BLACK-SCHOLES

---Modelling Asset Dynamics: The Stochastic Differential Equation (SDE)---

This framework is practically used for modelling the theoretical price of European-style derivatives and managing the risk of options portfolios. The core assumption in the Black-Scholes (BS) model is that the price of the underlying asset (*S_t*), follows a **Geometric Brownian Motion (GBM)**. This model is chosen because it guarantees that the asset prices are non-negative (as the real world) and that the *ln(S_t)* are distributed as a log-Normal distribution.

The dynamic of the price of the underlying asset is defined by the following Stochastic Differential Equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

(1)

The formula is composed of two primary terms, which, once combined, give life to a random path of the asset.

A. The Deterministic Term (Drift): $\mu S_t dt$

This represents the expected rate of return of the asset considering an infinitesimal time period (dt). In a real-world setting, 'μ' is the expected growth rate, but in the *Risk-Neutral* environment used for pricing, 'μ' is replaced by the risk-free rate '***r***'.

B. The Stochastic Term (Diffusion): $\sigma S_t dW_t$

This component is the cause of randomness-risk and it models the random fluctuations driven by the market. 'σ' is the **Volatility** (annualized *standard deviation* of returns) and *dW_t* is the **Wiener Process** (the increment of the GBM), representing the stochastic shock and it is normally distributed (N ~ (0,1)).

(The SDE is the parallelism and equivalent of simulating the random path of a particle, where the price movement is a random walk with a continuous drift.)

---Eliminating Risk: The Portfolio Construction and Ito's Lemma---

The BS model derives the option price (*C* for call options or *P* for put options) by constructing a hedged **risk-free portfolio** (π), primarily built thanks to the **no-arbitrage principle**. (That means the wallet we've created is no longer dependent on risk by balancing the amounts of derivatives and underlying stocks taking out the randomness of the market.)

A. The Hedged Portfolio : π

In this case the market maker (MM) sells one Call option (*C*) and simultaneously hedges the randomness of the asset by buying a quantity of the underlying asset (*S*) equal to the option's ***Delta(Δ):***

(2) $$\Pi = -C(S,t) + \Delta S$$

'*Δ*' is the sensitivity of the option price to changes in the underlying asset price (indeed *Δ=dC/dS*). By admitting this, the MM is delta hedging its portfolio.

B. The Risk-Neutral Condition

By building the portfolio in this way, the MM ensures that any instantaneous random change in the option's value (*dC*) is perfectly offset by an instantaneous random change in the stock's value (*dS*).

1. The infinitesimal change in the portfolio value (*d π*) is calculated by combining the (1) and (2) and applying **Itô's Lemma**. Itô's Lemma is fundamental as it leads to the risk elimination.
2. Since all market risk is eliminated (*dW=0*), the portfolio *π* must grow as by compounding the risk-free rate, r, obliged by being in a no-arbitrage state, resulting that:

(3) $$d\Pi = r\Pi dt$$

---The Pricing Equation: Partial Differential Equation (PDE)---

The algebra derived from the no-arbitrage condition brings directly to the **Black-Scholes Partial Differential Equation (PDE)**. This equation rules the price of any derivative that satisfies the assumptions of the model:

(4) $$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

(4.1) $$-\Theta - \frac{1}{2}\sigma^2 S^2 \Gamma = r(-C + \Delta S)$$

Interpretation of the Terms and Greeks (Theta, Gamma, Delta)

This PDE represents the balance of forces (greeks) required in order to have a no-arbitrage price:

- **Time Decay (*θ = dC/dt*):** This is the price decay due to the time passing. (As the time passes the option price decreases for long options.)
- **Convexity/Volatility Term (*0.5*σ²*S²*Γ = 0.5*σ²*S²*d²C/dS²*):** This term measures the instantaneous acceleration of the option's value due to variations of the underlying asset price (*S*), weighted by the *σ²*. (More simply the rises of volatility and variations of *S* increase the option price,)
- **Drift/Interest Rate Term (*r*S*Δ = r*S*dC/dS*):** This is the expected growth of the hedged position.

- **Discounting Term (*r*C*):** This is the continuous discounting of the option's value, so the cost of holding the position.

The equation essentially states that **Time Decay + Hedging Gains must equal the required risk-free return of the portfolio.** Otherwise by rearranging the terms as the (4.1), we can clearly visualize that under the *MM perspective* what gives return to its portfolio ($\pi = -C + \Delta S$) is exactly being *long* θ (dC/dt is always negative, so the first term '$-\theta$' positive) and *short* Γ (so that the price of the option decreases and the second term of the (4.1) is positive).

The Solution: Analytical Pricing Formula

Once we solve the PDE, under the condition of the *option's payoff at maturity* (so when the option expires) $C(S_T, T) = \max(S_T - K, 0)$ (for the call options) produces the closed-form BS formula for European Call option prices *C(S,t):*

(5) $$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

(6) $$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(7) $$d_2 = d_1 - \sigma\sqrt{T-t}$$

As we see *C* depends on the current asset price (*S*), the Strike (*K*), the expiration time (*T-t*), the risk-free rate (*r*), and the volatility (σ).

The formula is basically the difference between the expected present value of the stock multiplied by the probability of the option ending in-the-money (**ITM**), and the actualized expected value of the strike (*K*) multiplied by the probability of still ending ITM:

1. **Expected Stock Value (First Term):** $SN(d_1)$
   - *S*: Present value of the Stock.
   - *S*N(d_1)*: represents *the present value of the expected amount of underlying stock received* at expiration if the option ends ITM.
2. **Expected Cost of Strike (Second Term):** $Ke^{-r(T-t)}N(d_2)$
   - *K*exp(-r(T-t))*: *The value of the Strike price K actualized* (as we have to discount a future price (*K*) to the present in order to make it comparable with *S* that is current)
   - *N(d_2):* Represents the *probability that the option will expire ITM*, giving the expected cost of exercising the option.