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Contextual Background

1 Convex Constrained Optimisation

Convex constrained optimisation is a subproblem of the general theory of optimisation which is concerned with problems of the following form:

$$
\begin{aligned}
\text{Find the } \min_x \ & f_0(x) \\
\text{such that } & f_1(x) \leq 0 \\
& \vdots \\
& f_n(x) \leq 0 \\
& g_0(x) = 0 \\
& \vdots \\
& g_m(x) = 0
\end{aligned}
$$

Where $f_0, \cdots, f_n$ are convex functions $\mathbb{R}^p \to \mathbb{R}$ and $g_0, \cdots, g_m$ are affine functions $\mathbb{R}^p \to \mathbb{R}$.

Convex constrained optimisation problems are ubiquitous in engineering, economics and operations. Optimisation problems which are convex include linear programming, quadratic programming and semidefinite programming. Linear programming is a convex optimisation problem where f_0 through f_n and g_0 through g_n are linear. Linear programming problems range from portfolio optimisation through scheduling and planning and to statistical inference. find citation find citation

1.1 Example

A statistical example of a constrained convex optimisation problem is that of ℓ_1-norm regularised logsitic regression: Given some data $A = [a_1, \ldots, a_m]$ and the associated vector

$$\underset{v,w}{\text{minimise}} \ (1/m) \sum_{i=1}^{m} f(w^T a_i + v b_i) + \lambda \sum_{i=1}^{n} |w_i|$$

Where f is the logistic loss function. When the regularisation is incorporated into the objective as normal the objective is non-differentiable and therefore subgradient or ellipsoid methods must be used. However the regularisation can be factored out of the objective as a constraint as so:

$$
\begin{aligned}
\underset{u,v,w}{\text{minimise}} \ & (1/m) \sum_{i=1}^{m} f(w^T a_i + v b_i) \\
\text{such that } & -u_i \leq w_i \leq u_i, \ \ i = 1, ..n
\end{aligned}
$$

for which a convex constrained optimisation algorithm can be used.

1.2 Summary of Methods

A variety of methods exist to solve these problems. Prior to the mid-20$^{\text{th}}$ century inequality constraints had been largely ignored [1]. The first treatment of inequality constraints was by Kantorovich in 1939 who formalised linear programming problems in an attempt to solve an eningeering problem [2]. The next big step in constrained optimisation was obtained by Dantzig in 1947 through his simplex method [3]. The simplex method functions by observing that the feasible set for a linear programming problem constitues a convex polytope for which the vertices are basic feasible solutions. For a linear programming problem expressed in normal form if the objective function attains its maximum value in the feasible set then it attains this maximum value at its vertices. The simplex algorithm traverses from vertex to vertex along edges which increase/decrease the objective to find this maximum.

The next jump in solving convex constrained optimisation was the ellipsoid method first proposed by Yudin and Nemirovski in 1974 [4] and applied to linear programming by Khachiyan in 1979 [5].The significance of this method was that it had worst-case polynomial complexity for linear programming problems could be solved in polynomial time. The ellipsoid method functions by fitting an ellipsoid to the feasible set and using an oracle to find a cut through the center of the ellipsoid such that the optimal value is contained within one side. The ellipsoid of minimum volume covering the half of the ellipsoid is then fitted with this repeated with the center of the ellipsoid tending to the optimal value. Inequality constraints can be incorporated by making cuts to ensure the ellipsoid remains in the feasible set. The ellipsoid method suffers from numerical instability and poor performance in practice.

Non-linear methods were, during largely developed independently of linear programming methods during this period. In 1968 Fiacco and McCormick gave the founding principles of barrier methods. Barrier methods incorporate inequality constraints for general convex constrained problems into the objective as described in the lecture slides [6]. These methods proved to be non-optimal in comparison to existing methods at the time. This was the begining of interior point methods for optimisation which, rather than traversing the boundary of the fesaible set as with older methods, traversed the interior of the feasible set to the optimal value.

A practically efficient polynomial time interior point method for solving linear programming problems was invented in 1984 by Karmarkar [7]. Karmarkar's algorithm was later shown by Gill et al. to be particular case of projected Newton's method with a logarithmic barrier function. [8].

The currently preferred interior point method for solving convex constrained problems are primal-dual methods first proposed by Megiddo in 1989 [9]. Primal dual methods take newton steps and update both primal and dual variables to more quickly converge to minima at high accuracy.

References

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