Homework 1

Algebraic Topology 2024-2025

Due 26 February, 2025

EXERCISE 1

Let A be a topological space and $X, Y \subseteq A$ be subspaces such that $A = X \cup Y$; assume that X and Y are either both open or both closed in A. Let B be a topological space. Let

$$f_X : X \to B \text{ and } f_Y : Y \to B$$

be continuous maps that coincide on the intersection $X \cap Y$. Show that there exists a unique continuous map

$$f : A \to B$$

which restricts to f_X on X and to f_Y on Y.

EXERCISE 2

Let X be a topological space. We'll say that a map

$$F : X \times [0, 1] \to X$$

is a weak retraction of X to a subspace $A \subseteq X$ if for every $x \in X$ and $a \in A$,

$$F(x, 0) = 0, \quad F(x, 1) \in A \text{ and } F(a, 1) = a.$$

Show that the following conditions are equivalent:

- There exists a point $x \in X$ such that X weakly deformation retracts to x;
- For any point $x \in X$, X weakly deformation retracts to x;
- X is homotopy equivalent to the one-point topological space;

In this case, we will say that X is contractible. Show that if X is contractible, for any space Y, $X \times Y$ is homotopy equivalent to Y.