1) Calcolare l'integrale triplo

$$\iiint_E z \, dx\,dy\,dz$$

$$E = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, \ \operatorname{arctg} x \le y \le x \\ 0 \le z \le x\}$$

$$E = \{(x,y) \in D, \quad 0 \le z \le x\}$$

$$D = \{0 \le x \le 1, \ \operatorname{arctg} x \le y \le x\}$$

$$\iiint_E z \, dx\,dy\,dz = \iint_D \left(\int_0^x z \, dz \right) dx\,dy =$$

formule di riduzione

$$= \iint_D \frac{x^2}{2} \, dx\,dy$$

$$= \int_0^1 \left(\int_{\operatorname{arctg} x}^{x} \frac{x^2}{2} \, dy \right) dx = \int_0^1 \frac{x^2}{2} \left(x - \operatorname{arctg} x \right) dx$$

↳ formule di riduzione

$$= \int_0^1 \frac{x^3}{2} dx - \int_0^1 \frac{x^2}{2} \operatorname{arctg} x \, dx$$

$$= \frac{x^4}{8} \Big|_0^1 - \frac{1}{2} \left(\frac{x^3}{3} \operatorname{arctg} x \Big|_0^1 + \int_0^1 \frac{x^3}{3} \frac{1}{1+x^2} \right)$$

$$= \frac{1}{8} - \frac{1}{6} \operatorname{arctg} 1 + \frac{1}{6} \int_0^1 \frac{x^3 \, {\color{blue}\pm x}}{1+x^2} \, dx$$

$$= \frac{1}{8} - \frac{1}{6}\left(\frac{\pi}{4}\right) + \frac{1}{6} \int_0^1 \frac{x(x^2+1) - x}{1+x^2} \, dx$$

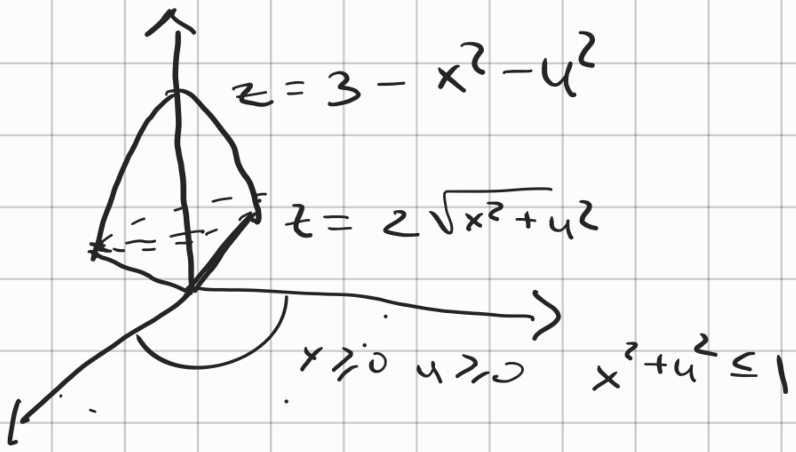
$$= \frac{1}{8} - \frac{\pi}{24} + \frac{1}{6} \left(\int_0^1 x - \frac{x}{1+x^2} \, dx \right)$$

$$= \frac{1}{8} - \frac{\pi}{24} + \frac{1}{12} x^2 \Big|_0^1 - \frac{1}{12} \log(1+x^2) \Big|_0^1$$

$$= \frac{1}{8} - \frac{\pi}{24} + \frac{1}{12} - \frac{1}{12} \log(2)$$

(2) Calcolare il volume del solido

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 : x \geq 0, \, y \geq 0, \, x^2+y^2 \leq 1, \; 2\sqrt{x^2+y^2} \leq z \leq 3 - x^2 - y^2 \right\}$$



$$\mathrm{Vol}(T) = \iiint_T 1 \, dx\,dy\,dz$$

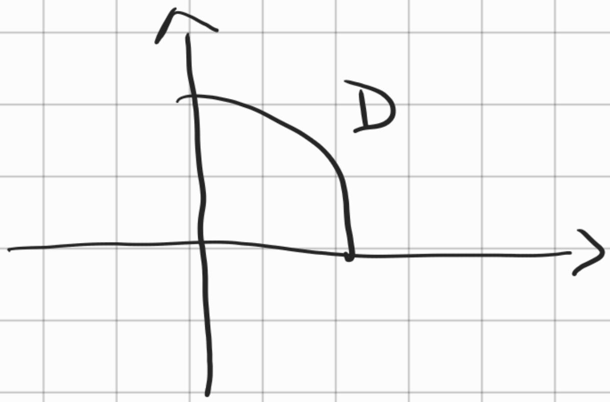
T è un dominio normale di $\mathbb{R}^3$

$$T = \left\{ (x,y) \in D \quad 2\sqrt{x^2+y^2} \le z \le 3 - x^2 - y^2 \right\}$$

$$D = \left\{ x \ge 0 \quad y \ge 0 \quad , \quad x^2 + y^2 \le 1 \right\}$$

$$\Rightarrow \iiint_T 1 \, dx\,dy\,dz = \iint_D 3 - x^2 - y^2 - 2\sqrt{x^2+y^2} \, dx\,dy$$

Passo a coordinate polari



$$D = \begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$\phi : \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \qquad J\phi = \rho$$

$$\mathrm{Vol}(T) = \int_0^{\pi/2} \left(\int_0^1 \left(\left(3 - \rho^2 - 2\sqrt{\rho^2}\right) \rho \, d\rho \right) d\theta \right.$$

$$= \left(\int_0^{\pi/2} d\theta \right) \int_0^1 3\rho - \rho^3 - 2\rho^2 \, d\rho$$

$$= \frac{\pi}{2} \cdot \left(\frac{3}{2}\rho^2 - \frac{\rho^4}{4} - \frac{2}{3}\rho^3\right)\Big|_0^1 =$$

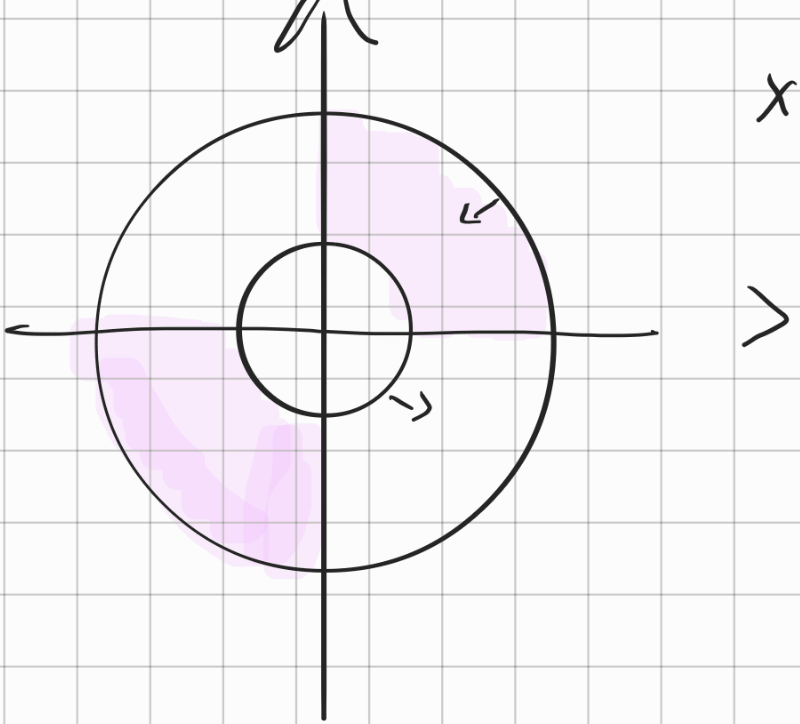
$$\frac{\pi}{2} \cdot \left(\frac{3}{2} - \frac{1}{4} - \frac{2}{3}\right) = \frac{\pi}{2} \cdot \frac{18 - 3 - 8}{12} =$$

$$= \frac{7\pi}{24}$$

③ Calcolare l'integrale doppio

$$\iint_D xy \, dx\, dy$$

$$D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2 \quad xy \ge 0\}$$



$xy \ge 0 \iff$ sono nel I o III quadrante

Passo a coordinate polari

$$D = \left\{ 1 < \rho \leq \sqrt{2} \quad 0 \leq \theta \leq \frac{\pi}{2} \ \cup \ \pi \leq \theta \leq \frac{3}{2}\pi \right\}$$

$$\iint_D xy \, dx\, dy = \int_1^{\sqrt{2}} \rho \left(\int_0^{\pi/2} \rho^2 \cos\theta \sin\theta \, d\theta + \int_{\pi}^{\frac{3}{2}\pi} \rho^2 \cos\theta \sin\theta \, d\theta \right) d\rho$$

sono uguali

perché $\sin(\theta+\pi)\cos(\theta+\pi) = \sin\theta \cos\theta$

$$= 2 \int_1^{\sqrt{2}} \rho^3 \left(\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \right) d\rho$$

$$= \left(\int_1^{\sqrt{2}} \rho^3 \, d\rho \right) \left(\int_0^{\pi/2} 2 \cos\theta \sin\theta \, d\theta \right)$$

$$= \frac{\rho^4}{4} \Big|_1^{\sqrt{2}} \cdot \sin^2\theta \Big|_0^{\frac{\pi}{2}} = \left(1 - \frac{1}{4}\right) \cdot 1$$

$$= \frac{3}{4}$$

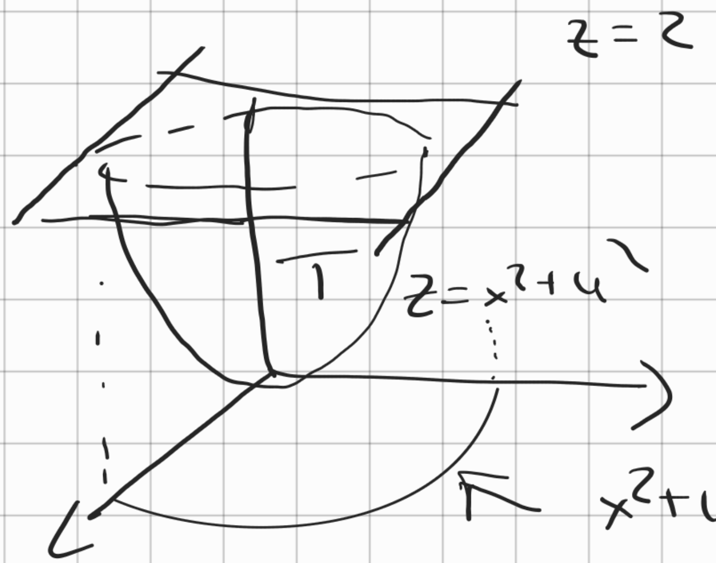
④ Calcolare

$$\iiint_T x^2 (2-z)\, dx\, dy\, dz$$

$$T = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le z \quad 0 \le z \le 2 \}$$

$$T = \{ (x,y) \in D \quad x^2+y^2 \le z \le 2 \}$$

$$D = \{ x^2+y^2 \le 2 \}$$



$$\iiint_T x^2 (2-z)\, dx\, dy\, dz = \iint_D x^2 \left(\int_{x^2+y^2}^{2} 2-z \, dz \right) dx dy$$

$$= \iint_D x^2 \cdot \left(- \frac{(2-z)^2}{2} \Bigg|_{x^2+y^2}^{2} \right) dx\,dy$$

$$= \iint_D x^2 \, \frac{(2 - x^2 - y^2)^2}{2} \, dx\,dy$$

Passo a coordinate polari

$$D = \{ 0 < \rho < \sqrt{2} \quad 0 \le \theta \le 2\pi \}$$

$$\iint_D x^2 \frac{(2-x^2-y^2)^2}{2} dx\,dy = \int_0^{2\pi} \left(\int_0^{\sqrt{2}} \frac{\rho^2 \cos^2\theta}{2} \cdot (2-\rho^2)^2 \rho \, d\rho \right) d\theta$$

$$= \frac{1}{2} \left(\int_0^{2\pi} \cos^2\theta \, d\theta \right) \left(\int_0^{\sqrt{2}} \rho^3 (2-\rho^2)^2 \, d\rho \right)$$

$\hookrightarrow \rho^2 = t \rightarrow 0 < t < 2$

$2\rho \, d\rho = dt$

$$= \frac{1}{2} \left(\frac{1}{2} (\theta + \sin\theta\cos\theta) \Big|_0^{2\pi} \right) \left(\frac{1}{2} \int_0^2 t(2-t)^2 \, dt \right)$$

$$= \frac{1}{2}\pi \cdot \int_0^2 t(2-t)^2\,dt =$$

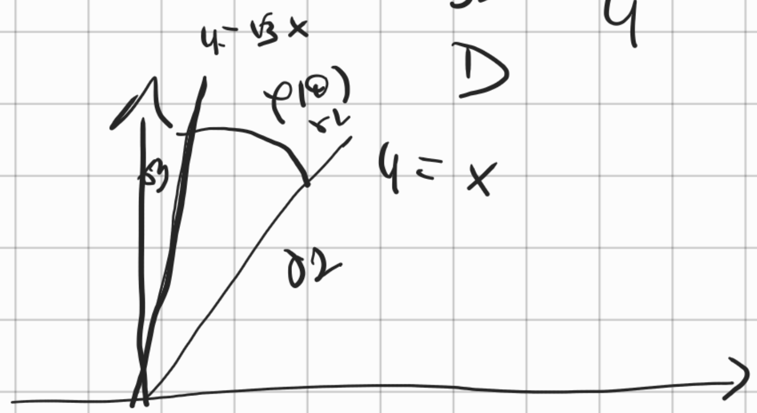
$$= \frac{\pi}{2}\left[-\frac{(2-t)^3}{3}t\Big|_0^2 + \int_0^2 \frac{(2-t)^3}{3}\right]$$

$$= \frac{\pi}{2}\left(-\frac{(2-t)^4}{12}\Big|_0^2\right) =$$

$$= \frac{\pi}{2}\left(+\frac{16}{12}\right) = \frac{2\pi}{3}$$

(5) Sia D il dominio nel piano contenuto nel primo quadrante, delimitato dalle rette $y = x$, $y = \sqrt{3}\,x$ e dalla curva $\rho(\theta) = 3$

Calcolare $\iint_D \frac{x}{y}\,dx\,dy$



$$\Rightarrow \iint_D \frac{x}{y}\,dx\,dy \overset{g.g.}{=} \int_{+\partial D} \frac{x^2}{2y}\,dy$$

$$+\partial D = \gamma_1 \cup \gamma_2 \cup (-\gamma_3)$$

$$\gamma_1 : \begin{cases} x = t \\ y = t \end{cases} \qquad 0 < t < 3 \cdot \frac{\sqrt{2}}{2}$$

$$\gamma_2 : \begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases} \qquad \frac{\pi}{4} \leq t \leq \frac{\pi}{3}$$

$$\gamma_3 : \begin{cases} x = t \\ y = \sqrt{3}t \end{cases} \qquad 0 \leq t \leq \frac{3}{2}$$

$$\int_{+\partial D} \frac{x^2}{2y}\,dy = \int_0^{\frac{3\sqrt{2}}{2}} \frac{t^2}{2\,t}\,dt + \int_{\frac{\pi}{4}}^{\pi/3} \frac{9}{2}\frac{\cos^2 t\cos t}{\sin t}$$

$$- \int_0^{3/2} \frac{1}{2}\frac{t^2}{\sqrt{3}\,t}\,\sqrt{3}\,dt$$

$$= \int_0^{3\frac{\sqrt{2}}{2}} \frac{t}{2}\,dt + \frac{9}{2}\int_{\frac{\pi}{4}}^{\pi/3} \frac{\cos t\,(1-\sin^2 t)}{\sin t}\,dt$$

$$- \frac{1}{2}\int_0^{3/2} t\,dt$$

$$= \frac{t^2}{4}\Big|_0^{3\frac{\sqrt{2}}{2}} - \frac{t^2}{4}\Big|_0^{\frac{3}{2}} + \frac{9}{2}\left(\int_{\frac{\pi}{4}}^{\pi/3} \frac{\cos(t)}{\sin t} - \int_{\pi/4}^{\pi/3} \sin(t)\cos t\right)$$

$$= \frac{1}{4}\left(\frac{9}{2} - \frac{9}{4}\right) + \frac{9}{2}\log(\sin t)\Big|_{\frac{\pi}{4}}^{\pi/3} - \frac{9}{4}\sin^2 t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$\frac{9}{16} + \frac{9}{2}\left(\log\left(\frac{\sqrt{3}}{2}\right) - \log\left(\frac{\sqrt{2}}{2}\right)\right) - \frac{9}{4}\left(\frac{3}{4} - \frac{1}{2}\right)$$

$$= \frac{9}{16} + \frac{9}{2}\left(\log\left(\frac{\sqrt{3}}{2}\right) - \log\left(\frac{\sqrt{2}}{2}\right)\right) - \frac{9}{4}\left(\frac{1}{4}\right)$$

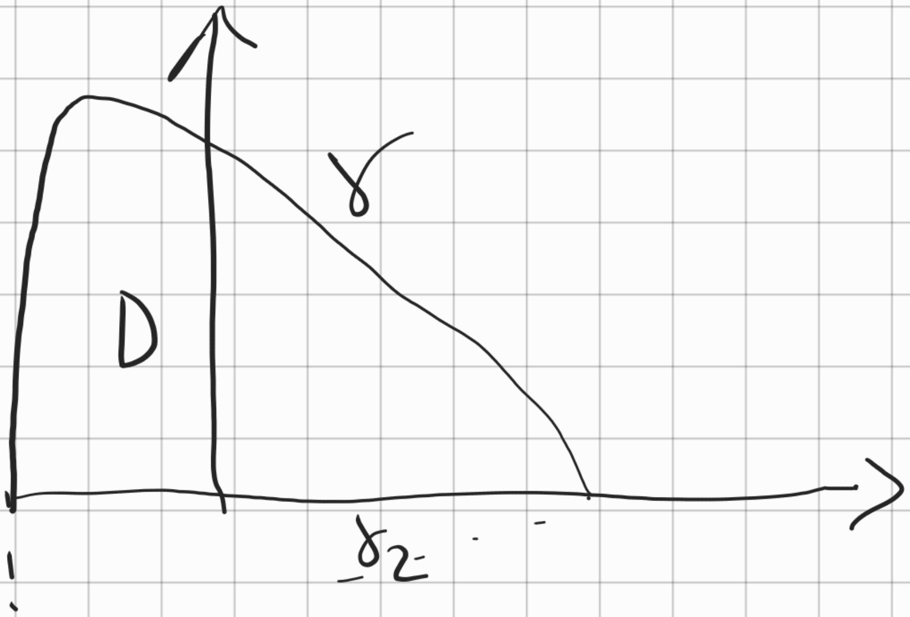
$$= \qquad + \frac{9}{2}\log\left(\frac{\sqrt{3}}{\sqrt{2}}\right)$$

(6) Data γ la curva: $\gamma: \begin{cases} x = \cos t \\ y = t \sin t \end{cases}$ $t \in [0, \pi]$

calcolare $\displaystyle\iint_D x^2 \, dx \, dy$

con D parte del piano delimitata da

γ e dall'asse x



$$\iint_D x^2 \, dx \, dy = -\int_{+\partial D} x^2 y \, dx$$

$$+\partial D = \gamma \cup \{\gamma_2\}$$

$$\gamma_2 = \begin{cases} x = t \\ y = 0 \end{cases} \quad -1 < t < 1$$

$$\iint_D x^2\,dx\,dy = -\int_{\gamma} x^2 y\,dx - \int_{\gamma_2} x^2 y\,dx$$

$$\int_{\gamma_2} x^2 y\,dx = 0 \quad \text{perché} \quad y=0 \text{ su } \gamma_2$$

$$= -\int_0^{\pi} \cos^2 t \cdot (t \sin t) \cdot (-\sin(t))\,dt$$

$$= +\int_0^{\pi} t \cos^2 t \sin^2 t\,dt$$

$$= \int_0^{\pi} t \,\frac{\sin(2t)^2}{4}\,dt \qquad \begin{aligned} 2t &= \theta \\ dt &= \frac{d\theta}{2} \end{aligned}$$

$$= \frac{1}{4}\int_0^{2\pi} \frac{\theta}{2} \sin^2(\theta) \cdot \frac{d\theta}{2} =$$

$$= \frac{1}{16}\int_0^{2\pi} \theta \sin^2\theta = \frac{1}{16}\theta \left(\frac{1}{2}(\theta - \sin\theta\cos\theta)\right)\Big|_0^{2\pi}$$

$$- \frac{1}{16}\int_0^{2\pi} \frac{1}{2}(\theta - \sin\theta\cos\theta)$$

$$= \frac{4\pi^2}{32} - \frac{1}{32}\left(\frac{\theta^2}{2}\Big|_0^{2\pi} + \frac{\cos^2\theta}{4}\Big|_0^{2\pi}\right)$$

$$= \frac{\pi^2}{8} - \frac{1}{32}\left(2\pi^2\right) = \pi^2\left(\frac{1}{8} - \frac{1}{16}\right) =$$

$$= \frac{\pi^2}{16}$$

Ex

Calcolare il flusso di

$$(x, z, y)$$

attraverso la superficie $+\partial E$ dove

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \frac{1}{\sqrt{1-x^2-y^2}} \leq z \leq 2 \\ x^2+y^2 \leq \frac{3}{4} \end{array} \right\}$$

Svolgimento

Dal teorema della divergenza

$$\iint_{+\partial E} (F \cdot \nu)\, d\sigma = \iiint_E \operatorname{div}(F)\, dx\, dy\, dz$$

$$\operatorname{div}(F) = 1$$

Chiamo $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq \frac{3}{4} \}$

$$\iiint_E dx\,dy\,dz = \iint_D \int_{\frac{1}{\sqrt{1-x^2-y^2}}}^{2} dz\,dx\,dy$$

$$= \iint_D \left(2 - \frac{1}{\sqrt{1-x^2-y^2}}\right) dx\,dy$$

coordinate polari

Jacobiano

$$= \int_0^{\frac{\sqrt{3}}{2}} d\rho \int_0^{2\pi} \left(2 - \frac{1}{\sqrt{1-\rho^2}}\right) \rho \, d\theta$$

$$= 2\pi \left(\int_0^{\frac{\sqrt{3}}{2}} 2\rho \, d\rho - \int_0^{\frac{\sqrt{3}}{2}} \frac{\rho}{\sqrt{1-\rho^2}} d\rho \right)$$

$$= 2\pi \left(\frac{3}{4} - \sqrt{1-\rho^2} \Big|_0^{\frac{\sqrt{3}}{2}} \right)$$

$$= 2\pi \left(\frac{3}{4} - \frac{1}{2} + 1 \right)$$

$$= \frac{5}{2}\pi$$

Ex

***

Calcolare

$$\iint_T x^2(y-x^3)\, e^{y+x^3}\, dx\,dy$$

in $T = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x^3 \leq y \leq 3 \\ x \geq 1 \end{array} \right\}$

Suggerimento: si utilizzi una sostituzione per semplificare la funzione integranda

Svolgimento

Utilizzo la sostituzione

$$\begin{cases} u = y - x^3 \\ v = y + x^3 \end{cases} \qquad \begin{array}{l} (u,v) = \phi(x,y) \\ du\,dv = \operatorname{Jac}\phi \; dx\,dy \end{array}$$

$$\operatorname{Jac}\phi = \left| \det \begin{pmatrix} -3x^2 & 1 \\ 3x^2 & 1 \end{pmatrix} \right| = 6x^2$$

Il dominio

$$T = \left\{ (x,y) \;\middle|\; \begin{array}{l} x^3 \leq y \leq 3 \\ x \geq 1 \end{array} \right\}$$

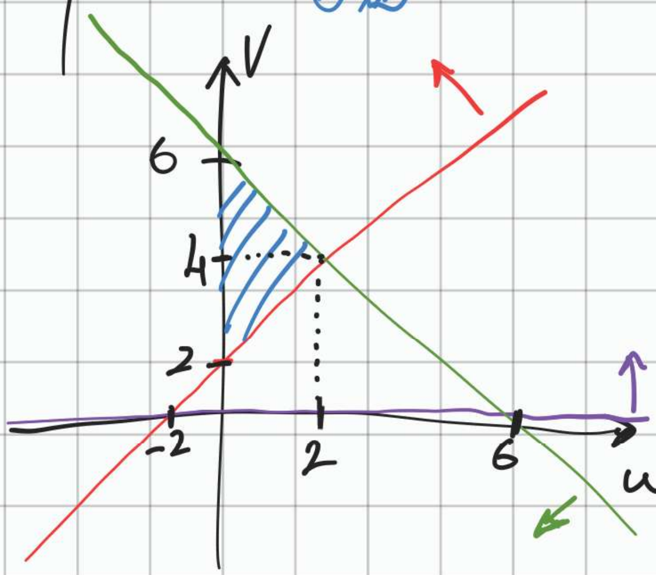
si trasforma usando le uguaglianze

$y = \frac{v+u}{2}$

$x^3 = y - u = \frac{v-u}{2}$

$$D = \left\{ (u,v) \;\middle|\; \begin{array}{l} \frac{v-u}{2} \leq \frac{v+u}{2} \leq 3 \\ \sqrt[3]{\frac{v-u}{2}} \geq 1 \end{array} \right\}$$

D



$\sqrt[3]{\frac{v-u}{2}} \geq 1 \iff v \geq 2+u$

$\frac{v-u}{2} \leq \frac{v+u}{2} \iff 0 \leq 2u \iff u \geq 0$

$\frac{v+u}{2} \leq 3 \iff v \leq 6-u$

Le rette $v = 2+u$ e $v = 6-u$ si incontrano nel punto (u,v) che risolve l'equazione

$$2+u = 6-u \Rightarrow 2u = 4 \Rightarrow u = 2$$

$$\Rightarrow v = 2+u = 4$$

$$D = \left\{ (u,v) \;\middle|\; \begin{array}{l} 0 \leq u \leq 2 \\ 2+u \leq v \leq 6-u \end{array} \right\}$$

$$\iint_T x^2 (y - x^3)\, e^{y+x^3}\, dx\,dy$$

$\left(\text{Ricordiamo} \quad 6x^2\, dx\,dy = du\,dv \right)$

$$\parallel$$

$$\frac{1}{6}\iint_D u e^v \, du\, dv$$

$$\parallel$$

$$\frac{1}{6}\int_0^2 du \int_{u+2}^{6-u} u e^v \, dv$$

$$\parallel$$

$$\frac{1}{6}\int_0^2 u\left(e^{6-u}-e^{u+2}\right) du$$

$$\parallel$$

$$\frac{e^6}{6}\int_0^2 u e^{-u}\, du - \frac{e^2}{6}\int_0^2 u e^{u}\, du$$

$$\boxed{\int t e^t \, dt = t e^t - \int e^t = (t-1)e^t + c}$$

$$\int_0^2 u e^{-u}\, du \overset{t=-u}{=} \int_0^{-2} -t e^t (-1)\, dt$$

$$= (t-1)e^t \Big|_0^{-2} = -3e^{-2}+1$$

$$\int_0^2 u e^u = (u-1)e^u \Big|_0^2 = e^2 + 1$$

Quindi il risultato è:

$$\frac{e^6}{6}\left(-3e^{-2}+1\right) - \frac{e^2}{6}\left(e^2+1\right) =$$

$$= -\frac{1}{2}e^4 + \frac{e^6}{6} - \frac{1}{6}e^4 - \frac{e^2}{6}$$

$$= \frac{e^6}{6} - \frac{2}{3}e^4 - \frac{e^2}{6}$$

Ex
**

Calcolare

$$I = \iint_D \frac{2x}{(x-y)^2+(x+y)^2}\,dx\,dy$$

dove $D = \{(x,y) \mid |y| \leq x \leq \frac{1}{2}\}$

Svolgimento

Sostituisco

$$\begin{cases} u = x - y \\ v = x + y \end{cases}$$

$(u,v) = \phi(x,y)$

$du\,dv = \mathrm{Jac}\,\phi \; dx\,dy$

$$\mathrm{Jac}\,\phi = \left|\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right| = 2$$

Inoltre, sommando e sottraendo le equazioni

$$\begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{v-u}{2} \end{cases}$$

Così che D si trasforma in

$$T = \left\{ (u,v) \;\middle|\; \begin{array}{l} \frac{u+v}{2} \leq \frac{1}{2} \\ \left|\frac{v-u}{2}\right| \leq \frac{u+v}{2} \end{array} \right\}$$

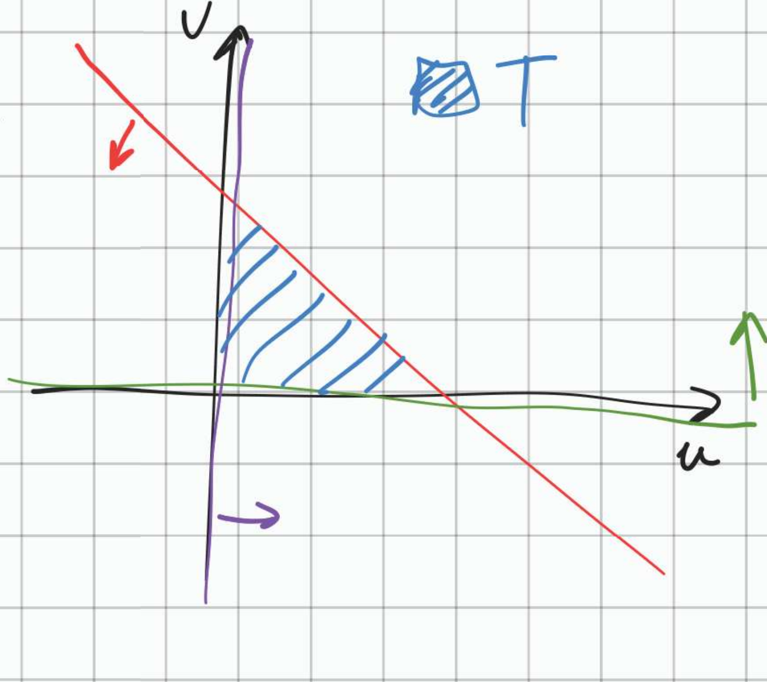
$\dfrac{u+v}{2} \leq \dfrac{1}{2} \iff v \leq 1-u$

$$\left|\frac{v-u}{2}\right| \leq \frac{u+v}{2} \iff \begin{cases} \dfrac{v-u}{2} \leq \dfrac{u+v}{2} \\ \dfrac{u-v}{2} \leq \dfrac{u+v}{2} \end{cases}$$

proprietà dei moduli

$v - u \leq u + v \iff u \geq 0$

$u - v \leq u + v \iff v \geq 0$



$$T = \left\{ (u,v) \;\middle|\; \begin{array}{c} u \geq 0 \\ v \geq 0 \\ v \leq 1-u \end{array} \right\}$$

$$\iint_D \frac{2x}{(x-y)^2+(x+y)^2}\,dx\,dy = \frac{1}{2}\iint_T \frac{2\,\frac{u+v}{2}}{u^2+v^2}\,du\,dv$$

In coordinate polari

$$v \leq 1-u \iff \rho\cos\theta \leq 1-\rho\sin\theta \iff \rho \leq \frac{1}{\sin\theta+\cos\theta}$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\sin\theta+\cos\theta}} \frac{1}{2}\,\frac{\cancel{\rho}(\cos\theta+\sin\theta)}{\cancel{\rho^2}}\,\cancel{\rho}\,d\rho$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cancel{\cos\theta+\sin\theta}}{\cancel{\cos\theta+\sin\theta}}\,d\theta = \frac{\pi}{4}$$

① $\iint_T x e^{xy}\, dx\, dy \qquad T = \left\{(x,y) \,/\, 0 \le x \le \frac{1}{y},\ 1 \le y \le 2\right\}$

Svolgimento

T è normale rispetto all'asse y, uso le formule di riduzione. $\left(T = \{(x,y) \,/\, y \in [c,d],\ \alpha(y) \le x \le \beta(y)\}\right)$

$$\iint_T f(x,y)\, dx\, dy = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} f(x,y)\, dx\right) dy$$

$$\Rightarrow \iint_T x e^{xy}\, dx\, dy = \int_1^2 \left(\int_0^{\frac{1}{y}} x e^{xy}\, dx\right) dy$$

integro per parti

$$= \int_1^2 \left(\frac{x}{y} e^{xy}\Big|_{x=0}^{x=\frac{1}{y}} - \int_0^{\frac{1}{y}} \frac{1}{y} e^{xy}\, dx\right) dy$$

$$= \int_1^2 \left(\frac{x}{y} e - \frac{1}{y^2} e^{xy}\Big|_{x=0}^{x=\frac{1}{y}}\right) dy$$

$$= \int_1^2 \left(\cancel{\frac{1}{y^2} e} - \cancel{\frac{1}{y^2} e} + \frac{1}{y^2}\right) dy$$

$$= \int_1^2 y^{-2}\, dy$$

$$= -y^{-1}\Big|_1^2$$

$$= -\left(\frac{1}{2} - 1\right) = \frac{1}{2}.$$

② Dato il campo

$$F = \left(\frac{x}{2}, \frac{y}{2} + e^{3x}, 1 \right)$$

calcolare il flusso di F uscente dalla superficie laterale del cilindro:

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2 + \frac{y^2}{4} \leq 1 \,,\, 0 \leq z \leq 1 \right\}$$

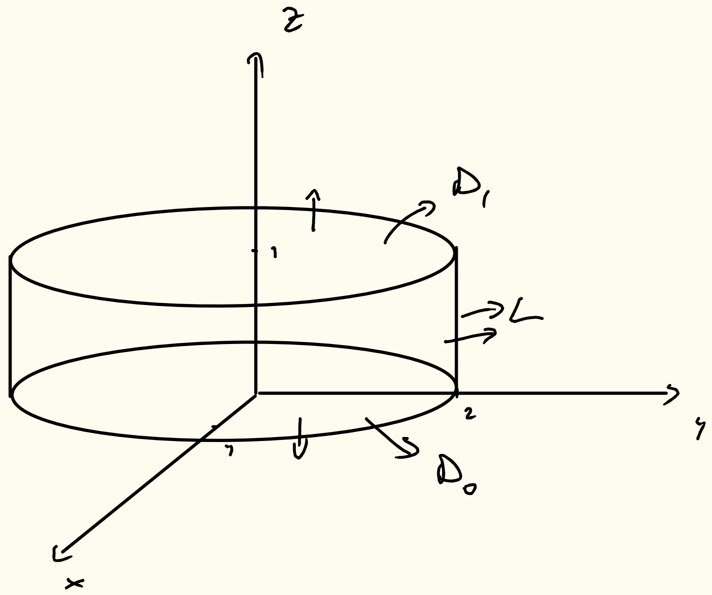
Svolgimento Utilizzando il Teorema della divergenza, abbiamo

$$\iiint_T \operatorname{div} F \, dx\,dy\,dz = \iint_{\partial T} \langle F, \nu^e \rangle \, d\sigma$$

dove $\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{1}{2} + \frac{1}{2} + 0 = 1$

e $\partial T = \underset{\substack{\uparrow \\ \text{superficie laterale}}}{L} + D_0 + D_1 \rightarrow$ ellisse ad altezza $z = 1$

$D_0 \rightarrow$ ellisse ad altezza $z = 0$



Da cui $\displaystyle\iint_L \langle F, \nu \rangle = \iiint_T \operatorname{div} F \, dx\,dy\,dz - \iint_{D_0} \langle F, \nu^e \rangle d\sigma - \iint_{D_1} \langle F, \nu^e \rangle d\sigma$

- $$\iiint_T \operatorname{div} F = \iiint_T 1\, dx\,dy\,dz = \overset{\text{misura di } T}{m(T)}$$

essendo T un cilindro a base ellittica $E = \{(x,y) / \ x^2 + \frac{y^2}{4} \le 1\}$

$$m(T) = m(E) \cdot \text{altezza} = 2\pi. \quad \Rightarrow \iiint_T \operatorname{div} F = 2\pi$$

- $$\iint_{D_0} \langle F \cdot \nu^e \rangle \, d\sigma$$

$$\varphi = \begin{cases} x = \rho\cos\theta & \rho = [0,1] \\ y = 2\rho\sin\theta & \theta \in [0, 2\pi] \\ z = 0 \end{cases}$$

$$\varphi_\rho \wedge \varphi_\theta = \begin{vmatrix} \cos\theta & -\rho\sin\theta & \underline{i} \\ 2\sin\theta & 2\rho\cos\theta & \underline{j} \\ 0 & 0 & \underline{k} \end{vmatrix} = (0, 0, 2\rho)$$

la normale esterna al cilindro in D_0 deve puntare verso il basso, mentre questa punta in alto, quindi dobbiamo cambiare l'orientamento e $\nu^e = (0,0,-2\rho)$

$$\Rightarrow \iint_{D_0} \langle F \cdot \nu^e \rangle \, d\sigma = \iint_{[0,1]\times[0,2\pi]} \left\langle \left(\frac{\rho\cos\theta}{2}, \frac{2\rho\sin\theta}{2} + 1, 1 \right) \cdot (0,0,-2\rho) \right\rangle d\rho\, d\theta$$

$$= -\iint_{[0,1]\times[0,2\pi]} 2\rho \, d\rho\, d\theta$$

$$= -\int_0^{2\pi} \left(\int_0^1 2\rho \, d\rho \right) d\theta = -1 \cdot 2\pi = -2\pi$$

• $\iint_{D_1} \langle F \cdot \nu^e \rangle \, d\sigma$ analogamente per D_0

$$\varphi = \begin{cases} x = \rho\cos\vartheta \\ y = 2\rho\sin\vartheta \\ z = 1 \end{cases} \qquad \begin{array}{l} \rho = [0,1] \\ \vartheta \in [0, 2\pi) \end{array}$$

$$\varphi_\rho \wedge \varphi_\vartheta = \begin{vmatrix} \cos\vartheta & -\rho\sin\vartheta & \underline{i} \\ 2\sin\vartheta & 2\rho\cos\vartheta & \underline{j} \\ 0 & 0 & \underline{k} \end{vmatrix} = (0, 0, 2\rho)$$

la normale esterna al cilindro in D_1 deve puntare verso l'alto, quindi è giusta

$$\Rightarrow \iint_{D_0} \langle F \cdot \nu^e \rangle \, d\sigma = \iint_{[0,1]\times[0,2\pi]} \left\langle \left(\frac{\rho\cos\vartheta}{2}, \frac{2\rho\sin\vartheta}{2} + e_1^3, 1 \right) \cdot (0,0,2\rho) \right\rangle d\rho\, d\vartheta$$

$$= \iint_{[0,1]\times[0,2\pi]} 2\rho \, d\rho \, d\vartheta$$

$$= \int_0^{2\pi} \left(\int_0^1 2\rho \, d\rho \right) d\vartheta = 2\pi$$

$$\Rightarrow \iint_L \langle F \cdot \nu^e \rangle = \iiint_T \operatorname{div} F \, d\sigma - \iint_{D_0} \langle F, \nu^e \rangle d\sigma - \iint_{D} \langle F, \nu^e \rangle \, d\sigma$$

$$= 2\pi - (-\cancel{2\pi}) - \cancel{2\pi} = 2\pi.$$

③ Dato il campo

$$F = (\operatorname{arctg} x + x^2 y, -y^2 x)$$

calcolare la circuitazione di F lungo il bordo della circonferenza di centro (1,0) e raggio 1, percorso in senso antiorario.

Svolgimento Utilizzando il Teorema di Stokes, otteniamo

$$\oint_{+\partial D} F\,dl = \int_{+\partial D} F_1\,dx + F_2\,dy = \iint_D \Bigg(\underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{\operatorname{rot} F} \Bigg) dx\,dy$$

$$\operatorname{rot} F = -x^2 - y^2$$

$$\Rightarrow \oint_{+\partial D} F\,dl = -\iint_D x^2 + y^2\,dx\,dy$$

dove $D = \{(x,y) \,/\, (x-1)^2 + y^2 \leq 1\}$.

$$\phi : \begin{cases} x = 1 + \rho\cos\theta & \rho \in [0,1] \\ y = \rho\sin\theta & \theta \in [0,2\pi] \end{cases}$$

$$\phi^{-1}(D) = [0,1] \times [0,2\pi] \qquad \operatorname{Jac}\phi = \rho$$

$$\iint_D x^2 + y^2\,dx\,dy = \int_0^1 \int_0^{2\pi} \left((1+\rho\cos\theta)^2 + \rho^2\sin^2\theta \right) \rho\,d\rho\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} \left(1 + 2\rho\cos\theta + \rho^2\cos^2\theta + \rho^2\sin^2\theta\right) \rho \, d\rho \, d\theta$$

$$= \int_0^{2\pi} \left(\int_0^1 \left(\rho + \rho^3 + 2\rho^2\cos\theta\right) d\rho \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{\rho^2}{2} + \frac{\rho^4}{4} + \frac{2\rho^3}{3}\cos\theta \right) \Big|_{\rho=0}^{\rho=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{4} + \frac{2}{3}\cos\theta \right) d\theta$$

$$= \frac{3}{\cancel{4}_2} \cancel{2}\pi + \frac{2}{3} \int_0^{2\pi} \cos\theta \, d\theta$$

$$= \frac{3}{2}\pi + \cancel{\frac{2}{3} \sin\theta \Big|_0^{2\pi}}^{\to 0} = \frac{3}{2}\pi$$

$$\iint_C \frac{\sqrt{x}}{(x^2+y^2)^{\frac{3}{4}}}\,dx\,dy$$

$$C = \{(x,y) / (x-1)^2+(y-1)^2 \leq 1\}$$

Svolgimento coordinate polari nell'origine

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$(\rho\cos\theta - 1)^2 + (\rho\sin\theta - 1)^2 \leq 1$$

$$\rho^2\cos^2\theta - 2\rho\cos\theta + 1 + \rho^2\sin^2\theta - 2\rho\sin\theta + \cancel{1} \leq \cancel{1}$$

$$\rho^2 - 2\rho(\cos\theta + \sin\theta) + 1 \leq 0$$

$$\Leftrightarrow \cos\theta + \sin\theta - \sqrt{(\cos\theta+\sin\theta)^2 - 1} \leq \rho \leq \cos\theta + \sin\theta + \sqrt{(\cos\theta+\sin\theta)^2-1}$$

$$\Rightarrow \iint_C \frac{\sqrt{x}}{(x^2+y^2)^{\frac{3}{4}}}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_{\cos\theta+\sin\theta-\sqrt{(\cos\theta+\sin\theta)^2-1}}^{\cos\theta+\sin\theta+\sqrt{(\cos\theta+\sin\theta)^2-1}} \frac{\sqrt{\rho}\,\sqrt{\cos\theta}}{(\rho^2)^{3/4}}\,\rho\,d\rho\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_{\cos\theta+\sin\theta-\sqrt{(\cos\theta+\sin\theta)^2-1}}^{\cos\theta+\sin\theta+\sqrt{(\cos\theta+\sin\theta)^2-1}} \frac{\cancel{\rho^{\frac{3}{2}}}}{\cancel{\rho^{3/2}}}\sqrt{\cos\theta}\,d\rho\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\left[\cancel{\cos\theta} + \cancel{\sin\theta} + \sqrt{(\cos\theta+\sin\theta)^2-1} - \cancel{\cos\theta} - \cancel{\sin\theta} + \sqrt{(\cos\theta+\sin\theta)^2-1}\right]d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\ \sqrt{(\cos\theta + \sin\theta)^2 - 1}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\ \sqrt{\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta - 1}\ d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\ \sqrt{\cancel{1} + 2\cos\theta\sin\theta - \cancel{1}}\ d\theta$$

$$= 2\sqrt{2}\int_0^{\frac{\pi}{2}} \cos\theta\ \sqrt{\sin\theta}\ d\theta$$

$$= 2\sqrt{2}\ \frac{(\sin\theta)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{\frac{\pi}{2}} = \frac{4}{3}\sqrt{2}$$