

The Cauchy-Kowalevski Theorem and Its Consequences

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Sofya Vasilyevna Kovalevskaya (1850-1891)

We assume the historical figure of Augustin-Louis Cauchy is well-known.

Kowalevski was:

- a Russian mathematician and student of Weierstrass
- the **first woman** to earn a doctorate (3 theses dating back to 1874) and to obtain a chair in Europe (in mathematics)

There are several **artistic representations** of her in both literature and cinema. The most notable are:

- An accurate biography: *Little Sparrow: A Portrait of Sophia Kovalevsky* (1983), Don H. Kennedy
- A short story: *Too Much Happiness* (2009), Alice Munro

Guiding Questions

*Is it possible for an analytical solution
of a PDE system
with Cauchy conditions to exist?*

The answer is affirmative, so we already ask:

- under what assumptions?
- is the solution unique?
- is the problem well-posed?
- what are the consequences of the obtained results?

Types of Equations (and Operators)

Equations of order k :

| | |
|--------|--|
| Linear | $\sum_{ \alpha \leq k} a_\alpha D^\alpha u = f$ |
|--------|--|

| | |
|--------------|--|
| Quasi-linear | $\sum_{ \alpha =k} a_\alpha(x, D^\beta u) D^\alpha u + a_0(x, D^\beta u) = f,$ $ \beta < k$ |
|--------------|--|

| | |
|------------|---|
| Non-linear | $F(x, D^\alpha u) = 0, \quad \alpha \leq k$ |
|------------|---|

| | |
|----------------|--|
| In normal form | $D_t^k u = G(x, D_x^\alpha D_t^j u), \quad \alpha + j \leq k, j < k$ |
|----------------|--|

Tools

- Characteristic surfaces
- Method of characteristics
- Cauchy problems
- Power series

Characteristic Surfaces for Linear Operators

L linear differential operator.

Definizione 2.1

Characteristic form of L :

$$\chi_L(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha \quad \text{with} \quad x, \xi \in \mathbb{R}^n$$

Definizione 2.2

Characteristic variety of L at x :

$$\text{char}_x(L) = \{\xi \neq 0 : \chi_L(x, \xi) = 0\}$$

Definizione 2.3

Γ characteristic surface for L at $x \iff \nu(x) \in \text{char}_x(L)$

Osservazione

Case of 1st order operator: $A = (a_1, \dots, a_n)$ tangent to Γ .
Useful for further generalizations.

Meaning

$$\xi \in \text{char}_x(L)$$

at x L is not “properly” of order k in the direction ξ .

Γ non-characteristic

given $D_\nu^i u$ ($i < k$) of a solution u on Γ it is possible to calculate all its partial derivatives on Γ .

1st Order Quasi-linear Operators

- $\gamma(s) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ local parametrization of Γ
- $u = \phi$ on Γ Cauchy data

Definizione 2.4

Γ non-characteristic at $x_0 = \gamma(s_0)$

$$\iff \det \underbrace{\begin{bmatrix} D_{s_1} \gamma_1 & \cdots & D_{s_{n-1}} \gamma_1 \\ \vdots & & \vdots \\ D_{s_1} \gamma_n & \cdots & D_{s_{n-1}} \gamma_n \end{bmatrix}}_{\text{span of the tangent plane}} \begin{bmatrix} a_1(\gamma, \phi(\gamma)) \\ \vdots \\ a_n(\gamma, \phi(\gamma)) \end{bmatrix} (s_0) \neq 0$$

Method of Characteristics

The following problems¹ are **equivalent**.

$$PDE : \begin{cases} \sum a_j(x, u) D_{x_j} u = b(x, u) \\ u = \phi \text{ on } \Gamma \end{cases} \quad (1)$$

$$ODE : \begin{cases} D_t x = A(x, y) \text{ }^2 \\ D_t y = b(x, y) \\ x(0) = x_0 \\ y(0) = \phi(x_0) \quad \forall x_0 \in \Gamma \end{cases} \quad (2)$$

Where $y = u(x)$ and $A(x, y) = [a_1(x, y), \dots, a_n(x, y)]$.

¹it can be generalized to the nonlinear case (1st order!)

²the solutions x are called *characteristic curves* ◀ ◻ ▶ ◀ ◻ ▶ ◀ ≡ ▶ ◀ ≡ ▶ ≡ ↺ 🔍 ↻

Teorema 2.1

| | |
|-------|--|
| Hp | Problem (1) $a_j, b, \phi, \Gamma \in C^1$ Γ non-characteristic |
| Ts | $\exists!$ unique C^1 solution in a neighborhood of Γ |
| Proof | using the local existence and uniqueness theorem for ODEs |

Cauchy Problem

- Often used when the data surface is **not** a boundary.
- It also requires the **normal derivatives** ($D_\nu^j u$) of the solution on the surface to uniquely determine it.
- It carries the risk of being **overdetermined** (good for uniqueness but less for the existence of the solution).

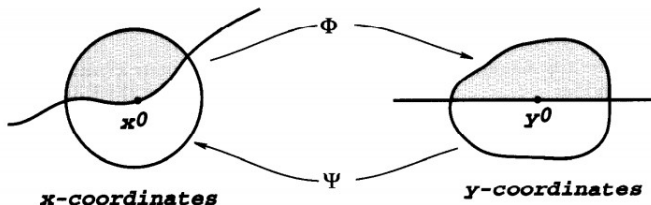
General Problem

$$\begin{cases} F^*(x, D^\alpha u^*) = 0 & |\alpha| \leq k, F^* \text{ at least } C^1 \\ D_\nu^j u^* = \phi_j^* & \text{on } \Gamma^* \text{ for } j < k \end{cases}$$

Mapping at $t = 0$

Let γ^* be the local parametrization of Γ^* , we apply the map:

$$\Phi(x) = [x_1 \quad \cdots \quad x_{n-1} \mid x_n - \gamma^*(x_1, \dots, x_{n-1})]$$



L. C. Evans, *Partial Differential Equations*

- 1 Select a privileged variable and call it “time”:

$$t \leftarrow x_n$$

$$x \leftarrow (x_1, \dots, x_{n-1})$$

- 2 Call $\Gamma_0 = \{t = 0\}$.
- 3 Indicate the derivatives as follows: $D_x^\alpha D_t^j u$.
- 4 Obtain the problem ($u^* = u(\Phi)$):

$$\begin{cases} F(x, t, D_x^\alpha D_t^j u) = 0 & |\alpha| + j \leq k \\ D_t^j u(x, 0) = \phi_j(x) & \text{for } j < k \end{cases}$$

Non-characteristic Surfaces in General

Definizione 2.5

Γ^* (or Γ_0) is non-characteristic \iff the equation on Γ_0 can be rewritten in **normal form** with respect to t .

Osservazione

It is shown to be consistent with previous definitions.

Osservazione

- Linear case \rightarrow condition on the coefficients.
- Non-linear case \rightarrow validity of implicit function theorem hypotheses on F .

Remarkable Power Series

Definizione 2.6

Majorizing function:

$$\mathcal{M}_{Cr}(x) = \frac{Cr}{r - (x_1 + \dots + x_n)}$$

Osservazione

By the multinomial theorem, if $|x| < r/n$ we have that

$$\frac{Cr}{r - (x_1 + \dots + x_n)} = C \sum_{\alpha} \frac{|\alpha|!}{\alpha! r^{|\alpha|}} x^{\alpha}.$$

Method of Majorants

Teorema 2.2 (utility of the majorant)

$$\begin{cases} g_\alpha \geq |f_\alpha| \\ \sum g_\alpha x^\alpha \text{ has a radius of conv. } R \end{cases} \implies \sum f_\alpha x^\alpha \text{ has a radius at least } R$$

In this case, we write: $\sum g_\alpha x^\alpha \gg \sum f_\alpha x^\alpha$.

Teorema 2.3 (construction of the majorant)

$\sum f_\alpha x^\alpha$ has radius $R \implies \exists r < R, C > 0$ such that

$$|f_\alpha| \leq C \frac{1}{r^{|\alpha|}} \leq C \frac{|\alpha|!}{\alpha! r^{|\alpha|}}$$

Outline of the Approach

Following the chronological order of discovery, we proceed by **progressive generalizations**:

- 1 ODEs
- 2 Quasi-linear PDEs
- 3 PDEs in normal form

ODE

Teorema 3.1

| | | |
|----|--|---|
| Hp | | $A \subseteq \mathbb{C}, B \subseteq \mathbb{C}^n$ open |
| | | $\Omega \subseteq A$ open, connected |
| | | $f : A \times B \rightarrow \mathbb{C}^n$ holomorphic |
| | | Pb: $\begin{cases} y' = f(x, y) & \forall x \in \Omega \\ y(x_0) = y_0 \end{cases}$ |
| Ts | | locally there exists a unique holomorphic solution |

Radius Estimate

Teorema 3.2

| | |
|----|--|
| Hp | <div>Assumptions of the previous theorem</div> <div>$\exists \overline{B_a(x_0)} \subseteq A, \overline{B_b(y_0)} \subseteq B$</div> |
| Ts | <div>The solution converges with at least radius³</div> <div>$\tilde{r} = a \left[1 - \exp \left(-\frac{b}{aM(n+1)} \right) \right]$</div> |

³ $M = \max_{B_a(x_0), B_b(y_0)} |f|$

Quasi-linear PDEs

Teorema 3.3

$$\begin{array}{l|l}
 \text{Hp} & \begin{array}{l} A_j, B \text{ analytic} \\ \text{Pb: } \begin{cases} D_t y = \sum_{j=1}^{n-1} A_j(x, y) D_{x_j} y + B(x, y) \\ y = 0 \quad \text{on } \Gamma_0 \end{cases} \end{array} \\
 \text{Ts} & \begin{array}{l} \exists! y(x, t) : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ analytic solution} \\ \text{in a neighborhood of the origin} \end{array}
 \end{array}$$

Proof

- 1 Assume $y_h = \sum c_h^{\alpha j} x^\alpha t^j$
- 2 Inserting the series of y, A_j, B we get:

$$c_h^{\alpha j} = Q_h^{\alpha j}(\text{coeff. of series of } A_j, B)$$

Q polynomial with non-negative coefficients

- 3 $\tilde{A}_i \gg A_i, \tilde{B} \gg B \implies \tilde{y} \gg y$ thanks to Q
- 4 Choose \tilde{A}_i, \tilde{B} so that \tilde{y} can be explicitly calculated as analytic with the method of characteristics

Majorizing System

As we already know, we majorize the series with

$$\mathcal{M}_{Cr}(x, y) \gg A_j, B$$

and solve the problem⁴:

$$\begin{cases} D_t \tilde{y}_h = \mathcal{M}_{Cr} \left[\sum_{i,j} D_{x_j} \tilde{y}_i + 1 \right] \\ \tilde{y}_h = 0 \quad \text{on } \Gamma_0 \end{cases}$$

⁴with $h = 1, \dots, m$

Majorant Solution

The previous system has the solution:

$$\tilde{y}_h(x, t) = u(x_1 + \cdots + x_n, t) \quad \forall h$$

with

$$u(s, t) = \frac{r - s - \sqrt{(r - s)^2 - 2tCr mn}}{mn},$$

whose radius of convergence we can study.

Radius of Convergence Estimate

Teorema 3.4

The solution of theorem 3.3 converges with radius at least

$$\tilde{r} = \frac{1}{n-1} \frac{r}{8Cmn} \text{ with } C \geq \frac{1}{2}$$

Let's observe its behavior⁵ with respect to r , knowing that:

$$r < \min\{\text{radii of conv. of the coefficients } a_{ml}^j, b_m\}$$

$$C \geq \max \left\{ \begin{array}{l} \max_{j,m,l,\alpha} |a_{ml}^j r^{|\alpha|}| \\ \max_{m,\alpha} |b_m r^{|\alpha|}| \end{array} \right\}$$

⁵trade-off Cr

PDE in Normal Form

Teorema 3.5

The following two problems are equivalent

$$\text{nonlinear : } \begin{cases} D_t^k u = G(x, D_x^\alpha D_t^j u) & |\alpha| + j \leq k, j < k \\ D_t^j u = \phi_j & \text{on } \Gamma_0, j < k \end{cases}$$

$$\text{quasi-linear : } \begin{cases} D_t y = \sum_{j=1}^{n-1} A_j(x, y) D_{x_j} y + B(x, y) \\ y = 0 & \text{on } \Gamma_0 \end{cases}$$

Proof

- 1 The system is constructed so that $y_{\alpha j} = D_x^\alpha D_t^j u$

The matrices A_j and B will then be derived from the expressions⁶:

$$D_t y_{\alpha j} = y_{\alpha(j+1)} \quad |\alpha| + j < k$$

$$D_t y_{\alpha j} = D_{x_i} y_{(\alpha-1_i)(j+1)} \quad |\alpha| + j = k, j < k$$

$$\begin{aligned} D_t y_{0k} = & D_t G + \sum_{|\alpha|+j < k} D_{y_{\alpha j}} G y_{\alpha(j+1)} \\ & + \sum_{|\alpha|+j=k, j < k} D_{y_{\alpha j}} G D_{x_i} y_{(\alpha-1_i)(j+1)} \end{aligned}$$

The Cauchy data will be:

$$y_{\alpha j}(x, 0) = D_x^\alpha \phi_j(x) \quad j < k$$

$$y_{0k}(x, 0) = G(x, 0, D_x^\alpha \phi_j(x)) \quad |\alpha| + j \leq k, j < k$$

⁶ $i(\alpha) = \min\{i : \alpha \neq 0\}$

- 2 removing $\phi : y(x, t) \leftarrow y(x, t) - \phi(x)$
- 3 removing t : the variable $y^0 = t$ is added (with its corresponding equation)

Holomorphic Version

As in the case of ODEs, everything extends in an **immediate** way to the complex case by assuming holomorphic data.

Examples

We now answer the questions with three examples:

- Lewy's example: importance of analyticity
- Kowalevski's example: importance of non-characteristicity
- Hadamard's example: the problem might not be well-posed

Lewy's Example

Definizione 4.1

$$\mathcal{L} = D_x + iD_y - 2i(x + iy)D_t$$

is called Lewy's operator.

Teorema 4.1

Hp

f continuous real-valued function
depending only on t
 $u \in C^1 : \mathcal{L}u = f$ in a neighborhood of the origin

Ts

f analytic in a neighborhood of $t = 0$

Proof

Schwarz reflection principle

The previous statement can be generalized as follows:

Teorema 4.2

Hp

 $A \subseteq \mathbb{R}^3$ open

Ts

 $\exists F \in C^\infty(\mathbb{R}^3, \mathbb{R}) : \nexists u \in C^1(A, \mathbb{R})$

such that $\begin{cases} \mathcal{L}u = F \text{ in } A \\ u_x, u_y, u_t \text{ satisfy} \\ \text{the Hölder condition} \end{cases}$

Proof

- 1 Translate the problem of the previous theorem so as to reduce it to the case of a generic point (x_0, y_0, t_0) , using the function $g(x, y, t) = f(t - 2xy_0 + 2x_0y)$ as the forcing function.
- 2 Construct a function $S_a \in C^\infty$ for each $a \in l^\infty$ using a series.
- 3 Construct closed sets $E_{j,n} \subseteq l^\infty$ with no interior using S_a and the Ascoli-Arzelà theorem.
- 4 Conclude the proof of the new theorem using the aforementioned lemmas to derive, by a contradiction argument, the equality $l^\infty = \bigcup E_{j,n}$, allowing the application of Baire's argument.

Kowalevski's Example

This problem admits no analytic solutions⁷ in a neighborhood of the origin:

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = \frac{1}{1+x^2} \quad \forall x \in \mathbb{R} \end{cases}$$

Osservazione

The surface is characteristic!

⁷proof by contradiction

Hadamard's Example

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x, 0) = 0 \\ u_y(x, 0) = n \sin(nx) e^{-\sqrt{n}} \text{ with } n \in \mathbb{N} \end{cases}$$

The solution to this problem is:

$$u_n(x, y) = \sin(nx) \underbrace{\sinh(ny) e^{-\sqrt{n}}}_{\xrightarrow{n \rightarrow \infty} \infty}$$

Alternative Versions

Abstract Version
(*Ovsyannikov classes*)



Classical Version
(*similar to local existence and uniqueness for ODEs*)



Invariant Version
(*non-characteristic surfaces*)

Classical Version

Teorema 5.1

| | | |
|----|--|--|
| Hp | | $\overline{\mathcal{O}}_0 \subseteq \mathcal{O}_1 \subseteq \mathbb{C}^n$ open connected bounded |
| | | A_j, f, y_0 holomorphic wrt z |
| Ts | | A_j, f continuous wrt t |
| | | Pb: $\begin{cases} D_t y = \sum A_j(z, t) D_{z_j} y + A_0(z, t) y + f(z, t) \\ y(z, 0) = y_0(z) \end{cases}$ |
| Ts | | $\exists \delta \in (0, T) : \exists ! y$ solution when $ t < T$ |
| | | – holomorphic wrt z |
| | | – C^1 wrt $t \quad \rightarrow (\neq \text{Holmgren})$ |

Consequences

The consequences of this theorem can be observed in various fields, including the main ones:

- theory of differential equations
- mathematical physics: emergence of numerous questions (what happens in reality if a local analytic solution exists?)
- differential geometry
- economic theory

Impact on the theory of differential equations:

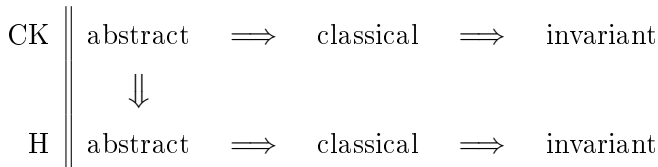
- refuting Weierstrass's conjecture
- Holmgren's theorem
- research on necessary and/or sufficient conditions for the existence of local solutions by Treves and Nirenberg
- Hörmander's theory of linear differential operators

Holmgren's Theorem

Result of **uniqueness** of solutions for linear PDEs.

Osservazione

Cauchy-Kowalevski theorem does not exclude the existence of other solutions that are not analytic!



Abstract Version

Any linear equation can be reduced to a **first-order system**.
We focus on this case.

Teorema 6.1

$$\begin{array}{l|l}
 \text{Hp} & \begin{array}{l} y \text{ distribution on } (\mathcal{O}_0 \cap \mathbb{R}^n) \times (-T, T) : \\ -K \subseteq \mathcal{O}_0 \cap \mathbb{R}^n \text{ compact: } y = 0 \text{ in } \mathcal{O}_0 \cap \mathbb{R}^n \setminus K \\ - \begin{cases} D_t y = \sum A_j(z, t) D_{z_j} y + A_0(z, t) y \\ y = 0 \text{ for } t < 0 \end{cases} \end{array} \\
 \text{Ts} & y = 0 \text{ in } (\mathcal{O}_0 \cap \mathbb{R}^n) \times (-T, T)
 \end{array}$$

Classical Version

Teorema 6.2

$$\begin{array}{l|l}
 \text{Hp} & \begin{array}{l} \Omega \subseteq \mathbb{R}^n \text{ open} \\ A_j \text{ analytic} \\ y \in C^1(\Omega \times (-T, T)) : \\ \begin{cases} D_t y = \sum A_j(x, t) D_{x_j} y + A_0(x, t) y \\ y = 0 \text{ for } t = 0 \end{cases} \end{array} \\
 \text{Ts} & y = 0 \text{ in a neighborhood of } \Omega \times \{0\}
 \end{array}$$

Proof

It is an application of the abstract version to the function

$$\tilde{y}(x, t) = H(t) y(x, t),$$

which always satisfies a system of the same type.

Cartan-Kähler Theorem

A very important theorem in differential geometry:

- on the integrability of **exterior differential systems**
- which is proved using the Cauchy-Kowalevski theorem
- which has an application in the economic field (I. Ekeland, P.A. Chiappori)

Quoting Ekeland regarding the paper written in 1999 with Chiappori:

*This paper solves a basic problem in economic theory, which had remained open for **thirty years**, namely the characterization of market demand functions. The method of proof consists of reducing the problem to a system of nonlinear PDEs, for which convex solutions are sought. This is rewritten as an exterior differential system, and is solved by the Cartan-Kähler theorem, together with some algebraic manipulations to achieve **convexity**.*

Despite the research conducted in those years

- wasn't guided by immediate applications
- led to **disappointing** results compared to the expectations of Cauchy and Weierstrass

it has had a gigantic impact thanks to the understanding of solutions of PDE systems it allowed us to achieve.

In conclusion, a quote about the relationship between Weierstrass and Kowalevski:

All his life – he had difficulty saying this, as he admitted, being always wary of too much enthusiasm – all his life he had been waiting for such a student to come into this room. A student who would challenge him completely, who was not only capable of following the strivings of his own mind but perhaps of flying beyond them.

— Alice Munro, *Too Much Happiness*