Convex Optimisation

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Convex Optimisation?

- What is it?
  - Finding the maxima / minima of convex functions over convex sets with respect to convex or affine constraints
- Why do we care?
  - Convex functions display many theoretical properties that are suited to optimisation
  - Solving convex optimisation allows you to place a lower bound on non-convex functions
  - Many real world optimisations are convex

Convexity

A set $C \subseteq \mathbb{R}$ is convex if $\forall\, x, y \in C$, $\forall\, \theta \in [0, 1]$

$$\theta x + (1 - \theta) y \in C$$

i.e. all points on the line segment between x and y lie in C

add affine definition

Convex Functions

A function $f : C \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is convex on some convex set C if $\forall\, x,\, y \in C, \forall\, \lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq f(\lambda x) + f((1-\lambda)y)$$

add picture or better explanation and affine function definition, concave = convex -f

Minima and Maxima

$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ has a global minimum (maximum resp.) $x^\star$ if:

$$\forall\, x \in S,\ f(x^\star) \leq f(x),\ \ (f(x^\star) \geq f(x))$$

f has a local minimum (maximum resp.) around $x^\star$ if $\exists\, R \in \mathbb{R}$ such that

$$\forall x\ \in B(x^\star, R),\ f(x^\star) \leq f(x),\ \ (f(x^\star) \geq f(x))$$

add strict definitions

Convex Optimisation Problem

A convex optimisation problem is a min/maximisation problem in the following form:

$$\begin{aligned} \text{Find the } \min_x \; & f_0(x) \\ \text{such that } & f_1(x) \leq 0 \\ & \vdots \\ & f_n(x) \leq 0 \\ & g_0(x) = 0 \\ & \vdots \\ & g_m(x) = 0 \end{aligned}$$

Where $f_0, \cdots, f_n$ are convex functions and $g_0, \cdots, g_m$ are affine

Feasible Set

The feasible set $C \subseteq D = \bigcap_{i=0}^{m} \mathrm{dom}(f_i) \bigcap_{j=0}^{n} \mathrm{dom}(g_j)$ is the set of all $x \in D$ such that the constraints are satisfied.

The optimal value of problem is $\inf\{f_0(x) \,|\, x \in C\}$. If there is no lower bound the optimisation problem has optimal value $-\infty$.

Convexity and Optimality

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