README

After summarizing the Mathematical and theoretical concepts behind the Black-Scholes model in *'concepts_BS.pdf'*, this document serves to explain widely the core of the scripts, written in order to put in practice the main applications of the model and to visualize them concretely by plotting in graphs the results. The applications which are going to be taken in consideration are the (1) **Theoretical Convergence**, (2) **Volatility Surface** and (3) **Dynamic Hedging**.

Files contained in BS_simulation directory:

- (I) `MC_simulation.py`
- (II) `vol_surface.py`
- (III) `delta_hedging.py`
- (IV) `pricing.py`

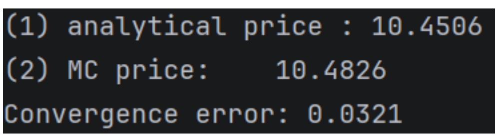
(1) Theoretical Convergence (I)

In this first demonstration we take the (I) file whose purpose is to confirm the relationship between the final result of the *PDE* and the *Monte Carlo simulation*.

- (1.a) **GBM Path Generation:** The script uses the SDE, imported from the (IV)(using the static variables declared) to generate 50,000 **asset price paths** (only 15 are shown in light grey). The red straight line represents the mean of the overall paths, thanks to which we're able to calculate the MC price of the call option.

(1.a)

- (1.b) **Convergence Error:** The analytical price (PDE solution, obtained from (IV)) is compared with the MC price (mean of discounted payoffs). The difference of the two quantities is definitively low, demonstrating the **equivalence** of the methods proposed.



(1.b)

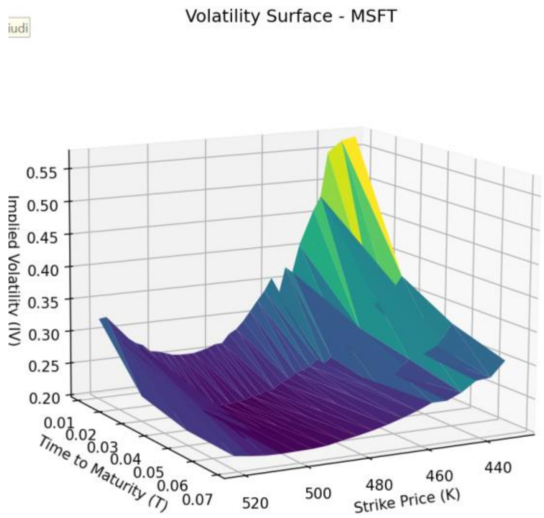
(2) Volatility Surface (II)

This application proves the limit of the theoretical assumption that states the volatility as constant, showing the skew of the IV (taking in consideration real market data).

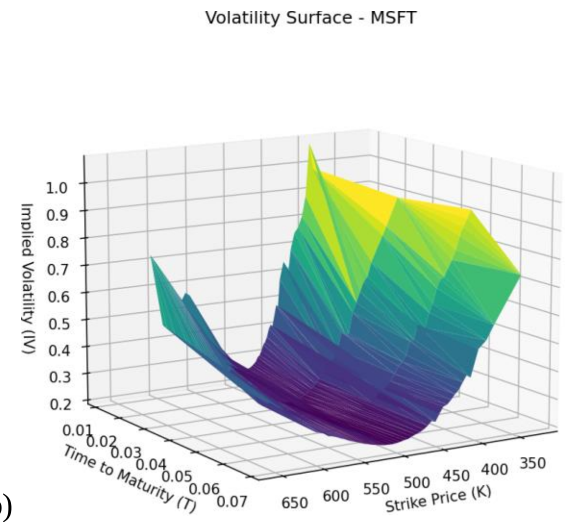
Procedure:

- **Market Data:** real option data for a chosen ticker (in this case Microsoft: 'MSFT') across multiple **Strikes (*K*)** and **Times to Maturity (*T*).**
- **IV Calculation:** for each option, the *implied_volatility method* from (IV) is executed. This method solves the SDE backward, as the only missing variable is σ, using the **Newton-Raphson algorithm**.
- **Filtering:**
  - (2.a) Near-The-Money options are taken, admitting only the ones that distance till 10% from the strike (default : line 23 of (II) , imposing 0.9 and 1.1)
  - (2.b) the range of options is increased, admitting the ones that distance till 50% from the strike (variation : imposing 0.5 and 1.5)

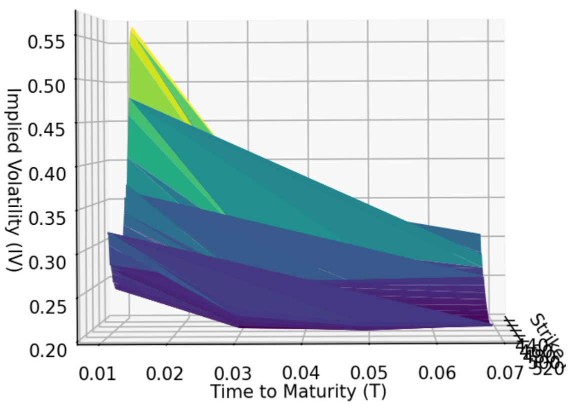
- **Visualization:** X-Y axes correspond to the tuple Strike Price and Time to Maturity of the option and Z-axis corresponds to its IV.



(2.a)



(2.b)



(2.c)

Analysis:

- **Volatility Skew:** (2.a) What emerges clearly is the **curve**, coinciding with the volatility skew, showing that the lowest IVs are located for ATM ($S \approx K$) contracts and the highest IVs for OTM/ITM ($S<K$ or $S>K$) contracts.
- **Term Structure:** (2.c) By taking options with equal-similar strike, as me move nearer to the maturity date, the IV tends to decrease, similarly as we increase the gap till the maturity date, the IV increases. This trend is justified by the fact that if the decay-time is wide then the volatility must be higher as the risk and uncertainties are higher, compared to the same option whose T is nearer to the current date, meaning that the price should be quite stable in that amount of time remaining.
- **Comparison:** By comparing the 2.a (limited view) and the 2.b (complete view), we see in the second case that the result is a more chaotic surface, caused by the inclusion of the majority of contracts (extremes included), stating that deep ITM/OTM options have exceptional highs in terms of IV, which is less reliable in these cases as there is less volume traded and spread are higher.
- **Conclusion:** This visualization provides concrete evidence that the assumption of **constant volatility** (a flat plane) is violated in real markets.

(3) Delta Hedging (III)

In the last section the core is implementing a **Delta-Neutral Strategy,** simulating the MM position. The objective is to dynamically hedge a MM portfolio by selling an option C and hedging it with the underlying asset S, by buying and selling it a certain number of times per day (in (III) in line 9, 100 is put). When it happens that cash is present in the portfolio, it compounds at r, and when we have to loan money in order to buy stocks, we take them at r interest.

Procedure:

- **Initialization (t_0):**
  - Short on 1 Call option (European), so we receive C_0
  - Calculation of Δ_0 and purchase of $\Delta * S$
  - π_0 is the sum of the net positions, so the cash residual (which should grow at r)
- **Discrete Rebalance:**
  - The cash account compounds at r
  - For each interval dt, the model recalculates Δ_t, in function of the new spot price S_t and the residual time $T-t$
  - Buy or sell of stocks in order to balance the new Δ_t, generating cash flows which increase/decrease the net position π_t
- **Liquidation (t=T):**
  - The stocks are liquidated
  - The option payoff $max(S_T - K, 0)$ is realized
  - π_T (**effective portfolio value**) is **compared** with $\pi_0 * e^{(rt)}$ (which is the portfolio initial net position capitalized at r for the overall period, or the **theoretical portfolio value**)

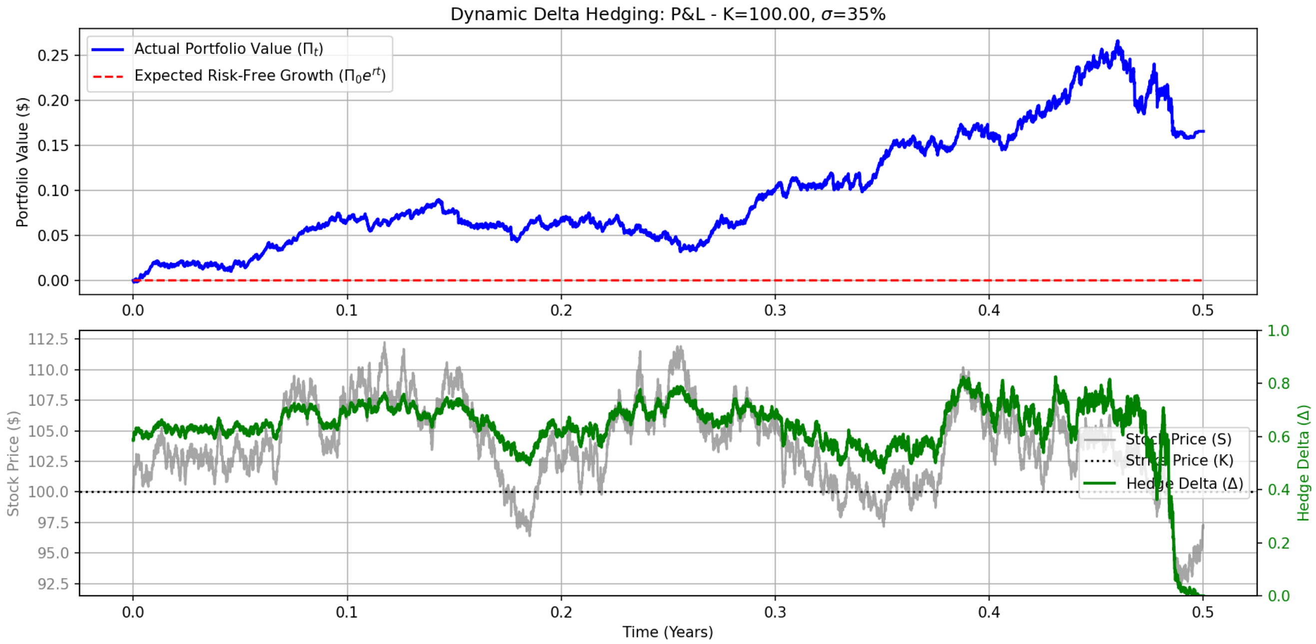
Analysis:

- **Discretization:** If we had been in a continuous regime the result of ***π _T-π_0*e^(rt)*** is perfectly null, but as we're in a discrete simulation, an **error** emerges and that derives from the second-order term of the Taylor expansion ***Γ*** (indeed the MM wallet as we have seen in '*concepts_BS'* derives from it, apart from ***θ***). That error/profit&loss is infact proportional to to *Γ* as the (3.a).

$$\text{P\&L Error} \approx -\frac{1}{2}\Gamma(\Delta S)^2$$
(3.a)

- **Comparison:**
  - (3.b): graphical output, (3.c): console output, values set: *hedges_per_day = 100, initial_capital= 0* by setting in in line 19)
  - (3.d), (3.e), values set: *hedges_per_day=1000, initial_capital= 0*
  - (3.f), (3.g), values set: *hedges_per_day=1000, initial_capital= 100*
- **Assumptions**: The bank fees are not considered in this simulation, otherwise there would be a trade-off between the commissions and the number of hedges per day
- **Conclusion:** As we clearly see from the given outputs , as we increase the density of hedges, the more the actual (blue line) and the expected portfolio (red line) converge together, tending to the continuous environment (indeed the error in 3.b >> 3.d-3.f). As we have said, *Γ* is responsable for the emerging error, as a consequence we're able to see it in the underlying graphs: the error that brings the blue line to cumulatively diverge from the red one is exactly in correspondence where the rate of change (indeed *Γ=dΔ/dS)* of the green function is higher as confirmed by (3.a).

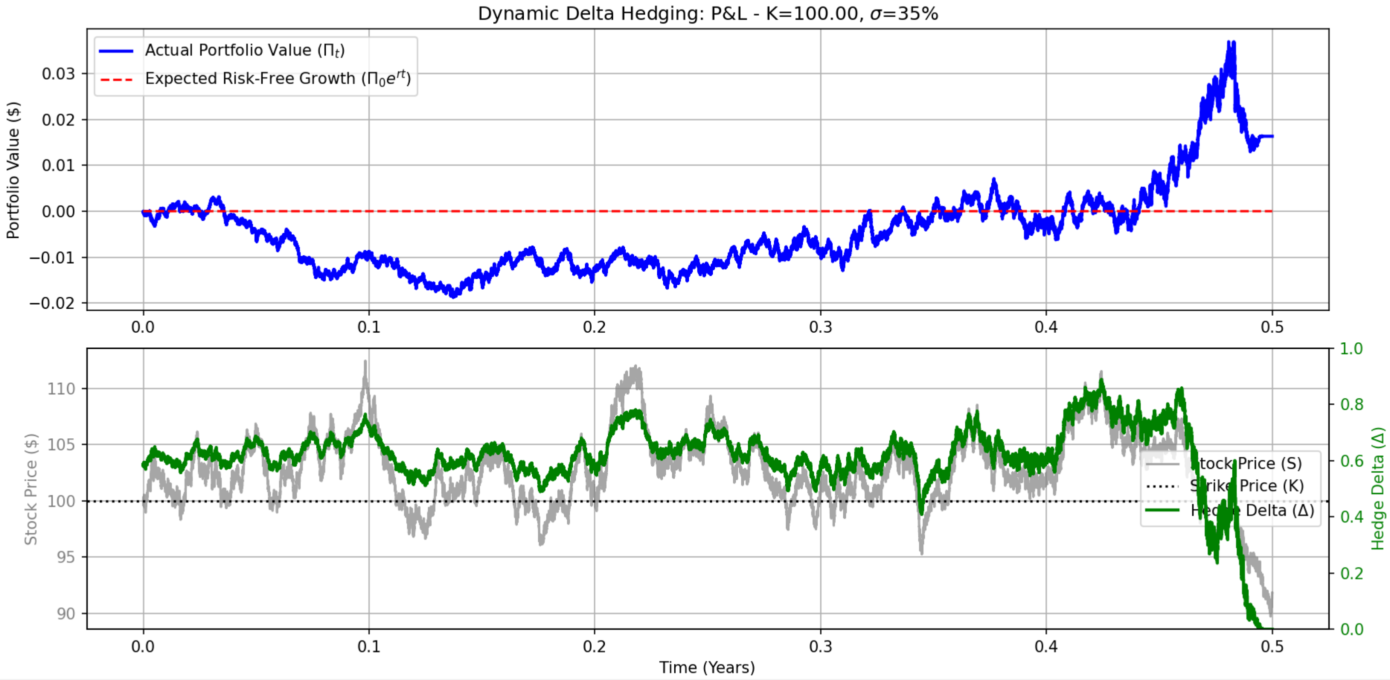
(3.b)



(3.c)

```
1. Final Value Actual (Pi_T): 0.165769
2. Final Value Expected (Pi_0 * exp(rT)): 0.000000
3. Error (Pi_T - Pi_0 * exp(rT)): 0.165769
```

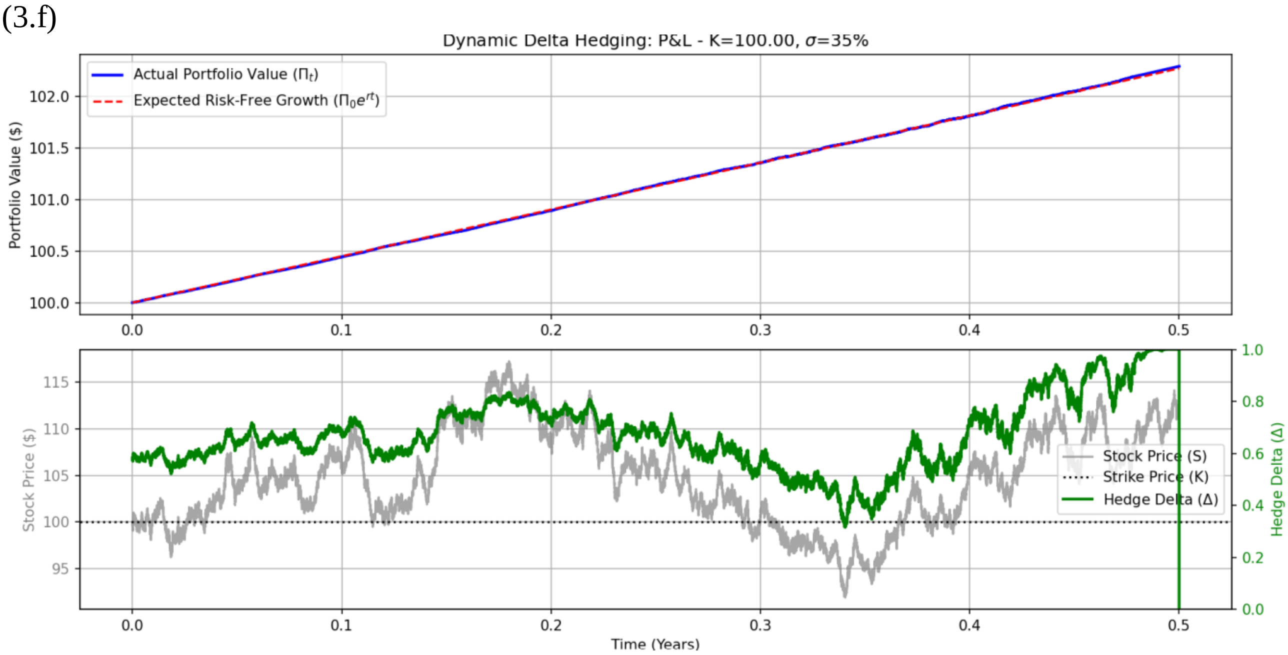
(3.d)



(3.e)

```
1. Final Value Actual (Pi_T): 0.016306
2. Final Value Expected (Pi_0 * exp(rT)): 0.000000
3. Error (Pi_T - Pi_0 * exp(rT)): 0.016306
```

(3.f)



(3.g)

```
1. Final Value Actual (Pi_T): 102.294857
2. Final Value Expected (Pi_0 * exp(rT)): 102.275503
3. Error (Pi_T - Pi_0 * exp(rT)): 0.019354
```