Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Functional Analysis 230 (2006) 382–418
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

# Mosco convergence of Dirichlet forms in infinite dimensions with changing reference measures☆

Alexander V. Kolesnikov

*Scuola Normale Superiore, Centro Di Ricerca Matematica Ennio De Giorgi, I-56100 Pisa, Italy*

Received 15 December 2004; accepted 4 June 2005
Communicated by L. Gross
Available online 30 August 2005

**Abstract**

Let $E$ be an infinite-dimensional locally convex space, let $\{\mu_n\}$ be a weakly convergent sequence of probability measures on $E$, and let $\{\mathcal{E}_n\}$ be a sequence of Dirichlet forms on $E$ such that $\mathcal{E}_n$ is defined on $L^2(\mu_n)$. General sufficient conditions for Mosco convergence of the gradient Dirichlet forms are obtained. Applications to Gibbs states on a lattice and to the Gaussian case are given. Weak convergence of the associated processes is discussed.


*Keywords:* Dirichlet forms; Mosco convergence; Convergence of stochastic processes; Gaussian measures; Gibbsian measures

## 1. Introduction

This paper continues the author's research [19] on the Mosco convergence. We recall that the Mosco convergence was introduced by Mosco in [26]. The main result of [26] states that the Mosco convergence of quadratic forms is equivalent to strong convergence of the corresponding semigroups. If the semigroups are associated with stochastic processes, then form convergence implies weak convergence of the finite-dimensional distributions of the corresponding processes.

☆ Supported in part by RFBR Grant 04–01–00748, DFG Grant 436 RUS 113/343/0(R), RFBR MAS Grant 03–01–06270, German Academic Exchange Service (DAAD).
*E-mail address:* sascha77@mail.ru.

0022-1236/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2005.06.002

Another important step was made by Zhikov in [41] and Kuwae and Shioya in [20]. In these works, the case of a sequence of Hilbert spaces was studied. More precisely, they introduced some natural convergence of a sequence of Hilbert spaces $\{H_n\}$ to a Hilbert space $H$. Kuwae and Shioya introduced the Mosco convergence of quadratic forms $\mathcal{E}_n \to \mathcal{E}$, where every $\mathcal{E}_n$ is defined on $H_n$. We emphasize that this situation is typical for applications, having in mind the basic example of a sequence of forms $\{\mathcal{E}_n\}$ defined by

$$\mathcal{E}_n(f) = \int_E |\nabla f|^2 \, d\mu_n.$$

Here $E$ is a finite- or infinite-dimensional space, $\{\mu_n\}$ is a weakly convergent sequence of probability measures, and $\nabla$ is some gradient operator on $E$ (e.g. the standard gradient on the finite-dimensional Euclidean space or the Malliavin gradient on Wiener space, etc.) and every form $\mathcal{E}_n$ is defined on $L^2(\mu_n)$.

In this paper we give applications of the Mosco convergence to some typical Dirichlet forms appearing in analysis and mathematical physics. In contrast to [19] we deal mainly with infinite-dimensional spaces. Some partial results on convergence of semigroups and processes in the infinite-dimensional case have been obtained in [12,21,22,36]. The Mosco convergence and convergence of stochastic processes in the finite-dimensional spaces have been studied in [4,18,19,23,24,28,29,37,38,40]. We refer the reader to [19] for a more detailed review.

It is well known that tightness of the finite-dimensional distributions of processes can be proved in many cases by a probabilistic method, the so-called Lyons–Zheng decomposition (see [16,35,39]). Therefore, we are able to prove convergence of forms if we identify the limiting point. This problem can be solved by applying the Mosco convergence techniques. We emphasize that the description of the limiting point can be rather non-trivial (for example, the following situation is possible: $\rho_n \to \rho$ in $L^1_{\mathrm{loc}}(dx)$ and $\mathcal{E}_n \to \mathcal{E}$ Mosco, where $\mathcal{E}_n(f) = \int_{\mathbb{R}^d} |\nabla f|^2 \rho_n \, dx$, but $\mathcal{E}(f) \neq \int_{\mathbb{R}^d} |\nabla f|^2 \rho \, dx$). In fact, it was shown in [19] that in general the Mosco-limits of classical Dirichlet forms on $R^1$ are non-local and defined on BV-functions. See also [8] for examples of Dirichlet forms of diffusion type which converge Mosco to a Dirichlet form with a non-trivial jumping part and [9] for the characterization of the Mosco limits of diffusion forms in $\mathbb{R}^d$, where $d \geqslant 3$.

The organization of the paper is as follows. In Section 2 we recall the main definitions and results from [20]. We also prove some useful lemmas. In Section 3 we prove the main result of the paper. We establish some sufficient conditions for the Mosco convergence of infinite-dimensional Dirichlet forms, which are easy to check in concrete applications. The Dirichlet forms considered in this paper are defined on a vector space $E$ of a quite general type (cf. [5], where some fundamental properties of Dirichlet forms were studied). However, the reader may assume that $E$ is a separable Banach space. According to [5], the closability of the partial Dirichlet forms is equivalent to some integrability conditions of the corresponding conditional densities (the Hamza condition). Our convergence result is established under an appropriate convergence requirement for the conditional densities. In order to demonstrate the power of the Mosco convergence method we formulate the following theorem, which is a direct consequence of the general results of the paper.

**Theorem 1.1.** *Let $m$ be a fully supported measure on $E$ and let $\{g_n\}$ be a sequence of $m$-a.e. positive functions such that $g_n\,dm$ are probability measures, $g_n\,dm \to g\,dm$ weakly and $g > 0$ $m$-a.e. Consider a sequence of forms $\{\mathcal{E}_n\}$, where each $\mathcal{E}_n$ is the maximal extension of $(\mathcal{E}_n, \mathcal{F}C_0^\infty)$, $\mathcal{E}_n(f) = \int_E |\nabla f|^2 g_n\,dm$ for every $f \in \mathcal{F}C_0^\infty$. Suppose that one of the following conditions holds*:

(1) *$E$ is $\mathbb{R}^d$, $m$ is Lebesgue measure on $\mathbb{R}^d$, $\nabla$ is the standard gradient, $\{e_i\}$ is an orthonormal basis, $m_i$ is the $(d-1)$-dimensional Lebesgue measure on the hyperplane $E_i := \{x : (x, e_i) = 0\}$*;

(2) *$E$ is a locally convex Polish space, $m$ is a centered Gaussian measure, $\nabla$ is the Malliavin gradient, $\{e_i\}$ is an orthonormal basis in the Cameron–Martin space $H$ such that every $\hat{e}_i \in E'$, $m_i$ is the projection of $m$ onto the space $E_i := \{x : \hat{e}_i(x) = 0\}$.*

*Suppose that $\{g_n\}$ is an $m$-equi-integrable sequence and for every $i \in \mathbb{N}$ and $m_i$-almost every $x \in E_i$ the sequence of locally finite one-dimensional measures $\left\{\frac{ds}{g_n(x+se_i)}\right\}$ vaguely converges to the measure $\frac{ds}{g(x+se_i)}$. Suppose in addition that $\mathcal{F}C_0^\infty$ is dense in $\mathcal{D}(\mathcal{E})$ with respect to the norm*

$$f \to (\|f\|_{L^2(g\,dm)} + \mathcal{E}(f))^{\frac{1}{2}}, \quad \mathcal{E}(f) = \int_E |\nabla f|^2 g\,dm.$$

*Then $\mathcal{E}_n \to \mathcal{E}$ Mosco.*

Note that many finite-dimensional results were obtained under the requirement of convergence $(g_n)^{-1} \to g^{-1}$ in $L^1_{\mathrm{loc}}(dx)$ or vaguely in the sense of measures (see [4,19,41]). Here we obtain convergence under much weaker assumptions. It is worth noting that convergence of measures considered in Theorem 1.1 is equivalent to vague convergence of the conditional measures.

Although all the measures in this theorem do admit densities with respect to some fixed measure, we emphasize that we are able to prove convergence also in cases when this property does not hold. In particular, we prove in Section 5 the Mosco convergence of the forms $f \to \int_E |\nabla_{H_n} f|^2_{H_n} \gamma_n$, where $\{\gamma_n\}$ is a weakly convergent sequence of Gaussian measures and $\nabla_{H_n}$ is the corresponding Malliavin gradient.

Although we are mainly interested in the case when the corresponding reference measures do not possess logarithmic derivatives, in Section 4 we give some applications especially for this case. We prove Mosco convergence in the finite-dimensional case under requirement that the logarithmic derivatives are bounded in the corresponding $L_2$-spaces.

Finally, in Section 6 we turn to essentially non-Gaussian cases. Namely, we give applications to a concrete model from statistical mechanics—Gibbs states on a lattice. In particular, we obtain that the Dirichlet form $\mathcal{E}_\mu = \sum_{k\in\mathbb{Z}^d} \mathcal{E}^k_\mu$ associated with a Gibbsian distribution $\mu$ on the configuration space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, where $\mathbb{Z}^d$ is the integer $d$-dimensional lattice, can be obtained as a Mosco limit of the essentially finite-dimensional forms $\mathcal{E}_{\mu,n} = \sum_{k\in\Lambda_n} \mathcal{E}^k_{\nu_{\Lambda_n}}$, where $\Lambda_n$ is an exhausting sequence of subsets in $\mathbb{Z}^d$ and $\{\nu_{\Lambda_n}\}$ is a sequence of the corresponding finite-dimensional Gibbsian distributions which converges to $\mu$ weakly.

Throughout the paper we assume that the limit form $(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu))$ satisfies the following property: the space of smooth cylindrical functions $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}_\mu), \mathcal{E}^1_\mu)$ (see the precise definitions below and some sufficient conditions for this to hold). This property is known for the Dirichlet form $\mathcal{E}_\mu$ associated with a Gibbsian measure $\mu$ on the lattice (see [1]). It is also known that the stochastic process corresponding to $\mathcal{E}_\mu$ exists (by [25]). In particular, we obtain an approximation of the process associated with a Gibbsian measure, by (essentially) finite-dimensional processes.

## 2. General results on Mosco convergence

Following [20] we define convergence of Hilbert spaces, vectors, operators, and forms. It should be noted that a close approach was developed earlier in [41] (cf. Lemma 2.7).

**Definition 2.1.** We say that a sequence of Hilbert spaces $\{H_n\}$ converges to a Hilbert space $H$ if there exists a dense subspace $C \subset H$ and a sequence of operators

$$\Phi_n : C \to H_n$$

with the following property:

$$\lim_{n\to\infty} \|\Phi_n u\|_{H_n} = \|u\|_H \tag{1}$$

for every $u \in C$.

**Definition 2.2** (*Strong convergence*). We say that a sequence of vectors $\{u_n\}$ with $u_n \in H_n$ strongly converges to a vector $u \in H$ if there exists a sequence $\{\tilde{u}_m\} \subset C$ with the following properties:

$$\|\tilde{u}_m - u\|_H \to 0$$

$$\lim_m \overline{\lim_n} \|\Phi_n \tilde{u}_m - u_n\|_{H_n} = 0.$$

**Definition 2.3** (*Weak convergence*). We say that a sequence of vectors $\{u_n\}$, $u_n \in H_n$ weakly converges to $u \in H$ if

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}$, $v_n \in H_n$ strongly convergent to $v \in H$.

**Definition 2.4.** We say that a sequence of bounded operators $\{B_n\}$, $B_n \in L(H_n)$ strongly converges to an operator $B \in L(H)$ if for every sequence $\{u_n\}$, $u_n \in H_n$, that is strongly convergent to $u \in H$, the sequence $\{B_n u_n\}$ strongly converges to $Bu$.

We define the space $\mathcal{H} := \bigcup_n H_n$ as the disjoint union of $H_n$ and define convergence in $\mathcal{H}$ according to Definition 2.1. Now we consider convergence of quadratic forms in $\mathcal{H}$. Recall that a quadratic form is a bilinear mapping $\mathcal{E} : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$, where $\mathcal{D}(\mathcal{E}) \subset H$ is some subspace of $H$. We will only consider non-negative and symmetric quadratic forms. Recall that a form $\mathcal{E}$ is closed if $\mathcal{D}(\mathcal{E})$ equipped with the inner product $\mathcal{E}^1(u) = (u, u)_H + \mathcal{E}(u)$ is complete. We identify a quadratic form $\mathcal{E}$ with the function

$$\mathcal{E}(u) : u \to \begin{cases} \mathcal{E}(u, u), & u \in \mathcal{D}(\mathcal{E}), \\ \infty, & u \notin \mathcal{D}(\mathcal{E}). \end{cases}$$

It is well known that $\mathcal{E}$ is closed if and only if $\mathcal{E} : H \to \overline{\mathbb{R}}$ is lower-semicontinuous (see [26]).

**Definition 2.5.** We say that a sequence $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ of quadratic forms Mosco converges to a quadratic form $\mathcal{E}$ on $H$ if the following conditions are fulfilled:

(M1) If a sequence $\{u_n\}$ with $u_n \in H_n$ weakly converges to $u \in H$, then

$$\mathcal{E}(u) \leqslant \underline{\lim}_n \mathcal{E}_n(u_n).$$

(M2) For every $u \in H$ there exists a strongly convergent sequence $u_n \to u$ with $u_n \in H_n$ such that

$$\mathcal{E}(u) = \lim_n \mathcal{E}_n(u_n).$$

With every closed form $\mathcal{E}$ we associate a non-negative self-adjoint operator $-A$ with $\mathcal{D}(\sqrt{-A}) = \mathcal{D}(\mathcal{E})$ such that $\mathcal{E}(u, v) = (-Au, v)$, $u, v \in \mathcal{D}(\mathcal{E})$. We will denote the associated semigroup $e^{tA}$, $t \geqslant 0$ by $\{T_t\}$ and the resolvent $(\beta - A)^{-1}$, $\beta > 0$, by $\{G_\beta\}$.

The main result of [20] is the following generalization of the Mosco theorem.

**Theorem 2.6.** *Let $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ be a sequence of closed forms and let $\mathcal{E}$ be a closed form on $H$. The following statements are equivalent*:

(1) $\{\mathcal{E}_n\}$ *Mosco converges to* $\mathcal{E}$,

(2) $\{G_{n,\beta}\}$ *strongly converges to* $G_\beta$ *for every* $\beta > 0$,
(3) $\{T_{n,t}\}$ *strongly converges to* $T_t$ *for every* $t > 0$.

The following lemma gives a simple criterion of strong convergence $u_n \to u$.

**Lemma 2.7.** *A sequence* $\{u_n\}$, $u_n \in H_n$, *converges to* $u \in H$ *if and only if* $\|u_n\|_{H_n} \to \|u\|_H$ *and* $(u_n, \Phi_n(\varphi))_{H_n} \to (u, \varphi)_H$ *for every* $\varphi \in C$.

**Proof.** Note that $\Phi_n(\varphi) \to \varphi$ strongly. Then the "only if" part follows from the results of [20]. Let us prove the "if"-part.

Let $\varphi_m \to u$ in $H$, $\varphi_m \in C$. Then

$$\lim_m \overline{\lim}_n \|u_n - \Phi_n(\varphi_m)\|^2_{H_n} = \lim_m (\|u\|^2_H - 2(u, \varphi_m)_H + \|\varphi_m\|^2_H)$$

$$= \lim_m \|u - \varphi_m\|^2_H = 0.$$

The proof is complete. □

**Lemma 2.8.** *Let a sequence of forms* $\{\mathcal{E}_n\}$ *satisfy the first condition of the Mosco convergence* (M1). *Suppose that there exists a set of vectors* $\tilde{C} \subset C$ *such that* $\tilde{C}$ *is dense in the Hilbert space* $(\mathcal{D}(\mathcal{E}), \mathcal{E} + \|\cdot\|^2_H)$, $\Phi_n(\varphi) \in \mathcal{D}(\mathcal{E}_n)$ *and* $\mathcal{E}_n(\Phi_n(\varphi)) \to \mathcal{E}(\varphi)$ *for every* $\varphi \in \tilde{C}$. *Then* $\mathcal{E}_n \to \mathcal{E}$ *Mosco.*

**Proof.** Let us take $u \in H$. It is enough to construct a sequence $\{u_m\}$ such that $u_m \in H_m$, $u_m \to u$ in $\mathcal{H}$ and $\mathcal{E}_m(u_m) \to \mathcal{E}(u)$. By [19, Proposition 7.2] the space $\mathcal{H} = \bigcup_{n=1}^\infty H_n$ is metrizable by some metric $d$. Let us choose a sequence $\{\tilde{u}_n\}$, $\tilde{u}_n \in \tilde{C}$, such that $\tilde{u}_n \to u$ in $H$ and $\mathcal{E}(\tilde{u}_n) \to \mathcal{E}(u)$. By the hypothesis of the lemma $\mathcal{E}_m(\Phi_m(\tilde{u}_n)) \to \mathcal{E}(\tilde{u}_n)$ if $m \to \infty$. Let $\{M(n)\}$ be a sequence of natural numbers such that $M(n+1) > M(n)$, $d(\tilde{u}_n, \Phi_m(\tilde{u}_n)) \leqslant \frac{1}{n}$ and $|\mathcal{E}_m(\Phi_m(\tilde{u}_n)) - \mathcal{E}(\tilde{u}_n)| \leqslant \dfrac{1}{n}$ for every $m > M(n)$. Now we construct the following sequence: $u_m = \Phi_m(\tilde{u}_{k(m)})$, where $k(m)$ is chosen in such a way that $M(k(m)) < m \leqslant M(k(m)+1)$ if $m > M(2)$ and $k(m) = 1$ if $m \leqslant M(2)$. The sequence $\{u_m\}$ possesses the desired properties. The proof is complete. □

Recall the important notion of Γ-convergence, introduced by De Giorgi (see [11] for review of related results).

**Definition 2.9.** We say that a sequence $\{\mathcal{E}_n : H_n \to \overline{\mathbb{R}}\}$ of quadratic forms Γ-converges to a quadratic form $\mathcal{E}$ if the following conditions are fulfilled:

(G1) If a sequence $\{u_n\}$ with $u_n \in H_n$ strongly converges to $u \in H$, then

$$\mathcal{E}(u) \leqslant \underline{\lim}_n \mathcal{E}_n(u_n).$$

(G2) For every $u \in H$ there exists a strongly convergent sequence $u_n \to u$ with $u_n \in H_n$ such that

$$\mathcal{E}(u) = \lim_n \mathcal{E}_n(u_n).$$

Obviously, $\Gamma$-convergence is weaker than the Mosco convergence.

## 3. Main result on Mosco convergence

Before we consider the problem in the most general setting, let us briefly discuss the one-dimensional case. Under the condition $\frac{1}{\rho_n} \in L^1_{\text{loc}}(dx)$ (a simplified version of the Hamza condition) the Dirichlet form $\mathcal{E}_n$, $\mathcal{E}_n(f) = \int_{\mathbb{R}} (f')^2 \rho_n \, dx$ is closable. It has been shown in [19] that if the measures $\dfrac{dx}{\rho_n}$ converge vaguely to some (not necessarily absolutely continuous!) measure $\mu$, then $\mathcal{E}_n \to \mathcal{E}$ Mosco, where $\mathcal{E}$ is associated with $\mu$ (see [19] for details). We emphasize that even if $\rho_n \to \rho$ a.e., $\mu$ may differ from $\frac{1}{\rho}$ and in that case $\mathcal{E}$ differs from $f \to \int_{\mathbb{R}} (f')^2 \rho \, dx$. In fact, the domain of definition of the limit form consists in general on the so-called BV functions. Recall that a function $f$ is called BV if the weak derivative of $f$ is a measure of bounded variation (see [13]). Some results on the BV functions from the point of view of the Dirichlet forms theory are obtained in [14,15].

Such examples can be easily generalized to higher dimensions. However, in this paper we are particularly interested in the case when the Mosco limit coincides with the natural "pointwise" limit. In the multi- and infinite-dimensional situation, the Mosco convergence of the gradient forms can be reduced in many cases to convergence of the corresponding partial forms. According to [5], the closability of an infinite-dimensional partial form follows from the Hamza condition for the conditional densities of the reference measure. In this section we generalize the one-dimensional result and obtain some sufficient conditions for the Mosco convergence in terms of convergence of the corresponding conditional densities.

We consider a Hausdorff locally convex space $E$. To have a nice measure theory, $E$ is supposed to be a Souslin space. For the sake of simplicity the reader may assume that $E$ is a separable Banach space. Let $\mathcal{B}(E)$ denote the Borel $\sigma$-field of $E$. The topological dual space will be denoted by $E'$.

Let $\{\mu_n\}$ be a sequence of Borel probability measures. Recall that a Borel measure $\mu$ is called $k$-quasi-invariant for some $k \in E$ if the "shifted" measure $\mu \circ \tau_{tk}^{-1}$ is absolutely continuous with respect to $\mu$ for every $t \in \mathbb{R}$, where $\tau_{tk}(z) = z - tk$. Throughout the paper we deal with tight sequences of measures. We warn the reader that the space $E$ in general may not be Prohorov, hence a weakly convergent sequence of measures may not be tight.

We say that a sequence of locally finite measures $\{m_n\}$ on $\mathbb{R}^d$ converges vaguely to a measure $m$ if $\int_{\mathbb{R}^d} \varphi \, dm_n \to \int_{\mathbb{R}^d} \varphi \, dm$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$.

The following assumptions hold throughout the paper.

**Assumption I.** $\mu_n \to \mu$ weakly.

**Assumption II.** There exists a dense set $\mathcal{K} \subset E$ such that $\mu$ is $k$-quasi-invariant for every $k \in \mathcal{K}$.

Assumption II implies, in particular, that $\mu$ has full support. We will apply the definition from Section 2 to the sequence $\{H_n\} = \{L^2(\mu_n)\}$. Set $C := \mathcal{F}C_0^\infty$, where

$$\mathcal{F}C_0^\infty := \text{linear span}\{u : E \to \mathbb{R} : \quad \text{there exist } l_1, \ldots, l_m \in E' \quad \text{and } f \in C_0^\infty(\mathbb{R}^m)$$
$$\text{such that} \quad u(z) = f(l_1(z), \ldots, l_m(z)), z \in E\}$$

and let $\Phi_n$ be the identity operator. Note that since $\operatorname{supp}(\mu) = E$, the operator $\Phi_n$ is well defined. Recall that the space $\cup_n H_n$ is denoted by $\mathcal{H}$.

Let us consider a weakly convergent sequence $k_n \to k$ of vectors from $E$, i.e., $l(k_n) \to l(k)$ for every $l \in E'$. We fix some $l \in E'$ such that $l(k_n) \neq 0$ if $k_n \neq 0$. We denote by $\pi_{k_n}$ the projection $\pi_{k_n} : E \to \pi_{k_n}(E) = E_0$, $k_n \neq 0$

$$\pi_{k_n}(z) := z - \frac{l(z)}{l(k_n)} k_n, \quad z \in E.$$

It is well known that every measure $\mu_n$ has conditional measures $\hat{\rho}_n(x, \cdot)$ on the real line such that letting $\hat{\nu}_n := \pi_{k_n}(\mu_n)$ be the image of $\mu_n$ under $\pi_{k_n}$ one has

$$\int_E u(z)\mu_n(dz) = \int_{E_0} \int_{\mathbb{R}} u(x + sk_n)\hat{\rho}_n(x, ds)\hat{\nu}_n(dx). \tag{2}$$

We use a more general form of this classical result, namely we do not assume what the conditional measures are normalized. This means that we consider a function $\rho_n : E_0 \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}^+$ such that for every bounded, $\mathcal{B}(E)$-measurable function $u : E \to \mathbb{R}$ one has

$$\int_E u(z)\mu_n(dz) = \int_{E_0} \int_{\mathbb{R}} u(x + sk_n)\rho_n(x, ds)\nu_n(dx), \tag{3}$$

where $\nu_n$ is a finite measure that is equivalent to $\pi_{k_n}(\mu_n)$.

**Remark 3.1.** We emphasize that unlike [5] the measures $\rho_n(x, ds)$, $\nu_n$ are not necessarily probability measures! The reader will see that an appropriate choice of non-normalized conditional measures is important. Hence representation (3) is not unique, but as soon as $\nu_n$ is fixed, $\rho_n(\cdot, ds)$ is $\nu_n$-uniquely determined.

In addition, for every $\mu_n$, we choose its own disintegration. We denote sets of the type

$$\{sk_n + v, s \in A \subset \mathbb{R}, v \in B \subset E_0\}$$

by $A \times B$. Suppose that we are given a Borel measure $\lambda$ on $\mathbb{R}$ and a Borel measure $\nu$ on $E_0$. Then there is a unique Borel measure $\mu$ defined by $\mu(A \times B) = \lambda(A) \times \nu(B)$. Let $\mu = \lambda \times \nu$. Note that in these formulas we do not explicitly indicate that the product is taken "along $k_n$", but it will be clear which direction $k_n$ is chosen. For instance, "$f(x + sk_n)\, ds\nu(dx)$" or "$f(x + sk_n)\, ds \cdot \nu(dx)$" means that we consider the measure given by its density $f$ with respect to the product of Lebesgue measure and $\nu$ taken "along $k_n$".

We define the following subspace $\mathcal{F}^l C_0^\infty \subset \mathcal{F} C_0^\infty$ by

$$\mathcal{F}^l C_0^\infty := \text{linear span}\{u \in \mathcal{F} C_0^\infty, u(z) = f(l(z), l_1(z), \ldots, l_m(z)),\\ f \in C_0^\infty(\mathbb{R}^{m+1}), z \in E\}.$$

Note that if $u \in \mathcal{F}^l C_0^\infty$, then $\operatorname{supp}(u) \subset \{z : |l(z)| \leqslant N\}$ for some $N > 0$.

We denote by $ds$ the one-dimensional Lebesgue measure and by $\chi_{[a,b]}\, ds$ the restriction of $ds$ to the interval $[a, b]$. The following lemma is well known (see [6]).

**Lemma 3.2.** *Let* $u \in \bigcap_{n=1}^\infty L^1(\chi_{[-n,n]}\, ds \times d\nu)$. *Then the following conditions are equivalent*:

(1) *For* $\nu$ *almost all* $x \in E_0$ $u_x := s \to u(x + ks)$ *has an absolutely continuous* $(ds)$*-version* $\tilde{u}_x$ *such that* $\left(\frac{d\tilde{u}_x}{ds}\right) \in \bigcap_{n=1}^\infty L^1(\chi_{[-n,n]}\, ds \times d\nu)$.

(2) *There exists a function* $v \in \bigcap_{n=1}^\infty L^1(\chi_{[-n,n]}\, ds \times \nu)$ *such that for every* $\varphi \in \mathcal{F}^l C_0^\infty$

$$\int_{E_0} \int_{\mathbb{R}} u \partial_s \varphi \, ds\, d\nu = -\int_{E_0} \int_{\mathbb{R}} v\varphi \, ds\, d\nu.$$

Now we consider the following sequence of partial forms:

$$\mathcal{D}(\mathcal{E}_{\mu_n}^{k_n}) := \Big\{u \in L^2(\mu_n) : \text{for } \nu_n\text{-a.e. } x \in E_0,\ s \to u(x + sk_n) \text{ has an absolutely}\\ \text{continuous } (ds)\text{-version } \tilde{u}_x \quad \text{and} \quad \frac{\partial u}{\partial k_n} := \left(\frac{d\tilde{u}(x + sk_n)}{ds}\right) \in L^2(\mu_n)\Big\}$$

$$\mathcal{E}_{\mu_n}^{k_n}(u, v) := \int_E \frac{\partial u}{\partial k_n} \frac{\partial v}{\partial k_n}\, d\mu_n, \quad u, v \in \mathcal{D}(\mathcal{E}_{\mu_n}).$$

If $k_n = 0$, we set $\mathcal{E}_{\mu_n}^{k_n} = 0$.

**Assumption III.** Every $\mu_n$ is $k_n$-quasi-invariant and $k_n \to k$ weakly.

This assumption implies (see [2]) that the conditional measures $\rho_n(x, ds)$ have densities with respect to Lebesgue measure, i.e., $\rho_n(x, ds) = \rho_n(x + sk_n)\, ds$ for $\nu_n$-a.e. $x$. It can be easily shown that one can choose a $\mu_n$-measurable version of the kernel $\rho_n(x + sk_n) : E_0 \times \mathbb{R} \to \mathbb{R}^+$.

**Assumption IV.** The following sequence of Borel measures

$$\tilde{\mu}_n^{k_n} := \frac{ds}{\rho_n(x + sk_n)} \nu_n(dx)$$

is uniformly bounded on all sets of the type

$$E_0^N := \{z : |l(z)| \leqslant N\},$$

i.e., the sequence

$$\int_E u(z)\, d\tilde{\mu}_n^{k_n} := \int_{E_0} \int_{\mathbb{R}} u(x + sk_n) \frac{ds}{\rho_n(x + sk_n)}\, \nu_n(dx)$$

is bounded for every bounded Borel function $u : E \to \mathbb{R}$ with support in $E_0^N$.

In particular, for $\nu_n$-almost every $x$ (hence, $\pi_{k_n}(\mu_n)$-a.e.) the function $\dfrac{1}{\rho_n(\cdot, x)}$ is locally integrable. This implies that the form $\mathcal{E}_{\mu_n}^{k_n}$ is closed (see [5, Theorem 3.2]). Obviously, $\mathcal{E}_{\mu_n}^{k_n}$ is a closed extension of the form $(\mathcal{E}_{\mu_n}^{k_n}, \mathcal{F}C_0^\infty)$. This extension is usually called "maximal".

In what follows we consider a sequence of forms

$$\mathcal{E}_{\mu_n} = \sum_{i=1}^{\infty} \mathcal{E}_{\mu_n}^{k_n^i}$$

with the domain of definition $\mathcal{D}(\mathcal{E}_{\mu_n}) = \bigcap_{i=1}^{n} \mathcal{D}(\mathcal{E}_{\mu_n}^{k_n^i})$ ("maximal" extension). It is well known that the sum of closed form is closed. If, in addition, $\sum_{i=1}^{\infty} l^2(k_n^i) < \infty$ for every $l \in E'$, then $\mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E}_{\mu_n})$ (see [25]).

Let us introduce some notation. Consider a closed quadratic form $\mathcal{E}$ on $L^2(m)$ such that $\mathcal{F}C_0^\infty \subset \mathcal{D}(\mathcal{E})$. Denote by $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ the minimal closed extension of $(\mathcal{E}, \mathcal{F}C_0^\infty)$. It is assumed throughout the paper that for a (Mosco) convergent sequence of forms $\mathcal{E}_n$, the limiting form $\mathcal{E}$ has the property $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0)) = (\mathcal{E}, \mathcal{D}(\mathcal{E}))$. This means that $\mathcal{F}C_0^\infty$ is dense in the space $\mathcal{D}(\mathcal{E})$ with the norm $\mathcal{E}_1^{1/2}$, where $\mathcal{E}_1(f) = \int_X f^2\, dm + \mathcal{E}(f)$. This assumption turns out to be very helpful for verifying condition (M2) of the Mosco convergence.

This property holds true for many forms considered below (see [1]), see also [12] for a survey on Markov uniqueness, and [10,33,34]. In [41], some counter-examples in finite dimensions can be found. Let us mention some sufficient conditions in finite dimensions for $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0)) = (\mathcal{E}, \mathcal{D}(\mathcal{E}))$ to hold. Let $\mathcal{E}(\varphi) = \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 \rho(x)\, dx$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0)) = (\mathcal{E}, \mathcal{D}(\mathcal{E}))$ if $\rho$ satisfies one of the following conditions:

(I) The Muckenhoupt condition

$$\sup_B \Big(\frac{1}{|B|}\int_B \rho\, dx\Big)\Big(\frac{1}{|B|}\int_B \frac{dx}{\rho}\Big) < \infty.$$

Here the supremum is taken over all balls $B \in \mathbb{R}^d$ and $|B|$ means Lebesgue measure of $B$.

(II) The function $\sqrt{\rho}$ belongs to the Sobolev space $W^{2,1}(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \rho\, dx + \int_{\mathbb{R}^d} \frac{|\nabla\rho|^2}{\rho}\, dx < \infty.$$

The first fact follows from the Muckenhoupt inequality for maximal functions (see [27,41]). The second fact was proved in [34], see also [10] for a short proof.

We note that the sum of two non-closable partial forms may be closable. A highly nontrivial example was constructed in [31].

In the following lemma we prove that $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}_\mu^k), (\mathcal{E}_\mu^k)_1^{1/2})$ for every partial form $\mathcal{E}_\mu^k$. This was verified in [33] for the case when the reference measure admits the logarithmic derivative in the corresponding direction.

**Lemma 3.3.** *Let $(\mathcal{E}_\mu^k, \mathcal{D}(\mathcal{E}_\mu^k))$ be a partial form as defined above and $(\mathcal{E}_{\mu,0}^k, \mathcal{D}(\mathcal{E}_{\mu,0}^k))$ be the minimal closed extension of $(\mathcal{E}_\mu^k, \mathcal{F}C_0^\infty)$. Then*

$$(\mathcal{E}_{\mu,0}^k, \mathcal{D}(\mathcal{E}_{\mu,0}^k)) = (\mathcal{E}_\mu^k, \mathcal{D}(\mathcal{E}_\mu^k)).$$

**Proof.** Let $f \in \mathcal{D}(\mathcal{E}_\mu^k)$. One can approximate $f$ by functions of the form $(f \wedge n) \vee (-n)$. Hence we may assume without loss of generality that $|f| < K$ for some $K > 0$. Approximating $f$ by functions of the type $f\varphi$, where $\varphi \in \mathcal{F}^l C_0^\infty$, we may assume that $\mathrm{supp}(f) \subset E_0^N$ for some $N > 0$. Choose a $\mu$-version $\tilde{f}$ of $f$ such that $\tilde{f}(x,\cdot)$ is absolutely continuous $\nu$-almost everywhere. Let $\varphi_n \to \frac{\partial \tilde{f}(x+sk)}{\partial s}$ in $L^2(\mu)$, where $\varphi_n \in \mathcal{F}^l C_0^\infty$ and $\mathrm{supp}(\varphi_n) \subset E_0^N$.

Let us consider the following sequence of functions:

$$\psi_n^K = \Big(\int_{-N}^{s} \varphi_n(x+tk)\, dt \wedge K\Big) \vee (-K).$$

Note that for every $A \in \mathbb{R}^+$ and $s \in [-N, A]$ one has

$$\int_{-A}^{A} \int_{E_0} \Big| \int_{-N}^{s} \varphi_n(x+tk)\,dt - \tilde{f}(x+sk) \Big|\, ds \cdot \nu(dx)$$

$$\leqslant \int_{-A}^{A} \int_{E_0} \Big[ \int_{-N}^{A} \Big| \varphi_n(x+tk)\,dt - \frac{\partial \tilde{f}(x+tk)}{\partial t} \Big|\, dt \Big]\, ds \cdot \nu(dx)$$

$$= 2A \int_{E_0} \int_{-N}^{A} \Big| \varphi_n(x+tk)\,dt - \frac{\partial \tilde{f}(x+tk)}{\partial t} \Big|\, dt \cdot \nu(dx)$$

$$\leqslant 2A \Big[ \int_{E} \Big| \varphi_n - \frac{\partial \tilde{f}(x+tk)}{\partial t} \Big|^2 \mu(dz) \Big]^{\frac{1}{2}} \Big[ \int_{E_0} \int_{-N}^{A} \frac{dt}{\rho(t,x)}\, \nu(dx) \Big]^{\frac{1}{2}} \to 0.$$

This means that $\int_{-N}^{s} \varphi_n(t,x)\,dt \to f$ in $L^1(\chi_{[-A,A]}(s)\,ds \cdot \nu(dx))$, hence one can extract a subsequence (denoted again by $\varphi_n$) such that $\int_{-N}^{s} \varphi_n(t,x)\,dt \to f$ $\mu$-a.e. Hence $\psi_n^K \to f$ $\mu$-a.e. Since $|\psi_n^K|$ are uniformly bounded by $K$, we have $\psi_n^K \to f$ in $L^2(\mu)$.

Obviously,

$$\frac{\partial \psi_n^k}{\partial s} = \begin{cases} \varphi_n, & |\int_{-N}^{s} \varphi_n(x+tk)\,dt| < K, \\ 0, & |\int_{-N}^{s} \varphi_n(x+tk)\,dt| \geqslant K \end{cases}$$

almost everywhere with respect to $\mu$.

Since $\chi_{\{x : |\int_{-N}^{s} \varphi_n(x+tk)\,dt| \geqslant K\}} \to 0$ $\mu$-a.e. and $\varphi_n \to \frac{\partial \tilde{f}}{\partial s}$ in $L^2(\mu)$, we obtain

$$\frac{\partial \psi_n^K}{\partial s} \to \frac{\partial \tilde{f}}{\partial s}$$

in $L^2(\mu)$. This yields that $\psi_n^K \to f$ in $D^1(\mathcal{E}_\mu^k)$.

It remains to approximate every $\psi_n^K$ by $\mathcal{F}C_0^\infty$-functions. One can easily verify that every function $\int_{-N}^{s} \varphi_n(x+tk)\,dt$ can be represented as

$$g_n(l_{i_n^1}(z), \ldots, l_{i_n^m}(z))$$

for some $l_{i_n^1}, \ldots, l_{i_n^m} \in E'$ and a smooth bounded function $g_n : \mathbb{R}^m \to \mathbb{R}$ such that $\mathrm{Im}(l_{i_n^1}(z), \ldots, l_{i_n^m}(z)) = \mathbb{R}^m$. Hence $\psi_n^K = g_n^K(l_{i_n^1}(z), \ldots, l_{i_n^m}(z))$, where

$$g_n^K = \Big( g_n \wedge K \Big) \vee (-K).$$

Since $\mathcal{K}$ is dense, the image of $\mu$ under the finite-dimensional mapping

$$z \to (l_{i_n^1}(z), \ldots, l_{i_n^m}(z))$$

has a density with respect to Lebesgue measure. Note that $g_n^K$ is bounded along with its first derivatives. Hence there exists a sequence of functions $\phi_n^N \in C_0^\infty(\mathbb{R}^m)$ that are uniformly bounded along with their first derivatives such that $\phi_n^N \to g_n^K$ and $\partial_i \phi_n^N \to \partial_i g_n^K$, $i \in \{1, \ldots, m\}$ a.e. with respect to Lebesgue measure if $N \to \infty$ (this sequence can be constructed by the standard technique of smoothing convolutions). Hence $\phi_n^N(l_{i_n^1}(z), \ldots, l_{i_n^m}(z)) \to g_n^K(l_{i_n^1}(z), \ldots, l_{i_n^m}(z))$ in $D^1(\mathcal{E}_\mu^k)$. $\square$

The main theorem of this paper gives a sufficient condition for the Mosco convergence of the partial and gradient Dirichlet forms. This condition can be easily verified for the concrete applications considered below.

**Theorem 3.4.** *Suppose that there exist disintegrations $\mu_n = \rho_n(x + sk_n)\, ds \cdot \nu_n(dx)$ such that*

(1) *$\mu_n \to \mu$ weakly,*
(2) *$\nu_n \to \nu$ weakly,*
(3) *there exists an increasing sequence of numbers $\{n_i\}$, $n_i \to \infty$ such that $\{\chi_{E_0^{n_i}} \tilde{\mu}_n^{k_n}\}$ is tight and, moreover,*

$$\chi_{E_0^{n_i}} \tilde{\mu}_n^{k_n} \to \chi_{E_0^{n_i}} \tilde{\mu}^k$$

*weakly for every $n_i$ as $n \to \infty$.*

*Then $\mathcal{E}_{\mu_n}^{k_n} \to \mathcal{E}_\mu^k$ Mosco.*

**Proof.** Let $\{f_n\}$, $f_n \in L^2(\mu_n)$ be a sequence of functions such that

$$c := \underline{\lim}_n \mathcal{E}_{\mu_n}(f_n) < \infty.$$

Choose a subsequence (denoted again by $\{f_n\}$) such that $c = \lim_n \mathcal{E}_{\mu_n}(f_n)$. Since the sequence of measures $\chi_{E_0^{n_i}} \tilde{\mu}_n^{k_n}$ is tight, one can find for every $\varepsilon > 0$ a compact set $K \subset E_0^{n_i}$ such that $\tilde{\mu}_n^{k_n}(E_0^{n_i} \setminus K) < \varepsilon$ for every $n$. Then by the Cauchy inequality

$$\left[\int_{E_0^{n_i} \setminus K} \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x)\right]^2 \leqslant \int_E \left(\frac{\partial f_n}{\partial k_n}\right)^2 d\mu_n \int_{E_0^{n_i} \setminus K} d\tilde{\mu}_n^{k_n} < c\varepsilon.$$

This implies that the sequence of measures $\chi_{E_0^{n_i}} \left|\frac{\partial f_n}{\partial k_n}\right| ds\, d\nu_n(x)$ is tight, hence by the Prohorov theorem and by the standard diagonal procedure one can extract a subsequence of measures (denoted again by $\chi_{E_0^{n_i}} \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x)$) such that

$$\chi_{E_0^{n_i}} \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x)$$

weakly converges to a finite measure $\chi_{E_0^{n_i}} m$ for every $n_i$.

Now let $\varphi \in \mathcal{F}^l C_0^\infty$, $\operatorname{supp}(\varphi) \subset E_0^{n_i}$. Then

$$\left[\int_{E_0^{n_i}} \varphi\, dm\right]^2 = \lim_n \left[\int_{E_0^{n_i}} \varphi \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x)\right]^2 \leqslant c \lim_n \int_{E_0^{n_i}} \varphi^2\, d\tilde{\mu}_n^{k_n}$$
$$= c \int_{E_0^{n_i}} \varphi^2\, d\tilde{\mu}^k. \tag{4}$$

This implies that $m$ is absolutely continuous with respect to $\tilde{\mu}^k$, consequently, with respect to $ds\, d\nu(x)$.

Now suppose that $\sup_n \|f_n\|^2_{L^2(\mu_n)} < \infty$. We can do the same with the sequence of measures $\{f_n\, ds\, d\nu_n(x)\}$. Finally, we obtain that there exist $ds\, d\nu$-measurable functions $f$ and $g$ on $E$ such that

$$\chi_{E_0^{n_i}} f_n\, ds\, d\nu_n(x) \to \chi_{E_0^{n_i}} f\, ds\, d\nu(x) \tag{5}$$

and

$$\chi_{E_0^{n_i}} \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x) \to \chi_{E_0^{n_i}} g\, ds\, d\nu(x)$$

weakly for every $E_0^{n_i}$. Let us take $\varphi \in \mathcal{F}^l C_0^\infty$ with support in $E_0^{n_i}$. Then

$$\int_{E_0} \int_{\mathbb{R}} \varphi g\, ds\, d\nu(x) = \lim_n \int_{E_0} \int_{\mathbb{R}} \varphi \left(\frac{\partial f_n}{\partial k_n}\right) ds\, d\nu_n(x)$$
$$= -\lim_n \int_{E_0} \int_{\mathbb{R}} \partial_s \varphi f_n\, ds\, d\nu_n(x) = -\int_{E_0} \int_{\mathbb{R}} \partial_s \varphi f\, ds\, d\nu(x).$$

By Lemma 3.2 we obtain that for $\nu$-almost all $x$ the function $s \to f(s,x) = f(sk+x)$ has an absolutely continuous version $\tilde{f}(.,x)$ such that $\frac{\partial f}{\partial k} := \partial_s \tilde{f}(s,x) = g(s,x)$. From

(4) we obtain

$$\Big[\int_E \varphi\left(\frac{\partial f}{\partial k}\right) ds\, d\nu(x)\Big]^2 \leqslant c \int_E \varphi^2\, d\tilde{\mu}^k.$$

Let $\{\varphi_n\}$ be a uniformly bounded sequence of $\mathcal{F}^l C_0^\infty$-functions such that

$$\varphi_n \to \frac{\partial f}{\partial k}(s,x)\rho(s,x)\chi_{M_N}$$

$ds\, d\nu$-almost everywhere, where

$$M_N = \left\{z : \left|\frac{\partial f(s,x)}{\partial k}\right| \rho(s,x) < N\right\}.$$

By the Lebesgue dominated convergence theorem we obtain

$$\int_{E\cap M_N} \left(\frac{\partial f}{\partial k}\right)^2 d\mu \leqslant c = \underline{\lim} \int_E \left(\frac{\partial f_n}{\partial k_n}\right)^2 d\mu_n.$$

Letting $N$ to infinity we have

$$\int_E \left(\frac{\partial f}{\partial k}\right)^2 d\mu \leqslant \underline{\lim}_n \int_E \left(\frac{\partial f_n}{\partial k_n}\right)^2 d\mu_n.$$

In the same way we show that $f \in L^2(\mu)$. Hence $f \in \mathcal{D}(\mathcal{E}^k_\mu)$ and $\mathcal{E}^k_\mu(f) \leqslant \underline{\lim}_n \mathcal{E}^{k_n}_{\mu_n}(f_n)$.

Now let us prove (M1). Suppose in addition that $f_n \to \tilde{f}$ weakly for some $\tilde{f} \in L^2(\mu)$.

We have to show that $\tilde{f} = f$ $\mu$-a.e. Indeed, set: $\psi_n = \dfrac{\varphi(x+sk_n)}{\rho_n(x+sk_n)}$, where $\varphi \in \mathcal{F}^l C_0^\infty$.

Let us show that $\psi_n \to \dfrac{\varphi(x+sk_n)}{\rho(x+sk_n)} = \psi$ strongly. Note that weak convergence of vectors $k_n \to k$ and weak convergence of measures $\mu_n \to \mu$ imply that

$$\int_E \varphi\, ds\, d\nu_n \to \int_E \varphi\, ds\, d\nu$$

for every $\varphi \in \mathcal{F}^l C_0^\infty$. Take $\tilde{\varphi} \in \mathcal{F}C_0^\infty$. Then

$$\int_E \psi_n \tilde{\varphi}\, d\mu_n = \int_E \frac{\varphi(x+sk_n)}{\rho_n(x+sk_n)} \tilde{\varphi}\, d\mu_n = \int_E \varphi\tilde{\varphi}\, ds\, d\nu_n \to \int_E \varphi\tilde{\varphi}\, ds\, d\nu$$
$$= \int_E \psi\tilde{\varphi}\, d\mu$$

and

$$\int_E \psi_n^2\, d\mu_n = \int_E \frac{\varphi^2(x+sk_n)}{\rho_n^2(x+sk_n)}\, d\mu_n = \int_E \varphi^2\, d\tilde{\mu}_n^{k_n} \to \int_E \varphi^2\, d\tilde{\mu}^k = \int_E \psi^2\, d\mu.$$

By Lemma 2.7 we have $\psi_n \to \psi$ strongly. Hence $\int_E f_n\psi_n\, d\mu_n \to \int_E \tilde{f}\psi\, d\mu$. Note that (5) implies

$$\int_E f_n\psi_n\, d\mu_n = \int_{E_0}\int_{\mathbb{R}} f_n\varphi\, ds\, d\nu_n(x) \to \int_{E_0}\int_{\mathbb{R}} f\varphi\, ds\, d\nu(x) = \int_E f\psi\, d\mu.$$

Hence $f = \tilde{f}$ $\mu$-a.e.

(M2) follows easily from the fact that $\mathcal{E}_{\mu_n}^{k_n}(\varphi) \to \mathcal{E}_\mu^k(\varphi)$ for $\varphi \in \mathcal{F}C_0^\infty$, Lemma 2.8 and Lemma 3.3. $\square$

In the following corollary we consider a sequence of forms

$$\mathcal{E}_{\mu_n} = \sum_{i=1}^\infty \mathcal{E}_{\mu_n}^{k_n^i}.$$

Recall that the domain of definition is defined by $\mathcal{D}(\mathcal{E}_{\mu_n}) = \bigcap_{i=1}^n \mathcal{D}(\mathcal{E}_{\mu_n}^{k_n^i})$.

**Corollary 3.5.** *Let $\{\mu_n\}$ and $\{k_n^i\}$, $i, n \in \mathbb{N}$ satisfy conditions* (1)–(3) *of Theorem* 3.4 *for every i. Suppose that $(\mathcal{E}_{\mu,0}, \mathcal{D}(\mathcal{E}_{\mu,0})) = (\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu))$ and*

$$\sup_n \sum_{i=1}^\infty l^2(k_n^i) < \infty \tag{6}$$

*for every $l \in E'$. Then $\mathcal{E}_{\mu_n} \to \mathcal{E}_\mu$ Mosco.*

**Proof.** Condition (M1) follows from the fact that (M1) is fulfilled for every sequence of partial forms $\{\mathcal{E}_{\mu_n}^{k_n^i}\}$. Let us verify (M2). Since $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}_\mu), (\mathcal{E}_\mu)_1^{\frac{1}{2}})$, Lemma 2.8 implies that it suffices to show that $\mathcal{E}_{\mu_n}(f) \to \mathcal{E}_{\mu}(f)$ for every $f \in \mathcal{F}C_0^{\infty}$. Take $f = \varphi(l_1(x), \ldots, l_d(x))$, $\varphi \in C_0^{\infty}(\mathbb{R}^d)$; then

$$\mathcal{E}_{\mu_n}(f) = \sum_{i=1}^{\infty} \int_E \sum_{j_1, j_2=1}^{d} \varphi_{j_1}(l_1, \ldots, l_d)\varphi_{j_2}(l_1, \ldots, l_d) l_{j_1}(k_n^i) l_{j_2}(k_n^i)\, d\mu_n.$$

The claim follows from the Cauchy inequality, weak convergence $\mu_n \to \mu$ and (6). □

In the following theorem we consider a partial case of the general situation, namely, we suppose that the measures are given by densities with respect to some fixed measure.

**Theorem 3.6.** *Let $m$ be a finite fully supported measure on a Prohorov space $E$, $k \in \mathcal{K}$, $\mathcal{K}$ is dense in $E$, $m_k := \pi_k(m)$ and $\rho_k(s,x)$ be the normalized conditional density*:

$$m(ds\, dx) = \rho_k(s,x)\, ds \cdot m_k(dx).$$

*Let $\{g_n\}$ be a sequence of probability densities such that $\{g_n\}$ is $m$-equi-integrable on every set $E_0^N$. Suppose in addition that $g_n\, dm \to g\, dm$ weakly and for $\pi_k(m)$-almost all $x \in E_0$*

$$\frac{ds}{g_n(s,x)\rho_k(s,x)} \to \frac{ds}{g(s,x)\rho_k(s,x)}$$

*vaguely in the sense of one-dimensional measures. Then $\mathcal{E}^k_{g_n\, dm} \to \mathcal{E}^k_{g\, dm}$ Mosco.*

**Proof.** Let us apply Theorem 3.4. It follows from the proof that Theorem 3.4 works under weaker assumptions. Namely, it is enough to show that *for every fixed $N$* all the measures $\{\mu_n\}$ admit a decomposition depending on $N$

$$\mu_n = \rho_{n,N}(x+sk)\, ds \cdot \nu_{n,N}(dx)$$

such that $\nu_{n,N} \to \nu_N$ weakly and $\chi_{E_0^N} \tilde{\mu}^k_{n,N} \to \chi_{E_0^N} \tilde{\mu}^k_N$ weakly, where $\tilde{\mu}^k_{n,N} = \frac{ds}{\rho_{n,N}(x+sk)} \cdot \nu_{n,N}(dx)$. Let us fix $N > 0$ and define for every $x \in E_0$

$$F_{n,N}(x) = \frac{1}{\int_{-N}^{N} \frac{ds}{g_n(s,x)\rho_k(s,x)}}.$$

Set:

$$\mu_n := g_n\, dm, \quad \nu_{n,N} = F_{n,N}(x)\, dm_k, \quad \tilde{\mu}^k_{n,N} = \frac{F^2_{n,N}(x)}{g_n(s,x)\rho_k(s,x)}\, dm_k.$$

It is enough to show that $\nu_{n,N} \to \nu_N$ weakly and $\int_X \varphi\, d\tilde{\mu}^k_{n,N} \to \int_X \varphi\, d\tilde{\mu}^k_N$ for every $\varphi \in C$ with $\mathrm{supp}(\varphi) \Subset E_0^N$. Let us show to this end that $F_{n,N} \to F_N$ in $L^1(m_k)$. Indeed, by the hypothesis of the theorem $F_{n,N} \to F_N$ $m_k$-a.e. (this follows from vague convergence and the fact that the limit measure has no atoms). It is enough to show that $\{F_{n,N}\}$ is an $m_k$-equi-integrable sequence. Note that for every $B \in E_0$ by the Cauchy inequality

$$4N^2 = \left(\int_{-N}^{N} dx\right)^2 \leqslant \int_{-N}^{N} \frac{ds}{g_n \rho_k} \int_{-N}^{N} g_n \rho_k\, ds,$$

hence

$$\int_B F_{n,N}\, dm_k \leqslant \frac{1}{4N^2} \int_B \int_{-N}^{N} g_n(s,x)\rho_k(s,x)\, ds\, dm_k(x)$$
$$= \frac{1}{4N^2} \int_{B\times[-N,N]} g_n(s,x)\, dm.$$

Hence the equi-integrability of $\{F_{n,N}\}$ follows from the equi-integrability of $\{g_n\}$. Now let us fix $\varphi \in C$, $\mathrm{supp}(\varphi) \Subset E_0^N$. Set

$$\psi_n(s,x) = F_{n,N}(x) \int_{-N}^{N} \frac{\varphi(s,x)}{g_n(s,x)\rho_k(s,x)}\, ds.$$

By the definition of $F_{n,N}$ the sequence $\{\psi_n(s,x)\}$ is uniformly bounded by $\sup(\varphi)$. Moreover, it converges to $F_N(x) \int_{-N}^{N} \frac{\varphi(s,x)}{g(s,x)\rho_k(s,x)}\, ds$ $m_k$-a.e. Hence it follows from the $L_1(m_k)$-convergence of $\{F_{n,N}\}$ that

$$\int_E \varphi\, d\tilde{\mu}_{k,n} = \int_{E_0} \psi_n(x) F_{n,N}(x)\, dm_k(x) \to \int_{E_0} \psi(x) F_N(x)\, dm_k(x) = \int_E \varphi\, d\tilde{\mu}_k.$$

The proof is complete. □

The following corollary can be proved exactly in the same way as Corollary 3.5.

**Corollary 3.7.** *Suppose that* $\{\mu_n\} = \{g_n\, dm\}$ *and* $\{k^i\}$, $i \in \mathbb{N}$, *satisfy the conditions of Theorem* 3.6 *for every* $i$. *Suppose that* $(\mathcal{E}_{\mu,0}, \mathcal{D}(\mathcal{E}_{\mu,0})) = (\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu))$ *and*

$$\sum_{i=1}^{\infty} l^2(k^i) < \infty \tag{7}$$

*for every* $l \in E'$. *Then* $\mathcal{E}_{\mu_n} \to \mathcal{E}_\mu$ *Mosco.*

As a direct consequence of Theorem 3.4 we give the following simple example for product measures. See Sections 5–6 for some examples of non-product cases.

**Example 3.8.** Let $E = \mathbb{R}^\infty$ and $\mu_n = \prod_{k=1}^{\infty} \mu_n^k$. We suppose that every $\mu_n^k$ is a probability measure on $\mathbb{R}^1$ with a density $\rho_n^k$ such that $\rho_n^k > 0$ almost everywhere, $\frac{1}{\rho_n^k} \in L^1_{\mathrm{loc}}(ds)$, $\rho_n^k(s)\,ds \to \rho^k(s)\,ds$ weakly and $\frac{ds}{\rho_n^k(s)} \to \frac{ds}{\rho^k(s)}$ vaguely. For every $n$ consider the maximal extension $\mathcal{E}_n$ of the form $(\mathcal{E}_n, \mathcal{F}C_0^\infty)$, where

$$\mathcal{E}_n(f) = \sum_{i=1}^{\infty} \int_E \left(\frac{\partial f}{\partial x_i}\right)^2 d\mu_n.$$

Suppose that $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}), (\mathcal{E})_1^{1/2})$, where

$$\mathcal{E}(f) = \sum_{i=1}^{\infty} \int_E \left(\frac{\partial f}{\partial x_i}\right)^2 d\mu, \quad \mu = \prod_{k=1}^{\infty} \rho^k(x_k)\,dx_k.$$

Then $\mathcal{E}_n \to \mathcal{E}$ Mosco.

**Remark 3.9.** One can ask what happens if $\tilde{\mu}_n^{k_n}$ has a limit which does not coincide with $\tilde{\mu}^k$. In this case the Mosco limit always differs from $\mathcal{E}_\mu^k = \mathcal{E}_\mu$ if $d = 1$ (see [19]). Following the proof of [19], the reader can easily verify that the same holds for the partial forms in the multidimensional and infinite-dimensional case. The situation with the gradient forms is not so obvious, since the gradient forms may converge even when the partial forms do not converge (see [31,19]).

## 4. Approach via logarithmic derivatives

In this section we discuss some sufficient conditions for the Mosco convergence in the case of measures with logarithmic derivatives. Assume that a sequence of probability measures $\{\mu_n\}$ on $E$ is tight and converges weakly to a fully supported probability measure $\mu$.

We recall that a measure $\mu$ admits a logarithmic derivative along $h$ if there exists a measurable function $\beta_h^\mu \in L^1(\mu)$ such that

$$\int_E \partial_h \varphi\, d\mu = -\int_E \varphi \beta_h^\mu\, d\mu$$

for every $\varphi \in \mathcal{F}C_0^\infty$.

The techniques of Mosco convergence provides a simple proof (given in the proposition below) of the well-known fact that $L^2$-convergence of the logarithmic derivatives of measures implies strong convergence of the corresponding semigroups. We fix some $h \in H$ such that every $\mu_n$ has the logarithmic derivative $\beta_h^{\mu_n} \in L^2(\mu_n)$ along $h$ and consider the sequence of partial forms $\{\mathcal{E}_{\mu_n}^h\}$ defined by

$$\mathcal{E}_{\mu_n}^h(f, g) = \int_E \frac{\partial f}{\partial h} \frac{\partial g}{\partial h}\, d\mu_n$$

for $f, g \in \mathcal{F}C_0^\infty$. The assumption $\beta_h^{\mu_n} \in L^2(\mu_n)$ implies the closability of these forms (see [25]). As usual, the maximal closure of $\{\mathcal{E}_{\mu_n}^h, \mathcal{F}C_0^\infty\}$ is considered. It was proved in [35] that $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}_\mu^h), (\mathcal{E}_\mu^h)_1^{1/2})$ for every partial form $\mathcal{E}_\mu^h$ if $\mu$ admits a logarithmic derivative along $h$.

**Proposition 4.1.** *Let* $\sup_n \|\beta_h^{\mu_n}\|_{L^2(\mu_n)} < \infty$. *Then* $\mu$ *possesses a logarithmic derivative and* $\{\mathcal{E}_{\mu_n}^h\}$ $\Gamma$*-converges to* $\mathcal{E}_\mu^h$. *If, in addition,* $\|\beta_h^{\mu_n}\|_{L^2(\mu_n)} \to \|\beta_h^\mu\|_{L^2(\mu)}$, *then* $\mathcal{E}_{\mu_n}^h \to \mathcal{E}_\mu^h$ *Mosco.*

**Proof.** Condition (2) of the Mosco convergence can be verified as in Lemma 2.8. Let us verify condition (1). Extract from $\{\beta_h^{\mu_n}\}$ an $\mathcal{H}$-weakly convergent subsequence (in the sense of convergent Hilbert spaces), denoted in the following again by $\{\beta_h^{\mu_n}\}$, such that $\beta_h^{\mu_n} \to \beta \in L^2(\mu)$. Then by the properties of weak convergence in $\mathcal{H}$

$$\int_E \varphi\beta\, d\mu = \lim_n \int_E \varphi\beta_h^{\mu_n}\, d\mu_n = -\lim_n \int_E \varphi_h\, d\mu_n = -\int_E \varphi_h\, d\mu$$

for every smooth $\varphi$. Hence $\mu$ has the logarithmic derivative $\beta_h^\mu := \beta \in L^2(\mu)$ and, moreover, $\beta_h^{\mu_n} \to \beta_h^\mu$ $\mathcal{H}$-weakly. Now let $f_n \to f$ strongly in $\mathcal{H}$. The tightness of measures $\{\mu_n\}$ and the Cauchy inequality

$$\left(\int_{E\setminus K} \left|\frac{\partial f_n}{\partial h}\right| d\mu_n\right)^2 \leqslant \mu_n(E \setminus K) \int_E \left(\frac{\partial f_n}{\partial h}\right)^2 d\mu_n$$

imply that the sequence of measures $\{\nu_n\} = \left\{\frac{\partial f_n}{\partial h}\mu_n\right\}$ is tight. Extract a weakly convergent sequence (denoted in the following again by $\{\nu_n\}$) $\nu_n \to \nu$. In the same way as in Theorem 3.4 one can show that $\nu$ is absolutely continuous with respect to $\mu$. Since $f_n \to f$ strongly and $\beta_h^{\mu_n} \to \beta_h^\mu$ weakly, we get

$$\int_E \varphi\, d\nu = \lim_n \int_E \varphi \frac{\partial f_n}{\partial h}\, d\mu_n = -\int_E \varphi_h f_n\, d\mu_n - \int_E \varphi f_n \beta_h^{\mu_n}\, d\mu_n \to$$
$$-\int_E \varphi_h f\, d\mu - \int_E \varphi f \beta_h^\mu\, d\mu.$$

This yields that $f$ admits a weak derivative along $h$ and, moreover, $\frac{d\nu}{d\mu} = \frac{\partial f}{\partial h}$. Hence

$$\int_E \varphi \frac{\partial f}{\partial h}\, d\mu = \lim_n \int_E \varphi \frac{\partial f_n}{\partial h}\, d\mu_n \leqslant \Bigl(\underline{\lim}_n \int_E \Bigl(\frac{\partial f_n}{\partial h}\Bigr)^2 d\mu_n\Bigr)^{1/2} \Bigl(\int_E \varphi^2\, d\mu\Bigr)^{1/2}.$$

Choosing a sequence $\varphi_n \to \frac{\partial f}{\partial h}$ in $L^2(\mu)$ one can easily complete the proof.

It can be easily seen from the proof that the stronger assumption $\|\beta_h^{\mu_n}\|_{L^2(\mu_n)} \to \|\beta_h^{\mu}\|_{L^2(\mu)}$ implies that $\beta_h^{\mu_n} \to \beta_h^{\mu}$ $\mathcal{H}$-strongly and $\mathcal{E}_h^{\mu_n} \to \mathcal{E}_h^{\mu}$ in the Mosco sense. $\square$

In the following result we obtain simple sufficient conditions for Mosco convergence in the finite-dimensional case. Note that unlike Proposition 4.1 we don't assume the strong $L^2$-convergence of the logarithmic gradients. We recall that the Sobolev space $W^{p,1}(\mathbb{R}^d)$ consists on functions possessing partial Sobolev derivatives and, in addition, $f \in L^p(\mathbb{R}^d)$ and $|\nabla f| \in L^p(\mathbb{R}^d)$. We say that $f \in W^{p,1}_{\mathrm{loc}}$ if $\varphi f \in W^{p,1}(\mathbb{R}^d)$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$.

**Theorem 4.2.** *Let $\{\mu_n\} = \{\rho_n\, dx\}$ be a sequence of probability measures on $\mathbb{R}^d$ such that $\rho_n\, dx \to \rho\, dx$ weakly, $\rho_0 := \rho$ and $\rho_n > 0$-a.e. for $n \geqslant 0$. Suppose that*

$$\|\sqrt{\rho_n}\|_{W^{2,1}(\mathbb{R}^d)} = \Bigl(\int_{\mathbb{R}^d} \rho_n\, dx\Bigr)^{\frac{1}{2}} + \Bigl(\int_{\mathbb{R}^d} \frac{(\nabla \rho_n)^2}{\rho_n}\, dx\Bigr)^{\frac{1}{2}} < C.$$

*Then $\mathcal{E}_n \to \mathcal{E} := \mathcal{E}_0$ Mosco, where $\mathcal{E}_n(f) = \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_n$.*

**Proof.** Let us fix a ball $B \subset \mathbb{R}^d$. Since $\sqrt{\rho_n}$ is bounded in $W^{2,1}(\mathbb{R}^d)$, by the compactness embedding theorem $\sqrt{\rho_n}$ has a subsequence $\sqrt{\rho_{n_m}}$ which converges in $L^2(B)$. Since this can be done for every $B$, by the standard diagonal procedure one can choose $\{\sqrt{\rho_{n_m}}\}$ in such a way that $\{\sqrt{\rho_{n_m}}\}$ converges in $L^2_{\mathrm{loc}}$ and almost everywhere. Hence $\{\rho_{n_m}\}$ converges in $L^1_{\mathrm{loc}}$ and the limit coincides with $\rho$. This implies that the initial sequence $\{\rho_n\}$ converges to $\rho$ in $L^1_{\mathrm{loc}}$.

By the definition of the logarithmic derivative

$$\beta_h^{\mu_n} = \frac{\partial_h \rho_n}{\rho_n}, \qquad \|\beta_h^{\mu_n}\|^2_{\mu_n} = \int_{\mathbb{R}^d} \frac{(\partial_h \rho_n)^2}{\rho_n}\, dx.$$

Note that all the forms are closed and according to [34] (see also [10]) the condition $\int_{\mathbb{R}^d} \frac{(\nabla \rho_n)^2}{\rho_n}\, dx < \infty$ implies that

$$(\mathcal{E}_n, \mathcal{D}(\mathcal{E}_n)) = \bigl((\mathcal{E}_n)_0, \mathcal{D}((\mathcal{E}_n)_0)\bigr).$$

By Proposition 4.1 the partial forms $\mathcal{E}_n^h$ $\Gamma$-converges to $\mathcal{E}^h$. Hence, condition (G1) of $\Gamma$-convergence holds also for sums of partial forms, i.e. for $\{\mathcal{E}_n\}$. Then following the arguments from Lemma 2.8 one can easily prove that $\{\mathcal{E}_n\}$ $\Gamma$-converges to $\mathcal{E}$. Now let us show that in fact $\mathcal{E}_n \to \mathcal{E}$ Mosco. It remains to prove (M1). To this end we fix a $\mathcal{H}$-weakly convergent sequence $f_n \to f$. We note that $f_n\rho_n$ is bounded in $W^{1,1}(\mathbb{R}^d)$. Indeed,

$$\sup_n\Big(\int_{\mathbb{R}^d} |f_n\rho_n|\,dx\Big)^2 \leqslant \sup_n\Big(\int_{\mathbb{R}^d} f_n^2\rho_n\,dx\Big) < \infty$$

and

$$\sup_n\Big(\int_{\mathbb{R}^d} |\nabla\big[f_n\rho_n\big]|\,dx\Big)^2 \leqslant 2\sup_n\Big(\int_{\mathbb{R}^d} |\nabla f_n|^2\rho_n\,dx + \int_{\mathbb{R}^d} f_n^2\rho_n\,dx \int_{\mathbb{R}^d} \frac{(\nabla\rho_n)^2}{\rho_n}\,dx\Big) < \infty.$$

By the compactness embedding of $W^{1,1}(B) \to L^1(B)$ and the same arguments as above we can assume that some subsequence $f_{n_m}\rho_{n_m}$ converges in $L^1_{\mathrm{loc}}$. In addition, by the weak $\mathcal{H}$-convergence $\int_{\mathbb{R}^d} \varphi f_{n_m}\rho_{n_m}\,dx \to \int_{\mathbb{R}^d} \varphi f\rho\,dx$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$. Hence $f_n\rho_n \to f\rho$ in $L^1_{\mathrm{loc}}$. Since $\rho_n > 0$ and $\rho > 0$ almost everywhere, one can extract a subsequence (denoted again by $\{f_n\}$) such that $f_n \to f$ almost everywhere. Fix some $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi \leqslant 1$ and $\psi = 1$ on $B_1$. Denote $\psi_K(x) := \psi(x/K)$. Now set: $f_n^{N,K} = \big(f_n\psi_K \wedge N\big) \vee -N$. Obviously, $f_n^{N,K} \to f^{N,K}$ $\mathcal{H}$-strongly. Hence by the $\Gamma$-convergence

$$\mathcal{E}(f^{N,K}) \leqslant \underline{\lim}_n \mathcal{E}_n(f_n^{N,K}).$$

In addition, by the contraction properties (see [25]) $\mathcal{E}_n(f_n^{N,K}) \leqslant \mathcal{E}_n(f_n\psi_K)$. Obviously, $\mathcal{E}(f^{N,K}) \to \mathcal{E}(f\psi_K)$ as $N \to \infty$. Hence $\mathcal{E}(f\psi_K) \leqslant \underline{\lim}_n \mathcal{E}_n(f_n\psi_K)$. Since $\mathcal{E}(f\psi_K) \to \mathcal{E}(\psi)$ as $K \to \infty$ and $|\mathcal{E}_n(f_n\psi_K) - \mathcal{E}_n(f_n)| \leqslant A/K$, where constant $A$ depends only on $\sup_n \|f_n\|_{L^2(\mu_n)}$, $\sup_n \mathcal{E}_n(f_n)$ and on the uniform bounds of $\psi, \nabla\psi$, one can easily obtain that

$$\mathcal{E}(f) \leqslant \underline{\lim}_n \mathcal{E}_n(f_n)$$

The proof is complete. $\square$

## 5. Gaussian case. Applications to measures absolutely continuous with respect to a Gaussian measure

Now we give applications of Theorems 3.4, 3.6 to the case of weakly convergent measures which are Gaussian or absolutely continuous with respect to a Gaussian measure. First, we recall some facts about Radon Gaussian measures (see [7] for details). Let $\gamma$ be a centered Radon Gaussian measure on $E$ with a covariance operator $Q : E^{'} \to E$. The space

$$H = H(\gamma) = \{h : \gamma(\cdot + h) \quad \text{is absolutely continuous with respect to } \gamma\}$$

of all vectors of quasi-invariance of $\gamma$ is called the Cameron–Martin space. One can associate with every $h \in H$ a function $\hat{h}$ that belongs to the closure of $E'$ in $L^2(\gamma)$ such that $h = Q\hat{h}$ and $l(h) = \int_E l(x)\hat{h}(x)\,d\gamma(x)$ for every $l \in E'$. The natural Hilbert inner product on $H$ is introduced by $(h_1, h_2)_H = \int_E \hat{h}_1(x)\hat{h}_2(x)\,d\gamma(x)$.

**Theorem 5.1.** *Let $\{\gamma_n\}$ be a tight sequence of centered Gaussian measures with covariance operators $Q_n$ weakly converging to a fully supported Gaussian measure $\gamma$. Suppose that $h_n \in H_n$ for every $n$ and one of the following conditions holds*:

(1) $\|h_n\|_{H_n} \to \|h\|_H$ *and for every* $l \in E'$ $\quad l(h_n) \to l(h), \quad l(h) \neq 0$;
(2) *for every* $l \in E'$ $l(h_n) \to 0$.

*Then* $\mathcal{E}_{\gamma_n}^{h_n} \to \mathcal{E}_{\gamma}^{h}$ *Mosco. We have* $\mathcal{E}_{\gamma}^{h} = 0$ *if condition* (2) *holds.*

**Proof.** Suppose that $\|h_n\|_{H_n} \to \|h\|_H$ and $\|h\|_H \neq 0$. Since $\mathcal{E}_{\gamma_n}^{h_n} = \frac{1}{c^2}\,\mathcal{E}_{\gamma_n}^{ch_n}$ for $c \neq 0$, we can assume without loss of generality that $\|h_n\|_{H_n} = 1$, hence $\hat{h}_n(h_n) = 1$. Fix some $l \in E'$ such that $l(h) \neq 0$. Denote by $\gamma_n^0$ the projection of $\gamma_n$ on $E_0 = \{x : l(x) = 0\}$. Apply the following disintegration formula from [7]:

$$\int_E u(z)\gamma_n(dz) = \frac{1}{\sqrt{2\pi}} \int_{E_0} \int_{\mathbb{R}} u(x + sh_n) e^{-\frac{(s-\hat{h}_n(x))^2}{2}}\, ds\gamma_n^0(dx),$$

where $z = x + sh_n$ and $s = \frac{l(z)}{l(h_n)}$, $x \in E_0$.

Let us apply Theorem 3.4. Take

$$\nu_n = e^{-\frac{(\hat{h}_n(x))^2}{2}}\gamma_n^0(dx), \quad \rho_n(x + sh_n) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{s^2 - 2s\hat{h}_n(x)}{2}}.$$

Then

$$\frac{1}{\sqrt{2\pi}}\tilde{\gamma}_n = \exp\Big(\frac{s^2 - 2s\hat{h}_n(x) - (\hat{h}_n(x))^2}{2}\Big)\, ds\gamma_n^0(dx) = e^{s^2} e^{-\frac{(\hat{h}_n(x)+s)^2}{2}}\, ds\gamma_n^0(dx).$$

Changing the variables $s \to -s$ we obtain by the disintegration formula the following relation for every measurable set $A \subset E$ and every function $\varphi \in \mathcal{F}C_0^\infty$:

$$\int_A \varphi\, d\tilde{\gamma}_n = \sqrt{2\pi}\int_A \varphi e^{s^2} e^{-\frac{(\hat{h}_n(x)+s)^2}{2}}\, ds\gamma_n^0(dx) = \sqrt{2\pi}\int_{A(\cdot-)} \varphi(z - 2sh_n)e^{s^2}\gamma_n(dz).$$

Here $A(\cdot-) := \{z : z - 2sh_n \in A\}$. Obviously, $\{\chi_{\{s:|s|\leqslant k\}}\tilde{\gamma}_n\}$ is tight for every $k > 0$. Taking $\varphi \in \mathcal{F}^l C_0^\infty$ we obtain

$$\begin{aligned}\int_E \varphi\, d\tilde{\gamma}_n &= \sqrt{2\pi}\int_E \varphi(z - 2sh_n)e^{s^2}\gamma_n(dz) \to \sqrt{2\pi}\int_E \varphi(z - 2sh)e^{s^2}\gamma(dz)\\ &= \int_E \varphi\, d\tilde{\gamma}.\end{aligned} \tag{8}$$

Indeed, let

$$\varphi = \psi\Big(l(z), l_1(z), \ldots, l_k(z)\Big).$$

Convergence in (8) follows from the fact that weak convergence is preserved by continuous mapping $z \to (l(z), l_1(z), \ldots, l_k(z)) := (t_0, t_1, \ldots, t_k)$, and uniform convergence of

$$(t_0, \ldots, t_n) \to \psi\Big(-t_0, t_1 - 2l_1(h_n)\frac{t_0}{l(h_n)}, \ldots, t_k - 2l_k(h_n)\frac{t_0}{l(h_n)}\Big)e^{\left(\frac{t_0}{l(h_n)}\right)^2}$$

to

$$(t_0, \ldots, t_n) \to \psi\Big(-t_0, t_1 - 2l_1(h)\frac{t_0}{l(h)}, \ldots, t_k - 2l_k(h)\frac{t_0}{l(h)}\Big)e^{\left(\frac{t_0}{l(h)}\right)^2}$$

on $\mathbb{R}^k$. Hence conditions (1) and (3) of Theorem 3.4 are fulfilled.

It remains to show that $\nu_n \to e^{-\frac{(\hat{h}(x))^2}{2}}\gamma^0(dx) = \nu$ weakly. Let us compute $\hat{\nu}_n$. Take $a \in E'$ and set $\tilde{l}_n = \frac{l}{l(h_n)}$. One can easily show (for instance, by choosing an appropriate basis in the Cameron–Martin space) that the measure $\sqrt{1 + \|h\|_H^2} e^{-\frac{(\hat{h}(z))^2}{2}}\, d\gamma$

is Gaussian with the covariance operator $l \mapsto Ql - \frac{1}{1+\|h\|_H^2} l(h)h$. Hence

$$\begin{aligned}\int_{E_0} e^{i\langle a,x\rangle} \nu_n(dx) &= \int_{E_0} e^{i\langle a,x\rangle} e^{-\frac{(\hat{h}_n(x))^2}{2}} \gamma_n^0(dx)\\ &= \int_E e^{i\langle a,z-\tilde{l}_n(z)h\rangle} e^{-\frac{(\hat{h}_n(z)-\tilde{l}_n(z)h))^2}{2}} \gamma_n(dz)\\ &= \int_E e^{i\langle a,z-\tilde{l}_n(z)h\rangle} e^{-\frac{(\hat{h}_n(z)-\widehat{Q_n\tilde{l}_n(z)})^2}{2}} \gamma_n(dz)\\ &= \frac{1}{\sqrt{1+\|h_n - Q_n\tilde{l}_n\|_{H_n}^2}} e^{-\frac{1}{2}\langle \tilde{Q}_n a, a\rangle},\end{aligned}$$

where $\tilde{Q}_n a = Q_n a - \frac{1}{1+\|h_n - Q_n\tilde{l}_n\|_{H_n}^2}\left\langle h_n - Q_n\tilde{l}_n, a\right\rangle (h_n - Q_n\tilde{l}_n)$.

Let $\alpha_n = 1 + \|h_n - Q_n\tilde{l}_n\|_{H_n}^2$. Let us show that $\alpha_n \to \alpha$. Indeed,

$$\begin{aligned}\alpha_n &= 1 + \|h_n - Q_n\tilde{l}_n\|_{H_n}\\ &= 1 + \|h_n\|_{H_n} - 2\left\langle h_n, Q_n\tilde{l}_n\right\rangle_{H_n} + \left\langle Q_n\tilde{l}_n, Q_n\tilde{l}_n\right\rangle_{H_n}\\ &= 2 - 2\tilde{l}_n(h_n) + Q_n(\tilde{l}_n)\tilde{l}_n = Q_n(\tilde{l}_n)\tilde{l}_n \to Q\tilde{l}(\tilde{l}) = \alpha \neq 0.\end{aligned}$$

Hence $\nu_n(E) \to \nu(E)$. The tightness of $\{\nu_n\}$ follows from the tightness of $\{\gamma_n^0\}$. Note that every weak limiting point of $\nu_n$ coincides with $\nu$, since $\left\langle \tilde{Q}_n a, a\right\rangle \to \left\langle \tilde{Q}a, a\right\rangle$ for every $a$. Hence, $\nu_n \to \nu$ weakly.

Suppose that $\left\langle l, h_n\right\rangle \to 0$ for every $l \in E'$. Then condition (M1) is obviously fulfilled. Clearly, $\mathcal{E}_{\gamma_n}^{h_n}(\varphi) \to 0$ for every $\varphi \in \mathcal{F}^l C_0^\infty$. Hence Lemma 2.8 implies (M2). The proof is complete. □

**Remark 5.2.** It is possible to apply the proof of Theorem 5.1 to the case when the reference measures have the form $\mu_n = g_n\, d\gamma_n$. For example, the reader can easily verify that the Mosco convergence holds if every $g_n$ is continuous, $g_n \to g$ uniformly on $X$ and $g_n > c > 0$ for every $n$. However, we do not formulate more general results, since the optimal conditions on $\{g_n\}$ for the Mosco convergence to hold are not clear. We just emphasize that the case of measures without logarithmic derivatives can be investigated using this technique. We remind that the case of $\mu_n = g_n\, d\gamma$ is considered in Theorem 1.1.

**Corollary 5.3.** *Let $\{\gamma_n\}$ satisfy the assumptions of Theorem* 5.1 *and*

$$\mathcal{E}_{\gamma_n} = \sum_{i=1}^{\infty} \mathcal{E}_{\gamma_n}^{h_n^i}$$

*be a sequence of Dirichlet forms such that every sequence $h_n^i$ of vectors satisfies condition* (1) *or* (2) *of Theorem* 5.1. *Suppose in addition that $(\mathcal{E}_{\gamma,0}, \mathcal{D}(\mathcal{E}_{\gamma,0}),) = (\mathcal{E}_{\gamma}, \mathcal{D}(\mathcal{E}_{\gamma}))$ and*

$$\sup_n \sum_{i=1}^{\infty} l^2(h_n^i) < \infty \tag{9}$$

*for every $l \in E'$. Then $\mathcal{E}_{\gamma_n} \to \mathcal{E}_{\gamma}$ Mosco.*

**Proof.** The claim follows from Theorem 5.1 and Corollary 3.5. □

**Corollary 5.4.** *Let $\{\gamma_n\}$ be a tight weakly convergent sequence of centered Gaussian measures such that the limit measure $\gamma$ has full support and let $\nabla_{H_n}$ be the Malliavin gradient for $\gamma_n$. Then $\mathcal{E}_n \to \mathcal{E}$ Mosco, where $\mathcal{E}_n(f) = \int_E |\nabla_{H_n} f|_{H_n}^2 \, d\gamma_n$.*

**Proof.** Let us choose an orthogonal basis $\{h_i\}$ in $L^2(\gamma)$, consisting on functions from $\mathcal{F}C_0^{\infty}$. Then we construct an orthogonal basis $\{h_i^n\}$ in every $L^2(\gamma_n)$ in the following way. Let $N(n)$ be the biggest number such that the vectors $\{h_1, \cdots h_{N(n)}\}$ are linearly independent in $L^2(\gamma_m)$ for every $m \geqslant n$ ($N$ can be equal to $\infty$).

We apply to $\{h_1, \ldots, h_{N(n)}\}$ the standard orthogonalization procedure in $L^2(\gamma_n)$ and obtain

$$\tilde{l}_1^n = h_1, \quad \tilde{l}_2^n = h_2 - \tilde{l}_1^n \frac{(\tilde{l}_1^n, h_2)_{L^2(\gamma_n)}}{(\tilde{l}_1^n, \tilde{l}_1^n)_{L^2(\gamma_n)}},$$

$$\tilde{l}_3^n = h_3 - \tilde{l}_1^n \frac{(\tilde{l}_1^n, h_3)_{L^2(\gamma_n)}}{(\tilde{l}_1^n, \tilde{l}_1^n)_{L^2(\gamma_n)}} - \tilde{l}_2^n \frac{(\tilde{l}_2^n, h_3)_{L^2(\gamma_n)}}{(\tilde{l}_2^n, \tilde{l}_2^n)_{L^2(\gamma_n)}}, \cdots$$

and take $h_i^n = \dfrac{\tilde{l}_i^n}{\|\tilde{l}_i^n\|_{L^2(\gamma_n)}}$. Then we fix some orthogonal basis in $L^2(\gamma_n)$ such that the first $N(n)$ vectors coincide with $\{h_i^n\}$, $i \in \{1, \ldots, N(n)\}$ (we denote in the sequel this complete system again by $\{h_i^n\}$).

Weak convergence $\gamma_n \to \gamma$ implies that $h_n^i$ satisfy condition 1) of Theorem 5.1. The coincidence $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0)) = ((\mathcal{E}, \mathcal{D}(\mathcal{E}))$ of weak and strong Sobolev spaces is well known (see [7,12]). Condition (9) is satisfied, since $\sum_{i=1}^{\infty} l^2(Q_n h_n^i) = \|l\|_{L^2(\gamma_n)}$. The proof is complete. □

Finally, we turn to the case when the reference measures are absolutely continuous with respect to a fixed Gaussian measure. We prove Theorem 1.1 from the introduction.

**Proof of Theorem 1.1.** The Gaussian case follows directly from Corollary 3.7 and properties of the Gaussian measures. Indeed, the vague convergence $\frac{ds}{g_n(x+se_i)} \to \frac{ds}{g(x+se_i)}$ implies the vague convergence $\frac{ds}{g_n(x+se_i)\rho(x+se_i)} \to \frac{ds}{g(x+se_i)\rho(x+se_i)}$, where $\rho(s,x) := \rho(x+se_i)$ is the corresponding conditional density. It follows from the fact that the conditional densities for Gaussian measures are smooth and locally bounded away from zero. Corollary 3.7 is not directly applicable to the finite-dimensional case, since Lebesgue measure is not finite, but the analysis of the proof of Theorem 3.6 shows that the same arguments work also in this case if we take $(d-1)$-dimensional Lebesgue measure for $m_k$ and set: $\rho_k(s,x) = 1$. $\square$

## 6. Applications to Gibbs states on a lattice

In this section we apply the results from Section 2 to the model described in [3].

Let $\mathbb{Z}^d$, $d \in \mathbb{N}$ be the integer lattice with the Euclidean distance $|k-j|$, $k, j \in \mathbb{Z}^d \subset \mathbb{R}^d$. $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ be the configuration space equipped with the product topology.

Define the scale of Hilbert spaces

$$S_p := S_p(Z_d) := \left\{ x \in \Omega \middle| |x|_p := \left[ \sum_{k \in \mathbb{Z}^d} (1+|k|)^{2p} x_k^2 \right]^{\frac{1}{2}} < \infty \right\}, \quad p \in \mathbb{Z}^1,$$

and the mutually dual nuclear spaces

$$S := S(\mathbb{Z}^d) = \bigcap_{p=1} S_p(\mathbb{Z}^d), \quad S' := S'(\mathbb{Z}^d) = \bigcup_{p=1} S_{-p}(\mathbb{Z}^d)$$

with the tangent space

$$H := \Omega_0 = \left\{ x \in \Omega \middle| |x|_0 := \left[ \sum_{k \in \mathbb{Z}^d} x_k^2 \right]^{\frac{1}{2}} < \infty \right\}$$

and the orthonormal basis in $H$

$$e_k = \{\delta_{k,j}\}_{j \in \mathbb{Z}^d} \in \Omega_0.$$

The duality between $S$ and $S'$ can be expressed in the following way:

$$(\varphi, x) = (x, \varphi) := \sum_{k \in \mathbb{Z}^d} \varphi_k x_k, \quad \varphi \in S, \quad x \in S'.$$

We consider a sequence of energy functionals

$$E^n(x) = \sum_{\{k,j \in \mathbb{Z}^d\}} W^n_{k,j}(x_k, x_j) + \sum_{\{k \in \mathbb{Z}^d\}} V^n_k(x_k),$$
$$n \in \mathbb{N} \cup \{0\}, \quad W_{k,j} := W^0_{k,j}, V_k := V^0_k \tag{10}$$

and the associated Gibbs states (see [2,3] for details). Similarly to [3] we impose the following assumptions:

(A1) The two particle-interactions $W^n_{k,j}$ are continuously differentiable, symmetric and satisfy the polynomial growth condition, i.e.,

$$W^n_{k,j} = W^n_{j,k},$$

$$|W^n_{k,j}|(s_1, s_2) \leqslant J_{k,j}(1 + |s_1| + |s_2|)^N,$$

$$|\partial_{s_1} W^n_{k,j}(s_1, s_2)| \leqslant J_{k,j}(1 + |s_1| + |s_2|)^{N-1},$$

where $k \in \mathbb{Z}^d$, $N \geqslant 2$ and $J = \{J_{k,j}\}_{k,j \in \mathbb{Z}^d}$, $J_{k,j} = J_{j,k} \geqslant 0$.

(A2) For any $p \in \mathbb{N}$

$$\|J\|_p := \sup_{k \in \mathbb{Z}^d} \|\{J_{k,k+j}\}\|_p \leqslant \infty.$$

(A3) The self-interaction are continuously differentiable, satisfy the polynomial growth condition

$$|V^n_k(s)| \leqslant C(1 + |s|)^L, \quad \left| \frac{d}{ds} V^n_k(s) \right| \leqslant C(1 + |s|)^{L-1}, \quad s \in \mathbb{R}$$

and the coercitivity estimate

$$\frac{d}{ds} V^n_k(s) s \geqslant A|s|^{N+\sigma} - B$$

with some constants $A, B, C, \sigma > 0$, $L \geqslant 1$ uniformly for all $k \in \mathbb{Z}^d$, $n \in \mathbb{N}$ and $x \in \Omega$.

There exist different approaches to Gibbs states. The Dobrushin–Lanford–Ruelle (DLR) formalism gives a description through the so-called local specifications $\mu_\Lambda$, $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$; these are stochastic kernels, defined by the energy functional. The Gibbs states can be defined as the measures satisfying the DRL equilibrium equation (see [17,30] for details). Another approach describes Gibbs states via Radon–Nikodym derivatives with respect to local shifts of the configuration space $\Omega$ and the corresponding integration by parts formulas. The equivalence of these approaches was well known for many concrete models and was shown in the general setting by Albeverio, Kondratiev and Röckner in [2]. In this paper we consider the model from [3]. We call a Borel probability measure $\mu_n$ a Gibbs state for the energy functional $E^n$ (see (10)) if the following conditions are fulfilled:

(1) "Temperedness condition"

$$\mu_n(S_{-p}(\mathbb{Z}^d)) = 1, \quad \text{for some} \quad p > \frac{d}{2},$$

(2) $\mu_n$ is quasi-invariant with respect to all shifts $x \to x + te_k$, $t \in \mathbb{R}$, $k \in \mathbb{Z}^d$, with the Radon–Nikodym derivatives

$$\frac{d\mu_n(x + te_k)}{d\mu_n(x)} = \alpha^n_{te_k} := \exp\Big\{ - \sum_{j \in \mathbb{Z}^d} \big[ W^n_{k,j}(x_k + t, x_j) - W^n_{k,j}(x_k, x_j) \big] - \big[ V^n_k(x_k + t) - V^n_k(x_k) \big] \Big\}, \quad x \in S'. \tag{11}$$

The set of Gibbs states with energy $E^n$ will be denoted by $\mathcal{M}^{\alpha^n}_t$.

The logarithmic derivative of $\mu_n \in \mathcal{M}^{\alpha^n}_t$ along $e_k$ is defined by

$$\beta^n_k(x) := (a^n_{te_k})'_{t=0} = - \sum_{j \in \mathbb{Z}^d} \partial_k W^n_{k,j}(x_k, x_j) - \partial_k V^n_k(x_k), \quad x \in S'.$$

Assumptions (A1)–(A3) imply that every $\beta^n_k$ is continuous on the balls in $S_{-p}$ for $p > \frac{d}{2}$. For every nice function $f$ the following integration by parts formula holds:

$$\int_\Omega \partial_k f(x)\, d\mu_n(x) = - \int_\Omega f(x) \beta^n_k(x)\, d\mu_n(x).$$

Let us take $p_1 > \frac{d}{2}$ and $p_2 > p_1\gamma + \frac{d}{2}$, $\gamma := \max\{L, N\} - 1$. For every $k_0 \in \mathbb{Z}^d$ we introduce a family of equivalent Hilbert norms defined by

$$|x|_{-p_2,k_0} := \Big[ \sum_{k \in \mathbb{Z}^d} (1 + |k - k_0|)^{-2p_2} x_k^2 \Big]^{\frac{1}{2}}, \quad k_0 \in \mathbb{Z}^d.$$

The main result of [3] is the following a priori estimate: for every $n$

$$\sup_{\mu_n \in \mathcal{M}_t^{\alpha_n}} \sup_{k_0 \in \mathbb{Z}^d} \int_\Omega e^{\lambda |x|^N_{-p_2, k_0}} \, d\mu_n(x) < \infty, \quad \forall \lambda > 0 \tag{12}$$

(see the proof of Theorem 3.2 in [3]). In particular, this result implies that every $\mu_n \in \mathcal{M}_t^{\alpha_n}$ is supported by $\bigcap_{p > \frac{d}{2}} S_{-p}$.

We consider the sequence of Dirichlet forms $\mathcal{E}^k_{\mu_n}(f, g) = \int_\Omega \frac{\partial f}{\partial k} \frac{\partial g}{\partial k} \, d\mu_n$, where $\{\mu_n\}$ is a sequence of probability measures such that $\mu_n \in \mathcal{M}_t^{\alpha_n}$. Let $\mathcal{F}C_0^\infty$ denote the set of smooth cylinder functions of the type $f(x) = f_N(x_{k_1}, \ldots, x_{k_N})$, $x \in \mathbb{R}^{\mathbb{Z}^d}$, with some $N \in \mathbb{N}$, $\{k_1, \ldots, k_N \subset \mathbb{Z}^d\}$ and $f_N \in C_0^\infty(\mathbb{R}^N)$.

Now we prove the main result of this section.

**Remark 6.1.** We emphasize that we do not use the existence of logarithmic derivatives in the proof but only a priory estimate (12) and the bounds on $W_{k,j}$, $V_k$.

**Theorem 6.2.** *Let $E_n$ be a sequence of energy functionals satisfying assumptions* (A1)–(A3), *$k \in \mathbb{Z}$ and $p > \frac{d}{2}$. Suppose that a sequence of measures $\{\mu_n\} \in \mathcal{M}_t^{\alpha_n}$, considered as measures on the Hilbert space $S_{-p}$, converges weakly to a measure $\mu \in \mathcal{M}_t^\alpha$ on $S_{-p}$. Suppose in addition that*

$$\sum_{j \in \mathbb{Z}^d} W^n_{k,j}(x_k, x_j) \to \sum_{j \in \mathbb{Z}^d} W_{k,j}(x_k, x_j)$$

*uniformly for all $x = (x_i)_{i \in \mathbb{Z}^d}$ on balls in $S_{-p}$ and that for some sequence of numbers $n_i$, $n_i < n_{i+1}$, $n_i \to \infty$*

$$e^{V_k^n(s)} \chi_{\{s: |s| \leqslant n_i\}} \, ds \to e^{V_k(s)} \chi_{\{s: |s| \leqslant n_i\}} \, ds \tag{13}$$

*weakly on $\mathbb{R}$. Then $\mathcal{E}^k_{\mu_n} \to \mathcal{E}^k_\mu$* Mosco.

**Proof.** Let $\pi_k(x) := z = x - x_k e_k$ be the projection onto hyperspace $L_k = (x, e_k) = 0$ and $\tilde{\nu}^k_n = \mu_n \circ \pi_k^{-1}$. Then the following disintegration formula holds (see [2] for the proof):

$$\int_\Omega u(x) \, d\mu_n(x) = \int_{L_k} \int_{\mathbb{R}} u(z + x_k e_k) \tilde{\rho}_n(z, x_k) \, dx_k \, d\tilde{\nu}^k_n(z),$$

where

$$\tilde{\rho}_n(z, x_k) = \left(\int_{\mathbb{R}} \alpha^n_{te_k}\, dt\right)^{-1} = \frac{\exp\left\{-\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j) - V^n_k(x_k)\right\}}{\int_{\mathbb{R}} \exp\left\{-\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k+t, x_j) - V^n_k(x_k+t)\right\} dt}. \tag{14}$$

We apply Theorem 3.4 and define the measure $\nu^k_n$ as the measure given by its Radon–Nykodim density $\left[\int_{\mathbb{R}} \exp\left\{-\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k+t, x_j) - V^n_k(x_k+t)\right\} dt\right]^{-1}$ with respect to $\tilde{\nu}^k_n$, i.e.,

$$\nu^k_n = \frac{1}{\int_{\mathbb{R}} \exp\left\{-\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k+t, x_j) - V^n_k(x_k+t)\right\} dt}\, d\tilde{\nu}^k_n.$$

The conditional measures $\rho^k_n$ for $\mu_n$ and $\nu^k_n$ are defined by the formula

$$\rho^k_n = \exp\left\{-\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j) - V^n_k(x_k)\right\},$$

$$\mu_n(dx) = \rho^k_n(z, x_k)\, dx_k \cdot \nu^k_n(dz).$$

According to Theorem 3.4 we have to show that for every $n_i$

$$\nu^k_n = (\rho^k_n)^{-1}\, d\mu_n \to (\rho^k)^{-1}\, d\mu = \nu^k \tag{15}$$

and

$$\chi_{\{x_k:|x_k|\leqslant n_i\}}(\rho^k_n)^{-2}\, d\mu_n \to \chi_{\{x_k:|x_k|\leqslant n_i\}}(\rho^k)^{-2}\, d\mu \tag{16}$$

weakly (considered as measures on $S_{-p}$). Indeed, we show first that

$$\exp\Big(\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\Big)\, d\mu_n \to \exp\Big(\sum_{j\in\mathbb{Z}^d} W_{k,j}(x_k, x_j)\Big)\, d\mu$$

weakly. We obtain from (A1)–(A3) (see also [3]) that $\forall p \in \mathbb{N}$

$$\left|\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\right| \leqslant C_1 + C_2|x|^N_{-p},$$

where $C_1, C_2$ depend from $N, p, J$. Together with (12) this implies that for every $A \subset \Omega$ one has

$$\Big[\int_A \exp\Big(\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\Big)\, d\mu_n\Big]^2 \leqslant \mu^k_n(A) \int_A e^{2(C_1 + C_2|x|^N_{-p_2})}\, d\mu_n \leqslant C' \mu^k_n(A).$$

Since $\{\mu_n\}$ is tight in $S_{-p}$, this yields that the sequence of measures

$$\Big\{\exp\Big(\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\Big)\, d\mu_n\Big\}$$

is tight in $S_{-p}$. Taking a subsequence we may assume that

$$\Big\{\exp\Big(\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\Big)\, d\mu_n\Big\}$$

converges weakly to some measure $\gamma$. Let us take a function $g$ that is continuous and has bounded support in the topology of $S_{-p}$. Then

$$\exp\Big(\sum_j W^n_{k,j}(x_k, x_j)\Big) g(x) \to \exp\Big(\sum_j W_{k,j}(x_k, x_j)\Big) g(x)$$

uniformly on $S_{-p}$, hence

$$\int_{S_{-p}} \exp\Big(\sum_j W^n_{k,j}(x_k, x_j)\Big) g(x)\, d\mu_n \to \int_{S_{-p}} \exp\Big(\sum_j W_{k,j}(x_k, x_j)\Big) g(x)\, d\mu$$

and $\int_{S_{-p}} g\, d\gamma = \int_{S_{-p}} \exp\Big(\sum_j W_{k,j}(x_k, x_j)\Big) g\, d\mu$. This yields that

$$\gamma = \exp\Big(\sum_{j\in\mathbb{Z}^d} W_{k,j}(x_k, x_j)\Big)\, d\mu.$$

Note that

$$\exp\Big(\sum_{j\in\mathbb{Z}^d} W_{k,j}(x_k, x_j)\Big)\, d\mu$$

is the product of the measures $\nu^k_n$ and $e^{-V^n_k(x_k)}\, dx_k$. Hence $\nu^k_n \to \nu^k$ weakly in $S_{-p}$.

In the same way as above we show that conditions (A1)–(A3) and (12) imply the tightness of the sequence of measures in (16). Condition (13) implies that

$$m_n := \chi_{\{x_k:|x_k|\leqslant n_i\}} e^{V_n^k(x_k)} \times \nu_n^k \to \chi_{\{x_k:|x_k|\leqslant n_i\}} e^{V^k(x_k)} \times \nu^k = m$$

weakly. Uniform convergence of $\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)$ to $\sum_{j\in\mathbb{Z}^d} W_{k,j}(x_k, x_j)$ on the balls in $S_{-p}$ implies that every limiting point of

$$\{\chi_{\{x_k:|x_k|\leqslant n_i\}} (\rho_n^k)^{-2}\, d\mu_n\} = \Big\{\exp\Big(\sum_{j\in\mathbb{Z}^d} W^n_{k,j}(x_k, x_j)\Big)\, dm_n\Big\}$$

coincides with

$$\exp\Big(\sum_{j\in\mathbb{Z}^d} W_{k,j}(x_k, x_j)\Big)\, dm = \chi_{\{x_k:|x_k|\leqslant n_i\}} (\rho^k)^{-2}\, d\mu.$$

This means that (16) holds. The proof is complete. $\square$

As above, we assume that the class $\mathcal{F}C_0^\infty$ is dense in $(\mathcal{D}(\mathcal{E}_\mu), (\mathcal{E}_\mu)_1^{1/2})$. Note that in this section we understand $\mathcal{F}C_0^\infty$ in the sense of the product topology on $\Omega$, i.e., we set $\mathcal{F}C_0^\infty = \{\varphi(x_{k_1}, \ldots, x_{k_n}), \varphi \in C_0^\infty(\mathbb{R}^n)\}$. However, one can easily verify that the equality $\big(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu)\big) = \big((\mathcal{E}_\mu)_0, \mathcal{D}((\mathcal{E}_\mu)_0)\big)$ holds in the sense of topology of any $S_p$ if and only if it holds in the sense of the product topology. The next corollary follows immediately from Theorem 6.2 and Corollary 3.5.

**Corollary 6.3.** *Suppose that a sequence of energy functionals and measures $\mu_n$ satisfies the hypotheses of Theorem* 6.2. *Consider the sequence of forms $\mathcal{E}_{\mu_n} = \sum_{k\in\mathbb{Z}^d} \mathcal{E}^k_{\mu_n}$. Suppose that $(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu)) = (\mathcal{E}_{\mu,0}, \mathcal{D}(\mathcal{E}_{\mu,0}))$. Then $\mathcal{E}_{\mu_n} \to \mathcal{E}_\mu$ Mosco.*

**Remark 6.4.** Some sufficient conditions for the equality

$$(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu)) = (\mathcal{E}_{\mu,0}, \mathcal{D}(\mathcal{E}_{\mu,0}))$$

are given in [1] and [12, Chapter 5e] .

Another important application of Theorem 6.2 is a construction of an approximating sequence of finite-dimensional Dirichlet forms for the gradient form $\mathcal{E}_\mu$. Following [3] we take a sequence of finite-dimensional Gibbs distributions $\{\nu_{\Lambda_n}(dx_{\Lambda_n}|y)\}_{n=1}^\infty$ with finite sets $\Lambda_n \subset \mathbb{Z}^d$, $\Lambda_n \subset \Lambda_{n+1}$ , $\cup_n \Lambda_n = \mathbb{Z}^d$, and a fixed boundary condition $y \in S'$ such that $\sup_j |y|_j \leqslant \infty$:

$$\nu_\Lambda(dx_\Lambda|y) := \frac{1}{Z_\Lambda} e^{-E_\Lambda(x_\Lambda \times y_{\Lambda^c})} \times_{k\in\Lambda} dx_k,$$

where

$$E_\Lambda(x_\Lambda \times y_{\Lambda^c}) := \sum_{\{k,j\}\subset\Lambda} W_{k,j}(x_k, x_j) + \sum_{k\in\Lambda,\, j\in\Lambda^c} W_{k,j}(x_k, y_j) + \sum_{k\in\Lambda} V_k(x_k),$$

and

$$Z_\Lambda(y) := \int_{\mathbb{R}^\Lambda} \exp\{-E_\Lambda(x_\Lambda \times y_{\Lambda^c})\} \times_{k\in\Lambda} dx_k.$$

For simplicity we take $y = 0$. Then every $E_{\Lambda_n}$ can be considered as a sequence of energy functionals with two particle-interactions

$$W^n_{k,j}(x_k, x_j) := \begin{cases} W_{k,j}(x_k, x_j), & k, j \in \Lambda \\ 0, & k \in \Lambda^c \quad \text{or} \quad j \in \Lambda^c. \end{cases} \tag{17}$$

and self-interactions $V^n_k(x_k) = V_k(x_k) + \sum_{j\in\Lambda^c} W_{k,j}(x_k, 0)$.

One can easily verify that the sequence of energy functionals $E_{\Lambda_n}$ satisfy Assumptions (A1)–(A3) (possibly, with different constants $A, B, C$) uniformly in $n$.

It was shown in [3] that $\nu_{\Lambda_n} \to \mu$ weakly on $S_{-p'}$ for some sufficiently large $p'$ (see Theorems 2.3 and 3.1) and some $\mu \in \mathcal{M}^\alpha_t$.

**Corollary 6.5.** *Suppose that* $(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu)) = (\mathcal{E}_{\mu,0}, \mathcal{D}(\mathcal{E}_{\mu,0}))$ (*see Remark* 6.4). *Then the sequence of forms* $\{\mathcal{E}_n\}$, *where*

$$\mathcal{E}_n = \sum_{k\in\Lambda_n} \mathcal{E}^k_{\nu_{\Lambda_n}},$$

*converges Mosco to* $\mathcal{E}_\mu$.

**Proof.** We show first that $\mathcal{E}^k_{\nu_{\Lambda_n}} \to \mathcal{E}^k_\mu$. Let us verify that the hypotheses of Theorem 6.2 are fulfilled. Take $k \in \Lambda_n$. Then $\sum_j W_{k,j}(x_k, x_j) - \sum_j W^n_{k,j}(x_k, x_j) = \sum_{j\in\Lambda^c} W_{k,j}(x_k, x_j)$. Let us fix some $p \in \mathbb{N}$ and a ball $B_{R,p} = \{\|x\|_{-p} \leqslant R\} \subset S_{-p}$. Then assumptions (A1)–(A3) imply

$$\begin{aligned} \Big|\sum_j W_{k,j}(x_k, x_j) - \sum_j W^n_{k,j}(x_k, x_j)\Big| &\leqslant \sum_{j\in\Lambda^c} J_{k,j}(1 + |x_k| + |x_j|)^N \\ &\leqslant C(N, B_{R,p}) \sum_{j\in\Lambda^c} J_{k,j}(1 + |x_j|^N) \\ &\leqslant C(N, B_{R,p})\Big[C|P_{\Lambda^c} J_{k,j}|_D + |P_{\Lambda^c} J_{k,j}|_{Np} \end{aligned}$$

$$\times\Big(\sum_{j\in\Lambda^c}\frac{|x_j|^{2N}}{(1+|j|)^{Np}}\Big)^{\frac{1}{2}}\Big]$$

$$\leqslant C(N,B_{R,p})\Big[C|P_{\Lambda^c}J_{k,j}|_D$$

$$+|P_{\Lambda^c}J_{k,j}|_{Np}|P_{\Lambda^c}x|_{-p}^{N}\Big],$$

where $P_{\Lambda^c}$ is the projection to $\mathbb{R}^{\Lambda^c}$, $D>\frac{d}{2}$ and $C=\Big(\sum_j\frac{1}{(1+|j|)^{2D}}\Big)^{\frac{1}{2}}<\infty$. This estimate implies that $\sum_j W_{k,j}(x_k,x_j)\to\sum_j W^n_{k,j}(x_k,x_j)$ uniformly on balls in $S_{-p}$ for every $p\in\mathbb{N}$. A similar estimate shows that $V^n(x_k)\to V(x_k)$ uniformly on every set $\{x:|x_k|\leqslant R\}$. Hence, Theorem 6.2 implies that $\mathcal{E}^k_{\nu_{\Lambda_n}}\to\mathcal{E}^k_\mu$ Mosco. The claim readily follows by Corollary 3.5. □

## 7. Convergence of laws

Now we briefly discuss convergence of the distributions of the associated processes. Let us show that the Mosco convergence implies weak convergence of the finite-dimensional distributions of the associated processes.

Indeed, let $\{\mathcal{E}_n\}$ be a Mosco convergent sequence of quasi-regular (see [25]) Dirichlet forms on $\mathcal{H}=\bigcup_n L^2(E;\mu_n)$, where every $\mu_n$ be a probability measure, and let

$$(\Omega,\mathcal{F},(X^n_t)_{t\geqslant 0},(P^n_x))$$

be the associated stochastic processes. We suppose that $\Omega=C([0,\infty)\to E)$, i.e., the trajectories of the processes $(X^n_t)_{t\geqslant 0}$ are continuous and these processes are conservative. Then the finite-dimensional distributions of the measure

$$P^n_{\mu_n}=\int_E P^n_x\,\mu_n(dx)$$

converge vaguely to $P_\mu=\int_E P_x\,\mu(dx)$.

Indeed, this follows from the formula

$$\int f_0(X^n_0)f_1(X^n_{t_1})f_2(X^n_{t_1+t_2})\cdots f_m(X^n_{t_1+\cdots+t_m})\,dP_{\mu_n}$$

$$=\int f_0T^n_{t_1}(f_1T^n_{t_2}(f_2\cdots T^n_{t_m}(f_m))\cdots)\,d\mu_n,$$

applied to $f_0,f_1,\ldots,f_m\in\mathcal{F}C_0^\infty(E)$ strong convergence of $T^n_t$ in $\mathcal{H}$, and the estimate

$$\sup_n\|T^n_t\|_{L^\infty(H_n)}<\infty.$$

Here we use the fact that $u_n \to u$ in $\mathcal{H}$ implies $(u_n, v)_{H_n} \to (u, v)_H$ in $\mathcal{H}$ for $v \in C = \mathcal{F}C_0^\infty(E)$.

If, in addition, we know that the sequence $\{P^n_{\mu_n}\}$ is tight, then using the standard subsequence argument we obtain that $P^n_{\mu_n} \to P_\mu$ weakly. The tightness of $\{P^n_{\mu_n}\}$ can be established in many cases with the help of the well-known probabilistic method—the so called Lyons–Zheng decomposition (see [16,35,39]).

**Remark 7.1.** According to [32], under the assumption that $(\mathcal{E}, \mathcal{D}(\mathcal{E})) = (\mathcal{E}_0, \mathcal{D}(\mathcal{E})_0)$, all Dirichlet forms in this paper are quasi-regular if $E$ is a separable Banach space.

## Acknowledgments

The author thanks Professor Michael Röckner for the formulation of the problem and fruitful discussions. The author also thanks Professor Vladimir Bogachev for helpful comments. The paper was completed during the author's very pleasant stay at Pisa, where he was supported by the Junior Visiting Position of the Centro di Ricerca Matematica Ennio De Giorgi, Scuole Normale Superiore di Pisa.

## References

[1] S. Albeverio, Yu.G. Kondratiev, M. Röckner, Dirichlet operators via stochastic analysis, J. Funct. Anal. 128 (1995) 102–138.
[2] S. Albeverio, Yu.G. Kondratiev, M. Röckner, Ergodicity for the stochastic dynamics of quasi-invariant measures with applications to Gibbs states, J. Funct. Anal. 149 (1997) 415–469.
[3] S. Albeverio, Yu.G. Kondratiev, M. Röckner, T.V. Tsikalenko, A priori estimates for symmetrizing measures and their applications to Gibbs states, J. Funct. Anal. 171 (2000) 366–400.
[4] S. Albeverio, S. Kusuoka, L. Streit, Convergence of Dirichlet forms and associated Schrödinger operators, J. Funct. Anal. 68 (1986) 130–148.
[5] S. Albeverio, M. Röckner, Classical Dirichlet forms on topological vector spaces—closability and a Cameron–Martin formula, J. Funct. Anal. 88 (1990) 395–436.
[6] V.I. Bogachev, Differentiable measures and the Malliavin calculus, J. Math. Sci. 87 (5) (1997) 3577–3731.
[7] V.I. Bogachev, Gaussian Measures, American Mathematical Society, Rhode Island, 1998.
[8] M. Briane, N. Tchou, Fibered microstructures for some nonlocal Dirichlet forms, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) (2001) 681–711.
[9] M. Camar-Eddine, P. Seppecher, Closure of the set of diffusion functionals with respect to the Mosco-convergence, Math. Models Methods Appl. Sci. 12 (8) (2002) 1153–1176.
[10] P. Cattiaux, M. Fradon, Entropy, reversible diffusion processes and Markov uniqueness, J. Funct. Anal. 138 (1996) 243–272.
[11] G. Dal Maso, An introduction to $\Gamma$-convergence, Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkhäuser, Boston, Inc., Boston, MA, 1993.
[12] A. Eberle, Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators, Lecture Notes in Mathematics, vol. 1718, Springer, Berlin, 1999.
[13] L.S. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, New York, Tokyo, 1992.
[14] M. Fukushima, BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, J. Funct. Anal. 174 (2000) 227–249.
[15] M. Fukushima, M. Hino, On the space of BV functions and a related stochastic calculus in infinite dimensions, J. Funct. Anal. 183 (2001) 245–268.

[16] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet Forms and Symmetric Markov Processes, De Greyter, Berlin, 1994.
[17] H.-O. Georgii, Gibbs measures and phase transitions, Studies in Mathematics, vol. 9, De Greyter, Berlin, New York, 1988.
[18] M. Hino, Convergence of non-symmetric forms, J. Math. Kyoto Univ. 38 (2) (1998) 329–341.
[19] A.V. Kolesnikov, Convergence of Dirichlet forms with changing speed measures on $\mathbb{R}^d$, Forum. Math. 17 (2005) 225–259.
[20] K. Kuwae, T. Shioya, Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry, Comm. Anal. Geom. 11 (4) (2003) 599–673.
[21] K. Kuwae, T. Uemura, Weak convergence of symmetric diffusion processes II, in: Proceedings of the 7th Japan–Russia Symposium on Probability Theory and Mathematical Statistics, World Scientific, 1996, pp. 256–265.
[22] K. Kuwae, T. Uemura, Weak convergence of symmetric diffusion processes, Probab. Theory Relat. Fields 109 (1997) 159–182.
[23] T.J. Lyons, T.S. Zhang, Note on convergence of Dirichlet processes, Bull. London Math. Sci. 25 (1993) 353–356.
[24] T.J. Lyons, T.S. Zhang, Convergence of non-symmetric Dirichlet processes, Stoch. Stoch. Reports 57 (1996) 159–167.
[25] Z.-M. Ma, M. Röckner, Introduction to the Theory of (non-symmetric) Dirichlet Forms, Springer, Berlin, 1992.
[26] U. Mosco, Composite media and Dirichlet forms, J. Funct. Anal. 123 (1994) 368–421.
[27] B. Muckenhoupt, Weighted norm inequalities for Hardy maximal functions, Trans. Amer. Math. Soc. 165 (1972) 207–226.
[28] Y. Ogura, M. Tomisaki, M. Tsuchiya, Convergence of local type Dirichlet forms to a non-local type one, Ann. Inst. H. Poincaré-PR 38 (4) (2002) 507–556.
[29] A. Posilicano, Convergence of distortet Brownian motions and singular Hamiltonians, Potential Analysis 5 (1996) 241–271.
[30] C. Preston, Random fields, Lecture Notes in Mathematics, vol. 534, Springer, Berlin, New York, 1976.
[31] O. Pugachev, On closability of classical Dirichlet forms, J. Funct. Anal. 207 (2) (2004) 330–343.
[32] M. Röckner, B. Schmuland, Tightness of general $C_{1,p}$-capacities on Banach space, J. Funct. Anal. 108 (1992) 1–12.
[33] M. Röckner, T.S. Zhang, Uniqueness of generalized Schrödinger operators and applications, J. Funct. Anal. 105 (1992) 187–231.
[34] M. Röckner, T.S. Zhang, Uniqueness of generalized Schrödinger operators and applications II, J. Funct. Anal. 119 (1994) 455–467.
[35] M. Röckner, T.S. Zhang, Finite dimensional approximations of diffusion processes on infinite dimensional state spaces, Stoch. Stoch. Reports 57 (1996) 37–55.
[36] M. Röckner, T.S. Zhang, Convergence of operators semigroups generated by elliptic operators, Osaka J. Math. 34 (1997) 923–932.
[37] K.T. Sturm, Analysis on local Dirichlet spaces I. Recurrence, conservativeness and $L_p$-Liouville properties, J. Reine Angew. Math. 456 (1994) 173–196.
[38] K.T. Sturm, Analysis on local Dirichlet spaces II. Upper Gaussian estimates for the fundamental solution of parabolic equations, Osaka J. Math. 32 (1995) 275–312.
[39] M. Takeda, On a martingal method for symmetric diffusion processes and its applications, Osaka J. Math. 26 (1989) 605–623.
[40] T. Uemura, On weak convergence of diffusion processes generated by energy forms, Osaka J. Math. 32 (1995) 861–868.
[41] V.V. Zhikov, Weighted Sobolev spaces, Sbornik: Mathematics 189 (8) (1998) 1139.