases} \int d\pi^\mu(x)\, \frac{|\nabla \varrho^\mu|^2(x)}{\varrho^\mu(x)} & \text{if } \mu \ll \pi^\mu \\ +\infty & \text{otherwise} \end{cases}$$ e:qp2

Since U is smooth, S has a unique minimizer on $\mathcal{P}(M)$ that we denote $\bar{\pi}$. It satisfies $\bar{\pi}(dx) := dx\, \exp(-U[\pi])$.

If $\mu \in \mathcal{P}(M)$ and $\xi \in \mathcal{V}(M)$, the current $\mu\,\xi \in \mathcal{J}(M)$ is understood as

$$\langle \mu\,\xi,\, \omega\rangle = \int d\mu(x)\, \langle \xi,\, \omega\rangle_x \qquad \texttt{e:curvec}$$

Given $\mu \in \mathcal{P}(M)$, define the currents $\bar{J}^{a,\mu}$, $\bar{J}^{s,\mu}$ as

$$\begin{aligned} \bar{J}^{a,\mu} &:= \mu\left(K[\mu] + (\nabla U[\mu])^\sharp\right) = \mu\, c[\mu] \\ \bar{J}^{s,\mu} &:= \bar{J}^{\mu} - \bar{J}^{a,\mu} = -\mu(\nabla U[\mu])^\sharp - \tfrac{1}{2}\nabla\mu \end{aligned} \qquad \texttt{e:jmu}$$

Then setting $\bar{\pi}^\mu(dx) := \exp(-U[\mu])dx$, and $h^\mu(x) := (\frac{d\mu}{d\bar{\pi}^\mu}(x))^{1/2} =: (\varrho^\mu)^{1/2}$

$$\begin{aligned} \langle \bar{J}^{s,\mu},\, \bar{J}^{s\mu}\rangle_\mu &= \int d\mu(x)\, \langle \nabla U[\mu] + \tfrac{1}{2}\nabla \log\mu,\, \nabla U[\mu] + \tfrac{1}{2}\nabla\log\mu\rangle_x \\ &= \tfrac{1}{4}\int d\pi^\mu(x)\frac{1}{\varrho^\mu(x)}\langle(\nabla\varrho^\mu)^\sharp,\, \nabla\varrho^\mu\rangle_x \\ &= \int d\pi^\mu(x)\langle(\nabla h^\mu)^\sharp,\, \nabla h^\mu\rangle_x \end{aligned} \qquad \texttt{e:scal1}$$

$$\begin{aligned} \langle \bar{J}^{a,\mu},\, \bar{J}^{s\mu}\rangle_\mu &= -\int d\mu(x)\langle c[\mu],\, \nabla U[\mu] + \tfrac{1}{2}\nabla\log\mu\rangle_x \\ &= \int d\mu(x)\, ((\nabla\cdot c[\mu])(x) - \langle c[\mu],\, \nabla U[\mu]\rangle_x) = 0 \end{aligned} \qquad \texttt{e:scal2}$$

$$\begin{aligned} \langle J,\, \bar{J}^{s,\mu_t}\rangle_\mu &= -\langle J,\, \nabla U[\mu] + \tfrac{1}{2}\nabla\log\mu\rangle \\ &= \langle \nabla\cdot J,\, U[\mu] + \tfrac{1}{2}\log\mu\rangle = \tfrac{1}{2}\langle\nabla\cdot J,\, \log\varrho^\mu\rangle \end{aligned} \qquad \texttt{e:scal3a}$$

In particular if $(\mu, J) \in \mathcal{X}$

$$\begin{aligned} \int_{[0,T]} dt\, \langle J_t,\, \bar{J}^{s,\mu_t}\rangle_\mu &= -\tfrac{1}{2}\int_{[0,T]} dt\, \langle \partial_t\mu_t,\, U[\mu] + \tfrac{1}{2}\log\mu\rangle \\ &= \tfrac{1}{2}S(\mu_T) - \tfrac{1}{2}S(\mu_0) \end{aligned} \qquad \texttt{e:scal3b}$$

$$\begin{aligned} \int d\mu(x)(\nabla\cdot\omega^\sharp)(x) &= \int d\pi(x)\,\varrho(x)\,(\nabla\cdot\omega^\sharp)(x) \\ &= -\int d\pi(x)\,\langle\nabla\varrho,\, \omega^\sharp)\rangle_x - \int d\pi(x)\,\langle\nabla\log\frac{d\pi}{dx},\, \omega^\sharp\rangle_x \\ &= -\int d\mu(x)\,\langle\nabla\log\varrho,\, \omega^\sharp)\rangle_x - \int d\mu(x)\,\langle\nabla\log\frac{d\pi}{dx},\, \omega^\sharp\rangle_x \end{aligned} \qquad \texttt{e:conto1}$$

Then $\bar{J}^{s,\mu} = \bar{J}^\mu - \bar{J}^{a,\mu} = \mu\,(\nabla\log\varrho)^\flat$.
From (A3), if one defines $c \in C(M; \mathcal{V}(M))$ as

$$c(x,y) := K(x,y) + 2(\nabla U)^\flat(x,y) \qquad \texttt{e:kdec1}$$

then

$$\sum_{i,j,k} [2\,\langle c(x_i,x_j),\, (\nabla U)(x_k,x_i)\rangle_{x_i} - \nu(\nabla\cdot c)(x_i,x_j)] = 0 \qquad \texttt{e:ort1}$$

Let $\mathbf{K}$ be the tangent field $\mathbf{K}_i(\underline{x}) := \frac{1}{N}\sum_j K(x_i, x_j)$ and define for $\mu \in \mathcal{P}(M^{\otimes N})$ the current $\bar{J}_N^{\mu}$ as

$$\langle \bar{J}_N^{\mu},\, \omega\rangle := \int d\mu(\underline{x})\langle \mathbf{K},\, \omega\rangle + \frac{\nu}{2}(\nabla\cdot\omega^{\sharp})(x)$$

e:jntyp

Define $I_N\colon \mathcal{X}_N \to [0,+\infty]$ as

$$I_N(\mu, J) := \sup_{\omega\in C([0,T];\Omega_N)} \int_{[0,T]} dt\, \Big[\langle J - \bar{J}_N^{\mu_t},\, \omega_t\rangle - \tfrac{\nu}{2}\int_{M^{\otimes N}} d\mu_t(\underline{x})\langle \omega_t^{\sharp},\, \omega_t\rangle\Big]$$

e:in1

$$S_N(\mu) := \frac{1}{N} H(\mu|\pi_N)$$

e:entn

$$\mathcal{E}_N(\mu) := \int d\pi_N(x)$$

e:diricn

p:decn

Proposition 2.5. *Let $(\mu, J) \in \mathcal{X}_N$ be such that $I_N(\mu, J) < +\infty$. Then*

$$I_N(\mu, J) = S_N(\mu_T) - S_N(\mu_0) + \tfrac{1}{2}\int_{[0,T]} dt\, \mathcal{E}_N(\mu_t) + \tfrac{1}{2}\int_{[0,T]} dt\, \|J_t - J_N^{a,\mu_t}\|_{\mu_t}^2 \tag{2.4}$$

e:decn

ss:2.2

2.2. **Notation and preliminaries.** A Polish space is a completely metrizable, separable topological space. Recall that closed and open subsets of Polish spaces are Polish w.r.t. the relative topology. Given a Polish space $\mathcal{M}$, we denote by $\mathcal{P}(\mathcal{M})$ the set of Borel probability measures on $\mathcal{M}$, and $\mathcal{P}(\mathcal{M})$ is itself a Polish space when equipped with the narrow topology, and it is compact iff $\mathcal{M}$ is.

Hereafter (M, g) is a smooth Riemmanian manifold with smooth boundary ∂_M, and we fix $T > 0$ to be interpreted as a final time. Elements of M are denoted with latin letters x, y etc. Fixed $N \in \mathbb{N}^+$, elements of the product manifold $M^{\otimes N}$ are denoted $\underline{x} \equiv (x_1, \dots, x_N)$. We suppose that we are given Σ_N as a closed subset of $M^{\otimes N}$, and

$$M_N := M^{\otimes N} \setminus \Sigma_N$$ e:mn

The case we have in mind is

$$\Sigma_N := \cup_{i \neq j} \left\{ \underline{x} \in M^{\otimes N} \,:\, x_i \neq x_j \right\}$$ e:mn2

M_N can be regarded both as a (non-complete) Riemmanian manifold, and as a Polish space (being an open subset of the Polish space $M^{\otimes N}$). This is a slightly delicate point. On the one hand M_N is equipped with a Riemmanian distance $d_{g,N}$ induced from the action of the tensor $g_N = g^{\otimes N}$; we always refer to this distance when considering local properties of M_N (smoothness, differential operators etc). However such a distance is not complete on M_N; so when dealing with global topological issues (in particular, uniformly continuous functions) we will rather refer to a complete, totally bounded distance d_N on M_N. For instance one may take

$$\begin{aligned} d_N(\underline{x}, \underline{y}) &:= \frac{\tilde{d}_N(\underline{x}, \underline{y})}{1 + \tilde{d}_N(\underline{x}, \underline{y})} \\ \tilde{d}_{N,g} &:= d_{N,g}(\underline{x}, \underline{y}) + \left| \tfrac{1}{d_{N,g}(\underline{x}, \Sigma_N)} - \tfrac{1}{d_{N,g}(\underline{y}, \Sigma_N)} \right| \end{aligned}$$ e:dn

We denote by Ω_N the set of infinitely differentiable 1-forms over M_N, uniformly continuous and with uniformly continuous derivates w.r.t. d_N. We denote by $\mathcal{J}(M_N)$ the set of distributional 1-currents on M_N, that is the dual of Ω_N equipped with the weak topology. ∇, $\nabla\cdot$ are the co??????variant gradient and divergence operators on M_N.

The map

$$\begin{aligned} &\theta_N \colon M_N \to \mathcal{P}(M) \\ &\theta_N \colon \underline{x} \mapsto \theta_N^{\underline{x}}(dy) := \tfrac{1}{N} \sum_{i=1}^N \delta_{x_i}(dy) \end{aligned}$$ e:emp1

is called the *empirical measure*. It naturally lifts to a map

$$C([0,T]; M_N) \mapsto C([0,T]; \mathcal{P}(M))$$ e:emp2

and with a little abuse of notation we still denote with θ_N such a map. θ_N induces a pushforward both on $\mathcal{P}(M_N)$ and $\mathcal{J}(M_N)$. With some abuse of notation, we denote by Θ_N the pushforward on both spaces, as well as on their product, namely:

$$\Theta_N \colon \mathcal{P}(M_N) \times \mathcal{J}(M_N) \to \mathcal{P}(\mathcal{P}(M)) \times \mathcal{P}(\mathcal{J}(M))$$
$$\Theta_N \colon (\mu, J) \mapsto \Theta_N(\mu, J) =: (\Theta_N \mu, \Theta_N J)$$

e:push1

$$\Theta_N \mu := \mu \circ \theta_N^{-1} \qquad \langle \Theta_N J,\, \omega\rangle := \tfrac{1}{N} \sum_{i=1}^{N} \langle J,\, d^* \Pi_i \omega\rangle$$

where $d^*\Pi_i$ is the co-differential of the i-th canonical projection $\Pi_i \colon M_N \to M$.

ss:2.3

2.3. A functional formulation of differential equations.

ss:2.4

2.4. The convergence criterium.

p:tight1 **Proposition 2.6.** *Assume equicoercivity of ??????. Then tightness ??????.*

t:crit **Theorem 2.7.** *Assume that*

(A) *The sequence $\Theta_N \pi^N$ converges to π in $\mathcal{P}(M)$.*

(B) *For any sequence $(m^N, \jmath^N) \in \mathcal{P}(M_N) \times \mathcal{J}(M_N)$ such that*

$$\lim_{N\to+\infty} (\Theta_N m^N, \Theta_N \jmath^N) = \delta_{(\mu,J)} \qquad \text{in } \mathcal{P}(\mathcal{P}(M) \times \mathcal{J}(M))$$ e:recinf

it holds

$$\varliminf_{N\to+\infty} \mathcal{E}_N(m^N) \geq \mathcal{E}(\mu)$$ e:gammaliminf1

$$\varliminf_{N\to+\infty} \|\jmath^N - \|_{m^N}\| \geq \|J - \|_\mu$$ e:gammaliminf2

Then the convergence ?????? holds. Assume furthermore that the conditions ?????? of Proposition 2.6 hold. Then ??????.

s:3

3. Metastability

The rest is mostly crap. The idea is that maybe one can prove the result as in Pelletier-Savaré, which would give really the limit of the process, not just estimates of hitting times. The boundary above comes from the fact that K may have singularities, not a real boundary.

For fixed N, K writes as $K = -\nabla U + c$. Just U and c depend on the invariant measure. π_N satisfies an SPDE

$$\partial_t u + \nabla \cdot (u\, K[u]) = \frac{\nu}{2} \Delta u + \sqrt{1/N} M$$ e:pi

where M is a martingale with quadratic variation $[M(\phi), M(\phi)]_t = \int_0^t \int du |\nabla\phi|^2 ds$. Then the quadratic decomposition (2.4) should hopefully work as in Peletier (which is however reversible). As a start, can we do it when there is no interaction?

References

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