Problem 24
Part 2

December 31, 2023

Assume the rock has starting position (x, y, z) and velocities (u, v, s). Denote the hailstones positions and velocities as (x_i, y_i, z_i) and (u_i, v_i, s_i) for $i = 1 \dots N$ where N is the number of hailstones. The problem states that the rock's trajectory intersects the hailstones' trajectories at some times $t_1 \dots t_N$ such that the intersection points all belong to the same line.
Note that the intersection times and the rock's position and velocity are unknowns.

For each hailstone i it holds that:

$$\begin{aligned} x + ut_i &= x_i + u_i t_i \\ y + vt_i &= y_i + v_i t_i \\ z + st_i &= z_i + s_i t_i \end{aligned}$$

We have that:

$$t_i = -\frac{x - x_i}{u - u_i} = -\frac{y - y_i}{v - v_i} = -\frac{z - z_i}{s - s_i}$$

The last equation allows us to remove the unknowns t_i and establish relationships for each plane xy, xz and yz. Starting with the xy plane, we have for each hailstone:

$$-\frac{x - x_i}{u - u_i} = -\frac{y - y_i}{v - v_i}$$

We can refactor it as:

$$xv - yu = xv_i - yu_i - y_i u + x_i v + (y_i u_i - x_i v_i)$$

Note that the LHS of the equation is constant w.r.t. i. This means that, given another hailstone j:

$$0 = x(v_i - v_j) - y(u_i - u_j) - (y_i - y_j)u + (x_i - x_j)v + (y_i u_i - x_i v_i - y_j u_j + x_j v_j)$$

This allows us to reframe the problem as a linear system of equations. Without loss of generality, assume $j = i + 1$ and denote the differentials as $\Delta x_i \equiv x_{i+1} - x_i$ and so on. We can establish the following linear system of four unknowns:

$$\begin{bmatrix} -\Delta v_1 & \Delta u_1 & \Delta y_1 & -\Delta x_1 \\ -\Delta v_2 & \Delta u_2 & \Delta y_2 & -\Delta x_2 \\ -\Delta v_3 & \Delta u_3 & \Delta y_3 & -\Delta x_3 \\ -\Delta v_4 & \Delta u_4 & \Delta y_4 & -\Delta x_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

with $b_i = y_{i+1}u_{i+1} - x_{i+1}v_{i+1} - y_i u_i + x_i v_i$.
Note that 5 hailstones are required to solve the system (since we are taking differentials, they provide 4 equations). If more than 5 hailstones are present, the system is overdetermined.
The system can be solved by any conventional solver or method to yield the unique solution (x, y, u, v).

We have similar linear systems for the other two remaining planes, xz and yz. The former system will yield the solution (x', z, u', s) and the latter will yield the solution (y', z', v', s').
Theoretically $x' = x$ and so on, but they can have slight differences due to numerical errors within the solver. To improve accuracy, we can present the problem solution as $R = \frac{1}{2}(x + x' + y + y' + z + z')$.

The example data only has 5 hailstones, therefore there is only one set of linear systems. To further improve accuracy for the input data, we can make use of all the extra hailstones and solve a set of linear systems for any block of 5 consecutive hailstones. Each block will yield a solution R_k with $k = 1 \dots M$ where M is the number of possible 5-long blocks of hailstones.
A simple average $R = \langle R_k \rangle$ provides the correct result. The standard deviation of the distribution $\{R_k\}$ for my input is $\sigma = 1.78$.