Electronic
Formulas and utilities

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Problem 6. a) Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy

$$|C_k| \leq \frac{M}{R^k}.$$

b) Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

Solution Part a): Since f is an entire function it can be expressed as an infinite power series, i.e.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} C_k z^k.$$

If we recall Cauchy's Integral we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw,$$

carefully notice that $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}}$ can be written as a geometric series. We have

$$\begin{aligned}
\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw &= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(\frac{1}{1-\frac{z}{w}} \right) \right\} dw \\
&= \frac{1}{2\pi i} \int_\gamma \left\{ \frac{f(w)}{w} \cdot \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \frac{z^3}{w^3} + \cdots \right) \right\} dw \\
&= \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w}\, dw \right) z^0 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^2}\, dw \right) z^1 + \left(\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^3}\, dw \right) z^2 \cdots
\end{aligned}$$

Now take the modulus of C_k to get

$$|C_k| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w^{k+1}}\, dw \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|f(w)|}{|w^{k+1}|}\, |dw| \leq \frac{M}{2\pi} \int_\gamma \frac{|dw|}{|w^{k+1}|}$$

Then integrate along $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, 2\pi]$ to get

$$|C_k| \leq \frac{M}{2\pi} \int_0^{2\pi} \frac{|iRe^{i\theta}\ d\theta|}{|R^{k+1}e^{ik\theta}|} = \frac{M}{2\pi \cdot R^k} \int_0^{2\pi}\ d\theta = \frac{M}{R^k}.$$

Hence, $|C_k| \leq \frac{M}{R^k}$.