Integrali

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Primitive delle funzioni elementari

$$\int dx = x + c$$

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} \quad \forall \alpha \in \mathbb{R} \setminus \{1\}$$

$$\int \frac{1}{x} dx = \log|x| + c$$

$$\int \sin(x) dx = -\cos(x) + c$$

$$\int \cos(x) dx = \sin(x) + c$$

$$\int \frac{1}{\cos^2(x)} dx = tg(x) + c$$

$$\int \frac{1}{\sin^2(x)} dx = cotg(x) + c$$

$$\int tg\, x \, dx = -\log(|\cos(x)|) + c$$

$$\int cotg\, x \, dx = -\log|\sin(x)| + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + c$$

$$\int \frac{1}{1+x^2} dx = arctg\, x + c$$

$$\int e^x = e^x + c$$

$$\int a^x = \frac{1}{\log(a)} a^x + c \quad a>0, a \neq 1$$

Formula di integrazione per parti

$$\int_a^b f g' \, dx = f \cdot g \Big|_a^b - \int_a^b f' g$$

Formule di sostituzione

$\varphi : [a,b] \to [\alpha, \beta]$ $\qquad$ $f : [\alpha, \beta] \to \mathbb{R}$ $\qquad$ φ invertibile

$$\int_a^b f(\varphi(t))\varphi'(t) \, dt = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(s) \, ds$$

Integrali coi fratti semplici

$$\int \frac{P(x)}{Q(x)}\,dx \qquad P(x) \quad Q(x) \quad \text{polinomi}$$

supponiamo che
grado(P) $<$ grado(Q)

(Altrimenti so che $\exists$ $S(x)$ $R(x)$: $P(x) = Q(x)\cdot S(x) + R(x)$
e grado(R) $<$ grado Q)

$\frac{P(x)}{Q(x)}$ Scomponiamo $Q(x)$:

$$Q(x) = (x-a_1)(x-a_2)\dots(x-a_m)(x^2+b_1x+c_1)\dots(x^2+b_nx+c_n)$$

Per essere scomposto ai minimi termini i polinomi di secondo grado hanno $\Delta < 0$

$$\Rightarrow \frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \dots + \frac{A_m}{(x-a_m)} + \frac{B_1x+C_1}{(x^2+b_1x+c_1)} + \dots + \frac{B_nx+C_n}{(x^2+b_nx+c_n)}$$

Per determinare le costanti si fa il mcm al II membro e si sfrutta il Principio di Identità dei polinomi

se $a_0 + a_1x + \dots + a_mx^m = b_0 + b_1x + \dots + b_mx^m \quad \forall x \in \mathbb{R}$

$$\Rightarrow a_i = b_i \quad \forall\, i = 0\dots m$$

Esercizi : Integrali per sostituzione :

1) $\int (4-5x)^3\,dx$

$4-5x = y \Rightarrow -5x = y-4 \Rightarrow x = \frac{4-y}{5} \qquad dx = -\frac{1}{5}dy$

$$\int (4-5x)^3\,dx = -\frac{1}{5}\int y^3\,dy = -\frac{1}{5}\frac{y^4}{4} + c = -\frac{1}{20}(4-5x)^4 + c$$

2) $\int \frac{2}{\sqrt{1-4x^2}}\,dx \qquad 2x = y \qquad 2dx = dy$

$$\int \frac{2}{\sqrt{1-4x^2}} = \int \frac{dy}{\sqrt{1-y^2}} = \operatorname{arctg}(y) + c = \operatorname{arctg}(2x) + c$$

3) $\int \frac{\log x}{x} \, dx$ $\qquad$ $\log(x) = y \Rightarrow \frac{1}{x} dx = dy$

$$\int \frac{\log x}{x} \, dx = \int y \, dy = \frac{y^2}{2} + c = \frac{(\log(x))^2}{2} + c$$

4) $\int \cos(x) \sin(x) \, dx$ $\qquad$ $\sin(x) = y$ $\qquad$ $\cos(x) \, dx = dy$

$$\int y \, dy = \frac{y^2}{2} + c = \frac{(\sin(x))^2}{2}$$

5) $\int \frac{(1 + \sin x)}{(x - \cos x)^3} \, dx$ $\qquad$ $x - \cos x = y$, $\quad (1 + \sin(x)) dx = dy$

$$\int \frac{dy}{y^3} = -\frac{1}{2} \frac{1}{y^2} + c = -\frac{1}{2} \frac{1}{(x - \cos x)^2}$$

6) $\int \frac{x + \operatorname{arctg}(x)}{1 + x^2} \, dx = \underbrace{\int \frac{x}{1+x^2} \, dx}_{1} + \underbrace{\int \frac{\operatorname{arctg}(x)}{1+x^2}}_{2}$

1 $\int \frac{x}{1+x^2} \, dx$ $\qquad$ $x^2 = y$ $\qquad$ $2x \, dx = dy$

$$\frac{1}{2} \int \frac{2x}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{1+y} \, dy = \frac{1}{2} \log|1+y| + c$$

$$= \frac{1}{2} \log(1 + x^2) + c$$

Levo le | | perchè è positivo

2) $\int \frac{\operatorname{arctg} x}{1 + x^2}$ $\qquad$ $\operatorname{arctg} x = y$, $\quad \frac{1}{1+x^2} dx = dy$

$$\int y \, dy = \frac{y^2}{2} + c = \frac{(\operatorname{arctg}(x))^2}{2} + c$$

$$\Rightarrow \int \frac{x + \operatorname{arctg} x}{1+x^2} = \frac{1}{2}\left(\log(1+x^2) + (\operatorname{arctg}(x))^2\right) + c$$

7) $\int \frac{\cos x}{\sin^2 x + 1}\, dx$ $\quad \sin x = y \quad \cos(x)\,dx = dy$

$$\overset{\shortparallel}{\int \frac{dy}{1+y^2}} = \operatorname{arctg}(y) + c = \operatorname{arctg}(\cos(x)) + c$$

8) $\int \frac{\sqrt{\log x}}{x}\, dx$ $\quad \log x = y \quad \frac{1}{x}\,dx = dy$

$$\overset{\shortparallel}{\int \sqrt{y}\, dy} = \frac{y^{3/2}}{3/2} + c = \frac{2}{3}(\sqrt{y})^3 + c$$

$$= \frac{2}{3}\left(\sqrt{\log x}\right)^3 + c$$

9) $\int \frac{\cos(1+\sqrt{x})}{\sqrt{x}}\, dx$ $\quad 1+\sqrt{x} = y \quad \frac{1}{\sqrt{x}}\,dx = y$

$$\int \cos(y)\, dy = \sin(y) + c = \sin(1+\sqrt{x})$$

10) $\int \log(1-\sqrt{x})\, dx$ $\quad 1-\sqrt{x} = y \quad \sqrt{x} = 1-y$

$x = (1-y)^2$

$dx = 2(1-y)\,dy$

$$\overset{\shortparallel}{\int 2\log(y)(1-y)}$$

$\hookrightarrow$ integro per parti $\quad y = g' \quad \log(y) = f$

$$\overset{\shortparallel}{2\int 1\cdot\log(y)} - 2\int y\log(y) =$$

$\hookrightarrow$ e uso integrazione per parti con $1 = g'$ e $\log(y) = f$

$$= 2\left(y\log(y) - \int y\frac{1}{y}\right) - 2\left(\frac{y^2}{2}\log(y) - \int \frac{y^2}{2}\frac{1}{y}\right)$$

$$= 2\left(y\log(y) - y + c\right) - 2\left(\frac{y^2}{2}\log(y) - \frac{y^2}{4} + c\right)$$

$$\Rightarrow \int \log(1-\sqrt{x})\,dx = 2(1-\sqrt{x})\left(\log(1-\sqrt{x}) - 1\right) - (1-\sqrt{x})^2\log(1-\sqrt{x})$$
$$+ \frac{(1-\sqrt{x})^2}{2} + c$$

In generale, possiamo ricordarci le seguenti formule di sostituzione

X $\int R(a^x)\,dx$ (R è una funzione razionale nel seguito)

si pone $a^x = t$ $\quad x = \log_a t$

ES $\int \frac{e^x}{e^{2x}-3e^x+2}\,dx \longrightarrow \int \frac{dt}{t^2-3t+2}$

X $\int R(x, \sqrt[n]{ax+b})\,dx$

si pone $\sqrt[n]{ax+b} = t$ $\quad x = \frac{t^n - b}{a}$

ES $\int \frac{\sqrt{x+1}}{x+\sqrt{x+1}}\,dx$ $\quad t = \sqrt{x+1}$ $\quad x = t^2-1$ $\quad dx = 2t\,dt$

$\downarrow$

$\int \frac{2t^2}{t^2-1+t}$

X $\int R(x, \sqrt[n_1]{ax+b}, \sqrt[n_2]{ax+b}, \ldots \sqrt[n_k]{ax+b})$ $\quad n = m.c.m(n_1, n_2 \ldots n_k)$

$t = \sqrt[n]{ax+b}$

ES $\int \frac{\sqrt[3]{x}}{\sqrt{x}+1}$ $\quad \sqrt[6]{x} = t$ $\quad x = t^6$ $\quad dx = 6t^5$

$\downarrow$

$\int \frac{6t^7}{t^3+1}$

X $\int R(x, \sqrt{ax^2+bx+c})$ $\quad \Delta = \sqrt{b^2-4ac} \neq 0$ $\quad a \neq 0$

X $\int R\left(x, \sqrt{ax^2+bx+c}\right)$ $\qquad \Delta = \sqrt{b^2-4ac} \neq 0 \quad a \neq 0$

Si pone $\sqrt{ax^2+bx+c} = t - \sqrt{a}\, x$

ES $\int \frac{1}{\sqrt{x^2+1}}\, dx$

$$\sqrt{x^2+1} = t - x$$

$$\cancel{x^2}+1 = t^2 + \cancel{x^2} - 2xt$$

$$x = \frac{t^2-1}{2t} \qquad dx = \frac{2t^2+2}{4t^2}$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+1}}\, dx = \int \frac{1}{t - \frac{t^2-1}{2t}} \cdot \frac{2t^2+2}{4t^2}$$

X $\int R(\sin(x), \cos(x))\, dx$

$$t = tg\left(\frac{x}{2}\right) \qquad \sin(x) = \frac{2t}{t^2+1}$$

$$\cos(x) = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\, dt$$

ES $\int \frac{tg(x)}{\sin(x)+tg(x)}\, dx$ $\qquad tg(x) = \frac{2t}{1-t^2}$

$\downarrow$

$$\int \frac{2t}{1-t^2} \cdot \frac{1}{\frac{2t}{t^2+1} + \frac{2t}{1-t^2}} \cdot \frac{2}{1+t^2}\, dt$$

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INTEGRALI DI FUNZIONI RAZIONALI

1) $\int \frac{5}{x^2+5x-6}\,dx$

$x^2+5x-6=0$
$\Delta = 25+24=49 \Rightarrow x_{1,2} = \frac{-5\pm 7}{2} = \begin{cases} 1 \\ -6 \end{cases}$

TRAMITE I FRATTI SEMPLICI:

$$\frac{5}{x^2+5x-6} = \frac{5}{(x-1)(x+6)} = \frac{A}{x-1} + \frac{B}{x+6} = \frac{Ax+6A+Bx-B}{(x-1)(x+6)}$$

$$= \frac{x(A+B)+6A-B}{(x-1)(x+6)}$$

PER IL PRINCIPIO DI IDENTITÀ DEI POLINOMI:

$$\begin{cases} A+B=0 \\ 6A-B=5 \end{cases} \Rightarrow \begin{cases} A=-B \\ 6A+A=5 \end{cases} \Rightarrow \begin{cases} B=-\frac{5}{7} \\ A=\frac{5}{7} \end{cases}$$

SEGUE CHE

$$\int \frac{5}{x^2+5x-6}\,dx = \frac{5}{7}\int \frac{dx}{x-1} - \frac{5}{7}\int \frac{dx}{x+6} = \frac{5}{7}\log|x-1| - \frac{5}{7}\log|x+6|+c$$

$$= \frac{5}{7}\log\left|\frac{x-1}{x+6}\right| + c$$

2) $\int \frac{x+1}{x^3+1}$

$P(x)=x^3+1=0 \Rightarrow x^3=-1 \Rightarrow x=-1$

(IL POLINOMIO HA UNA SOLA RADICE REALE)

$$\begin{array}{rrrr|l} x^3 & +0 & +0 & +1 & x+1 \\ x^3 & +x^2 & & & x^2-x+1 \\ \hline // & -x^2 & +0 & +1 & \\ & -x^2 & -x & & \\ \hline & // & x & +1 & \\ & & x & +1 & \\ \hline & & & 0 & \end{array}$$

POLINOMIO DI 2° GRADO CON $\Delta<0$

$$\Rightarrow x^3+1 = (x+1)(x^2-x+1)$$

(NOTIAMO CHE SI POTEVA ANCHE RICORDARE LA FORMULA DELLA SOMMA DI DUE CUBI)

QUINDI

QUINDI

$$\int \frac{x+1}{x^3+1}\,dx = \int \frac{x+1}{(x+1)(x^2-x+1)}\,dx = \int \frac{1}{x^2-x+1}\,dx$$

SCRIVIAMO:

$$x^2-x+1 = (x+a)^2+b = x^2+2ax+a^2+b$$

$$\Rightarrow \begin{cases} 2a=-1 \\ a^2+b=1 \end{cases} \Rightarrow \begin{cases} a=-\frac{1}{2} \\ b=1-\frac{1}{4}=\frac{3}{4} \end{cases}$$

$$\Rightarrow \quad x^2-x+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x-\frac{1}{2}\right)^2+1\right] =$$

$$= \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1\right]$$

PER cui

$$\int \frac{1}{x^2-x+1}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2}\,dx =$$

$$= \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{2/\sqrt{3}}{1+\left(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)^2}\,dx = \frac{2\sqrt{3}}{3}\int \frac{d}{dx}\left[\operatorname{arctg}\left(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)\right]dx$$

$$= \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right) + c$$

3) $$\int \frac{x^3+x+1}{x^4+x^2}\,dx$$

$$\frac{x^3+x+1}{x^4+x^2} = \frac{x^3+x+1}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1} =$$

$$= \frac{A\cdot x(x^2+1)+B(x^2+1)+x^2(Cx+D)}{x^2(x^2+1)} = \frac{Ax^3+Ax+Bx^2+B+Cx^3+Dx^2}{x^2(x^2+1)}$$

$$= \frac{x^3(A+C)+x^2(B+D)+Ax+B}{x^2(x^2+1)}$$

$$= \frac{x^3(A+C) + x^2(B+D) + Ax + B}{x^2(x^2+1)}$$

$$\Rightarrow \begin{cases} A+C = 1 \\ B+D = 0 \\ A = 1 \\ B = 1 \end{cases} \Rightarrow \begin{cases} C = 0 \\ D = -1 \\ A = 1 \\ B = 1 \end{cases}$$

Per cui:

$$\int \frac{x^3+x+1}{x^4+x^2}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx - \int \frac{1}{x^2+1}\,dx =$$

$$= \log|x| - \frac{1}{x} - \operatorname{arctg} x + c$$

4) $$\int \frac{4}{(2x-1)^2}\,dx = 2\int \frac{2}{(2x-1)^2} = 2\int \frac{d}{dx}\left[-\frac{1}{2x-1}\right]dx =$$

$$= -\frac{2}{2x-1} + c$$

5) $$\int \frac{x^2}{\sqrt{1+x}}\,dx$$

Sia $y = \sqrt{1+x} \Rightarrow y^2 - 1 = x \Rightarrow x^2 = (y^2-1)^2$

e $dx = 2y\,dy$, quindi:

$$\int \frac{x^2}{\sqrt{1+x}}\,dx = 2\int \frac{(y^2-1)^2}{y}\cdot y\,dy = 2\int (y^2-1)^2\,dy =$$

$$= 2\int (y^4 - 2y^2 + 1)\,dy = \frac{2}{5}y^5 - \frac{4}{3}y^3 + 2y + c$$

$$= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{4}{3}(1+x)^{\frac{3}{2}} + 2(1+x)^{\frac{1}{2}} + c$$

6) $$\int \frac{1}{x^2+2x+2}\,dx$$

$\Delta = 4 - 8 = -4 < 0 \Rightarrow$ COMPLETIAMO IL QUADRATO:

$$x^2+2x+2 = x^2+2x+1+1 = (x+1)^2+1$$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{1+(x+1)^2}\,dx = \arctan(x+1) + c$$

7) $$\int \frac{x}{x^2-6x+9}\,dx$$

$\frac{\Delta}{4} = 9 - 9 = 0 \Rightarrow x_{1,2} = 3 \Rightarrow x^2-6x+9 = (x-3)^2$

$$\int \frac{x}{x^2-6x+9}\,dx = \int \frac{x-3+3}{(x-3)^2}\,dx = \int \frac{1}{x-3}\,dx + \int \frac{3}{(x-3)^2}\,dx$$

$$= \log|x-3| - \frac{3}{x-3} + c$$

8) $$\int \frac{(x+1)(2x-1)}{x^3-1}\,dx$$

$x_0 = 1$ É RADICE DI x^3-1; PER RUFFINI:

$$\begin{array}{c|ccc|c} & 1 & 0 & 0 & -1 \\ 1 & & 1 & -1 & 1 \\ \hline & \underbrace{1 \quad -1 \quad +1}_{x^2-x+1} & & & 0 \end{array} \Rightarrow x^3-1 = (x+1)(\overbrace{x^2-x+1}^{\Delta<0})$$

$$\int \frac{(x+1)(2x-1)}{(x+1)(x^2-x+1)}\,dx = \int \frac{2x-1}{x^2-x+1}\,dx =$$

$$= \int \frac{d}{dx}\left[\log(x^2-x+1)\right]dx = \log(x^2-x+1) + c$$

NIENTE VALORE ASSOLUTO: IL POLINOMIO È SEMPRE >0.

9) $\int \frac{4}{3-7x} dx = -\frac{4}{7} \cdot \int \frac{-7}{3-7x} dx =$

$= -\frac{4}{7} \log|3-7x| + c$

10) $\int \frac{\sqrt{x-1}}{x} dx$

Sia $y = \sqrt{x-1} \Rightarrow x = y^2+1 \ , \ dx = 2y$

$\int \frac{\sqrt{x-1}}{x} dx = 2\int \frac{y^2}{y^2+1} = 2\int \frac{y^2+1-1}{y^2+1} =$

$= 2\int dy - 2\int \frac{1}{1+y^2} dy = 2y - 2 \arctan y + c =$

$= 2\left[\sqrt{x-1} - \arctan\sqrt{x-1}\right] + c$

Integrali

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Integrazione per parti

1) $\int e^x (x^2-x)\,dx = e^x(x^2-x) - \int e^x(2x-1) = e^x(x^2-x) - \left[e^x(2x-1) - \int 2e^x\right]$

$= e^x(x^2-x) - e^x(2x-1) + 2\int e^x$

$= e^x(x^2-x) - e^x(2x-1) + 2e^x$

$= e^x(x^2-x-2x+1+2) = e^x(x^2-3x+3)$

2) $\int x^2 \operatorname{sen} x\,dx = -x^2\cos x + 2\int x\cos x = -x^2\cos x + 2\left[x\operatorname{sen}x - \int \operatorname{sen}x\right]$

$= -x^2\cos x + 2x\operatorname{sen}x + 2\cos x$

3) $\int x^2\cos(2x) = \frac{1}{2}x^2\operatorname{sen}(2x) - \frac{1}{2}2\int x\operatorname{sen}(2x) =$

$= \frac{1}{2}x^2\operatorname{sen}(2x) - \left[-\frac{1}{2}x\cos(2x) + \frac{1}{2}\int\cos(2x)\right]$

$= \frac{1}{2}x^2\operatorname{sen}(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{2}\cdot\frac{1}{2}\operatorname{sen}(2x)$

$= \frac{1}{2}x^2\operatorname{sen}(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\operatorname{sen}(2x)$

4) $\int e^x\cos x\,dx = e^x\cos x + \int e^x\operatorname{sen}x = e^x\cos x + e^x\operatorname{sen}x - \int e^x\cos x$

$\Rightarrow \cancel{2}\int e^x\cos x\,dx = \frac{e^x(\cos x+\operatorname{sen}x)}{2}$

5) $\int x^2 e^{3x}\,dx = \frac{1}{3}x^2e^{3x} - \frac{1}{3}\cdot 2\int x e^{3x} =$

$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\right]$

$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{3}\cdot\frac{1}{9}e^{3x} = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x}$

6) $\int x^3\log x\,dx = \frac{x^4}{4}\log x - \frac{1}{4}\int x^{\cancel{4}3}\cdot\frac{1}{\cancel{x}} = \frac{x^4}{4}\log x - \frac{x^4}{16}$

7) $\int(\log x)^2\,dx = \int \log x \log x \overset{*}{=} (x\log x - x)\log x - \int(x\log x - x)\frac{1}{x}$

$\left(* \int\log x = x\log x - \int x\cdot\frac{1}{x} = x\log x - x\right)$ $= (x\log x - x)\log x - \int \log x - 1$

$$\boxed{* \int \log x = x \log x - \int x \cdot \frac{1}{x} = x \log x - x}$$

$$= (x\log x - x)\log x - \int \log x - 1$$

$$= (x\log x - x)\log x - x\log x + x + x$$

$$= x\left(\log^2 x - 2\log x + 2\right)$$

8) $$\int x^3 \log(x+1)\,dx = \frac{x^4}{4}\log(x+1) - \frac{1}{4}\int \frac{x^4}{x+1}$$

$$\frac{x^4}{x+1} = \frac{(x^4-1)+1}{x+1} = \frac{(x^2-1)(x^2+1)}{x+1} + \frac{1}{x+1}$$

$$= \frac{(x-1)\cancel{(x+1)}(x^2+1)}{\cancel{(x+1)}} + \frac{1}{x+1}$$

$$= (x-1)(x^2+1) + \frac{1}{x+1}$$

$$\Rightarrow \int \frac{x^4}{x+1} = \int (x-1)(x^2+1) + \int \frac{1}{x+1} = \int x^3 - x^2 + x - 1\,dx + \log|x+1|$$

$$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + \log|x+1|$$

$$\Rightarrow \int x^3 \log(x+1)\,dx = \frac{x^4}{4}\log(x+1) - \frac{1}{4}\left(\frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + \log(x+1)\right)$$

$$= \frac{x^4}{4}\log(x+1) - \frac{x^4}{16} + \frac{x^3}{12} - \frac{x^2}{8} + \frac{x}{4} - \frac{1}{4}\log(x+1)$$

$$= \frac{1}{48}\left(12(x^4-1)\log(x+1) + x(-3x^3 + 4x^2 - 6x + 12)\right)$$

Esercizi di riepilogo

1) $$\int \frac{1}{1+e^x}\,dx$$

$$e^x = y \Rightarrow x = \log y \Rightarrow dx = \frac{1}{y}\,dy$$

$$\int \frac{1}{y(1+y)}\,dy$$

$$\frac{1}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y} = \frac{A(1+y)+By}{(1+y)y} = \frac{A+(A+B)y}{y(1+y)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \quad \begin{cases} B=-A=-1 \\ A=1 \end{cases}$$

$$\int \frac{1}{y(1+y)}\,dy = \int \frac{1}{y}\,dy + \int -\frac{1}{1+y}\,dy$$

$$= \log|y| - \log|1+y| = \log e^x - \log|e^x+1|$$

$$= x - \log|e^x+1|$$

2) $$\int x \operatorname{arctg} x = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2}\int \frac{x^2}{1+x^2}$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2}\left[\int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right]$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x$$

3) $$\int \operatorname{sen}(\log x)\, dx$$

$\log x = y$

$x = e^y \Rightarrow dx = e^y\, dy$

$$\int e^y \operatorname{sen} y\, dy = e^y \operatorname{sen} y - \int e^y \cos y =$$

$$= e^y \operatorname{sen} y - e^y \cos y + \int e^y(-\operatorname{sen} y)$$

$$\Rightarrow 2\int e^y \operatorname{sen} y\, dy = \frac{1}{2}(e^y \operatorname{sen} y - e^y \cos y)$$

$$= \frac{1}{2} e^y (\operatorname{sen} y - \cos y)$$

$$= \frac{1}{2} e^{\log x}(\operatorname{sen}(\log x) - \cos(\log x))$$

$$= \frac{1}{2} x (\operatorname{sen}(\log x) - \cos(\log x))$$

4) $$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} = \int \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \int \frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \int \frac{e^{2x}-1}{e^{2x}+1}$$

$$= \int \frac{e^{2x}}{e^{2x}+1} - \int \frac{1}{e^{2x}+1}$$

$$= \frac{1}{2}\log(1+e^{2x}) - \underbrace{\int \frac{1}{1+e^{2x}}}_{\alpha}$$

*) $$\int \frac{1}{1+e^{2x}}\, dx$$

$e^{2x} = y \Rightarrow 2x = \log y \Rightarrow x = \frac{1}{2}\log y$

$dx = \frac{1}{2y}$

$\Rightarrow$

$$dx = \frac{1}{2y}$$

$$\Rightarrow \frac{1}{2}\int \frac{1}{(1+y)y}\,dy$$

$$\frac{1}{(1+y)y} = \frac{A}{y} + \frac{B}{1+y} = \frac{A+Ay+By}{y(1+y)} = \frac{A+(A+B)y}{y(1+y)}$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \quad \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\frac{1}{2}\int \frac{1}{(1+y)y} = \frac{1}{2}\left[\int\frac{1}{y} - \int\frac{1}{1+y}\right] = \frac{1}{2}\log y - \frac{1}{2}\log(1+y)$$

$$= \frac{1}{2}\log e^{2x} - \frac{1}{2}\log(1+e^{2x})$$

$$= x - \frac{1}{2}\log(e^{2x}+1)$$

$$\Rightarrow \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx = \log(e^{2x}-1) - x$$

5) $$\int \frac{\cos x}{4+\operatorname{sen}^2 x} = \int \frac{\cos x}{4\left[1+\left(\frac{\operatorname{sen} x}{2}\right)^2\right]} = \frac{1}{4}2\int\frac{1}{2}\frac{\cos x}{1+\left(\frac{\operatorname{sen} x}{2}\right)^2}$$

$$= \frac{2}{4}\frac{1}{2}\operatorname{arctg}\left(\frac{\operatorname{sen} x}{2}\right)$$

6) $$\int \operatorname{sen}^2(3x+5) = \int \operatorname{sen}(3x+5)\operatorname{sen}(3x+5) =$$

$$= -\frac{1}{3}\cos(3x+5)\operatorname{sen}(3x+5) + \frac{1}{3}\cdot 3\int \cos^2(3x+5)$$

$$= -\frac{1}{3}\cos(3x+5)\operatorname{sen}(3x+5) + \int 1 - \operatorname{sen}^2(3x+5)$$

$$= -\frac{1}{6}\operatorname{sen}(6x+10) + x - \int \operatorname{sen}^2(3x+5)$$

$$\Rightarrow \cancel{2}\int \operatorname{sen}^2(3x+5) = -\frac{1}{6}\operatorname{sen}(6x+10)+x$$

$$= -\frac{1}{12}\operatorname{sen}(6x+10) + \frac{1}{2}x$$

7) $$\int e^{2x}\log(e^{2x}+1) =$$

$$e^{2x} = y \Rightarrow 2x = \log y \Rightarrow x = \frac{1}{2}\ln u \Rightarrow dx = \frac{1}{2u}du$$

$$e^{2x} = y \Rightarrow 2x = \log y \Rightarrow x = \frac{1}{2}\log y \Rightarrow dx = \frac{1}{2y}\,dy$$

$$\int y \log(y+1)\,\frac{1}{2y}\,dy = \frac{1}{2}\int \log(y+1)$$

$$= \frac{1}{2}\left(y\log(y+1) - \int \frac{y}{y+1} \right)$$

$$= \frac{1}{2}\left[y\log(y+1) - \int \frac{y+1-1}{y+1} \right]$$

$$= \frac{1}{2}\left[y\log(y+1) - y + \log(y+1) \right]$$

$$= \frac{1}{2}\left\{ (e^{2x}+1)\log(e^{2x}+1) - e^{2x} \right]$$