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Traffic Theory - 054657

Homework Markov Processes

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1 The limit of $\lim_{n\to\infty} P^n$

We start by analyze the limit of the transition probability matrix P2 to check if there is a convergence where P2 is given by:

$$P_2 = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.1 & 0.7 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \tag{1}$$

$$lim_{n\to\infty} P_2^n = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.3571 & 0.6429 \\ 0 & 0 & 0 & 0.3571 & 0.6429 \end{bmatrix} \tag{2}$$

As we can see, the limit of the transition probability matrix P2 exists which means we have a limiting distribution. Therefore, we can compute the limiting distribution for different initial state distribution $\pi(0)$. Furthermore, we can observe for (2) that the transient probabilities to transient states are null. For states in the same communicating class, we know the rows of those same communicating class are equal. However since the rows aren't the same, we can conclude there is no steady-state distribution.

2 The limiting distributions starting from different initial state distributions

Here is a proposition of different initial state distributions:

$$\begin{aligned} \pi(0)_1 = \begin{bmatrix} 0.1 & 0.3 & 0.4 & 0.2 & 0 \end{bmatrix} \quad \pi(0)_2 = \begin{bmatrix} 0 & 0.7 & 0.1 & 0.2 & 0 \end{bmatrix} \\ \pi(0)_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \pi(0)_4 = \begin{bmatrix} 0.4 & 0.4 & 0 & 0.2 & 0 \end{bmatrix} \end{aligned} \tag{3}$$

In order to check if the initial state distributions represent a probability distribution, we can sum all the elements of the row and check if it's equal to 1. In our case, they represent a probability distribution. Let's analyze the limiting distributions $\pi = lim_{n\to\infty}\ \pi(n) = lim_{n\to\infty}\ \pi(0)P^n$ starting from different initial state distributions given here below:

$$\begin{aligned} \pi_1 = \begin{bmatrix} 0 & 0 & 0.8 & 0.0714 & 0.1286 \end{bmatrix} \quad \pi_2 = \begin{bmatrix} 0 & 0 & 0.8 & 0.0714 & 0.1286 \end{bmatrix} \\ \pi_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \pi_4 = \begin{bmatrix} 0 & 0 & 0.8 & 0.0714 & 0.1286 \end{bmatrix} \end{aligned} \tag{4}$$

Looking at the limiting distributions given by (4), we can see that not every initial state probability distribution evolves to the same state probability distribution.

3 Does the steady-state distribution exist ?

No, there is not a steady state distribution because the rows of matrix P2 to its limit aren't the same and we can clearly see that there is a dependence on the initial distribution $\pi(0)$ while computing the limiting distribution.

4 Computation of the Stationary distribution

Since our matrix P2 is reducible, it means it has multiple stationary distributions. Indeed, any linear combination of a stationary distribution that we can find is itself a stationary distribution which makes a lot of stationary distributions.

4.1 Using GBE

It's impossible to compute the stationary distribution by solving the **Global Balance Equation** (GBE) because when adding the normalization equation and trying to solve the new linear system given by: $A_n \pi^T = b_n$ in Matlab, the matrix we want to compute π proportional to the inverse of matrix A_n has determinant $det(A_n) = 0$. A_n is thus a singular matrix and hence the inverse of that matrix does not exist.

4.2 Uniqueness of the stationary distribution

Sincere there are infinitely many choices, we have infinite number of stationary distributions. Any linear combination of a stationary distribution that we can find is itself a stationary distribution. A function has been made in Matlab in order to compute a stationary distribution and verify it.