

TOWARDS A SUBSTITUTION CALCULUS FOR DINATURAL TRANSFORMATIONS

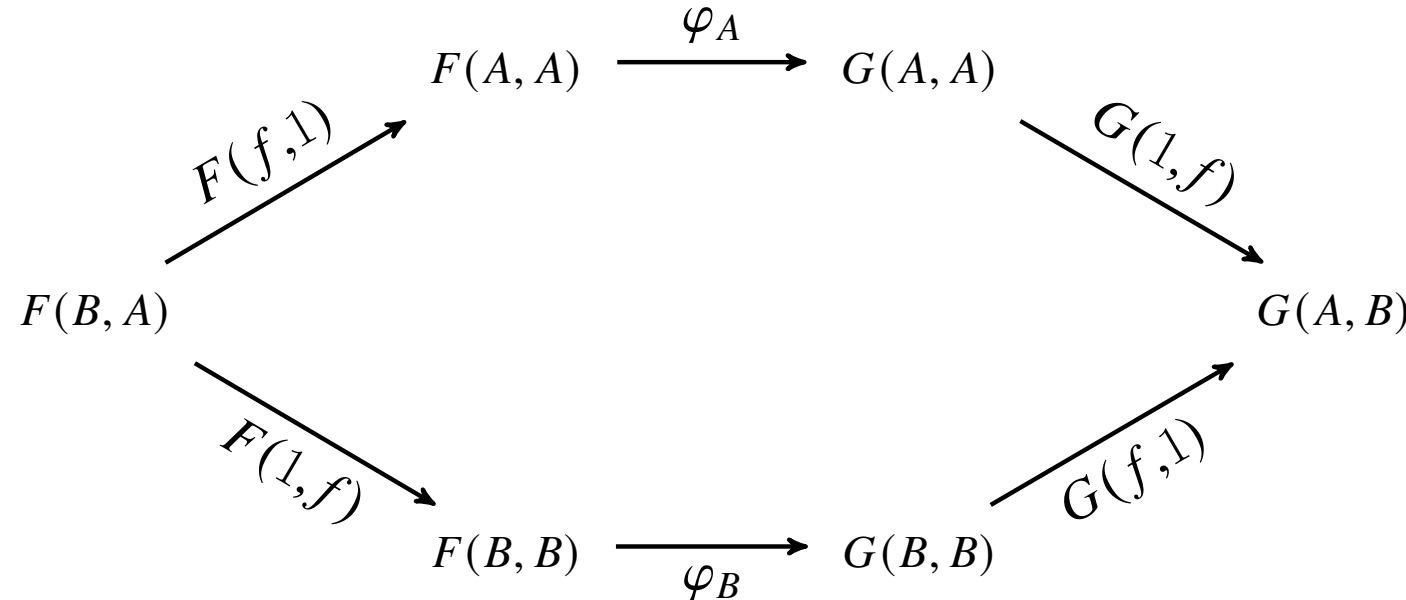
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DINATURAL TRANSFORMATIONS

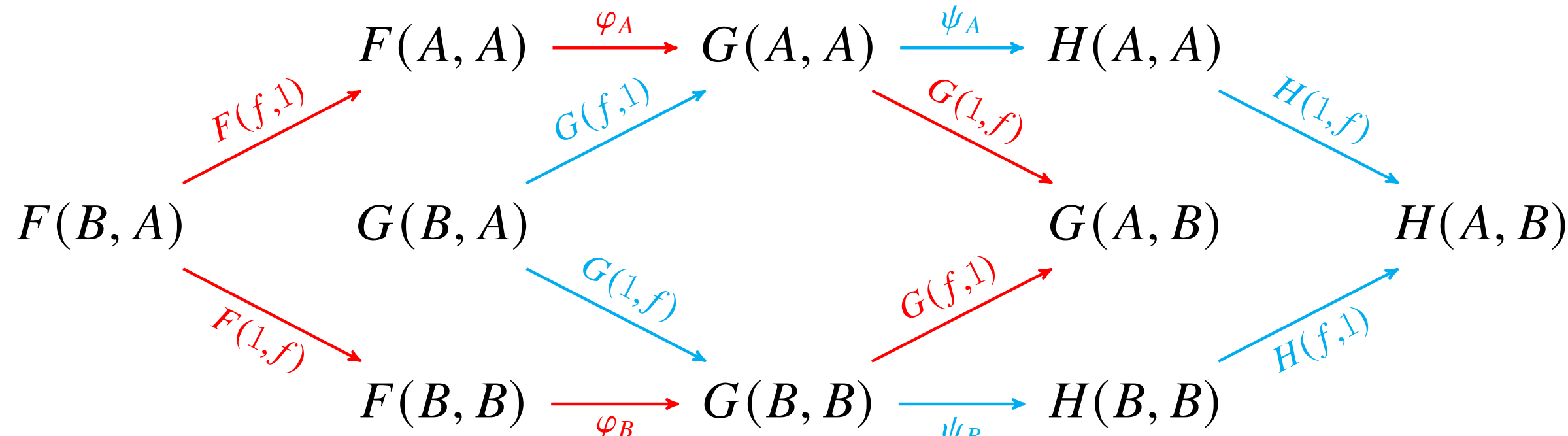
Dinatural transformations [4] generalise natural ones for mixed-variance functors.

Let $F, G: \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \mathbb{D}$ be functors. A family $\varphi = (\varphi_A: F(A, A) \rightarrow G(A, A))_{A \in \mathbb{C}}$ is a *dinatural transformation* $\varphi: F \rightarrow G$ iff for all $f: A \rightarrow B$ in \mathbb{C} the following commutes:



Dinatural transformations arise in many different contexts: they define the notion of *end* [13, 18] of certain functors, which in appropriate enriched contexts allows to compute Kan extensions [5, 9]; they correspond to proofs in intuitionistic and multiplicative linear logic [2, 7, 15]; they provide a semantics for the notion of *parametric polymorphism* in the second-order λ -calculus [1, 3, 6, 17]; they characterise fixed point operators on many categories of domains [16]; and very recently they were proved to provide an operational semantics [8] as well as to model subtyping and bounded quantification in a game semantics [11, 12] for higher-order languages.

Yet, they suffer from a troublesome shortcoming: they do not compose.

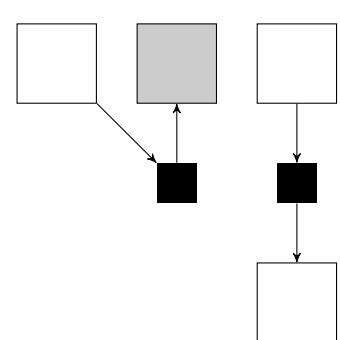


There is no way, in general, to infer the commutativity of the outer hexagon from that of the two inner ones. If either φ or ψ are natural, or if the middle diamond above is a pushout or a pullback, then $\psi \circ \varphi$ is dinatural. However, these are far from being satisfactory solutions for the compositionality problem, for either they are too restrictive (as in the first case), or they speak of properties enjoyed not by φ and ψ themselves, but rather by other structures, namely one of the functors involved.

VERTICAL COMPOSITION

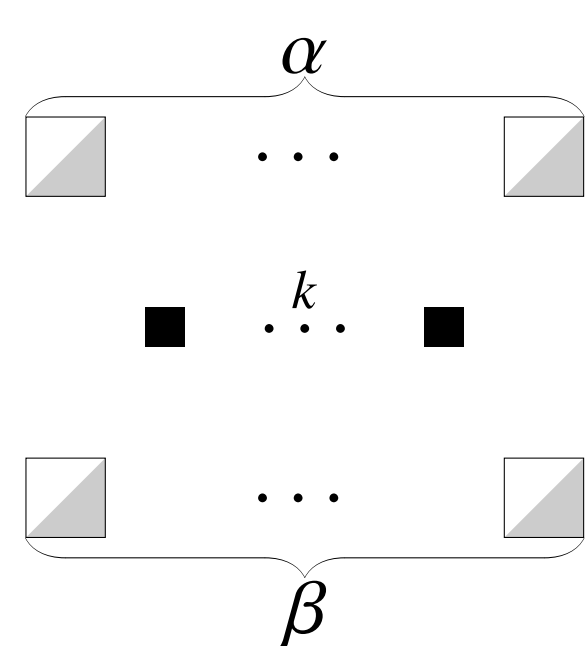
Let \mathbb{C} be a cartesian closed category, and consider the evaluation morphism $\text{eval}_{A,B}: A \times (A \Rightarrow B) \rightarrow B$. We see it as a transformation (i.e. a family of morphisms) $\text{eval}: F \rightarrow \text{id}_{\mathbb{C}}$ dinatural in A and natural in B , where $\mathbb{C} \times \mathbb{C}^{\text{op}} \times \mathbb{C} \xrightarrow{F} \mathbb{C}$ and $(X, Y, Z) \mapsto X \times (Y \Rightarrow Z)$.

We can associate to eval a *graph* that reflects its signature:



The 3 upper boxes correspond to the arguments of F , the 1 lower box to the argument of $\text{id}_{\mathbb{C}}$, the 2 middle black squares to the variables of eval . The edges tell us which arguments of F and $\text{id}_{\mathbb{C}}$ we have to equate to each other when we write down the general component eval_{AB} of the transformation eval . The direction of the edges and the colour shade of the upper/lower boxes keep track of mixed variance.

In general, we consider arbitrary categories \mathbb{B} and \mathbb{C} , functors $F: \mathbb{B}^{\alpha} \rightarrow \mathbb{C}$ and $G: \mathbb{B}^{\beta} \rightarrow \mathbb{C}$, with $\alpha, \beta \in \text{List}\{+, -\}$, and transformations $\varphi = (\varphi_{A_1, \dots, A_k}): F \rightarrow G$ in k variables *together with a graph* $\Gamma(\varphi)$ that reflects its signature:



In the above, the i -th upper/lower box is linked with the j -th middle black square if the i -th argument of F/G is A_j when we write down the general component $\varphi_{A_1, \dots, A_k}$ of φ .

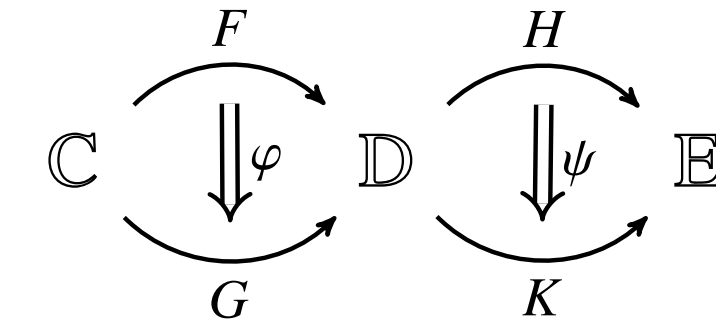
Theorem (Vertical Compositionality), cf.[14]

Let $\varphi: F \rightarrow G$ and $\psi: G \rightarrow H$ be transformations in many variables as above, dinatural in all their variables. If the composite graph $\Gamma(\psi) \circ \Gamma(\varphi)$, obtained by glueing together $\Gamma(\psi)$ and $\Gamma(\varphi)$ along the G -boxes, is *acyclic*, then $\psi \circ \varphi$ is dinatural in all its variables.

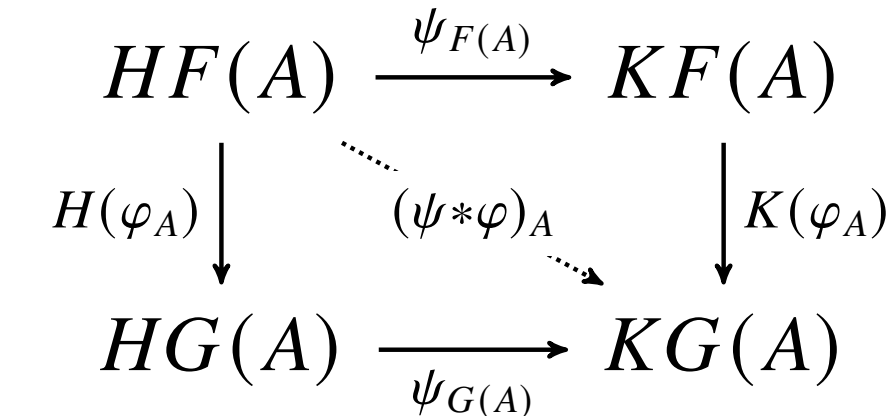
Notice: if φ and ψ are natural, their composite graph is always acyclic. If instead they are both dinatural and of the form $\varphi_A: F(A, A) \rightarrow G(A, A)$ and $\psi_A: G(A, A) \rightarrow H(A, A)$, then $\Gamma(\psi) \circ \Gamma(\varphi)$ is *never* acyclic! We need to consider many-variable functors and transformations to unlock compositionality.

HORIZONTAL COMPOSITION

Natural transformations compose also horizontally. In the following situation:

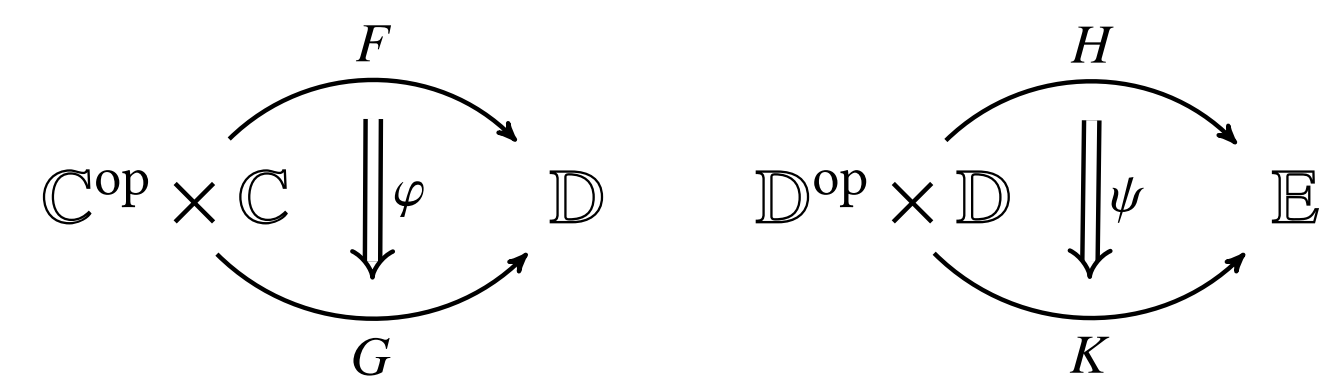


Their horizontal composition $\psi * \varphi: HF \rightarrow KG$ is the natural transformation whose A -th component, $A \in \mathbb{C}$, is either leg of

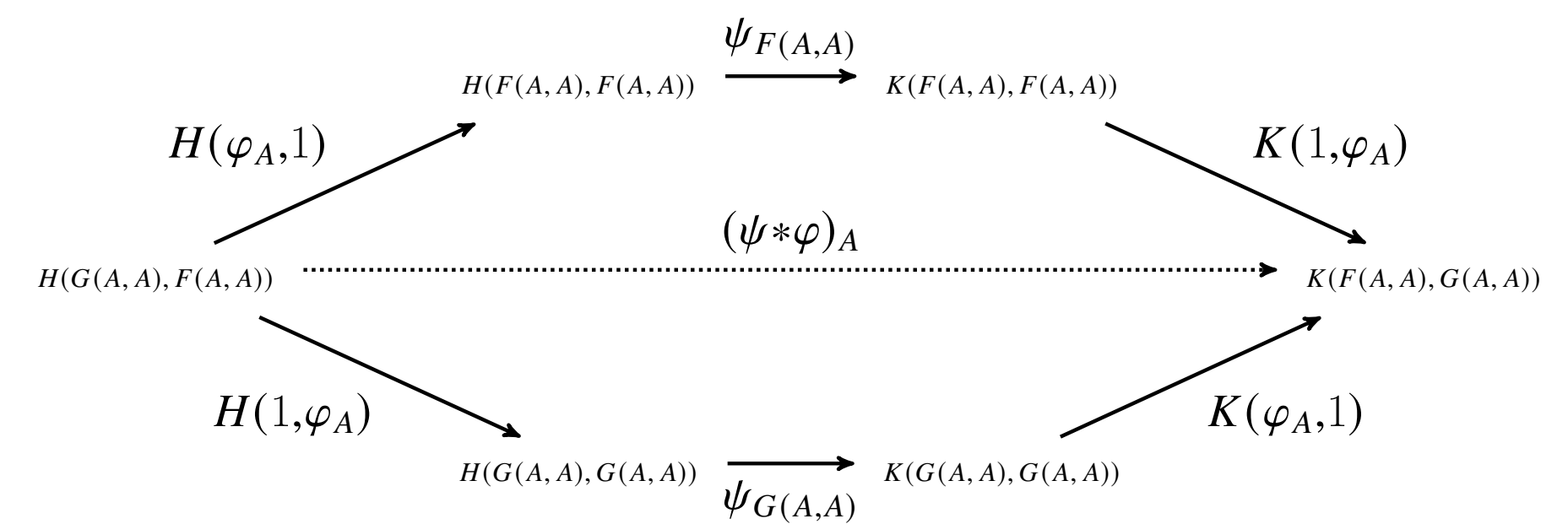


$*$ is associative, unitary, and satisfies an interchange law with vertical composition.

Let now φ, ψ be dinatural transformations.



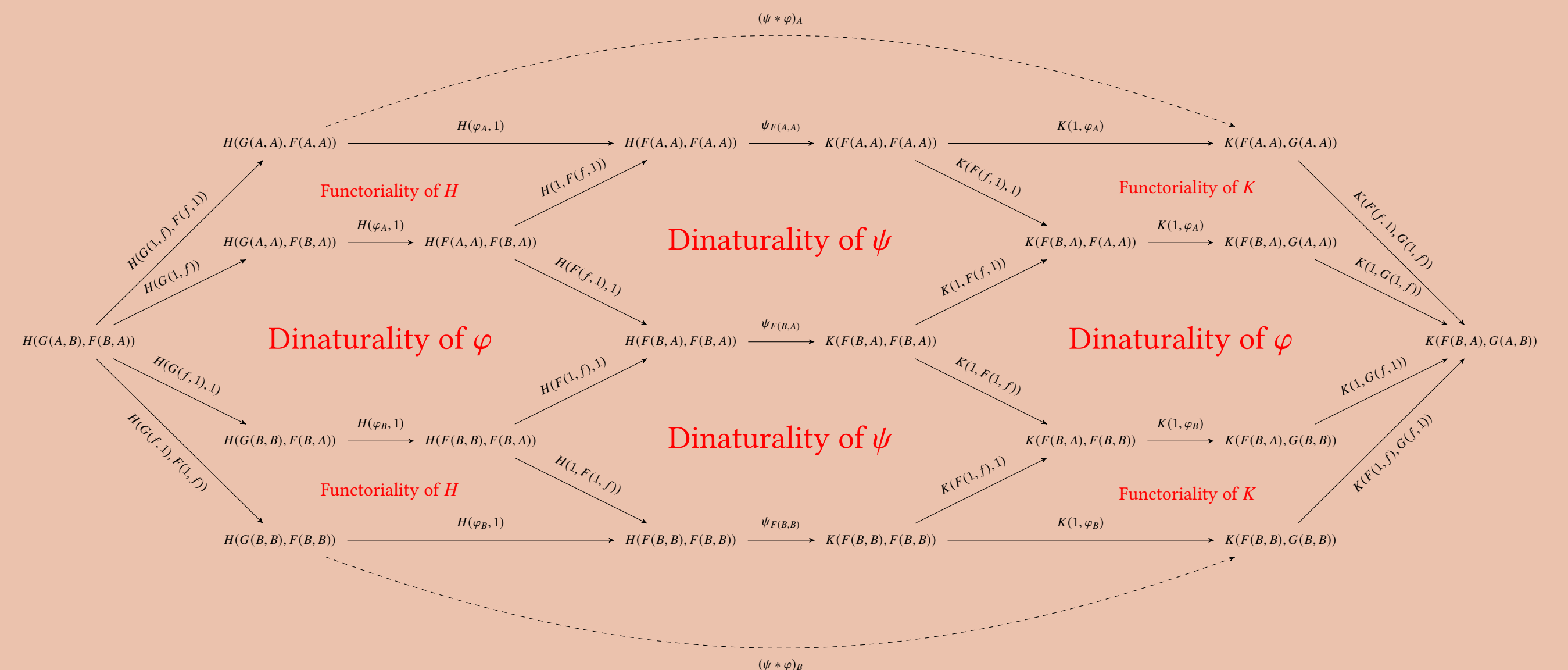
$\psi * \varphi: H(G^{\text{op}}, F) \rightarrow K(F^{\text{op}}, G)$ is the family of morphisms whose A -th component, $A \in \mathbb{C}$, is either leg of



Theorem (Horizontal compositionality)

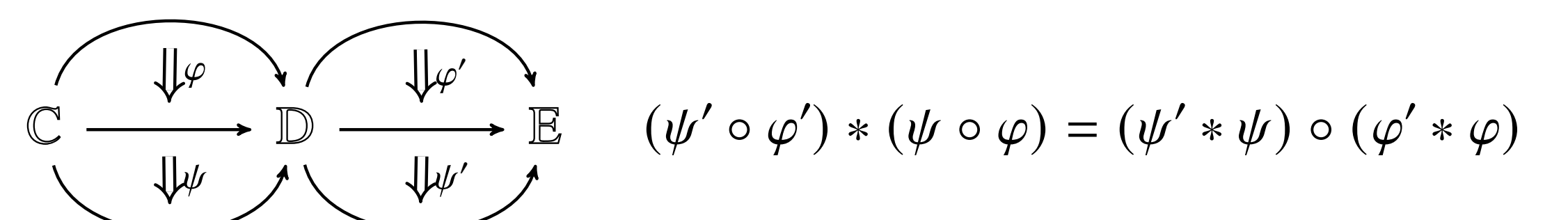
If φ and ψ are dinatural, so is $\psi * \varphi$.

PROOF. The outer hexagon below is the dinaturality condition of $\psi * \varphi$:

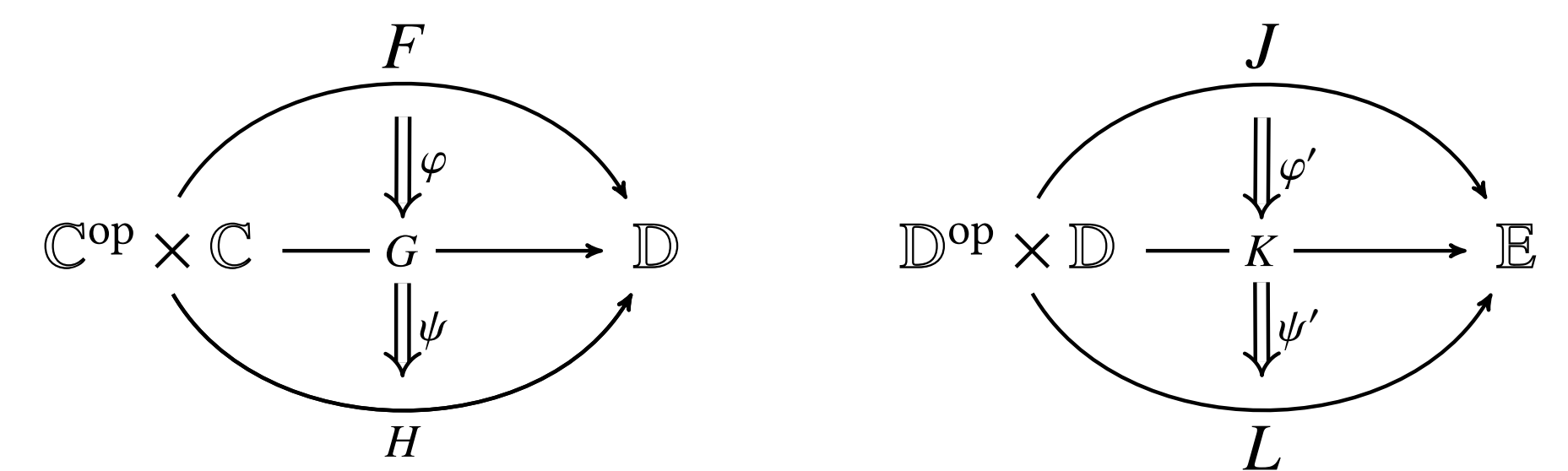


$*$ is associative and unitary. How about an interchange law with vertical composition?

Recall in the natural case:



Let now $\varphi, \psi, \varphi', \psi'$ be dinatural:



such that $\psi \circ \varphi$ and $\psi' \circ \varphi'$ are dinatural. Then

$$\varphi' * \varphi: J(G^{\text{op}}, F) \rightarrow K(F^{\text{op}}, G) \quad \psi' * \psi: K(H^{\text{op}}, G) \rightarrow L(G^{\text{op}}, H)$$

They are not composable at all, not even as transformations!

TOWARDS A SUBSTITUTION CALCULUS

The usual calculus of natural transformations stems from the cartesian closedness of Cat , as it is embodied in the functor $M: [\mathbb{B}, \mathbb{C}] \times [\mathbb{A}, \mathbb{B}] \rightarrow [\mathbb{A}, \mathbb{C}]$. We aim to develop a calculus of many-variable functors and dinatural transformations centred around the notion of *substitution* as originally envisioned by Kelly [10]. Substitution generalises ordinary composition of functors and horizontal composition of (di)natural transformations for the many-variable case. We shall define a generalised functor category $\{\mathbb{B}, \mathbb{C}\}$ over an appropriate category \mathbb{G} of graphs, and prove that $\{\mathbb{B}, -\}$ has a left adjoint $- \circ \mathbb{B}$, making $\text{Cat}_{\mathbb{G}}/\mathbb{G}$ monoidal closed. The analogue of the functor M above, now of the form $\{\mathbb{B}, \mathbb{C}\} \circ \{\mathbb{A}, \mathbb{B}\} \rightarrow \{\mathbb{A}, \mathbb{C}\}$, will yield the desired calculus.

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