PROVA 14 LUGLIO 2025

1. Studiare convergenza puntuale e uniforme di

$$f_n(x) = nx\, e^{-nx^2} \qquad n \in \mathbb{N},\ x \in \mathbb{R}$$

$$\lim_{n \to +\infty} \frac{nx}{e^{nx^2}} = 0 \qquad \forall\, x \in \mathbb{R}$$

f_n converge puntualmente a 0 in tutto $\mathbb{R}$

conv uniforme:

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \frac{n|x|}{e^{nx^2}}$$

$$g(x) = \frac{n|x|}{e^{nx^2}}$$

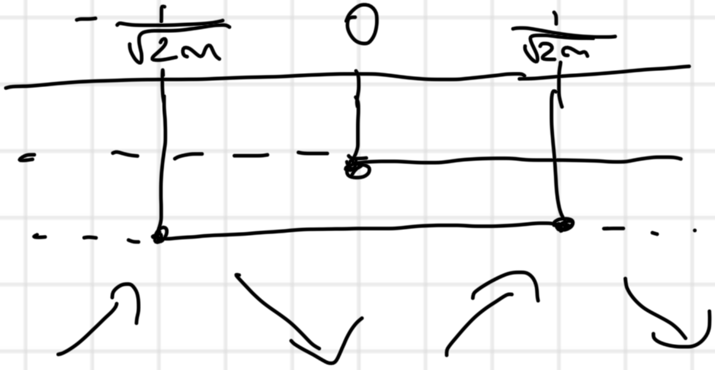
$$g'(x) = \frac{n \frac{x}{|x|} e^{nx^2} - n^2|x| \cdot 2x\, e^{nx^2}}{e^{2nx^2}}$$

$$g'(x) = \frac{nx\, e^{nx^2} - 2n^2 x^3 e^{nx^2}}{|x|\, e^{2nx^2}}$$

$$= \frac{n\, e^{nx^2}}{e^{2nx^2}}\, x \left(1 - 2n x^2\right) \geq 0$$

$x \geq 0$

$$-\frac{1}{\sqrt{2n}} \leq x \leq \frac{1}{\sqrt{2n}}$$



$$x = \pm \frac{1}{\sqrt{2n}}$$

(più semplicemente si poteva osservare che f è dispari e studiarla solo per $x>0$)

$$g\left(\pm\frac{1}{\sqrt{2n}}\right) = \frac{n \cdot \frac{1}{\sqrt{2n}}}{e^{n\frac{1}{2n}}} = \frac{1}{2}\sqrt{\frac{2n}{e}}$$

$$\lim_{n\to+\infty} |f_n(x) - f(x)| = \lim_{n\to+\infty} \frac{1}{2}\sqrt{\frac{2n}{e}} = +\infty$$

non ho convergenza uniforme in $\mathbb{R}$.

(2) trovare i punti critici di f e classificarli

$$f(x,y) = (y+y^3)\log(1+x^2)$$

$$f_x = (y+y^3)\frac{2x}{1+x^2}$$

$$f_y = (1+3y^2)\log(1+x^2)$$

$$\begin{cases} (y+y^3)\dfrac{2x}{1+x^2} = 0 \\ (1+3y^2)\log(1+x^2) = 0 \end{cases}$$

$$\begin{cases} y(1+y^2)\dfrac{2x}{1+x^2} = 0 \\ (1+3y^2)\log(1+x^2) = 0 \end{cases}$$

$$\begin{cases} y=0 \\ \log(1+x^2) = 0 \Rightarrow x=0 \end{cases}$$

$$\begin{cases} x=0 \\ y \in \mathbb{R} \end{cases} \Rightarrow (0, y_0) \text{ punti critici}$$

$$f_{xx} = (y+y^3)\left[\frac{2(1+x^2) - 6x^2}{(1+x^2)^2}\right]$$

$$f_{yy} = 6y\log(1+x^2)$$

$$f_{xy} = f_{yx} = (1+3y^2)\frac{2x}{1+x^2}$$

$$Hf(0, y_0) = \begin{vmatrix} 2(y_0+y_0^3) & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Studiamo il segno:

$$f(x,y) - f(0,y_0) = f(x,y) = y(1+y^2)\log(1+x^2)$$

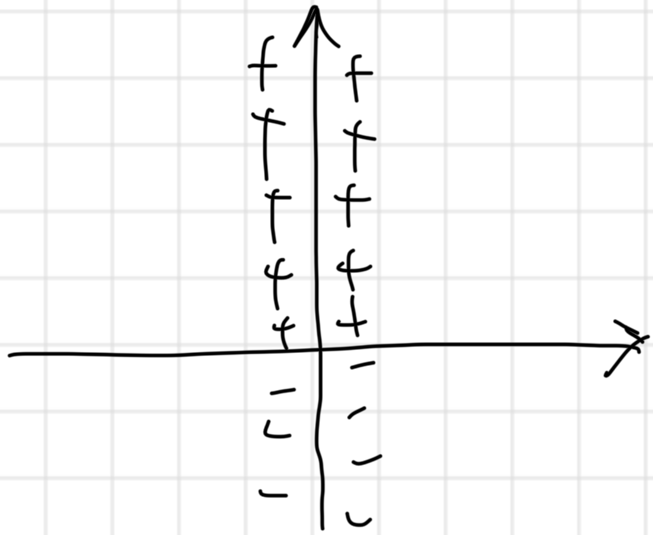
$$f(x,y) \geq 0 \quad \forall y > 0$$

$$f(x,y) < 0 \quad \forall y < 0$$

$(0,0)$ sella

$y_0 > 0$ $\quad (0, y_0)$ min

$y_0 < 0$ $\quad (0, y_0)$ max



3) $$\omega = \frac{x}{1+x^2+y^4}\,dx + \frac{2y^3}{1+x^2+y^4}\,dy$$

$$\frac{\partial a}{\partial y} = -\frac{4y^3 x}{(1+x^2+y^4)^2} \qquad \frac{\partial b}{\partial x} = -\frac{4xy^3}{(1+x^2+y^4)^2}$$

in $\mathbb{R}^2$ semplicemente connesso $\rightarrow$ chiusa $\Rightarrow$ esatta

$$F(x,y) = \int \frac{x}{1+x^2+y^4}\,dx$$

$$= \frac{1}{2}\log(1+x^2+y^4) + g(y)$$

$$F_y = \frac{2y^3}{1+x^2+y^4} + g'(y) = \frac{2y^3}{1+x^2+y^4}$$

$g(y) = \text{cost}$

$$F(x,y) = \frac{1}{2}\log(1+x^2+y^4) + C$$

$$\int_\gamma \omega$$

$$\gamma : \begin{cases} x = t^2 \\ y = t \end{cases} \qquad t \in [0,1]$$

$$= F(1,1) - F(0,0) = \frac{1}{2} \log 3$$

opp. senza primitiva:

$$\int_0^1 \frac{t^2}{1+2t^4} \cdot 2t + \frac{2t^3}{1+2t^4} \cdot 1 \, dt$$

$$= \int_0^1 \frac{4t^3}{1+2t^4} \, dt = \frac{4}{8} \int_0^1 \frac{8t^3}{1+2t^4} \, dt$$

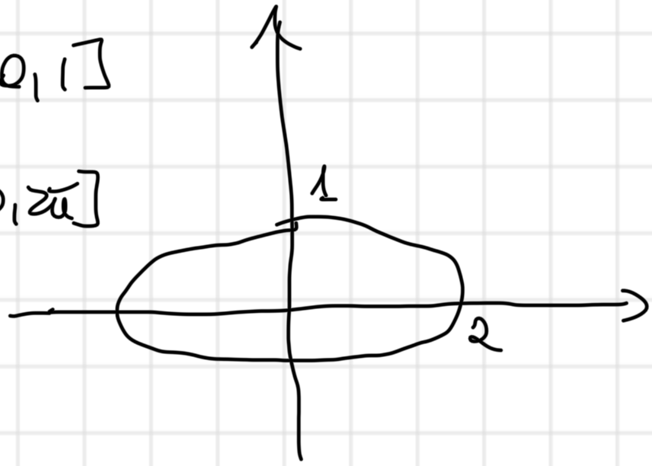
$$= \frac{1}{2} \log(1+2t^4)$$

$$= \frac{1}{2} \log 3$$

4 $\iint_D (x^2+4y^2+6x)\,dx\,dy$

$$D=\left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{4}+y^2\le 1\right\}$$

$$\begin{cases} x = 2\rho\cos\theta \\ y=\rho\,\mathrm{sen}\,\theta \end{cases}\qquad \begin{aligned} &\rho\in[0,1]\\ &\theta\in[0,2\pi]\end{aligned}$$



$$\int_0^{2\pi} d\theta\int_0^1 \left(4\rho^2\cos^2\theta+4\rho^2\,\mathrm{sen}^2\theta+12\rho\cos\theta\right)2\rho\,d\rho$$

$$=\int_0^{2\pi} d\theta\int_0^1 8\rho^3+24\rho^2\cos\theta\,d\rho$$

$$=\int_0^{2\pi} 2+8\cos\theta\,d\theta = 4\pi+8\,\mathrm{sen}\,\theta\Big|_0^{2\pi}$$

$$=4\pi.$$

5) Risolvere il seguente P.C

$$\begin{cases} y' = \frac{y}{x} - xy^2 \\ y(1) = 1 \end{cases}$$

divido per y^2

$$\frac{y'}{y^2} = \frac{y^{-1}}{x} - x$$

$$y^{-1} = z \Rightarrow z' = -\frac{y'}{y^2}$$

$$-z' = \frac{z}{x} - x \Rightarrow z' + \frac{z}{x} = x$$

$$z(x) = e^{-\int \frac{1}{x} dx} \left[\int x e^{\int \frac{1}{x} dx} dx + c \right]$$

$$\Rightarrow z(x) = e^{-\log|x|} \left[\int x e^{\log|x|} + c \right]$$

$$= \frac{1}{|x|} \left[\int x|x| + c \right]$$

$x>0$

$$= \frac{1}{x} \left(\frac{x^3}{3} + c \right)$$

$$y^{-1} = \frac{1}{x} \left(\frac{x^3}{3} + c \right)$$

$$y(x) = \frac{x}{\left(\frac{x^3}{3} + c\right)}$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{\frac{1}{3} + C}$$

$$\Rightarrow \frac{1}{3} + C = 1 \Rightarrow C = \frac{2}{3}$$

$$y(x) = \frac{x}{\frac{x^3}{3} + \frac{2}{3}} = \frac{3x}{x^3 + 2}$$