Categorical deformations of higher categories

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Motivation: the curvature problem

Let A be a dg-algebra over a field k. It is a well known problem in deformation theory [KL09] that (first order) deformations of A as a dg-algebra do not suffice to span its whole second Hochschild cohomology. It turns out [Low08; Leh24] that $\mathrm{HH}^2(A)$ parametrizes first order deformations of A as a *curved* dg algebra. This is a significant issue, since cdg algebras do not have classical derived categories [Pos10]; in particular, there is no obvious deformation of $D(A)$ corresponding to a curved deformation of A [KLN10].
The same problem was observed by Lurie in a different setting: he essentially showed in [Lur11] that the functor

$$\mathrm{Def}_A\colon \mathrm{dgart}_k \to \mathrm{Set}$$

which associates to a (dg) local artinian k-algebra R the set of R-deformations of A as a dg-algebra is not a (derived) deformation functor.
The main question that we aim to answer is the following:

Which deformation of $D(A)$ corresponds to a curved deformation of A?

Behind this one lies a more fundamental question: *which notion* of deformation of a triangulated category allows for the question above to have a positive answer? The usual one – which roughly corresponds to reducing the hom-sets of an opportune resolution – cannot work: to a curved deformation of A corresponds a curved deformation of the dg-category $D(A)$, which does not have an underlying triangulated category. This question is a crucial step towards obtaining a satisfactory deformation theory for noncommutative schemes.

cdg algebras

A cdg algebra $\mathcal{A}$ over a commutative ring R is a triple $(\mathcal{A}^\#, d_\mathcal{A}, c)$ where:

- $\mathcal{A}^\#$ is a graded R-algebra;
- $d_\mathcal{A}\colon \mathcal{A}^\# \to \mathcal{A}^\#$ is a degree 1 derivation;
- $c \in \mathcal{A}^\#$ is a degree 2 element such that $d_\mathcal{A} c = 0$ and $d_\mathcal{A}^2 = [c, -]$.

A cdg module M over a cdg algebra $\mathcal{A}$ is a pair $(M^\#, d_M)$ where $M^\#$ is a graded $\mathcal{A}^\#$-module and

$$d_M\colon M^\# \to M^\#$$

is a degree 1 derivation such that $d_M^2 m = cm$.

Key point: if M and N are cdg $\mathcal{A}$-modules, then

$$\mathrm{Hom}_\mathcal{A}(M, N)$$

is a *complex*, so $\mathcal{A}$-Mod is a (pretriangulated) dg-category.

However, cdg modules have no cohomology, so there is no obvious notion of a derived category of $\mathcal{A}$ [Pos10; KLN10].

The n-derived category

Let A_n be a cdg deformation of A over $k[t]/(t^{n+1})$. A cdg A_n-module M is n-acyclic if its associated graded with respect to the t-adic filtration is acyclic; this makes sense since each graded piece is a complex. The n-derived category $D^n(A_n)$ is the Verdier quotient $D^n(A_n) = H^0 A_n\text{-Mod}/n\text{-Ac}(A_n)$.

Theorem 1 ([LL24])

- The category $D^n(A_n)$ is generated by $n+1$ explicit compact objects $\Gamma_0, \dots, \Gamma_n$ and the projection $H^0 A_n\text{-Mod} \to D^n(A_n)$ admits both adjoints;
- Calling A_i the induced deformation of order $i \le n$, the restriction functor $A_i\text{-Mod} \to A_n\text{-Mod}$ induces a system of fully faithful embeddings
$$D(A) = D^0(A_0) \xhookrightarrow{i_1} D^1(A_1) \hookrightarrow \dots \hookrightarrow D^{n-1}(A_{n-1}) \xhookrightarrow{i_n} D^n(A_n);$$
- The abelian category $Z^0 A_n$-Mod admits a model structure presenting $D^n(A_n)$;

The classical case

If A_n has no curvature, one can consider the classical derived category $D(A_n)$. One sees that there are strictly more n-acyclics than acyclics; it turns out that $D^n(A_n)$ can be seen as a (partial) categorical resolution of $D(A_n)$. Indeed:

- There is an embedding
$$D(A_n) \hookrightarrow D^n(A_n);$$
- The (dg) category $D^n(A_n)$ is smooth if and only if $D(A)$ is smooth.

$D^n(A_n)$ as a categorical extension

The embedding $D^{n-1}(A_{n-1}) \xhookrightarrow{i_n} D^n(A_n)$ admits both a left adjoint $\mathrm{Ker}\, t^n$ and a right adjoint $\mathrm{Coker}\, t^n$. We also have the functor $\mathrm{Im}\, t^n\colon D^n(A_n) \to D(A)$. These functors do not need to be derived, since they preserve n-acyclics.

Theorem 2 ([LL24])

There is a recollement

$$\begin{tikzcd} D^{n-1}(A_{n-1}) \arrow[r, hook, "i_n"] & D^n(A_n) \arrow[l, bend right, "\mathrm{Ker}\, t^n"'] \arrow[l, bend left, "\mathrm{Coker}\, t^n"] \arrow[r, "\mathrm{Im}\, t^n"] & D(A). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

In particular, the functor

$$\mathrm{Im}\, t^n\colon D^n(A_n) \to D(A)$$

induces an equivalence between the quotient $D^n(A_n)/D^{n-1}(A_{n-1})$ and $D(A)$.

Inductively, we see that $D^n(A_n)$ is obtained by gluing $n+1$ copies of $D(A)$.

Categorical deformations

All triangulated categories and functors between them are appropriately enhanced. Let $\mathcal{T}$ be a triangulated category. A first order categorical deformation of $\mathcal{T}$ is the datum of a recollement

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "i"] & \mathcal{T}_\varepsilon \arrow[l, bend right, "K"'] \arrow[l, bend left, "Q"] \arrow[r, "I"] & \mathcal{T} \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

together with an appropriately defined Yoneda extension of functors

$$0 \to I \xrightarrow{\delta_1} K \xrightarrow{\alpha} Q \xrightarrow{\delta_2} I \to 0.$$

Intuitively: the natural transformation α induces an isomorphism between the fibre of δ_1 and the cofibre of δ_2. Here, the map α is determined by the semiorthogonal decomposition, while δ_1 and δ_2 are extra data. These abstract the exact sequence

$$0 \to tM \hookrightarrow \mathrm{Ker}\, t_M \to \mathrm{Coker}\, t_M \xrightarrow{t} tM \to 0$$

defined for any $k[\varepsilon]$-module M. Denote with $\mathrm{Def}^{\mathrm{cat}}_{k[\varepsilon]}(\mathcal{T})$ the set of first order deformation of $\mathcal{T}$ up to equivalence.

Main result: a commutative diagram of deformations

Theorem 3 ([LL25])

Let $\mathcal{T}$ be a triangulated category. There is a bijection

$$\mathrm{Def}^{\mathrm{cat}}_{k[\varepsilon]}(\mathcal{T}) \xleftrightarrow{\kappa} \mathrm{HH}^2(\mathcal{T})$$

between first order deformations of $\mathcal{T}$ as a triangulated category and its second Hochschild cohomology.

The point of the bijection is that the gluing functor is the cone of the Hochschild cocycle.

Theorem 4 ([LL25])

There is a commutative diagram of bijections

$$\begin{tikzcd} \mathrm{cDef}^{\mathrm{Mor}}_{k[\varepsilon]}(A) \arrow[r] \arrow[d, "D^1(-)"'] & \mathrm{HH}^2(A) \arrow[d, "\chi_A"] \\ \mathrm{Def}^{\mathrm{cat}}_{k[\varepsilon]}(D(A)) \arrow[r, "\kappa"] & \mathrm{HH}^2(D(A)) \end{tikzcd}$$

Where $\mathrm{cDef}^{\mathrm{Mor}}_{k[\varepsilon]}(A)$ is the set of curved deformations of A up to equivalence of 1-derived categories; the upper arrow was introduced in [Leh24], while χ_A is the characteristic morphism from e.g. [Low08].

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