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Dipartimento di Informatica

Technical Report

Tape diagrams for rig categories with finite biproducts

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April 21, 2022

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Abstract

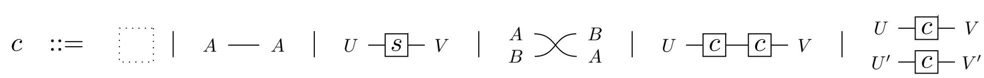
Rig categories with finite biproducts are categories with two monoidal products $\oplus$ and $\otimes$, with the former being a biproduct and the latter distributing over the other. In this report we present tape diagrams, a sound and complete diagrammatic language for rig categories with finite biproducts, which can be thought intuitively as string diagrams of string diagrams.

1 Introduction

Scientists and engineers of different fields often rely on diagrammatic notations to model and design systems: Petri nets, flowcharts, message sequence charts, signal flow graphs, different sorts of circuits and state machines are just a few of the many graphical languages commonly used by computer scientists and software engineers.

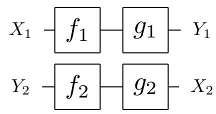
On the one hand, diagrams provide an intuitive and human-readable specification of the relationships and the interactions occurring amongst the various components of a system: for instance, when facing some sort of network, a picture can offer an immediate description of its topology, distribution and connectivity. On the other hand, diagrammatic languages often lack a formal semantics or, when given, this is rarely compositional: usually diagrams cannot be composed as one composes sentences in English, formulas in mathematics, or programs in a programming language.

Motivated by the interest in dealing with languages that are, at the same time, diagrammatic, formal and compositional, a growing number of works exploits *string diagrams* [18, 27], a graphical notation that emerged in the field of category theory. Formally, string diagrams are arrows of a strict symmetric monoidal category freely generated by a monoidal signature. Graphically, they are depicted as the circuits generated by the following context free grammar:



(1)

Every circuits has on the left and on the right several ports, denoted by $A, B \dots$. These ports appear in a precise order, so that the left and the right boundaries of any circuit consist of a word (i.e, a list) of ports, denoted by $U, V \dots$. Formally, the ports are called *basic sorts*, namely elements of some fixed set $\mathcal{S}$; words, i.e. elements of $\mathcal{S}^*$, are objects of the category; circuits are arrows of the category having as source the left boundary and as target the right one. Reading the grammar (1) from left to right, a circuit may be: either the empty circuit (in categorical language, the identity of the monoidal unit 1); or a wire connecting a port A on the left to itself on the right (the identity of A); or some box s (a symbol of the signature); or a circuit switching the order of two ports (a symmetry); or the horizontal composition of two circuits (composition ; of arrows) or the vertical composition of two circuits (the monoidal product $\otimes$ of arrows).



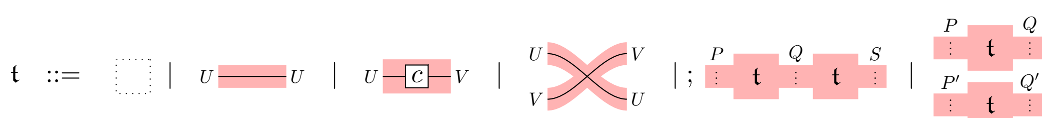
The above picture represents a circuit having as left boundary the word X_1Y_2 and as right boundary Y_1X_2. In technical jargon, it is a string diagram with source object X_1Y_2 and target Y_1X_2. Observe that the above diagram can be regarded as both $(f_1; g_1) \otimes (f_2; g_2)$ and $(f_1 \otimes f_2); (g_1 \otimes g_2)$. Such ambiguity is not an issue, since in any monoidal category the law $(f_1; g_1) \otimes (f_2; g_2) = (f_1 \otimes f_2); (g_1 \otimes g_2)$ holds for all arrows f_1, f_2, g_1, g_2.

More generally, it turns out that the diagrammatic representation identifies exactly *all and only the laws of monoidal categories* [18]. This is the key feauture of string diagrams. Indeed, by virtue of this fact, one can safely exploit diagrams to make proofs, which in this way often amount to simple and suggestive manipulations of diagrams that abstract away from the laws of monoidal categories.

By carefully crafting the monoidal signature, one obtains the syntaxes of several languages specifying a large variety of systems: quantum protocols [9], linear dynamical systems [5], Petri nets [6], concurrent connectors [7], digital circuits [15], automata [24], or conjunctive queries [4] used in databases. However, in different occasions the string diagrammatic syntax seems to be too restrictive and, to avoid such restrictions, several authors started to use a mixture of diagrammatic and algebraic syntax. For instance, in [3] $\sqcup$-props were introduced as a way to enrich string diagrams with a join operation $\sqcup$ expressing, somehow, union of diagrams. A similar operation seems necessary, for instance, to extend the aforementioned conjunctive queries [4] to conjunctive-disjunctive queries.

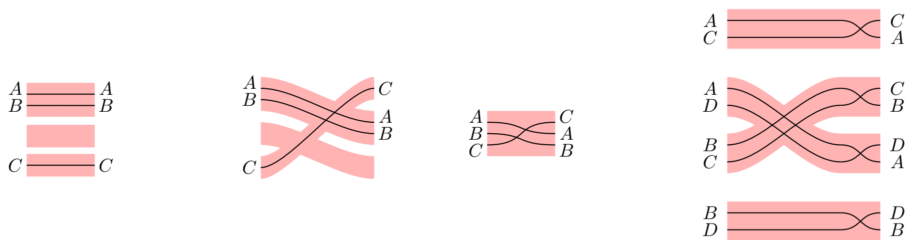
To expand the expressivity of string diagrams, a more radical shift consists in moving from monoidal categories to *rig categories* [20, 17], roughly categories equipped with two monoidal products: $\otimes$ and $\oplus$. The main challenge in depicting arrows of a rig category is given by the possibility of composing them not only with ; (horizontally) and $\otimes$ (vertically), but also with the novel monoidal product $\oplus$. The natural solution consists in exploiting 3 dimensions. This is the approach taken by *sheet diagrams* [12], certain topological manifolds that, modulo a notion of isotopy, capture exactly the laws of rig categories.

In this report, we introduce *tape diagrams* as a way to depict arrows not of arbitrary rig categories but only of those where $\oplus$ is a *biproduct* [22, 10]. The payoff of this restriction in expressiveness is a better usability: tape diagrams are two dimensional pictures and for this reason they are, in our opinion, more intuitive and more drawable than three dimensional diagrams. A second important novelty is that we do not need to define ad-hoc topological structures and transformations, since tape diagrams are, literally, *string diagrams of string diagrams*. Indeed, in a first approximation they are graphically depicted as illustrated by the following context free grammar.



(2)

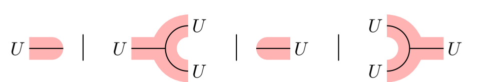
Note that the grammar above is similar to (1), but now diagrams are surrounded by a coloured (in pink or, when black/white printed, gray) area: this is what we call a *tape*. Note moreover that in the third production of the above grammar (U $\boxed{c}$ V) the tape is containing a circuit c generated by (1). This allows us to have two monoidal products in 2 dimensions: vertical composition within a tape represents $\otimes$, while vertical composition of tapes represents $\oplus$. Consider, for instance, the leftmost of the four tape diagrams depicted below. This is obtained as the vertical composition of three tapes: $\substack{A\\B}$ $\substack{A\\B}$ $\oplus$ $\oplus$ C C . The first tape contains a circuit that is the vertical composition of two wires: A — A $\otimes$ B — B , the second tape contains the empty circuit, while the third tape contains a circuit that is a single wire C — C .



Before describing the remaining three diagrams above, it is convenient to observe that also tape diagrams, like string diagrams, have left and right boundaries but these are *words of words of sorts*, i.e. elements of $(\mathcal{S}^*)^*$, rather than simply words in $\mathcal{S}^*$. For instance, the left and the right boundaries of the leftmost diagram are the word of words $AB, 1, C$. Indeed such tape diagram formally represents the identity for the object $AB, 1, C$.

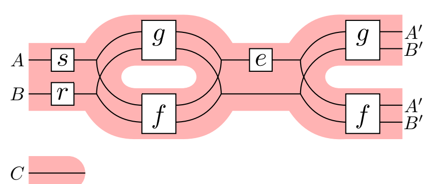
The second (from left to right) diagram has left boundary $AB, 1, C$ and right boundary $C, AB, 1$. Note that this diagram is simply switching the order of tapes so, categorically speaking, it is a symmetry for $\oplus$. The third diagram, with left boundary ABC and right boundary CAB, is switching the order of wires within a tape: this is a symmetry for $\otimes$. One can combine switching of tapes and switching of wires within a tape as illustrated in the rightmost diagram: this has left boundary AC, AD, BC, BD and right boundary CA, CB, DA, DB.

In order to increase expressivity of tape diagrams we extend the grammar in (2) with the following four items, which give the ability of splitting, joining and cutting tapes.



(3)

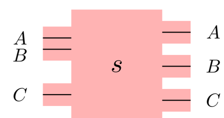
An example of a tape diagram is illustrated below.



By imposing certain axioms on the diagrams in (3), notably those of (co)monoids and bialgebras in Figure 3 (p. 23), the category of tape diagrams turns out to be isomorphic to the rig category with finite biproducts freely generated by a monoidal signature: this is the statement of Theorem 4.24, the main result of this report.

The key for proving Theorem 4.24 is showing that tapes form a rig category: one basically needs to extend $\otimes$ from circuits to arbitrary tapes. This is carefully done by using an inductive definition of left and right whiskerings (Definitions 4.9 and 4.11), that can be readily implemented in a computer program. Despite the simplicity of this definition, the product of two tapes is usually rather big and not really handy. A nice result, Theorem 8.2, helps us in avoiding this issue: when doing diagrammatic reasoning on tapes, namely proofs by means of diagrammatic manipulations, one can safely forget about $\otimes$ on tapes.

At this point it is worth mentioning that both Theorem 4.24 and Theorem 8.2 hold not only for rig categories where $\oplus$ is a biproduct but also for those where $\oplus$ is either a product or a coproduct (see Remarks 4.25 and 8.4). However, in these cases, the results are kind of limited since they only apply when the starting signature is monoidal, whereas in general one would like to use arbitrary *rig signatures* [12]. For instance, one could think to have as symbol of the signature the following diagram,



which does not belong to a monoidal signature since its boundaries, the left one AB, C and right one A, B, C, are strings of strings rather than mere strings. This limitation does not appear when $\oplus$ is a biproduct since, in this case, any rig signature can be reduced to an equivalent monoidal signature: this is the statement of Theorem 7.1. Thus, one can safely state that tape diagrams provide a universal diagrammatic language for rig categories with finite biproducts, a fact that justifies the title of this report.

The proof of Theorem 7.1 is based on a nice result characterising tape diagrams as matrices of multisets of string diagrams (Corollary 6.9). Interestingly enough $\oplus$ of tapes corresponds to direct sum of matrices, while $\otimes$ to their Kronecker product (Theorem 6.12). We conclude this report by testing the expressiveness of our diagrammatic language: we show how the aforementioned $\sqcup$-props can be comfortably translated into tape diagrams so to obtain a purely graphical calculus that avoids the use of algebraic operators.

Structure of the report. The very first issue that we have to deal with is *strictness*: while in monoidal categories natural isomorphisms like associators and unitors can be safely forgotten, the situation is far more complicated for rig categories. For this reason, we are obliged to consider at the beginning not-necessarily strict monoidal/rig categories. To avoid to fill the first pages with lots of diagrams illustrating coherence axioms we defer them to several figures reported in Appendix A. Similarly, the axioms for monoidal/rig categories freely generated by some signature are displayed in Table 4 in between Sections 2 and 3, while the axioms for the freely generated *strict* cases are displayed in the natural order.

A meaningful technical contribution of this report, that we avoided to describe so far, is the introduction of the notion of *sesquistrict* rig category generated by some signature. This notion allows us to significantly simplify various proofs and state Theorem 4.24 as a proper isomorphism, rather than a mere equivalence. The notion of sesquistrict is somehow orthogonal to the results appearing here and, in our opinion, it represents an interesting application of term rewriting to the issue of categorical coherence and strictness. For this reason, the main technical tools allowing to introduce sesquistrict rig categories are developed at the very end of the report, after the various appendixes, in what we named the *Ghost Track*.

The rest of the report is organised as follows. Section 2 recalls monoidal categories, symmetric monoidal categories, finite biproduct categories and the associated languages of string diagrams. Section 3 describes rig categories following the presentation in [17], explains the issue of strictness and introduces the notion of sesquistrict rig category. It concludes by illustrating a few properties of rig categories where $\oplus$ is a biproduct. Section 4 introduces the category of tape diagrams and proves that it is isomorphic to the sesquistrict rig category freely generated by a signature (Theorem 4.24). An alternative proof of the same result is illustrated in Section 5. Section 6 illustrates the matrix calculus underlying tape diagrams, while Section 7 exploits such calculus to show that, in the presence of finite biproducts, any rig signature can be reduced to a monoidal one (Theorem 7.1). In Section 8 we show how contextual reasoning with tapes can be safely performed without using the (sometimes inconvenient) monoidal product $\otimes$ of tapes (Theorem 8.2). Finally, Section 9 illustrates how tape diagrams provide a purely graphical calculus for $\sqcup$-props.

Acknowledgments. We would like to thank Dusko Pavlovic and Pawel Sobocinski for the encouraging comments on the very early ideas underlying these results (several years ago). Moreover, Donald Yau and Samuel Mimram provided some hints for the material appearing in the Ghost Track.

2 Monoidal categories and string diagrams

Definition 2.1. A monoidal category consists of a category $\mathbf{C}$, a bifunctor $\odot \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, an object I and natural isomorphisms

$$\alpha_{X,Y,Z} \colon (X \odot Y) \odot Z \to X \odot (Y \odot Z) \qquad \lambda_X \colon I \odot X \to X \qquad \rho_X \colon X \odot I \to X$$

satisfying the coherence axioms in Figure 4 (p. 66). A monoidal category is said to be *strict* when α, λ and ρ are all identity natural isomorphisms.

The bifunctor $\odot$ and the object I are called *monoidal product* and *monoidal unit*, while the natural isomorphisms α, λ and ρ are called the *associator*, the *left* and the *right unitor*. The strictification theorem [22] allows us to forget about the often-annoying bureaucracy of structural isomorphisms: it states that every monoidal category is monoidally equivalent to a strict one. Unfortunately, the issue of strictness for rig categories is slightly more subtle, as we will see in Section 3, and thus it is useful to consider, from the beginning, not necessarily strict categories.

Since string diagrams are arrows of a strict monoidal category generated by a signature, through this report we will consider different kinds of signatures. We start with the most elementary one: a (single-sorted) *cartesian signature* consists of a set Σ and a function $ar \colon \Sigma \to \mathbb{N}$. For a set X, we denote by $\mathcal{T}_\Sigma(X)$ the sets of Σ-terms with variables in X and just by $\mathcal{T}_\Sigma$ the set of terms without variables. Particularly relevant for this report is $\mathtt{M}$, the signature of monoids: $\mathtt{M} = \{\odot, I\}$ with $ar(\odot) = 2$ and $ar(I) = 0$.

Definition 2.2. A *monoidal signature* consists of a set $\mathcal{S}$, a set Σ and two functions

$$\mathcal{T}_{\mathtt{M}}(\mathcal{S}) \xleftarrow{ar} \Sigma \xrightarrow{coar} \mathcal{T}_{\mathtt{M}}(\mathcal{S}).$$

We call *sorts* the elements of $\mathcal{S}$, *generators* the elements of Σ, *arity* and *coarity* the functions *ar* and *coar*. We will often refer to a signature $(\mathcal{S}, \Sigma)$ just by the set of generators Σ, keeping implicit the set of sorts $\mathcal{S}$.

Given a monoidal signature Σ, one can freely generate a monoidal category from it. The objects of this category are terms in $\mathcal{T}_{\mathtt{M}}(\mathcal{S})$. For arrows, we first consider the set of Σ-terms, defined inductively as

$$\begin{array}{rcl} f & ::= & id_X \mid s \mid f;f \mid f \odot f \\ & & \alpha^{\odot}_{X,Y,Z} \mid \lambda^{\odot}_X \mid \rho^{\odot}_X \mid \alpha^{-\odot}_{X,Y,Z} \mid \lambda^{-\odot}_X \mid \rho^{-\odot}_X \end{array} \tag{4}$$

for $s \in \Sigma$ and arbitrary objects $X, Y, Z \in \mathcal{T}_{\mathtt{M}}(\mathcal{S})$. In the first row of (4), there are identities, generators $s \in \Sigma$, composition ; and monoidal product $\odot$. In the second row, the three natural isomorphism and their inverses. Each Σ-term can be regarded as an arrow by readily associating a domain and a codomain in the expected way: the not familiar reader can have a look to the first four rows of Table 8 (p. 70). Please note that the composition $f;g$ is possible only when the codomain of f is equal to the domain of g. The Σ-terms properly equipped with domain and codomain however are not yet arrows of a monoidal category: one has to first take equivalence classes of Σ-terms modulo some axioms, namely the coherence axioms in Figure 4 and the axioms expressing the laws of categories, $\odot$-functoriality, naturality and inverses (see the first six lines in Table 4, p. 12).

Remark 2.3. Through this report we will introduce other kinds of categories freely generated by signatures: every time we will use the same procedure as above. So, from now on, we will avoid to explain the full details and just illustrate Σ-terms and axioms.

Remark 2.4. In (4), we introduce an identity id_X for every object $X \in \mathcal{T}_{\mathtt{M}}(\mathcal{S})$ and then impose the functoriality axiom $id_{X \odot Y} = id_X \odot id_Y$. An alternative presentation, more convenient in various occasions, consists in dropping the aforementioned axiom and introducing id_X only when X is either I or some sort A in $\mathcal{S}$. This means replacing in the definition of Σ-terms in (4) id_X by id_I and id_A for all $A \in \mathcal{S}$. By using only these basic identities one can define id_X for all $X \in \mathcal{T}_{\mathtt{M}}(\mathcal{S})$ by induction as follows:

$$id_I \stackrel{\text{def}}{=} id_I \qquad id_A \stackrel{\text{def}}{=} id_A \qquad id_{X \odot Y} \stackrel{\text{def}}{=} id_X \odot id_Y$$

The reader can notice that the dropped axiom now holds by definition.

From a monoidal signature one can generate also a free *strict* monoidal category. The objects of this category are terms in $\mathcal{T}_{\mathtt{M}}(\mathcal{S})$ taken modulo associativity and unitarity:

$$(X \odot Y) \odot Z = X \odot (Y \odot Z) \quad I \odot X = X = X \odot I$$

In other words, the set of objects is $\mathcal{S}^\star$, the monoid freely generated by $\mathcal{S}$. Hereafter, we will use $U, V, W, \dots$ to denote elements of $\mathcal{S}^\star$: we will often keep implicit $\odot$, so that an arbitrary element $A_1 \odot A_2 \odot \dots \odot A_n$ would be represented as the word $A_1 A_2 \dots A_n$. Sometimes, it will be convenient to define families of arrows indexed by $\mathcal{S}^\star$ by means of induction: it will suffice to give the definition for I and for AW with $A \in \mathcal{S}$ and $W \in \mathcal{S}^\star$. For instance, the above inductive definition of id_X now becomes

$$id_I \stackrel{\text{def}}{=} id_I \qquad id_{AW} \stackrel{\text{def}}{=} id_A \odot id_W.$$

Arrows of the free strict monoidal category are the Σ-tems generated by the following grammar

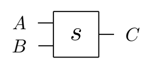
$$f \quad ::= \quad id_A \mid id_I \mid s \mid f;f \mid f \odot f \tag{5}$$

$$(f;g);h = f;(g;h) \qquad id_X;f = f = f;id_Y$$
$$(f_1 \odot f_2);(g_1 \odot g_2) = (f_1;g_1) \odot (f_2;g_2)$$
$$id_I \odot f = f = f \odot id_I \qquad (f \odot g) \odot h = f \odot (g \odot h)$$

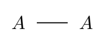
Table 1: Axioms for freely generated a strict monoidal categories

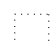
modulo the axioms in Table 1. Note that the axioms are significantly simplified compared to those appearing in Table 4: since the natural isomorphisms are all identities both the coherence axioms and the axioms forcing inverses trivially hold. The axioms expressing naturality of α, λ and ρ are replaced in Table 1 by the axioms in the third row asserting strict associativity and unitarity.

Arrows of a free strict monoidal category enjoy an elegant graphical representation in terms of string diagrams [18, 27]. A generator s in Σ with arity X and coarity Y is depicted as a *box* having *labelled wires* on the left and on the right representing, respectively, the words X and Y. For instance a generator $s\colon AB \to C$ of Σ is depicted as follows.

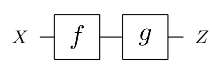


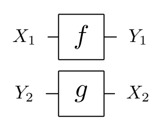
Moreover, $id_A\colon A \to A$ is displayed as a single wire while $id_I\colon I \to I$ as the empty diagram.



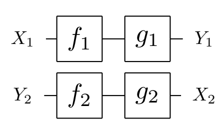


Finally, the composition $f;g$ is represented by connecting the right wires of f with the left wires of g when their labels match, while the monoidal product $f \odot g$ is depicted by stacking the corresponding diagrams on top of each other.





All the axioms of strict monoidal categories (Table 1) are implicit in the graphical representation. For instance, in the functoriality axiom $(f_1;g_1) \odot (f_2;g_2) = (f_1 \odot f_2);(g_1 \odot g_2)$, both the left and the right hand side of the equation are depicted as the same string diagram



2.1 Symmetric Monoidal Categories

Definition 2.5. A monoidal category is said to be *symmetric* when it is equipped with a natural isomorphism $\sigma_{X,Y}\colon X \odot Y \to Y \odot X$ called the *symmetry*, satisfying the axioms in Figure 5 (p. 66).

Like for monoidal categories, one can freely generate a symmetric monoidal category from a signature Σ. Beside the usual monoidal structure (in (4)), one has to add an arrow $\sigma^{\odot}_{X,Y}\colon X \odot Y \to Y \odot X$ for all $X, Y \in \mathcal{T}_{\mathtt{M}}(\mathcal{S})$ and impose the coherence axioms in Figure 5 and the axioms expressesing naturality and inverses of $\sigma_{X,Y}$ (seventh and eighth row in Table 4, p. 12).

$$\sigma_{A,B} ; \sigma_{B,A} = id_{A \odot B} \qquad (s \odot id_Z) ; \sigma_{Y,Z} = \sigma_{X,Z} ; (id_Z \odot s)$$

Table 2: Additional axioms for freely generated symmetric strict monoidal categories.

Also for symmetric monoidal categories the presentation is simplified significantly when generating the free symmetric strict monoidal category compared to the not strict case.. When α, λ and ρ are identities, the coherence axiom (S1) in Figure 5 asserts that $\sigma_{X,I} = id_X$ while axiom (S2) that $\sigma_{X \odot Y, Z} ; (\sigma_{Z,X} \odot id_Y) = id_X \odot \sigma_{Y,Z}$ that is $\sigma_{X \odot Y, Z} = (id_X \odot \sigma_{Y,Z}) ; (\sigma_{X,Y} \odot id_Y)$. One can thus add just $\sigma_{A,B}$ for all sorts $A, B \in \mathcal{S}$ and then define $\sigma_{X,Y}$ for all $X, Y \in \mathcal{S}^\star$ inductively as:

$$\begin{aligned}
\sigma_{I,U} &\stackrel{\text{def}}{=} id_U \\
\sigma_{AW,I} &\stackrel{\text{def}}{=} id_{AW} \\
\sigma_{AW,BU} &\stackrel{\text{def}}{=} (id_A \odot \sigma_{W,BU}) ; (\sigma_{A,B} \odot id_{UW}) ; (id_B \odot \sigma_{A,U} \odot id_W)
\end{aligned}$$

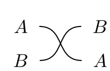
The reader can check that symmetries defined in this way satisfy both $\sigma_{X,I} = id_X$ and $\sigma_{X \odot Y, Z} = (id_X \odot \sigma_{Y,Z}) ; (\sigma_{X,Y} \odot id_Y)$. Moreover, assuming $\sigma_{A,B} ; \sigma_{B,A} = id_{A \odot B}$ for all $A, B \in \mathcal{S}$ is enough to prove that $\sigma_{X,Y} ; \sigma_{Y,X} = id_{X \odot Y}$ for all $X, Y \in \mathcal{S}^\star$. Finally, assuming $(s \odot id_Z) ; \sigma_{Y,Z} = \sigma_{X,Y} ; (id_Z \odot s)$ for all generators $s \in \Sigma$ with arity X and coarity Y is enough to prove that the equivalence holds for an arbitrary arrow f.

Definition 2.6. Given a monoidal signature Σ, we denote by $\mathbf{C}_\Sigma$ the free symmetric strict monoidal category generated by Σ: objects are elements of $\mathcal{S}^\star$; arrows are the Σ-terms inductively generated as

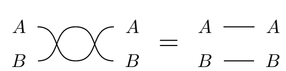
$$f \quad ::= \quad id_A \ \mid\ id_I \ \mid\ s \ \mid\ f ; f \ \mid\ f \odot f \ \mid\ \sigma^{\odot}_{A,B} \tag{6}$$

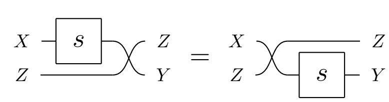
modulo the axioms in Tables 1 and 2.

Also the arrows of symmetric strict monoidal categories enjoy an elegant graphical representation in terms of string diagrams. The symmetry $\sigma_{A,B} \colon A \odot B \to B \odot A$ is depicted as



However the two axioms in Figure 2 are not implicit in the graphical representation. These are displayed as





2.2 Finite product, finite coproduct and finite biproduct categories

Within a symmetric monoidal category it is possible to define certain algebraic structures that can characterise $\odot$ as certain (co)limits: Fox's theorem [14] states that if every object of a symmetric monoidal category is equipped with a natural and commutative comonoid structure, then $\odot$ is the categorical product $\times$.

$$
\begin{tikzcd}
(X \odot X) \odot X \arrow[r, "\alpha_{X,X,X}"] \arrow[d, "\blacktriangleright_X \odot id_X"'] & X \odot (X \odot X) \arrow[r, "id_X \odot \blacktriangleright_X"] & X \odot X \arrow[d, "\blacktriangleright_X"] \\
X \odot X \arrow[rr, "\blacktriangleright_X"'] & & X
\end{tikzcd}
\tag{Mon1}
$$

$$
\begin{tikzcd}
I \odot X \arrow[r, "\text{¡}_X \odot id_X"] \arrow[dr, "\lambda_X"'] & X \odot X \arrow[d, "\blacktriangleright_X"] & X \odot I \arrow[l, "id_X \odot \text{¡}_X"'] \arrow[dl, "\rho_X"] \\
& X &
\end{tikzcd}
\tag{Mon2}
$$

$$
\begin{tikzcd}
X \odot X \arrow[rr, "\sigma_{X,X}"] \arrow[dr, "\blacktriangleright_X"'] & & X \odot X \arrow[dl, "\blacktriangleright_X"] \\
& X &
\end{tikzcd}
\tag{Mon3}
$$

Figure 1: Commutative monoid axioms

$$
\begin{tikzcd}
(X \odot X) \odot X & X \odot (X \odot X) \arrow[l, "\alpha^{-}_{X,X,X}"'] & X \odot X \arrow[l, "id_X \odot \blacktriangleleft_X"'] \\
X \odot X \arrow[u, "\blacktriangleleft_X \odot id_X"] & & X \arrow[ll, "\blacktriangleleft_X"] \arrow[u, "\blacktriangleleft_X"']
\end{tikzcd}
\tag{Com1}
$$

$$
\begin{tikzcd}
I \odot X & X \odot X \arrow[l, "!_X \odot id_X"'] \arrow[r, "id_X \odot !_X"] & X \odot I \\
& X \arrow[ul, "\lambda^{-}_X"] \arrow[u, "\blacktriangleleft_X"'] \arrow[ur, "\rho^{-}_X"'] &
\end{tikzcd}
\tag{Com2}
$$

$$
\begin{tikzcd}
X \odot X \arrow[rr, "\sigma_{X,X}"] & & X \odot X \\
& X \arrow[ul, "\blacktriangleleft_X"] \arrow[ur, "\blacktriangleleft_X"'] &
\end{tikzcd}
\tag{Com3}
$$

Figure 2: Cocommutative comonoid axioms

Definition 2.7. Let $\mathbf{C}$ be a symmetric monoidal category. A *commutative monoid* in $\mathbf{C}$ consists of an object X and arrows $\text{¡}_X : I \to X$ and $\blacktriangleright_X : X \odot X \to X$ making the diagrams in Figure 1 commute. A *cocommutative comonoid* in $\mathbf{C}$ is a commutative monoid in $\mathbf{C}^{op}$, namely, it is an object X with arrows $!_X : X \to I$ and $\blacktriangleleft_X : X \to X \odot X$ making the diagrams in Figure 2 commute.

Hereafter we will often avoid to specify (co)commutative and just write (co)monoids.

Definition 2.8. Let $(\mathbf{C}, \odot, I)$ be a symmetric monoidal category. $\mathbf{C}$ is called:

1. a *finite product* (fp, for short) *category* if every object has a comonoid structure $(\blacktriangleright_X, \text{¡}_X)$ satisfying the coherence axioms in Figure 6 (p. 67) and making $\blacktriangleright$ and ¡ natural transformations.
2. a *finite coproduct* (fc) *category* if every object has a monoid structure $(\blacktriangleleft_X, !_X)$ satisfying the coherence axioms in Figure 7 (p. 67) and making $\blacktriangleleft$ and $!$ natural transformations.
3. a *finite biproduct* (fb) *category* if every object has both natural (in the sense above) monoid and comonoid structures satisfying the coherence axioms in Figures 6 and 7.

Remark 2.9. 1. For any two objects X_1, X_2 of a fp-category $\mathbf{C}$, $X_1 \odot X_2$ is the categorical product $X_1 \times X_2$: the projections $\pi_1 : X_1 \odot X_2 \to X_1$ and $\pi_2 : X_1 \odot X_2 \to X_2$ are

$$X_1 \odot X_2 \xrightarrow{id_{X_1} \odot !_{X_2}} X_1 \odot I \xrightarrow{\rho_{X_1}} X_1 \quad \text{and} \quad X_1 \odot X_2 \xrightarrow{!_{X_1} \odot id_{X_2}} I \odot X_2 \xrightarrow{\lambda_{X_2}} X_2.$$

The unit I is the terminal object and $!_X$ the unique morphism of type $X \to I$. For $f_1 \colon Y \to X_1$, $f_2 \colon Y \to X_2$, their pairing $\langle f_1, f_2 \rangle \colon Y \to X_1 \odot X_2$ is given by

$$Y \xrightarrow{\blacktriangleleft_Y} Y \odot Y \xrightarrow{f_1 \odot f_2} X_1 \odot X_2.$$

2. In a fc category, $X_1 \odot X_2$ is instead a categorical coproduct, with injections $\mu_i \colon X_i \to X_1 \odot X_2$ and copairing $[f_1, f_2] \colon X_1 \odot X_2 \to Y$, for $f_1 \colon X_1 \to Y$ and $f_2 \colon X_2 \to Y$, in the dual way. Also dually I is the initial object and ¡_X the unique morphism of type $I \to X$.

3. In a fb category, $\odot$ is a categorical biproduct, I is a zero object and $!_X ; \text{¡}_Y \colon X \to Y$ is a zero morphism. Traditionally a category $\mathbf{C}$ with a zero object 0 is said to have finite biproducts if for every pair of objects X_1 and X_2 there exists a third object $X_1 \oplus X_2$ and morphisms $\pi_i \colon X_1 \oplus X_2 \to X_i$, $\mu_i \colon X_i \to X_1 \oplus X_2$ such that $(X_1 \oplus X_2, \pi_1, \pi_2)$ is a product, $(X_1 \oplus X_2, \mu_1, \mu_2)$ is a coproduct and
$$\mu_i ; \pi_j = \delta_{i,j}$$
where $\delta_{i,j} = id_{X_i}$ if $i = j$, otherwise $\delta_{i,j} = 0_{X_i, X_j}$ (i.e., $!_{X_i} ; \text{¡}_{X_j}$). However, asking instead that every object is equipped with natural and coherent structures of commutative monoid and comonoid over the same symmetric monoidal product $\oplus$ ensures not only that $\oplus$ is both a product and a coproduct, but also the third requirement $\mu_i ; \pi_j = \delta_{i,j}$. Indeed,

$$\begin{tikzcd}
X_1 \arrow[r, "\rho^-_{X_1}"] \arrow[rr, bend left, "\mu_1"] \arrow[ddrr, bend right, "id_{X_1}"'] & X_1 \oplus 0 \arrow[r, "id_{X_1} \oplus \text{¡}_{X_2}"] \arrow[dr, "id_{X_1} \oplus \text{¡}_{X_2} ; !_{X_2}"] \arrow[dr, bend right, "id_{X_1} \oplus id_0"'] & X_1 \oplus X_2 \arrow[d, "id_{X_1} \oplus !_{X_2}"] \arrow[dd, bend left=60, "\pi_1"] \\
& & X_1 \oplus 0 \arrow[d, "\rho_{X_1}"] \\
& & X_1
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "\lambda^-_{X_2}"] \arrow[rr, bend left, "\mu_2"] \arrow[ddrr, bend right, "id_{X_2}"'] & 0 \oplus X_2 \arrow[r, "\text{¡}_{X_1} \oplus id_{X_2}"] \arrow[dr, "\text{¡}_{X_1} ; !_{X_1} \oplus id_{X_2}"] \arrow[dr, bend right, "id_0 \oplus id_{X_2}"'] & X_1 \oplus X_2 \arrow[d, "!_{X_1} \oplus id_{X_2}"] \arrow[dd, bend left=60, "\pi_2"] \\
& & 0 \oplus X_2 \arrow[d, "\lambda_{X_2}"] \\
& & X_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_1 \arrow[r, "\rho^-_{X_1}"] \arrow[rr, bend left, "\mu_1"] \arrow[d, "!_{X_1}"'] & X_1 \oplus 0 \arrow[r, "id_{X_1} \oplus \text{¡}_{X_2}"] \arrow[d, "!_{X_1} \oplus id_0"] & X_1 \oplus X_2 \arrow[d, "!_{X_1} \oplus id_{X_2}"] \arrow[dd, bend left=60, "\pi_2"] \\
0 \arrow[r, "\rho^-_0"] \arrow[dr, "id_0"'] & 0 \oplus 0 \arrow[r, "id_0 \oplus \text{¡}_{X_2}"] \arrow[d, "\lambda_0"] & 0 \oplus X_2 \arrow[d, "\lambda_{X_2}"] \\
& 0 \arrow[r, "\text{¡}_{X_2}"] & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "\lambda^-_{X_2}"] \arrow[rr, bend left, "\mu_2"] \arrow[d, "!_{X_2}"'] & 0 \oplus X_2 \arrow[r, "\text{¡}_{X_1} \oplus id_{X_2}"] \arrow[d, "id_0 \oplus !_{X_2}"] & X_1 \oplus X_2 \arrow[d, "id_{X_1} \oplus !_{X_2}"] \arrow[dd, bend left=60, "\pi_1"] \\
0 \arrow[r, "\lambda^-_0"] \arrow[dr, "id_0"'] & 0 \oplus 0 \arrow[r, "\text{¡}_{X_1} \oplus id_0"] \arrow[d, "\rho_0"] & X_1 \oplus 0 \arrow[d, "\rho_{X_1}"] \\
& 0 \arrow[r, "\text{¡}_{X_1}"] & X_1
\end{tikzcd}$$

commute, using naturality of λ, ρ and the fact that $\lambda_0 = \rho_0$.

Remark 2.10. If $(\mathbf{C}, \odot, I)$ is a fb category, then $\mathbf{C}$ is automatically enriched over the category **CMon** of commutative monoids: given two morphisms $f, g \in \mathbf{C}[A, B]$, one can define

$$f + g = \Big(\ A \xrightarrow{\blacktriangleleft_A} A \odot A \xrightarrow{f \odot g} B \odot B \xrightarrow{\blacktriangleright_B} B \ \Big) \tag{7}$$

$$0_{A,B} = \Big(\ A \xrightarrow{!_A} I \xrightarrow{\text{¡}_B} B \ \Big) \tag{8}$$

and obtain that $(\mathbf{C}[A, B], +, 0_{A,B})$ is a commutative monoid, that $0_{A,B}$ is a zero morphism and, moreover, that composition distributes over addition of morphisms both on the left and on the right, that is for $u \colon A' \to A$ and $v \colon B \to B'$:

$$f; 0_{B,B'} = 0_{A,B'} \tag{9a}$$

$$0_{A',A}; f = 0_{A',B} \tag{9b}$$

$$(f + g); v = (f; v) + (g; v) \tag{9c}$$

$$u; (f + g) = (u; f) + (u; g) \tag{9d}$$

We also have the following property:

$$\sum_{k=1}^{n} (\pi_k; \mu_k) = id_{\bigoplus_k A_k} \tag{10}$$

Like for (symmetric) monoidal categories, we are interested in the freely generate fp (fc, fb) categories. We will illustrate only the case for fb categories, as fp and fc category can trivially be obtained from fb by just discarding the unnecessary (co)monoids.

To freely generate a fb category from a signature Σ one has to take the usual symmetric monoidal structure, add $!_X$, $\blacktriangleleft_X$, ¡_X, $\blacktriangleright_X$ for all $X \in \mathcal{T}_{\mathtt{M}}(\mathcal{S})$ and impose the axioms expressing naturality (ninth row in Table 4, p. 12), those for (comonoids) in Figures 1, 2 and those for coherence in Figures 6, 7 (pp. 67-67).

We now consider the problem of freely generating a strict fb category. Like for symmetries, it is enough to add as generators $!_A$, $\blacktriangleleft_A$, ¡_A, $\blacktriangleright_A$ for all $A \in \mathcal{S}$ and define $!_X$, $\blacktriangleleft_X$, ¡_X, $\blacktriangleright_X$ for all $X \in \mathcal{S}^*$ inductively as follows:

$$\begin{array}{rcl|rcl} !_I & \stackrel{\text{def}}{=} & id_I & \blacktriangleleft_I & \stackrel{\text{def}}{=} & id_I \\ !_{A \odot W} & \stackrel{\text{def}}{=} & !_A \odot !_W & \blacktriangleleft_{A \odot W} & \stackrel{\text{def}}{=} & (\blacktriangleleft_A \odot \blacktriangleleft_W); (id_A \odot \sigma_{A,W} \odot id_W) \end{array} \tag{11}$$

$$\begin{array}{rcl|rcl} \text{¡}_I & \stackrel{\text{def}}{=} & id_I & \blacktriangleright_I & \stackrel{\text{def}}{=} & id_I \\ \text{¡}_{A \odot W} & \stackrel{\text{def}}{=} & \text{¡}_A \odot \text{¡}_W & \blacktriangleright_{A \odot W} & \stackrel{\text{def}}{=} & (id_A \odot \sigma_{A,W} \odot id_W); (\blacktriangleright_A \odot \blacktriangleright_W) \end{array} \tag{12}$$

With these definitions the coherence axioms in Figures 6 and 7 are automatically satisfied. In the strict setting also the monoid and comonoid axioms get simplified, as illustrated in the first two rows of Table 3. Like for symmetries, to obtain naturality of of $!_X$, $\blacktriangleleft_X$, ¡_X, $\blacktriangleright_X$ for arbitrary arrows, it is enough to impose naturality just with respect to the generators in Σ. However, one has to take care also of naturality of $!_A$, $\blacktriangleleft_A$ with respect to ¡_A, $\blacktriangleright_A$: these are exactly the so called *bialgebra axioms* in the third row of Table 3.

In a nutshell, the free strict fb category generated by Σ is defined as follows. Objects are elements of $\mathcal{S}^\star$; arrows are the Σ-terms inductively generated as

$$f \quad ::= \quad id_A \ \mid \ id_I \ \mid \ s \ \mid \ f;f \ \mid \ f \odot f \ \mid \ \sigma^{\odot}_{A,B} \ \mid !_A \ \mid \blacktriangleleft_A \mid \text{¡}_A \ \mid \blacktriangleright_A \tag{13}$$

$$\begin{array}{cccc}
\blacktriangleleft_A ; (id_A \odot \blacktriangleleft_A) = \blacktriangleleft_A ; (\blacktriangleleft_A \odot id_A) & \blacktriangleleft_A ; (!_A \odot id_A) = id_A & \blacktriangleleft_A ; \sigma_{A,A} = \blacktriangleleft_A \\
(id_A \odot \blacktriangleright_A) ; \blacktriangleright_A = (\blacktriangleright_A \odot id_A) ; \blacktriangleright_A & (¡_A \odot id_A) ; \blacktriangleright_A = id_A & \sigma_{A,A} ; \blacktriangleright_A = \blacktriangleright_A \\
\blacktriangleright_A ; \blacktriangleleft_A = \blacktriangleleft_{A \odot A} ; (\blacktriangleright_A \odot \blacktriangleright_A) & ¡_A ; !_A = id_I \quad ¡_A ; \blacktriangleleft_A = ¡_{A \odot A} & \blacktriangleleft_A ; !_A = !_{A \odot A} \\
s ; !_Y = !_X \quad s ; \blacktriangleleft_Y = \blacktriangleleft_X ; (s \odot s) & ¡_X ; s = ¡_Y & \blacktriangleright_X ; s = (s \odot s) ; \blacktriangleright_Y
\end{array}$$

Table 3: Additional axioms for freely generated strict fb categories.

modulo the axioms in Tables 1, 2 and 3.

Also the arrows of free strict fb categories enjoy an elegant graphical representation in terms of string diagrams. However we are not going to illustrate it now since this would be redundant with tape diagrams.

$$\begin{array}{cr}
(f;g);h = f;(g;h) \qquad id_X ; f = f = f ; id_Y & \text{(cat)} \\
id_{X \odot Y} = id_X \odot id_Y \qquad (f_1 \odot f_2);(g_1 \odot g_2) = (f_1;g_1) \odot (f_2;g_2) & \text{(fun)} \\
(id_I \odot f);\lambda^{\odot}_Y = \lambda^{\odot}_X ; f \qquad (f \odot id_I);\rho^{\odot}_Y = \rho^{\odot}_X ; f & \text{(nat)} \\
(\,(f \odot g) \odot h);\alpha^{\odot}_{X_2,Y_2,Z_2} = \alpha^{\odot}_{X_1,Y_1,Z_1};(f \odot (g \odot h)\,) & \text{(nat)} \\
\lambda^{\odot}_X ; \lambda^{-\odot}_X = id_{I \odot X} \qquad \lambda^{-\odot}_X ; \lambda^{\odot}_X = id_X \qquad \rho^{\odot}_X ; \rho^{-\odot}_X = id_{X \odot I} \qquad \rho^{-\odot}_X ; \rho^{\odot}_X = id_X & \text{(inv)} \\
\alpha^{\odot}_{X,Y,Z} ; \alpha^{-\odot}_{X,Y,Z} = id_{(X \odot Y) \odot Z} \qquad \alpha^{-\odot}_{X,Y,Z} ; \alpha^{\odot}_{X,Y,Z} = id_{X \odot (Y \odot Z)} & \text{(inv)} \\
 & \\
(f \odot id_Z);\sigma_{Y,Z} = \sigma_{X,Y};(id_Z \odot f) & \text{(nat)} \\
\sigma_{X,Y};\sigma_{Y,X} = id_{X \odot Y} & \text{(inv)} \\
 & \\
f;!_Y = !_X \qquad f;\blacktriangleleft_Y = \blacktriangleleft_X ; (f \odot f) \qquad ¡_X ; f = ¡_Y \qquad \blacktriangleright_X ; f = (f \odot f);\blacktriangleright_Y & \text{(nat)} \\
 & \\
(f \otimes (g \oplus h));\delta^l_{X_2,Y_2,Z_2} = \delta^l_{X_1,Y_1,Z_1};((f \otimes g) \oplus (f \otimes h)) & \text{(nat)} \\
((f \oplus g) \otimes h);\delta^r_{X_2,Y_2,Z_2} = \delta^r_{X_1,Y_1,Z_1};((f \otimes h) \oplus (g \otimes h)) & \text{(nat)} \\
(f \otimes id_0);\lambda^{\bullet}_Y = \lambda^{\bullet}_X ; id_0 \qquad (id_0 \otimes f);\rho^{\bullet}_Y = \rho^{\bullet}_X ; id_0 & \text{(nat)} \\
\delta^l_{X,Y,Z};\delta^{-l}_{X,Y,Z} = id_{X \otimes (Y \oplus Z)} \qquad \delta^{-l}_{X,Y,Z};\delta^l_{X,Y,Z} = id_{(X \otimes Y) \oplus (X \otimes Z)} & \text{(inv)} \\
\delta^r_{X,Y,Z};\delta^{-r}_{X,Y,Z} = id_{(X \oplus Y) \otimes Z} \qquad \delta^{-l}_{X,Y,Z};\delta^l_{X,Y,Z} = id_{(X \otimes Z) \oplus (Y \otimes Z)} & \text{(inv)} \\
\lambda^{\bullet}_X ; \lambda^{-\bullet}_X = id_{0 \otimes X} \qquad \lambda^{-\bullet}_X ; \lambda^{\bullet}_X = id_0 \qquad \rho^{\bullet}_X ; \rho^{-\bullet}_X = id_{X \otimes 0} \qquad \rho^{-\bullet}_X ; \rho^{\bullet}_X = id_0 & \text{(inv)}
\end{array}$$

Table 4: Axioms for freely generated monoidal/symmetric monoidal/fb/rig categories. Here $f \colon X \to Y$.

3 Rig categories

Rig categories, also known as *bimonoidal categories*, involve two (symmetric) monoidal structures where one distributes over the other. They were first studied by Laplaza [20], who discovered two coherence results establishing which diagrams necessarily commute as a consequence of the axioms given in their definition. An extensive treatment was recently given by Johnson and Yau [17], from which we borrow most of the notation in this report.

Definition 3.1. A rig category is a category $\mathbf{C}$ with two symmetric monoidal structures $(\mathbf{C}, \otimes, 1, \sigma^{\otimes})$ and $(\mathbf{C}, \oplus, 0, \sigma^{\oplus})$ and natural isomorphisms

$$\delta^l_{X,Y,Z} \colon X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z) \qquad \lambda^\bullet_X \colon 0 \otimes X \to 0$$

$$\delta^r_{X,Y,Z} \colon (X \oplus Y) \otimes Z \to (X \otimes Z) \oplus (Y \otimes Z) \qquad \rho^\bullet_X \colon X \otimes 0 \to 0$$

satisfying the coherence axioms in Figure 8 (p. 68).

The natural isomorphisms δ^l (δ^r) and $\lambda^\bullet$ ($\rho^\bullet$) are called *left* (*right*) *distributor* and *annihilator* respectively. In the literature (see e.g. [17]) the category in Definition 3.1 is usually referred to as a *symmetric* rig category, due to both its monoidal structures being symmetric. For the sake of simplicity, in this report we will just use the term rig category.

Definition 3.2. A rig category is said to be *right* (respectively *left*) *strict* when both its monoidal structures are strict and $\lambda^\bullet, \rho^\bullet$ and δ^r (respectively δ^l) are all identity natural isomorphisms.

The reader may wonder why only one of the two distributors is forced to be the identity within a strict rig category. This can intuitively be explained as follows: imagine requiring that both distributors are identities. This would imply that both equations below should hold for all objects X, Y, Z of any such strict category:

$$(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z) \qquad X \otimes (Y \oplus Z) = (X \otimes Y) \oplus (X \otimes Z).$$

The coexistence of the above laws without the commutativity of $\oplus$ is however problematic since it holds at once that

$$\begin{aligned}(A \oplus B) \otimes (C \oplus D) &= ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D)\\ &= ((A \otimes C) \oplus (B \otimes C)) \oplus ((A \otimes D) \oplus (B \otimes D))\end{aligned}$$

and

$$\begin{aligned}(A \oplus B) \otimes (C \oplus D) &= (A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D))\\ &= ((A \otimes C) \oplus (A \otimes D)) \oplus ((B \otimes C) \oplus (B \otimes D)).\end{aligned}$$

Note that $(B \otimes C)$ and $(A \otimes D)$ are in the opposite order in the two terms. Through these notes, we will always consider right strict categories. Thus, hereafter, we will refer to a right strict rig category simply as a *strict rig category*.

Definition 3.3. A *rig signature* consists of a set $\mathcal{S}$, a set Σ and two functions

$$\mathcal{T}_{\mathtt{R}}(\mathcal{S}) \xleftarrow{ar} \Sigma \xrightarrow{coar} \mathcal{T}_{\mathtt{R}}(\mathcal{S})$$

where $\mathtt{R}$ is the (single-sorted) cartesian signature of *rigs*: $\mathtt{R} = \{\otimes, 1, \oplus, 0\}$ with $ar(\otimes) = ar(\oplus) = 2$ and $ar(1) = ar(0) = 1$

As for the case of monoidal categories, one can freely generate a rig category from a rig signature.

$$
\begin{gathered}
(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \qquad (X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) \\
1 \otimes X = X = X \otimes 1 \qquad 0 \oplus X = X = X \oplus 0 \\
0 \otimes X = 0 = X \otimes 0 \\
(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z)
\end{gathered}
$$

Table 5: Equations for the objects of a free strict rig categories

Definition 3.4. Given a rig signature Σ, we denote by $\mathbf{R}_\Sigma$ the free rig category generated by Σ: objects are terms in $\mathcal{T}_\Sigma(\mathcal{S})$; arrows are are the terms inductively generated as

$$
\begin{array}{rcl}
f & ::= & id_A \mid s \mid f;f \mid id_0 \mid f \oplus f \mid \sigma^{\oplus}_{X,Y} \mid id_1 \mid f \otimes f \mid \sigma^{\otimes}_{X,Y} \\
& & \alpha^{\oplus}_{X,Y,Z} \mid \lambda^{\oplus}_X \mid \rho^{\oplus}_X \mid \alpha^{-\oplus}_{X,Y,Z} \mid \lambda^{-\oplus}_X \mid \rho^{-\oplus}_X \\
& & \alpha^{\otimes}_{X,Y,Z} \mid \lambda^{\otimes}_X \mid \rho^{\otimes}_X \mid \alpha^{-\otimes}_{X,Y,Z} \mid \lambda^{-\otimes}_X \mid \rho^{-\otimes}_X \\
& & \delta^{l}_{X,Y,Z} \mid \delta^{r}_{X,Y,Z} \mid \lambda^{\bullet}_X \mid \rho^{\bullet}_X \mid \delta^{-l}_{X,Y,Z} \mid \delta^{-r}_{X,Y,Z} \mid \lambda^{-\bullet}_X \mid \rho^{-\bullet}_X
\end{array}
\tag{14}
$$

modulo the axioms in Table 4 and the coherence axioms in Figures 4, 5 (for both $\otimes$ and $\oplus$) and 8 (pp. 66, 68).

Recall that, as already mentioned for monoidal categories, domain and codomain of the terms generated by the grammar above can be easily inferred; nonetheless, we report their formal definition in Table 8 (p. 70).

For free strict rig categories, one has to take as objects the terms in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$ modulo the equations in Table 5. These equations come from requiring the natural isomorphisms to be identities. Note that only right distributivity holds, since the left distributor is not forced to be an identity. Most of the twelve coherence axioms in Figure 8 become trivial in a strict rig category, with the exception of (R1), (R2), (R5) and (R9). In particular axiom (R1) states that

$$
\delta^{l}_{X,Y,Z} = \Big(X \otimes (Y \oplus Z) \xrightarrow{\sigma^{\otimes}_{X,Y\oplus Z}} (Y \oplus Z) \otimes X = (Y \otimes X) \oplus (Z \otimes X) \xrightarrow{\sigma^{\otimes}_{Y,X} \oplus \sigma^{\otimes}_{Z,X}} (X \otimes Y) \oplus (X \otimes Z) \Big) \tag{15}
$$

so that one can avoid to add $\delta^{l}_{X,Y,Z}$ as a generator and rather take the above equation as its definition.

Definition 3.5. Given a rig signature Σ, we denote by $\mathbf{sR}_\Sigma$ the free strict rig category generated by Σ: objects are terms in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$ modulo the equations in Table 5 and arrows are inductively generated as

$$
f \quad ::= \quad id_A \mid s \mid f;f \mid id_0 \mid f \oplus f \mid \sigma^{\oplus}_{X,Y} \mid id_1 \mid f \otimes f \mid \sigma^{\otimes}_{X,Y} \tag{16}
$$

modulo the axioms in Table 6 and the coherence axioms in Figure 5 for both $\otimes$ and $\oplus$ and the coherence axioms (R2), (R5) and (R9) in Figure 8.

Observe that in (16), we have symmetries (for both $\oplus$ and $\otimes$) for arbitrary objects X and Y, while in (6) only for sorts $A, B \in \mathcal{S}$. Somehow, the objects of $\mathbf{sR}_\Sigma$ do not enjoy a handy representation as those of strict monoidal categories (namely, strings), and for this reason it is not that simple to construct symmetries for arbitrary objects so to make the coherence axioms in Figure 5 hold by definition.

$$(f;g);h = f;(g;h) \qquad id_X;f = f = f;id_Y$$
$$(f_1 \otimes f_2);(g_1 \otimes g_2) = (f_1;g_1) \otimes (f_2;g_2)$$
$$id_1 \otimes f = f = f \otimes id_1 \qquad (f \otimes g) \otimes h = f \otimes (g \otimes h)$$
$$\sigma^{\otimes}_{X,Y};\sigma^{\otimes}_{Y,X} = id_{X \otimes Y} \qquad (s \otimes id_Z);\sigma^{\otimes}_{Y,Z} = \sigma^{\otimes}_{X,Y};(id_Z \odot s)$$
$$(f_1 \oplus f_2);(g_1 \oplus g_2) = (f_1;g_1) \oplus (f_2;g_2)$$
$$id_0 \oplus f = f = f \oplus id_0 \qquad (f \oplus g) \oplus h = f \oplus (g \oplus h)$$
$$\sigma^{\oplus}_{X,Y};\sigma^{\oplus}_{Y,X} = id_{X \oplus Y} \qquad (s \oplus id_Z);\sigma^{\oplus}_{Y,Z} = \sigma^{\oplus}_{X,Y};(id_Z \oplus s)$$
$$(f \oplus g) \otimes h = (f \otimes h) \oplus (g \otimes h)$$
$$id_0 \otimes f = id_0 = f \otimes id_0$$

Table 6: Additional axioms for freely generating a strict rig category from a signature Σ.

3.1 The sesquistrict rig category generated by a signature

Recall that the objects of a free strict monoidal category are terms in $\mathcal{T}_{\mathtt{M}}(\mathcal{S})$ modulo associativity and unitarity and thus can be conveniently represented as strings. The objects of a free strict rig category are terms in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$ modulo the axioms in Table 5 but, unfortunately, they do not enjoy a very handy form like strings. To overcome this problem, we introduce now a stronger form of strictness for rig categories generated by a rig signature Σ that we call *sesquiscrictness*. The prefix 'sesqui' suggests full right but only partial left strictness.

The very first step consists in extending the algebraic theory in Table 5 with the equations

$$A \otimes (Y \oplus Z) = (A \otimes Y) \oplus (A \otimes Z) \tag{17}$$

for all $A \in \mathcal{S}$. This is the form of partial left strictness that we were referring to: left distributivity holds only when A is a basic sort in $\mathcal{S}$. It is useful to observe that the addition of these equations avoids the problem of using left and right strictness at the same time. Indeed $(A \oplus B) \otimes (C \oplus D)$ turns out to be equal to $(A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D)$ but not to $(A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D)$ as (17) does *not* allow to left distribute $(A \oplus B)$ over $(C \oplus D)$.

We denote with E the congruence relation generated by the equations in Table 5 plus (17) for each $A \in \mathcal{S}$. It turns out that terms in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$ modulo E enjoy a unique normal form that is extremely convenient. To introduce it, we first define n-ary sums and products as follows:

$$\bigoplus_{i=1}^{0} X_i \stackrel{\text{def}}{=} 0 \qquad \bigoplus_{i=1}^{1} X_i \stackrel{\text{def}}{=} X_1 \qquad \bigoplus_{i=1}^{n+1} X_i \stackrel{\text{def}}{=} X_1 \oplus (\bigoplus_{i=1}^{n} X_{i+1})$$
$$\bigotimes_{i=1}^{0} X_i \stackrel{\text{def}}{=} 1 \qquad \bigotimes_{i=1}^{1} X_i \stackrel{\text{def}}{=} X_1 \qquad \bigotimes_{i=1}^{n+1} X_i \stackrel{\text{def}}{=} X_1 \otimes (\bigotimes_{i=1}^{n} X_{i+1})$$

Definition 3.6. A term $X \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$ is said to be in *polynomial* form if there exist n, m_i and $A_{i,j} \in \mathcal{S}$ for $i = 1 \dots n$ and $j = 1 \dots m_i$ such that

$$X = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{m_i} A_{i,j}.$$

For instance 0, 1, $(A \otimes B) \oplus 1$ and $(A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (C \otimes D)$ are in polynomial form, while $(A \oplus B) \oplus (C \oplus D)$, $(A \otimes B) \oplus 0$ and $A \otimes 1$ are not.

We will always refer to terms in polynomial form as *polynomials* and, for a polynomial like X above, we will call *monomials* of X the n terms $\bigotimes_{j=1}^{m_i} A_{i,j}$. For instance the monomials of $(A \otimes B) \oplus 1$ are $A \otimes B$ and 1. Note that, differently from the polynomials we are used to deal with, here neither $\oplus$ nor $\otimes$ are commutative so, for instance, $(A \otimes B) \oplus 1$ is different from both $1 \oplus (A \otimes B)$ and $(B \otimes A) \oplus 1$. Note that this form of non commuative polynomials are in one to one correspondence with *words of words* over $\mathcal{S}$ while monomials are words over $\mathcal{S}$. For instance $(A \otimes B) \oplus 1$ is the word of words consisting of the word AB followed by the empty word 1. Through all these notes, and in particular when drawing tape diagrams, we will always implicitly identify polynomials with words of words.

Proposition 3.7. For every term $X \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$, there exists a unique term $X{\downarrow}$ that is in polynomial form and it is equivalent to X in E.

We call $X{\downarrow}$ the *normal form* of X. The fact that it is unique suggests that E is a rather well-behaved algebraic theory: indeed $X{\downarrow}$ can be used as the canonical representative for the E-equivalence class of X. Moreover, X and Y are equal in E if and only if $X{\downarrow} = Y{\downarrow}$.

In order to obtain a strict rig category where objects are in one to one correspondence with terms modulo E, which, thanks to Proposition 3.7, are in one-to one correspondence with polynomials, one has to force the axioms (17). This can be done by requiring the left distributor $\delta^l_{A,Y,Z}$ for $A \in \mathcal{S}$ to be the identity $id_{AY \oplus AZ}$. Since $\delta^l_{A,Y,Z}$ is defined, in $\mathbf{sR}_\Sigma$, as $\sigma^{\otimes}_{A,Y\oplus Z}; (\sigma^{\otimes}_{Y,A} \oplus \sigma^{\otimes}_{Z,A})$, this amounts to

$$\sigma^{\otimes}_{A,Y\oplus Z}; (\sigma^{\otimes}_{Y,A} \oplus \sigma^{\otimes}_{Z,A}) = id_{AY\oplus AZ}. \tag{SS}$$

Definition 3.8. Given a rig signature Σ, the sesquistrict rig category generated by Σ, hereafter denoted by $\mathbf{ssR}_\Sigma$, is the quotient of $\mathbf{sR}_\Sigma$ by (SS) for all $A \in \mathcal{S}$.

It is often more convenient to work with $\mathbf{ssR}_\Sigma$ rather than $\mathbf{sR}_\Sigma$, although they are equivalent. Moreover, as we will show in Theorem 4.24, $\mathbf{ssR}_\Sigma$ is isomorphic to the category of tape diagrams. There are some useful laws that hold in $\mathbf{ssR}_\Sigma$ but not in $\mathbf{sR}_\Sigma$.

Lemma 3.9. *For all monomials U, V and polynomials Y, Z, the following hold in $\mathbf{ssR}_\Sigma$:*

1. $\delta^l_{U,Y,Z} = id_{UY\oplus UZ}$.
2. *If* $Y = \bigoplus_i U_i$, *then* $\sigma^{\otimes}_{Y,V} = \bigoplus_i \sigma^{\otimes}_{U_i,V}$.

The proof of the above lemma is not particularly interesting and, for this reason, it is illustrated in Appendix C. Instead, the proof of Proposition 3.7 uses some techniques from term rewriting [1] that we will illustrate in the Ghost Track at the end of the report. The same techniques combined with Theorem 4.2 in [2] allow to prove an extremely useful result: a form of coherence for rig categories where only some of the natural isomorphisms of Definition 3.1 are taken into account.

Definition 3.10. Let $\mathbf{R}_\Sigma$ be the rig category freely generated by a signature Σ. *Structural isomorphisms* are arrows of $\mathbf{R}_\Sigma$ defined inductively by the following four conditions:

- Identities are structural isomorphisms.
- $\delta^r_{X,Y,Z}$, $\lambda^{\bullet}_X$, $\rho^{\bullet}_X$, $\alpha^{\oplus}_{X,Y,Z}$, $\lambda^{\oplus}_X$, $\rho^{\oplus}_X$, $\alpha^{\otimes}_{X,Y,Z}$, $\lambda^{\otimes}_X$, $\rho^{\otimes}_X$ and their inverses are structural isomorphisms.

- For all $A \in \mathcal{S}$, $\delta^l_{A,Y,Z}$ and $\delta^{-l}_{A,Y,Z}$ are structural isomorphisms.
- Structural isomorphisms are closed under ; , $\oplus$ and $\otimes$.

Hereafter we will refer to $\delta^r_{X,Y,Z}$, $\lambda^\bullet_X$, $\rho^\bullet_X$, $\alpha^\oplus_{X,Y,Z}$, $\lambda^\oplus_X$, $\rho^\oplus_X$, $\alpha^\otimes_{X,Y,Z}$, $\lambda^\otimes_X$, $\rho^\otimes_X$ as *basic structural isomorphisms*.

Note that symmetries and left distributors $\delta^l_{X,Y,Z}$ for $X \notin \mathcal{S}$ are not considered structural isomorphisms.

Theorem 3.11. *Let $\mathbf{R}_\Sigma$ be the rig category freely generated by a signature Σ. For any object X, there exists a unique structural isomorphism $n_X : X \to X{\downarrow}$. More generally, for any two objects X and Y of $\mathbf{R}_\Sigma$, there exists at most one structural isomorphism from X to Y.*

The proof of the above result is contained in the Ghost Track. Its relevance is provided by the following theorem proved in Appendix C.

Theorem 3.12. *There is an equivalence of rig categories*

$$\begin{tikzcd} \mathbf{R}_\Sigma \arrow[r, bend left, "F"] & \mathbf{ssR}_\Sigma \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $FG = id_{\mathbf{ssR}_\Sigma}$ and $n_X : X \to GF(X)$ is a natural isomorphism.

Remark 3.13. The use of polynomials for the objects of a strict rig category is not a novelty. For instance, the objects of the category of sheet diagrams [12] are indeed polynomials over $\mathcal{S}$, exactly like ours. Similarly, in the strictfication theorem for rig category [17], the objects of the strictified category are polynomials over the objects of the original category. However, to the best of our knowledge, the equation characterisation of sesquistrict rig category is a useful novelty, as it allows us to significantly simplify proofs as well as to state results like Theorem 4.24 as proper isomorphisms rather than equivalences.

3.2 Finite product, finite coproduct, finite biproduct rig categories

In many occasions, one is interested in rig categories where $\oplus$ has some additional structures. For instance, *distributive monoidal categories*, such as the one of sets and relations or the one of vectors spaces and linear maps, are defined to be rig categories where $\oplus$ is a coproduct. notice that in the two aforementioned examples $\oplus$ is actually a biproduct. It is thus worth extending the definitions of fp, fc, fb-categories (Section 2.2) to rig categories.

Definition 3.14. A rig category $\mathbf{C}$ is said to be a finite biproduct (finite product/finite coproduct) rig category if $(\mathbf{C}, \oplus, 0, \sigma^\oplus)$ is a finite biproduct (finite product/finite coproduct) category.

By definition, every object X of a fp-rig category is equipped with a comonoid $\blacktriangleleft_X : X \to X \oplus X$ and $!_X : X \to 0$ and, similarly, in a fc-rig category with a monoid $\blacktriangleright_X : X \oplus X \to X$ and $¡_X : 0 \to X$. How do these structures interact with $\otimes$? The following proposition gives us a satisfactory answer.

Proposition 3.15. In any fp-rig category the following diagrams commute:

$$
\begin{tikzcd}
X \otimes (Y \oplus Y) \arrow[dr, "\delta^l_{X,Y,Y}"] & \\
X \otimes Y \arrow[u, "id_X \otimes \blacktriangleleft_Y"] \arrow[r, "\blacktriangleleft_{X \otimes Y}"] \arrow[d, "\blacktriangleleft_X \otimes id_Y"'] & (X \otimes Y) \oplus (X \otimes Y) \\
(X \oplus X) \otimes Y \arrow[ur, "\delta^r_{X,X,Y}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes 0 \arrow[dr, "\rho^\bullet_X"] & \\
X \otimes Y \arrow[u, "id_X \otimes !_Y"] \arrow[r, "!_{X \otimes Y}"] \arrow[d, "!_X \otimes id_Y"'] & 0 \\
0 \otimes Y \arrow[ur, "\lambda^\bullet_Y"'] &
\end{tikzcd}
$$

Dually, in any fc-rig category the following diagrams commute:

$$
\begin{tikzcd}
X \otimes (Y \oplus Y) \arrow[dr, "id_X \otimes \blacktriangleright_Y"] & \\
(X \otimes Y) \oplus (X \otimes Y) \arrow[u, "\delta^{-l}_{X,Y,Y}"] \arrow[r, "\blacktriangleright_{X \otimes Y}"] \arrow[d, "\delta^{-r}_{X,Y,Y}"'] & X \otimes Y \\
(X \oplus X) \otimes Y \arrow[ur, "\blacktriangleright_X \otimes id_Y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes 0 \arrow[dr, "id_X \otimes \text{¡}_Y"] & \\
0 \arrow[u, "\rho^{-\bullet}_X"] \arrow[r, "\text{¡}_{X \otimes Y}"] \arrow[d, "\lambda^{-\bullet}_Y"'] & X \otimes Y \\
0 \otimes Y \arrow[ur, "\text{¡}_X \otimes id_Y"'] &
\end{tikzcd}
$$

The diagrammatic language that we are introducing in these notes is particularly well-behaved for fb-rig categories: diagrams are in one-to-one correspondence with arrows of the fb-rig category generated by *any* rig signature Σ. The one-to-one correspondence also holds for fp and fc rig categories but only for *some* rig signatures. For this reason we illustrate our construction for fb-rig categories, but Theorems 4.24 and 8.2 hold also for fp- and fc-rig categories by simply forgetting the unnecessary (co)monoids.

Definition 3.16. Given a rig signature Σ, we denote by $\mathbf{R}^\flat_\Sigma$ the *free fb-rig category* generated by Σ: objects are terms in $\mathcal{T}_\Sigma(\mathcal{S})$; arrows are terms inductively generated by (14) and

$$!_X \;\mid \blacktriangleleft_X \mid \text{¡}_X \;\mid \blacktriangleright_X \tag{18}$$

modulo the axioms expressing naturality (ninth row in Table 4, p. 12), those for (co)monoids in Figures 1, 2 (p. 9) and those for coherence in Figures 6, 7 (p. 67).

The *free strict fb-rig category* generated by Σ has as objects terms in $\mathcal{T}_\Sigma(\mathcal{S})$ modulo the equations in Table 5 (p. 14) and arrows are terms inductively generated by (16) and (18) subject to the axioms in Table 3 (p. 12).

By quotienting the strict fb-rig category by axiom (SS), one obtains the *sesquistrict fb-rig category* generated by Σ, hereafter denoted by $\mathbf{ssR}^\flat_\Sigma$.

The following useful result is an immediate consequence of Proposition 3.15 and Lemma 3.9.

Lemma 3.17. *Let U be a monomial and Y a polynomial. The following holds in $\mathbf{ssR}^\flat_\Sigma$:*

1. $id_U \otimes \blacktriangleleft_Y = \blacktriangleleft_{UY} = \blacktriangleleft_U \otimes id_Y$.

2. $id_U \otimes !_Y = !_{UY} = !_U \otimes id_Y$.

3. $id_U \otimes \blacktriangleright_Y = \blacktriangleright_{UY} = \blacktriangleright_U \otimes id_Y$.

4. $id_U \otimes ¡_Y = ¡_{UY} = ¡_U \otimes id_Y$.

Theorem 3.18. *There is an equivalence of rig categories*

$$\begin{tikzcd} \mathbf{R}^\flat_\Sigma \arrow[r, bend left, "F"] & \mathbf{ssR}^\flat_\Sigma \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $FG = id_{\mathbf{ssR}^\flat_\Sigma}$ *and* $n_X \colon X \to GF(X)$ *is a natural isomorphism.*

4 Tape diagrams

We have seen in Section 2 that string diagrams provide a convenient graphical language for strict monoidal categories. In this section, we introduce tape diagrams to graphically represent arrows of strict rig categories.

In order to introduce our main idea, it is convenient to recall that there is an adjunction between **CAT**, the category of categories and functors, and **SMC**, the category of symmetric strict monoidal categories and functors.

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "F_1"] \arrow[r, phantom, "\perp"] & \mathbf{SMC} \arrow[l, bend left, "U_1"] \end{tikzcd}$$

It turns out that, for any symmetric strict monoidal category $(\mathbf{M}, \otimes, 1, \sigma^{\otimes})$, $F_1U_1(\mathbf{M})$, namely the symmetric monoidal category obtained by first forgetting the monoidal structure of $\mathbf{M}$ and then freely adding a novel monoidal structure $(\oplus)$, is a strict rig category (cf. Remark 4.16). In particular, if $\mathbf{M}$ is $\mathbf{C}_\Sigma$, the free strict monoidal category generated by a monoidal signature Σ, $F_1U_1(\mathbf{C}_\Sigma)$ is isomorphic to $\mathbf{ssR}_\Sigma$ (Definition 3.8). Now, the arrows of $\mathbf{C}_\Sigma$ are string diagrams, while the arrows of $F_1U_1(\mathbf{C}_\Sigma)$ can be intuitively thought as string diagrams of string diagrams: a graphical language for $\mathbf{ssR}_\Sigma$.

Rather than making this statement more precise, it is worth saying that $F_1U_1(\mathbf{C}_\Sigma)$ is not a particularly interesting rig category, since the signature Σ is just monoidal, not a rig signature, and the additional monoidal structure does not really interact with Σ. We thus rather consider the adjunction between **Cat** and **FBC**, the category of strict finite biproduct categories and functors.

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathbf{FBC} \arrow[l, bend left, "U_2"] \end{tikzcd} \tag{19}$$

Definition 4.1. Let $\mathbf{C}$ be a category. The strict fb category freely generated by $\mathbf{C}$, hereafter denoted by $F_2(\mathbf{C})$, has as objects words of objects of $\mathbf{C}$. Arrows are terms inductively generated by the following grammar, where A, B and c range over arbitrary objects and arrows of $\mathbf{C}$:

$$f \ ::= \ id_A \ | \ id_I \ | \ \overline{\underline{c}} \ | \ f;f \ | \ f \odot f \ | \ \sigma^{\odot}_{A,B} \ | \ !_A \ | \ \blacktriangleleft_A \ | \ ¡_A \ | \ \blacktriangleright_A \tag{20}$$

modulo the axioms in Tables 1, 2 and 3 (pp. 7, 8, 12) and the following two:

$$\overline{\underline{id_A}} = id_A \qquad \overline{\underline{c;d}} = \overline{\underline{c}};\overline{\underline{d}} \tag{Tape}$$

The assignment $\mathbf{C} \mapsto F_2(\mathbf{C})$ easily extends to functors $H \colon \mathbf{C} \to \mathbf{D}$. The unit of the adjunction $\eta \colon Id_{\mathbf{Cat}} \Rightarrow F_2 U_2$ is defined for each category $\mathbf{C}$ as the functor

$$\underline{\overline{\cdot}} \colon \mathbf{C} \to U_2 F_2(\mathbf{C})$$

which is the identity on objects and maps every arrow c in $\mathbf{C}$ into the arrow $\underline{\overline{c}}$ of $U_2 F_2(\mathbf{C})$. Observe that $\underline{\overline{\cdot}}$ is indeed a functor, namely an arrow in $\mathbf{Cat}$, thanks to the axioms (Tape). We will refer hereafter to this functor as the *taping functor*.

Given a strict fb category $\mathbf{D}$ and functor $H \colon \mathbf{C} \to U_2(\mathbf{D})$, one can define the fb-functor $H^\sharp \colon F_2(\mathbf{C}) \to \mathbf{D}$ inductively on objects of $F_2(\mathbf{C})$ as

$$H^\sharp(I) = I \qquad H^\sharp(AW) = H(A) \odot H^\sharp(W)$$

and on arrows as

$$H^\sharp(id_I) = id_I \qquad H^\sharp(id_A) = H(id_A) \qquad H^\sharp(\underline{\overline{c}}) = H(c)$$

$$H^\sharp(f;g) = H^\sharp(f); H^\sharp(g) \qquad H^\sharp(f \odot g) = H^\sharp(f) \odot H^\sharp(g) \qquad H^\sharp(\sigma_{A,B}) = \sigma_{H(A),H(B)}$$

$$H^\sharp(!_A) = !_{H(A)} \qquad H^\sharp(\blacktriangleleft_A) = \blacktriangleleft_{H(A)} \qquad H^\sharp(¡_A) = ¡_{H(A)} \qquad H^\sharp(\blacktriangleright_A) = \blacktriangleright_{H(A)}$$

Observe that $H^\sharp$ is well-defined:

$$\begin{aligned} H^\sharp(\underline{\overline{id_A}}) &= H(id_A) && (\text{Def. } H^\sharp) \\ &= H^\sharp(id_A) && (\text{Def. } H^\sharp) \end{aligned}$$

$$\begin{aligned} H^\sharp(\underline{\overline{c;d}}) &= H(c;d) && (\text{Def. } H^\sharp) \\ &= H(c);H(d) && (\text{Fun. } H) \\ &= H^\sharp(\underline{\overline{c}}); H^\sharp(\underline{\overline{d}}) && (\text{Def. } H^\sharp) \\ &= H^\sharp(\underline{\overline{c}}; \underline{\overline{d}}) && (\text{Fun. } H^\sharp) \end{aligned}$$

The axioms in Tables 1, 2 and 3 are preserved by $H^\sharp$, since they hold in $\mathbf{D}$. By definition, $\underline{\overline{\cdot}}; H^\sharp = H$. Moreover $H^\sharp$ is the unique strict fb functor satisfying this equation. Thus indeed $F_2 \dashv U_2$.

Now, given a symmetric strict monoidal category $(\mathbf{M}, \otimes, 1, \sigma^{\otimes})$, the fb-category obtained by forgetting the monoidal structure and freely adding the fb-structure, namely $F_2 U_1(\mathbf{M})$, is a strict fb-rig category. Moreover when $\mathbf{M}$ is $\mathbf{C}_\Sigma$, $F_2 U_1(\mathbf{C}_\Sigma)$ is isomorphic to $\mathbf{ssR}^\flat_\Sigma$. This is the main result of this section: Theorem 4.24.

To motivate our interest in such result, let us give a closer look to $F_2 U_1(\mathbf{C}_\Sigma)$, which we will hereafter refer to as $\mathbf{T}_\Sigma$.

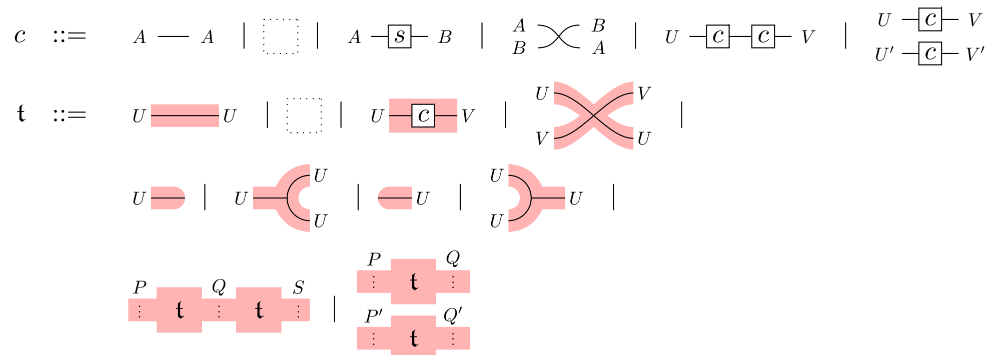
Recall that the set of objects of $\mathbf{C}_\Sigma$ is $\mathcal{S}^*$, i.e. words of sorts in $\mathcal{S}$. The set of objects of $\mathbf{T}_\Sigma$ is thus $(\mathcal{S}^*)^*$, namely words of words of sorts in $\mathcal{S}$. Following the convention adopted since Section 2, we will denote by $A, B, C \dots$ the sorts in $\mathcal{S}$, by $U, V, W, \dots$ the words in $\mathcal{S}^*$ and by $P, Q, R, S, \dots$ the words of words in $(\mathcal{S}^*)^*$.

For arrows, consider the following two-layer grammar where $s \in \Sigma$, $A, B \in \mathcal{S}$ and $U, V \in \mathcal{S}^*$.

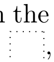
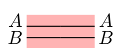
$$\begin{array}{rcl} c & ::= & id_A \mid id_1 \mid s \mid c;c \mid c \otimes c \mid \sigma_{A,B} \\ \mathfrak{t} & ::= & id_U \mid id_0 \mid \underline{\overline{c}} \mid \mathfrak{t};\mathfrak{t} \mid \mathfrak{t} \oplus \mathfrak{t} \mid \sigma^{\oplus}_{U,V} \mid !_U \mid \blacktriangleleft_U \mid ¡_U \mid \blacktriangleright_U \end{array} \tag{21}$$

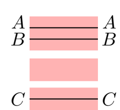
The terms of the first row, denoted by c, are called *circuits*. Modulo the axioms in Tables 1 and 2 (after replacing $\odot$ with $\otimes$), these are exactly the arrows of $\mathbf{C}_\Sigma$ (see Definition 2.6). The terms of the second row, denoted by $\mathfrak{t}$, are called *tapes*. Modulo the axioms in Tables 1, 2, 3 and (Tape) (after replacing $\odot$ with $\oplus$ and A with U), these are exactly the arrows of $F_2U_1(\mathbf{C}_\Sigma)$, i.e., $\mathbf{T}_\Sigma$.

Now, since circuits are arrows of $\mathbf{C}_\Sigma$, these can be graphically represented as string diagrams. Also tapes can be represented as string diagrams, since they satisfy all the axioms of strict fb-categories. Note however that *inside* tapes, there are string diagrams, which justifies the motto *tape diagrams are string diagrams of string diagrams*. We can thus render graphically the grammar in (21):

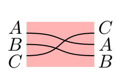
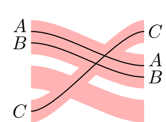


(22)

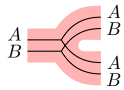
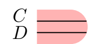
We now explain the graphical language starting from the identities. The identity id_0 is rendered as the empty tape , while id_1 is : a tape filled with the empty circuit. For a monomial $U = \bigotimes_{i=1}^n A_i$, id_U is depicted as a tape containing n wires labelled by A_i. For instance, id_{AB} is rendered as . When clear from the context, we will simply represent it as a single wire  with the appropriate label. Similarly, for a polynomial $P = \bigoplus_{i=1}^n U_i$, id_P is obtained as a vertical composition of tapes. For example, $id_{AB\oplus 1\oplus C}$ is



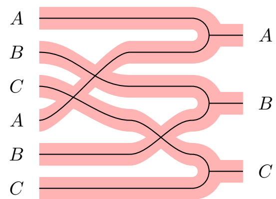
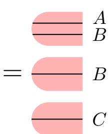
Similarly to identities, we can render graphically the symmetries $\overline{\sigma_{U,V}}\colon UV \to VU$ and $\sigma^{\oplus}_{P,Q}\colon P \oplus Q \to Q \oplus P$ as crossings of wires and crossings of tapes, respectively. For example, $\overline{\sigma_{AB,C}}\colon ABC \to CAB$ and $\sigma^{\oplus}_{AB\oplus 1,C}\colon AB \oplus 1 \oplus C \to C \oplus AB \oplus 1$ are

$\overline{\sigma_{AB,C}} =$  $\qquad \sigma^{\oplus}_{AB\oplus 1,C} =$ 

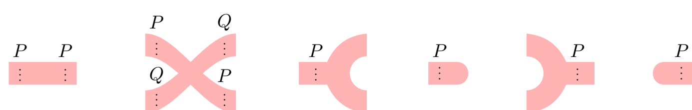
To conclude, we show how to depict the fb-structure. The diagonal $\blacktriangleleft_U\colon U \to U \oplus U$ is represented as a splitting of tapes, while the bang $!_U\colon U \to 0$ is a tape closed on its right boundary. For example, $\blacktriangleleft_{AB}\colon AB \to AB \oplus AB$ and $!_{CD}\colon CD \to 0$ are

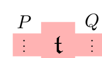
$\blacktriangleleft_{AB}=$  $\qquad !_{CD} =$ 

Codiagonals and cobangs are represented in the same way but mirrored about the y-axis. Moreover, exploiting their inductive definitions in (11) and (12), we can construct (co)diagonals and (co)bangs for arbitrary polynomials P. For example, computing $\blacktriangleright_{A\oplus B\oplus C}$ and $¡_{AB\oplus B\oplus C}$ yields the following diagrams:

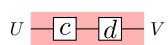
$\blacktriangleright_{A\oplus B\oplus C}=$  $\qquad ¡_{AB\oplus B\oplus C} =$ 

When the structure inside a tape is not relevant the graphical language can be "compressed" in order to simplify the diagrammatic reasoning. For example, for arbitrary polynomials P, Q we represent $id_P, \sigma^{\oplus}_{P,Q}, \blacktriangleleft_P, !_P, \blacktriangleright_P, ¡_P$ as follows:



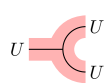
Moreover, for an arbitrary tape diagram $\mathfrak{t}\colon P \to Q$ we write .

It is important to observe that the graphical representation takes care of the two axioms in (Tape): both sides of the leftmost axioms are decipcted as  while both sides of the rightmost axiom as



The axioms of monoidal categories (in Table 1) are also implicit in the graphical representation, while those for symmetries and the fb-structure (in Tables 2 and 3) have to be depicted explicitly as in Figure 3. In particular, the diagrams in the first row express the inverse law and naturality of $\sigma^{\oplus}$. In the second and third row there are the (co)monoid axioms and in the fourth row the biaglebra ones. Finally, the last two rows express naturality of the (co)diagonals and (co)bangs.

Remark 4.2. The graphical representation of the tape $\blacktriangleleft_U$ does *not* contain any morphism of $\mathbf{C}_\Sigma$ within itself. The black wire inside

$\blacktriangleleft_U=$  $: U \to U \oplus U$

is only a graphical expedient that we adopted by a design choice to reflect the fact that $\blacktriangleleft_U$ forks out the tape id_U, which *does* contain the circuit id_U of $\mathbf{C}_\Sigma$ since $id_U = \overline{id_U}$, into two parallel ones $id_U \oplus id_U$, but $\blacktriangleleft_U$ is not of the form $\overline{c}$ for any $c \in \mathbf{C}_\Sigma$. (And rightly so: $\blacktriangleleft_U$ is one of those

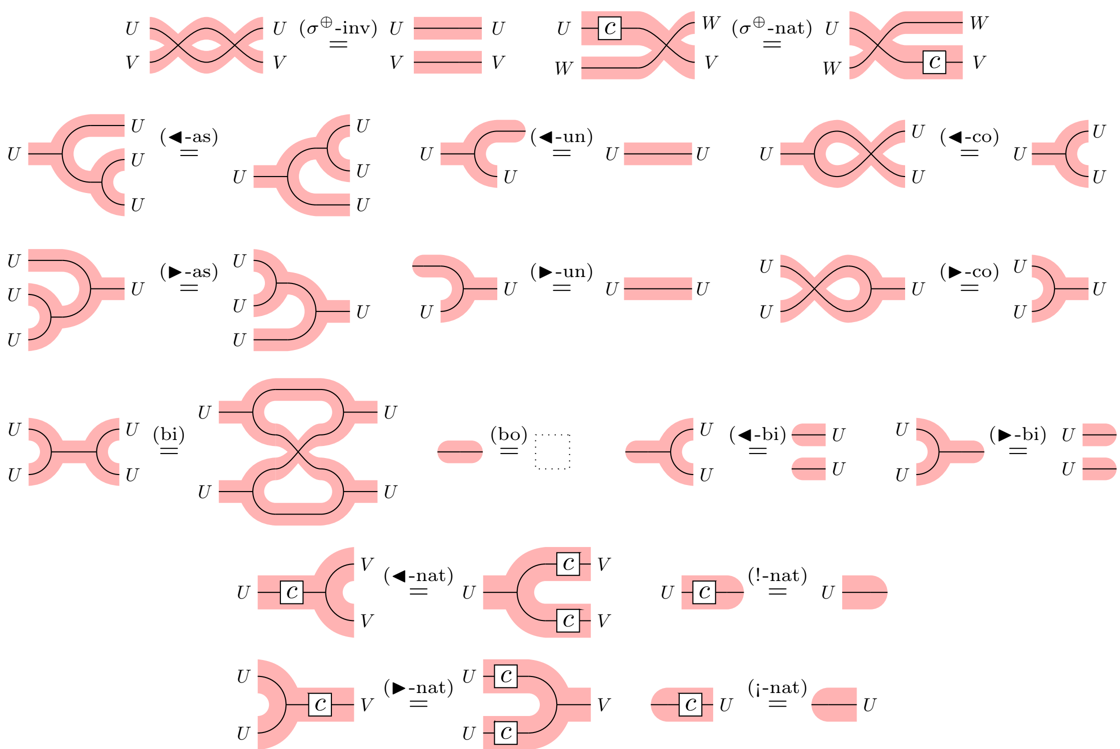
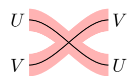


Figure 3: Axioms for tape diagrams

morphisms that we have *freely* added to $\mathbf{C}_\Sigma$ in order to make the new operation $\oplus$, also freely added, a biproduct.) The fact that this is a happy notation is witnessed by how the naturality law of $\blacktriangleleft$ appears graphically in equation ($\blacktriangleleft$-nat) of Figure 3, showing how any circuit in a tape preceding $\blacktriangleleft_U$ can be duplicated by passing through $\blacktriangleleft_U$ and appear twice after it.

The same remark holds for $!_U$, $\blacktriangleright_U$, $¡_U$ and $\sigma^{\oplus}_{U,V}$. Notice in particular that the two crossing wires in

$$\sigma^{\oplus}_{U,V} =$$


are only "formally" crossing to give the (correct!) idea that the two monomials U and V get swapped by $\sigma^{\oplus}_{U,V} \colon U \oplus V \to V \oplus U$, and that every internal string diagram c of $\mathbf{C}_\Sigma$ can slide through them as in the equation ($\sigma^{\oplus}$-nat) of Figure 3. Indeed, $\sigma^{\oplus}_{U,V}$ changes the order of the addends in the polynomial $U \oplus V$ but does not perturb the order of the factors of the monomials U and V.

Remark 4.3. By removing the second line from the grammar in (22) we obtain a graphical language for $F_1U_1(\mathbf{C}_\Sigma)$. The claim that this category is not particularly interesting is thus justified by the graphical rendering of its morphisms. These are depicted indeed as mere vertical compositions of single tapes which fail to highlight any real *interaction* among the string diagrams living inside them. Moreover, as explained later in Remark 4.25, if we were to consider only the productions for the diagonals and their bangs (resp. codiagonals and thier cobangs) we would get a language for rig categories with finite products (resp. coproducts).

4.1 The finite biproduct rig structure of tapes

By definition $\mathbf{T}_\Sigma$ is equipped with a monoidal product $\oplus$ that is a biproduct. We now show that $\mathbf{T}_\Sigma$ carries a second monoidal product $\otimes$ which makes it a strict rig category. In order to do this, we first define $\otimes$ on objects, then we introduce $\otimes$-symmetries and left-distributors which are essential ingredients for illustrating the algebras of whiskerings. Then, the definition of $\otimes$ on arrows just amounts to the expected combination of left and right whiskerings.

Let $P = \bigoplus_i U_i$ and $Q = \bigoplus_j V_j$ be arbitrary objects of $\mathbf{T}_\Sigma$. We define $\otimes$ on objects as

$$P \otimes Q \stackrel{\text{def}}{=} \bigoplus_i \bigoplus_j U_i V_j$$

For instance, $(A \oplus B) \otimes (C \oplus D)$ is the polynomial $AC \oplus AD \oplus BC \oplus BD$ and not $AC \oplus BC \oplus AD \oplus BD$. This is justified by the fact that we want $\mathbf{T}_\Sigma$ to be a right strict rig category and thus $\otimes$ distributes over $\oplus$ only on the right. Indeed we have the following remark.

Remark 4.4. For all polynomials P_1, P_2 and Q in $\mathbf{T}_\Sigma$, $(P_1 \oplus P_2) \otimes Q = (P_1 \otimes Q) \oplus (P_2 \otimes Q)$. Moreover, if U is a monomial, then $U \otimes (P_1 \oplus P_2) = (U \otimes P_1) \oplus (U \otimes P_2)$. Finally, if V is also a monomial, then $U \otimes V = UV$.

Distributivity on the left is possible, but has to be made explicit through left distributors which are built just from identities and $\oplus$-symmetries: in the previous example $\delta^l_{A \oplus B, C, D} \colon AC \oplus AD \oplus BC \oplus BD \to AC \oplus BC \oplus AD \oplus BD$ is exactly what one would expect.

Definition 4.5. $\delta^l_{P,Q,R} \colon P \otimes (Q \oplus R) \to (P \otimes Q) \oplus (P \otimes R)$ is defined by induction on P as:

$$\begin{aligned} \delta^l_{0,Q,R} &\stackrel{\text{def}}{=} id_0 \\ \delta^l_{U \oplus P',Q,R} &\stackrel{\text{def}}{=} (id_{U\otimes(Q\oplus R)} \oplus \delta^l_{P',Q,R}); (id_{U\otimes Q} \oplus \sigma^{\oplus}_{U\otimes R, P'\otimes Q} \oplus id_{P'\otimes R}) \end{aligned}$$

where we used that $(U \oplus P') \otimes (Q \oplus R) = \big(U \otimes (Q \oplus R)\big) \oplus \big(P' \otimes (Q \oplus R)\big)$ and $(U \otimes Q) \oplus (P' \otimes Q) \oplus (U \otimes R) \oplus (P' \otimes R) = \big((U \oplus P') \otimes Q\big) \oplus \big((U \oplus P') \otimes R\big)$ by Remark 4.4.

Notice that $\delta^l_{P,Q,R}$ is an isomorphism. Similarly, arbitrary $\otimes$-symmetries can be built from left distributors and the symmetries within tapes $\underline{\overline{\sigma_{U,V}}}$ for all monomials U, V.

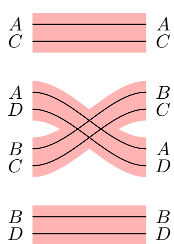
Definition 4.6. Let $P = \bigoplus_i U_i$. $\sigma^{\otimes}_{P,Q} \colon P \otimes Q \to Q \otimes P$ is defined by induction on Q as:

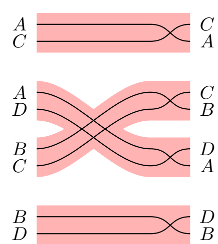
$$\begin{aligned} \sigma^{\otimes}_{P,0} &\stackrel{\text{def}}{=} id_0 \\ \sigma^{\otimes}_{P,V\oplus Q'} &\stackrel{\text{def}}{=} \delta^l_{P,V,Q'}; (\textstyle\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'}). \end{aligned}$$

where we used that $P \otimes V = (\bigoplus_i U_i) \otimes V = \bigoplus_i (U_i V)$, similarly $V \otimes P = \bigoplus_i (V U_i)$, and that $(V \otimes P) \oplus (Q' \otimes P) = (V \oplus Q') \otimes P$ by Remark 4.4.

From now on we will often simply write PQ instead of $P \otimes Q$: because of Remark 4.4 there is no possibility of confusion between $\otimes$ for polynomials and the monoidal product of $\mathbf{C}_\Sigma$.

Example 4.7. Following Definitions 4.5 and 4.6 and the syntax given in (22), one can easily compute the graphical representation of δ^l and $\sigma^{\otimes}$. For example, $\delta^l_{A\oplus B,C,D} \colon (A \oplus B)(C \oplus D) \to (A \oplus B)C \oplus (A \oplus B)D$ and $\sigma^{\otimes}_{A\oplus B,C\oplus D} \colon (A \oplus B)(C \oplus D) \to (C \oplus D)(A \oplus B)$ are depicted as follows:

$\delta^l_{A\oplus B,C,D} =$ 

$\sigma^{\otimes}_{A\oplus B,C\oplus D} =$ 

The following lemma provides a sanity check for our inductive definitions.

Lemma 4.8. *For all P, P', Q, R, monomials U, V the followings hold:*

1. $\delta^l_{U,Q,R} = id_{U(Q\oplus R)}$
2. $\delta^l_{P\oplus P',Q,R} = (\delta^l_{P,Q,R} \oplus \delta^l_{P',Q,R}); (id_{PQ} \oplus \sigma^{\oplus}_{PR,P'Q} \oplus id_{P'R})$
3. $\delta^l_{P,0,R} = id_{PR}$
4. $\delta^l_{P,Q,0} = id_{PQ}$
5. $\sigma^{\otimes}_{P,Q}; \sigma^{\otimes}_{Q,P} = id_{PQ}$

6. $\sigma^{\otimes}_{P,Q\oplus R} = \delta^{l}_{P,Q,R}; (\sigma^{\otimes}_{P,Q} \oplus \sigma^{\otimes}_{P,R})$

7. $\sigma^{\otimes}_{P,1} = id_P$

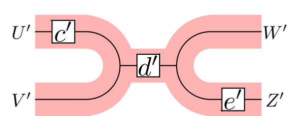
8. $\sigma^{\otimes}_{P,V} = \bigoplus_i \underline{\overline{\sigma_{U_i,V}}}$ *for* $P = \bigoplus_i U_i$*. In particular* $\sigma^{\otimes}_{U,V} = \underline{\overline{\sigma_{U,V}}}$.

We can now introduce left and right whiskering. First we give a restricted definition, just for those objects U that are monomials (i.e., unary sums).

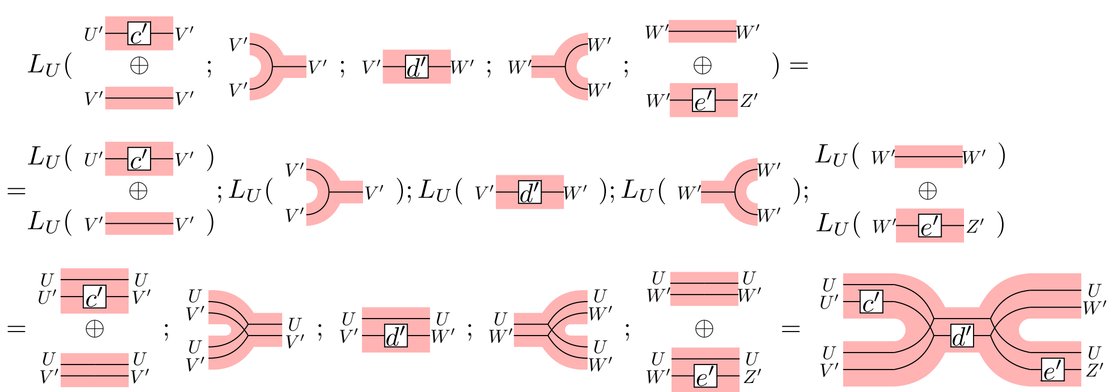
Definition 4.9. Let U be a monomial in $\mathbf{T}_\Sigma$. The *left* and *right whiskering* (with respect to U) are two functors $L_U, R_U \colon \mathbf{T}_\Sigma \to \mathbf{T}_\Sigma$ which are defined on objects as $L_U(P) \stackrel{\text{def}}{=} U \otimes P$ and $R_U(P) \stackrel{\text{def}}{=} P \otimes U$ and on arrows as:

$$
\begin{array}{rcl|rcl}
L_U(id_0) & \stackrel{\text{def}}{=} & id_0 & R_U(id_0) & \stackrel{\text{def}}{=} & id_0 \\
L_U(\underline{\overline{c}}) & \stackrel{\text{def}}{=} & \underline{\overline{id_U \otimes c}} & R_U(\underline{\overline{c}}) & \stackrel{\text{def}}{=} & \underline{\overline{c \otimes id_U}} \\
L_U(\sigma^{\oplus}_{V,W}) & \stackrel{\text{def}}{=} & \sigma^{\oplus}_{UV,UW} & R_U(\sigma^{\oplus}_{V,W}) & \stackrel{\text{def}}{=} & \sigma^{\oplus}_{VU,WU} \\
L_U(\blacktriangleleft_V) & \stackrel{\text{def}}{=} & \blacktriangleleft_{UV} & R_U(\blacktriangleleft_V) & \stackrel{\text{def}}{=} & \blacktriangleleft_{VU} \\
L_U(!_V) & \stackrel{\text{def}}{=} & !_{UV} & R_U(!_V) & \stackrel{\text{def}}{=} & !_{VU} \\
L_U(\blacktriangleright_V) & \stackrel{\text{def}}{=} & \blacktriangleright_{UV} & R_U(\blacktriangleright_V) & \stackrel{\text{def}}{=} & \blacktriangleright_{VU} \\
L_U(¡_V) & \stackrel{\text{def}}{=} & ¡_{UV} & R_U(¡_V) & \stackrel{\text{def}}{=} & ¡_{VU} \\
L_U(\mathfrak{t}_1 ; \mathfrak{t}_2) & \stackrel{\text{def}}{=} & L_U(\mathfrak{t}_1); L_U(\mathfrak{t}_2) & R_U(\mathfrak{t}_1 ; \mathfrak{t}_2) & \stackrel{\text{def}}{=} & R_U(\mathfrak{t}_1); R_U(\mathfrak{t}_2) \\
L_U(\mathfrak{t}_1 \oplus \mathfrak{t}_2) & \stackrel{\text{def}}{=} & L_U(\mathfrak{t}_1) \oplus L_U(\mathfrak{t}_2) & R_U(\mathfrak{t}_1 \oplus \mathfrak{t}_2) & \stackrel{\text{def}}{=} & R_U(\mathfrak{t}_1) \oplus R_U(\mathfrak{t}_2)
\end{array}
$$

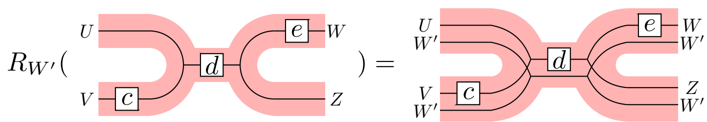
Example 4.10. The meaning of the monomial whiskering is quite immediate in graphical terms. Given a tape diagram $\mathfrak{t}$, $L_U(\mathfrak{t})$ *thickens* the tapes of which $\mathfrak{t}$ is made of, by stacking the wires of id_U inside them. As an example consider the following diagram

$\mathfrak{s} =$ 

The whiskering $L_U(\mathfrak{s})$ is computed as follows:



The right whiskering works analogously except that the additional wires are stacked at the bottom of the single tapes. As an example, consider the following diagram $\mathfrak{t}$ and its right whiskering $R_{W'}(\mathfrak{t})$:



We can extend Definition 4.9 to arbitrary polynomials S as follows.

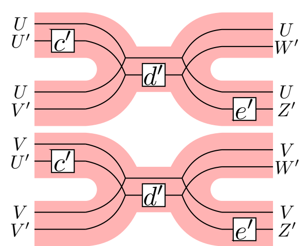
Definition 4.11. For each polynomial S, $L_S, R_S \colon \mathbf{T}_\Sigma \to \mathbf{T}_\Sigma$ are defined on objects as $L_S(P) \stackrel{\text{def}}{=} S \otimes P$ and $R_S(P) \stackrel{\text{def}}{=} P \otimes S$ and on arrows $\mathfrak{t} \colon P \to Q$ by induction on S:

$$L_0(\mathfrak{t}) \stackrel{\text{def}}{=} id_0 \qquad\qquad R_0(\mathfrak{t}) \stackrel{\text{def}}{=} id_0$$

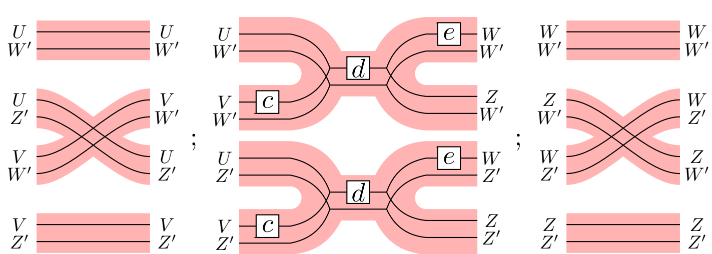
$$L_{W \oplus S'}(\mathfrak{t}) \stackrel{\text{def}}{=} L_W(\mathfrak{t}) \oplus L_{S'}(\mathfrak{t}) \qquad\qquad R_{W \oplus S'}(\mathfrak{t}) \stackrel{\text{def}}{=} \delta^l_{P,W,S'} ; (R_W(\mathfrak{t}) \oplus R_{S'}(\mathfrak{t})) ; \delta^{-l}_{Q,W,S'}$$

Observe that there is an asymmetry in the definition of left and right whiskerings for polynomials: again, this is justified by the fact that $\otimes$ distributes over $\oplus$ on the right. Suppose for instance to whisk on the right a tape $\mathfrak{t} \colon A \oplus B \to A' \oplus B'$ with the polynomial $C \oplus D$: $R_{C \oplus D}(\mathfrak{t})$ should go from $(A \oplus B) \otimes (C \oplus D) = AC \oplus AD \oplus BC \oplus BD$ to $(A' \oplus B') \otimes (C \oplus D) = A'C \oplus A'D \oplus B'C \oplus B'D$, while $R_C(\mathfrak{t}) \oplus R_D(\mathfrak{t})$ goes from $AC \oplus BC \oplus AD \oplus BD$ to $A'C \oplus B'C \oplus A'D \oplus B'D$.

Example 4.12. By the definition above, the *polynomial* left whiskering is obtained by the parallel composition of the monomial whiskerings. Consider $\mathfrak{s}$ in the Example 4.10, then $L_{U \oplus V}(\mathfrak{s})$ is rendered as:



However, for the *polynomial* right whiskering this is not enough. Indeed, by Definition 4.11, we have to precompose and postcompose the monomial whiskerings with left distributors. For example $R_{W' \oplus Z'}(\mathfrak{t})$ is represented as follows:



Left and right whiskerings enjoy some useful properties that are convenient to illustrate now. From the definition, it follows that, for all S, both $L_S(-)$ and $R_S(-)$ are functors, i.e.,

$$L_S(id_P) = id_{SP} \qquad R_S(id_P) = id_{PS} \tag{W1}$$

$$L_S(\mathfrak{t}_1;\mathfrak{t}_2) = L_S(\mathfrak{t}_1);L_S(\mathfrak{t}_2) \qquad R_S(\mathfrak{t}_1;\mathfrak{t}_2) = R_S(\mathfrak{t}_1);R_S(\mathfrak{t}_2) \tag{W2}$$

for all objects P and arrows $\mathfrak{t}_1 \colon P \to Q$ and $\mathfrak{t}_2 \colon Q \to R$. Moreover,

$$L_1(\mathfrak{t}) = \mathfrak{t} \qquad\qquad R_1(\mathfrak{t}) = \mathfrak{t} \tag{W3}$$

$$L_0(\mathfrak{t}) = id_0 \qquad\qquad R_0(\mathfrak{t}) = id_0 \tag{W4}$$

for all arrows $\mathfrak{t}$. It happens that $R_S(-)$ strictly preserves $\oplus$, while $L_S(-)$ only up-to δ^l:

$$L_S(\mathfrak{t}_1 \oplus \mathfrak{t}_2) = \delta^l_{S,P_1,P_2};(L_S(\mathfrak{t}_1) \oplus L_S(\mathfrak{t}_2));\delta^{-l}_{S,Q_1,Q_2} \quad R_S(\mathfrak{t}_1 \oplus \mathfrak{t}_2) = R_S(\mathfrak{t}_1) \oplus R_S(\mathfrak{t}_2) \tag{W5}$$

for $\mathfrak{t}_1 \colon P_1 \to Q_1$ and $\mathfrak{t}_2 \colon P_2 \to Q_2$. Such asymmetry is justified by the fact that we are in a right strict rig category where δ^r is identity, while δ^l is not. Another consequence of this fact is the asymmetry in the following laws

$$L_{S\oplus T}(\mathfrak{t}) = L_S(\mathfrak{t}) \oplus L_T(\mathfrak{t}) \qquad R_{S\oplus T}(\mathfrak{t}) = \delta^l_{P,S,T};(R_S(\mathfrak{t}) \oplus R_T(\mathfrak{t}));\delta^{-l}_{Q,S,T} \tag{W6}$$

holding for all objects S and T and arrows $\mathfrak{t} \colon P \to Q$.

The leftmost equation of (W6) and the rightmost of (W5) are essential to prove the *sliding law*: for all $\mathfrak{t}_1 \colon P \to Q$ and $\mathfrak{t}_2 \colon R \to S$

$$L_P(\mathfrak{t}_2);R_S(\mathfrak{t}_1) = R_R(\mathfrak{t}_1);L_Q(\mathfrak{t}_2). \tag{W7}$$

Intuitively, this law allows to swap the order of consecutive left and right whiskerings whenever $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have the appropriate sources and targets. The proof of these laws proceed by induction on $\mathfrak{t}_1$ and crucially rely on naturality of $\blacktriangleleft_U$, $\blacktriangleright_U$, $!_U$, $¡_U$, $\sigma^{\oplus}_{P,Q}$ and the following laws.

$$R_S(\blacktriangleleft_U) = \blacktriangleleft_{US} \quad R_S(\blacktriangleright_U) = \blacktriangleright_{US} \tag{W8}$$

$$R_S(!_U) = !_{US} \quad R_S(¡_U) = ¡_{US} \tag{W9}$$

$$R_S(\sigma^{\oplus}_{P,Q}) = \sigma^{\oplus}_{PS,QS} \tag{W10}$$

Like (W7), two more laws rule the interaction of left and right whiskering: for all $\mathfrak{t} \colon P \to Q$

$$R_S(\mathfrak{t});\sigma^{\otimes}_{Q,S} = \sigma^{\otimes}_{P,S};L_S(\mathfrak{t}) \tag{W11}$$

$$L_S(R_T(\mathfrak{t})) = R_T(L_S(\mathfrak{t})) \tag{W12}$$

The leftmost law allows to exchange left and right whiskering via a $\otimes$-symmetry: we will see that this corresponds exactly to naturality of $\sigma^{\otimes}_{P,Q}$. The rightmost law allows to exchange the order in which a left and a right whiskerings are applied to some arrow $\mathfrak{t}$. Similar to the latter are the following:

$$L_{ST}(\mathfrak{t}) = L_S(L_T(\mathfrak{t})) \tag{W13}$$

$$R_{TS}(\mathfrak{t}) = R_S(R_T(\mathfrak{t})) \tag{W14}$$

We conclude with three more useful laws.

$$R_S(\delta^l_{P,Q,R}) = \delta^l_{P,QS,RS} \tag{W15}$$

$$L_S(\delta^l_{P,Q,R}) = \delta^l_{SP,Q,R};\delta^{-l}_{S,PQ,PR} \tag{W16}$$

$$\sigma^{\otimes}_{PQ,S} = L_P(\sigma^{\otimes}_{Q,S});R_Q(\sigma^{\otimes}_{P,S}) \tag{W17}$$

The proof for each of these rules is given in Appendix D.3.

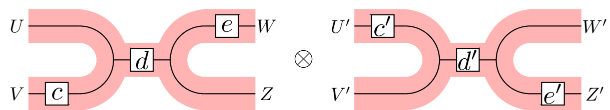
Lemma 4.13. *The laws* (W1)-(W17) *hold.*

With the definition of left and right whiskerings we can now define $\otimes$ on arrows $\mathfrak{t}_1 \colon P \to Q, \mathfrak{t}_2 \colon R \to S$ as

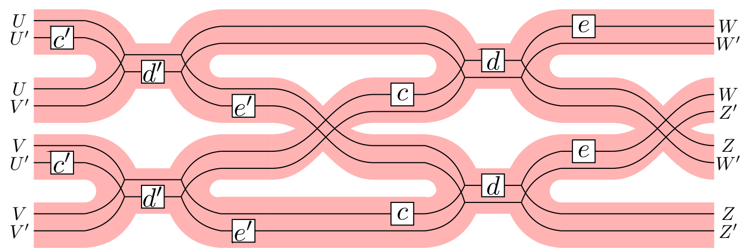
$$\mathfrak{t}_1 \otimes \mathfrak{t}_2 \stackrel{\text{def}}{=} L_P(\mathfrak{t}_2); R_S(\mathfrak{t}_1) \tag{23}$$

which, by (W7), is the same as $R_R(\mathfrak{t}_1); L_Q(\mathfrak{t}_2)$.

Example 4.14. The graphical representation of $\otimes$ follows from its definition. Indeed, it is enough to render the diagrams for the two polynomial whiskerings and compose them together in sequence. For example, consider $\mathfrak{t}$ and $\mathfrak{s}$ from Example 4.10, then their product



is simply the sequential composition of $L_{U\oplus V}(\mathfrak{s})$ and $R_{W'\oplus Z'}(\mathfrak{t})$ shown in Example 4.12:



Theorem 4.15. $\mathbf{T}_\Sigma$ *is a strict finite biproduct rig category.*

Proof. Observe that for all objects S and arrows $\mathfrak{t} \colon P \to Q$, it holds that

$$\mathfrak{t} \otimes id_S = R_S(\mathfrak{t}) \qquad\qquad id_S \otimes \mathfrak{t} = L_S(\mathfrak{t}) \tag{24}$$

as shown below:

$$\begin{aligned} \mathfrak{t} \otimes id_S &= L_P(id_S); R_S(\mathfrak{t}) && \text{(Def. } \otimes) & \qquad id_S \otimes \mathfrak{t} &= L_S(\mathfrak{t}); R_Q(id_S) && \text{(Def. } \otimes) \\ &= R_S(\mathfrak{t}) && \text{(W1)} & &= L_S(\mathfrak{t}) && \text{(W1)} \end{aligned}$$

Thus $id_P \otimes id_Q \stackrel{(24)}{=} L_P(id_Q) \stackrel{(\text{W1})}{=} id_{P\otimes Q}$. To conclude functoriality of $\otimes$, the key is the sliding law (W7): for all $\mathfrak{t}_1 \colon P \to Q, \mathfrak{t}_2 \colon Q \to S, \mathfrak{t}_3 \colon P' \to Q', \mathfrak{t}_4 \colon Q' \to S'$,

$$\begin{aligned} (\mathfrak{t}_1; \mathfrak{t}_2) \otimes (\mathfrak{t}_3; \mathfrak{t}_4) &= L_P(\mathfrak{t}_3; \mathfrak{t}_4); R_{S'}(\mathfrak{t}_1; \mathfrak{t}_2) && \text{(Def. } \otimes) \\ &= L_P(\mathfrak{t}_3); L_P(\mathfrak{t}_4); R_{S'}(\mathfrak{t}_1); R_{S'}(\mathfrak{t}_2) && \text{(W2)} \\ &= L_P(\mathfrak{t}_3); R_{Q'}(\mathfrak{t}_1); L_Q(\mathfrak{t}_4); R_{S'}(\mathfrak{t}_2) && \text{(W7)} \\ &= (\mathfrak{t}_1 \otimes \mathfrak{t}_3); (\mathfrak{t}_2 \otimes \mathfrak{t}_4) && \text{(Def. } \otimes) \end{aligned}$$

By (24) and (W3) it immediately follows that

$$id_1 \otimes \mathfrak{t} \;=\; \mathfrak{t} \;=\; \mathfrak{t} \otimes id_1$$

For associativity of $\otimes$, observe that for all $\mathsf{t}_1 \colon P_1 \to Q_1, \mathsf{t}_2 \colon P_2 \to Q_2, \mathsf{t}_3 \colon P_3 \to Q_3$,

$$\begin{aligned}
((\mathsf{t}_1 \otimes \mathsf{t}_2) \otimes \mathsf{t}_3) &= L_{P_1P_2}(\mathsf{t}_3); R_{Q_3}(L_{P_1}(\mathsf{t}_2); R_{Q_2}(\mathsf{t}_1)) && \text{(Def. } \otimes) \\
&= L_{P_1P_2}(\mathsf{t}_3); R_{Q_3}(L_{P_1}(\mathsf{t}_2)); R_{Q_3}(R_{Q_2}(\mathsf{t}_1)) && \text{(W2)} \\
&= L_{P_1}(L_{P_2}(\mathsf{t}_3)); L_{P_1}(R_{Q_3}(\mathsf{t}_2)); R_{Q_2Q_3}(\mathsf{t}_1) && \text{((W13),(W12),(W14))} \\
&= L_{P_1}(L_{P_2}(\mathsf{t}_3); R_{Q_3}(\mathsf{t}_2)); R_{Q_2Q_3}(\mathsf{t}_1) && \text{(W2)} \\
&= (\mathsf{t}_1 \otimes (\mathsf{t}_2 \otimes \mathsf{t}_3)) && \text{(Def. } \otimes)
\end{aligned}$$

So far, we proved that $(\mathbf{T}_\Sigma, \otimes, 1)$ is a strict monoidal category. For symmetries, observe that naturality follows immediately by (24) and (W11). Inverses is Lemma 4.8.5 and the coherence axioms (S1) and (S2) holds by Lemma 4.8.7 and (W17).

By construction $(\mathbf{T}_\Sigma, \oplus, 0, \sigma^\oplus)$ is a symmetric strict monoidal category. We are left to prove a few laws. First,

$$id_0 \otimes \mathsf{t} \;=\; id_0 = \mathsf{t} \otimes id_0$$

holds by (24) and (W4). Regarding right distributivity of $\otimes$ over $\oplus$, observe that for all $\mathsf{t}_1 \colon P_1 \to Q_1$, $\mathsf{t}_2 \colon P_2 \to Q_2$, $\mathsf{t}_3 \colon P_3 \to Q_3$ it holds that:

$$\begin{aligned}
(\mathsf{t}_1 \oplus \mathsf{t}_2) \otimes \mathsf{t}_3 &= L_{P_1 \oplus P_2}(\mathsf{t}_3); R_{Q_3}(\mathsf{t}_1 \oplus \mathsf{t}_2) && \text{(Def. } \otimes) \\
&= (L_{P_1}(\mathsf{t}_3) \oplus L_{P_2}(\mathsf{t}_3)); (R_{Q_3}(\mathsf{t}_1) \oplus R_{Q_3}(\mathsf{t}_2)) && \text{((W6), (W5))} \\
&= (L_{P_1}(\mathsf{t}_3); R_{Q_3}(\mathsf{t}_1)) \oplus (L_{P_2}(\mathsf{t}_3); R_{Q_3}(\mathsf{t}_2)) && \text{(Funct. } \oplus) \\
&= (\mathsf{t}_1 \otimes \mathsf{t}_3) \oplus (\mathsf{t}_2 \otimes \mathsf{t}_3) && \text{(Def. } \otimes)
\end{aligned}$$

(This is tantamount to saying that $\mathbf{T}_\Sigma$ has a natural right distributor whose components $\delta^r_{P,Q,R}$ are all identity morphisms).

To conclude we have to check the axioms of coherence for strict rig categories that amounts just to (R1), (R2), (R5) and (R9) from Figure 8 (p. 68). Axiom (R1) is Lemma 4.8.6. Axiom (R2) follows by (24) and (W10). Axiom (R5) is Lemma 4.8.2. And Axiom (R9) holds by Definition 4.6. Finally, from Axiom (R1) and the naturality of the right distributor and of $\sigma^\otimes$ we get that also δ^l is natural. This proves that $\mathbf{T}_\Sigma$ is a strict rig category. $(\mathbf{T}_\Sigma, \oplus, 0, \sigma^\oplus)$ is a finite biproduct category by definition. □

Remark 4.16. In order to prove that $\mathbf{T}_\Sigma = F_2U_1(\mathbf{C}_\Sigma)$ is a strict fb rig category, in Theorem 4.15, we only used the symmetric monoidal structure of $\mathbf{C}_\Sigma$, and not the fact that $\mathbf{C}_\Sigma$ is freely generated by a signature Σ. More specifically, we have always worked with monomials and circuits (that is, generic objects and morphisms of $\mathbf{C}_\Sigma$) to define the rig structure of $\mathbf{T}_\Sigma$, without ever referring to sorts in $\mathcal{S}$ and generators in the signature Σ. This means that, in fact, we can say something more general: for any symmetric monoidal category $\mathbf{M}$, $F_2U_1(\mathbf{M})$ is a strict fb rig category.

Lemma 4.17. *In $\mathbf{T}_\Sigma$,* (SS) *holds.*

Proof. By Lemma 4.8.6, it holds that $\delta^l_{P,Q,R} = \sigma^\otimes_{P,Q\oplus R}; (\sigma^\otimes_{Q,P} \oplus \sigma^\otimes_{R,P})$ and by Lemma 4.8.1 that $\delta^l_{A,Q,R} = id_{A(Q\oplus R)}$. Thus $\sigma^\otimes_{A,Q\oplus R}; (\sigma^\otimes_{Q,A} \oplus \sigma^\otimes_{R,A}) = id_{A(Q\oplus R)}$. □

4.2 The isomorphism theorem

We conclude this section by showing that $\mathbf{T}_\Sigma$ is ismorphic to $\mathbf{ssR}^\flat_\Sigma$.

First of all recall that, by Definition 3.8, $\mathbf{ssR}^\flat_\Sigma$ is the quotient by (SS) of the strict fb rig category freely generated by Σ. Since $\mathbf{T}_\Sigma$ is a fb-rig category (Theorem 4.15) which additionally satisfies (SS) (Lemma 4.17), there exists a unique strict fb-rig functor $F\colon \mathbf{ssR}^\flat_\Sigma \to \mathbf{T}_\Sigma$ extending the assignment $s \mapsto \underline{\overline{s}}$ for all $s \in \Sigma$. Such functor is the identity on objects and on arrows is defined inductively as

$$F(s) = \underline{\overline{s}}$$

$$F(id_0) = id_0 \qquad F(id_1) = id_1 \qquad F(id_A) = id_A$$
$$F(\sigma^{\otimes}_{X,Y}) = \sigma^{\otimes}_{X,Y} \qquad F(\sigma^{\oplus}_{X,Y}) = \sigma^{\oplus}_{X,Y}$$
$$F(\blacktriangleleft_X) = \blacktriangleleft_X \qquad F(!_X) = !_X \qquad F(\blacktriangleright_X) = \blacktriangleright_X \qquad F(¡_X) = ¡_X$$
$$F(f;g) = F(f);F(g) \qquad F(f \oplus g) = F(f) \oplus F(g) \qquad F(f \otimes g) = F(f) \otimes F(g)$$

In order to construct the inverse of F, recall that $\mathbf{T}_\Sigma = F_2(U_1(\mathbf{C}_\Sigma))$ and that the unit of the adjunction (19) is the taping functor $\underline{\overline{\cdot}}\colon \mathbf{C}_\Sigma \to \mathbf{T}_\Sigma$. Observe that there is an embedding functor $\iota\colon \mathbf{C}_\Sigma \to \mathbf{ssR}^\flat_\Sigma$. Thus by freeness of $\mathbf{T}_\Sigma$, there exists a unique fb-functor $G\colon \mathbf{T}_\Sigma \to \mathbf{ssR}^\flat_\Sigma$ extending ι, i.e., $G = \iota^\sharp$. More explicitely, G is an identity-on-objects functor, defined by induction on morphisms as:

$$G\,(id_0) = id_0 \qquad G\,(\underline{\overline{c}}) = c \qquad G(\sigma^{\oplus}_{P,Q}) = \sigma^{\oplus}_{P,Q}$$
$$G(\blacktriangleleft_U) = \blacktriangleleft_U \qquad G(!_U) = !_U \qquad G(\blacktriangleright_U) = \blacktriangleright_U \qquad G(¡_U) = ¡_U$$
$$G(\mathsf{t}_1;\mathsf{t}_2) = G(\mathsf{t}_1);G(\mathsf{t}_2) \qquad G(\mathsf{t}_1 \oplus \mathsf{t}_2) = G(\mathsf{t}_1) \oplus G(\mathsf{t}_2)$$

Observe that by definition G is a fb-functor. To prove that it is $\otimes$-monoidal, it is convenient to first prove that G preserves $\otimes$-symmetries and left distributors.

Lemma 4.18. *For all P, Q, R, the following hold:*

1. $G(\delta^l_{P,Q,R}) = \delta^l_{P,Q,R}$,
2. $G(\sigma^{\otimes}_{P,Q}) = \sigma^{\otimes}_{P,Q}$.

Lemma 4.19. $G(\mathsf{t}_1 \otimes \mathsf{t}_2) = G(\mathsf{t}_1) \otimes G(\mathsf{t}_2)$.

The above two lemmas, proved in Appendix D.4, ensure that $G\colon \mathbf{T}_\Sigma \to \mathbf{ssR}^\flat_\Sigma$ is a fb-rig functor. From this fact the next proposition follows immediately.

Proposition 4.20. $GF = id_{\mathbf{ssR}^\flat_\Sigma}$.

In order to prove that $FG = id_{\mathbf{T}_\Sigma}$, it is convenient to first show the following two results.

Lemma 4.21. *The taping functor $\underline{\overline{\cdot}}\colon \mathbf{C}_\Sigma \to \mathbf{T}_\Sigma$ is symmetric strict monoidal.*

Proof. By Definition 4.6, $\sigma^{\otimes}_{U,V} = \underline{\overline{\sigma_{U,V}}}$ holds for all U, V. Moreover, let $c\colon U_1 \to V_1$ and $d\colon U_2 \to V_2$. Then

$$\begin{aligned}
\underline{\overline{c}} \otimes \underline{\overline{d}} &= L_{U_1}(\underline{\overline{d}}); R_{V_2}(\underline{\overline{c}}) && \text{(Def. } \otimes\text{)}\\
&= \underline{\overline{id_{U_1} \otimes d}}; \underline{\overline{c \otimes id_{V_2}}} && \text{(Def. } L,\ R\text{)}\\
&= \underline{\overline{(id_{U_1} \otimes d); (c \otimes id_{V_2})}} && \text{(Tape)}\\
&= \underline{\overline{c \otimes d}} && \text{(Funct. } \otimes\text{)}
\end{aligned}$$

$\square$

Lemma 4.22. *The functors* $\mathbf{C}_\Sigma \xrightarrow{\iota} \mathbf{ssR}^\flat_\Sigma \xrightarrow{F} \mathbf{T}_\Sigma$ *and* $\mathbf{C}_\Sigma \xrightarrow{\underline{\overline{\cdot}}} \mathbf{T}_\Sigma$ *coincide.*

Proof. Since $\mathbf{C}_\Sigma$ is the symmetric strict monoidal category freely generated by Σ, and since both F and $\underline{\overline{\cdot}}$ are symmetric strict $\otimes$-monoidal functors, it is enough to check that for all $s \in \Sigma$, $F(s) = \underline{\overline{s}}$. But this is trivial by definition of F. □

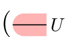
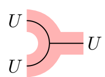
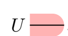
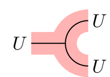
By the above lemma and freeness of $\mathbf{T}_\Sigma$, we immediately get the following proposition.

Proposition 4.23. $FG = id_{\mathbf{T}_\Sigma}$.

We can thus conclude the promised isomorphism theorem.

Theorem 4.24. *There is an isomorphism of fb-rig categories*

$$\begin{tikzcd} \mathbf{ssR}^\flat_\Sigma \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathbf{T}_\Sigma \arrow[l, bend left, "G"] \end{tikzcd}$$

Remark 4.25. The results of this section still hold when replacing in the adjunction (19) the category **FBC** with **FPC** or **FCC**, namely the categories of categories with finite (co)products. In order to obtain an isomorphism, one has to remove from the definition of $\mathbf{T}_\Sigma$ in (22) either the monoid ( , ) or the comonoid (, ). The proofs proceed exactly in the same way since, so far, we did not use any specific properties of finite biproduct categories. These will be essential instead in Sections 6 and 7.

5 Alternative proof

In this section we propose a different approach to the proof of Theorems 4.15 and 4.24: rather than defining a tensor on $\mathbf{T}_\Sigma$ directly, proving it makes $\mathbf{T}_\Sigma$ a strict rig category and using this to obtain the isomorphism $\mathbf{T}_\Sigma \cong \mathbf{ssR}^\flat_\Sigma$, we define an isomorphism of *fb categories* between $\mathbf{T}_\Sigma$ and $\mathbf{ssR}^\flat_\Sigma$ (Theorem 5.5) and then endow $\mathbf{T}_\Sigma$ with a tensor inherited from the one in $\mathbf{ssR}^\flat_\Sigma$. This will automatically make $\mathbf{T}_\Sigma$ a strict rig category (Theorem 5.6). The bulk of the proof consists in showing that every morphism in $\mathbf{ssR}^\flat_\Sigma$ can always be written in a way that is trivially translatable into a tape (Proposition 5.8).

Preliminaries. Recall that in $\mathbf{ssR}^\flat_\Sigma$ the left distributor $\delta^l_{X,Y,Z}$ is defined as in (15), that is:

$$X \otimes (Y \oplus Z) \xrightarrow{\sigma^{\otimes}_{X,Y\oplus Z}} (Y \oplus Z) \otimes X = (Y \otimes X) \oplus (Z \otimes X) \xrightarrow{\sigma^{\otimes}_{Y,X} \oplus \sigma^{\otimes}_{Z,X}} (X \otimes Y) \oplus (X \otimes Z).$$

Remark 5.1. δ^l is a natural isomorphism. From the limited left distributivity we get that, for all $A \in \mathcal{S}$,

$$id_A \otimes (g \oplus h) = (id_A \otimes g) \oplus (id_A \otimes h).$$

Moreover, if $X = \bigotimes_i A_i$ with $A_i \in \mathcal{S}$, then $\delta^l_{X,Y,Z} = id_{X(Y \oplus Z)}$ because of Axiom (R4). Therefore, by naturality of δ^l, for all $s \in \Sigma$ and for all g, h morphisms

$$s \otimes (g \oplus h) = (s \otimes g) \oplus (s \otimes h).$$

Definition 5.2. Given objects $X \neq 0, Y_1, \ldots, Y_m$ of $\mathbf{ssR}^\flat_\Sigma$, we define the *generalised left distributor*

$$\Delta^l_{X,Y_1,\ldots,Y_m} : X(\bigoplus_{j=1}^{m} Y_j) \to \bigoplus_{j=1}^{m} XY_j$$

as the following morphism, by induction on $m \geq 1$:

- if $m = 1$, then $\Delta^l_{X,Y_1} = id_{X\otimes Y}$,
- if $m \geq 2$, then $\Delta^l_{X,Y_1,\ldots,Y_m}$ is the composite

$$X(\bigoplus_{j=1}^m Y_j) \xrightarrow{\delta^l_{X,Y_1,\bigoplus_{j=2}^m Y_j}} (XY_1) \oplus (X(\bigoplus_{j=2}^m Y_j)) \xrightarrow{id_{XY_1} \oplus \Delta^l_{X,Y_2,\ldots,Y_m}} \bigoplus_{j=1}^m XY_j.$$

Remark 5.3. Δ^l is a natural isomorphism in all its variables. Moreover, if $m \geq 2$, then $\Delta^l_{X,Y_1,\ldots,Y_m}$ is also equal to the following composite:

$$X(\bigoplus_{j=1}^m Y_j) \xrightarrow{\delta^l_{X,\bigoplus_{j=1}^{m-1} Y_j, Y_m}} X(\bigoplus_{j=1}^{m-1} Y_j) \oplus (XY_m) \xrightarrow{\Delta^l_{X,Y_1,\ldots,Y_{m-1}} \oplus id_{XY_m}} \bigoplus_{j=1}^m XY_j$$

due to Axiom (R3). Finally, if $X = \bigotimes_i A_i$ with $A_i \in \mathcal{S}$, then $\Delta^l_{X,Y_1,\ldots,Y_m} = id_{X(\bigoplus_j Y_j)}$.

Lemma 5.4. *For all $X \neq 0$, $Y_1, \ldots, Y_m$ objects of $\mathbf{ssR}^\flat_\Sigma$, $\Delta^l_{X,Y_1,\ldots,Y_m}$ is equal to the following composite:*

$$X(\bigoplus_j Y_j) \xrightarrow{\sigma^\otimes_{X,\bigoplus_j Y_j}} (\bigoplus_j Y_j)X = \bigoplus_j XY_j \xrightarrow{\bigoplus_j \sigma^\otimes_{Y_j,X}} \bigoplus_j XY_j \tag{25}$$

Proof. We prove it by induction on m. Since there is no risk of confusion, as every symmetry in this proof is with respect to $\otimes$, we will simply write σ instead of $\sigma^\otimes$. Also we shall write Z for id_Z to make the notation lighter.

If $m = 1$, then $\Delta^l_{X,Y} = id_{XY}$ while (25) is $\sigma_{X,Y};\sigma_{Y,X}$ which is indeed id_{XY}.

If $m = 2$, then $\Delta^l_{X,Y_1,Y_2} = \delta^l_{X,Y_1,Y_2}$ which is equal to (25) by definition.

Now suppose $m \geq 2$ and that the statement holds for all $X, Y_1, \ldots, Y_m$. Consider now objects $X, Y_1, \ldots, Y_{m+1}$: if we call $Y = \bigoplus_{j=1}^{m+1} Y_j$ and $Y' = \bigoplus_{j=2}^{m+1} Y_j$ for short, we have that $\Delta^l_{X,Y_1,\ldots,Y_{m+1}}$ is equal to the upper leg of the following commutative diagram:

$$\begin{tikzcd}
XY \arrow[rr, "\delta^l_{X,Y_1,Y'}"] \arrow[d, "\sigma_{X,Y}"'] & *_1 & XY_1 \oplus XY' \arrow[rr, "XY_1 \oplus \Delta^l_{X,Y_2,\ldots,Y_{m+1}}"] \arrow[dr, "XY_1 \oplus \sigma_{X,Y'}"] & & \bigoplus_{j=1}^{m+1} XY_j \\
YX = Y_1X \oplus Y'X \arrow[ur, "\sigma_{Y_1,X} \oplus \sigma_{Y',X}"] \arrow[rrr, "\sigma_{Y_1,X} \oplus Y'X"] & & & XY_1 \oplus Y'X = XY_1 \oplus Y' \arrow[ur, "XY_1 \oplus \bigoplus_{j=2}^{m+1} \sigma_{Y_j,X}"'] & *_2
\end{tikzcd}$$

where $*_1$ commutes by definition of $\delta^l_{X,Y_1,Y'}$, $*_2$ by inductive hypothesis and the central triangle because $\sigma_{Y',X};\sigma_{X,Y'} = id_{Y'X}$. The lower leg is exactly $\sigma_{X,Y};\bigoplus_{j=1}^{m+1} \sigma_{Y_j,X}$, as required. $\square$

Morphisms $S^{n,m}$ in symmetric monoidal categories. Let $(\mathbf{C}, \oplus, 0, \sigma)$ be a strict symmetric monoidal category, $n, m \in \mathbb{N}$. Consider the functor

$$\begin{array}{ccc} \mathbf{C}^{n+m} & \xrightarrow{\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m}} & \mathbf{C} \\ (X_{ij})_{\substack{i=1\dots n \\ j=1\dots m}} & \longmapsto & \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} X_{ij} \end{array}$$

For $n = 0$ or $m = 0$ we have, by definition, that $\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} = K_0 \colon \mathbf{C}^0 \to \mathbf{C}$, the constant functor that points to 0. This is what Mac Lane [22] called a "permuted word" and, by the coherence theorem of symmetric monoidal categories [23], there exists a unique natural isomorphism $S^{n,m} \colon \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} \to \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n}$ built only using identities, symmetries and tensors. For the interested reader we present here an explicit definition of $S^{n,m}$.

We shall first discuss some preliminary cases on n and m: in each case we suppose we are not falling in a case previously discussed.

- **Case $n = 0$ or $m = 0$:** $S^{n,0} = S^{0,m} = id_{K_0} \colon K_0 \to K_0$.
- **Case $n = 1$:** $S^{1,m} = id_{\bigoplus_{j=1}^{m}} \colon \bigoplus_{j=1}^{m} \to \bigoplus_{j=1}^{m}$
- **Case $m = 1$:** $S^{n,1} = id_{\bigoplus_{i=1}^{n}} \colon \bigoplus_{i=1}^{n} \to \bigoplus_{i=1}^{n}$
- **Case $m = 2$:** we define $S^{n,2}$ by induction on $n \geq 2$. If $n = 2$, given $X = (X_{ij})_{\substack{i=1,2 \\ j=1,2}}$, we set $S^{n,2}_X$ to be:

$$\bigoplus_{i=1}^{2}\bigoplus_{j=1}^{2} X_{ij} = X_{11} \oplus X_{12} \oplus X_{21} \oplus X_{22} \xrightarrow{X_{11} \oplus \sigma_{X_{12},X_{21}} \oplus X_{22}} X_{11} \oplus X_{21} \oplus X_{12} \oplus X_{22} = \bigoplus_{j=1}^{2}\bigoplus_{i=1}^{2} X_{ij}$$

If $n \geq 2$ and $S^{n,2}$ is defined as a family of morphisms in $2n$ variables, let $X = (X_{ij})_{\substack{i=1\dots n+1 \\ j=1,2}}$ and X' be the subfamily $(X_{ij})_{\substack{i=1\dots n \\ j=1,2}}$; we define $S^{n+1,2}_X$ as:

$$\begin{array}{ccc} \bigoplus_{i=1}^{n+1}\bigoplus_{j=1}^{2} X_{ij} = \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{2} X_{ij} \oplus \bigoplus_{j=1}^{2} X_{n+1,j} & \xrightarrow{S^{n,2}_{X'} \oplus id} & \bigoplus_{j=1}^{2}\bigoplus_{i=1}^{n} X_{ij} \oplus \bigoplus_{j=1}^{2} X_{n+1,j} \\ & & \parallel \\ \bigoplus_{j=1}^{2}\bigoplus_{i=1}^{n+1} X_{ij} = \bigoplus_{i=1}^{n} X_{i1} \oplus X_{n+1,1} \oplus \bigoplus_{i=1}^{n} X_{i2} \oplus X_{n+1,2} & \xleftarrow{id \oplus \sigma \oplus id} & \bigoplus_{i=1}^{n} X_{i1} \oplus \bigoplus_{i=1}^{n} X_{i2} \oplus X_{n+1,1} \oplus X_{n+1,2} \end{array}$$

We have finished discussing the preliminary cases. Now we fix $n \geq 2$ and we define $S^{n,m}$ by induction on $m \geq 2$. The base case of $m = 2$ has already been treated above. If $m \geq 2$ and $S^{n,m}$ is defined as a family of morphisms in nm variables, consider $X = (X_{ij})_{\substack{i=1\dots n\\ j=1\dots m+1}}$ and its subfamily $X' = (X_{ij})_{\substack{i=1\dots n\\ j=1\dots m}}$. Then we have

$$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m+1} X_{ij} = \bigoplus_{i=1}^{n} \Big(\bigoplus_{j=1}^{m} X_{ij} \oplus X_{i,m+1} \Big).$$

Call $Y_{i1} = \bigoplus\limits_{j=1}^{m} X_{ij}$ and $Y_{i2} = X_{i,m+1}$. Then $Y = (Y_{ij})_{\substack{i=1,\dots,n\\ j=1,2}}$ is a family of $2n$ objects for which we have defined

$$S_Y^{n,2} \colon \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{2} Y_{ij} \to \bigoplus_{j=1}^{2} \bigoplus_{i=1}^{n} Y_{ij}.$$

Set then $S_X^{n,m+1}$ be equal to the composite:

$$\begin{tikzcd}
\bigoplus\limits_{i=1}^{n} \bigoplus\limits_{j=1}^{m+1} X_{ij} = \bigoplus\limits_{i=1}^{n} \Big(\bigoplus\limits_{j=1}^{m} X_{ij} \oplus X_{i,m+1} \Big) = \bigoplus\limits_{i=1}^{n} (Y_{i1} \oplus Y_{i2}) = \bigoplus\limits_{i=1}^{n} \bigoplus\limits_{j=1}^{2} Y_{ij} \arrow[r, "S_Y^{n,2}"] & \bigoplus\limits_{j=1}^{2} \bigoplus\limits_{i=1}^{n} Y_{ij} \arrow[d, equal] \\
\bigoplus\limits_{j=1}^{m+1} \bigoplus\limits_{i=1}^{n} X_{ij} = \bigoplus\limits_{j=1}^{m} \bigoplus\limits_{i=1}^{n} X_{ij} \oplus \bigoplus\limits_{i=1}^{n} X_{i,m+1} & \bigoplus\limits_{i=1}^{n} \bigoplus\limits_{j=1}^{m} X_{ij} \oplus \bigoplus\limits_{i=1}^{n} X_{i,m+1} \arrow[l, "S_{X'}^{n,m} \oplus id"']
\end{tikzcd}$$

This concludes the definition of $S^{n,m}$ for every $n, m \in \mathbb{N}$.

In the rest of this section we aim to prove the following results.

Theorem 5.5. *There is an isomorphism of fb categories*

$$\begin{tikzcd}
\mathbf{ssR}_\Sigma^\flat \arrow[r, bend left, "F"] \arrow[r, phantom, "\cong"] & \mathbf{T}_\Sigma \arrow[l, bend left, "G"]
\end{tikzcd} \tag{26}$$

Theorem 5.6. *The isomorphism* (26) *induces a structure of strict rig category on* $\mathbf{T}_\Sigma$*, which is preserved on the nose by* F *and* G*.*

The functor $G \colon \mathbf{T}_\Sigma \to \mathbf{ssR}_\Sigma^\flat$. Every object of $\mathbf{T}_\Sigma$ is in polynomial form, that is it is a formal sum of tensors of elements in $\mathcal{S}$, therefore it is also an object of $\mathbf{ssR}_\Sigma^\flat$. The functor G is thus defined as the identity on objects. It is defined on morphisms by induction, for $U = \bigotimes_i A_i$ and $V = \bigotimes_j B_j$ with A_i, $B_j \in \mathcal{S}$:

- $G(id_0) = id_0$
- $G\left(\underline{\overline{c}}\right) = c$
- $G(\sigma^{\oplus}_{U,V}) = \sigma^{\oplus}_{U,V}$

- $G(\blacktriangleleft_U) = \blacktriangleleft_U$, $G(!_U) = !_U$, $G(\blacktriangleright_U) = \blacktriangleright_U$, $G(¡_U) = ¡_U$ for $U = \bigotimes_i A_i$ with $A_i \in \mathcal{S}$
- $G(\mathsf{t}_1; \mathsf{t}_2) = G(\mathsf{t}_1); G(\mathsf{t}_2)$, $G(\mathsf{t}_1 \oplus \mathsf{t}_2) = G(\mathsf{t}_1) \oplus G(\mathsf{t}_2)$

Remark 5.7. Every tape diagram can be canonically embedded, via G, into $\mathbf{ssR}^\flat_\Sigma$ as a composite of sums of $\sigma^\oplus$, $\blacktriangleleft$, $!$, $\blacktriangleright$, $¡$ and c with c a morphism of $\mathbf{C}_\Sigma$.

Towards the functor $F\colon \mathbf{ssR}^\flat_\Sigma \to \mathbf{T}_\Sigma$. We have seen in the previous Remark how the image of G looks like. To build the inverse functor, we first show that any morphism of $\mathbf{ssR}^\flat_\Sigma$, in fact, falls into the image of G.

Proposition 5.8. For any $f \in \mathbf{ssR}^\flat_\Sigma$ ther exist $f^1_1, \dots, f^1_{l_1}, \dots, f^q_1, \dots, f^q_{l_q}$ such that

$$f = \bigoplus_{k=1}^{l_1} f^1_k; \dots; \bigoplus_{k=1}^{l_q} f^q_k$$

where $f^i_k = \sigma^\oplus_{U,V}, \blacktriangleleft_U, !_U, \blacktriangleright_U, ¡_U, c$ with $c \in \mathbf{C}_\Sigma$.

Proof. By induction on f. First of all notice that every object in $\mathbf{ssR}^\flat_\Sigma$ is equal to its normal form, which is of the form $\bigoplus_i \bigotimes_j A^i_j$ with $A^i_j \in \mathcal{S}$, as a consequence of Proposition 3.7.

Case $f \in \mathbf{C}_\Sigma$ or $f = id$: immediate.

Case $f = \sigma^\oplus_{P,Q}$: we have $P = \bigoplus_i U_i$ and $Q = \bigoplus_j V_j$ for appropriate U_i, V_j monomials, hence f is indeed a composite of identities and $\sigma^\oplus$ involving only the addends U_i and V_j by Axiom (S2) of symmetric monoidal categories.

Case $f = \sigma^\otimes_{P,Q}$: write $P = \bigoplus_{i=1}^n U_i$ and $Q = \bigoplus_{j=1}^m V_j$ for appropriate U_i, V_j monomials. Call $X = (V_j U_i)_{\substack{i=1\dots n \\ j=1\dots m}}$: X is a family of objects of $\mathbf{ssR}^\flat_\Sigma$. Then $\sigma^\otimes_{P,Q} = \bigoplus_i \bigoplus_j \sigma^\otimes_{U_i,V_j}; S^{n,m}_X$, because the following diagram commutes.

$$\begin{tikzcd}
\bigoplus_i \bigoplus_j U_i V_j \arrow[r, "\bigoplus_i \bigoplus_j \sigma^\otimes_{U_i,V_j}"] & \bigoplus_i \bigoplus_j V_j U_i \arrow[r, "S^{n,m}_X"] & \bigoplus_j \bigoplus_i V_j U_i \\
\bigoplus_i U_i Q \arrow[u, "\bigoplus_i \Delta^l_{U_i,V_1,\dots,V_m}"] \arrow[u, bend left=60, "id"] \arrow[r, "\bigoplus_i \sigma^\otimes_{U_i,Q}"] \arrow[ur, phantom, "*_1"] & \bigoplus_i Q U_i \arrow[u, equal] \arrow[r, phantom, "*_2"] & \bigoplus_j V_j P \arrow[u, "\bigoplus_j \Delta^l_{V_j,U_1,\dots,U_n}"'] \arrow[u, bend right=60, "id"'] \\
(\bigoplus_i U_i)(\bigoplus_j V_j) \arrow[u, equal] \arrow[rr, "\sigma^\otimes_{P,Q}"] \arrow[ur, phantom, "*_3"] & & (\bigoplus_j V_j)(\bigoplus_i U_i) \arrow[u, equal] \arrow[ul, "\Delta^l_{V,U_1,\dots,U_n}"]
\end{tikzcd}$$

Diagram $*_1$ commutes for every i by Lemma 5.4; $*_2$ commutes by a generalisation of Axiom (R5); $*_3$ again commutes by Lemma 5.4. Finally, we have that $\Delta^l_{U_i,V_1,\dots,V_m}$ e $\Delta^l_{V_j,U_1,\dots,U_n}$ are identities for all i and j because U_i and V_j are monomials, hence Remark 5.3 applies.

Case $f = g; h$ or $f = g \oplus h$: immediate from the inductive hypothesis.

Case $f = g \otimes h$: it is sufficient to prove cases $f = g \otimes id$ and $f = id \otimes h$ because $\otimes$ is a functor.

Now, by inductive hypothesis our proposition holds for g and h. Since $\oplus$ is a functor, without loss of generalisation we can therefore suppose that $g = id_P \oplus g' \oplus id_Q$ with $P, Q \in \mathbf{ssR}^\flat_\Sigma$ and

$g' = \sigma^{\oplus}, \blacktriangleleft, !, \blacktriangleright, ¡, c$ and similarly for h. To keep the presentation simple, we will also assume that P and Q are monomials, however it is of course possible to perform the following calculations for arbitrary polynomials P and Q with no major difficulties.

In the following all U_i and V_j are monomials. We will often write A instead of id_A.

Case 1a: $g = U_1 \oplus \sigma^{\oplus}_{U_2,U_3} \oplus U_4$, $h = id_Q$ with $Q = \bigoplus_{j=1}^{m} V_j$. Define $U_1' = U_1$, $U_2' = U_3$, $U_3' = U_2$, $U_4' = U_4$. Then

$$g \otimes h = \bigoplus_j U_i V_j \oplus \sigma^{\oplus}_{\bigoplus_j U_2 V_j, \bigoplus_j U_3 V_j} \oplus \bigoplus_j U_4 V_j$$

because the following diagram commutes:

$$\begin{tikzcd}
\left(\bigoplus_{i=1}^{4} U_i\right) Q \arrow[rr, "(U_1 \oplus \sigma^{\oplus}_{U_2,U_3} \oplus U_4)Q"] \arrow[d, equal] & & \left(\bigoplus_{i=1}^{4} U_i'\right) Q \arrow[d, equal] \\
\big(U_1 \oplus (U_2 \oplus U_3) \oplus U_4\big)Q \arrow[d, equal] & *_1 & \big(U_1 \oplus (U_3 \oplus U_2) \oplus U_4\big)Q \arrow[d, equal] \\
U_1Q \oplus (U_2 \oplus U_3)Q \oplus U_4Q \arrow[rr, "U_1Q \oplus \sigma^{\oplus}_{U_2,U_3}Q \oplus U_4Q"] \arrow[d, equal] & *_2 & U_1Q \oplus (U_2 \oplus U_3)Q \oplus U_4Q \arrow[d, equal] \\
\bigoplus_{i=1}^{4} U_iQ \arrow[rr, "U_1Q \oplus \sigma^{\oplus}_{U_2Q,U_3Q} \oplus U_4Q"] \arrow[d, equal] & *_3 & \bigoplus_{i=1}^{4} U_i'Q \arrow[d, equal] \\
\bigoplus_{i=1}^{4} \bigoplus_{j=1}^{m} U_iV_j \arrow[rr, "\bigoplus_j U_iV_j \oplus \sigma^{\oplus}_{\bigoplus_j U_2V_j, \bigoplus_j U_3V_j} \oplus \bigoplus_j U_4V_j"] & & \bigoplus_{i=1}^{4} \bigoplus_{j=1}^{m} U_i'V_j
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by Axiom (R2) $*_3$ by naturality of $\sigma^{\oplus}$.

Notice that $\sigma^{\oplus}_{\bigoplus_j U_2V_j, \bigoplus_j U_3V_j}$ is in fact a composite of symmetries only involving monomials because of Axiom (S2).

Case 1b: $g = id_P$, $h = V_1 \oplus \sigma^{\oplus}_{V_2,V_3} \oplus V_4$, with $P = \bigoplus_{i=1}^{n} U_i$. Call $V_1' = V_1$, $V_2' = V_3$, $V_3' = V_2$, $V_4' = V_4$. Then

$$g \otimes h = \bigoplus_{i=1}^{n} \Big(U_iV_1 \oplus \sigma^{\oplus}_{U_iV_2, U_iV_3} \oplus U_iV_4\Big)$$

because the following diagram commutes:

$$\begin{tikzcd}
P\left(\bigoplus_{j=1}^{4} V_j\right) \arrow[r, "P(V_1\oplus\sigma^{\oplus}_{V_2,V_3}\oplus V_4)"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & P\left(\bigoplus_{j=1}^{4} V'_j\right) \arrow[d, equal] \\
\bigoplus_{i=1}^{n}\left(A_i(\bigoplus_{j=1}^{4} V_j)\right) \arrow[r, "\bigoplus_i U_i(V_1\oplus\sigma^{\oplus}_{V_2,V_3}\oplus V_4)"] \arrow[d, equal] & \bigoplus_{i=1}^{n}\left(A_i(\bigoplus_{j=1}^{4} V'_j)\right) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} U_i(V_1\oplus(V_2\oplus V_3)\oplus V_4) \arrow[d, equal] \arrow[r, phantom, "*_2"] & \bigoplus_{i=1}^{n} U_i(V_1\oplus(V_3\oplus V_2)\oplus V_4) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} U_iV_1\oplus U_i(V_2\oplus V_3)\oplus U_iV_4 \arrow[r, "\bigoplus_i (U_iV_1\oplus U_i\sigma^{\oplus}_{V_2,V_3}\oplus U_iV_4)"] \arrow[d, equal] \arrow[dr, phantom, "*_3"] & \bigoplus_{i=1}^{n} U_iV_1\oplus U_i(V_3\oplus V_2)\oplus U_iV_4 \arrow[d, equal] \\
\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{4} U_iV_j \arrow[r, "\bigoplus_i \left(U_iV_1\oplus\sigma^{\oplus}_{U_iV_2,U_iV_3}\oplus U_iV_4\right)"] & \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{4} U_iV'_j
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by limited left distributivity (U_i is a monomial for each i, hence Remark 5.1 applies) and functoriality of $\oplus$, while diagram $*_3$ commutes by (36).

Case 2a: $g = U_1\oplus \blacktriangleleft_{U_2} \oplus U_3$, $h = id_Q$ with $Q = \bigoplus_{j=1}^{m} V_j$. Call $U'_1 = U_1$, $U'_2 = U_2$, $U'_3 = U_2$, $U'_4 = U_3$. Then

$$g\otimes h = \bigoplus_{j=1}^{m} U_1V_j\oplus \blacktriangleleft_{\bigoplus_j U_2V_j} \oplus \bigoplus_{j=1}^{m} U_3V_j$$

because the following diagram commutes:

$$\begin{tikzcd}
(U_1\oplus U_2\oplus U_3)Q \arrow[r, "(U_1\oplus\blacktriangleleft_{U_2}\oplus U_3)Q"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & (U_1\oplus U_2\oplus U_2\oplus U_3)Q \arrow[d, equal] \\
U_1Q\oplus U_2Q\oplus U_3Q \arrow[r, "U_1Q\oplus\blacktriangleleft_{U_2}Q\oplus U_3Q"] \arrow[dr, "U_1Q\oplus\blacktriangleleft_{U_2Q}\oplus U_3Q"'] \arrow[d, equal] & U_1Q\oplus(U_2\oplus U_2)Q\oplus U_3Q \arrow[d, equal, "*_2"'] \\
\bigoplus_{i=1}^{3}\bigoplus_{j=1}^{m} U_iV_j \arrow[r, phantom, "*_3"] \arrow[dr, "\bigoplus_j U_1V_j\oplus\blacktriangleleft_{\bigoplus_j U_2V_j}\oplus\bigoplus_j U_3V_j"'] & U_1Q\oplus U_2Q\oplus U_2Q\oplus U_3Q \arrow[d, equal] \\
 & \bigoplus_{i=1}^{4}\bigoplus_{j=1}^{m} U'_iV_j
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by Proposition 3.15 that says $\blacktriangleleft_X Y; \delta^r_{X,X,Y} = \blacktriangleleft_{XY}$, while $*_3$ by naturality of $\blacktriangleleft$ applied to the right distributor $U_2Q = \bigoplus_j U_2V_j$.

Notice that $\blacktriangleleft_{\bigoplus_j U_2V_j}$ is a composite of diagonals only involving monomials because of Axiom (FP1).

Case 2b: $g = id_P$, $h = V_1 \oplus \blacktriangleleft_{V_2} \oplus V_3$, with $P = \bigoplus_{i=1}^{n} U_i$. Then

$$g \otimes h = \bigoplus_{i=1}^{n} (U_iV_1 \oplus \blacktriangleleft_{U_iV_2} \oplus U_iV_3)$$

because the following diagram commutes:

$$\begin{tikzcd}
P(V_1 \oplus V_2 \oplus V_3) \arrow[r, "P(V_1 \oplus \blacktriangleleft_{V_2} \oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & P(V_1 \oplus V_2 \oplus V_2 \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} U_i(V_1 \oplus V_2 \oplus V_3) \arrow[r, "\bigoplus_i U_i(V_1 \oplus \blacktriangleleft_{V_2} \oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_2"] & \bigoplus_{i=1}^{n} U_i(V_1 \oplus V_2 \oplus V_2 \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} (U_iV_1 \oplus U_iV_2 \oplus U_iV_3) \arrow[r, "\bigoplus_{i=1}^{n} (U_iV_1 \oplus U_i \blacktriangleleft_{V_2} \oplus U_iV_3)"] \arrow[dr, "\bigoplus_i (U_iV_1 \oplus \blacktriangleleft_{U_i v_2} \oplus U_iV_3)"'] & \bigoplus_{i=1}^{n} (U_iV_1 \oplus U_i(V_2 \oplus V_2) \oplus U_iV_3) \arrow[d, equal] \arrow[d, phantom, "*_3"'] \\
 & \bigoplus_{i=1}^{n} (U_iV_1 \oplus U_iV_2 \oplus U_iV_2 \oplus U_iV_3)
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by limited left distributivity (as U_i is a monomial for each i), $*_3$ by Proposition 3.15 which says $X \blacktriangleleft_Y; \delta^l_{X,Y,Y} = \blacktriangleleft_{XY}$.

Case 3a: $g = U_1 \oplus !_{U_2} \oplus U_3$, $h = id_Q$ with $Q = \bigoplus_{j=1}^{m} V_j$. Then

$$g \otimes h = \bigoplus_{j=1}^{m} U_1V_j \oplus !_{\bigoplus_j U_2V_j} \oplus \bigoplus_{j=1}^{m} U_3V_j$$

because the following diagram commutes:

$$
\begin{tikzcd}
(U_1 \oplus U_2 \oplus U_3)Q \arrow[r, "(U_1 \oplus !_{U_2} \oplus U_3)Q"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & (U_1 \oplus 0 \oplus U_3)Q \arrow[d, equal] \\
U_1Q \oplus U_2Q \oplus U_3Q \arrow[r, "U_1Q \oplus !_{U_2}Q \oplus U_3Q"] \arrow[d, equal] \arrow[dr, "U_1Q \oplus !_{U_2Q} \oplus U_3Q"'] & U_1Q \oplus 0Q \oplus U_3Q \arrow[d, equal, "*_2"'] \\
\bigoplus_{i=1}^{3} \bigoplus_{j=1}^{m} U_iV_j \arrow[dr, "\bigoplus_j U_1V_j \oplus !_{\bigoplus_j U_2V_j} \oplus \bigoplus_j U_3V_j"'] \arrow[r, phantom, "*_3"] & U_1Q \oplus 0 \oplus U_3Q \arrow[d, equal] \\
 & \bigoplus_{j=1}^{m} U_1V_j \oplus \bigoplus_{j=1}^{m} U_3V_j
\end{tikzcd}
$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by Proposition 3.15 that says $!_X Y; \lambda^{\bullet}_Y = !_{XY}$, $*_3$ by limited left distributivity and naturality of ! applied to $\Delta^l_{U_2,V_1,\dots,V_m}$.

Notice that $!_{\bigoplus_j U_2V_j}$ is a sum of $!_{U_2V_j}$ by Axiom (FP2).

Case 3b: $g = id_P$, $h = V_1 \oplus !_{V_2} \oplus V_3$, with $P = \bigoplus_{i=1}^{n} U_i$. Then

$$
g \otimes h = \bigoplus_{i=1}^{n} (U_iV_1 \oplus !_{U_iV_2} \oplus U_iV_3)
$$

because the following diagram commutes:

$$
\begin{tikzcd}
P(V_1 \oplus V_2 \oplus V_3) \arrow[r, "P(V_1 \oplus !_{V_2} \oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & P(V_1 \oplus 0 \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} U_i(V_1 \oplus V_2 \oplus V_3) \arrow[r, "\bigoplus_i U_i(V_1 \oplus !_{V_2} \oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_2"] & \bigoplus_{i=1}^{n} U_i(V_1 \oplus 0 \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} (U_iV_1 \oplus U_iV_2 \oplus U_iV_3) \arrow[r, "\bigoplus_{i=1}^{n} (U_iV_1 \oplus U_i!_{V_2} \oplus U_iV_3)"] \arrow[dr, "\bigoplus_i (U_iV_1 \oplus !_{U_iv_2} \oplus U_iV_3)"'] & \bigoplus_{i=1}^{n} (U_iV_1 \oplus U_i0 \oplus U_iV_3) \arrow[d, equal, "*_3"'] \\
 & \bigoplus_{i=1}^{n} (U_iV_1 \oplus 0 \oplus U_iV_2 \oplus U_iV_3)
\end{tikzcd}
$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by limited left distributivity (as U_i is a monomial for each i), $*_3$ by Proposition 3.15 which says $X!_Y; \rho^{\bullet}_X = !_{XY}$.

Cases 4 and 5 involving ▶ and ¡ are analogous to cases 2 and 3.

Case 6a: $g = U_1 \oplus c \oplus U_3$ $h = id_Q$ with $Q = \bigoplus_{j=1}^{m} V_j$ and $c \colon U_2 \to U_2'$, $c \in \mathbf{C}_\Sigma$. Call $U_1' = U_1$

and $U_3' = U_3$. Then

$$g \otimes h = \bigoplus_{j=1}^{m} U_1 V_j \oplus \bigoplus_{j=1}^{m} cV_j \oplus \bigoplus_{j=1}^{m} U_3 V_j$$

because the following diagram commutes:

$$\begin{tikzcd}
(U_1 \oplus U_2 \oplus U_3)Q \arrow[r, "(U_1\oplus c\oplus U_3)Q"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & (U_1 \oplus U_2' \oplus U_3)Q \arrow[d, equal] \\
U_1Q \oplus U_2Q \oplus U_3Q \arrow[r, "U_1Q\oplus cQ\oplus U_3Q"] \arrow[d, equal] \arrow[dr, phantom, "*_2"] & U_1Q \oplus U_2'Q \oplus U_3Q \arrow[d, equal] \\
\bigoplus_{i=1}^{3} \bigoplus_{j=1}^{m} U_i V_j \arrow[r, "\bigoplus_j U_iV_j \oplus \bigoplus_j cV_j \oplus \bigoplus_j U_3V_j"] & \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{m} U_i' V_j
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity while $*_2$ by limited left distributivity.

Case 6b: $g = id_P$, $h = V_1 \oplus c \oplus V_3$ with $P = \bigoplus_{i=1}^{n} U_i$ and $\bigotimes_u g_u \colon V_2 \to V_2'$, $c \in \mathbf{C}_\Sigma$. Call $V_1' = V_1$ and $V_3' = V_3$. Then

$$g \otimes h = \bigoplus_{i=1}^{n} (U_i V_1 \oplus U_i c \oplus U_i V_3)$$

because the following diagram commutes:

$$\begin{tikzcd}
P(V_1 \oplus V_2 \oplus V_3) \arrow[r, "P(V_1\oplus c\oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_1"] & P(V_1 \oplus V_2' \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} U_i(V_1 \oplus V_2 \oplus V_3) \arrow[r, "\bigoplus_i U_i(V_1\oplus c\oplus V_3)"] \arrow[d, equal] \arrow[dr, phantom, "*_2"] & \bigoplus_{i=1}^{n} U_i(V_1 \oplus V_2' \oplus V_3) \arrow[d, equal] \\
\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{3} U_i V_j \arrow[r, "\bigoplus_i (U_iV_1\oplus U_ic\oplus U_iV_3)"] & \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{3} U_i V_j'
\end{tikzcd}$$

Diagram $*_1$ commutes by right distributivity, $*_2$ by limited left distributivity. $\square$

Proof of Theorem 5.5. We are now ready to define the functor $F \colon \mathbf{ssR}^{\flat}_{\Sigma} \to \mathbf{T}_\Sigma$: on objects, $F(X) = X{\downarrow}$; on morphisms it suffices to define F on $\sigma^{\oplus}_{U,V}$, $\blacktriangleleft_U$, $!_U$, $\blacktriangleright_U$, $¡_U$ and c (with $c \in \mathbf{C}_\Sigma$) for U, V monomials by virtue of Proposition 5.8, and then to extend F as a $\oplus$-preserving functor by setting $F(g \oplus h) = F(g) \oplus F(h)$, $F(g; h) = F(g); F(h)$, $F(id_U) = id_{F(U)}$. Just define

$$F(\sigma^{\oplus}_{U,V}) = \sigma^{\oplus}_{A\downarrow,B\downarrow} \qquad F(\blacktriangleleft_U) = \blacktriangleleft_{U\downarrow} \qquad F(!_U) = !_{U\downarrow}$$
$$F(c) = \underline{\overline{c}} \qquad F(\blacktriangleright_U) = \blacktriangleright_{U\downarrow} \qquad F(¡_U) = ¡_{U\downarrow},$$

and we get that $GF = id_{\mathbf{ssR}^\flat_\Sigma}$, because for any f in $\mathbf{ssR}^\flat_\Sigma$ we have

$$
\begin{aligned}
GF(f) &= GF\Big(\bigoplus_{k=1}^{l_1} f_k^1; \dots ; \bigoplus_{k=1}^{l_q} f_k^q\Big) && \text{(Proposition 5.8)}\\
&= \bigoplus_{k=1}^{l_1} GF(f_k^1); \dots \bigoplus_{k=1}^{l_q} GF(f_k^q) && \text{(F and G preserve $\oplus$; F, G, $\oplus$ are functors)}\\
&= \bigoplus_{k=1}^{l_1} f_k^1; \dots ; \bigoplus_{k=1}^{l_q} f_k^q && \text{(Immediate from the definition of F and G)}\\
&= f && \text{(Proposition 5.8)}
\end{aligned}
$$

Also the fact that $FG = id_{\mathbf{T}_\Sigma}$ is immediate from the definition of F and G. This ends the proof of Theorem 5.5.

Proof of Theorem 5.6. By definition $(\mathbf{T}_\Sigma, \oplus, 0)$ is symmetric monoidal (actually, it is a biproduct category). The isomorphism (26) allows us to endow $\mathbf{T}_\Sigma$ with an additional symmetric monoidal structure $\otimes_{\mathbf{T}_\Sigma}$, simply by defining:

$$
\begin{tikzcd}
\mathbf{T}_\Sigma \times \mathbf{T}_\Sigma \arrow[r, "\otimes_{\mathbf{T}_\Sigma}"] & \mathbf{T}_\Sigma \\
(P,Q) \arrow[r, mapsto] \arrow[d, "(\mathrm{t}_1,\mathrm{t}_2)"'] & F\big(G(P) \otimes_{\mathbf{ssR}^\flat_\Sigma} G(Q)\big) \arrow[d, "F\big(G(\mathrm{t}_1) \otimes_{\mathbf{ssR}^\flat_\Sigma} G(\mathrm{t}_2)\big)"] \\
(P',Q') \arrow[r, mapsto] & F\big(G(P') \otimes_{\mathbf{ssR}^\flat_\Sigma} G(Q')\big)
\end{tikzcd}
$$

The assignment $\otimes_{\mathbf{T}_\Sigma}$ is a functor because so are F, G and $\otimes_{\mathbf{ssR}^\flat_\Sigma}$. Its unit is $1_{\mathbf{T}_\Sigma}$, that is the unary sum of the zero-ary monomial. Now we show that $\mathbf{T}_\Sigma$ together with the structures $(\mathbf{T}_\Sigma, \oplus, 0)$ and $(\mathbf{T}_\Sigma, \otimes_{\mathbf{T}_\Sigma}, 1_{\mathbf{T}_\Sigma})$ is a strict rig category.

$\otimes_{\mathbf{T}_\Sigma}$ is strictly associative:

$$
\begin{aligned}
(P \otimes_{\mathbf{T}_\Sigma} Q) \otimes_{\mathbf{T}_\Sigma} R &= F\big(G(P \otimes_{\mathbf{T}_\Sigma} Q) \otimes GR\big) && \text{(Definition of $\otimes_{\mathbf{T}_\Sigma}$)}\\
&= F\big(GF(GP \otimes GQ) \otimes GR\big) && \text{(Definition of $\otimes_{\mathbf{T}_\Sigma}$)}\\
&= F\big((GP \otimes GQ) \otimes GR\big) && (GF = id)\\
&= F\big(GP \otimes (GQ \otimes GR)\big) && \text{($\otimes_{\mathbf{ssR}^\flat_\Sigma}$ is strictly associative)}\\
&= F\big(GP \otimes GF(GQ \otimes GR)\big) && (id = GF)\\
&= F\big(GP \otimes G(Q \otimes_{\mathbf{T}_\Sigma} R)\big) && \text{(Definition of $\otimes_{\mathbf{T}_\Sigma}$)}\\
&= P \otimes_{\mathbf{T}_\Sigma} (Q \otimes_{\mathbf{T}_\Sigma} R) && \text{(Definition of $\otimes_{\mathbf{T}_\Sigma}$)}
\end{aligned}
$$

$\otimes_{\mathbf{T}_\Sigma}$ is strictly unitary:

$$\begin{aligned}
P \otimes_{\mathbf{T}_\Sigma} 1_{\mathbf{T}_\Sigma} &= F(GP \otimes G1_{\mathbf{T}_\Sigma}) && \text{(Definition of } \otimes_{\mathbf{T}_\Sigma}) \\
&= F(GP \otimes 1) && (G(1_{\mathbf{T}_\Sigma}) = 1) \\
&= FGP && (\otimes_{\mathbf{ssR}^\flat_\Sigma} \text{ is strictly unitary}) \\
&= P && (FG = id)
\end{aligned}$$

We will now simply write $\otimes$ instead of $\otimes_{\mathbf{T}_\Sigma}$: the context will clarify when we are speaking of the newly defined tensor in $\mathbf{T}_\Sigma$ or in $\mathbf{ssR}^\flat_\Sigma$. For $P, Q \in \mathbf{T}_\Sigma$, define $\sigma^{\otimes}_{P,Q} \stackrel{\text{def}}{=} F(\sigma^{\otimes}_{GP,GQ})\colon P \otimes_{\mathbf{T}_\Sigma} Q \to Q \otimes_{\mathbf{T}_\Sigma} P$. Then $\sigma^{\otimes}$ is a natural isomorphism because it is the result of a whiskering:

$$\begin{tikzcd}
\mathbf{T}_\Sigma \times \mathbf{T}_\Sigma \arrow[r, "G \times G"] & \mathbf{ssR}^\flat_\Sigma \times \mathbf{ssR}^\flat_\Sigma \arrow[r, bend left, "\otimes"] \arrow[r, bend right, "swap;\otimes"'] \arrow[r, phantom, "\Downarrow \sigma^{\otimes}"] & \mathbf{ssR}^\flat_\Sigma \arrow[r, "F"] & \mathbf{T}_\Sigma
\end{tikzcd}$$

The symmetry axiom, which requires the commutativity of the following triangle:

$$\begin{tikzcd}
P \otimes Q \arrow[r, "\sigma^{\otimes}_{P,Q}"] \arrow[dr, "id"'] & Q \otimes P \arrow[d, "\sigma^{\otimes}_{Q,P}"] \\
& P \otimes Q
\end{tikzcd} \qquad \text{(in } \mathbf{T}_\Sigma)$$

is trivially satisfied, because the above is the image along F of

$$\begin{tikzcd}
GP \otimes GQ \arrow[r, "\sigma^{\otimes}_{GP,GQ}"] \arrow[dr, "id"'] & GQ \otimes GP \arrow[d, "\sigma^{\otimes}_{GQ,GP}"] \\
& GP \otimes GQ
\end{tikzcd} \qquad \text{(in } \mathbf{ssR}^\flat_\Sigma)$$

which commutes because $\sigma^{\otimes}$ is a symmetry for $\otimes$ in $\mathbf{ssR}^\flat_\Sigma$. Same holds for the other axioms of symmetric monoidal categories.

$\mathbf{T}_\Sigma$ inherits distributors and annihilators from $\mathbf{ssR}^\flat_\Sigma$ as well, by setting

- $\delta^r_{P,Q,R} = F(\delta^r_{GP,GQ,GR}) = F(id_{(GP \oplus GQ)GR}) = id_{(P \oplus Q)R}$
- $\delta^l_{P,Q,R} = F(\delta^l_{GP,GQ,GR})$
- $\lambda^{\bullet}_P = F(\lambda^{\bullet}_{GP}) = id_0$, $\rho^{\bullet}_P = F(\rho^{\bullet}_{GP}) = id_0$

and we have that $\delta^l_{A,Q,R} = id_{A \otimes_{\mathbf{T}_\Sigma}(Q \oplus R)}$ when $A \in \mathcal{S}$. $\mathbf{T}_\Sigma$ satisfies all the twelve axioms of the definition of rig category, because each of them requires a diagram to commute, and all those diagrams are image along F of diagrams in $\mathbf{ssR}^\flat_\Sigma$ that involve objects of the form GP, GQ, GR... and that commute because $\mathbf{ssR}^\flat_\Sigma$ is in fact a rig category. For instance, looking at Axiom (R1), we have

$$\begin{tikzcd} (P\oplus Q)\otimes R \arrow[r, "\delta^r_{P,Q,R}"] \arrow[d, "\sigma^{\otimes}_{P\oplus Q,R}"'] \arrow[dr, phantom, "\text{in } \mathbf{T}_\Sigma"] & (P\otimes R)\oplus(Q\otimes R) \arrow[d, "\sigma^{\otimes}_{P,R}\oplus\sigma^{\otimes}_{Q,R}"] \\ R\otimes(P\oplus Q) \arrow[r, "\delta^l_{R,P,Q}"] & (R\otimes P)\oplus(R\otimes Q) \end{tikzcd} = F\left[\begin{tikzcd} (GP\oplus GQ)\otimes GR \arrow[r, "\delta^r_{GP,GQ,GR}"] \arrow[d, "\sigma^{\otimes}_{GP\oplus GQ,GR}"'] \arrow[dr, phantom, "\text{in } \mathbf{ssR}^\flat_\Sigma"] & (GP\otimes GR)\oplus(GQ\otimes GR) \arrow[d, "\sigma^{\otimes}_{GP,GR}\oplus\sigma^{\otimes}_{GQ,GR}"] \\ GR\otimes(GP\oplus GQ) \arrow[r, "\delta^l_{GR,GP,GQ}"] & (GR\otimes GP)\oplus(GR\otimes GQ) \end{tikzcd}\right]$$

because

$$F(\sigma^{\otimes}_{GP\oplus GQ,GR}) = F(\sigma^{\otimes}_{G(P\oplus Q),GR}) = \sigma^{\otimes}_{P\oplus Q,R}$$

and

$$F(\sigma^{\otimes}_{GP,GR}\oplus\sigma^{\otimes}_{GQ,GR}) = F(\sigma^{\otimes}_{GP,GR})\oplus F(\sigma^{\otimes}_{GQ,GR}) = \sigma^{\otimes}_{P,R}\oplus\sigma^{\otimes}_{Q,R},$$

since F and G preserve sums.

Finally, F and G trivially preserve the structure of rig category of $\mathbf{T}_\Sigma$ and $\mathbf{ssR}^\flat_\Sigma$, by construction and the fact that they are mutually inverse. This concludes the proof of Theorem 5.6.

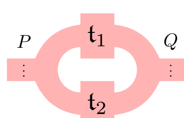
The tensor in $\mathbf{T}_\Sigma$, explicitly. How does one compute the tensor between two morphisms of $\mathbf{T}_\Sigma$? Since $\otimes_{\mathbf{T}_\Sigma}$ is a functor, it suffices to describe $\mathfrak{t}\otimes_{\mathbf{T}_\Sigma} id_Q$ and $id_P\otimes_{\mathbf{T}_\Sigma}\mathfrak{t}$, with

$$\mathfrak{t} = id_X\oplus\mathfrak{t}'\oplus id_Y \quad \text{where } \mathfrak{t}' = \underline{c}, \sigma^{\oplus}_{C,D}, \blacktriangleleft_C, !_C, \blacktriangleright_C, ¡_C$$

for $X=\bigoplus_{k=1}^p X_k$, $Y=\bigoplus_{k=1}^q Y_k$, $P=\bigoplus_{i=1}^n U_i$, $Q=\bigoplus_{j=1}^m V_j$ and C,D monomials. We do this in Table 7. Then one can prove that the tensor product of $\mathbf{T}_\Sigma$ defined in (23) coincides with $\otimes_{\mathbf{T}_\Sigma}$ defined in this section by checking that for each case of $\mathfrak{t}'$ one obtains the same table as Table 7.

6 Tapes as Matrices

Like any category with finite biproducts, $\mathbf{T}_\Sigma$ is enriched over **CMon**, the category of commutative monoids (Remark 2.10). For all polynomials P,Q, the homset $\mathbf{T}_\Sigma[P,Q]$ carries a commutative monoid structure defined as

$\mathfrak{t}_1+\mathfrak{t}_2 \stackrel{\text{def}}{=}$  (i.e. $\mathfrak{t}_1+\mathfrak{t}_2 \stackrel{\text{def}}{=} \blacktriangleleft_P;(\mathfrak{t}_1\oplus\mathfrak{t}_2);\blacktriangleright_Q$)

$\mathfrak{o}_{P,Q} \stackrel{\text{def}}{=}$  (i.e. $\mathfrak{o}_{P,Q} \stackrel{\text{def}}{=} !_P;¡_Q$)

for all $\mathfrak{t}_1,\mathfrak{t}_2\colon P\to Q$. This structure distributes not only over composition but also with respect to the monoidal product $\otimes$.

Proposition 6.1. Let $\mathfrak{t}_1,\mathfrak{t}_2\colon P\to Q$ and $\mathfrak{s}\colon R\to S$. It holds that

1. $(\mathfrak{t}_1+\mathfrak{t}_2);\mathfrak{s} = (\mathfrak{t}_1;\mathfrak{s})+(\mathfrak{t}_2;\mathfrak{s})$
2. $\mathfrak{s};(\mathfrak{t}_1+\mathfrak{t}_2) = (\mathfrak{s};\mathfrak{t}_1)+(\mathfrak{s};\mathfrak{t}_2)$
3. $\mathfrak{o};\mathfrak{s} = \mathfrak{o} = \mathfrak{s};\mathfrak{o}$
4. $(\mathfrak{t}_1+\mathfrak{t}_2)\otimes\mathfrak{s} = (\mathfrak{t}_1\otimes\mathfrak{s})+(\mathfrak{t}_2\otimes\mathfrak{s})$
5. $\mathfrak{s}\otimes(\mathfrak{t}_1+\mathfrak{t}_2) = (\mathfrak{s}\otimes\mathfrak{t}_1)+(\mathfrak{s}\otimes\mathfrak{t}_2)$
6. $\mathfrak{o}\otimes\mathfrak{s} = \mathfrak{o} = \mathfrak{s}\otimes\mathfrak{o}$

Table 7: $\otimes_{\mathbf{T}_\Sigma}$ explicitly, with $\mathfrak{t} = \bigoplus_{k=1}^{p} X_k \oplus \mathfrak{t}' \oplus \bigoplus_{k=1}^{q} Y_k$, $P = \bigoplus_{i=1}^{n} U_i$, $Q = \bigoplus_{j=1}^{m} V_j$, C and D monomials.

$\mathfrak{t}'$	$\mathfrak{t} \otimes_{\mathbf{T}_\Sigma} id_Q$	$id_P \otimes_{\mathbf{T}_\Sigma} \mathfrak{t}$
$\underline{\overline{c}}$	$\bigoplus_{k=1}^{p} \bigoplus_{j=1}^{m} X_k V_j \oplus \bigoplus_{j=1}^{m} \underline{\overline{c \otimes V_j}} \oplus \bigoplus_{k=1}^{q} \bigoplus_{j=1}^{m} Y_k V_j$	$\bigoplus_{i=1}^{n} \left(\bigoplus_{k=1}^{p} U_i X_k \oplus \underline{\overline{U_i \otimes c}} \oplus \bigoplus_{k=1}^{q} U_i Y_k \right)$
$\sigma^{\oplus}_{C,D}$	$\sigma^{\oplus}_{\bigoplus_j CV_j, \bigoplus_j DV_j}$	$\sigma^{\oplus}_{U_i C, U_i D}$
$◀_C$	$◀_{\bigoplus_j CV_j}$	$◀_{U_i C}$
$!_C$	$!_{\bigoplus_j CV_j}$	$!_{U_i C}$
$▶_C$	$▶_{\bigoplus_j CV_j}$	$▶_{U_i C}$
$¡_C$	$¡_{\bigoplus_j CV_j}$	$¡_{U_i C}$

Proof. The first three points hold in any category with finite biproduct. To prove the remaing properties it is important to choose the good order. It is indeed convenient to first prove the fourth one, as one can exploit some laws that only hold for the right whiskering.

$$\begin{aligned}
(\mathfrak{t}_1 + \mathfrak{t}_2) \otimes \mathfrak{s} &= L_P(\mathfrak{s}); R_S(\mathfrak{t}_1 + \mathfrak{t}_2) && \text{(Def. } \otimes\text{)}\\
&= L_P(\mathfrak{s}); R_S(\blacktriangleleft_P; (\mathfrak{t}_1 \oplus \mathfrak{t}_2); \blacktriangleright_Q) && \text{(Def. } +\text{)}\\
&= L_P(\mathfrak{s}); R_S(\blacktriangleleft_P); (R_S(\mathfrak{t}_1) \oplus R_S(\mathfrak{t}_2)); R_S(\blacktriangleright_Q) && \text{((W2), (W5))}\\
&= L_P(\mathfrak{s}); \blacktriangleleft_{PS}; (R_S(\mathfrak{t}_1) \oplus R_S(\mathfrak{t}_2)); \blacktriangleright_{QS} && \text{(Lemma 3.17)}\\
&= \blacktriangleleft_{PR}; (L_P(\mathfrak{s}) \oplus L_P(\mathfrak{s})); (R_S(\mathfrak{t}_1) \oplus R_S(\mathfrak{t}_2)); \blacktriangleright_{QS} && \text{(Nat. } \blacktriangleleft_{PS}\text{)}\\
&= \blacktriangleleft_{PR}; (\,(L_P(\mathfrak{s}); R_S(\mathfrak{t}_1)) \oplus (L_P(\mathfrak{s}); R_S(\mathfrak{t}_2))\,); \blacktriangleright_{QS} && \text{(Funct. } \oplus\text{)}\\
&= \blacktriangleleft_{PR}; (\,(\mathfrak{t}_1 \otimes \mathfrak{s}) \oplus (\mathfrak{t}_2 \otimes \mathfrak{s})\,); \blacktriangleright_{QS} && \text{(Def. } \otimes\text{)}\\
&= (\mathfrak{t}_1 \otimes \mathfrak{s}) + (\mathfrak{t}_2 \otimes \mathfrak{s}) && \text{(Def. } +\text{)}
\end{aligned}$$

To prove the fifth property, one can now reuse the fourth one and exploits symmetries.

$$\begin{aligned}
\mathfrak{s} \otimes (\mathfrak{t}_1 + \mathfrak{t}_2) &= \sigma^{\otimes}_{R,P}; \sigma^{\otimes}_{P,R}; (\,\mathfrak{s} \otimes (\mathfrak{t}_1 + \mathfrak{t}_2)\,) && \text{(Inv.)}\\
&= \sigma^{\otimes}_{R,P}; (\,(\mathfrak{t}_1 + \mathfrak{t}_2) \otimes \mathfrak{s}\,); \sigma^{\otimes}_{Q,S} && \text{(Nat. } \sigma^{\otimes}\text{)}\\
&= \sigma^{\otimes}_{R,P}; (\,(\mathfrak{t}_1 \otimes \mathfrak{s}) + (\mathfrak{t}_2 \otimes \mathfrak{s})\,); \sigma^{\otimes}_{Q,S} && \text{(Prop. 6.1.4)}\\
&= (\,(\sigma^{\otimes}_{R,P}; (\mathfrak{t}_1 \otimes \mathfrak{s})) + (\sigma^{\otimes}_{R,P}; (s \otimes \mathfrak{t}_2))\,); \sigma^{\otimes}_{Q,S} && \text{(+ enric.)}\\
&= (\,((\mathfrak{s} \otimes \mathfrak{t}_1); \sigma^{\otimes}_{S,Q}) + ((s \otimes \mathfrak{t}_2); \sigma^{\otimes}_{S,Q})\,); \sigma^{\otimes}_{Q,S} && \text{(Nat. } \sigma^{\otimes}\text{)}\\
&= (\,(\mathfrak{s} \otimes \mathfrak{t}_1) + (s \otimes \mathfrak{t}_2)\,); \sigma^{\otimes}_{S,Q}; \sigma^{\otimes}_{Q,S} && \text{(+ enric.)}\\
&= (\mathfrak{s} \otimes \mathfrak{t}_1) + (s \otimes \mathfrak{t}_2) && \text{(Inv.)}
\end{aligned}$$

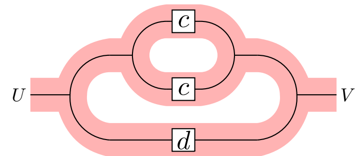
The proof of the leftmost equivalence of the sixth point follows a path similar to the fourth one.

$$\begin{aligned}
\mathfrak{o} \otimes \mathfrak{s} &= L_P(\mathfrak{s}); R_S(\mathfrak{o}) && \text{(Def. } \otimes\text{)}\\
&= L_P(\mathfrak{s}); R_S(!_P; ¡_Q) && \text{(Def. } \mathfrak{o}\text{)}\\
&= L_P(\mathfrak{s}); R_S(!_P); R_S(¡_Q) && \text{((W2), (W5))}\\
&= L_P(\mathfrak{s}); !_{PS}; ¡_{QS} && \text{(Lemma 3.17)}\\
&= !_{PR}; ¡_{QS} && \text{(Nat. !)}\\
&= \mathfrak{o} && \text{(Def. } \mathfrak{o}\text{)}
\end{aligned}$$

The proof for the rightmost equivalence is completely analogous to the leftmost one. □

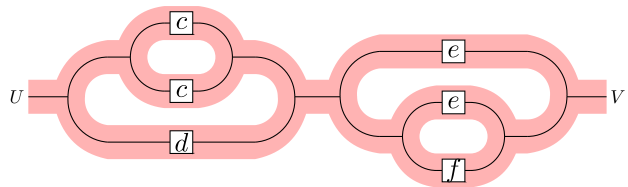
In this section, we illustrate how such monoidal enrichment can be exploited for defining a matrix calculus of tapes. First of all we need to identify the entries of these matrices. Let **Mnm** be the full subcategory of $\mathbf{T}_{\Sigma}$ whose objects are just monomials (i.e., unary sums). It is immediate to see that **Mnm** is a symmetric strict monoidal category (with respect to $\otimes$), because if U and V are two monomials, then so is $U \otimes V$ by definition of $\otimes$. It is also clearly enriched over **CMon**.

Here is an example of a morphism in **Mnm**:



$$(27)$$

Notice that it is in fact $\underline{\overline{c}} + \underline{\overline{c}} + \underline{\overline{d}}$, with c, $d \in \mathbf{C}_\Sigma$. Now, by definition, a morphism in **Mnm** is a tape of $\mathbf{T}_\Sigma$ with only one 'input' and one 'output', but in between these two it can be arbitrarily complicated. For instance, also the following tape is in **Mnm**:



$$(28)$$

However, it turns out that every tape in **Mnm** can be written as in (27), that is, as a local sum $\sum_i \underline{\overline{c_i}}$. This is a consequence of the fact that, as we will see in Corollary 6.13, **Mnm** is isomorphic to $\mathbf{C}_\Sigma^+$: the free **CMon**-enriched category generated by $\mathbf{C}_\Sigma$.

Definition 6.2. Let $\mathbf{C}$ be any category. The *free* **CMon**-*enriched category generated by* $\mathbf{C}$, denoted as $\mathbf{C}^+$, is the category whose objects are those of $\mathbf{C}$, while $\mathbf{C}^+[A, B]$ is the free commutative monoid generated by $\mathbf{C}[A, B]$: a morphism from $A \to B$ in $\mathbf{C}^+$ is a finite multiset of morphisms $A \to B$ of $\mathbf{C}$. We write multisets as $\{\!|\, a_1, \dots, a_n \,|\!\}$ where the a_i are not necessarily distinct.

The identity morphism for A in $\mathbf{C}^+$ is $\left\{\!\middle|\, id_A^{\mathbf{C}} \,\middle|\!\right\}$ (the singleton multiset of id_A in $\mathbf{C}$), while if $f \colon A \to B$ and $g \colon B \to C$ are morphisms of $\mathbf{C}^+$, then

$$f ; g \stackrel{\text{def}}{=} \{\!|\, a ; b \mid a \in f,\, b \in g \,|\!\}$$

($a; b$ has multiplicity equal to the product of the multiplicities of a in the multiset f and b in the multiset g). Addition of multisets is given by union, with the empty multiset being the neutral element.

The isomorphism between **Mnm** and $\mathbf{C}_\Sigma^+$ assigns to each arrow $\mathfrak{m} \colon U \to V$ in **Mnm**, hereafter referred to as *monomial tapes*, a multiset whose elements are the morphisms of $\mathbf{C}_\Sigma$ that appear in a path in the diagrammatic representation of $\mathfrak{m}$. For instance, $\mathfrak{o}_{U,V}$ corresponds to the empty multiset $\{\!|\ |\!\}$, the monomial tape (27) corresponds to the multiset $\{\!|\, c, c, d \,|\!\}$, while (28) to

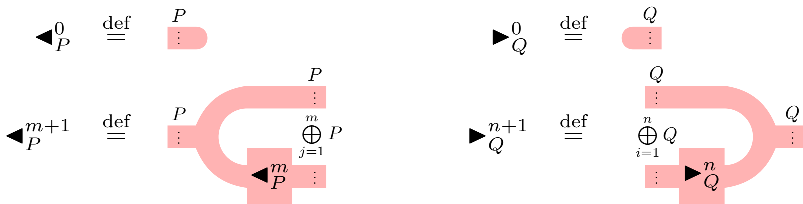
$$\{\!|\, \underline{\overline{c; e}}, \underline{\overline{c; e}}, \underline{\overline{c; f}}, \underline{\overline{c; e}}, \underline{\overline{c; e}}, \underline{\overline{c; f}}, \underline{\overline{d; e}}, \underline{\overline{d; e}}, \underline{\overline{d; f}} \,|\!\} .$$

Vice versa, every multiset $\{\!|\, c_1, \dots, c_n \,|\!\}$ of arrows in $\mathbf{C}_\Sigma$ corresponds to the monomial tape $\mathfrak{m} = \sum_{i=1}^n \underline{\overline{c_i}}$.

We can now consider an arbitrary tape $\mathfrak{t} \colon \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_j$. This is said to be in *matrix normal form* if there exist monomial tapes $\mathfrak{m}_{ji} \colon U_i \to V_j$ such that

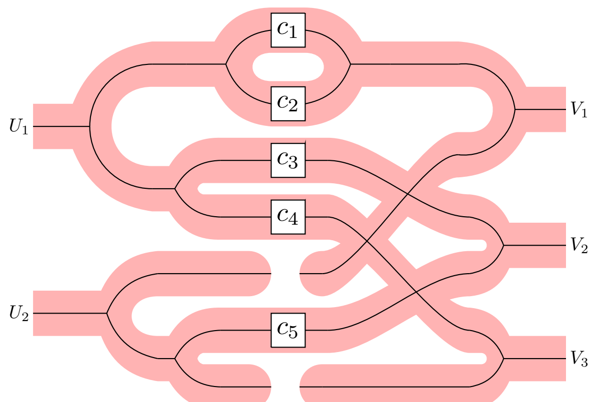
$$\mathfrak{t} = \bigoplus_{i=1}^{n} \blacktriangleleft_{U_i}^{m} ; \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathfrak{m}_{ji} ; \blacktriangleright^{n}_{\bigoplus_{j=1}^m V_j}$$

where the generalised diagonals and codiagonals $\blacktriangleleft^m_P \colon P \to \bigoplus^m P$, $\blacktriangleright^n_Q \colon Q \to \bigoplus^n Q$ are defined as:



(29)

For instance, the following tape $\mathfrak{t} \colon U_1 \oplus U_2 \to V_1 \oplus V_2 \oplus V_3$ is in matrix normal form.

$\mathfrak{t} =$ 

Every tape $\mathfrak{t} \colon \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_j$ in matrix normal form corresponds now to a *matrix* $\mathcal{F}(\mathfrak{t})$ of multisets of arrows in $\mathbf{C}_\Sigma$, of size $m \times n$. The entry at row j and column i is the multiset corresponding to the monomial tape $\mathfrak{m}_{ji}$ given by the matrix normal form of $\mathfrak{t}$. For example the matrix corresponding to the tape $\mathfrak{t}$ above is:

$$\mathcal{F}(\mathfrak{t}) = \begin{array}{c} \leftharpoondown \\ V_1 \\ V_2 \\ V_3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} \;\;U_1\;\; & \;\;U_2\;\; \end{array} \\ \begin{pmatrix} \{\!|\, c_1, c_2 \,|\!\} & \emptyset \\ \{\!|\, c_3 \,|\!\} & \{\!|\, c_5 \,|\!\} \\ \{\!|\, c_4 \,|\!\} & \emptyset \end{pmatrix} \end{array}$$

Now, we will see in Proposition 6.6 that every tape $\mathfrak{t}$ can in fact be written in matrix normal form, where $\mathfrak{m}_{ji}$ is $\mu_i ; \mathfrak{t} ; \pi_j$. Therefore, if $\mathfrak{t} \colon \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_j$, then the entry at row j and column i of $\mathcal{F}(\mathfrak{t})$ is the multiset collecting all the morphisms in $\mathbf{C}_\Sigma$ that appear in some path from the i-th input to the j-th output of $\mathfrak{t}$, thus of type $U_i \to V_j$.

We will show in Corollary 6.9 that the correspondence $\mathcal{F}$ defines an isomorphism between the category of tapes $\mathbf{T}_\Sigma$ and the category of matrices of multisets of arrows in $\mathbf{C}_\Sigma$. Interestingly enough, $\oplus$ in $\mathbf{T}_\Sigma$ corresponds to the direct sum of matrices, while $\otimes$ to the Kronecker product. We now set all this in detail.

6.1 Matrix calculus for categories with finite biproducts

It is well known that morphisms in a fb category have a matrix representation such that composition corresponds to the usual matrix multiplication (see, for example, [22]). Here we recall how the correspondence is defined and a few useful properties. For the rest of this section we denote the composite of two morphisms $f_1 \colon X \to Y$ and $f_2 \colon Y \to Z$ as $f_2 \circ f_1 \colon X \to Z$.

Definition 6.3. Let $f \colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{m} B_k$ in $\mathbf{C}$. The *matrix associated to* f is the $m \times n$ matrix $\mathcal{M}(f)$ whose (j,i) entry (row j, column i), for $j = 1,\dots,m$ and $i = 1,\dots,n$, is

$$f_{ji} \overset{\text{def}}{=} \Big(A_i \xrightarrow{\mu_i} \bigoplus_{k=1}^{n} A_k \xrightarrow{f} \bigoplus_{k=1}^{m} B_k \xrightarrow{\pi_j} B_j \Big).$$

For $f \colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{m} B_k$, we have

$$\mathcal{M}(f) = \begin{array}{cc} & \begin{array}{ccc} A_1 & \cdots & A_n \end{array} \\ \begin{array}{c} B_1 \\ \vdots \\ B_m \end{array} & \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} \end{array}$$

Notice how all morphisms in the i-th column have domain A_i, hence it makes sense to consider their pairing $\langle f_{1i},\dots,f_{mi}\rangle \colon A_i \to \bigoplus_k B_k$. Ranging over $i = 1,\dots,n$, we can now consider their copairing $[\langle f_{11},\dots,f_{m1}\rangle,\dots,\langle f_{1n},\dots,f_{mn}\rangle]$, which is now a morphism of type $\bigoplus_k A_k \to \bigoplus_k B_k$, just like f. Also, all morphisms in the j-th row have codomain B_j, hence we can take their copairing $[f_{j1},\dots,f_{jn}] \colon \bigoplus_k A_k \to B_j$ and then, ranging over $j = 1,\dots,m$, their collective pairing $\langle [f_{11},\dots,f_{1n}],\dots,[f_{m1},\dots,f_{mn}]\rangle$, which is also of type $\bigoplus_k A_k \to \bigoplus_k B_k$. It turns out these morphisms are all equal to each other.

Lemma 6.4. *Let* $f \colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{m} B_k$. *Then*

$$\begin{aligned} f &= [\langle f_{11},\dots,f_{m1}\rangle,\dots,\langle f_{1n},\dots,f_{mn}\rangle] \\ &= \langle [f_{11},\dots,f_{1n}],\dots,[f_{m1},\dots,f_{mn}]\rangle \end{aligned}$$

Proof. For the first equality we need to prove that $\mu_i ; f = \langle f_{1i},\dots,f_{mi}\rangle$ for all $i \in \{1,\dots,n\}$. This holds for any i because $(\mu_i ; f) ; \pi_j = f_{ji}$ for all $j \in \{1,\dots,m\}$.

Similarly, for the second equality we need to prove that for all $j \in \{1,\dots,m\}$ we have $f ; \pi_j = [f_{j1},\dots,j_{jn}]$, which is true because for all $i \in \{1,\dots,n\}$ it holds that $\mu_i ; (f ; \pi_j) = f_{ji}$. $\square$

Corollary 6.5. *Let* f *and* g *have same domain and codomain. Then* $\mathcal{M}(f) = \mathcal{M}(g)$ *if and only if* $f = g$.

We denote with $\blacktriangleleft_X^m \colon X \to \bigoplus_{j=1}^{m} X$ the generalised diagonal of X, inductively defined in the obvious way:

$$\blacktriangleleft_X^0 \overset{\text{def}}{=} !_X \qquad \blacktriangleleft_X^{m+1} \overset{\text{def}}{=} \Big(X \xrightarrow{\blacktriangleleft_X} X \oplus X \xrightarrow{id_X \oplus \blacktriangleleft_X^m} \bigoplus_{j=1}^{m+1} X \Big) \tag{30}$$

Similarly, we denote with $\blacktriangleright^n_X \colon \bigoplus_{i=1}^{n} X \to X$ the generalised codiagonal of X, inductively defined as:

$$\blacktriangleright^0_X \overset{\text{def}}{=} ¡_X \qquad \blacktriangleright^{n+1}_X \overset{\text{def}}{=} \Big(\bigoplus_{i=1}^{n+1} X \xrightarrow{id_X \oplus \blacktriangleright^n_X} X \oplus X \xrightarrow{\blacktriangleright_X} X \Big). \tag{31}$$

Observe that these definitions generalise (29) to an arbitrary category $\mathbf{C}$ with finite biproducts.

Proposition 6.6. Let $f \colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{m} B_k$. Then

$$\begin{aligned} f &= \Big(\bigoplus_{i=1}^{n} A_i \xrightarrow{\bigoplus_{i=1}^{n} \blacktriangleleft^m_{A_i}} \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} A_i \xrightarrow{\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} f_{ji}} \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} B_j \xrightarrow{\blacktriangleright^n_{\bigoplus_{j=1}^{m} B_j}} \bigoplus_{j=1}^{m} B_j \Big) \\ &= \Big(\bigoplus_{i=1}^{n} A_i \xrightarrow{\blacktriangleleft^m_{\bigoplus_{i=1}^{n} A_i}} \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n} A_i \xrightarrow{\bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n} f_{ji}} \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n} B_j \xrightarrow{\bigoplus_{j=1}^{m} \blacktriangleright^n_{B_j}} \bigoplus_{j=1}^{m} B_j \Big) \end{aligned}$$

Proof. In general, if $u_j \colon X \to Y_j$, with $j = 1, \dots, m$, we have

$$\langle u_1, \dots, u_m \rangle = \Big(X \xrightarrow{\blacktriangleleft^m_X} \bigoplus_{j=1}^{m} X \xrightarrow{\bigoplus_j u_j} \bigoplus_{j=1}^{m} U_j \Big)$$

(one can prove it by showing that the above satisfies the same universal property of $\langle u_1, \dots, u_m \rangle$). Hence, for any $i \in \{1, \dots, n\}$,

$$\langle f_{1i}, \dots, f_{mi} \rangle = \Big(A_i \xrightarrow{\blacktriangleleft^m_{A_i}} \bigoplus_{j=1}^{m} A_j \xrightarrow{\bigoplus_{j=1}^{m} f_{ji}} \bigoplus_{j=1}^{m} B_j \Big)$$

Analogously, if $v_i \colon W_i \to Z$, with $i = 1, \dots, n$,

$$[v_1, \dots, v_n] = \Big(\bigoplus_{i=1}^{n} W_i \xrightarrow{\bigoplus_{i=1}^{n} v_i} \bigoplus_{i=1}^{n} Z \xrightarrow{\blacktriangleright^n_Z} Z \Big)$$

Using these facts and Lemma 6.4, we get the two equations of the statement. $\square$

Example 6.7. Let $A = \bigoplus_{k=1}^{n} A_k$ and $B = \bigoplus_{k=1}^{m} B_k$. Then

$$\mathcal{M}(id_A) = \Big(\delta_{A_i,A_j}\Big) = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ A_n \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & \cdots & A_{n-1} & A_n \end{array} \\ \begin{pmatrix} id_{A_1} & 0_{A_2,A_1} & \cdots & \cdots & 0_{A_n,A_1} \\ 0_{A_1,A_2} & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0_{A_n,A_{n-1}} \\ 0_{A_1,A_n} & \cdots & \cdots & 0_{A_{n-1},A_n} & id_{A_n} \end{pmatrix} \end{array}$$

while

$$\mathcal{M}(0_{A,B}) = \begin{pmatrix} 0_{A_i,B_j} \end{pmatrix} = \begin{array}{c} \\ B_1 \\ \vdots \\ B_m \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & \cdots\cdots & A_n \end{array} \\ \begin{pmatrix} 0_{A_1,B_1} & \cdots & 0_{A_n,B_1} \\ \vdots & \ddots & \vdots \\ 0_{A_1,B_m} & \cdots & 0_{A_n,B_m} \end{pmatrix} \end{array}$$

For this reason we will sometimes simply write $0_{A,B}$ for the matrix $\mathcal{M}(0_{A,B})$ and id_A for the matrix $\mathcal{M}(id_A)$. The context they appear in will make clear whether we are referring to a matrix or to a morphism of $\mathbf{C}$.

The matrix representation of morphisms in fb categories enjoys several nice properties, for instance the fact that $\mathcal{M}(g \circ f) = \mathcal{M}(g) \cdot \mathcal{M}(f)$, which we recall in Appendix E.

6.2 Biproduct completion

Given a **CMon**-enriched category $\mathbf{S}$, we can form the *biproduct completion* (to use the terminology from [10], see also [22]) of $\mathbf{S}$, denoted as $\mathbf{Mat}(\mathbf{S})$, which we now describe. Objects are words of objects of $\mathbf{S}$, which we write as follows:

- 0 is the empty word,
- $\bigoplus_{k=1}^{n} A_k$ is the word $A_1, \ldots, A_n$.

A morphism $M\colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{m} B_k$ is a $m \times n$ matrix where $M_{ji} \in \mathbf{S}[A_i, B_j]$. The identity morphism of $\bigoplus_{k=1}^{n} A_k$ is given by the $n \times n$ "identity matrix" whose entry at row j and column i is id_{A_i} if $i = j$ and $0_{A_i,A_j}$ otherwise. Composition of two morphisms

$$\bigoplus_{k=1}^{n} A_k \xrightarrow{M} \bigoplus_{k=1}^{m} B_k \xrightarrow{N} \bigoplus_{k=1}^{l} C_k$$

is the matrix $N \circ M \stackrel{\text{def}}{=} N \cdot M$ where $N \cdot M$ is the result of matrix multiplication, i.e. it is the $l \times n$ matrix whose entry at row j and column i is

$$\sum_{k=1}^{m} N_{jk} \circ M_{ki}$$

(sum performed in the commutative monoid $\mathbf{S}[A_i, C_j]$).

$\mathbf{Mat}(\mathbf{S})$ has a zero object, given by 0, the empty word. The biproduct of two words is simply the concatenated word. Given two matrices $M\colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{n'} A'_k$ and $N\colon \bigoplus_{k=1}^{m} B_k \to \bigoplus_{k=1}^{m'} B'_k$,

$$M \oplus N \stackrel{\text{def}}{=} \begin{pmatrix} M & 0_{n' \times m} \\ 0_{m' \times n} & N \end{pmatrix} \quad \text{of size } (n' + m') \times (n + m).$$

Finally, for $A = \bigoplus_{k=1}^{n} A_k$,

$$\blacktriangleleft_A = \begin{pmatrix} id_A \\ id_A \end{pmatrix} \text{ of size } (n+n)\times n \qquad \blacktriangleright_A = \begin{pmatrix} id_A & id_A \end{pmatrix} \text{ of size } n \times (n+n)$$

$$!_A = \emptyset_{0\times n} \text{ of size } 0\times n \qquad ¡_A = \emptyset_{n\times 0} \text{ of size } n \times 0$$

The inherited structure of commutative monoid on $\mathbf{Mat}(\mathbf{S})[\bigoplus_k A_k, \bigoplus_k B_k]$ is given by the usual matrix sum (which makes use of the **CMon**-enrichment of $\mathbf{S}$), and the zero morphism $0_{\bigoplus_k A_k, \bigoplus_k B_k}$ is the null matrix, whose entries are all zero morphisms in $\mathbf{S}$.

We are now ready to present the main result of this section, which states that the strict fb category freely generated by an arbitrary category $\mathbf{C}$ can be obtained in two steps: first by forming the free **CMon**-enriched category on $\mathbf{C}$, and then by biproduct-completing the result.

Theorem 6.8. *Let $\mathbf{C}$ be any category and $F_2(\mathbf{C})$ be the strict fb category freely generated by $\mathbf{C}$ as in Definition 4.1. Then there is an isomorphism of categories with biproducts*

$$\begin{tikzcd} F_2(\mathbf{C}) \arrow[rr, bend left, "\mathcal{F}"] \arrow[rr, phantom, "\cong"] && \mathbf{Mat}(\mathbf{C}^+) \arrow[ll, bend left, "\mathcal{G}"] \end{tikzcd} \tag{32}$$

Proof. By [22, Exercises VIII.2.5-6], we have a pair of adjunctions

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathbf{CMonCat} \arrow[l, bend left] \arrow[r, bend left, "\mathbf{Mat}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{FBC} \arrow[l, bend left] \end{tikzcd}$$

where the right adjoints are forgetful functors. Since adjunctions compose, the composite functor $\mathbf{Mat}((-)^+)$ is left adjoint to the forgetful functor $U\colon \mathbf{FBC} \to \mathbf{Cat}$, as is F_2: hence they are naturally isomorphic. □

This result applies, in particular, to our category $\mathbf{T}_\Sigma = F_2(\mathbf{C}_\Sigma)$:

Corollary 6.9. *Let Σ be a monoidal signature and $\mathbf{C}_\Sigma$ the free strict symmetric monoidal category generated by Σ. Then $\mathbf{T}_\Sigma$ and $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$ are isomorphic as categories with biproducts.*

We present here the two functors $\mathcal{F}$ and $\mathcal{G}$ of Theorem 6.8 explicitly.

$\mathcal{F}$ is the identity on objects (and so will be $\mathcal{G}$). We define $\mathcal{F}$ on morphisms by induction:

- $\mathcal{F}(\underline{c}) = \begin{pmatrix} \{\!| c |\!\} \end{pmatrix}$ (the 1×1 matrix of the multiset consisting of one copy of c)
- $\mathcal{F}(id) = id$, $\mathcal{F}(\mathsf{t}_2 \circ \mathsf{t}_1) = \mathcal{F}(\mathsf{t}_2) \circ \mathcal{F}(\mathsf{t}_1)$
- Given $A, B \in \mathbf{C}$, $\mathcal{F}(\sigma_{A,B}) = \begin{pmatrix} \emptyset & \{\!| id_B |\!\} \\ \{\!| id_A |\!\} & \emptyset \end{pmatrix}$ of size 2×2,

$$\mathcal{F}(\blacktriangleleft_A) = \begin{pmatrix} \{\!| id_A |\!\} \\ \{\!| id_A |\!\} \end{pmatrix} \qquad \mathcal{F}(\blacktriangleright_A) = \begin{pmatrix} \{\!| id_A |\!\} & \{\!| id_A |\!\} \end{pmatrix}$$

$$\mathcal{F}(!_A) = \emptyset_{0\times 1} \qquad \mathcal{F}(¡_A) = \emptyset_{1\times 0}$$

- For $f_1 \colon \bigoplus_{i=1}^{n} A_i \to \bigoplus_{i=1}^{n'} A'_i$ and $f_2 \colon \bigoplus_{i=1}^{m} B_j \to \bigoplus_{i=1}^{m'} B'_j$,

$$\mathcal{F}(f_1 \oplus f_2) = \begin{pmatrix} \mathcal{F}(f_1) & \emptyset_{n' \times m} \\ \emptyset_{m' \times n} & \mathcal{F}(f_2) \end{pmatrix}$$

It follows that if $U = \bigoplus_{i=1}^{n} A_i$ then

$$\mathcal{F}(id_U) = \begin{pmatrix} \{\!|\, id_{A_1} \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, id_{A_n} \,|\!\} \end{pmatrix}$$

Since $\mathcal{F}$ sends symmetries, (co)diagonals and (co)bangs into symmetries, (co)diagonals and (co)bangs of $\mathbf{Mat}(\mathbf{C}^+)$, if $U = \bigoplus_{i=1}^{n} A_i$ and $V = \bigoplus_{j=1}^{m} B_j$ then:

$$\mathcal{F}(\sigma_{U,V}) = \begin{pmatrix} \emptyset_{m \times n} & \mathcal{F}(id_V) \\ \mathcal{F}(id_U) & \emptyset_{n \times m} \end{pmatrix}$$

$$\mathcal{F}(\blacktriangleleft_U) = \begin{pmatrix} \mathcal{F}(id_U) & \mathcal{F}(id_U) \end{pmatrix} \qquad \mathcal{F}(\blacktriangleright_U) = \begin{pmatrix} \mathcal{F}(id_U) \\ \mathcal{F}(id_U) \end{pmatrix}$$

$$\mathcal{F}(!_U) = \emptyset_{0 \times n} \qquad \mathcal{F}(¡_U) = \emptyset_{n \times 0}$$

Remark 6.10. Since $\mathbf{C}$ has all finite biproducts, $\mathbf{C}[U, V]$ is a commutative monoid where addition is given by (7). The action of $\mathcal{F}$ on $f_1 + f_2 \colon U \to V$, with $U = \bigoplus_{i=1}^{n} A_i$ and $V = \bigoplus_{j=1}^{m} B_j$, is the usual entry-by-entry addition of matrices:

$$\begin{aligned} \mathcal{F}(f_1 + f_2) &= \mathcal{F}(\blacktriangleright_V) \circ \mathcal{F}(f_1 \oplus f_2) \circ \mathcal{F}(\blacktriangleleft_U) \\ &= \begin{pmatrix} \mathcal{F}(id_V) & \mathcal{F}(id_V) \end{pmatrix} \begin{pmatrix} \mathcal{F}(f_1) & \emptyset_{m \times n} \\ \emptyset_{m \times n} & \mathcal{F}(f_2) \end{pmatrix} \begin{pmatrix} \mathcal{F}(id_U) \\ \mathcal{F}(id_U) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{F}(f_1) & \mathcal{F}(f_2) \end{pmatrix} \begin{pmatrix} \mathcal{F}(id_U) \\ \mathcal{F}(id_U) \end{pmatrix} \\ &= \mathcal{F}(f_1) + \mathcal{F}(f_2). \end{aligned}$$

Remark 6.11. If $f_1 \colon W \to U$ and $f_2 \colon W \to V$ then

$$
\begin{aligned}
\mathcal{F}(\langle f_1, f_2 \rangle) &= \mathcal{F}((f_1 \oplus f_2) \circ \langle id_W, id_W \rangle) \\
&= \mathcal{F}(f_1 \oplus f_2) \circ \mathcal{F}(\blacktriangleleft_W) \\
&= \begin{pmatrix} \mathcal{F}(f_1) & \emptyset \\ \emptyset & \mathcal{F}(f_2) \end{pmatrix} \begin{pmatrix} \mathcal{F}(id_W) \\ \mathcal{F}(id_W) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{F}(f_1) \\ \mathcal{F}(f_2) \end{pmatrix}
\end{aligned}
$$

Analogously, if $f_1 \colon U \to W$ and $f_2 \colon V \to W$ then $\mathcal{F}([f_1, f_2]) = \begin{pmatrix} \mathcal{F}(f_1) & \mathcal{F}(f_2) \end{pmatrix}$.

We now consider how to define the functor $\mathcal{G} \colon \mathbf{Mat}(\mathbf{C}^+) \to F_2(\mathbf{C})$. On objects it simply is the identity. On morphisms: let

$$
M = (M_{ji})_{\substack{j=1\dots m \\ i=1\dots n}} \colon \bigoplus_{i=1}^{n} A_i \to \bigoplus_{j=1}^{m} B_j \text{ in } \mathbf{Mat}(\mathbf{C}^+).
$$

If $m = 0$, then define $\mathcal{G}(M) = !_{\bigoplus_j B_j}$; if $n = 0$ define instead $\mathcal{G}(M) = ¡_{\bigoplus_i A_i}$. (This is consistent with the case $n = 0 = m$, because $!_0 = ¡_0 = id_0$ by Axioms (FP4) and (FFC4).)

Suppose now that $n \neq 0 \neq m$. We have that M_{ji} is a multiset of morphisms of type $A_i \to B_j$ in $\mathbf{C}$. M_{ji} determines a morphism in $F_2(\mathbf{C})$:

$$
\overline{\mathcal{G}}(M_{ji}) \stackrel{\text{def}}{=} \sum_{a \in M_{ji}} \underline{a} \colon A_i \to B_j
$$

Notice that for $M_{ji} = \emptyset$ we get $\overline{\mathcal{G}}(\emptyset) = 0_{A_i, B_j}$. Now, $\big(\overline{\mathcal{G}}(M_{ji})\big)_{j=1\dots m, i=1\dots n}$ is a matrix of morphisms of $F_2(\mathbf{C})$, to which corresponds a unique arrow of the same type of M by Corollary 6.5 and Proposition 6.6:

$$
\mathcal{G}(M) \stackrel{\text{def}}{=} \Big(\bigoplus_{i=1}^{n} A_i \xrightarrow{\bigoplus_{i=1}^{n} \blacktriangleleft_{A_i}^{m}} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} A_i \xrightarrow{\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \overline{\mathcal{G}}(M_{ji})} \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} B_j \xrightarrow{\blacktriangleright^{n}_{\bigoplus_{j=1}^{m} B_j}} \bigoplus_{j=1}^{m} B_j \Big).
$$

The interested reader can find a proof of the fact that $\mathcal{G}$ is a functor of fb categories and that it is the inverse of $\mathcal{F}$ in Appendix E.

6.3 Kronecker product in $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$

In the category $\mathbf{C}_\Sigma^+$, not only we can add and compose multisets of morphisms in $\mathbf{C}_\Sigma$ as one does in any $\mathbf{C}^+$ for arbitrary categories $\mathbf{C}$, but also tensorise them, using the monoidal structure of $\mathbf{C}_\Sigma$. Indeed, if $f \colon A \to A'$ and $g \colon B \to B'$ are in $\mathbf{C}_\Sigma^+$, then we can define

$$
f \otimes g \stackrel{\text{def}}{=} \{\!|\, a \otimes_{\mathbf{C}_\Sigma} b \mid a \in f,\, b \in g \,|\!\} : A \otimes_{\mathbf{C}_\Sigma} B \to A' \otimes_{\mathbf{C}_\Sigma} B'. \tag{33}
$$

This allows us to define a tensor product in the biproduct completion of $\mathbf{C}_\Sigma^+$ *à la* Kronecker. We will denote it as $\circledast$. On objects it is set as follows:

$$\Big(\bigoplus_{i=1}^{n} U_i\Big) \circledast \Big(\bigoplus_{j=1}^{m} V_j\Big) \stackrel{\text{def}}{=} \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} U_iV_j.$$

If $M\colon \bigoplus_{i=1}^{n} U_i \to \bigoplus_{i'=1}^{n'} U'_{i'}$ and $N\colon \bigoplus_{j=1}^{m} V_j \to \bigoplus_{j'=1}^{m'} V'_{j'}$, define $M \circledast N$ as the block matrix:

$$M \circledast N = \begin{array}{cc} & \begin{array}{ccccccc} U_1V_1 & \cdots & U_1V_m & \cdots & U_nV_1 & \cdots & U_nV_m \end{array} \\ \begin{array}{c} U'_1V'_1 \\ \vdots \\ U'_1V'_{m'} \\ \vdots \\ U'_{n'}V'_1 \\ \vdots \\ U'_{n'}V'_{m'} \end{array} & \left(\begin{array}{c|c|c} M_{11} \otimes N & \cdots & M_{1n} \otimes N \\ \hline \cdots & & \cdots \\ \hline M_{n'1} \otimes N & \cdots & M_{n'n} \otimes N \end{array}\right) \end{array}$$

where

$$M_{i'i} \otimes N = \begin{pmatrix} M_{i'i} \otimes N_{11} & \cdots & M_{i'i} \otimes N_{1m} \\ \vdots & \ddots & \vdots \\ M_{i'i} \otimes N_{m'1} & \cdots & M_{i'i} \otimes N_{m'n'} \end{pmatrix}$$

In other words, $M \circledast N$ is a matrix of size $n'm' \times nm$, whose rows are indexed by $(i', j') \in \{1, \ldots, n'\} \times \{1, \ldots, m'\}$ and whose columns by $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$; then we have

$$(M \circledast N)_{(i',j'),(i,j)} = M_{i'i} \otimes N_{j'j} = \{\!|\, a \otimes_{\mathbf{C}_\Sigma} b \mid a \in M_{i'i},\, b \in N_{j'j} \,|\!\}\,.$$

We have that $\circledast$ coincides with the tensor product inherited from $\mathbf{T}_\Sigma$, which therefore makes $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$ a strict rig category. The proof can be found in Appendix E.

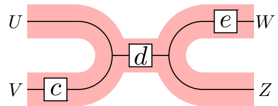
Theorem 6.12. $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$ *inherits the structure of a strict rig category from* $\mathbf{T}_\Sigma$ *via the isomorphism* (32)*, where the tensor product coincides with* $\circledast$*, making* (32) *an isomorphism of rig categories.*

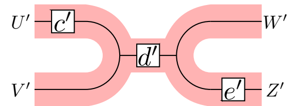
We conclude with a simple observation.

Corollary 6.13. $\mathbf{Mnm}$*, the full subcategory of* $\mathbf{T}_\Sigma$ *whose objects are monomials, is isomorphic to* $\mathbf{C}_\Sigma^+$ *as symmetric monoidal categories. In particular, if* $\mathcal{F}(\mathfrak{m})$ *is the* 1×1 *matrix consisting of the multiset* $\{\!|\, c_1, \ldots, c_n \,|\!\}$*, then* $\mathfrak{m} = \sum_{i=1}^{n} \underline{\overline{c_i}}$.

Proof. The functor $\mathcal{F}\colon \mathbf{T}_\Sigma \to \mathbf{Mat}(\mathbf{C}_\Sigma^+)$ restricts and corestricts to an isomorphism of symmetric monoidal categories (with respect to $\otimes$) between $\mathbf{Mnm}$ and the subcategory of $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$ consisting only of 1×1 matrices, which is monoidally isomorphic to $\mathbf{C}_\Sigma^+$. $\square$

Example 6.14. Consider again, from Example 4.14, the two tapes

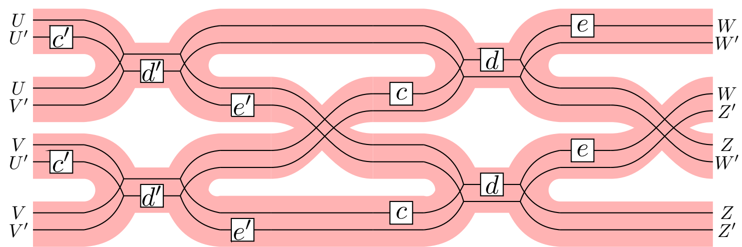
$\mathfrak{t}_1 =$ 

and $\mathfrak{t}_2 =$ 

Then

$$\mathcal{F}(\mathfrak{t}_1) = \begin{array}{c} \\ W \\ Z \end{array}\!\!\begin{array}{c} \begin{array}{cc} U & V \end{array} \\ \begin{pmatrix} \{\!|\, \boxed{d}\boxed{e} \,|\!\} & \{\!|\, \boxed{c}\boxed{d}\boxed{e} \,|\!\} \\ \{\!|\, \boxed{d} \,|\!\} & \{\!|\, \boxed{c}\boxed{d} \,|\!\} \end{pmatrix} \end{array} \quad \text{and} \quad \mathcal{F}(\mathfrak{t}_2) = \begin{array}{c} \\ W' \\ Z' \end{array}\!\!\begin{array}{c} \begin{array}{cc} U' & V' \end{array} \\ \begin{pmatrix} \{\!|\, \boxed{c'}\boxed{d'} \,|\!\} & \{\!|\, \boxed{d'} \,|\!\} \\ \{\!|\, \boxed{c'}\boxed{d'}\boxed{e'} \,|\!\} & \{\!|\, \boxed{d'}\boxed{e'} \,|\!\} \end{pmatrix} \end{array}$$

Their tensor,

$\mathfrak{t}_1 \otimes \mathfrak{t}_2 =$ 

corresponds to the Kronecker product of $\mathcal{F}(\mathfrak{t}_1)$ and $\mathcal{F}(\mathfrak{t}_2)$, which is indeed:

$$\mathcal{F}(\mathfrak{t}_1 \otimes \mathfrak{t}_2) = \begin{array}{c|cccc} & UU' & UV' & VU' & VV' \\ \hline WW' & \left\{\left|\begin{smallmatrix} \boxed{d}\boxed{e} \\ \boxed{c'}\boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{d}\boxed{e} \\ \boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d}\boxed{e} \\ \boxed{c'}\boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d}\boxed{e} \\ \boxed{d'} \end{smallmatrix}\right|\right\} \\ WZ' & \left\{\left|\begin{smallmatrix} \boxed{d}\boxed{e} \\ \boxed{c'}\boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{d}\boxed{e} \\ \boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d}\boxed{e} \\ \boxed{c'}\boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d}\boxed{e} \\ \boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} \\ ZW' & \left\{\left|\begin{smallmatrix} \boxed{d} \\ \boxed{c'}\boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{d} \\ \boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d} \\ \boxed{c'}\boxed{d'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d} \\ \boxed{d'} \end{smallmatrix}\right|\right\} \\ ZZ' & \left\{\left|\begin{smallmatrix} \boxed{d} \\ \boxed{c'}\boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{d} \\ \boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d} \\ \boxed{c'}\boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} & \left\{\left|\begin{smallmatrix} \boxed{c}\boxed{d} \\ \boxed{d'}\boxed{e'} \end{smallmatrix}\right|\right\} \end{array}$$

The entry at row j and column i of $\mathcal{F}(\mathfrak{t}_1 \otimes \mathfrak{t}_2)$ is the multiset collecting the morphisms of $\mathbf{C}_\Sigma$ obtained by following the paths from the i-th input to the j-th output of the tape $\mathfrak{t}_1 \otimes \mathfrak{t}_2$. Since, in this particular case, there is exactly one path between any given input and output, all the entries of $\mathcal{F}(\mathfrak{t}_1 \otimes \mathfrak{t}_2)$ are singletons.

7 From rig to monoidal signatures

Thanks to Theorems 4.24, we know that for any monoidal signature Σ, the category of tapes is isomorphic to $\mathbf{ssR}^\flat_\Sigma$, the sesquistrict finite biproduct rig category generated by Σ (and thus, by Theorem 3.18, equivalent to the fb-rig category freely generated by Σ). In this section we show that tape diagrams are actually a universal language for fb-rig categories since any rig signature Σ can be safely transformed into a monoidal signature Σ_M. By safely we mean that $\mathbf{ssR}^\flat_\Sigma$ is isomorphic to $\mathbf{ssR}^\flat_{\Sigma_M}$, the sesquistrict rig category generated by Σ_M. The underlying idea of this transformation is similar to the one allowing to pass from monoidal signature to a cartesian signature when considering finite product categories.

Let Σ be a rig signature and let s be a generator in Σ. By Theorem 3.18, we can safely assume that arity and coarity of s are polynomials. So, let us say that

$$s\colon \bigoplus_{i=1}^{n_s} U_i \to \bigoplus_{j=1}^{m_s} V_j$$

where U_i, V_j are monomials for $i = 1 \dots n_s$ and $j = 1 \dots m_s$. For each (i, j), we define a new generator

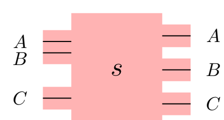
$$s_{j,i} \colon U_i \to V_j$$

having as arity the monomial U_i and as coarity the monomial V_j. These $s_{j,i}$'s are now formal symbols that we use to define a signature Σ_M, obtained from Σ by replacing each $s \in \Sigma$ with all the $s_{j,i}$'s:

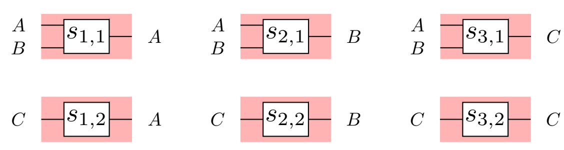
$$\Sigma_M \stackrel{\text{def}}{=} \{s_{j,i} \mid s \in \Sigma,\ j = 1 \dots m_s,\ i = 1 \dots n_s\}.$$

Observe that Σ_M is a monoidal signature since it only contains symbols having as arity and coarity some monomials.

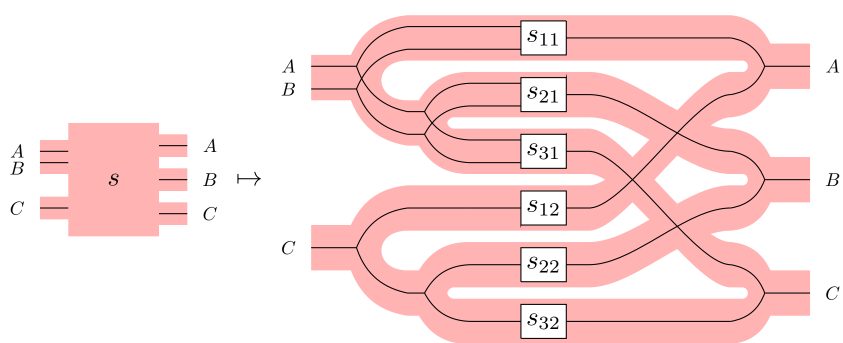
For example, assume that Σ consists of a single generator $s\colon AB \oplus C \to A \oplus B \oplus C$ depicted as follows:



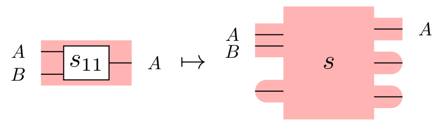
Then Σ_M is the signature consisting of the following six generators:



There is a canonical way to encode $s \in \mathbf{ssR}^\flat_\Sigma$ within $\mathbf{ssR}^\flat_{\Sigma_M}$, using the fb-structure:



Similarly, each $s_{j,i} \in \mathbf{ssR}^\flat_{\Sigma_M}$ can be encoded within $\mathbf{ssR}^\flat_{\Sigma}$ by using s, ! and ¡. For instance,



These two assignments induce two functors that, thanks to Proposition 6.6, can be easily shown to be isomorphisms.

Theorem 7.1. *Let Σ be a rig signature and Σ_M the corresponding monoidal signature. Then there is an isomorphism of rig categories*

$$\begin{tikzcd} \mathbf{ssR}^\flat_{\Sigma} \arrow[r, bend left, "F"] & \mathbf{ssR}^\flat_{\Sigma_M} \arrow[l, bend left, "G"] \end{tikzcd}$$

Proof. The functor F is the unique strict fb-rig functor mapping every $s \in \Sigma$ with arity $\bigoplus_i^n U_i$ and coarity $\bigoplus_j^m V_j$ into

$$\bigoplus_{i=1}^n U_i \xrightarrow{\bigoplus_{i=1}^n \blacktriangleleft^m_{U_i}} \bigoplus_{i=1}^n\bigoplus_{j=1}^m U_i \xrightarrow{\bigoplus_{i=1}^n\bigoplus_{j=1}^m s_{ji}} \bigoplus_{i=1}^n\bigoplus_{j=1}^m V_j \xrightarrow{\blacktriangleright^n_{\bigoplus_{j=1}^m V_j}} \bigoplus_{j=1}^m V_j$$

Similarly, the functor G is the unique strict fb-rig functor mapping every $s_{ji} \in \Sigma_M$ into

$$U_i \xrightarrow{\mu_i} \bigoplus_{i=1}^n U_i \xrightarrow{s} \bigoplus_{j=1}^m V_j \xrightarrow{\pi_j} V_j$$

To prove that $GF = Id_{\mathbf{ssR}^\flat_\Sigma}$, it is enough to check that $GF(s) = s$. Since G preserves the finite biproduct rig structure, it holds that $GF(s)$ is

$$\bigoplus_{i=1}^n U_i \xrightarrow{\bigoplus_{i=1}^n \blacktriangleleft^m_{U_i}} \bigoplus_{i=1}^n\bigoplus_{j=1}^m U_i \xrightarrow{\bigoplus_{i=1}^n\bigoplus_{j=1}^m G(s_{ji})} \bigoplus_{i=1}^n\bigoplus_{j=1}^m V_j \xrightarrow{\blacktriangleright^n_{\bigoplus_{j=1}^m V_j}} \bigoplus_{j=1}^m V_j$$

which, by Proposition 6.6, is exactly s.

To prove that $FG(s_{ji}) = s_{ji}$, just observe that $FG(s_{ji})$ is the upper leg of the following diagram:

$$\begin{tikzcd}
 & & \bigoplus_{h=1}^n\bigoplus_{k=1}^m U_h \arrow[r, "\bigoplus_{h=1}^n\bigoplus_{k=1}^m s_{kh}"] & \bigoplus_{h=1}^n\bigoplus_{k=1}^m V_k \arrow[dr, "\blacktriangleright^n_{\bigoplus_k V_k}"] & & \\
 & \bigoplus_{h=1}^n U_h \arrow[ur, "\bigoplus_{h=1}^n \blacktriangleleft^m_{U_h}"] & \bigoplus_{k=1}^m U_i \arrow[u, "\mu_i"] \arrow[r, "\bigoplus_{k=1}^m s_{ki}"] \arrow[d, "\pi_j"] & \bigoplus_{k=1}^m V_k \arrow[u, "\mu_i"] \arrow[r, "id"] \arrow[d, "\pi_j"] & \bigoplus_{k=1}^m V_k \arrow[dr, "\pi_j"] & \\
U_i \arrow[ur, "\mu_i"] \arrow[urr, "\blacktriangleleft^m_{U_i}"] \arrow[rr, "id"] & & U_i \arrow[r, "s_{ji}"] & V_j \arrow[rr, "id"] & & V_j
\end{tikzcd}$$

which commutes because of the naturality of the injections and the projections for the biproduct $\oplus$, as well as the universal property of $\blacktriangleright^{n}_{\bigoplus_k V_k}$ as the copairing of n copies of $id_{\bigoplus_k V_k}$ (in the top-right triangle). $\square$

8 Diagrammatic reasoning with tapes

String diagrams allow for graphical reasoning on arrows of (symmetric) monoidal categories. In particular, graphical proofs are significantly simpler, since the diagrammatic representation implicitly embodies the laws of strict monoidal categories (Table 1, p. 7). In this section, we show that tape diagrams allow for the same kind of graphical reasoning as string diagrams. However, this fact is not completely obvious because of the peculiar role played by $\otimes$ in tape diagrams.

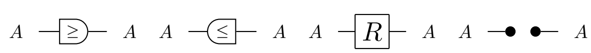
The usual way of reasoning through string diagrams is based on monoidal theories, namely a signature plus a set of axioms: either equations or inequations. Similarly a *tape theory* is a pair $(\Sigma, \mathcal{I})$ where Σ is a monoidal signature (or by the results of the previous section even a rig signature) and $\mathcal{I}$ is a set of axioms, namely a set of pairs of tapes with same domain and codomain. Herafter, we think to each pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ as an inequation $\mathfrak{t}_1 \leq \mathfrak{t}_2$, but the results that we develop in this section trivially hold also for equations: it is enough to add in $\mathcal{I}$ a pair $(\mathfrak{t}_2, \mathfrak{t}_1)$ for each $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathcal{I}$.

In the following, we write $\mathfrak{t}_1 \mathcal{I} \mathfrak{t}_2$ for $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathcal{I}$ and $\leq_{\mathcal{I}}$ for the smallest precongruence (w.r.t. $\oplus$, $\otimes$ and ;) generated by $\mathcal{I}$, i.e., the relation inductively generated as

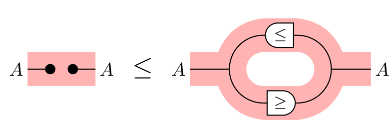
$$\frac{\mathfrak{t}_1 \mathcal{I} \mathfrak{t}_2}{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2}\,(\mathcal{I}) \qquad \frac{-}{\mathfrak{t} \leq_{\mathcal{I}} \mathfrak{t}}\,(r) \qquad \frac{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \quad \mathfrak{t}_2 \leq_{\mathcal{I}} \mathfrak{t}_3}{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_3}\,(t)$$
$$\frac{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{s}_2}{\mathfrak{t}_1 \oplus \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \oplus \mathfrak{s}_2}\,(\oplus) \qquad \frac{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{s}_2}{\mathfrak{t}_1 \otimes \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \otimes \mathfrak{s}_2}\,(\otimes) \qquad \frac{\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{s}_2}{\mathfrak{t}_1 ; \mathfrak{s}_1 \leq_{\mathcal{I}} \mathfrak{t}_2 ; \mathfrak{s}_2}\,(;)$$

Now, given two tape diagrams $\mathfrak{s}$ and $\mathfrak{t}$, one would like to prove that $\mathfrak{s} \leq_{\mathcal{I}} \mathfrak{t}$ through some graphical manipulation involving the axioms in $\mathcal{I}$ and the one in Figure 3 (p. 23). Unfortunately, this is not completely obvious with tapes, as illustrated by the following example.

Example 8.1. Let $\mathcal{S} = \{A\}$ and Σ be the monoidal signature consisting of the following generators.

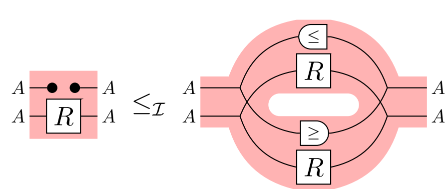


Let $\mathcal{I}$ be the set consiting only of the following axiom.



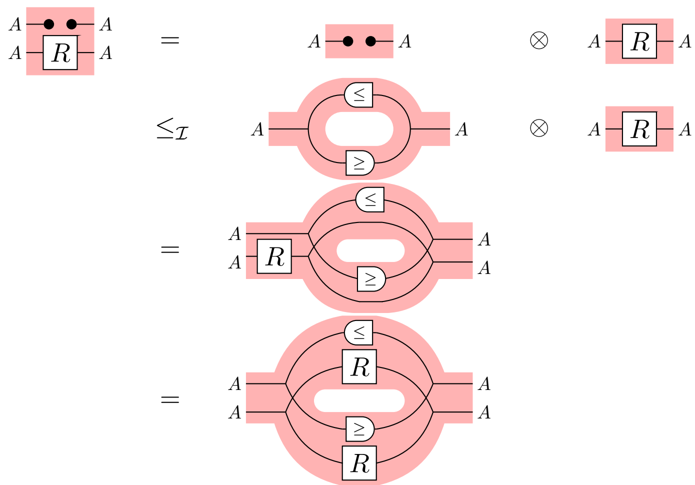
(TO)

It holds that



(34)

but, to prove it, it is necessary to decompose the two diagrams above through $\otimes$:



(Def. $\otimes$)

(TO)

(Def. $\otimes$)

($\blacktriangleleft$-nat)

The proof of (34) is not entirely graphical since one has to decompose the tape diagrams by $\otimes$. Hereafter, we show that one can easily avoid this incovenient decomposition by taking the right wiskering, for all monomials U, of each of the axioms in $\mathcal{I}$. In other words, rather than $\mathcal{I}$, we consider the following set of axioms

$$\hat{\mathcal{I}} = \{(R_U(\mathfrak{t}_1), R_U(\mathfrak{t}_2)) \mid (\mathfrak{t}_1, \mathfrak{t}_2) \in \mathcal{I} \text{ and } U \in \mathcal{S}^*\}$$

and we write $\lesssim_{\hat{\mathcal{I}}}$ for the smallest precongruence (w.r.t. $\oplus$ and ;) generated by $\hat{\mathcal{I}}$, i.e., the relation inductively defined as

$$\frac{\mathfrak{t}_1 \hat{\mathcal{I}} \mathfrak{t}_2}{\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2}\ (\hat{\mathcal{I}}) \qquad \frac{-}{\mathfrak{t} \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}}\ (r) \qquad \frac{\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 \quad \mathfrak{t}_2 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_3}{\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_3}\ (t)$$

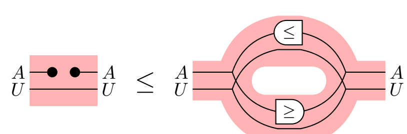
$$\frac{\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{s}_2}{\mathfrak{t}_1 \oplus \mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 \oplus \mathfrak{s}_2}\ (\oplus) \qquad \frac{\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{s}_2}{\mathfrak{t}_1 ; \mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 ; \mathfrak{s}_2}\ (;)$$

Observe that in the above definition, we do not close $\lesssim_{\hat{\mathcal{I}}}$ by $\otimes$ but, as stated by the following theorem, $\lesssim_{\hat{\mathcal{I}}}$ coincides with $\leq_{\mathcal{I}}$.

Theorem 8.2. *For all tapes* $\mathfrak{t}_1, \mathfrak{t}_2$, $\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2$ *if and only if* $\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2$

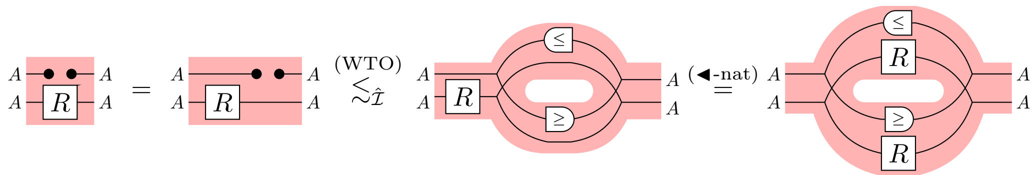
The proof of the above result is rather trivial using the laws of wiskering. The curious reader can check in Appendix F.

Example 8.3. The set $\hat{\mathcal{I}}$ corresponding to $\mathcal{I}$ in Exampe 8.1 consists of the following pair of tapes



(WTO)

for all monomials U. With $\hat{\mathcal{I}}$, one can prove whatever holds in $\leq_{\mathcal{I}}$ but without any $\otimes$-decomposition. For instance, (34), can be proved in a purely graphical way, by exploiting $\lesssim_{\hat{\mathcal{I}}}$.

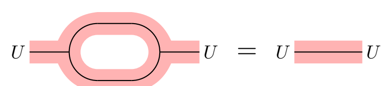


Remark 8.4. The proof of Theorem 8.2 does not rely on any specific properties of biproducts, so like Theorem 4.24, the result holds also when $\oplus$ is just a product or a biproduct. See Remark 4.25.

9 A case study: $\sqcup$-props

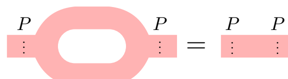
In this section, we illustrate the first application of tape diagrams: $\sqcup$-props. These structures, recently introduced in [3] to model piecewise linear systems, e.g., diodes, rely on a mixture of algebraic and diagrammatic syntax. Instead, as we illustrate in this section, tape diagrams provide a purely graphical calculus for $\sqcup$-props.

We begin by considering the following axiom, which is often referred to as the *special* axiom:



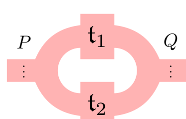
$$(\text{i.e., } \blacktriangleleft_U ; \blacktriangleright_U = id_U) \qquad \text{(Spe)}$$

It is easy to prove, using the inductive definitions in (12), that imposing (Spe) for all monomials U is actually enough to prove that



$$(\blacktriangleleft_P ; \blacktriangleright_P = id_P) \qquad (35)$$

for all polynomials P. Now, recall from Section 6, that $\mathbf{T}_\Sigma$ is **CMon**-enriched, with the addition on homset $\mathbf{T}_\Sigma[P, Q]$, given by

$\mathfrak{t}_1 + \mathfrak{t}_2 \stackrel{\text{def}}{=}$ 

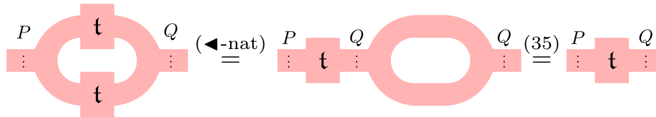
$$(\mathfrak{t}_1 + \mathfrak{t}_2 \stackrel{\text{def}}{=} \blacktriangleleft_P ; (\mathfrak{t}_1 + \mathfrak{t}_2) ; \blacktriangleright_Q\ )$$

and unit element

$\mathfrak{o} \stackrel{\text{def}}{=}$ 

$$(\mathfrak{o} \stackrel{\text{def}}{=} !_P ; ¡_Q)$$

By quotienting $\mathbf{T}_\Sigma$ by (Spe), one obtains a category, hereafter referred to as $\mathbf{T}_\Sigma^S$, enriched over the category of join semilattices with bottom (**JSL**). Indeed, the addition on homset becomes now idempotent:



$$\left(\begin{array}{ll} \mathfrak{t}+\mathfrak{t} & = \blacktriangleleft_P ; (\mathfrak{t}+\mathfrak{t}); \blacktriangleright_Q \\ & = \mathfrak{t}; \blacktriangleleft_Q ; \blacktriangleright_Q \\ & = \mathfrak{t} \end{array}\right)$$

With this observation and Corollary 6.9 it is easy to see that $\mathbf{T}_\Sigma^S$ is isomorphic to the biproduct completion of the **JSL**-category freely generated by $\mathbf{C}_\Sigma$: for an arbitrary category $\mathbf{C}$, the **JSL**-category freely generated by $\mathbf{C}$, hereafter denoted as $\mathbf{C}^{\sqcup}$, is defined as $\mathbf{C}^{+}$ but homsets $[A, B]$ are finite *sets* of arrows in $\mathbf{C}[A, B]$ rather than finite multisets.

Proposition 9.1. $\mathbf{T}_\Sigma^S$ and $\mathbf{Mat}((\mathbf{C}_\Sigma)^{\sqcup})$ are isomorphic as categories with biproducts.

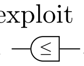
The second step to move toward $\sqcup$-props consists in considering signatures where the set of sorts $\mathcal{S}$ is the singleton set $\{X\}$. In this case, objects of $\mathbf{T}_\Sigma^S$ are polynomials of the form $\bigoplus_i^n \bigotimes_j^{m_i} X$, i.e.,

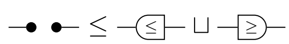
$$\bigoplus_{i=1}^{n} X^{m_i}$$

for some $n, m_1, \dots m_n \in \mathbb{N}$. Such polynomials can be easily seen to be in one-to-one correspondence with words over natural numbers: $\bigoplus_i^n X^{m_i} \mapsto m_1 m_2 \dots m_n$. In particular monomials are natural numbers and $\otimes$ on monomials is simply addition of natural numbers. Thus, by taking the full subcategory of $\mathbf{T}_\Sigma^S$ whose objects are just monomials, that we called in Section 6 **Mnm**, one obtains a *prop*, namely, a symmetric strict monoidal category where the monoid of objects is given by natural numbers with addition.

It is immediate to see that **Mnm** is a $\sqcup$-*prop*, i.e., a prop enriched over **JSL**: every homset carries a join semilattice with bottom, which is preserved by composition and monoidal product (Proposition 6.1).

In particular, **Mnm** is the $\sqcup$-prop freely generated by Σ, as defined in [3]. Indeed every arrow $n \to m$ of **Mnm** corresponds to a 1×1 matrix. The single entry of this matrix is an arrow of $\mathbf{C}_\Sigma^{\sqcup}[X^n, X^m]$, i.e., a finite set of of arrows in $\mathbf{C}_\Sigma[X^n, X^m]$. Composition in **Mnm**, corresponds to multiplication of 1×1 matrices, which is just composition of arrows in $\mathbf{C}_\Sigma^{\sqcup}$: $f; g = \{a; b \mid a \in f,\ b \in g\}$ for all f, g finite sets of arrows of $\mathbf{C}_\Sigma$. The monoidal product is defined as in (33) modulo the fact that we are dealing with sets rather than multisets: $f \otimes g = \{a \otimes b \mid a \in f,\ b \in g\}$.

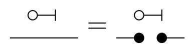
Therefore, whenever one is interested in the $\sqcup$-prop freely generated by some monoidal signature Σ, one can rather embed it into $\mathbf{T}_\Sigma^S$ and exploit the graphical calculus of tapes. For instance, to express the totality of an ordering relation , the authors of [3] exploit diagrams and the join structure $\sqcup$:



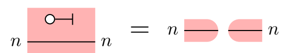
With tape diagrams, such axiom is rendered purely graphically as (TO) in Example 8.1.

Before concluding, it is worth mentioning that the $\sqcup$-props introduced in [3] are enriched over the category of join semilattices, not necessarily with bottom. Therefore the freely generated one has as arrows *non-empty* finite sets. The empty set is present in **Mnm**, since it is the tape

The presence of this structure seems to us a feature rather than an issue. For instance the axiom $\varnothing$ of [3]

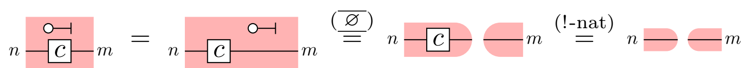


could be rather depicted with tapes as



$(\overline{\varnothing})$

to deduce



for all string diagrams $c\colon n \to m$.

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A Coherence Axioms

$$
\begin{tikzcd}
(X \odot I) \odot Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \odot id_Y"'] & & X \odot (1 \odot Y) \arrow[dl, "id_X \odot \lambda_Y"] \\
& X \odot Y &
\end{tikzcd}
\tag{M1}
$$

$$
\begin{tikzcd}
& (X \odot Y) \odot (Z \odot W) \arrow[dr, "\alpha_{X,Y,Z \odot W}"] & \\
((X \odot Y) \odot Z) \odot W \arrow[ur, "\alpha_{X \odot Y,Z,W}"] \arrow[d, "\alpha_{X,Y,Z} \odot id_W"'] & & X \odot (Y \odot (Z \odot W)) \\
(X \odot (Y \odot Z)) \odot W \arrow[rr, "\alpha_{X,Y \odot Z,W}"'] & & X \odot ((Y \odot Z) \odot W) \arrow[u, "id_X \odot \alpha_{Y,Z,W}"']
\end{tikzcd}
\tag{M2}
$$

Figure 4: Coherence axioms of monoidal categories

$$
\begin{tikzcd}
X \odot I \arrow[r, "\sigma_{X,I}"] \arrow[d, "\rho_X"'] & I \odot X \arrow[d, "\lambda_X"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\tag{S1}
$$

$$
\begin{tikzcd}
& X \odot (Y \odot Z) \arrow[r, "id_X \odot \sigma_{Y,Z}"] & X \odot (Z \odot Y) \arrow[dr, "\alpha^{-}_{X,Z,Y}"] & \\
(X \odot Y) \odot Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "\sigma_{X \odot Y,Z}"'] & & & (X \odot Z) \odot Y \\
& Z \odot (X \odot Y) \arrow[r, "\alpha^{-}_{Z,X,Y}"'] & (Z \odot X) \odot Y \arrow[ur, "\sigma_{Z,X} \odot id_Y"'] &
\end{tikzcd}
\tag{S2}
$$

Figure 5: Coherence axioms of symmetric monoidal categories

$$
\begin{tikzcd}
X \odot Y \arrow[r, "\blacktriangleleft_{X \odot Y}"] \arrow[d, "\blacktriangleleft_X \odot \blacktriangleleft_Y"'] & (X \odot Y) \odot (X \odot Y) \\
(X \odot Y) \odot (Y \odot Y) \arrow[d, "\alpha_{X,X,Y \odot Y}"'] & \\
X \odot (X \odot (Y \odot Y)) \arrow[d, "id_X \odot \alpha^{-}_{X,Y,Y}"'] & X \odot (Y \odot (X \odot Y)) \arrow[uu, "\alpha^{-}_{X,Y,X \odot Y}"'] \\
X \odot ((X \odot Y) \odot Y) \arrow[r, "id_X \odot (\sigma_{X,Y} \odot id_Y)"] & X \odot ((Y \odot X) \odot Y) \arrow[u, "id_X \odot \alpha_{Y,X,Y}"']
\end{tikzcd}
\tag{FP1}
$$

$$
\begin{tikzcd}
X \odot Y \arrow[r, "!_{X \odot Y}"] \arrow[d, "!_X \odot !_Y"'] & I \\
I \odot I \arrow[ur, "\lambda_I"'] &
\end{tikzcd}
\tag{FP2}
$$

$$
\begin{tikzcd}
I \arrow[r, shift left, "\blacktriangleleft_I"] \arrow[r, shift right, "\lambda^{-}_I"'] & I \odot I
\end{tikzcd}
\tag{FP3}
$$

$$
\begin{tikzcd}
I \arrow[r, shift left, "!_I"] \arrow[r, shift right, "id_I"'] & I
\end{tikzcd}
\tag{FP4}
$$

Figure 6: Coherence axioms for fp categories

$$
\begin{tikzcd}
(X \odot Y) \odot (X \odot Y) \arrow[r, "\blacktriangleright_{X \odot Y}"] \arrow[dd, "\alpha_{X,Y,X \odot Y}"'] & X \odot Y \\
& (X \odot X) \odot (Y \odot Y) \arrow[u, "\blacktriangleright_X \odot \blacktriangleright_Y"'] \\
X \odot (Y \odot (X \odot Y)) \arrow[d, "id_X \odot \alpha^{-}_{Y,X,Y}"'] & X \odot (X \odot (Y \odot Y)) \arrow[u, "\alpha^{-}_{X,X,Y \odot Y}"'] \\
X \odot ((Y \odot X) \odot Y) \arrow[r, "id_X \odot (\sigma_{Y,X} \odot id_Y)"] & X \odot ((X \odot Y) \odot Y) \arrow[u, "id_X \odot \alpha_{X,Y,Y}"']
\end{tikzcd}
\tag{FC1}
$$

$$
\begin{tikzcd}
I \arrow[r, "¡_{X \odot Y}"] \arrow[d, "\lambda^{-}_I"'] & X \odot Y \\
I \odot I \arrow[ur, "¡_X \odot ¡_Y"'] &
\end{tikzcd}
\tag{FC2}
$$

$$
\begin{tikzcd}
I \odot I \arrow[r, shift left, "\blacktriangleright_I"] \arrow[r, shift right, "\lambda_I"'] & I
\end{tikzcd}
\tag{FC3}
$$

$$
\begin{tikzcd}
I \arrow[r, shift left, "¡_I"] \arrow[r, shift right, "id_I"'] & I
\end{tikzcd}
\tag{FFC4}
$$

Figure 7: Coherence axioms for fc categories

$$\begin{tikzcd}
(X \oplus Y)Z \arrow[r, "\delta^r_{X,Y,Z}"] \arrow[d, swap, "\sigma^{\otimes}_{X \oplus Y, Z}"] & XZ \oplus YZ \arrow[d, "\sigma^{\otimes}_{X,Z} \oplus \sigma^{\otimes}_{Y,Z}"] \\
Z(X \oplus Y) \arrow[r, swap, "\delta^l_{Z,X,Y}"] & ZX \oplus ZY
\end{tikzcd} \tag{R1}$$

$$\begin{tikzcd}
(X \oplus Y)Z \arrow[r, "\delta^r_{X,Y,Z}"] \arrow[d, swap, "\sigma^{\oplus}_{X,Y} \otimes id_Z"] & XZ \oplus YZ \arrow[d, "\sigma^{\oplus}_{XZ,YZ}"] \\
(Y \oplus X)Z \arrow[r, swap, "\delta^r_{Y,X,Z}"] & YZ \oplus XZ
\end{tikzcd} \tag{R2}$$

$$\begin{tikzcd}
((X \oplus Y) \oplus Z)W \arrow[r, "\delta^r_{X \oplus Y, Z, W}"] \arrow[d, swap, "\alpha^{\oplus}_{X,Y,Z} \otimes id_W"] & (X \oplus Y)W \oplus ZW \arrow[r, "\delta^r_{X,Y,W} \otimes id_{ZW}"] & (XW \oplus YW) \oplus ZW \arrow[d, "\alpha^{\oplus}_{XW,YW,ZW}"] \\
(X \oplus (Y \oplus Z))W \arrow[r, swap, "\delta^r_{X, Y \oplus Z, W}"] & XW \oplus (Y \oplus Z)W \arrow[r, swap, "id_{XW} \otimes \delta^r_{Y,Z,W}"] & XW \oplus (YW \oplus ZW)
\end{tikzcd} \tag{R3}$$

$$\begin{tikzcd}
((X \oplus Y)Z)W \arrow[r, "\delta^r_{X,Y,Z} \otimes id_W"] \arrow[d, swap, "\alpha^{\otimes}_{X \oplus Y, Z, W}"] & (XZ \oplus YZ)W \arrow[r, "\delta^r_{XZ,YZ,W}"] & (XZ)W \oplus (YZ)W \arrow[d, "\alpha^{\otimes}_{X,Z,W} \oplus \alpha^{\otimes}_{Y,Z,W}"] \\
(X \oplus Y)(ZW) \arrow[rr, swap, "\delta^r_{X,Y,ZW}"] & & X(ZW) \oplus Y(ZW)
\end{tikzcd} \tag{R4}$$

$$\begin{tikzcd}
(X \oplus Y)(Z \oplus W) \arrow[r, "\delta^r_{X,Y,Z \oplus W}"] \arrow[d, swap, "\delta^l_{X \oplus Y, Z, W}"] & X(Z \oplus W) \oplus Y(Z \oplus W) \arrow[d, "\delta^l_{X,Z,W} \oplus \delta^l_{Y,Z,W}"] \\
(X \oplus Y)Z \oplus (X \oplus Y)W \arrow[d, swap, "\delta^r_{X,Y,Z} \oplus \delta^r_{X,Y,W}"] & (XZ \oplus XW) \oplus (YZ \oplus YW) \arrow[d, "\alpha^{\oplus}_{XZ,XW,YZ \oplus YW}"] \\
(XZ \oplus YZ) \oplus (XW \oplus YW) \arrow[d, swap, "\alpha^{\oplus}_{XZ,YZ,XW \oplus YW}"] & XZ \oplus (XW \oplus (YZ \oplus YW)) \arrow[d, "id_{XZ} \oplus \alpha^{-\oplus}_{XW,YZ,YW}"] \\
XZ \oplus (YZ \oplus (XW \oplus YW)) \arrow[d, swap, "id_{XZ} \oplus \alpha^{-\oplus}_{YZ,XW,YW}"] & XZ \oplus ((XW \oplus YZ) \oplus YW) \arrow[d, "id_{XZ} \oplus (\sigma^{\oplus}_{XW,YZ} \oplus id_{YW})"] \\
XZ \oplus ((YZ \oplus XW) \oplus YW) \arrow[r, equal] & XZ \oplus ((YZ \oplus XW) \oplus YW)
\end{tikzcd} \tag{R5}$$

$$\begin{tikzcd}
0 \otimes 0 \arrow[r, shift left, "\lambda^{\bullet}_0"] \arrow[r, shift right, swap, "\rho^{\bullet}_0"] & 0
\end{tikzcd} \tag{R6}$$

$$\begin{tikzcd}
(X \oplus Y)0 \arrow[r, "\delta^r_{X,Y,0}"] \arrow[d, swap, "\rho^{\bullet}_{X \oplus Y}"] & X0 \oplus Y0 \arrow[d, "\rho^{\bullet}_X \oplus \rho^{\bullet}_Y"] \\
0 & 0 \oplus 0 \arrow[l, "\lambda^{\oplus}_0"]
\end{tikzcd} \tag{R7}$$

$$\begin{tikzcd}
0 \otimes 1 \arrow[r, shift left, "\lambda^{\bullet}_1"] \arrow[r, shift right, swap, "\rho^{\otimes}_0"] & 0
\end{tikzcd} \tag{R8}$$

$$\begin{tikzcd}
X \otimes 0 \arrow[rr, "\sigma^{\otimes}_{X,0}"] \arrow[dr, swap, "\rho^{\bullet}_X"] & & 0 \otimes X \arrow[dl, "\lambda^{\bullet}_X"] \\
& 0 &
\end{tikzcd} \tag{R9}$$

$$\begin{tikzcd}
(XY)0 \arrow[r, "\alpha^{\otimes}_{X,Y,0}"] \arrow[d, swap, "\rho^{\bullet}_{XY}"] & X(Y0) \arrow[d, "id_X \otimes \rho^{\bullet}_Y"] \\
0 & X0 \arrow[l, "\rho^{\bullet}_X"]
\end{tikzcd} \tag{R10}$$

$$\begin{tikzcd}
(0 \oplus X)Y \arrow[r, "\delta^r_{0,X,Y}"] \arrow[d, swap, "\lambda^{\oplus}_X \otimes id_Y"] & 0Y \oplus XY \arrow[d, "\lambda^{\bullet}_Y \oplus id_{XY}"] \\
XY & 0 \oplus XY \arrow[l, "\lambda^{\oplus}_{XY}"]
\end{tikzcd} \tag{R11}$$

$$\begin{tikzcd}
(X \oplus Y)1 \arrow[rr, "\delta^r_{X,Y,1}"] \arrow[dr, swap, "\rho^{\otimes}_{X \oplus Y}"] & & X1 \oplus Y1 \arrow[dl, "\rho^{\otimes}_X \oplus \rho^{\otimes}_Y"] \\
& X \oplus Y &
\end{tikzcd} \tag{R12}$$

Figure 8: Coherence Axioms of symmetric rig categories

$$\begin{tikzcd}
X(Y \oplus Z) \arrow[r, "\delta^l_{X,Y,Z}"] \arrow[d, "id_X \otimes \sigma^{\oplus}_{Y,Z}"'] & XY \oplus XZ \arrow[d, "\sigma^{\oplus}_{XY,XZ}"] \\
X(Z \oplus Y) \arrow[r, "\delta^l_{X,Z,Y}"'] & XZ \oplus XY
\end{tikzcd} \tag{36}$$

$$\begin{tikzcd}
X[(Y \oplus Z) \oplus W] \arrow[r, "\delta^l_{X,Y \oplus Z,W}"] \arrow[d, "id_X \otimes \alpha^{\oplus}_{Y,Z,W}"'] & X(Y \oplus Z) \oplus XW \arrow[r, "\delta^l_{X,Y,Z} \oplus id_{XW}"] & (XY \oplus XZ) \oplus XW \arrow[d, "\alpha^{\oplus}_{XY,XZ,XW}"] \\
X[Y \oplus (Z \oplus W)] \arrow[r, "\delta^l_{X,Y,Z \oplus W}"] & XY \oplus X(Z \oplus W) \arrow[r, "id_{XY} \oplus \delta^l_{X,Z,W}"] & XY \oplus (XZ \oplus XW)
\end{tikzcd} \tag{37}$$

$$\begin{tikzcd}
(XY)(Z \oplus W) \arrow[rr, "\delta^l_{XY,Z,W}"] \arrow[d, "\alpha^{\otimes}_{X,Y,Z \oplus W}"'] & & (XY)Z \oplus (XY)W \arrow[d, "\alpha^{\otimes}_{X,Y,Z} \oplus \alpha^{\otimes}_{X,Y,W}"] \\
X(Y(Z \oplus W)) \arrow[r, "id_X \otimes \delta^l_{Y,Z,W}"'] & X(YZ \oplus YW) \arrow[r, "\delta^l_{X,YZ,YW}"'] & X(YZ) \oplus X(YW)
\end{tikzcd} \tag{38}$$

$$\begin{tikzcd}
0(X \oplus Y) \arrow[r, "\delta^l_{0,X,Y}"] \arrow[d, "\lambda^{\bullet}_{X \oplus Y}"'] & 0X \oplus 0Y \arrow[d, "\lambda^{\bullet}_X \oplus \lambda^{\bullet}_Y"] \\
0 & 0 \oplus 0 \arrow[l, "\lambda^{\oplus}_0"]
\end{tikzcd} \tag{39}$$

$$\begin{tikzcd}
X(0 \oplus Y) \arrow[r, "\delta^l_{X,0,Y}"] \arrow[d, "id_X \otimes \lambda^{\oplus}_Y"'] & X0 \oplus XY \arrow[d, "\rho^{\bullet}_X \oplus id_{XY}"] \\
XY & 0 \oplus XY \arrow[l, "\lambda^{\oplus}_{XY}"]
\end{tikzcd} \tag{40}$$

$$\begin{tikzcd}
X(Y \oplus 0) \arrow[r, "\delta^l_{X,Y,0}"] \arrow[d, "id_X \otimes \rho^{\oplus}_Y"'] & XY \oplus X0 \arrow[d, "id_{XY} \oplus \rho^{\bullet}_X"] \\
XY & XY \oplus 0 \arrow[l, "\rho^{\oplus}_{XY}"]
\end{tikzcd} \tag{41}$$

$$\begin{tikzcd}
(X \oplus 0)Y \arrow[r, "\delta^r_{X,0,Y}"] \arrow[d, "\rho^{\oplus}_X \otimes id_Y"'] & XY \oplus 0Y \arrow[d, "id_{XY} \oplus \lambda^{\bullet}_Y"] \\
XY & XY \oplus 0 \arrow[l, "\rho^{\oplus}_{XY}"]
\end{tikzcd} \tag{42}$$

$$\begin{tikzcd}
1(X \oplus Y) \arrow[rr, "\delta^l_{1,X,Y}"] \arrow[dr, "\lambda^{\oplus}_{X \oplus Y}"'] & & 1X \oplus 1Y \arrow[dl, "\lambda^{\oplus}_X \oplus \lambda^{\oplus}_Y"] \\
& X \oplus Y &
\end{tikzcd} \tag{43}$$

Figure 9: Derived laws of symmetric rig categories

B Typing rules

$$\frac{s\colon X \to Y \in \Sigma}{s\colon X \to Y} \qquad \frac{f\colon X_1 \to Y_1 \qquad g\colon X_2 \to Y_2}{f \odot g\colon X_1 \odot X_2 \to Y_1 \odot Y_2} \qquad \frac{f\colon X \to Y \qquad g\colon Y \to Z}{f ; g\colon X \to Z}$$

$$id_X\colon X \to X \qquad \sigma^{\odot}_{X,Y}\colon X \odot Y \to Y \odot X$$

$$\alpha^{\odot}_{X,Y,Z}\colon (X \odot Y) \odot Z \to X \odot (Y \odot Z) \qquad \lambda^{\odot}_X\colon I \odot X \to X \qquad \rho^{\odot}_X\colon X \odot I \to X$$

$$\alpha^{-\odot}_{X,Y,Z}\colon X \odot (Y \odot Z) \to (X \odot Y) \odot Z \qquad \lambda^{-\odot}_X\colon X \to I \odot X \qquad \rho^{-\odot}_X\colon X \to X \odot I$$

$$\delta^{l}_{X,Y,Z}\colon X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z)$$

$$\delta^{r}_{X,Y,Z}\colon (X \oplus Y) \otimes Z \to (X \otimes Z) \oplus (Y \otimes Z)$$

$$\lambda^{\bullet}_X\colon X \otimes 0 \to 0 \quad \rho^{\bullet}_X\colon 0 \otimes X \to 0$$

$$\delta^{-l}_{X,Y,Z}\colon (X \otimes Y) \oplus (X \otimes Z) \to X \otimes (Y \oplus Z)$$

$$\delta^{-r}_{X,Y,Z}\colon (X \otimes Z) \oplus (Y \otimes Z) \to (X \oplus Y) \otimes Z$$

$$\lambda^{-\bullet}_X\colon 0 \to X \otimes 0 \quad \rho^{-\bullet}_X\colon 0 \to 0 \otimes X$$

$$!_X\colon X \to I \qquad \blacktriangleleft_X\colon X \to X \odot X \qquad ¡_X\colon I \to X \qquad \blacktriangleright_X\colon X \odot X \to X$$

Table 8: Typing Rules

C Proofs of Section 3

In this appendix, we illustrate the proofs of the results in Section 3 that are missing in the main text, with the exception of Proposition 3.7 and Theorem 3.11. Indeed, these two results are based on some rewriting techniques that we describe in full details in Appendix G. Particularly important is the proof of Theorem 3.12 which relies on Proposition 3.7 and Theorem 3.11. Moreover, the proofs of Lemma 3.9 and of Proposition 3.15 rely on the derived laws in Figure 9 that hold in any rig category [17].

Proof of Lemma 3.9. We want to prove that for all monomials U, V and polynomials Y, Z, the following hold in $\mathbf{ssR}_{\Sigma}$:

1. $\delta^{l}_{U,Y,Z} = id_{UY \oplus UZ}$.
2. If $Y = \bigoplus_i U_i$, then $\sigma^{\otimes}_{Y,V} = \bigoplus_i \sigma^{\otimes}_{U_i,V}$

We prove the first item by induction on U.

Case $U = 1$: $\delta^{l}_{1,Y,Z} \overset{(43)}{=} id_{Y \oplus Z}$.

Case $U = AU'$:

$$\begin{aligned}
\delta^{l}_{AU',Y,Z} &= (id_A \otimes \delta^{l}_{U',Y,Z}); \delta^{l}_{A,U'Y,U'Z} && (38)\\
&= (id_A \otimes id_{U'Y \oplus U'Z}); id_{AU'Y \oplus AU'Z} && \text{(Ind. hp., (SS))}\\
&= id_{AU'Y \oplus AU'Z}
\end{aligned}$$

We prove the second point by induction on Y.
Case $Y = 0$**:** it follows from axiom (R9).
Case $Y = U \oplus Y'$**:** Let $Y' = \bigoplus_{i'} U'_{i'}$

$$\begin{aligned}
\sigma^{\otimes}_{U\oplus Y',V} &= (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{Y',V}); \delta^{-l}_{V,U,Y'} && \text{(R1)}\\
&= \sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{Y',V} && \text{(Lemma 3.9.1)}\\
&= \sigma^{\otimes}_{U,V} \oplus \bigoplus_{i'} \sigma^{\otimes}_{U'_{i'},V} && \text{(Ind. hp.)}
\end{aligned}$$

□

Proof of Theorem 3.12. In this proof and in the next one, for X a term in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$, we will write $[X]_E$ for the equivalence class of X modulo the congruence E generated by the axioms in Table 5 and (17). The objects of $\mathbf{ssR}_\Sigma$ are exactly such equivalence classes.

We want to prove that there is an equivalence of rig categories

$$\begin{tikzcd} \mathbf{R}_\Sigma \arrow[r, bend left, "F"] & \mathbf{ssR}_\Sigma \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $FG = id_{\mathbf{ssR}_\Sigma}$ and $n_X \colon X \to GF(X)$ is a natural isomorphism.

Let $X \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$ and Y a term obtained from X by replacing any subterm Z of X by $Z{\downarrow}$. Then $X{\downarrow} = Y{\downarrow}$, because E is a congruence and the normal form is unique. For example, consider $X = X_1 \oplus X_2$. By Proposition 3.7 we have $[X_1]_E = [X_1{\downarrow}]_E$ and $[X_2]_E = [X_2{\downarrow}]_E$, therefore $[X_1 \oplus X_2]_E = [(X_1{\downarrow} \oplus X_2{\downarrow})]_E$. This means that both $(X_1 \oplus X_2){\downarrow}$ and $(X_1{\downarrow} \oplus X_2{\downarrow}){\downarrow}$ are terms in polynomial form and both are equivalent to $X_1 \oplus X_2$ in E: by Proposition 3.7 they must be equal. In symbols

$$(X_1 \oplus X_2){\downarrow} = (X_1{\downarrow} \oplus X_2{\downarrow}){\downarrow}. \tag{44}$$

We can now start to prove the equivalence.

$F \colon \mathbf{R}_\Sigma \to \mathbf{ssR}_\Sigma$ is the unique rig functor, given by freeness of $\mathbf{R}_\Sigma$, that extends the assignment $F(X) = [X]_E$ and $F(s) = s$ where X is a term in $\mathcal{T}_{\mathtt{R}}(\mathcal{S})$ and $s \in \Sigma$. In particular we have that

$$F(\delta^l_{X,Y,Z}) = \sigma^{\otimes}_{X,Y\oplus Z}; (\sigma^{\otimes}_{Y,X} \oplus \sigma^{\otimes}_{Z,X}) \qquad F(\delta^{-l}_{X,Y,Z}) = (\sigma^{\otimes}_{X,Y} \oplus \sigma^{\otimes}_{X,Z}); \sigma^{\otimes}_{Y\oplus Z,X}$$

and for ι a basic structural isomorphism or an inverse of a basic structural isomorphism, $F(\iota)$ is the identity of the appropriate object, e.g. $F(\lambda^{\bullet}_X) = id_{[0\otimes X]_E} = id_0$.

Next we define the functor $G \colon \mathbf{ssR}_\Sigma \to \mathbf{R}_\Sigma$. On objects $G([X]_E) = X{\downarrow}$. On arrows, it is defined inductively as:

$$G(s) = n^-; s; n \qquad G(id_{[X]_E}) = id_{X\downarrow} \qquad G(f;g) = G(f); G(g)$$

$$G(f \oplus g) = n^-; G(f) \oplus G(g); n \qquad G(\sigma^{\oplus}_{[X]_E,[Y]_E}) = n^-; \sigma^{\oplus}_{X\downarrow,Y\downarrow}; n$$

$$G(f \otimes g) = n^-; G(f) \otimes G(g); n \qquad G(\sigma^{\otimes}_{[X]_E,[Y]_E}) = n^-; \sigma^{\otimes}_{X\downarrow,Y\downarrow}; n$$

Above and in the rest of this proof, we avoid to specify the object in the morphism n. The reader can however easily infer such object from the definition. For instance, if $f\colon [X_1]_E \to [Y_1]_E$ and $g\colon [X_2]_E \to [Y_2]_E$, then $G(f \oplus g)$ is the morphism

$$(X_1{\downarrow} \oplus X_2{\downarrow}){\downarrow} \xrightarrow{n^-_{X_1\downarrow \oplus X_2\downarrow}} X_1{\downarrow} \oplus X_2{\downarrow} \xrightarrow{G(f)\oplus G(g)} Y_1{\downarrow} \oplus Y_2{\downarrow} \xrightarrow{n_{Y_1\downarrow \oplus Y_2\downarrow}} (Y_1{\downarrow} \oplus Y_2{\downarrow}){\downarrow}$$

Please note that this morphism has the appropriate source object: $G([X_1 \oplus X_2]_E) = (X_1 \oplus X_2){\downarrow} = (X_1{\downarrow} \oplus X_2{\downarrow}){\downarrow}$, see (44).

To show that G is well defined, we need to show that the axioms additionally satisfied by $\mathbf{ssR}_\Sigma$ are preserved by G. The most challenging axiom is

$$\sigma^{\otimes}_{[A]_E,[Y\oplus Z]_E} ; (\sigma^{\otimes}_{[Y]_E,[A]_E} \oplus \sigma^{\otimes}_{[Z]_E,[A]_E}) = id_{[A\otimes(Y\oplus Z)]_E} \tag{45}$$

for $A \in \mathcal{S}$. Consider the following diagram in $\mathbf{R}_\Sigma$ for $X, Y, Z \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$: the two triangles on the left part commute by Theorem 3.11. The rightmost square also commutes by Theorem 3.11. As far as the two squares on the middle are concerned, the top one commutes by naturality of symmetries, the bottom one commutes by axiom (R1).

$$\begin{tikzcd}
(X(Y\oplus Z)){\downarrow} \arrow[r, "n^-"] \arrow[dr, "n^-"] & X{\downarrow}(Y\oplus Z){\downarrow} \arrow[r, "\sigma"] \arrow[d, "id\otimes n^-"] & (Y\oplus Z){\downarrow}X{\downarrow} \arrow[r, "n"] \arrow[d, "n^-\otimes id"] & ((Y\oplus Z)X){\downarrow} \arrow[d, equal] \\
(XY\oplus XZ){\downarrow} & X{\downarrow}(Y{\downarrow}\oplus Z{\downarrow}) \arrow[r, "\sigma"] \arrow[d, "\delta^l_{X\downarrow,Y\downarrow,Z\downarrow}"] & (Y{\downarrow}\oplus Z{\downarrow})X{\downarrow} \arrow[d, "\delta^r_{X\downarrow,Y\downarrow,Z\downarrow}"] & ((YX)\oplus(ZX)){\downarrow} \arrow[d, "n^-"] \\
XY{\downarrow}\oplus XZ{\downarrow} \arrow[u, "n"] & (X{\downarrow}Y{\downarrow})\oplus(X{\downarrow}Z{\downarrow}) \arrow[l, "n\oplus n"'] \arrow[ul, "n"'] & (Y{\downarrow}X{\downarrow})\oplus(Z{\downarrow}X{\downarrow}) \arrow[l, "\sigma\oplus\sigma"'] & (YX){\downarrow}\oplus(ZX){\downarrow} \arrow[l, "n^-\oplus n^-"']
\end{tikzcd}$$

Observe that by definition of G, the outer border of the above diagram is exactly

$$G(\sigma^{\otimes}_{[X]_E,[Y\oplus Z]_E} ; (\sigma^{\otimes}_{[Y]_E,[X]_E} \oplus \sigma^{\otimes}_{[Z]_E,[X]_E})).$$

Therefore the commutativity of the diagram informs us that, for any $X, Y, Z \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$,

$$G(\sigma^{\otimes}_{[X]_E,[Y\oplus Z]_E} ; (\sigma^{\otimes}_{[Y]_E,[X]_E} \oplus \sigma^{\otimes}_{[Z]_E,[X]_E})) = n^- ; \delta^l_{X\downarrow,Y\downarrow,Z\downarrow} ; n. \tag{46}$$

Now, fix X to be $A \in \mathcal{S}$. By definition, $\delta^l_{A,Y\downarrow,Z\downarrow}$ is a structural iso and thus $n^- ; \delta^l_{A,Y\downarrow,Z\downarrow} ; n = id_{(A(Y\oplus Z))\Downarrow}$ by Theorem 3.11 and, by definition of G, $id_{(A(Y\oplus Z))\Downarrow} = G(id_{[A(Y\oplus Z)]_E})$. This proves that (45) is preserved by G.

The proofs for the other axioms follow a regular pattern. We illustrate only the proof for $G((f \oplus g) \otimes h) = G((f \otimes h) \oplus (g \otimes h))$ so that the reader can easily check the others. The following diagram commutes: the triangles commute by Theorem 3.11, the square by naturality of

$\delta^r_{X_1\downarrow, X_2\downarrow, X_3\downarrow}$.

$$
\begin{tikzcd}
& (X_1\downarrow \oplus X_2\downarrow)X_3\downarrow \arrow[r, "(Gf\oplus Gg)\otimes G(h)"] \arrow[dd, "\delta^r_{X_1\downarrow, X_2\downarrow, X_3\downarrow}"] & (Y_1\downarrow \oplus Y_2\downarrow)Y_3\downarrow \arrow[dd, "\delta^r_{Y_1\downarrow, Y_2\downarrow, Y_3\downarrow}"] \arrow[dr, "n"] & \\
((X_1\oplus X_2)X_3)\downarrow \arrow[ur, "n^-"] \arrow[dr, "n^-"'] & & & ((Y_1\oplus Y_2)Y_3)\downarrow \\
& (X_1\downarrow X_3\downarrow) \oplus (X_2\downarrow X_3\downarrow) \arrow[r, "(Gf\oplus Gh)\oplus(Gg\otimes Gh)"] & (Y_1\downarrow Y_3\downarrow) \oplus (Y_2\downarrow Y_3\downarrow) \arrow[ur, "n"'] &
\end{tikzcd}
$$

By definition of G and Theorem 3.11, one can readily see that $G((f\oplus g)\otimes h) = n^-; (\,(Gf\oplus Gg)\otimes Gh\,); n$ and $G((f\otimes h)\oplus(g\otimes h)) = n^-; (\,(Gf\otimes Gh)\oplus(Gg\otimes Gh)\,); n$. So the commutativity of the above diagram proves that the axiom is preserved by G.

So far, we proved that G is well defined. By definition, it is immediate to see that G is a functor. To show that it is a strong monoidal $\otimes$-functor, one has to exhibit a morphism $\epsilon\colon 1 \to G([1]_E)$ and a natural isomorphism $\mu\colon G([X]_E)\otimes G([Y]_E) \to G([X\otimes Y]_E)$ making certain diagrams (e.g. (1.2.14) and (1.2.15) in [17]) commute. For the former, it is enough to observe that $G([1]_E) = 1$, so one can take ϵ as id_1. For μ, one takes the isomorphism $n_{X\downarrow\otimes Y\downarrow}\colon X\downarrow \otimes Y\downarrow \to (X\otimes Y)\downarrow$ which is trivially natural by definition of $G(f\otimes g)$. Commutativity of the aforementioned diagrams is given immediately by Theorem 3.11, since all the involved arrows are structural isomorphisms. Proving that G is a strong $\oplus$-functor is completely analogous. To conclude that G is a strong rig functor one has to check that two additional diagrams commute (see e.g. (5.1.2) and (5.1.3) in [17]) but, again, this is immediate by Theorem 3.11.

To prove that F and G provide an equivalence, we first illustrate that $FG = id_{\mathbf{ssR}_\Sigma}$. On objects, this is trivial since $FG([X]_E) = F(X\downarrow) = [X]_E$. For arrows, one proves by induction that $FG(f) = f$ for all arrows f. The proof is an easy consequence of the fact that $F(n_X\colon X \to X\downarrow) = id_{[X]_E}$.

The last step consists in proving that $n_X\colon X \to GFX$ is a natural isomorphism. The fact that it is an iso is ensured by Theorem 3.11. To prove that it is natural one has to check the commutativity of the following square for all arrows $f\colon X\to Y$ of $\mathbf{R}_\Sigma$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "n_X"'] & Y \arrow[d, "n_Y"] \\
X\downarrow \arrow[r, "GF(f)"'] & Y\downarrow
\end{tikzcd}
$$

The proof proceeds by induction on f. The base cases id_X and s are trivial. The cases for basic structural isomorphism or their inverses follow immediately by Theorem 3.11. The cases for the symmetries $\sigma^{\oplus}_{X,Y}$ and $\sigma^{\otimes}_{X,Y}$ follow easily by naturality and Theorem 3.11. The only challenging base case is $\delta^l_{X,Y,Z}$: by definition $GF(\delta^l_{X,Y,Z}) = G(\sigma^{\otimes}_{[X]_E,[Y\oplus Z]_E}; (\sigma^{\otimes}_{[Y]_E,[X]_E} \oplus \sigma^{\otimes}_{[Z]_E,[X]_E}))$. However, by (46), the latter is $n^-; \delta^l_{X\downarrow,Y\downarrow,Z\downarrow}; n$. Thus, for $f = \delta^l_{X,Y,Z}$, the diagram above becomes the outer part of the following one:

$$\begin{tikzcd}
& X(Y \oplus Z) \arrow[r, "\delta^l_{X,Y,Z}"] \arrow[dl, "n"'] \arrow[dd, "n\otimes(n\oplus n)"] & (XY) \oplus (XZ) \arrow[dr, "n"] \arrow[dd, "(n\otimes n)\oplus(n\otimes n)"'] & \\
(X(Y \oplus Z)){\downarrow} \arrow[dr, "n^-"'] & & & ((XY) \oplus (XZ)){\downarrow} \\
& X{\downarrow}(Y{\downarrow} \oplus Z{\downarrow}) \arrow[r, "\delta^l_{X{\downarrow},Y{\downarrow},Z{\downarrow}}"] & (X{\downarrow}Y{\downarrow}) \oplus (X{\downarrow}Z{\downarrow}) \arrow[ur, "n"'] &
\end{tikzcd}$$

which commutes by Theorem 3.11 and naturality of $\delta^l_{X,Y,Z}$.

We are now left to prove the inductive cases. For $f;g$ consider the following derivation.

$$\begin{aligned}
GF(f;g) &= GF(f);GF(g) && \text{(Functoriality of } GF\text{)} \\
&= n^-;f;n;n^-;g;n && \text{(Induction Hypothesis)} \\
&= n^-;(f;g);n && \text{(Inverses)}
\end{aligned}$$

For $f \oplus g$ consider the following derivation.

$$\begin{aligned}
GF(f \oplus g) &= G(F(f) \oplus F(g)) && \text{(Definition of } F\text{)} \\
&= n^-;(\,GF(f) \oplus GF(g)\,);n && \text{(Definition of } G\text{)} \\
&= n^-;(\,(n^-;f;n) \oplus (n^-;g;n)\,);n && \text{(Induction Hypothesis)} \\
&= n^-;(n^- \oplus n^-);(f \oplus g);(n \oplus n);n && \text{(Functoriality of } \oplus\text{)} \\
&= n^-;(f \oplus g);n && \text{(Theorem 3.11)}
\end{aligned}$$

The case for $f \otimes g$ is completely analogous. □

Proof of Theorem 3.18. We define F and G by simply extending their respective counterparts in Theorem 3.12 on the new generators:

$$\begin{array}{llll}
F(\blacktriangleleft_X) = \blacktriangleleft_{[X]_E} & F(!_X) = !_{[X]_E} & G(\blacktriangleleft_{[X]_E}) = n^-_X;\blacktriangleleft_X;n_{X\oplus X} & G(!_{[X]_E}) = !_{X\downarrow} \\
F(\blacktriangleright_X) = \blacktriangleright_{[X]_E} & F(¡_X) = ¡_{[X]_E} & G(\blacktriangleright_{[X]_E}) = n^-_{X\oplus X};\blacktriangleright_X;n_X & G(¡_{[X]_E}) = ¡_{X\downarrow}
\end{array}$$

It is immediate to check that it still holds that $FG = id_{\mathbf{ssfbR_\Sigma}}$ and that $n_X \colon X \to GF$ is natural on $\blacktriangleleft_X$, $\blacktriangleright_X$, $!_X$ and $¡_X$. □

Proof of Proposition 3.15. The commutativity of the diagrams for $!_{X\otimes Y}$ and $¡_{X\otimes Y}$ is immediate from the fact that 0 is a terminal or initial object respectively.

We show that $\blacktriangleleft_{X\otimes Y} = (\blacktriangleleft_X \otimes id_Y);\delta^r_{X,X,Y}$. To do that, we prove that the morphism $(\blacktriangleleft_X \otimes id_Y);\delta^r_{X,X,Y}$ satisfies the same universal property of $\blacktriangleleft_{X\otimes Y} = \langle id_{X\otimes Y}, id_{X\otimes Y}\rangle$; in other words we show that

1. $(\blacktriangleleft_X \otimes id_Y);\delta^r_{X,X,Y};\pi_1 = id_{X\otimes Y}$,

2. $(\blacktriangleleft_X \otimes id_Y); \delta^r_{X,X,Y}; \pi_2 = id_{X \otimes Y}$

where $\pi_1, \pi_2 \colon (X \otimes Y) \oplus (X \otimes Y) \to X \otimes Y$. Recall that $\pi_1 = (id_{X \otimes Y} \oplus !_{X \otimes Y}); \rho^{\oplus}_{X \otimes Y}$ and that $\pi_2 = (!_{X \otimes Y} \oplus id_{X \otimes Y}); \lambda^{\oplus}_{X \otimes Y}$. We have that the following diagram commutes:

$$\begin{tikzcd}
& (X \oplus X) \otimes Y \arrow[r, "\delta^r_{X,X,Y}"] \arrow[d, "(id_X \oplus !_X) \otimes id_Y"'] \arrow[dr, phantom, "*_1"] & (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "id_{X \otimes Y} \oplus !_{X \otimes Y}"] \arrow[d, "(id_X \otimes id_Y) \oplus (!_X \otimes id_Y)"'] \arrow[dr, phantom, "*_2"] & (X \otimes Y) \oplus 0 \arrow[dr, "\rho^{\oplus}_{X \otimes Y}"] & \\
X \otimes Y \arrow[ur, "\blacktriangleleft_X \otimes id_Y"] \arrow[rrrr, bend right=40, "id_{X \otimes Y}"'] \arrow[rrrr, phantom, bend right=20, "*_4"] & (X \oplus 0) \otimes Y \arrow[r, "\delta^r_{X,0,Y}"] \arrow[rrr, bend right, "\rho^{\oplus}_X \otimes id_Y"'] & (X \otimes Y) \oplus (0 \otimes X) \arrow[ur, "id_{X \otimes Y} \oplus \lambda^{\bullet}_X"'] \arrow[rr, phantom, "*_3"] & & X \otimes Y
\end{tikzcd}$$

because of the naturality of δ^r $(*_1)$, terminality of 0 $(*_2)$, the derived law (42) $(*_3)$ and the fact that $\blacktriangleleft_X; \pi_1 = id_X$ $(*_4)$. This proves the first equality. As per the second, for similar reasons the following diagram commutes:

$$\begin{tikzcd}
& (X \oplus X) \otimes Y \arrow[r, "\delta^r_{X,X,Y}"] \arrow[d, "(!_X \oplus id_X) \otimes id_Y"'] & (X \otimes Y) \oplus (X \otimes Y) \arrow[r, "!_{X \otimes Y} \oplus id_{X \otimes Y}"] \arrow[d, "(!_X \otimes id_Y) \oplus (id_X \otimes id_Y)"'] & 0 \oplus (X \otimes Y) \arrow[dr, "\lambda^{\oplus}_{X \otimes Y}"] & \\
X \otimes Y \arrow[ur, "\blacktriangleleft_X \otimes id_Y"] \arrow[rrrr, bend right=40, "id_{X \otimes Y}"'] & (0 \oplus X) \otimes Y \arrow[r, "\delta^r_{0,X,Y}"] \arrow[rrr, bend right, "\lambda^{\oplus}_X \otimes id_Y"'] & (0 \otimes Y) \oplus (X \otimes Y) \arrow[ur, "\lambda^{\bullet}_Y \oplus id_{X \otimes Y}"'] \arrow[rr, phantom, "*_5"] & & X \otimes Y
\end{tikzcd}$$

where now $*_5$ commutes because of Axiom (R11).

The fact that $\blacktriangleright_{X \otimes Y} = \delta^{-r}_{X,X,Y}; (\blacktriangleright_X \otimes id_Y)$ can be proved in a similar way using the derived laws (40) and (41). $\square$

D Proofs of Section 4

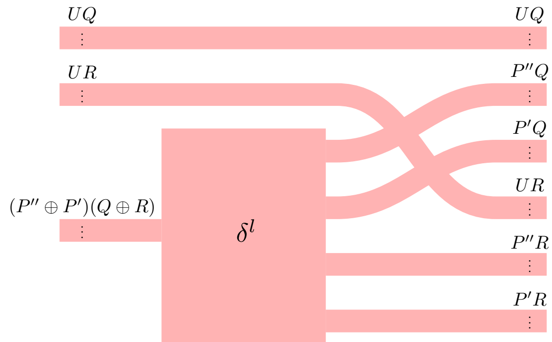
D.1 Distributors and symmetries

Proof of Lemma 4.8.2. We prove that $\delta^l_{P \oplus P', Q, R} = (\delta^l_{P,Q,R} \oplus \delta^l_{P',Q,R}); (id_{PQ} \oplus \sigma^{\oplus}_{PR, P'Q} \oplus id_{P'R})$ by induction on P.

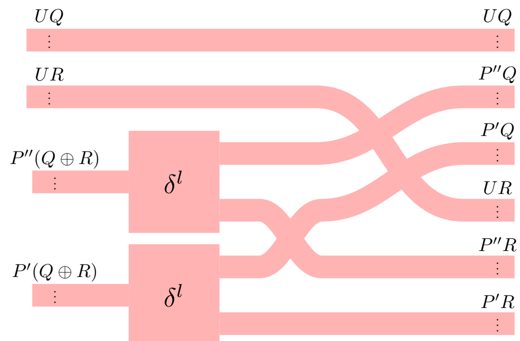
Case $P = 0$**:** it holds by Definition 4.5.

Case $P = U \oplus P''$**:**

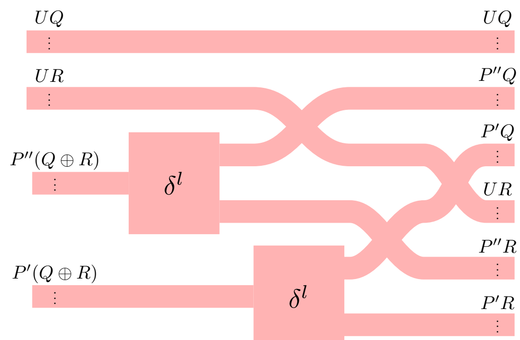
$$\begin{aligned}
\delta^l_{U \oplus P'' \oplus P', Q, R} &= \delta^l_{U \oplus (P'' \oplus P'), Q, R} \\
&= (id_{U(Q \oplus R)} \oplus \delta^l_{P'' \oplus P', Q, R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR, P''Q \oplus P'Q} \oplus id_{P''R \oplus P'R}) \qquad \text{(Def. } \delta^l)
\end{aligned}$$

$=$ 

(Rem. 4.4)

$=$ 

(Ind. hp.)

$=$ 

(Funct. $\oplus$)

$$= (((id_{U(Q\oplus R)} \oplus \delta^l_{P'',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P''Q} \oplus id_{P''R})) \oplus \delta^l_{P',Q,R});$$
$$(id_{UQ\oplus P''Q} \oplus \sigma^{\oplus}_{UR\oplus P''R,P'Q} \oplus id_{P'R})$$
$$= (\delta^l_{U\oplus P'',Q,R} \oplus \delta^l_{P',Q,R}); (id_{UQ\oplus P''Q} \oplus \sigma^{\oplus}_{UR\oplus P''R,P'Q} \oplus id_{P'R}) \qquad \text{(Def. } \delta^l\text{)}$$

□

Proof of Lemma 4.8.3. We prove that $\delta^l_{P,0,R} = id_{PR}$ by induction on P.

Case $P = 0$**:** it holds by Definition 4.5.

Case $P = U \oplus P'$**:**

$$
\begin{aligned}
&\delta^l_{U\oplus P',0,R} \\
&= (id_{U(0\oplus R)} \oplus \delta^l_{P',0,R}); (id_{U0} \oplus \sigma^{\oplus}_{UR,P'0} \oplus id_{P'R}) && \text{(Def. } \delta^l\text{)} \\
&= (id_{UR} \oplus id_{P'R}); (id_0 \oplus \sigma^{\oplus}_{UR,0} \oplus id_{P'R}) && \text{(Ind. hp.)} \\
&= (id_{UR} \oplus id_{P'R}); (id_{UR} \oplus id_{P'R}) && \text{(S1)} \\
&= id_{UR\oplus P'R} && \text{(Funct. } \oplus\text{)} \\
&= id_{(U\oplus P')R} && \text{(Rem. 4.4)}
\end{aligned}
$$

□

Proof of Lemma 4.8.5. We want to prove that $\sigma^{\otimes}_{P,Q}; \sigma^{\otimes}_{Q,P} = id_{PQ}$. First we prove the following three equations:

$$
\begin{aligned}
\sigma^{\otimes}_{0,Q} &= id_0 && (*_1) \\
\sigma^{\otimes}_{U\oplus P',V} &= (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}); \delta^{-l}_{V,U,P'} && (*_{2.1}) \\
\sigma^{\otimes}_{U\oplus P',Q} &= (\bigoplus_j \underline{\overline{\sigma_{U,V_j}}} \oplus \sigma^{\otimes}_{P',Q}); \delta^{-l}_{Q,U,P'} \text{ with } Q = \bigoplus_j V_j && (*_2)
\end{aligned}
$$

For $(*_1)$ we proceed by induction on Q:
Case $Q = 0$**:** it holds by Definition 4.6.
Case $Q = V \oplus Q'$**:**

$$
\begin{aligned}
\sigma^{\otimes}_{0,V\oplus Q'} &= \delta^l_{0,V,Q'}; (\bigoplus_{i=1}^{0} \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{0,Q'}) && \text{(Def. } \sigma^{\otimes}\text{)} \\
&= id_0 && \text{(Def. } \delta^l\text{, ind. hp.)}
\end{aligned}
$$

For $(*_{2.1})$, suppose that $P' = \bigoplus_{i'} U'_{i'}$ and observe that the following holds:

$$
\begin{aligned}
\sigma^{\otimes}_{U\oplus P',V} &= \delta^l_{U\oplus P',V,0}; (\underline{\overline{\sigma_{U,V}}} \oplus \bigoplus_{i'} \underline{\overline{\sigma_{U'_{i'},V}}} \oplus \sigma^{\otimes}_{U,0}) && \text{(Def. } \sigma^{\otimes}\text{)} \\
&= \delta^l_{U\oplus P',V,0}; (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}) && \text{(Def. } \sigma^{\otimes}\text{)} \\
&= (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}) && \text{(Lemma 4.8.3)} \\
&= (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}); \delta^{-l}_{V,U,P'} && \text{(Lemma 4.8.1)}
\end{aligned}
$$

For $(*_2)$ we proceed by induction on Q :
Case $Q = 0$**:**

$$
\begin{aligned}
\sigma^{\otimes}_{U\oplus P',0} &= id_0 && \text{(Def. } \sigma^{\otimes}\text{)} \\
&= (\bigoplus_{i=1}^{0} \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{0,Q'}); \delta^l_{0,V,Q'} && \text{(Def. } \delta^l\text{)}
\end{aligned}
$$

Case $Q = V \oplus Q'$**:**

$$\begin{aligned}
\sigma^{\otimes}_{U\oplus P',V\oplus Q'} &= \delta^{l}_{U\oplus P',V,Q'}; (\overline{\underline{\sigma_{U,V}}} \oplus \bigoplus_{i'} \overline{\underline{\sigma_{U'_{i'},V}}} \oplus \sigma^{\otimes}_{U\oplus P',Q'}) && (\text{Def. } \sigma^{\otimes})\\
&= \delta^{l}_{U\oplus P',V,Q'}; (\sigma^{\otimes}_{U\oplus P',V} \oplus \sigma^{\otimes}_{U\oplus P',Q'}) && (\text{Lemma 4.8.8})\\
&= \delta^{l}_{U\oplus P',V,Q'}; (((\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}); \delta^{-l}_{V,U,P'}) \oplus ((\sigma^{\otimes}_{U,Q'} \oplus \sigma^{\otimes}_{P',Q'}); \delta^{-l}_{Q',U,P'}))\\
& && ((*_{2.1}), \text{ ind. hp.})\\
&= (\delta^{l}_{U,V,Q'} \oplus \delta^{l}_{P',V,Q'}); (id_{UV} \oplus \sigma^{\oplus}_{UQ',P'V} \oplus id_{P'Q'}); && (\text{Def. } \delta^{l})\\
&\qquad (((\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V}); \delta^{-l}_{V,U,P'}) \oplus ((\sigma^{\otimes}_{U,Q'} \oplus \sigma^{\otimes}_{P',Q'}); \delta^{-l}_{Q,U,P'}))\\
&= (\delta^{l}_{U,V,Q'} \oplus \delta^{l}_{P',V,Q'}); (id_{UV} \oplus \sigma^{\oplus}_{UQ',P'V} \oplus id_{P'Q'}); && (\text{Funct. } \oplus)\\
&\qquad (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{P',V} \oplus \sigma^{\otimes}_{U,Q'} \oplus \sigma^{\otimes}_{P',Q'}); (\delta^{-l}_{V,U,P'} \oplus \delta^{-l}_{Q',U,P'})\\
&= (\delta^{l}_{U,V,Q'} \oplus \delta^{l}_{P',V,Q'}); (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{U,Q'} \oplus \sigma^{\otimes}_{P',V} \oplus \sigma^{\otimes}_{P',Q'}); && (\text{Nat. } \sigma^{\oplus})\\
&\qquad\quad (id_{VU} \oplus \sigma^{\oplus}_{Q'U,VP'} \oplus id_{Q'P}); (\delta^{-l}_{V,U,P'} \oplus \delta^{-l}_{Q',U,P'})\\
&= (\delta^{l}_{U,V,Q'} \oplus \delta^{l}_{P',V,Q'}); (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{U,Q'} \oplus \sigma^{\otimes}_{P',V} \oplus \sigma^{\otimes}_{P',Q'}); \delta^{-l}_{V\oplus Q',U,P'} && (\text{Def. } \delta^{l})\\
&= (\delta^{l}_{U,V,Q'}; (\sigma^{\otimes}_{U,V} \oplus \sigma^{\otimes}_{U,Q'})) \oplus (\delta^{l}_{P',V,Q'}; (\sigma^{\otimes}_{P',V} \oplus \sigma^{\otimes}_{P',Q'})); \delta^{-l}_{V\oplus Q',U,P'} && (\text{Funct. } \oplus)\\
&= (\sigma^{\otimes}_{U,V\oplus Q'} \oplus \sigma^{\otimes}_{P',V\oplus Q'}); \delta^{-l}_{V\oplus Q',U,P'} && (\text{Def. } \sigma^{\otimes}, \text{ Lemma 4.8.8})
\end{aligned}$$

The rest of the proof is by induction on Q:
Case $Q = 0$

$$\begin{aligned}
\sigma^{\otimes}_{P,0}; \sigma^{\otimes}_{0,P} &= id_0; \sigma^{\otimes}_{0,P} && (\text{Def. } \sigma^{\otimes})\\
&= id_0; id_0 && (*_1)\\
&= id_0
\end{aligned}$$

Case $Q = V \oplus Q'$

$$\begin{aligned}
\sigma^{\otimes}_{P,V\oplus Q'}; \sigma^{\otimes}_{V\oplus Q',P} &= \delta^{l}_{P,V,Q'}; (\bigoplus_{i} \overline{\underline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'}); (\bigoplus_{i} \overline{\underline{\sigma_{V,U_i}}} \oplus \sigma^{\otimes}_{Q',P}); \delta^{-l}_{P,V,Q'} && (\text{Def. } \sigma^{\otimes}, (*_2))\\
&= \delta^{l}_{P,V,Q'}; ((\bigoplus_{i} \overline{\underline{\sigma_{U_i,V}}}; \bigoplus_{i} \overline{\underline{\sigma_{V,U_i}}}) \oplus (\sigma^{\otimes}_{P,Q'}; \sigma^{\otimes}_{Q',P})); \delta^{-l}_{P,V,Q'} && (\text{Funct. } \oplus)\\
&= \delta^{l}_{P,V,Q'}; (id_{PV} \oplus id_{PQ'}); \delta^{-l}_{P,V,Q'} && (\text{SMC, ind. hp.})\\
&= id_{P(V\oplus Q')} && (\text{Iso})
\end{aligned}$$

$\square$

Lemma D.1. *The following holds for all monomials V and polynomials P, Q, R:*

$$\delta^{l}_{P,V\oplus Q,R}; (\delta^{l}_{P,V,Q} \oplus id_{PR}) = \delta^{l}_{P,V,Q\oplus R}; (id_{PV} \oplus \delta^{l}_{P,Q,R})$$

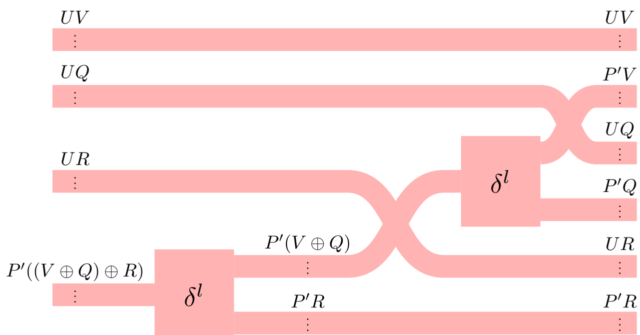
Proof. By induction on P.

Case $P = 0$: it holds since both sides of the equation are id_0 by definition of δ^l.

Case $P = U \oplus P'$: For the inductive case we will mix the classical syntax with tape diagrams to ease the reading.

$$
\begin{aligned}
&\delta^l_{U\oplus P',V\oplus Q,R};(\delta^l_{U\oplus P',V,Q} \oplus id_{UR\oplus P'R}) \\
&= (id_{U((V\oplus Q)\oplus R)} \oplus \delta^l_{P',V\oplus Q,R});(id_{UV\oplus UQ} \oplus \sigma^{\oplus}_{UR,P'(V\oplus Q)} \oplus id_{P'R}); && (\text{Def. } \delta^l) \\
&\qquad\qquad (\delta^l_{U\oplus P',V,Q} \oplus id_{UR\oplus P'R}) \\
&= (id_{U((V\oplus Q)\oplus R)} \oplus \delta^l_{P',V\oplus Q,R});(id_{UV\oplus UQ} \oplus \sigma^{\oplus}_{UR,P'(V\oplus Q)} \oplus id_{P'R}); && (\text{Def. } \delta^l) \\
&\qquad (((id_{U(V\oplus Q)} \oplus \delta^l_{P',V,Q});(id_{UV} \oplus \sigma^{\oplus}_{UQ,P'V} \oplus id_{P'Q})) \oplus id_{UR\oplus P'R})
\end{aligned}
$$

$=$



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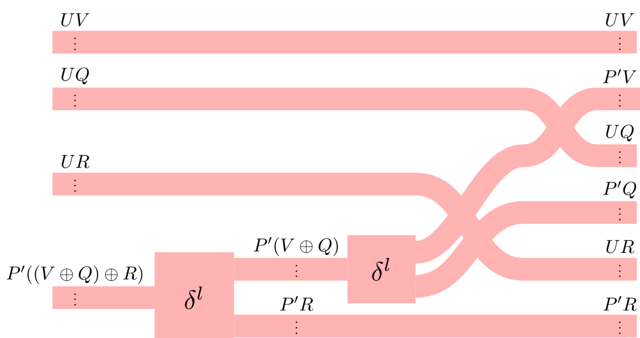
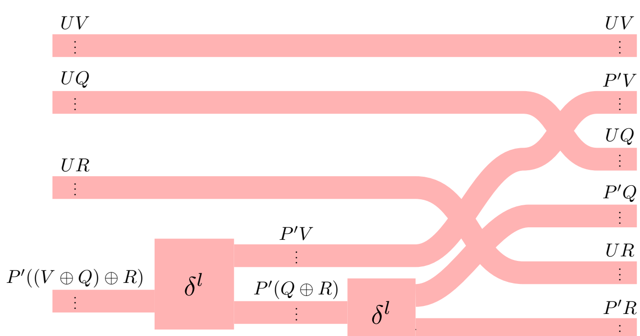
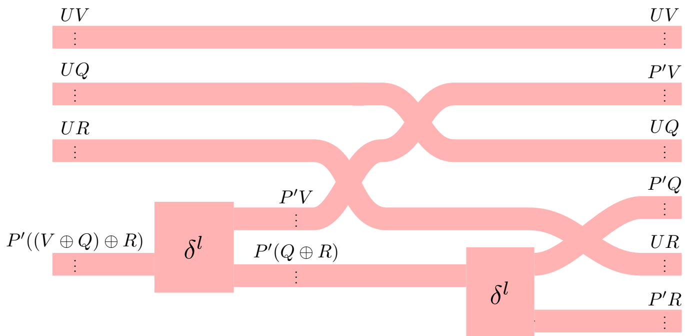


($\sigma^{\oplus}$-nat)

$=$



(Ind. hp.)

$=$ 

(Funct. $\oplus$)

$$= (id_{U(V\oplus(Q\oplus R))} \oplus \delta^l_{P',V,Q\oplus R}); (id_{UV} \oplus \sigma^{\oplus}_{U(Q\oplus R),P'V} \oplus id_{P'(Q\oplus R)});$$
$$(id_{UV\oplus P'V} \oplus ((id_{U(Q\oplus R)} \oplus \delta^l_{P',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P'Q} \oplus id_{P'R})))$$
$$= \delta^l_{U\oplus P',V,Q\oplus R}; (id_{UV\oplus P'V} \oplus \delta^l_{U\oplus P',Q,R}) \qquad \text{(Def. } \delta^l\text{)}$$

□

Proof of Lemma 4.8.6. We want to prove that $\sigma^{\otimes}_{P,Q\oplus R} = \delta^l_{P,Q,R}; (\sigma^{\otimes}_{P,Q} \oplus \sigma^{\otimes}_{P,R})$ by induction on Q.

Case $Q = 0$**:** By Lemma 4.8.3, $\delta^l_{P,0,R} = id_R$ and by Definition 4.6, $\sigma^{\otimes}_{P,0} = id_0$. The right-hand side reduces therefore to $\sigma^{\oplus}_{P,R}$, as desired.

Case $Q = V \oplus Q'$**:**

$$\begin{aligned}
&\sigma^{\otimes}_{P,V\oplus Q'\oplus R} \\
&= \delta^l_{P,V,Q'\oplus R}; (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'\oplus R}) && \text{(Def. } \sigma^{\otimes}\text{)} \\
&= \delta^l_{P,V,Q'\oplus R}; (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus (\delta^l_{P,Q',R}; (\sigma^{\otimes}_{P,Q'} \oplus \sigma^{\otimes}_{P,R}))) && \text{(Ind. hp.)} \\
&= \delta^l_{P,V,Q'\oplus R}; (id_{PV} \oplus \delta^l_{P,Q',R}); (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'} \oplus \sigma^{\otimes}_{P,R}) && \text{(Funct. } \oplus\text{)} \\
&= \delta^l_{P,V\oplus Q',R}; (\delta^l_{P,V,Q'} \oplus id_{PR}); (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'} \oplus \sigma^{\otimes}_{P,R}) && \text{(Lemma D.1)} \\
&= \delta^l_{P,V\oplus Q',R}; ((\delta^l_{P,V,Q'}; (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'})) \oplus \sigma^{\otimes}_{P,R}) && \text{(Funct. } \oplus\text{)} \\
&= \delta^l_{P,V\oplus Q',R}; (\sigma^{\otimes}_{P,V\oplus Q'} \oplus \sigma^{\otimes}_{P,R}) && \text{(Def. } \sigma^{\otimes}\text{)}
\end{aligned}$$

□

Proof of Lemma 4.8.7. We want to prove that $\sigma^{\otimes}_{P,1} = id_P$ by induction on P.

Case $P = 0$**:** it holds by Definition 4.6.

Case $P = U \oplus P'$**:**

$$\begin{aligned}
\sigma^{\otimes}_{U\oplus P',1} &= \sigma^{-\otimes}_{1,U\oplus P'} && \text{(Lemma 4.8.5)}\\
&= (\overline{\underline{\sigma_{U,1}}} \oplus \sigma^{\otimes}_{P',1}); \delta^{-l}_{1,U,P'} && (\text{Def. } \sigma^{\otimes})\\
&= (\overline{\underline{\sigma_{U,1}}} \oplus \sigma^{\otimes}_{P',1}); id_{UP'} && \text{(Lemma 4.8.1)}\\
&= id_U \oplus id_{P'} && \text{((S1), ind. hp.)}\\
&= id_{U\oplus P'}
\end{aligned}$$

□

D.2 Additional lemmas for monomial whiskerings

Lemma D.2. *Let* $\mathfrak{t}\colon \bigoplus_i U_i \to \bigoplus_j V_j$ *be a tape diagram, then the following holds*

$$R_W(\mathfrak{t}) = \bigoplus_i \overline{\underline{\sigma_{U_i,W}}}; L_W(\mathfrak{t}); \bigoplus_j \overline{\underline{\sigma_{W,V_j}}}$$

Proof. By induction on $\mathfrak{t}$.

Case $\mathfrak{t} = \boxed{}$**:** Notice that $\bigoplus_{i=1}^{0} \overline{\underline{\sigma_{U_i,W}}} = \bigoplus_{j=1}^{0} \overline{\underline{\sigma_{W,V_j}}} = \boxed{}$ and thus $R_W(\boxed{}) = \boxed{} = L_W(\boxed{})$.

Case $\mathfrak{t} = \overline{\underline{c}}$**:** Suppose $c\colon U \to V$, then

$$\begin{aligned}
R_W(\overline{\underline{c}}) &= \overline{\underline{c \otimes id_W}} && \text{(Def. } R)\\
&= \overline{\underline{c \otimes id_W}}; \overline{\underline{\sigma_{V,W}}}; \overline{\underline{\sigma_{W,V}}} && \text{(SMC)}\\
&= \overline{\underline{\sigma_{U,W}}}; \overline{\underline{id_W \otimes c}}; \overline{\underline{\sigma_{W,V}}} && (\text{Nat. } \overline{\underline{\sigma}})\\
&= \overline{\underline{\sigma_{U,W}}}; L_W(\overline{\underline{c}}); \overline{\underline{\sigma_{W,V}}} && \text{(Def. } L)
\end{aligned}$$

Case $\mathfrak{t} = \sigma^{\oplus}_{U,V}$**:**

$$\begin{aligned}
R_W(\sigma^{\oplus}_{U,V}) &= \sigma^{\oplus}_{UW,VW} && \text{(Def. } R)\\
&= \sigma^{\oplus}_{UW,VW}; ((\overline{\underline{\sigma_{V,W}}}; \overline{\underline{\sigma_{W,V}}}) \oplus (\overline{\underline{\sigma_{U,W}}}; \overline{\underline{\sigma_{W,U}}})) && \text{(SMC)}\\
&= (\overline{\underline{\sigma_{U,W}}} \oplus \overline{\underline{\sigma_{V,W}}}); \sigma^{\oplus}_{WU,WV}; (\overline{\underline{\sigma_{W,V}}} \oplus \overline{\underline{\sigma_{W,U}}}) && (\text{Nat. } \sigma^{\oplus})\\
&= (\overline{\underline{\sigma_{U,W}}} \oplus \overline{\underline{\sigma_{V,W}}}); L_W(\sigma^{\oplus}_{U,V}); (\overline{\underline{\sigma_{W,V}}} \oplus \overline{\underline{\sigma_{W,U}}}) && \text{(Def. } L)
\end{aligned}$$

Case $\mathfrak{t} = \blacktriangleleft_U$**:** (similarly for $\blacktriangleright_U$)

$$\begin{aligned}
R_W(\blacktriangleleft_U) &= \blacktriangleleft_{UW} && \text{(Def. } R)\\
&= \blacktriangleleft_{UW}; ((\overline{\underline{\sigma_{U,W}}}; \overline{\underline{\sigma_{W,U}}}) \oplus (\overline{\underline{\sigma_{U,W}}}; \overline{\underline{\sigma_{W,U}}})) && \text{(SMC)}\\
&= \overline{\underline{\sigma_{U,W}}}; \blacktriangleleft_{WU}; (\overline{\underline{\sigma_{W,U}}} \oplus \overline{\underline{\sigma_{W,U}}}) && (\text{Nat. } \blacktriangleleft)\\
&= \overline{\underline{\sigma_{U,W}}}; L_W(\blacktriangleleft_U); (\overline{\underline{\sigma_{W,U}}} \oplus \overline{\underline{\sigma_{W,U}}}) && \text{(Def. } L)
\end{aligned}$$

Case $\mathsf{t} = !_U$**:** (similarly for $¡_U$)

$$
\begin{aligned}
R_W(!_U) &= !_{UW} && \text{(Def. } R\text{)}\\
&= \overline{\underline{\sigma_{U,W}}}; \overline{\underline{\sigma_{W,U}}}; !_{UW} && \text{(SMC)}\\
&= \overline{\underline{\sigma_{U,W}}}; !_{WU} && \text{(Nat. !)}\\
&= \overline{\underline{\sigma_{U,W}}}; L_W(!_U) && \text{(Def. } L\text{)}
\end{aligned}
$$

Case $\mathsf{t} = \mathsf{t}_1; \mathsf{t}_2$**:** Suppose $\mathsf{t}_1 \colon \bigoplus_i U_i \to \bigoplus_l Z_l, \mathsf{t}_2 \colon \bigoplus_l Z_l \to \bigoplus_j V_j$, then

$$
\begin{aligned}
R_W(\mathsf{t}_1;\mathsf{t}_2) &= R_W(\mathsf{t}_1); R_W(\mathsf{t}_2) && \text{(Funct. } R\text{)}\\
&= \bigoplus_i \overline{\underline{\sigma_{U_i,W}}}; L_W(\mathsf{t}_1); \bigoplus_l \overline{\underline{\sigma_{W,Z_l}}}; \bigoplus_l \overline{\underline{\sigma_{Z_l,W}}}; L_W(\mathsf{t}_2); \bigoplus_j \overline{\underline{\sigma_{W,V_j}}} && \text{(Ind. hp.)}\\
&= \bigoplus_i \overline{\underline{\sigma_{U_i,W}}}; L_W(\mathsf{t}_1); L_W(\mathsf{t}_2); \bigoplus_j \overline{\underline{\sigma_{W,V_j}}} && \text{(SMC)}\\
&= \bigoplus_i \overline{\underline{\sigma_{U_i,W}}}; L_W(\mathsf{t}_1;\mathsf{t}_2); \bigoplus_j \overline{\underline{\sigma_{W,V_j}}} && \text{(Funct. } L\text{)}
\end{aligned}
$$

Case $\mathsf{t} = \mathsf{t}_1 \oplus \mathsf{t}_2$**:** Suppose $\mathsf{t}_1 \colon \bigoplus_i U_i \to \bigoplus_j V_j, \mathsf{t}_2 \colon \bigoplus_{i'} U'_{i'} \to \bigoplus_{j'} V'_{j'}$, then

$$
\begin{aligned}
R_W(\mathsf{t}_1 \oplus \mathsf{t}_2) &= R_W(\mathsf{t}_1) \oplus R_W(\mathsf{t}_2) && \text{(Funct. } R\text{)}\\
&= (\bigoplus_i \overline{\underline{\sigma_{U_i,W}}}; L_W(\mathsf{t}_1); \bigoplus_j \overline{\underline{\sigma_{W,V_j}}}) \oplus (\bigoplus_{i'} \overline{\underline{\sigma_{U'_{i'},W}}}; L_W(\mathsf{t}_2); \bigoplus_{j'} \overline{\underline{\sigma_{W,V'_{j'}}}}) && \text{(Ind. hp.)}\\
&= (\bigoplus_i \overline{\underline{\sigma_{U_i,W}}} \oplus \bigoplus_{i'} \overline{\underline{\sigma_{U'_{i'},W}}}); (L_W(\mathsf{t}_1) \oplus L_W(\mathsf{t}_2); (\bigoplus_j \overline{\underline{\sigma_{W,V_j}}} \oplus \bigoplus_{j'} \overline{\underline{\sigma_{W,V'_{j'}}}}) && \text{(Funct. } \oplus\text{)}\\
&= (\bigoplus_i \overline{\underline{\sigma_{U_i,W}}} \oplus \bigoplus_{i'} \overline{\underline{\sigma_{U'_{i'},W}}}); (L_W(\mathsf{t}_1 \oplus \mathsf{t}_2)); (\bigoplus_j \overline{\underline{\sigma_{W,V_j}}} \oplus \bigoplus_{j'} \overline{\underline{\sigma_{W,V'_{j'}}}}) && \text{(Funct. } L\text{)}
\end{aligned}
$$

□

Lemma D.3. *For all* $\overline{\underline{c}} \colon U \to V, \mathsf{t} \colon P \to Q$*,* $L_U(\mathsf{t}); R_Q(\overline{\underline{c}}) = R_P(\overline{\underline{c}}); L_V(\mathsf{t})$.

Proof. We fix $P = \bigoplus_k W_k$ and $Q = \bigoplus_l Z_l$. By definition of $R_P(-)$ and δ^l, it is immediate that $R_Q(\overline{\underline{c}}) = \bigoplus_l \overline{\underline{c \otimes id_{Z_l}}}$ and $R_P(\overline{\underline{c}}) = \bigoplus_k \overline{\underline{c \otimes id_{W_k}}}$.

We proceed by induction on t.

Case $\mathsf{t} = id_0$**:** $L_U(id_0); \bigoplus_{l=1}^0 \overline{\underline{c \otimes id_{Z_l}}} = id_0 = (\bigoplus_{k=1}^0 \overline{\underline{c \otimes id_{W_k}}}); L_V(id_0)$

Case $\mathsf{t} = \overline{\underline{d}}$**:** Suppose $d \colon W \to Z$, then

$$
\begin{aligned}
L_U(\overline{\underline{d}}); \overline{\underline{c \otimes id_Z}} &= \overline{\underline{id_U \otimes d}}; \overline{\underline{c \otimes id_Z}} && \text{(Def. } L\text{)}\\
&= \overline{\underline{c \otimes d}} && \text{(Funct. } \overline{\underline{\otimes}}\text{)}\\
&= \overline{\underline{c \otimes id_W}}; \overline{\underline{id_V \otimes d}} && \text{(Funct. } \overline{\underline{\otimes}}\text{)}\\
&= \overline{\underline{c \otimes id_W}}; L_V(\overline{\underline{d}}) && \text{(Def. } L\text{)}
\end{aligned}
$$

Case $\mathsf{t} = \sigma^{\oplus}_{W,Z}$**:**

$$\begin{aligned}
L_U(\sigma^{\oplus}_{W,Z}); (\overline{\underline{c \otimes id_Z}} \oplus \overline{\underline{c \otimes id_W}}) &= \sigma^{\oplus}_{UW,UZ}; (\overline{\underline{c \otimes id_Z}} \oplus \overline{\underline{c \otimes id_W}}) && \text{(Def. } L) \\
&= (\overline{\underline{c \otimes id_W}} \oplus \overline{\underline{c \otimes id_Z}}); \sigma^{\oplus}_{VW,VZ} && \text{(Funct. } \sigma^{\oplus}) \\
&= (\overline{\underline{c \otimes id_W}} \oplus \overline{\underline{c \otimes id_Z}}); L_V(\sigma^{\oplus}_{W,Z}) && \text{(Def. } L)
\end{aligned}$$

Case $\mathsf{t} = \blacktriangleleft_W$**:** (similarly for $\blacktriangleright_W$)

$$\begin{aligned}
L_U(\blacktriangleleft_W); (\overline{\underline{c \otimes id_W}} \oplus \overline{\underline{c \otimes id_W}}) &= \blacktriangleleft_{UW}; (\overline{\underline{c \otimes id_W}} \oplus \overline{\underline{c \otimes id_W}}) && \text{(Def. } L) \\
&= \overline{\underline{c \otimes id_W}}; \blacktriangleleft_{VW} && \text{(Nat. } \blacktriangleleft) \\
&= \overline{\underline{c \otimes id_W}}; L_V(\blacktriangleleft_W) && \text{(Def. } L)
\end{aligned}$$

Case $\mathsf{t} = !_W$**:** (similarly for $¡_W$)

$$\begin{aligned}
L_U(!_W); \bigoplus_{l=1}^{0} \overline{\underline{c \otimes id_{Z_l}}} &= !_{UW} && \text{(Def. } L) \\
&= \overline{\underline{c \otimes id_W}}; !_{VW} && \text{(Nat. } !) \\
&= \overline{\underline{c \otimes id_W}}; L_V(!_W) && \text{(Def. } L)
\end{aligned}$$

Case $\mathsf{t} = \mathsf{t}_1; \mathsf{t}_2$**:** Suppose $\mathsf{t}_1 \colon \bigoplus_k W_k \to \bigoplus_j V_j, \mathsf{t}_2 \colon \bigoplus_j V_j \to \bigoplus_l Z_l$, then

$$\begin{aligned}
L_U(\mathsf{t}_1; \mathsf{t}_2); \bigoplus_l \overline{\underline{c \otimes id_{Z_l}}} &= L_U(\mathsf{t}_1); L_U(\mathsf{t}_2); \bigoplus_l \overline{\underline{c \otimes id_{Z_l}}} && \text{(Funct. } L) \\
&= L_U(\mathsf{t}_1); \bigoplus_j \overline{\underline{c \otimes id_{V_j}}}; L_V(\mathsf{t}_2) && \text{(Ind. hp.)} \\
&= \bigoplus_k \overline{\underline{c \otimes id_{W_k}}}; L_V(\mathsf{t}_1); L_V(\mathsf{t}_2) && \text{(Ind. hp.)} \\
&= \bigoplus_k \overline{\underline{c \otimes id_{W_k}}}; L_V(\mathsf{t}_1; \mathsf{t}_2) && \text{(Funct. } L)
\end{aligned}$$

Case $\mathsf{t} = \mathsf{t}_1 \oplus \mathsf{t}_2$**:** Suppose $\mathsf{t}_1 \colon \bigoplus_k W_k \to \bigoplus_l Z_l, \mathsf{t}_2 \colon \bigoplus_{k'} W'_{k'} \to \bigoplus_{l'} Z'_{l'}$, then

$$
\begin{aligned}
& L_U(\mathsf{t}_1 \oplus \mathsf{t}_2); (\bigoplus_l \underline{\overline{c \otimes id_{Z_l}}} \oplus \bigoplus_{l'} \underline{\overline{c \otimes id_{Z'_{l'}}}}) \\
&= (L_U(\mathsf{t}_1) \oplus L_U(\mathsf{t}_2)); (\bigoplus_l \underline{\overline{c \otimes id_{Z_l}}} \oplus \bigoplus_{l'} \underline{\overline{c \otimes id_{Z'_{l'}}}}) && \text{(Funct. } L) \\
&= (L_U(\mathsf{t}_1); \bigoplus_l \underline{\overline{c \otimes id_{Z_l}}}) \oplus (L_U(\mathsf{t}_2); \bigoplus_{l'} \underline{\overline{c \otimes id_{Z'_{l'}}}}) && \text{(Funct. } \oplus) \\
&= (\bigoplus_k \underline{\overline{c \otimes id_{W_k}}}; L_V(\mathsf{t}_1)) \oplus (\bigoplus_{k'} \underline{\overline{c \otimes id_{W'_{k'}}}}; L_V(\mathsf{t}_2)) && \text{(Ind. hp.)} \\
&= (\bigoplus_k \underline{\overline{c \otimes id_{W_k}}} \oplus \bigoplus_{k'} \underline{\overline{c \otimes id_{W'_{k'}}}}); (L_V(\mathsf{t}_1) \oplus L_V(\mathsf{t}_2)) && \text{(Funct. } \oplus) \\
&= (\bigoplus_k \underline{\overline{c \otimes id_{W_k}}} \oplus \bigoplus_{k'} \underline{\overline{c \otimes id_{W'_{k'}}}}); L_V(\mathsf{t}_1 \oplus \mathsf{t}_2) && \text{(Funct. } L)
\end{aligned}
$$

□

D.3 The algebra of whiskerings

Proof of Lemma 4.13. The properties of the left whiskering are all trivial, thus we show only the proofs for their right counterpart.

Let $P = \bigoplus_i U_i, Q = \bigoplus_j V_j, S = \bigoplus_k W_k, T = \bigoplus_l Z_l$.

EQUATION (W1): $R_S(id_P) = id_{PS}$. We prove it by induction on S.

Case $S = 0$**:** it holds by Definition 4.11.

Case $S = W \oplus S'$**:**

$$
\begin{aligned}
R_{W \oplus S'}(id_Q) &= \delta^l_{Q,W,S'}; (R_W(id_Q) \oplus R_{S'}(id_Q)); \delta^{-l}_{Q,W,S'} && \text{(Def. } R) \\
&= \delta^l_{Q,W,S'}; (id_{QW} \oplus id_{QS'}); \delta^{-l}_{Q,W,S'} && \text{(Def. } R\text{, ind. hp.)} \\
&= id_{Q(W \oplus S')} && \text{(Iso)}
\end{aligned}
$$

EQUATION (W2): $R_S(\mathsf{t}_1; \mathsf{t}_2) = R_S(\mathsf{t}_1); R_S(\mathsf{t}_2)$. Let $\mathsf{t}_1 \colon P \to Q, \mathsf{t}_2 \colon Q \to T$, we prove it by induction on S.

Case $S = 0$**:** it holds by Definition 4.11.

Case $S = W \oplus S'$**:**

$$
\begin{aligned}
& R_{W \oplus S'}(\mathsf{t}_1; \mathsf{t}_2) \\
&= \delta^l_{P,W,S'}; (R_W(\mathsf{t}_1; \mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_1; \mathsf{t}_2)); \delta^{-l}_{T,W,S'} && \text{(Def. } R) \\
&= \delta^l_{P,W,S'}; ((R_W(\mathsf{t}_1); R_W(\mathsf{t}_2)) \oplus (R_{S'}(\mathsf{t}_1); R_{S'}(\mathsf{t}_2))); \delta^{-l}_{T,W,S'} && \text{(Def. } R\text{, ind. hp.)} \\
&= \delta^l_{P,W,S'}; ((R_W(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_1)); (R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_2))); \delta^{-l}_{T,W,S'} && \text{(Funct. } \oplus) \\
&= \delta^l_{P,W,S'}; ((R_W(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_1)); \delta^{-l}_{Q,W,S'}; \delta^l_{Q,W,S'}; (R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_2))); \delta^{-l}_{T,W,S'} && \text{(Iso)} \\
&= R_{W \oplus S'}(\mathsf{t}_1); R_{W \oplus S'}(\mathsf{t}_2) && \text{(Def. } R)
\end{aligned}
$$

Equation (W3): $R_1(\mathsf{t}) = \mathsf{t}$. We prove it by induction on t. All the base cases are trivial and the inductive cases follow by inductive hypothesis and functoriality of R and $\oplus$.

Equation (W4): $R_0(\mathsf{t}) = id_0$. It holds by Definition 4.11.

Equation (W6): $R_{S\oplus T}(\mathsf{t}) = \delta^l_{P,S,T}; (R_S(\mathsf{t}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,S,T}$. Let $\mathsf{t}\colon P \to Q$, we prove it by induction on S.

Case $S = 0$: it holds by Definition 4.11.

Case $S = W \oplus S'$:

$$
\begin{aligned}
& R_{W\oplus S'\oplus T}(\mathsf{t}) \\
&= \delta^l_{P,W,S'\oplus T}; (R_W(\mathsf{t}) \oplus R_{S'\oplus T}(\mathsf{t})); \delta^{-l}_{Q,W,S'\oplus T} && \text{(Def. } R) \\
&= \delta^l_{P,W,S'\oplus T}; (R_W(\mathsf{t}) \oplus (\delta^l_{P,S',T}; (R_{S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,S',T})); \delta^{-l}_{Q,W,S'\oplus T} && \text{(Ind. hp.)} \\
&= \delta^l_{P,W,S'\oplus T}; (id_{PW} \oplus \delta^l_{P,S',T}); (R_W(\mathsf{t}) \oplus ((R_{S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,S',T})); \delta^{-l}_{Q,W,S'\oplus T} && \text{(Funct. } \oplus) \\
&= \delta^l_{P,W\oplus S',T}; (\delta^l_{P,W,S'} \oplus id_{PT}); (R_W(\mathsf{t}) \oplus ((R_{S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,S',T})); \delta^{-l}_{Q,W,S'\oplus T} && \text{(Lemma D.1)} \\
&= \delta^l_{P,W\oplus S',T}; (\delta^l_{P,W,S'} \oplus id_{PT}); (R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); (id_{QW} \oplus \delta^{-l}_{Q,S',T}); \delta^{-l}_{Q,W,S'\oplus T} \\
& && \text{(Funct. } \oplus) \\
&= \delta^l_{P,W\oplus S',T}; (\delta^l_{P,W,S'} \oplus id_{PT}); (R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); (\delta^{-l}_{Q,W,S'} \oplus id_{QT}); \delta^{-l}_{Q,W\oplus S',T} \\
& && \text{(Lemma D.1)} \\
&= \delta^l_{P,W\oplus S',T}; ((\delta^l_{P,W,S'}; (R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t})); \delta^{-l}_{Q,W,S'}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,W\oplus S',T} && \text{(Funct. } \oplus) \\
&= \delta^l_{P,W\oplus S',T}; (R_{W\oplus S'}(\mathsf{t}) \oplus R_T(\mathsf{t})); \delta^{-l}_{Q,W\oplus S',T} && \text{(Def. } R)
\end{aligned}
$$

Equation (W5): $R_S(\mathsf{t}_1 \oplus \mathsf{t}_2) = R_S(\mathsf{t}_1) \oplus R_S(\mathsf{t}_2)$. Let $\mathsf{t}_1\colon P \to Q, \mathsf{t}_2\colon P' \to Q'$, we prove it by induction on S.

Case $S = 0$: it holds by Definition 4.11.

Case $S = W \oplus S'$:

$$
\begin{aligned}
& R_{W\oplus S'}(\mathsf{t}_1 \oplus \mathsf{t}_2) \\
&= \delta^l_{P\oplus P',W,S'}; (R_W(\mathsf{t}_1 \oplus \mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_1 \oplus \mathsf{t}_2)); \delta^{-l}_{Q\oplus Q',W,S'} && \text{(W6)} \\
&= \delta^l_{P\oplus P',W,S'}; (R_W(\mathsf{t}_1) \oplus R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_2)); \delta^{-l}_{Q\oplus Q',W,S'} && \text{(Def. } R\text{, ind. hp.)} \\
&= (\delta^l_{P,W,S'} \oplus \delta^l_{P',W,S'}); (id_{PW} \oplus \sigma^{\oplus}_{PS',P'W} \oplus id_{P'S'}); && \text{(Lemma 4.8.2)} \\
&\qquad\quad (R_W(\mathsf{t}_1) \oplus R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_2)); \\
&\qquad (id_{QW} \oplus \sigma^{\oplus}_{Q'W,QS'} \oplus id_{Q'S'}); (\delta^{-l}_{Q,W,S'} \oplus \delta^{-l}_{Q',W,S'}) \\
&= (\delta^l_{P,W,S'} \oplus \delta^l_{P',W,S'}); (R_W(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_1) \oplus R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_2)); (\delta^{-l}_{Q,W,S'} \oplus \delta^{-l}_{Q',W,S'}) && \text{(Nat. } \sigma^{\oplus}) \\
&= (\delta^l_{P,W,S'}; (R_W(\mathsf{t}_1) \oplus R_{S'}(\mathsf{t}_1)); \delta^{-l}_{Q,W,S'}) \oplus (\delta^l_{P',W,S'}; (R_W(\mathsf{t}_2) \oplus R_{S'}(\mathsf{t}_2)); \delta^{-l}_{Q',W,S'}) && \text{(Funct. } \oplus) \\
&= R_{W\oplus S'}(\mathsf{t}_1) \oplus R_{W\oplus S'}(\mathsf{t}_2) && (W6)
\end{aligned}
$$

Equation (W8): $R_S(\blacktriangleleft_U) = \blacktriangleleft_{US}$. We prove it by induction on S.

Case $S = 0$: it holds by Definition 4.11 and definition of $\blacktriangleleft$.

Case $S = W \oplus S'$**:**

$$\begin{aligned}
&R_{W\oplus S'}(\blacktriangleleft_U)\\
&= \delta^l_{U,W,S'}; (R_W(\blacktriangleleft_U) \oplus R_{S'}(\blacktriangleleft_U)); \delta^{-l}_{U\oplus U,W,S'} && \text{(W6)}\\
&= (R_W(\blacktriangleleft_U) \oplus R_{S'}(\blacktriangleleft_U)); (id_{UW} \oplus \sigma^{\oplus}_{UW,US'} \oplus id_{US'}); (id_{U(W\oplus S')} \oplus \delta^{-l}_{U,W,S'})\\
& && \text{(Lemma 4.8.1, def. } \delta^l)\\
&= (R_W(\blacktriangleleft_U) \oplus R_{S'}(\blacktriangleleft_U)); (id_{UW} \oplus \sigma^{\oplus}_{UW,US'} \oplus id_{US'}) && \text{(Lemma 4.8.1)}\\
&= (\blacktriangleleft_{UW} \oplus \blacktriangleleft_{US'}); (id_{UW} \oplus \sigma^{\oplus}_{UW,US'} \oplus id_{US'}) && \text{(Def. } R\text{, ind. hp.)}\\
&= \blacktriangleleft_{UW\oplus US'} && \text{(Def. } \blacktriangleleft)\\
&= \blacktriangleleft_{U(W\oplus S')} && \text{(Rem. 4.4)}
\end{aligned}$$

The case for $\blacktriangleright_U$ is completely analogous.

Equation (W9): $R_S(!_U) = !_{US}$. We prove it by induction on S.

Case $S = 0$**:** it holds by Definition 4.11 and definition of !.

Case $S = W \oplus S'$**:**

$$\begin{aligned}
R_{W\oplus S'}(!_U) &= \delta^l_{U,W,S'}; (R_W(!_U) \oplus R_{S'}(!_U)); \delta^{-l}_{0,W,S'} && \text{(W6)}\\
&= R_W(!_U) \oplus R_{S'}(!_U) && \text{(Lemma 4.8.1, def. } \delta^l)\\
&= !_{UW} \oplus !_{US'} && \text{(Def. } R\text{, ind. hp.)}\\
&= !_{UW\oplus US'} && \text{(Def. !)}\\
&= !_{U(W\oplus S')} && \text{(Rem. 4.4)}
\end{aligned}$$

The case for $¡_U$ is completely analogous.

Equation (W10): $R_S(\sigma^{\oplus}_{P,Q}) = \sigma^{\oplus}_{PS,QS}$. We prove it by induction on P.

Case $P = 0$**:**

$$\begin{aligned}
R_S(\sigma^{\oplus}_{0,Q}) &= R_S(id_Q) && \text{(S1)}\\
&= id_{QS} && \text{(W1)}\\
&= \sigma^{\oplus}_{0,QS} && \text{(S1)}
\end{aligned}$$

Case $P = U \oplus P'$**:**

$$\begin{aligned}
R_S(\sigma^{\oplus}_{U\oplus P',Q}) &= R_S((id_U \oplus \sigma^{\oplus}_{P',Q}); (\sigma^{\oplus}_{U,Q} \oplus id_{P'})) && \text{(S2)}\\
&= (R_S(id_U) \oplus R_S(\sigma^{\oplus}_{P',Q})); (R_S(\sigma^{\oplus}_{U,Q}) \oplus R_S(id_{P'})) && \text{((W2), (W5))}\\
&= (id_{US} \oplus \sigma^{\oplus}_{P'S,QS}); (\sigma^{\oplus}_{US,QS} \oplus id_{P'S}) && \text{((W1), ind. hp)}\\
&= \sigma^{\oplus}_{US\oplus P'S,QS} && \text{(S2)}\\
&= \sigma^{\oplus}_{(U\oplus P')S,QS} && \text{(Rem. 4.4)}
\end{aligned}$$

Equation (W15): $R_S(\delta^l_{P,Q,R}) = \delta^l_{P,QS,RS}$. We prove it by induction on P.

Case $P = 0$**:** it follows from Definitions 4.11 and 4.5.

Case $P = U \oplus P'$**:**

$$
\begin{aligned}
& R_S(\delta^l_{U\oplus P',Q,R}) \\
& = R_S((id_{U(Q\oplus R)} \oplus \delta^l_{P',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P'Q} \oplus id_{P'R})) && \text{(Def. } \delta^l\text{)} \\
& = (R_S(id_{U(Q\oplus R)}) \oplus R_S(\delta^l_{P',Q,R})); (R_S(id_{UQ}) \oplus R_S(\sigma^{\oplus}_{UR,P'Q}) \oplus R_S(id_{P'R})) && \text{((W2), (W5))} \\
& = (R_S(id_{U(Q\oplus R)}) \oplus \delta^l_{P',QS,RS}); (R_S(id_{UQ}) \oplus R_S(\sigma^{\oplus}_{UR,P'Q}) \oplus R_S(id_{P'R})) && \text{(Ind. hp.)} \\
& = (id_{U(Q\oplus R)S} \oplus \delta^l_{P',QS,RS}); (id_{UQS} \oplus \sigma^{\oplus}_{URS,P'QS} \oplus id_{P'RS}) && \text{((W1), (W10))} \\
& = \delta^l_{U\oplus P',QS,RS} && \text{(Def. } \delta^l\text{)}
\end{aligned}
$$

Equation (W17): $\sigma^{\otimes}_{PQ,S} = L_P(\sigma^{\otimes}_{Q,S}); R_Q(\sigma^{\otimes}_{P,S})$. First we prove the following equations:

$$
\begin{aligned}
L_U(\overline{\underline{\sigma_{V,W}}}); R_V(\overline{\underline{\sigma_{U,W}}}) &= \overline{\underline{\sigma_{UV,W}}} && (*_1) \\
L_U(\bigoplus_j \overline{\underline{\sigma_{V_j,W}}}); R_Q(\overline{\underline{\sigma_{U,W}}}) &= \bigoplus_j \overline{\underline{\sigma_{UV_j,W}}} \qquad \text{where } Q = \bigoplus_j V_j && (*_2) \\
L_P(\sigma^{\otimes}_{Q,W}); R_Q(\sigma^{\otimes}_{P,W}) &= \sigma^{\otimes}_{PQ,W} && (*_3)
\end{aligned}
$$

For $(*_1)$ observe that the following holds:

$$
\begin{aligned}
L_U(\overline{\underline{\sigma_{V,W}}}); R_V(\overline{\underline{\sigma_{U,W}}}) &= \overline{\underline{id_U \otimes \sigma_{V,W}}}; \overline{\underline{\sigma_{U,W} \otimes id_V}} && \text{(Def. } L, R\text{)} \\
&= \overline{\underline{\sigma_{UV,W}}} && \text{(SMC)}
\end{aligned}
$$

For $(*_2)$ we proceed by induction on Q:

$$
L_U(\bigoplus_j \overline{\underline{\sigma_{V_j,W}}}); R_Q(\overline{\underline{\sigma_{U,W}}}) = \bigoplus_j \overline{\underline{\sigma_{UV_j,W}}} \qquad \text{where } Q = \bigoplus_j V_j \tag{47}
$$

Case $Q = 0$**:** it follows from Definitions 4.9 and 4.11 and from the fact that $\bigoplus_{j=1}^{0} \overline{\underline{\sigma_{V_j,W}}} = id_0$.

Case $Q = V \oplus Q'$**:** Let $Q' = \bigoplus_{j'} V'_{j'}$, then

$$
\begin{aligned}
& L_U(\overline{\underline{\sigma_{V,W}}} \oplus \bigoplus_{j'} \overline{\underline{\sigma_{V'_{j'},W}}}); R_{V\oplus Q'}(\overline{\underline{\sigma_{U,W}}}) \\
& = (L_U(\overline{\underline{\sigma_{V,W}}}) \oplus L_U(\bigoplus_{j'} \overline{\underline{\sigma_{V'_{j'},W}}})); \delta^l_{UW,V,Q'}; (R_V(\overline{\underline{\sigma_{U,W}}}) \oplus R_{Q'}(\overline{\underline{\sigma_{U,W}}})); \delta^{-l}_{WU,V,Q'} && \text{(Def. } L\text{, (W6))} \\
& = (L_U(\overline{\underline{\sigma_{V,W}}}) \oplus L_U(\bigoplus_{j'} \overline{\underline{\sigma_{V'_{j'},W}}})); (R_V(\overline{\underline{\sigma_{U,W}}}) \oplus R_{Q'}(\overline{\underline{\sigma_{U,W}}})) && \text{(Lemma 4.8.1)} \\
& = (L_U(\overline{\underline{\sigma_{V,W}}}); R_V(\overline{\underline{\sigma_{U,W}}})) \oplus (L_U(\bigoplus_{j'} \overline{\underline{\sigma_{V'_{j'},W}}}); R_{Q'}(\overline{\underline{\sigma_{U,W}}})) && \text{(Funct. } \oplus\text{)} \\
& = \overline{\underline{\sigma_{UV,W}}} \oplus \bigoplus_{j'} \overline{\underline{\sigma_{UV'_{j'},W}}} && \text{((}*_1\text{), ind. hp.)}
\end{aligned}
$$

For $(*_3)$ observe that the following holds:

$$
\begin{aligned}
& L_P(\sigma^{\otimes}_{Q,W}); R_Q(\sigma^{\otimes}_{P,W}) \\
&= L_P(\bigoplus_j \underline{\overline{\sigma_{V_j,W}}}); R_Q(\bigoplus_i \underline{\overline{\sigma_{U_i,W}}}) && \text{(Lemma 4.8.8)} \\
&= \bigoplus_i L_{U_i}(\bigoplus_j \underline{\overline{\sigma_{V_j,W}}}); \bigoplus_i R_Q(\underline{\overline{\sigma_{U_i,W}}}) && \text{(Def. } L\text{,(W5))} \\
&= \bigoplus_i (L_{U_i}(\bigoplus_j \underline{\overline{\sigma_{V_j,W}}}); R_Q(\underline{\overline{\sigma_{U_i,W}}})) && \text{(Funct. } \oplus) \\
&= \bigoplus_i \bigoplus_j \underline{\overline{\sigma_{U_iV_j,W}}} && (*_2) \\
&= \sigma^{\otimes}_{PQ,W} && \text{(Lemma 4.8.8)}
\end{aligned}
$$

Now we are ready to prove Equation (W17) by induction on S:
Case $S = 0$: it follows from Definitions 4.9, 4.11 and 4.6.
Case $S = W \oplus S'$:

$$
\begin{aligned}
& L_P(\sigma^{\otimes}_{Q,W\oplus S'}); R_Q(\sigma^{\otimes}_{P,W\oplus S'}) \\
&= L_P(\delta^l_{Q,W,S'}; (\sigma^{\otimes}_{Q,W} \oplus \sigma^{\otimes}_{Q,S'})); R_Q(\delta^l_{P,W,S'}; (\sigma^{\otimes}_{P,W} \oplus \sigma^{\otimes}_{P,S'})) && \text{(Lemma 4.8.6)} \\
&= L_P(\delta^l_{Q,W,S'}); \delta^l_{P,QW,QS'}; (L_P(\sigma^{\otimes}_{Q,W}) \oplus L_P(\sigma^{\otimes}_{Q,S'})); \delta^{-l}_{P,WQ,S'Q}; && \text{((W2), (W5))} \\
&\qquad R_Q(\delta^l_{P,W,S'}); (R_Q(\sigma^{\otimes}_{P,W}) \oplus R_Q(\sigma^{\otimes}_{P,S'})) \\
&= \delta^l_{PQ,W,S'}; \delta^{-l}_{P,QW,QS'}; \delta^l_{P,QW,QS'}; (L_P(\sigma^{\otimes}_{Q,W}) \oplus L_P(\sigma^{\otimes}_{Q,S'})); \delta^{-l}_{P,WQ,S'Q}; && \text{((W16), (W15))} \\
&\qquad \delta^l_{P,WQ,S'Q}; (R_Q(\sigma^{\otimes}_{P,W}) \oplus R_Q(\sigma^{\otimes}_{P,S'})) \\
&= \delta^l_{PQ,W,S'}; (L_P(\sigma^{\otimes}_{Q,W}) \oplus L_P(\sigma^{\otimes}_{Q,S'})); (R_Q(\sigma^{\otimes}_{P,W}) \oplus R_Q(\sigma^{\otimes}_{P,S'})) && \text{(Iso)} \\
&= \delta^l_{PQ,W,S'}; ((L_P(\sigma^{\otimes}_{Q,W}); R_Q(\sigma^{\otimes}_{P,W})) \oplus (L_P(\sigma^{\otimes}_{Q,S'}); R_Q(\sigma^{\otimes}_{P,S'}))) && \text{(Funct. } \oplus) \\
&= \delta^l_{PQ,W,S'}; (\sigma^{\otimes}_{PQ,W} \oplus \sigma^{\otimes}_{PQ,S'}) && ((*_3)\text{, ind. hp.)} \\
&= \sigma^{\otimes}_{PQ,W\oplus S'} && \text{(Lemma 4.8.6)}
\end{aligned}
$$

EQUATION (W11): $R_S(\mathsf{t}); \sigma^{\otimes}_{Q,S} = \sigma^{\otimes}_{P,S}; L_S(\mathsf{t})$. First observe that when S is a monomial W, the following holds:

$$R_W(\mathsf{t}); \sigma^{\otimes}_{Q,W} = \sigma^{\otimes}_{P,W}; L_W(\mathsf{t}) \tag{$*$}$$

$$
\begin{aligned}
R_W(\mathsf{t}); \sigma^{\otimes}_{Q,W} &= R_W(\mathsf{t}); \bigoplus_j \underline{\overline{\sigma_{V_J,W}}} && \text{(Lemma 4.8.8)} \\
&= \bigoplus_i \underline{\overline{\sigma_{U_i,W}}}; L_W(\mathsf{t}) && \text{(Lemma D.2)} \\
&= \sigma^{\otimes}_{P,W}; L_W(\mathsf{t}) && \text{(Lemma 4.8.8)}
\end{aligned}
$$

Then we proceed by induction on S.

Case $S = 0$ it follows from Definitions 4.11 and 4.6.

Case $S = W \oplus S'$**:**

$$
\begin{aligned}
& R_{W\oplus S'}(\mathfrak{t}); \sigma^{\otimes}_{Q,W\oplus S'} \\
&= \delta^{l}_{P,W,S'}; (R_W(\mathfrak{t}) \oplus R_{S'}(\mathfrak{t})); \delta^{-l}_{Q,W,S'}; \delta^{l}_{Q,W,S'}; (\sigma^{\otimes}_{Q,W} \oplus \sigma^{\otimes}_{Q,S'}) && \text{((W6), Lemma 4.8.6)} \\
&= \delta^{l}_{P,W,S'}; (R_W(\mathfrak{t}) \oplus R_{S'}(\mathfrak{t})); (\sigma^{\otimes}_{Q,W} \oplus \sigma^{\otimes}_{Q,S'}) && \text{(Iso)} \\
&= \delta^{l}_{P,W,S'}; ((R_W(\mathfrak{t}); \sigma^{\otimes}_{Q,W}) \oplus (R_{S'}(\mathfrak{t}); \sigma^{\otimes}_{Q,S'})) && (\text{Funct. } \oplus) \\
&= \delta^{l}_{P,W,S'}; ((\sigma^{\otimes}_{P,W}; L_W(\mathfrak{t})) \oplus (\sigma^{\otimes}_{P,S'}; L_{S'}(\mathfrak{t}))) && ((*), \text{ ind. hp.}) \\
&= \delta^{l}_{P,W,S'}; (\sigma^{\otimes}_{P,W} \oplus \sigma^{\otimes}_{P,S'}); (L_W(\mathfrak{t}) \oplus L_{S'}(\mathfrak{t})) && (\text{Funct. } \oplus) \\
&= \sigma^{\otimes}_{P,W\oplus S'}; L_{W\oplus S'}(\mathfrak{t}) && \text{(Lemma 4.8.6, (W6))}
\end{aligned}
$$

Equations (W13): $L_{ST}(\mathfrak{t}) = L_S(L_T(\mathfrak{t}))$, (W12): $L_S(R_T(\mathfrak{t})) = R_T(L_S(\mathfrak{t}))$, (W14):$R_{TS}(\mathfrak{t}) = R_S(R_T(\mathfrak{t}))$. First we prove the particular cases in which S and T are monomials:

$$
\begin{aligned}
L_W(L_Z(\mathfrak{t})) &= L_{WZ}(\mathfrak{t}) && (*_1) \\
L_W(R_Z(\mathfrak{t})) &= R_Z(L_W(\mathfrak{t})) && (*_2) \\
R_W(R_Z(\mathfrak{t})) &= R_{ZW}(\mathfrak{t}) && (*_3)
\end{aligned}
$$

All the three equations are proved by induction on $\mathfrak{t}$ as follows: the base cases $\mathfrak{t} = \boxed{}, \sigma^{\oplus}_{U,V}, \blacktriangleleft_U, !_U, \blacktriangleright_U, ¡_U$ are trivial by definition of L and R. For $\mathfrak{t} = \underline{\overline{c}}$ we apply the definition of L and R and the fact that $\otimes$ is associative inside a tape. The inductive cases follow from functoriality of L and R and the inductive hypothesis.

For the general cases, we start with (W13):

$$
\begin{aligned}
L_S(L_T(\mathfrak{t})) &= \bigoplus_k \bigoplus_l L_{W_k}(L_{Z_l}(\mathfrak{t})) && (\text{Def. } L) \\
&= \bigoplus_k \bigoplus_l L_{W_k Z_l}(\mathfrak{t}) && (*_1) \\
&= L_{ST}(\mathfrak{t}) && (\text{Def. } L)
\end{aligned}
$$

Equation (W12) First observe that the following holds:

$$
R_W(L_T(\mathfrak{t})) = L_T(R_W(\mathfrak{t})) \qquad (*)
$$

indeed:

$$
\begin{aligned}
R_W(L_T(\mathfrak{t})) &= R_W(\bigoplus_l L_{Z_l}(\mathfrak{t})) && (\text{Def. } L) \\
&= \bigoplus_l R_W(L_{Z_l}(\mathfrak{t})) && (\text{Def. } R) \\
&= \bigoplus_l L_{Z_l}(R_W(\mathfrak{t})) && (*_2) \\
&= L_T(R_W(\mathfrak{t})) && (\text{Def. } L)
\end{aligned}
$$

Then we proceed by induction on S:
Case $S = 0$**:** if follows from Definition 4.11 and (W4).
Case $S = W \oplus S'$**:**

$$\begin{aligned}
& R_{W\oplus S'}(L_T(\mathsf{t})) \\
& = \delta^{l}_{TP,W,S'}; (R_W(L_T(\mathsf{t})) \oplus R_{S'}(L_T(\mathsf{t}))); \delta^{-l}_{TQ,W,S'} && \text{(Def. } R\text{)} \\
& = \delta^{l}_{TP,W,S'}; (L_T(R_W(\mathsf{t})) \oplus L_T(R_{S'}(\mathsf{t}))); \delta^{-l}_{TQ,W,S'} && \text{((}*\text{), ind. hp.)} \\
& = \delta^{l}_{TP,W,S'}; \delta^{-l}_{T,PW,PS'}; L_T(R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t})); \delta^{l}_{T,QW,QS'}; \delta^{-l}_{TQ,W,S'} && \text{(W5)} \\
& = L_T(\delta^{l}_{P,W,S'}); L_T(R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t})); L_T(\delta^{-l}_{Q,W,S'}) && \text{(W16)} \\
& = L_T(\delta^{l}_{P,W,S'}; (R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t})); \delta^{-l}_{Q,W,S'}) && \text{(W2)} \\
& = L_T(R_{W\oplus S'}(\mathsf{t})) && \text{(W6)}
\end{aligned}$$

Equation (W14) is proved by means of the other two:

$$\begin{aligned}
R_{TS}(\mathsf{t}) &= \sigma^{\otimes}_{P,TS}; L_{TS}(\mathsf{t}); \sigma^{\otimes}_{TS,Q} && \text{(W11)} \\
&= \sigma^{\otimes}_{P,TS}; L_T(L_S(\mathsf{t})); \sigma^{\otimes}_{TS,Q} && \text{(W13)} \\
&= R_S(\sigma^{\otimes}_{P,T}); L_T(\sigma^{\otimes}_{P,S}); L_T(L_S(\mathsf{t})); L_T(\sigma^{\otimes}_{S,Q}); R_S(\sigma^{\otimes}_{T,Q}) && \text{(W17)} \\
&= R_S(\sigma^{\otimes}_{P,T}); L_T(\sigma^{\otimes}_{P,S}; L_S(\mathsf{t}); \sigma^{\otimes}_{S,Q}); R_S(\sigma^{\otimes}_{T,Q}) && \text{(W2)} \\
&= R_S(\sigma^{\otimes}_{P,T}); L_T(R_S(\mathsf{t})); R_S(\sigma^{\otimes}_{T,Q}) && \text{(W11)} \\
&= R_S(\sigma^{\otimes}_{P,T}); R_S(L_T(\mathsf{t})); R_S(\sigma^{\otimes}_{T,Q}) && \text{(W12)} \\
&= R_S(\sigma^{\otimes}_{P,T}; L_T(\mathsf{t}); \sigma^{\otimes}_{T,Q}) && \text{(W2)} \\
&= R_S(R_T(\mathsf{t})) && \text{(W11)}
\end{aligned}$$

Equation (W7): $L_P(\mathsf{t}_2); R_S(\mathsf{t}_1) = R_R(\mathsf{t}_1); L_Q(\mathsf{t}_2)$. Let $\mathsf{t}_1 \colon P \to Q, \mathsf{t}_2 \colon R \to S$, then we show by induction on t_1 that

$$L_P(\mathsf{t}_2); R_S(\mathsf{t}_1) = R_R(\mathsf{t}_1); L_Q(\mathsf{t}_2)$$

Case $\mathsf{t}_1 = id_0$**:** by (W1), $R_S(id_0) = id_0 = R_R(id_0)$. By definition, $L_0(\mathsf{t}_2) = id_0$.
Case $\mathsf{t}_1 = \underline{c}$**:** it holds by Lemma D.3.
Case $\mathsf{t}_1 = \sigma^{\oplus}_{U,V}, \blacktriangleleft_U, !_U$**:** these cases all follow the same pattern, thus we show only the one for $\blacktriangleleft_U$:

$$\begin{aligned}
L_U(\mathsf{t}_2); R_S(\blacktriangleleft_U) &= L_U(\mathsf{t}_2); \blacktriangleleft_{US} && \text{(W8)} \\
&= \blacktriangleleft_{UR}; (L_U(\mathsf{t}_2) \oplus L_U(\mathsf{t}_2)) && \text{(Nat. } \blacktriangleleft\text{)} \\
&= R_R(\blacktriangleleft_U); (L_U(\mathsf{t}_2) \oplus L_U(\mathsf{t}_2)) && \text{(W8)} \\
&= R_R(\blacktriangleleft_U); (L_{U\oplus U}(\mathsf{t}_2)) && \text{(Def. } L\text{)}
\end{aligned}$$

Case $\mathsf{t}_1 = \mathsf{t}_1'; \mathsf{t}_1''$**:** Suppose $\mathsf{t}_1' \colon P \to P', \mathsf{t}_1'' \colon P' \to Q, \mathsf{t}_2 \colon R \to S$

$$\begin{aligned}
L_P(\mathsf{t}_2); R_S(\mathsf{t}_1'; \mathsf{t}_1'') &= L_P(\mathsf{t}_2); R_S(\mathsf{t}_1'); R_S(\mathsf{t}_1'') && \text{(W2)}\\
&= R_R(\mathsf{t}_1'); L_{P'}(\mathsf{t}_2); R_S(\mathsf{t}_1'') && \text{(Ind. hp.)}\\
&= R_R(\mathsf{t}_1'); R_R(\mathsf{t}_1''); L_Q(\mathsf{t}_2) && \text{(Ind. hp.)}\\
&= R_R(\mathsf{t}_1'; \mathsf{t}_1''); L_Q(\mathsf{t}_2) && \text{(W2)}
\end{aligned}$$

Case $\mathsf{t}_1 = \mathsf{t}_1' \oplus \mathsf{t}_1''$**:** Suppose $\mathsf{t}_1' \colon P_1 \to Q_1, \mathsf{t}_1'' \colon P' \to Q', \mathsf{t}_2 \colon R \to S$

$$\begin{aligned}
L_{P_1 \oplus P'}(\mathsf{t}_2); R_S(\mathsf{t}_1' \oplus \mathsf{t}_1'') &= (L_{P_1}(\mathsf{t}_2) \oplus L_{P'}(\mathsf{t}_2)); (R_S(\mathsf{t}_1') \oplus R_S(\mathsf{t}_1'')) && \text{((W6), (W5))}\\
&= (L_{P_1}(\mathsf{t}_2); R_S(\mathsf{t}_1')) \oplus (L_{P'}(\mathsf{t}_2); R_S(\mathsf{t}_1'')) && \text{(Funct. } \oplus\text{)}\\
&= (R_R(\mathsf{t}_1'); L_{Q_1}(\mathsf{t}_2)) \oplus (R_R(\mathsf{t}_1''); L_{Q'}(\mathsf{t}_2)) && \text{(Ind. hp.)}\\
&= (R_R(\mathsf{t}_1') \oplus R_R(\mathsf{t}_1'')); (L_{Q_1}(\mathsf{t}_2) \oplus L_{Q'}(\mathsf{t}_2)) && \text{(Funct. } \oplus\text{)}\\
&= R_R(\mathsf{t}_1' \oplus \mathsf{t}_1''); L_{Q_1 \oplus Q'}(\mathsf{t}_2) && \text{((W6), (W5))}
\end{aligned}$$

□

D.4 Isomorphism

Proof of Lemma 4.18. For the first point ($G(\delta^l_{P,Q,R}) = \delta^l_{P,Q,R}$) we proceed by induction on P:
Case $P = 0$**:**

$$\begin{aligned}
G(\delta^l_{0,Q,R}) &= G(id_0) && \text{(Def. } \delta^l\text{)}\\
&= id_0 && \text{(Def. } G\text{)}\\
&= \delta^l_{0,Q,R} && \text{(39)}
\end{aligned}$$

Case $P = U \oplus P'$**:**

$$\begin{aligned}
&G(\delta^l_{U\oplus P',Q,R})\\
&= G((id_{U(Q\oplus R)} \oplus \delta^l_{P',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P'Q} \oplus id_{P'R})) && \text{(Def. } \delta^l\text{)}\\
&= (G(id_{U(Q\oplus R)}) \oplus G(\delta^l_{P',Q,R})); (G(id_{UQ}) \oplus G(\sigma^{\oplus}_{UR,P'Q}) \oplus G(id_{P'R})) && \text{(Funct. } G\text{)}\\
&= (id_{U(Q\oplus R)} \oplus \delta^l_{P',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P'Q} \oplus id_{P'R}) && \text{(Def. } G\text{)}\\
&= (\delta^l_{U,Q,R} \oplus \delta^l_{P',Q,R}); (id_{UQ} \oplus \sigma^{\oplus}_{UR,P'Q} \oplus id_{P'R}) && \text{(Def. } \delta^l\text{)}\\
&= \delta^l_{U\oplus P',Q,R} && \text{(R5)}
\end{aligned}$$

For the second point ($G(\sigma^{\otimes}_{P,Q}) = \sigma^{\otimes}_{P,Q}$), let $Q = \bigoplus_j V_j$. We proceed by induction on Q.
Case $Q = 0$**:** it follows from Definition 4.6 and definition of G.

Case $Q = V \oplus Q'$**:**

$$
\begin{aligned}
G(\sigma^{\otimes}_{P,V\oplus Q'}) &= G(\delta^l_{P,V,Q'}; (\bigoplus_i \underline{\overline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'})) && \text{(Def. } \sigma^{\otimes}\text{)}\\
&= G(\delta^l_{P,V,Q'}); (\bigoplus_i G(\underline{\overline{\sigma_{U_i,V}}}) \oplus G(\sigma^{\otimes}_{P,Q'})) && \text{(Def. } G\text{)}\\
&= \delta^l_{P,V,Q'}; (\bigoplus_i G(\underline{\overline{\sigma_{U_i,V}}}) \oplus G(\sigma^{\otimes}_{P,Q'})) && \text{(Lemma 4.18.1)}\\
&= \delta^l_{P,V,Q'}; (\bigoplus_i \sigma^{\otimes}_{U_i,V} \oplus \sigma^{\otimes}_{P,Q'}) && \text{(Def. } G\text{, ind. hp.)}\\
&= \delta^l_{P,V,Q'}; (\sigma^{\otimes}_{P,V} \oplus \sigma^{\otimes}_{P,Q'}) && \text{(Lemma 3.9)}\\
&= \sigma^{\otimes}_{P,V\oplus Q'} && \text{(R1)}
\end{aligned}
$$

□

Lemma D.4. *Let* $\mathsf{t}\colon P \to Q$ *be a tape diagram, then* $G(R_W(\mathsf{t})) = G(\mathsf{t}) \otimes id_W$

Proof. By induction on t.
Case $\mathsf{t} = id_0$**:** $G(R_W(id_0)) = G(id_0) = id_0 = id_0 \otimes id_W = G(id_0) \otimes id_W$
Case $\underline{c}$**:** $G(R_W(\underline{c})) = G(\underline{\overline{c \otimes id_W}}) = c \otimes id_W = G(\underline{c}) \otimes id_W$
Case $\sigma^{\oplus}_{U,V}$**:** $G(R_W(\sigma^{\oplus}_{U,V})) = G(\sigma^{\oplus}_{UW,VW}) = \sigma^{\oplus}_{UW,VW} \stackrel{\text{(R2)}}{=} \sigma^{\oplus}_{U,V} \otimes id_W = G(\sigma^{\oplus}_{U,V}) \otimes id_W$
Case $\blacktriangleleft_U$**:** $G(R_W(\blacktriangleleft_U)) = G(\blacktriangleleft_{UW}) = \blacktriangleleft_{UW} \stackrel{Prop.3.15}{=} \blacktriangleleft_U \otimes id_W = G(\blacktriangleleft_U) \otimes id_W$ and analogously for $!_U, \blacktriangleright_U, ¡_U$.
Case $\mathsf{t} = \mathsf{t}_1; \mathsf{t}_2$**:** $G(R_W((\mathsf{t}_1; \mathsf{t}_2))) = G(R_W(\mathsf{t}_1)); G(R_W(\mathsf{t}_2)) = (G(\mathsf{t}_1) \otimes id_W); (G(\mathsf{t}_2) \otimes id_W) = (G(\mathsf{t}_1); G(\mathsf{t}_2)) \otimes id_W = G(\mathsf{t}_1; \mathsf{t}_2) \otimes id_W$
Case $\mathsf{t} = \mathsf{t}_1 \oplus \mathsf{t}_2$**:** $G(R_W((\mathsf{t}_1 \oplus \mathsf{t}_2))) = G(R_W(\mathsf{t}_1)) \oplus G(R_W(\mathsf{t}_2)) = (G(\mathsf{t}_1) \otimes id_W) \oplus (G(\mathsf{t}_2) \otimes id_W) = (G(\mathsf{t}_1) \oplus G(\mathsf{t}_2)) \otimes id_W = G(\mathsf{t}_1 \oplus \mathsf{t}_2) \otimes id_W$. □

Lemma D.5. *Let* $\mathsf{t}\colon P \to Q$ *be a tape diagram, then*

1. $G(R_S(\mathsf{t})) = G(\mathsf{t}) \otimes id_S$
2. $G(L_S(\mathsf{t})) = id_S \otimes G(\mathsf{t})$

Proof. We prove the first point by induction on S:
Case $S = 0$**:** $G(R_0(\mathsf{t})) = G(id_0) = id_0 = G(\mathsf{t}) \otimes id_0$.
Case $S = W \oplus S'$**:**

$$
\begin{aligned}
G(R_{W\oplus S'}(\mathsf{t})) &= G(\delta^l_{P,W,S'}; (R_W(\mathsf{t}) \oplus R_{S'}(\mathsf{t})); \delta^l_{Q,W,S'}) && \text{(W6)}\\
&= G(\delta^l_{P,W,S'}); (G(R_W(\mathsf{t})) \oplus G(R_{S'}(\mathsf{t}))); G(\delta^l_{Q,W,S'}) && \text{(Def. } G\text{)}\\
&= \delta^l_{P,W,S'}; (G(R_W(\mathsf{t})) \oplus G(R_{S'}(\mathsf{t}))); \delta^l_{Q,W,S'} && \text{(Lemma 4.18.1)}\\
&= \delta^l_{P,W,S'}; ((G(\mathsf{t}) \otimes id_W) \oplus (G(\mathsf{t}) \otimes id_{S'})); \delta^l_{Q,W,S'} && \text{(Lemma D.4, ind. hp.)}\\
&= G(\mathsf{t}) \otimes (id_W \oplus id_{S'}) && \text{(Nat. } \delta^l\text{)}\\
&= G(\mathsf{t}) \otimes id_{W\oplus S'} && \text{(Funct. } \oplus\text{)}
\end{aligned}
$$

For the second point, we use the first point and naturality of $\otimes$-symmetries.

$$\begin{aligned}
G(L_S(\mathfrak{t})) &= G(\sigma^{\otimes}_{S,P}; \sigma^{\otimes}_{P,S}; L_S(\mathfrak{t})) && \text{(Lemma 4.8.5)}\\
&= G(\sigma^{\otimes}_{S,P}; R_S(\mathfrak{t}); \sigma^{\otimes}_{Q,S}) && \text{(W11)}\\
&= G(\sigma^{\otimes}_{S,P}); G(R_S(\mathfrak{t})); G(\sigma^{\otimes}_{Q,S}) && \text{(Funct. } G\text{)}\\
&= \sigma^{\otimes}_{S,P}; G(R_S(\mathfrak{t})); \sigma^{\otimes}_{Q,S} && \text{(Lemma 4.18.2)}\\
&= \sigma^{\otimes}_{S,P}; (G(\mathfrak{t}) \otimes id_S); \sigma^{\otimes}_{Q,S} && \text{(Lemma D.5.1)}\\
&= id_S \otimes G(\mathfrak{t}) && \text{(Nat. } \sigma^{\otimes}\text{)}
\end{aligned}$$

□

Proof of Lemma 4.19. Let $\mathfrak{t}_1 \colon P \to Q, \mathfrak{t}_2 \colon R \to S$ be tape diagrams. Then

$$\begin{aligned}
G(\mathfrak{t}_1 \otimes \mathfrak{t}_2) &= G(L_P(\mathfrak{t}_2); R_S(\mathfrak{t}_1)) && \text{(Def. } \otimes\text{)}\\
&= G(L_P(\mathfrak{t}_2)); G(R_S(\mathfrak{t}_1)) && \text{(Funct. } G\text{)}\\
&= (id_P \otimes G(\mathfrak{t}_2)); (G(\mathfrak{t}_1) \otimes id_S) && \text{(Lemma D.5)}\\
&= G(\mathfrak{t}_1) \otimes G(\mathfrak{t}_2) && \text{(Funct. } \otimes\text{)}
\end{aligned}$$

□

Proof of Proposition 4.20. GF is the identity on objects since G and F are both identity-on-objects functors. We prove it on morphisms by induction:

Case $f = id_0$**:** $GF(id_0) = G(id_0) = id_0$.

Case $f = id_1$**:** $GF(id_1) = G(id_1) = id_1$.

Case $f = id_A$**:** $GF(id_A) = G(id_A) = id_A$.

Case $f = s$**:** $GF(s) = G(\underline{s}) = s$.

Case $f = \sigma^{\otimes}_{P,Q}$**:** $GF(\sigma^{\otimes}_{P,Q}) = G(\sigma^{\otimes}_{P,Q}) = \sigma^{\otimes}_{P,Q}$.

Case $f = \sigma^{\oplus}_{P,Q}$**:** $GF(\sigma^{\oplus}_{P,Q}) = G(\sigma^{\oplus}_{P,Q}) = \sigma^{\oplus}_{P,Q}$.

Case $f = \blacktriangleleft_X$**:** $GF(\blacktriangleleft_X) = G(\blacktriangleleft_X) = \blacktriangleleft_X$ and analogously for $!_X, \blacktriangleright_X, ¡_X$.

The inductive cases $f = g; h$ and $f = g \oplus h$ directly follow from the inductive hypothesis and functoriality of G and F.

Case $f = g \otimes h$**:** Let $g \colon X_1 \to Y_1, h \colon X_2 \to Y_2$, then

$$\begin{aligned}
GF(g \otimes h) &= G(F(g) \otimes F(h)) && \text{(Def. } F\text{)}\\
&= G(F(g)) \otimes G(F(h)) && \text{(Lemma 4.19)}\\
&= g \otimes h && \text{(Ind. hp.)}
\end{aligned}$$

□

Proof of Proposition 4.23. FG is the identity on objects since both F and G are identity on objects. We prove it on morphisms by induction:

Case $\mathfrak{t} = id_0$**:** Follows immediately from the fact that both F and G are functors.

Case $\mathfrak{t} = \underline{c}$**:** By definition of G and Lemma 4.22, it holds that $FG(\underline{c}) = F(c) = \underline{c}$.

Case $\mathfrak{t} = \sigma^{\oplus}_{P,Q}$**:** Follows immediately from the fact that both F and G are symmetric strict $\oplus$-monoidal functors.

Case $\mathsf{t} = \blacktriangleleft_U$**:** $FG(\blacktriangleleft_U) = F(\blacktriangleleft_U) = \blacktriangleleft_U$ and analogously for $!_U, \blacktriangleright_U, ¡_U$.

The inductive cases $\mathsf{t} = \mathsf{t}_1 ; \mathsf{t}_2$ and $\mathsf{t} = \mathsf{t}_1 \oplus \mathsf{t}_2$ directly follow from the inductive hypothesis and functoriality of F and G. □

E Appendix for the matrix calculus

We begin by listing some useful properties of the matrix representation of morphisms in a fb category **C**.

Proposition E.1. If $\bigoplus_{k=1}^{n} A_k \xrightarrow{f} \bigoplus_{k=1}^{m} B_k \xrightarrow{g} \bigoplus_{k=1}^{l} C_k$ then $\mathcal{M}(g \circ f) = \mathcal{M}(g) \cdot \mathcal{M}(f)$, where $\cdot$ is the usual matrix multiplication.

Proof. For $j \in \{1, \dots, l\}$ and $i \in \{1, \dots, n\}$, entry (j, i) of $\mathcal{M}(g) \cdot \mathcal{M}(f)$ is $\sum_{k=1}^{m} g_{jk} \circ f_{ki}$. Using the first equation of Lemma 6.4 for f and the second equation for g, we have the following chain of equalities, where μ_k and π_k are the k-th injection and projection of B_k into/from $\bigoplus_{k=1}^{m} B_k$:

$$\begin{aligned}
\sum_{k=1}^{m} g_{jk} \circ f_{ki} &= \sum_{k=1}^{m} \Big([g_{j1}, \dots, g_{jm}] \circ \mu_k \circ \pi_k \circ \langle f_{i1}, \dots, f_{mi} \rangle \Big) && \text{(Definition of (co)pairing)} \\
&= [g_{j1}, \dots, g_{jm}] \circ \Big(\sum_{k=1}^{m} \mu_k \circ \pi_k \Big) \circ \langle f_{i1}, \dots, f_{mi} \rangle && (9) \\
&= [g_{j1}, \dots, g_{jm}] \circ \langle f_{i1}, \dots, f_{mi} \rangle && (10) \\
&= \pi_j \circ g \circ f \circ \mu_i && \text{(Lemma 6.4)} \\
&= (g \circ f)_{ji}.
\end{aligned}$$

In the penultimate equality we used the second equation of Lemma 6.4 for g and the first equation of the same lemma for f. □

Next we list a few useful facts about matrices associated to morphisms built using the biproduct structure of **C**.

Proposition E.2. For f and g morphisms of **C** of appropriate type:

1. $\mathcal{M}(f \oplus g) = \begin{pmatrix} \mathcal{M}(f) & 0 \\ 0 & \mathcal{M}(g) \end{pmatrix}$

2. $\mathcal{M}(\langle f, g \rangle) = \begin{pmatrix} \mathcal{M}(f) \\ \mathcal{M}(g) \end{pmatrix}$

3. $\mathcal{M}([f, g]) = \begin{pmatrix} \mathcal{M}(f) & \mathcal{M}(g) \end{pmatrix}$

4. $\mathcal{M}(\mu_i \colon A_i \to \bigoplus_{k=1}^{n} A_k) = \begin{pmatrix} \delta_{i,1} \\ \vdots \\ \delta_{i,n} \end{pmatrix}$, $\mathcal{M}(\pi_j \colon \bigoplus_{k=1}^{n} A_k \to A_j) = \begin{pmatrix} \delta_{1,j} & \cdots & \delta_{m,j} \end{pmatrix}$

5. For $A = \bigoplus_{k=1}^{n} A_k$ and $B = \bigoplus_{k=1}^{m} B_k$,

$$\mathcal{M}(\sigma_{A,B}) = \begin{pmatrix} \mathcal{M}(0_{B,A}) & \mathcal{M}(id_B) \\ \mathcal{M}(id_A) & \mathcal{M}(0_{A,B}) \end{pmatrix}$$

matrix of size $(m+n) \times (n+m)$.

Proposition E.3. $\mathcal{G} \colon \mathbf{Mat}(\mathbf{C}^+) \to F_2(\mathbf{C})$ is a functor that preserves biproducts.

Proof. Let

$$\bigoplus_{i=1}^{n} A_i \xrightarrow{M} \bigoplus_{k=1}^{m} V_k \xrightarrow{N} \bigoplus_{j=1}^{l} W_j.$$

We aim to compute $\mathcal{G}(N \circ M)$. We have

$$\begin{aligned} (N \circ M)_{ji} = (N \cdot M)_{ji} &= \sum_{k=1}^{m} N_{jk} \circ M_{ki} && \text{(Sum and composition in } \mathbf{C}^+_\Sigma) \\ &= \bigcup_{k=1}^{m} \{\!\!\{ b \circ_{\mathbf{C}} a \mid a \in M_{ki},\, b \in N_{jk} \}\!\!\} && \text{(Union of multisets)} \end{aligned}$$

therefore

$$\overline{\mathcal{G}}\big((N \circ M)_{ji}\big) = \sum_{k=1}^{m} \sum_{\substack{a \in M_{ki} \\ b \in N_{jk}}} \overline{\underline{b \circ a}}$$

Now, $\mathcal{G}(N) \circ \mathcal{G}(M)$ is a morphism in $F_2(\mathbf{C})$ whose associated matrix, by Proposition E.1, is

$$\big(\overline{\mathcal{G}}(N_{wv})\big)_{\substack{w=1\ldots l \\ v=1\ldots m}} \cdot \big(\overline{\mathcal{G}}(M_{vu})\big)_{\substack{v=1\ldots m \\ u=1\ldots n}}$$

whose entry (j,i), for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$, is the following sum of compositions in

$F_2(\mathbf{C})$:

$$
\begin{aligned}
\big(\mathcal{M}(\mathcal{G}(N)\circ\mathcal{G}(M))\big)_{ji} &= \sum_{k=1}^{m} \overline{\mathcal{G}}(N_{jk})\circ\overline{\mathcal{G}}(M_{ki}) \\
&= \sum_{k=1}^{m} \Big(\sum_{b\in N_{jk}} \underline{\overline{b}}\Big)\circ\Big(\sum_{a\in M_{ki}} \underline{\overline{a}}\Big) \\
&= \sum_{k=1}^{m} \sum_{a\in M_{ki}} \Big(\sum_{b\in N_{jk}} \underline{\overline{b}}\Big)\circ\underline{\overline{a}} && \text{(9c)} \\
&= \sum_{k=1}^{m} \sum_{a\in M_{ki}} \sum_{b\in N_{jk}} \underline{\overline{b}}\circ\underline{\overline{a}} && \text{(9d)} \\
&= \sum_{k=1}^{m} \sum_{a\in M_{ki}} \sum_{b\in N_{jk}} \underline{\overline{b\circ a}} && \text{(Tape)} \\
&= \overline{\mathcal{G}}\big((N\circ M)_{ji}\big) \\
&= \mathcal{M}(\mathcal{G}(N\circ M))_{ji}.
\end{aligned}
$$

Hence $\mathcal{G}(N)\circ\mathcal{G}(M) = \mathcal{G}(N\circ M)$ by Corollary 6.5.

Regarding preservation of biproducts, if $M\colon \bigoplus_{k=1}^{n} A_k \to \bigoplus_{k=1}^{n'} A'_k$ and $N\colon \bigoplus_{k=1}^{m} B_k \to \bigoplus_{k=1}^{m'} B_k$ are any two matrices, we have that

$$
\mathcal{G}(M\oplus N) = \mathcal{G}\begin{pmatrix} M & \emptyset \\ \emptyset & N \end{pmatrix} \tag{Block matrix}
$$

thus $\mathcal{G}(M\oplus N)$ is the morphism in $\mathbf{T}_\Sigma$ whose associated matrix is

$$
\begin{pmatrix}
\overline{\mathcal{G}}(M_{11}) & \cdots & \overline{\mathcal{G}}(M_{1n}) & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\overline{\mathcal{G}}(M_{n'1}) & \cdots & \overline{\mathcal{G}}(M_{n'n}) & 0 & \cdots & 0 \\
0 & \cdots & 0 & \overline{\mathcal{G}}(N_{11}) & \cdots & \overline{\mathcal{G}}(N_{1m}) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \overline{\mathcal{G}}(N_{m'1}) & \cdots & \overline{\mathcal{G}}(N_{m'm})
\end{pmatrix}
$$

which is exactly the block matrix $\begin{pmatrix} \mathcal{M}(\mathcal{G}(M)) & 0 \\ 0 & \mathcal{M}(\mathcal{G}(N)) \end{pmatrix}$, which in turn is $\mathcal{M}(\mathcal{G}(M)\oplus\mathcal{G}(N))$ by Proposition E.2 (1). Hence $\mathcal{G}(M\oplus N) = \mathcal{G}(M)\oplus\mathcal{G}(N)$.

In light of this, in order to check that $\mathcal{G}$ preserves identities it suffices to make sure that $\mathcal{G}(id_A) = id_A$ for A object in $\mathbf{C}$ (seen as a unary list). And indeed $\mathcal{G}(id_A)$ is the morphism whose associated matrix is the 1×1 matrix $\begin{pmatrix} \{\!|\, id_A \,|\!\} \end{pmatrix}$, hence $\mathcal{G}(id_A) = id_A$. $\square$

Proposition E.4. $\mathcal{F}$ and $\mathcal{G}$ are mutually inverse.

Proof. It is not difficult to prove by induction on f that $\mathcal{GF}(f) = f$, using Proposition E.2. We show here that $\mathcal{FG} = id_{\mathbf{Mat(C^+)}}$: let $M \colon \bigoplus_{i=1}^{n} A_i \to \bigoplus_{j=1}^{m} B_j$. First notice that

$$\mathcal{G}(M) = [\langle \overline{\mathcal{G}}(M_{11}), \ldots, \overline{\mathcal{G}}(M_{n1}) \rangle, \ldots, \langle \overline{\mathcal{G}}(M_{1m}), \ldots, \overline{\mathcal{G}}(M_{mn}) \rangle]$$

by Lemma 6.4 and Proposition 6.6. Then

$$\begin{aligned}
\mathcal{FG}(M) &= \mathcal{F}\Big([\langle \sum_{f \in M_{11}} \underline{\overline{f}}, \ldots, \sum_{f \in M_{n1}} \underline{\overline{f}} \rangle, \ldots, \langle \sum_{f \in M_{1m}} \underline{\overline{f}}, \ldots, \sum_{f \in M_{mn}} \underline{\overline{f}} \rangle]\Big) && \text{(Definition of } \mathcal{G}) \\
&= \begin{pmatrix} \mathcal{F}\big(\sum_{f \in M_{11}} \underline{\overline{f}}\big) & \cdots & \mathcal{F}\big(\sum_{f \in M_{1m}} \underline{\overline{f}}\big) \\ \vdots & \ddots & \vdots \\ \mathcal{F}\big(\sum_{f \in M_{n1}} \underline{\overline{f}}\big) & \cdots & \mathcal{F}\big(\sum_{f \in M_{mn}} \underline{\overline{f}}\big) \end{pmatrix} && \text{(Remark 6.11)} \\
&= M
\end{aligned}$$

because for every $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$

$$\begin{aligned}
\mathcal{F}\big(\sum_{f \in M_{ji}} \underline{\overline{f}} \big) &= \sum_{f \in M_{ji}} \mathcal{F}(\underline{\overline{f}}) && \text{(Remark 6.10)} \\
&= \sum_{f \in M_{ji}} \begin{pmatrix} \{\!|\, f \,|\!\} \end{pmatrix} \\
&= M_{ji}
\end{aligned}$$

where above $\sum_{f \in M_{ji}} \begin{pmatrix} \{\!|\, f \,|\!\} \end{pmatrix}$ is a entry-by-entry sum of 1×1 matrices, which yields the matrix whose only entry is $\sum_{f \in M_{ji}} \{\!|\, f \,|\!\}$: a sum of multisets equal to M_{ji} itself. $\square$

E.1 Proof of Theorem 6.12

We begin by proving some preliminary results.

Proposition E.5. Let $M \colon P \to P'$ and $N \colon Q \to Q'$, with $P = \bigoplus_{i=1}^{n} U_i$, $P' = \bigoplus_{i'=1}^{n'} U'_{i'}$, $Q = \bigoplus_{j=1}^{m} V_j$ and $Q' = \bigoplus_{j'=1}^{m'} V'_{j'}$. Then

$$\begin{aligned}
M \circledast N &= (id_{P'} \circledast N) \circ (M \circledast id_Q) \\
&= (M \circledast id_{Q'}) \circ (id_P \circledast N)
\end{aligned}$$

Proof. We only show the first equality: the other is analogous. We have:

$$id_{P'} \circledast N = \begin{pmatrix} \{\!| id_{U'_1} |\!\} \otimes N & \emptyset \otimes N & \cdots & \emptyset \otimes N \\ \emptyset \otimes N & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \emptyset \otimes N \\ \emptyset \otimes N & \cdots & \emptyset \otimes N & \left\{\!\left| id_{U'_{n'}} \right|\!\right\} \otimes N \end{pmatrix}$$

Notice that $\emptyset \otimes N$ is $\emptyset_{m' \times m}$, the $m' \times m$ matrix whose every entry is the empty multiset. Next,

$$M \circledast id_Q = \begin{pmatrix} M_{11} \otimes id_Q & \cdots & M_{1n} \otimes id_Q \\ \vdots & \ddots & \vdots \\ M_{n'1} \otimes id_Q & \cdots & M_{n'n} \otimes id_Q \end{pmatrix}$$

(in the above $id_{P'}$ and id_Q are identity morphisms in $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$, hence they are matrices). To save space, we shall drop the multiset brackets in $\left\{\!\left| id_{U'_{i'}} \right|\!\right\}$, since it is a singleton, and simply write $id_{U'_{i'}}$. Therefore $(id_{P'} \circledast N) \circ (M \circledast id_Q)$ is equal to:

$$\begin{pmatrix} (id_{U'_1} \otimes N) \circ (M_{11} \otimes id_Q) & (id_{U'_1} \otimes N) \circ (M_{12} \otimes id_Q) & \cdots & (id_{U'_1} \otimes N) \circ (M_{1n} \otimes id_Q) \\ (id_{U'_2} \otimes N) \circ (M_{21} \otimes id_Q) & (id_{U'_2} \otimes N) \circ (M_{22} \otimes id_Q) & \cdots & (id_{U'_2} \otimes N) \circ (M_{2n} \otimes id_Q) \\ \vdots & \vdots & \ddots & \vdots \\ (id_{U_1n''} \otimes N) \circ (M_{n'1} \otimes id_Q) & (id_{U'_{n'}} \otimes N) \circ (M_{n'2} \otimes id_Q) & \cdots & (id_{U'_{n'}} \otimes N) \circ (M_{n'n} \otimes id_Q) \end{pmatrix}$$

In other words, $(id_{P'} \circledast N) \circ (M \circledast id_Q)$ is a matrix consisting of $n'n$ blocks (the entries in the matrix above), each of size $m' \times m$, where the block at row i' and column i is

$$B^{i'i} = \underbrace{(id_{U'_{i'}} \otimes N)}_{m' \times m} \circ \underbrace{(M_{i'i} \otimes id_Q)}_{m' \times m}.$$

The entry (j', j) of the above block is the following sum of composite of multisets:

$$\begin{aligned} B^{i'i}_{j'j} &= \sum_{k=1}^{m} (id_{U'_{i'}} \otimes N)_{j'k} \circ (M_{i'i} \otimes id_Q)_{kj} \\ &= \sum_{k=1}^{m} (id_{U'_{i'}} \otimes N_{j'k}) \circ (M_{i'i} \otimes (id_Q)_{kj}) \end{aligned}$$

Now, if $k \neq j$ then $(id_Q)_{kj} = \emptyset$, hence $M_{i'i} \otimes \emptyset = \emptyset$ and $(id_{U'_{i'}} \otimes N_{j'k}) \circ \emptyset = \emptyset$. Therefore,

$$
\begin{aligned}
B^{i'i}_{j'j} &= (id_{U'_{i'}} \otimes N_{j'j}) \circ (M_{i'i} \otimes id_{V_j}) \\
&= \left\{\!\!\left|\, id_{U'_{i'}} \otimes_{\mathbf{C}_\Sigma} b \mid b \in N_{j'j} \,\right|\!\!\right\} \circ \{\!|\, a \otimes_{\mathbf{C}_\Sigma} id_{V_j} \mid a \in M_{i'i} \,|\!\} && (\text{Definition of } \otimes_{\mathbf{C}^+_\Sigma}) \\
&= \left\{\!\!\left|\, (id_{U'_{i'}} \otimes_{\mathbf{C}_\Sigma} b) \circ (a \otimes_{\mathbf{C}_\Sigma} id_{V_j}) \mid a \in M_{i'i},\, b \in N_{j'j} \,\right|\!\!\right\} && (\text{Definition of } \circ \text{ in } \mathbf{C}^+_\Sigma) \\
&= \{\!|\, a \otimes_{\mathbf{C}_\Sigma} b \mid a \in M_{i'i},\, b \in N_{j'j} \,|\!\} && (\text{Functoriality of } \otimes_{\mathbf{C}_\Sigma}) \\
&= (M \circledast N)_{(i',j'),(i,j)}. && \square
\end{aligned}
$$

Proposition E.6. Let $P = \bigoplus_{i=1}^{n} U_i$ and $Q = \bigoplus_{j=1}^{m} V_j$. Then $id_P \circledast id_Q = id_{P \circledast Q}$.

Proof. We have:

$$
\begin{aligned}
id_P \circledast id_Q &= \begin{pmatrix} (id_P)_{11} \otimes id_Q & \cdots & (id_P)_{1n} \otimes id_Q \\ \vdots & \ddots & \vdots \\ (id_P)_{n1} \otimes id_Q & \cdots & (id_P)_{nn} \otimes id_Q \end{pmatrix} \\
&= \begin{pmatrix} \{\!|\, id_{U_1} \,|\!\} \otimes id_Q & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, id_{U_n} \,|\!\} \otimes id_Q \end{pmatrix} \\
&= \begin{pmatrix} \{\!|\, id_{U_1} \otimes id_{V_1} \,|\!\} & \emptyset & \cdots & \cdots & \emptyset \\ \emptyset & \ddots & \{\!|\, id_{U_1} \otimes id_{V_m} \,|\!\} & \ddots & \vdots \\ \vdots & \ddots & \{\!|\, id_{U_n} \otimes id_{V_1} \,|\!\} & \ddots & \emptyset \\ \emptyset & \cdots & \cdots & \emptyset & \{\!|\, id_{U_n} \otimes id_{V_m} \,|\!\} \end{pmatrix} \\
&= id_{\bigoplus_i \bigoplus_j U_i V_j} \\
&= id_{P \circledast Q}. \qquad \square
\end{aligned}
$$

Next, observe that $\mathbf{Mat}(\mathbf{C}^+_\Sigma)$ inherits a tensor functor $\otimes_{\mathbf{Mat}}$ from $\mathbf{T}_\Sigma$, via the isomorphism (32), defined as

$$M \otimes_{\mathbf{Mat}} N \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{G}(M) \otimes_{\mathbf{T}_\Sigma} \mathcal{G}(N)).$$

This makes $\mathbf{Mat}(\mathbf{C}^+_\Sigma)$ a strict rig category using the same argument of the proof of Theorem 5.6. Now we show that $\otimes_{\mathbf{Mat}} = \circledast$.

On objects: if $P = \bigoplus_{i=1}^{n} U_i$ and $Q = \bigoplus_{j=1}^{m} V_j$ we have $P \otimes_{\mathbf{Mat}} Q = \mathcal{F}(P \otimes_{\mathbf{T}_\Sigma} Q) = \mathcal{F}(\bigoplus_i \bigoplus_j U_i V_j = P \circledast Q$.

Claim: $\mathcal{F}(\mathfrak{t} \otimes_{\mathbf{T}_\Sigma} id) = \mathcal{F}(\mathfrak{t}) \circledast id$ and $\mathcal{F}(id \otimes_{\mathbf{T}_\Sigma} \mathfrak{t}) = id \circledast \mathcal{F}(\mathfrak{t})$ for every $\mathfrak{t} \in \mathbf{T}_\Sigma$.

The claim will allow us to conclude the proof, because we will have:

$$\begin{aligned}
M \otimes_{\mathbf{Mat}} N &= \mathcal{F}(\mathcal{G}(M) \otimes_{\mathbf{T}_\Sigma} \mathcal{G}(N)) && \text{(Definition of } \otimes_{\mathbf{Mat}}\text{)} \\
&= \mathcal{F}\big((id \otimes_{\mathbf{T}_\Sigma} \mathcal{G}(N)) \circ (\mathcal{G}(M) \otimes_{\mathbf{T}_\Sigma} id)\big) && \text{(Functoriality of } \otimes_{\mathbf{T}_\Sigma}\text{)} \\
&= \mathcal{F}\big((id \otimes_{\mathbf{T}_\Sigma} \mathcal{G}(N))\big) \circ \mathcal{F}\big(\mathcal{G}(M) \otimes_{\mathbf{T}_\Sigma} id\big) && \text{(Functoriality of } \mathcal{F}\text{)} \\
&= \big(id \circledast \mathcal{F}\mathcal{G}(N)\big) \circ \big(\mathcal{F}\mathcal{G}(M) \circledast id\big) && \text{(Claim)} \\
&= (id \circledast N) \circ (M \circledast id) && (\mathcal{F}\mathcal{G} = id) \\
&= M \circledast N && \text{(Proposition E.5)}
\end{aligned}$$

We now proceed to proving the claim. First of all, if $\mathfrak{t} = id_P$ say, then

$$\begin{aligned}
\mathcal{F}(id_P \otimes_{\mathbf{T}_\Sigma} id_R) &= \mathcal{F}(id_{P \otimes_{\mathbf{T}_\Sigma} R}) && \text{(Functoriality of } \otimes_{\mathbf{T}_\Sigma}\text{)} \\
&= id_{P \otimes_{\mathbf{Mat}} R} && \text{(Functoriality of } \mathcal{F}\text{)} \\
&= id_{P \circledast R} && (\otimes_{\mathbf{Mat}} \text{ and } \circledast \text{ coincide on objects)} \\
&= id_P \circledast id_R \quad . && \text{(Proposition E.6)}
\end{aligned}$$

therefore the claim holds.

Next, suppose $\mathfrak{t} \neq id$. Then $\mathfrak{t}$ can be written as a composite of morphisms of the form $id_P \oplus \mathfrak{t}' \oplus id_Q$ with $\mathfrak{t}' = \underline{\overline{c}}$, $\sigma^{\oplus}_{U,V}$, $\blacktriangleleft_U$, $!_U$, $\blacktriangleright_U$, $¡_U$ with U, V tensors of basic sorts. Now, since $\otimes_{\mathbf{T}_\Sigma}$ is right distributive,

$$(id_P \oplus \mathfrak{t}' \oplus id_Q) \otimes_{\mathbf{T}_\Sigma} id_R = (id_P \otimes_{\mathbf{T}_\Sigma} id_R) \oplus (\mathfrak{t}' \otimes_{\mathbf{T}_\Sigma} id_R) \oplus (id_Q \otimes_{\mathbf{T}_\Sigma} id_R).$$

Since $\mathcal{F}$ preserves biproducts, it suffices to prove that

$$\mathcal{F}(\mathfrak{t} \otimes id_R) = \mathcal{F}(\mathfrak{t}) \circledast id_R$$

for $\mathfrak{t} = \underline{\overline{c}}$, $\sigma^{\oplus}_{U,V}$, $\blacktriangleleft_U$, $!_U$, $\blacktriangleright_U$ $¡_U$ and $R = \bigoplus_{k=1}^{l} Z_k$ in order to prove the first half of the claim. We will refer to Table 7 throughout the proof.

Case 1: $\mathfrak{t} = \underline{\overline{c}}\colon U \to U'$. We have that

$$\mathfrak{t} \otimes_{\mathbf{T}_\Sigma} id_R = \bigoplus_{k=1}^{l} \underline{\overline{c \otimes_{\mathbf{C}_\Sigma} id_{Z_k}}} \colon \bigoplus_{k=1}^{l} U Z_k \to \bigoplus_{k=1}^{l} U' Z_k.$$

Therefore

$$
\begin{aligned}
\mathcal{F}(\mathfrak{t} \otimes_{\mathbf{T}_\Sigma} id_R) &= \begin{pmatrix} \mathcal{F}(\overline{\underline{c \otimes_{\mathbf{C}_\Sigma} id_{Z_1}}}) & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \mathcal{F}(\overline{\underline{c \otimes_{\mathbf{C}_\Sigma} id_{Z_l}}}) \end{pmatrix} \\
&= \begin{pmatrix} \{\!\!|\, c \otimes_{\mathbf{C}_\Sigma} id_{Z_1} \,|\!\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!\!|\, c \otimes_{\mathbf{C}_\Sigma} id_{Z_l} \,|\!\!\} \end{pmatrix} \\
&= \left(\{\!\!|\, c \,|\!\!\} \right) \circledast \begin{pmatrix} \{\!\!|\, id_{Z_1} \,|\!\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!\!|\, id_{Z_l} \,|\!\!\} \end{pmatrix} \\
&= \mathcal{F}(\overline{\underline{c}}) \circledast id_R.
\end{aligned}
$$

Case 2: $\mathsf{t} = \sigma^{\oplus}_{U,V}$. We have:

$$
\begin{aligned}
\mathcal{F}(\mathsf{t} \otimes_{\mathbf{T}_\Sigma} id_R) &= \mathcal{F}(\sigma^{\oplus}_{\bigoplus_k UZ_k, \bigoplus_k VZ_k}) \\
&= \begin{pmatrix} \emptyset_{l\times l} & id_{\bigoplus_k VZ_k} \\ id_{\bigoplus_k UZ_k} & \emptyset_{l\times l} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{F}(\overline{\underline{c \otimes_{\mathbf{C}_\Sigma} id_{Z_1}}}) & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \mathcal{F}(\overline{\underline{c \otimes_{\mathbf{C}_\Sigma} id_{Z_l}}}) \end{pmatrix} \\
&= \left(\begin{array}{c|c} \emptyset & \begin{matrix} \{\!|\, id_{VZ_1} \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, id_{VZ_l} \,|\!\} \end{matrix} \\ \hline \begin{matrix} \{\!|\, id_{UZ_1} \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, id_{UZ_l} \,|\!\} \end{matrix} & \emptyset \end{array}\right) \\
&= \begin{pmatrix} \emptyset & \{\!|\, id_V \,|\!\} \\ \{\!|\, id_U \,|\!\} & \emptyset \end{pmatrix} \circledast \begin{pmatrix} \{\!|\, id_{Z_1} \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, id_{Z_l} \,|\!\} \end{pmatrix} \\
&= \mathcal{F}(\sigma^{\oplus}_{U,V}) \circledast id_R.
\end{aligned}
$$

Case 3: $\mathfrak{t} = \blacktriangleleft_U$. We have:

$$\begin{aligned}
\mathcal{F}(\blacktriangleleft_U \otimes_{\mathbf{T}_\Sigma} id_R) &= \mathcal{F}(\blacktriangleleft_{\bigoplus_k UZ_k}) \\
&= \begin{pmatrix}
\{\!|\, id_{UZ_1} \,|\!\} & \emptyset & \cdots & \emptyset \\
\emptyset & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \emptyset \\
\emptyset & \cdots & \emptyset & \{\!|\, id_{UZ_l} \,|\!\} \\
\{\!|\, id_{UZ_1} \,|\!\} & \emptyset & \cdots & \emptyset \\
\emptyset & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \emptyset \\
\emptyset & \cdots & \emptyset & \{\!|\, id_{UZ_l} \,|\!\}
\end{pmatrix} \\
&= \begin{pmatrix}
\{\!|\, id_U \,|\!\} \\
\{\!|\, id_U \,|\!\}
\end{pmatrix} \circledast \begin{pmatrix}
\{\!|\, id_{UZ_1} \,|\!\} & \emptyset & \cdots & \emptyset \\
\emptyset & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \emptyset \\
\emptyset & \cdots & \emptyset & \{\!|\, id_{UZ_l} \,|\!\}
\end{pmatrix} \\
&= \mathcal{F}(\blacktriangleleft_U) \circledast id_R.
\end{aligned}$$

Case 4: $\mathfrak{t} = !_U$. We have:

$$\begin{aligned}
\mathcal{F}(!_U \otimes_{\mathbf{T}_\Sigma} id_R) = \mathcal{F}(!_{\bigoplus_k UZ_k}) = \emptyset_{0\times l} = \emptyset_{0\times 1} \circledast & \begin{pmatrix}
\{\!|\, id_{Z_1} \,|\!\} & \emptyset & \cdots & \emptyset \\
\emptyset & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \emptyset \\
\emptyset & \cdots & \emptyset & \{\!|\, id_{Z_l} \,|\!\}
\end{pmatrix} \\
= \mathcal{F}(!_U) \circledast id_R. &
\end{aligned}$$

The other cases concering $\blacktriangleright_U$ and $¡_U$ are analogous. Now we prove that

$$\mathcal{F}(id_R \otimes_{\mathbf{T}_\Sigma} (id_P \oplus \mathfrak{t} \oplus id_Q)) = id_R \circledast \mathcal{F}(id_P \oplus \mathfrak{t} \oplus Q)$$

with $P = \bigoplus_{i=1}^{n} X_i$, $Q = \bigoplus_{j=1}^{m} Y_j$, $R = \bigoplus_{k=1}^{l} Z_k$. To lighten notation, for the rest of the proof we will often denote identity morphisms with their corresponding objects, that is we will simply write X instead of id_X.

Case 1: $\mathfrak{t} = \overline{\underline{c}}\colon U \to V$.

$$
\begin{aligned}
\mathcal{F}(R \otimes_{\mathbf{T}_\Sigma} (P \oplus \mathfrak{t} \oplus Q)) &= \mathcal{F}\Big(\bigoplus_{k=1}^{l} (\bigoplus_{i=1}^{n} Z_k X_i \oplus \overline{\underline{Z_k \otimes_{\mathbf{C}_\Sigma} c}} \oplus \bigoplus_{j=1}^{m} Z_k Y_j)\Big) \\
&= \bigoplus_{k=1}^{l} \left[\bigoplus_{i=1}^{n} \Big(\{\!| \, Z_k X_i \, |\!\} \Big) \oplus \Big(\{\!| \, Z_k \otimes_{\mathbf{C}_\Sigma} c \, |\!\} \Big) \oplus \bigoplus_{j=1}^{m} \Big(\{\!| \, Z_k Y_j \, |\!\} \Big) \right] \\
&= \bigoplus_{k=1}^{l} \begin{pmatrix}
\{\!| \, Z_k X_1 \, |\!\} & \emptyset & \cdots & \cdots & \emptyset \\
\emptyset & \{\!| \, Z_k X_n \, |\!\} & & & \vdots \\
\vdots & & \{\!| \, Z_k \otimes_{\mathbf{C}_\Sigma} c \, |\!\} & & \vdots \\
\vdots & & & \{\!| \, Z_k Y_1 \, |\!\} & \emptyset \\
\emptyset & \cdots & \cdots & \emptyset & \{\!| \, Z_k Y_m \, |\!\}
\end{pmatrix} \\
&= id_R \circledast \left(\begin{array}{ccc:c:ccc}
 & & & \emptyset & & & \\
 & id_P & & \vdots & & \emptyset & \\
 & & & \emptyset & & & \\ \hdashline
\emptyset & \cdots & \emptyset & c & \emptyset & \cdots & \emptyset \\ \hdashline
 & & & \emptyset & & & \\
 & \emptyset & & \vdots & & id_Q & \\
 & & & \emptyset & & &
\end{array} \right) \\
&= id_R \circledast \mathcal{F}(P \oplus \mathfrak{t} \oplus Q).
\end{aligned}
$$

Case 2: $\mathfrak{t} = \sigma^{\oplus}_{U,V}$.

$$
\begin{aligned}
&\mathcal{F}(R \otimes_{\mathbf{T}_\Sigma} (P \oplus \mathfrak{t} \oplus Q)) \\
&= \bigoplus_{k=1}^{l} \left[\bigoplus_{i=1}^{n} \Big(\{\!|\, Z_k X_i \,|\!\} \Big) \oplus \begin{pmatrix} \emptyset & \{\!|\, Z_k V \,|\!\} \\ \{\!|\, Z_k U \,|\!\} & \emptyset \end{pmatrix} \oplus \bigoplus_{j=1}^{m} \Big(\{\!|\, Z_k Y_j \,|\!\} \Big) \right] \\
&= \bigoplus_{k=1}^{l} \left(\begin{array}{c:c:c}
\begin{matrix} \{\!|\, Z_k X_1 \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, Z_k X_n \,|\!\} \end{matrix} & \emptyset & \emptyset \\ \hdashline
\emptyset & \begin{matrix} \emptyset & \{\!|\, Z_k V \,|\!\} \\ \{\!|\, Z_k U \,|\!\} & \emptyset \end{matrix} & \emptyset \\ \hdashline
\emptyset & \emptyset & \begin{matrix} \{\!|\, Z_k Y_1 \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, Z_k Y_m \,|\!\} \end{matrix}
\end{array} \right) \\
&= id_R \circledast \left(\begin{array}{c:c:c}
\begin{matrix} \{\!|\, X_1 \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, X_n \,|\!\} \end{matrix} & \emptyset & \emptyset \\ \hdashline
\emptyset & \begin{matrix} \emptyset & \{\!|\, V \,|\!\} \\ \{\!|\, U \,|\!\} & \emptyset \end{matrix} & \emptyset \\ \hdashline
\emptyset & \emptyset & \begin{matrix} \{\!|\, Y_1 \,|\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!|\, Y_m \,|\!\} \end{matrix}
\end{array} \right) \\
&= id_R \circledast \mathcal{F}(P \oplus \mathfrak{t} \oplus Q).
\end{aligned}
$$

Case 3: $\mathsf{t} = \blacktriangleleft_U$.

$$\mathcal{F}(R \otimes_{\mathbf{T}_\Sigma} (P \oplus \mathsf{t} \oplus Q))$$

$$= \bigoplus_{k=1}^{l} \left[\bigoplus_{i=1}^{n} \Big(\{\!| Z_k X_i |\!\} \Big) \oplus \begin{pmatrix} \{\!| Z_k U |\!\} \\ \{\!| Z_k U |\!\} \end{pmatrix} \oplus \bigoplus_{j=1}^{m} \Big(\{\!| Z_k Y_j |\!\} \Big) \right]$$

$$= \bigoplus_{k=1}^{l} \left(\begin{array}{c:c:c} \begin{matrix} \{\!| Z_k X_1 |\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!| Z_k X_n |\!\} \end{matrix} & \emptyset & \emptyset \\ \hdashline \emptyset & \begin{matrix} \{\!| Z_k U |\!\} \\ \{\!| Z_k U |\!\} \end{matrix} & \emptyset \\ \hdashline \emptyset & \emptyset & \begin{matrix} \{\!| Z_k Y_1 |\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!| Z_k Y_m |\!\} \end{matrix} \end{array} \right)$$

$$= id_R \circledast \left(\begin{array}{c:c:c} \begin{matrix} \{\!| X_1 |\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!| X_n |\!\} \end{matrix} & \emptyset & \emptyset \\ \hdashline \emptyset & \begin{matrix} \{\!| U |\!\} \\ \{\!| U |\!\} \end{matrix} & \emptyset \\ \hdashline \emptyset & \emptyset & \begin{matrix} \{\!| Y_1 |\!\} & \emptyset & \cdots & \emptyset \\ \emptyset & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \emptyset \\ \emptyset & \cdots & \emptyset & \{\!| Y_m |\!\} \end{matrix} \end{array} \right)$$

$$= id_R \circledast \mathcal{F}(P \oplus \mathsf{t} \oplus Q).$$

Case 4: $\mathsf{t} = !_U$.

$$
\begin{aligned}
&\mathcal{F}(R \otimes_{\mathbf{T}_\Sigma} (P \oplus \mathsf{t} \oplus Q))\\
&= \bigoplus_{k=1}^{l} \left[\bigoplus_{i=1}^{n} \Big(\{\!|\, Z_k X_i \,|\!\} \Big) \oplus \emptyset_{0\times 1} \oplus \bigoplus_{j=1}^{m} \Big(\{\!|\, Z_k Y_j \,|\!\} \Big) \right]\\
&= \bigoplus_{k=1}^{l} \left(\begin{array}{cccc:cccc}
\{\!|\, Z_k X_1 \,|\!\} & \emptyset & \cdots & \emptyset & & & & \\
\emptyset & \ddots & \ddots & \vdots & & & & \\
\vdots & \ddots & \ddots & \emptyset & & \multicolumn{2}{c}{\emptyset} & \\
\emptyset & \cdots & \emptyset & \{\!|\, Z_k X_n \,|\!\} & & & & \\
\hdashline
 & & & & \{\!|\, Z_k Y_1 \,|\!\} & \emptyset & \cdots & \emptyset \\
 & \multicolumn{2}{c}{\emptyset} & & \emptyset & \ddots & \ddots & \vdots \\
 & & & & \vdots & \ddots & \ddots & \emptyset \\
 & & & & \emptyset & \cdots & \emptyset & \{\!|\, Z_k X Y_m \,|\!\}
\end{array} \right)\\
&= id_R \circledast (P \oplus \emptyset_{0\times 1} \oplus Q).
\end{aligned}
$$

The other cases concerning $\mathsf{t} = \blacktriangleright_U$ and $\mathsf{t} = ¡_U$ are analogous. This concludes the proof of the claim, thus of the Theorem.

F Proofs of Section 8

In this appendix, we illustrate the proof of Theorem 8.2 in Section 8. First, it is convenient to prove two lemmas.

Lemma F.1. *If $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$, then $R_U(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} R_U(\mathsf{t}_2)$ for all monomials U.*

Proof. We proceed by induction on the rule generating $\lesssim_{\hat{\mathcal{I}}}$.

If $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$ by rule $(\hat{\mathcal{I}})$, then $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$, i.e., there exists $(\mathsf{t}_1', \mathsf{t}_2') \in \mathcal{I}$ and $V \in S^*$ such that $\mathsf{t}_i = R_V(\mathsf{t}_i')$. Thus $R_U(\mathsf{t}_i) = R_U(R_V(\mathsf{t}_i')) \stackrel{(W14)}{=} R_{VU}(\mathsf{t}_i')$. Since $(\mathsf{t}_1', \mathsf{t}_2') \in \mathcal{I}$, by definition of $\hat{\mathcal{I}}$, it holds that $\mathsf{t}_1 \hat{\mathcal{I}} \mathsf{t}_2$ and thus, by rule $(\hat{\mathcal{I}})$, it holds that $R_U(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} R_U(\mathsf{t}_2)$

If $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$ by rule (t), then there exists t such that $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}$ and $\mathsf{t} \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$. By induction hypothesis, it holds that $R_U(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} R_U(\mathsf{t})$ and $R_U(\mathsf{t}) \lesssim_{\hat{\mathcal{I}}} R_U(\mathsf{t}_2)$. Thus by rule (t), $R_U(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} R_U(\mathsf{t}_2)$.

The cases for ; and $\oplus$ exploit induction in the same way and the laws (W2) and (W5) expressing the fact that $R_U(-)$ preserves both ; and $\oplus$. The case for (r) is trivial. $\square$

Lemma F.2. *If $\mathsf{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathsf{t}_2$, then $R_S(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} R_S(\mathsf{t}_2)$ and $L_S(\mathsf{t}_1) \lesssim_{\hat{\mathcal{I}}} L_S(\mathsf{t}_2)$ for all polynomials S.*

Proof. We first show the case for $R_S(-)$. The proof proceed by induction on S.

Case $S = 0$: $R_0(\mathsf{t}_1) = id_0 \lesssim_{\hat{\mathcal{I}}} id_0 = R_0(\mathsf{t}_2)$.

Case $S = W \oplus S'$**:**

$$
\begin{aligned}
R_{W\oplus S'}(\mathfrak{t}_1) &= \delta^l_{P,W,S'}; (R_W(\mathfrak{t}_1) \oplus R_{S'}(\mathfrak{t}_1)); \delta^{-l}_{Q,W,S'} && \text{(Def. } R\text{)}\\
&\lesssim_{\hat{\mathcal{I}}} \delta^l_{P,W,S'}; (R_W(\mathfrak{t}_2) \oplus R_{S'}(\mathfrak{t}_1)); \delta^{-l}_{Q,W,S'} && \text{(Lemma F.1, Rules } (r), (;), (\oplus))\\
&\lesssim_{\hat{\mathcal{I}}} \delta^l_{P,W,S'}; (R_W(\mathfrak{t}_2) \oplus R_{S'}(\mathfrak{t}_2)); \delta^{-l}_{Q,W,S'} && \text{(Ind. Hyp., Rules } (r), (;), (\oplus))\\
&= R_{W\oplus S'}(\mathfrak{t}_2) && \text{(Def. } R\text{)}
\end{aligned}
$$

The proof for $L_S(-)$, it trivial using the result for $R_S(-)$ and (W11). □

Proof of Theorem 8.2. To prove that $\leq_{\mathcal{I}} \subseteq \lesssim_{\hat{\mathcal{I}}}$, first observe that $\mathcal{I} \subseteq \hat{\mathcal{I}}$ by (W3). Thus, to conclude it is enough to show that $\lesssim_{\hat{\mathcal{I}}}$ is closed under $\otimes$, i.e., that if if $\mathfrak{t}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2$ and $\mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{s}_2$, then $\mathfrak{t}_1 \otimes \mathfrak{s}_1 \lesssim_{\hat{\mathcal{I}}} \mathfrak{t}_2 \otimes \mathfrak{s}_2$. But this is immediate using the definition of $\otimes$ and Lemma F.2.

To prove that $\lesssim_{\hat{\mathcal{I}}} \subseteq \leq_{\mathcal{I}}$, it is enough to show that $\hat{\mathcal{I}} \subseteq \leq_{\mathcal{I}}$: if $\mathfrak{t}_1 \hat{\mathcal{I}} \mathfrak{t}_2$, there there exists $(\mathfrak{t}_1', \mathfrak{t}_2') \in \mathcal{I}$ such that $R_U(\mathfrak{t}_i') = \mathfrak{t}_i$ for some monomial U. Now, by (24), $R_U(\mathfrak{t}_i) = \mathfrak{t}_i \otimes id_U$, thus by rules (r) and $(\otimes)$, $\mathfrak{t}_1 \leq_{\mathcal{I}} \mathfrak{t}_2$. □

G Ghost Track: coherence by term rewriting

The relevance of term rewriting to the issues of coherence has been hinted since at least the eighties (see e.g. [25]) however, only recently this has been made formal in two independent works: [2] and [11]. In this Ghost Track, we recall the basics of term rewriting (Section G.1), we illustrate the theory in [2] (Section G.2) and we exploit such theory to prove Theorem 3.11 (Section G.3).

As a side effect of our investigation, we retrieve a sort of coherence theorem for rig categories. Such result, which we named *right coherence*, is weaker than the two Laplaza coherence theorems [20], but it enjoys a more readily formulation and a simpler proof. Moreover, our approach provides a simple algorithm to construct the normalization morphism $n_X \colon X \to N(X)$: just apply as much as possible associators, unitors, annihilators, right distributors and partial left distributors. Independently from the order, this algorithm will always *terminate* and will always produce the same result. Termination, a concept which is central in computer science but sometimes underestimated in mathematics, is a key ingredient of our approach.

We decided to illustrate this material as a Ghost Track at the end of this report since we believe that it is interesting on its own and the only application for the theory of tapes is in providing a proof of Theorem 3.11. Thus, the uninterested reader can safely skip it. On the other hand, through what follows we will use terminology and notation introduced in Sections 2 and 3, so the reader is recommended to have a look at them first.

Our starting point is the notion of Lawvere theory. Recall that the *Lawvere theory* [21] generated by a single sorted cartesian signature Σ, hereafter denoted by L_Σ, is the free strict finite product category generated by Σ. More explicitly, objects of L_Σ are natural numbers and arrows are tuples of terms over a countable set of variables $V = \{x_1, x_2, \dots\}$: arrows from n to m are tuples $\langle t_1, \dots, t_m\rangle$ where each term t_i has variables in $\{x_1, \dots, x_n\}$. Composition is defined by (simultaneous) substitution: the composition of $\langle t_1, \dots, t_m\rangle \colon n \to m$ with $\langle s_1, \dots, s_l\rangle \colon m \to l$ is the tuple $\langle u_1, \dots, u_l\rangle \colon n \to l$ where $u_i = s_i[^{t_1 \dots t_m}/_{x_1 \dots x_m}]$ for all $i = 1, \dots, l$. One can readily check that L_Σ is a symmetric monoidal category having $(\mathbb{N}, +, 0)$ as the monoid of objects. Given $\langle t_1, \dots, t_m\rangle \colon n \to m$ and $\langle s_1, \dots, s_k\rangle \colon j \to k$, $\langle t_1, \dots, t_m\rangle \odot \langle s_1, \dots, s_k\rangle$ is defined as the tuple $\langle t_1, \dots, t_m, s_1', \dots, s_k'\rangle \colon n+j \to m+k$ where $s_i' = s_i[^{x_{n+1} \dots x_{n+j}}/_{x_1 \dots x_j}]$ for all $i = 1, \dots, k$. Identities

id_n and symmetries $\sigma_{n,m}$ are defined as expected; $\blacktriangleleft_n : n \to n{+}n$ is the tuple $\langle x_1, \dots, x_n, x_1, \dots, x_n\rangle$ thus acting as a *duplicator* of variables; $!_n : n \to 0$ is the empty tuple $\langle\rangle$, acting as a *discharger*.

G.1 A short introduction to term rewriting

A *rewriting rule* consists of a pair of arrows $(l, r) : n \to 1$ in L_Σ for some $n \in \mathbb{N}$. A *rewriting systems* is a set $\mathcal{R}$ of rewriting rules. Any rewriting system $\mathcal{R}$ defines, for each $m \in \mathbb{N}$, a relation $\Rightarrow_{\mathcal{R}} \subseteq L_\Sigma[m,1] \times L_\Sigma[m,1]$ as follows: $s \Rightarrow_{\mathcal{R}} t$ iff there exists $(l,r) : n \to 1$ in $\mathcal{R}$, $o \in \mathbb{N}$, $f : m \to n{+}o$ and $g : 1 + o \to 1$ such that

$$s = f; (l \times id_o); g \quad \text{and} \quad f; (r \times id_o); g = t.$$

Hereafter, we will always drop the subscript $\mathcal{R}$ when clear from the context, we will write $\Rightarrow^*$ for the reflexive and transitive closure of $\Rightarrow$ and $s \not\Rightarrow$ if there exists no t such that $s \Rightarrow t$. For a rewrting rule $(l, r) : n \to 1$ we will often write $l \rightsquigarrow r$. Since rewriting rules will later correspond to natural transformations, we often label them with a greek letter, e.g., $\iota : l \rightsquigarrow r$.

$$\alpha^{\otimes}_{x_1,x_2,x_3} : (x_1 \otimes x_2) \otimes x_3 \rightsquigarrow x_1 \otimes (x_2 \otimes x_3) \qquad \alpha^{\oplus}_{x_1,x_2,x_3} : (x_1 \oplus x_2) \oplus x_3 \rightsquigarrow x_1 \oplus (x_2 \oplus x_3)$$

$$\lambda^{\otimes}_{x_1} : 1 \otimes x_1 \rightsquigarrow x_1 \qquad \rho^{\otimes}_{x_1} : x_1 \otimes 1 \rightsquigarrow x_1 \qquad \lambda^{\oplus}_{x_1} : 0 \oplus x_1 \rightsquigarrow x_1 \qquad \rho^{\oplus}_{x_1} : x_1 \oplus 0 \rightsquigarrow x_1$$

$$\lambda^{\bullet}_{x_1} : 0 \otimes x_1 \rightsquigarrow 0 \qquad \rho^{\bullet}_{x_1} : x_1 \otimes 0 \rightsquigarrow 0$$

$$\delta^{r}_{x_1,x_2,x_3} : (x_1 \oplus x_2) \otimes x_3 \rightsquigarrow (x_1 \otimes x_3) \oplus (x_2 \otimes x_3)$$

Table 9: The rewriting system $\mathcal{R}_{rs}$ for right strict rig categories

Example G.1. Table 9 illustrates the rewriting system $\mathcal{R}_{rs}$ over the signature of rigs $\mathtt{R}$. To understand the definition of $\Rightarrow$, it is convenient to consider as an example the following rewriting step:

$$(0 \otimes (x_1 \oplus x_3)) \oplus x_2 \Rightarrow 0 \oplus x_2.$$

The involved rewriting rule is the one labeled by $\lambda^{\bullet}_{x_1}$, i.e., $(0 \otimes x_1, 0) : 1 \to 1$, o is 1, the arrow f is $\langle x_1 \oplus x_3, x_2\rangle : 3 \to 2$ and g is $\langle x_1 \oplus x_2\rangle : 2 \to 1$. In standard rewriting jargon, g acts as context and f as a substitution. The role of id_o is the one of allowing the context g to access the variables present in the substitution f but not in the rule l: in this case, the variable is x_2.

A span $t_1 \Leftarrow s \Rightarrow t_2$ is said to be *joinable*, if there exists a term u such that $t_1 \Rightarrow^* u \;{}^*\!\Leftarrow t_2$. A rewriting system is *locally confluent* if every span is joinable. A rewriting system is *terminating* if there do not exist infinitely many terms t_i such that $t_0 \Rightarrow t_1 \Rightarrow \dots \Rightarrow t_i \Rightarrow \dots$ A rewriting system that is both terminating and locally confluent is *convergent* in the sense that for every term t there exists a unique term $N(t)$ such that $t \Rightarrow^* N(t)$ and $N(t) \not\Rightarrow$. Such unique $N(t)$ is called the *normal form* of t.

Convergent rewriting systems provide an algorithmic procedure to check equivalence of terms in some algebraic theory (Σ, E). Indeed any equation $l = r$ in E can be oriented either as the rewriting rule $l \rightsquigarrow r$ or as $r \rightsquigarrow l$ and, if the resulting system is congervent, it holds that any two terms s and t are provably equal in E iff $N(s) = N(t)$. For instance, equivalence in the algebraic theory in

Table 5 can be checked with the rewriting system $\mathcal{R}_{rs}$ which, as we will see later, is confluent and terminating. Unfortunately, it is not possible to find a convergent rewriting systems for many sets of equations E.

Example G.2. Let us consider the rewriting system $\mathcal{R}_{rs}$ extended with the rule

$$\delta^l_{x_1,x_2,x_3} : \; x_1 \otimes (x_2 \oplus x_3) \rightsquigarrow (x_1 \otimes x_2) \oplus (x_1 \otimes x_3)$$

that corresponds to left distributivity. Such extended rewriting system is terminating, but not locally confluent. Consider indeed the following span:

$$((x_1 \oplus x_2) \otimes x_3) \oplus ((x_1 \oplus x_2) \otimes x_4) \Leftarrow (x_1 \oplus x_2) \otimes (x_3 \oplus x_4) \Rightarrow (x_1 \otimes (x_3 \oplus x_4)) \oplus (x_2 \otimes (x_3 \oplus x_4))$$

The leftmost step is obtained by using $\delta^l_{x_1,x_2,x_3}$, while the rightmost by $\delta^r_{x_1,x_2,x_3}$. Unfortunately, there is no term u such that

$$((x_1 \oplus x_2) \otimes x_3) \oplus ((x_1 \oplus x_2) \otimes x_4) \Rightarrow^* u \;{}^*\!\Leftarrow (x_1 \otimes (x_3 \oplus x_4)) \oplus (x_2 \otimes (x_3 \oplus x_4))$$

Indeed,

$$((x_1 \oplus x_2) \otimes x_3) \oplus ((x_1 \oplus x_2) \otimes x_4) \Rightarrow^* (x_1 \otimes x_3) \oplus ((x_2 \otimes x_3) \oplus ((x_1 \otimes x_4) \oplus (x_2 \otimes x_4))) \not\Rightarrow$$

while

$$(x_1 \otimes (x_3 \oplus x_4)) \oplus (x_2 \otimes (x_3 \oplus x_4)) \Rightarrow^* (x_1 \otimes x_3) \oplus ((x_1 \otimes x_4) \oplus ((x_2 \otimes x_3) \oplus (x_2 \otimes x_4))) \not\Rightarrow$$

In order to make the system locally confluent, it would be enough to add the following rewriting rule

$$\sigma^{\oplus}_{x_1,x_2} : \; x_1 \oplus x_2 \rightsquigarrow x_2 \oplus x_1$$

asserting that $\oplus$ is commutative. However, such system is clearly not terminating since

$$x_1 \otimes x_2 \Rightarrow x_2 \otimes x_1 \Rightarrow x_1 \otimes x_2 \Rightarrow x_2 \otimes x_1 \Rightarrow \dots$$

Several techniques have been developed to prove termination and local confluence. We refer the interested reader to [1] for an introduction to the subject. The proofs of termination usually exploit some well-founded orderings and there are several automatic tools guaranteeing termination of rewriting systems.[1]

The proofs of local confluence rely on Knuth's critical pair lemma [19]: to guarantee local confluence one has to check joinability for just a few minimal spans $t_1 \Leftarrow s \Rightarrow t_2$ that are called *critical pairs*. To illustrate this notion, consider two rules $(l_1, r_1) \colon n \to 1$ and $(l_2, r_2) \colon m \to 1$ in $\mathcal{R}$ and assume that l_1 factors as $n \xrightarrow{x} 1+o \xrightarrow{f} 1$. Whenever there exists $g_1 \colon p \to n$ and $g_2 \colon p \to m+o$ for which the following diagram commutes:

$$\begin{tikzcd} p \arrow[r, "g_1"] \arrow[d, "g_2"'] & n \arrow[d, "x"] \\ m+o \arrow[r, "l_2 \odot id_o"'] & 1+o \end{tikzcd} \tag{48}$$

[1]Since termination is an undecidable property, such tools are sound but not complete: there might be some terminating rewriting systems for which a tool cannot prove termination.

it holds that $g_1; l_1 = g_2; (l_2 \odot id_o); f$ and we have the following span:

$$g_1; r_1 \Leftarrow g_1; l_1 = g_2; (l_2 \odot id_o); f \Rightarrow g_2; (r_2 \odot id_o); f$$

Such span is a critical pair whenever x is not an isomorphism and the square in (48) is a pullback. The condition on x is necessary to avoid to consider trivial cases: indeed, whenever x is an iso, the above span is always joinable. As far as pullbacks are concerned, observe that they do not exist in general in L_Σ (e.g. for $2 \stackrel{\langle x_1 \otimes x_2 \rangle}{\longrightarrow} 1 \stackrel{\langle x_1 \oplus x_2 \rangle}{\longleftarrow} 2$). However, whenever a cospan $n_1 \xrightarrow{f_1} m \xleftarrow{f_2} n_2$ can be closed to a commutative square (namely there exists g_i such that $g_1; f_1 = g_2; f_2$), then it admits a pullback and this is exactly the *most general unifier* [16, 8] which can be computed via the so called unification algorithm [26]. More generally, given a rewriting system $\mathcal{R}$, the set of all critical pairs can be computed automatically and, if $\mathcal{R}$ is terminating, local confluence can be decided automatically.

We have used the tools in [28, 13] to automatically prove termination and local confluence of $\mathcal{R}_{rs}$: the proof of termination relies on the lexicographic path order [1]; the proof of local confluence checks joinability of the 22 critical pairs generated by the 9 rules of $\mathcal{R}_{rs}$. It is noteworthy that the diagrams joining these critical pairs are, in a sense that we will make precise later, some of the coherence axioms and derived laws in Figures 8 and 9 (pp. 68,69).

Proposition G.3. The rewriting system $\mathcal{R}_{rs}$ is terminating and locally confluent.

Proof. The result have been proven automatically with the tool in [28, 13]. The interested reader can however easily retrieve the proof of such result from the one of Proposition G.10. ☐

Example G.4. There is only one critical pair for the rules

$$\delta^r_{x_1,x_2,x_3} : (x_1 \oplus x_2) \otimes x_3 \rightsquigarrow (x_1 \otimes x_3) \oplus (x_2 \otimes x_3) \qquad \alpha^\oplus_{x_1,x_2,x_3} : (x_1 \oplus x_2) \oplus x_3 \rightsquigarrow x_1 \oplus (x_2 \oplus x_3)$$

The left hand side of $\delta^r_{x_1,x_2,x_3}$, the arrow $(x_1 \oplus x_2) \otimes x_3 \colon 3 \to 1$, can be factorised as $3 \stackrel{\langle x_1 \oplus x_2, x_3 \rangle}{\longrightarrow} 1+1 \stackrel{\langle x_1 \otimes x_2 \rangle}{\longrightarrow} 1$. Moreover, the following diagram is a pullback in L_Σ.

$$\begin{tikzcd}
4 \arrow[r, "\langle x_1 \oplus x_2, x_3, x_4 \rangle"] \arrow[d, "id_4"'] & 3 \arrow[d, "\langle x_1 \oplus x_2, x_3 \rangle"] \\
3+1 \arrow[r, "\langle (x_1 \oplus x_2) \oplus x_3, x_4 \rangle"'] & 1+1
\end{tikzcd}$$

Since $\langle x_1 \oplus x_2, x_3 \rangle \colon 3 \to 1+1$ is not an iso, the span

$$((x_1 \oplus x_2) \otimes x_4) \oplus (x_3 \otimes x_4) \Leftarrow ((x_1 \oplus x_2) \oplus x_3) \otimes x_4 \Rightarrow (x_1 \oplus (x_2 \oplus x_3)) \otimes x_4$$

is a critical pair. Observe that this span can be joined as illustrated by the following diagram and that such diagram closely resemble axiom (R3) in Figure 8.

$$\begin{tikzcd}
((x_1 \oplus x_2) \oplus x_3) \otimes x_4 \arrow[r] \arrow[d] & ((x_1 \oplus x_2) \otimes x_4) \oplus (x_3 \otimes x_4) \arrow[r] & ((x_1 \otimes x_4) \oplus (x_2 \otimes x_4)) \oplus (x_3 \otimes x_4) \arrow[d] \\
(x_1 \oplus (x_2 \oplus x_3)) \otimes x_4 \arrow[r] & (x_1 \otimes x_4) \oplus ((x_2 \oplus x_3) \otimes x_4) \arrow[r] & (x_1 \otimes x_4) \oplus ((x_2 \otimes x_4) \oplus (x_3 \otimes x_4))
\end{tikzcd} \tag{49}$$

G.2 Coherent 2-theories

The main intuition of [2] is that, given a terminating and locally confluent rewriting system $\mathcal{R}$, if *all* diagrams joining critical pairs, like for instance (49), correspond to some commutative diagram, e.g., (R3), then a certain form of coherence holds. To make this precise, it is necessary to introduce some auxiliary definitions. The *graph of terms* of $\mathcal{R}$ is the graph having terms as vertexes and edges from s to t are rewriting steps $s \Rightarrow t$. A *2-theory* is a triple $(\Sigma, \mathcal{R}, C)$ where Σ is a cartesian signature, $\mathcal{R}$ is a rewriting system and C is a set of pairs (p_1, p_2) of paths in the graph of terms: p_1 and p_2 must have the same initial and final vertexes. For instance, diagram (49) is exactly a pair of such paths with initial vertex $((x_1 \oplus x_2) \oplus x_3) \otimes x_4$ and final vertex $(x_1 \otimes x_4) \oplus ((x_2 \otimes x_4) \oplus (x_3 \otimes x_4))$.

A (categorical) interpretation $\mathcal{I}$ of Σ consists of a category $\mathbf{C}$ together with a functor $\mathcal{I}(f)\colon \mathbf{C}^n \to \mathbf{C}$ for each $f \in \Sigma$ of arity n. In terms of Lawvere's functorial sematics [21], one can define an interpretation as a product preserving functor $\mathcal{I}\colon L_\Sigma \to \mathbf{Cat}$ where $\mathbf{Cat}$ is the category of categories and functors. It is thus easy to see that, given an interpretation $\mathcal{I}$, every term $t\colon m \to 1$ in L_Σ corresponds to a functor $\mathcal{I}(t)\colon \mathbf{C}^m \to \mathbf{C}$. A (2-categorical) *interpretation* $\mathcal{I}$ of $(\Sigma, \mathcal{R})$ consists of an interpretation of Σ together with a natural transformation $\mathcal{I}(\iota)\colon \mathcal{I}(l) \Rightarrow \mathcal{I}(r)$ for each rewriting rule $\iota\colon l \rightsquigarrow r$ in $\mathcal{R}$. Such interpretation can be seen as product preserving 2-functor $\mathcal{I}\colon L_\Sigma^{\mathcal{R}} \to \mathbf{Cat}$ where $\mathbf{Cat}$ is the 2-category of categories, functors and natural transformations and $L_\Sigma^{\mathcal{R}}$ is the 2-category freely generated by adding to L_Σ the rules in $\mathcal{R}$ as 2-cells (see e.g. Construction 2.2 in [25]). Given an interpretation $\mathcal{I}$, every rewriting step $s \Rightarrow t$ corresponds to a natural transformation $\mathcal{I}(s) \Rightarrow \mathcal{I}(t)$. More generally, every path $p\colon s \Rightarrow^* t$ in the graph of terms corresponds to a natural transformation $\mathcal{I}(p)\colon \mathcal{I}(s) \Rightarrow \mathcal{I}(t)$. If, for each pair of paths $(p_1, p_2) \in C$, it holds that $\mathcal{I}(p_1) = \mathcal{I}(p_2)$, then $\mathcal{I}$ is said to be a *model* of $(\Sigma, \mathcal{R}, C)$.

Hereafter, we will be interested in models assigning to each rule in $\mathcal{R}$ a natural isomorphism. In this case, any interpretation $\mathcal{I}$ for $\mathcal{R}$ trivially extends to an interpretation for $\mathcal{R} \cup \mathcal{R}^-$ where $\mathcal{R}^- = \{\iota^-\colon r \rightsquigarrow l \mid \iota\colon l \rightsquigarrow r \in \mathcal{R}\}$. Any path $p\colon s \Rightarrow^* t$ in the graph of terms corresponding to $\mathcal{R} \cup \mathcal{R}^-$ is mapped by $\mathcal{I}$ into a natural isomorphism. We call such natural isomorphisms $(\Sigma, \mathcal{R})$-*natural isomorphisms*. Exploiting the characterisation of L_Σ as the free strict fp-category generated by Σ, one can give a simple inductive definition of $(\Sigma, \mathcal{R})$-natural isomorphisms. First, we need to define their source and target functors.

Definition G.5. Let $\mathcal{I}$ be an interpretation for a signature Σ and let $\mathbf{C}$ be the category underlying $\mathcal{I}$ (that is $\mathcal{I}(1) = \mathbf{C}$).

- The identity functor $Id_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}$,
- the functor $\sigma_{\mathbf{C},\mathbf{C}}\colon \mathbf{C^2} \to \mathbf{C^2}$ mapping $(z_1\colon X_1 \to Y_1, z_2\colon X_2 \to Y_2)$ into $(z_2\colon X_2 \to Y_2, z_1\colon X_1 \to Y_1)$,
- the functor $!_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}^0$ defined by finality of $\mathbf{C}^0$,
- the functor $\blacktriangleleft_{\mathbf{C}}\colon \mathbf{C}^2 \to \mathbf{C}$ defined as $\langle Id_{\mathbf{C}}, Id_{\mathbf{C}} \rangle$,
- the functor $\mathcal{I}(f)\colon \mathbf{C}^n \to \mathbf{C}$ for $f \in \Sigma$ of arity n

are called *generating* Σ-*functors*. A Σ-*functor* is either a generating Σ-functor or the product (in $\mathbf{Cat}$, $\times$) or the composition (;) of two Σ-functors.

Definition G.6. Let $\mathcal{I}$ be an interpretation for $(\Sigma, \mathcal{R})$. The class of $(\Sigma, \mathcal{R})$-natural isomorphisms for $\mathcal{I}$ is inductively defined as follows.

- For g a generating Σ-functor, the identity natural transformation $1_g \colon g \Rightarrow g$ is a $(\Sigma, \mathcal{R})$-structural isomorphism.
- For $\iota \colon l \rightsquigarrow r$, $\mathcal{I}(\iota) \colon \mathcal{I}(l) \Rightarrow \mathcal{I}(r)$ is a $(\Sigma, \mathcal{R})$-structural isomorphism.
- For $\iota \colon l \rightsquigarrow r$, $\mathcal{I}(\iota)^- \colon \mathcal{I}(r) \Rightarrow \mathcal{I}(l)$ is a $(\Sigma, \mathcal{R})$-structural isomorphism.
- $(\Sigma, \mathcal{R})$-natural isomorphisms are closed under product, horizontal and vertical composition.

Example G.7. Consider the signature $\mathtt{R}$ and the rewriting system $\mathcal{R}_{rs}$. An interpretation for $(\mathtt{R}, \mathcal{R}_{rs})$ consists of a category $\mathbf{C}$, functors $\oplus, \otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, functors $0, 1 \colon \mathbf{1} \to \mathbf{C}$ and the natural transformations $\alpha^{\otimes}_{X,Y,Z}$, $\alpha^{\oplus}_{X,Y,Z}$, $\lambda^{\otimes}_X$, $\rho^{\otimes}_X$, $\lambda^{\oplus}_X$, $\rho^{\oplus}_X$, $\lambda^{\bullet}_X$, $\rho^{\bullet}_X$, $\delta^r_{X,Y,Z}$ of the type required by Definition 3.1.

Every $\mathtt{R}$-term corresponds to a $\mathtt{R}$-functor: for instance, $(x_1 \oplus x_2) \otimes x_1$ is the functor $(\blacktriangleleft_{\mathbf{C}} \times Id_{\mathbf{C}}); (Id_{\mathbf{C}} \times \sigma_{\mathbf{C},\mathbf{C}}); (\oplus \times Id_{\mathbf{C}}); \otimes \colon \mathbf{C}^2 \to \mathbf{C}$. Every rewriting step $\Rightarrow_{\mathcal{R}_{rs}}$ corresponds to a $(\mathtt{R}, \mathcal{R}_{rs})$-natural isomorphism: a rewriting step using the rule $\iota \colon l \rightsquigarrow r$, substitution f and context g is mapped to the $(\Sigma, \mathcal{R})$-natural isomorphism

$$1_f \cdot (\iota \times 1_{Id_{\mathbf{C}^o}}) \cdot 1_g \colon f; (l \times Id_{\mathbf{C}^o}); g \Rightarrow f; (r \times Id_{\mathbf{C}^o}); g$$

where $\times$ and $\cdot$ stand for product and horizontal composition of natural transformations. For instance, the rewriting step in Example G.1 corresponds to the $(\Sigma, \mathcal{R})$-natural isomorphisms $1_{\langle x_1 \oplus x_3, x_2 \rangle} \cdot (\lambda^{\bullet}_X \times 1_{id_1}) \cdot 1_{\langle x_1 \oplus x_2 \rangle} \colon (0 \otimes (x_1 \oplus x_3)) \oplus x_2 \Rightarrow 0 \oplus x_2$, that is just what we usually write as $\lambda^{\bullet}_{x_1 \oplus x_3} \oplus x_2$. Vertical compositions and identity natural transformations allow for taking care of arbitrary paths $p \colon s \Rightarrow t$ in the graph of terms. For instance, the pair of paths in (49) corresponds exactly to the two $(\mathtt{R}, \mathcal{R}_{rs})$-natural isomorphisms of axiom (R3) in Table 8 (p. 68).

Now, consider the 2-theory $(\mathtt{R}, \mathcal{R}_{rs}, C)$ where C is the set containing only the pair of paths in (49). A model for $(\mathtt{R}, \mathcal{R}_{rs}, C)$ is an interpretation such that diagram (R3) commutes. Observe that any rig category is a model for $(\mathtt{R}, \mathcal{R}_{rs}, C)$.

A 2-theory $(\Sigma, \mathcal{R}, C)$ is said to be *coherent* if for all models $\mathcal{I}$ assigning to each rule in $\mathcal{R}$ a natural isomorphism and for all Σ-functors s, t there exists at most one $(\Sigma, \mathcal{R})$-natural isomorphism from s to t. This kind of coherence is a strong requirement but seems to be in the spirit of Mac Lane's dictum [22] that coherence theorems state that all diagrams well-formed from the data commute – provided the diagrammatic axioms do. We can finally state Theorem 4.2 of [2].

Theorem G.8 ([2])**.** *Let Σ be a (single sorted) cartesian signature and $\mathcal{R}$ be a terminating and locally confluent rewriting system. Let C be the set containing for each critical pair $t_1 \Leftarrow s \Rightarrow t_2$, one pair of path $(s \Rightarrow t_1 \Rightarrow^* N(s), s \Rightarrow t_2 \Rightarrow^* N(s))$. Then $(\Sigma, \mathcal{R}, C)$ is a coherent 2-theory.*

The above theorem can be easily exploited to prove some form of coherence for rig categories. The two Laplaza coherence theorems for rig categories impose restrictions either on the objects (namely *regularity*) or on the arrows (having the same *distortion*). Our theorem, which can also be proved by the second Laplaza coherence theorem, instead holds for all arrows generated by those in $\mathcal{R}_{rs}$. In essence, rather than imposing restrictions as Laplaza, we remove from our set of generators both symmetries and left distributors. Note that these are exactly the natural isomorphisms that are *not* forced to be identities in right strict rig categories.

Theorem G.9 (Right Coherence)**.** *Let $\mathbf{C}$ be a rig category and $s, t \colon \mathbf{C}^n \to \mathbf{C}$ be two $\mathtt{R}$-functors. There exists at most one $(\mathtt{R}, \mathcal{R}_{rs})$-natural isomorphism from s to t.*

This theorem is weaker than Laplaza coherence for rig categories, but we believe that it has its own interest: it can be more easily formulated and thus more readily exploited. Most importantly, it has a far simpler proof.

Proof. Recall from Proposition G.3 that $\mathcal{R}_{rs}$ is terminating and locally confluent. By Theorem G.8, it is enough to check that all diagrams given by the pairs of paths joining the critical pairs commute in any rig category. Such check is delayed to the proof of Theorem 3.11. $\square$

In essence, the proof only exploits cartesianity of the 2-category **Cat** and all the combinatorics is delegated to the Knuth-Bendix critical pairs lemma.

G.3 Sesqui coherence

Theorem G.9 provides us with a form of coherence that is not sufficient to prove Theorem 3.11, as the left distributors $\delta^l_{A,Y,Z}$ for $A \in \mathcal{S}$ are not taken into account. Our strategy to prove Theorem 3.11 consists in extending the rewriting system $\mathcal{R}_{rs}$ with $\delta^l_{A,Y,Z}$, prove that the resulting system is terminating and locally confluent and then use Theorem G.8.

First, we fix the signature $\mathtt{R} \cup \mathcal{S}$ where each sort in $\mathcal{S}$ is regarded as a constant, i.e., a symbol of arity 0. For all $A \in \mathcal{S}$, we define the following rewriting rule

$$\delta^l_{A,x_1,x_2} \colon A \otimes (x_1 \oplus x_2) \rightsquigarrow (A \otimes x_1) \oplus (A \otimes x_2)$$

and we call $\mathcal{R}_{ss}$ the rewriting system obtained by adding to $\mathcal{R}_{rs}$ the rule δ^l_{A,x_1,x_2} for each $A \in \mathcal{S}$. In symbols,

$$\mathcal{R}_{ss} = \mathcal{R}_{rs} \cup \{\delta^l_{A,x_1,x_2} \mid A \in \mathcal{S}\}.$$

The intuition is that δ^l_{A,x_1,x_2} allows for a limited form of left-distributivity that does not conflict with right-distributivity: for instance the critical pair illustrated in Example G.2 is not possible within $\mathcal{R}_{ss}$, since the leftmost rewriting step, exploiting $\delta^l_{x_1,x_2,x_3}$, cannot be performed with δ^l_{A,x_1,x_2}. Observe that all the rules in $\mathcal{R}_{ss}$ correspond to the natural isomorphisms that are forced to be identities in the definition of sesquistrict rig category.

Proposition G.10. The rewriting system $\mathcal{R}_{ss}$ is terminating and locally confluent.

Proof. The result has been proven automatically with the tool in [28, 13]. However, we report below the proofs of termination and confluence for the curious reader.

For termination, the strategy is to find some strict order relation $\succ$ on terms that is well founded and such that if $s \Rightarrow t$ then $s \succ t$. Clearly if such $\succ$ exists, then $\mathcal{R}_{ss}$ is terminating. To identify such $\succ$, we fist consider the following strict order on $\mathtt{R}\cup\mathcal{S}$: $\otimes > A > 0 > 1 > \oplus$. We then take $>_{lpo}$ on $\mathcal{T}_{\mathtt{R}\cup\mathcal{S}}(V)$, the set of $\mathtt{R}\cup\mathcal{S}$-terms with variables in V, to be the lexicographic path order [1] generated by $>$. By well-known results in rewriting [1], it holds that $>_{lpo}$ is a reduction order, namely it is well-founded, closed under substitution and context. This guarantees in particular that if $l >_{lpo} r$ for all $l \rightsquigarrow r$ in $\mathcal{R}_{ss}$, then the desired properties of $\succ$ hold and $\mathcal{R}_{ss}$ is proved to be terminating. So, all that is left to do is to check that $l >_{lpo} r$ for all $l \rightsquigarrow r$ in $\mathcal{R}_{ss}$, which amounts to rather tedious computations that are reported in Section G.4 together with the definition of lexicographic path order for the interested reader.

The proof of local confluence relies on the Knuth's critical pair lemma [19]: to prove local confluence it is enough to check that all critical pairs are joinable. Table 10 (p. 117) summarises the critical pairs analysis for $\mathcal{R}_{ss}$. Since there are 10 rules in $\mathcal{R}_{ss}$, there are 55 unordered pairs of rules. Each of these unordered pairs corresponds to a row in Table 10: each row displays on the left the rules and, on the right, either *none* or the reference to some *commutative* diagram. In the former case, 'none' means that the pair of rules does not generate critical pairs. In the latter case, each referred diagram contains on the top-left corner a span that can be easily seen to be a critical pair of the two rules. Most importantly, each of the referred diagram can be seen as the pair of paths joining the critical pair.

For instance, the unordered pair of rule $\{\alpha^{\oplus}_{x_1,x_2,x_3}, \delta^{r}_{x_1,x_2,x_3}\}$ refers in Table 10 only to diagram (R3). This means that there is only one critical pair generated by the two rules. The critical pair is the span on the top-left corner of (R3):

$$((x_1 \oplus x_2) \otimes x_4) \oplus (x_3 \otimes x_4) \Leftarrow ((x_1 \oplus x_2) \oplus x_3) \otimes x_4 \Rightarrow (x_1 \oplus (x_2 \oplus x_3)) \otimes x_4$$

Since all the arrows in the diagram correspond to some rewriting steps, the whole diagram can be thought as a pair of paths joining the critical pair, namely (49). $\square$

From Proposition G.10 it immediately follows that every term t has a unique normal form $N(t)$. More importantly, when t is a closed term, namely $t \in \mathcal{T}_{\mathtt{R}\cup\mathcal{S}}$ which means $t \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$, $N(t)$ is in polynomial form.

Proposition G.11. For every term $t \in \mathcal{T}_{\mathtt{R}}(\mathcal{S})$, $N(t)$ is in polynomial form. Moreover, if t is in polynomial form, then $t = N(t)$.

Proof. By definition of normal form, it holds that $N(t) \not\Rightarrow$. Consider an arbitrary occurrence of $t_1 \otimes t_2$ within $N(t)$. It holds that

- t_1 cannot be of the form $s_1 \otimes s_2$, otherwise $N(t) \Rightarrow$ by rule $\alpha^{\otimes}_{x_1,x_2,x_3}$;
- t_1 cannot be of the form 1, otherwise $N(t) \Rightarrow$ by rule $\lambda^{\otimes}_{x_1}$;
- t_1 cannot be of the form 0, otherwise $N(t) \Rightarrow$ by rule $\lambda^{\bullet}_{x_1}$;
- t_1 cannot be of the form $s_1 \oplus s_2$, otherwise $N(t) \Rightarrow$ by rule $\delta^{r}_{x_1,x_2,x_3}$.

Therefore $t_1 = A$ for some $A \in \mathcal{S}$. It follows that

- t_2 cannot be of the form $s_1 \oplus s_2$, otherwise $N(t) \Rightarrow$ by rule δ^{l}_{A,x_1,x_2}.
- t_2 cannot be of the form 1, otherwise $N(t) \Rightarrow$ by rule $\rho^{\otimes}_{x_1}$;
- t_2 cannot be of the form 0, otherwise $N(t) \Rightarrow$ by rule $\rho^{\bullet}_{x_1}$.

Therefore t_2 can be either A, for some $A \in \mathcal{S}$ or $s_1 \otimes s_2$. Thus any occurrence of $\otimes$ in $N(t)$ cannot be followed by $\oplus$, 0 and 1. More precisely, if t' is a subterm of $N(t)$ of the form $t_1 \otimes t_2$ then $t' = \bigotimes_{j=1}^{m} A_j$ for some $m > 1$ and $A_j \in \mathcal{S}$ for $j = 1 \dots m$.

Let us now consider an arbitrary occurrence of $t_1 \oplus t_2$ within $N(t)$. It holds that

- t_1 cannot be of the form $s_1 \oplus s_2$, otherwise $N(t) \Rightarrow$ by rule $\alpha^{\oplus}_{x_1,x_2,x_3}$;
- t_1 cannot be of the form 0, otherwise $N(t) \Rightarrow$ by rule $\lambda^{\oplus}_{x_1}$.
- t_2 cannot be of the form 0, otherwise $N(t) \Rightarrow$ by rule $\rho^{\oplus}_{x_1}$.

Thus if t' is a subterm of $N(t)$ of the form $t_1 \oplus t_2$ then t_1 can be either 1 or $A \in \mathcal{S}$, or a product, that is $\bigotimes_{j=1}^{m} A_j$ for $m > 1$. This can be summarised by saying that $t_1 = \bigotimes_{j=1}^{m} A_j$ for $m \in \mathbb{N}$ and $A_j \in \mathcal{S}$ for $j = 1 \dots m$. The term t_2 instead can additionally be some sum. In summary, $t' = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{m_i} A_{i,j}$ for some $n > 1$, m_i and $A_{i,j} \in \mathcal{S}$ for $i = 1 \dots n$ and $j = 1 \dots m_i$.

Now, if $N(t)$ is

- 0, then $N(t) = \bigoplus_{i=1}^{0} \bigotimes_{j=1}^{m_i} A_{i,j}$;
- $A \in \mathcal{S}$, then $N(t) = \bigoplus_{i=1}^{1} \bigotimes_{j=1}^{1} A_{i,j}$;
- 1, then $N(t) = \bigoplus_{i=1}^{1} \bigotimes_{j=1}^{0} A_{i,j}$;
- of the form $t_1 \oplus t_2$, then $N(t) = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{m_i} A_{i,j}$ for $n > 1$;
- of the form $t_1 \otimes t_2$, then $N(t) = \bigoplus_{i=1}^{1} \bigotimes_{j=1}^{m_1} A_{i,j}$ for $m_1 > 1$.

This prove the first part of the statement. For the second part, simply observe that if t is in polynomial form then none of the rules in $\mathcal{R}_{ss}$ can be applied, thus $t \not\Rightarrow$, hence $t = N(t)$. $\square$

With the above result, it is easy to prove Proposition 3.7.

Proof of Proposition 3.7. Since every rule $l \to r$ in $\mathcal{R}_{ss}$ is an equation $l = r$ in the set E and, viceversa, by orienting an equation in E one obtains a rule in $\mathcal{R}_{ss}$, by general results in rewriting it holds that for all terms s and t, s is E-equivalent to t iff $N(s) = N(t)$. Thus every term t is E-equivalent to $N(t)$ which is in polynomial form by the first part of Proposition G.11. To prove uniqueness, assume that there is some term s in polynomial form and E-equivalent to t. By the second part of Proposition G.11, it holds that $s = N(s)$ and, since s is E-equivalent to t, $N(s) = N(t)$. Thus $s = N(t)$. To have the statement of Proposition 3.7 it is enough to replace $N(t)$ by $t{\downarrow}$. $\square$

To prove Theorem 3.11 we can exploit again Theorem G.8.

Proof of Theorem 3.11. First, observe that the rig category $\mathbf{R}_\Sigma$ freely generated by a rig signature Σ with sorts $\mathcal{S}$ is an interpretation for $(\mathtt{R} \cup \mathcal{S}, \mathcal{R}_{ss})$. Then, observe that all structural isomorphisms (Definition 3.10) are the compontents of $(\mathtt{R} \cup \mathcal{S}, \mathcal{R}_{ss})$-natural isomorphisms. Since $\mathcal{R}_{ss}$ is terminating and locally confluent, by Theorem G.8, the proof of Theorem 3.11 only consists in checking that all diagrams given by the pairs of paths joining critical pairs of $\mathcal{R}_{ss}$ commutes in $\mathbf{R}_\Sigma$. Such check is summarised in Table 10 where, for each unordered pairs of rules in $\mathcal{R}_{ss}$ we refer either to the commuting diagrams joining the critical pairs generated by the two rules or we write 'none' if the two rules do not generate critical pairs. $\square$

G.4 Details for the proof of termination of $\mathcal{R}_{ss}$

Decades of research in rewriting have produced a useful tool-box for analysing several properties of term rewriting systems. One of the most effective tools for proving terminations is the so-called lexicographic path order [1], defined as follows.

Definition G.12. Let Σ be a single sorted cartesian signature and $>$ be a strict order on Σ. The *lexicographic path order* $>_{lpo}$ on $\mathcal{T}_\Sigma(V)$ generated by $>$ is defined as follows: $s >_{lpo} t$ iff

$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \alpha^{\otimes}_{x_1,x_2,x_3} \mid\}$	(M2)
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \alpha^{\oplus}_{x_1,x_2,x_3} \mid\}$	*none*
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \lambda^{\otimes}_{x_1} \mid\}$	(1.2.7) in [17]
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \rho^{\otimes}_{x_1} \mid\}$	(M1) and (1.2.7) in [17]
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \lambda^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \rho^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \lambda^{\bullet}_{x_1} \mid\}$	(2.1.22) in [17]
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \rho^{\bullet}_{x_1} \mid\}$	(R10) and (2.1.21) in [17]
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(R4)
$\{\mid \alpha^{\otimes}_{x_1,x_2,x_3}, \delta^{l}_{A,x_1,x_2} \mid\}$	(38)
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \alpha^{\oplus}_{x_1,x_2,x_3} \mid\}$	(M2)
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \lambda^{\otimes}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \rho^{\otimes}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \lambda^{\oplus}_{x_1} \mid\}$	(1.2.7) in [17]
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \rho^{\oplus}_{x_1} \mid\}$	(M1) and (1.2.7) in [17]
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \lambda^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \rho^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(R3)
$\{\mid \alpha^{\oplus}_{x_1,x_2,x_3}, \delta^{l}_{A,x_1,x_2} \mid\}$	(37)
$\{\mid \lambda^{\otimes}_{x_1}, \lambda^{\otimes}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\otimes}_{x_1}, \rho^{\otimes}_{x_1} \mid\}$	(1.2.6) in [17]
$\{\mid \lambda^{\otimes}_{x_1}, \lambda^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\otimes}_{x_1}, \rho^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\otimes}_{x_1}, \lambda^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\otimes}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	(2.1.18) in [17]
$\{\mid \lambda^{\otimes}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	*none*
$\{\mid \lambda^{\otimes}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*
$\{\mid \rho^{\otimes}_{x_1}, \rho^{\otimes}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\otimes}_{x_1}, \lambda^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\otimes}_{x_1}, \rho^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\otimes}_{x_1}, \lambda^{\bullet}_{x_1} \mid\}$	(R8)
$\{\mid \rho^{\otimes}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\otimes}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(R12)
$\{\mid \rho^{\otimes}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*
$\{\mid \lambda^{\oplus}_{x_1}, \lambda^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\oplus}_{x_1}, \rho^{\oplus}_{x_1} \mid\}$	(1.2.6) in [17]
$\{\mid \lambda^{\oplus}_{x_1}, \lambda^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\oplus}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\oplus}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(R11)
$\{\mid \lambda^{\oplus}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	(2.1.23) in [17]
$\{\mid \rho^{\oplus}_{x_1}, \rho^{\oplus}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\oplus}_{x_1}, \lambda^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\oplus}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\oplus}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(2.1.26) in [17]
$\{\mid \rho^{\oplus}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	(2.1.25) in [17]
$\{\mid \lambda^{\bullet}_{x_1}, \lambda^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \lambda^{\bullet}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	(R6)
$\{\mid \lambda^{\bullet}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	*none*
$\{\mid \lambda^{\bullet}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*
$\{\mid \rho^{\bullet}_{x_1}, \rho^{\bullet}_{x_1} \mid\}$	*none*
$\{\mid \rho^{\bullet}_{x_1}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	(R7)
$\{\mid \rho^{\bullet}_{x_1}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*
$\{\mid \delta^{r}_{x_1,x_2,x_3}, \delta^{r}_{x_1,x_2,x_3} \mid\}$	*none*
$\{\mid \delta^{r}_{x_1,x_2,x_3}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*
$\{\mid \delta^{l}_{A,x_2,x_3}, \delta^{l}_{A,x_1,x_2} \mid\}$	*none*

Table 10: Critical pair analysis for $\mathcal{R}_{ss}$

- (LPO1) t is a variable of s and $t \neq s$, or
- (LPO2) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$ and
  - (LPO2a) there exists $i \in \{1 \dots m\}$, such that $s_i \geq_{lpo} t$, or
  - (LPO2b) $f > g$ and, for all $j \in 1 \dots n$, $s >_{lpo} t_j$, or
  - (LPO2c) $f = g$ and, for all $j \in 1 \dots n$, $s >_{lpo} t_j$ and there exists $i \in \{1 \dots m\}$ such that $s_1 = t_1, \dots, s_{i-1} = t_{i-1}$ and $s_i >_{lpo} t_i$.

Above $\geq_{lpo}$ stands for the reflexive closure of $>_{lpo}$.

As mentioned earlier for $\mathcal{R}_{ss}$ we take as signature $\mathtt{R} \cup \mathcal{S}$ and as generating strict order

$$\otimes \;>\; A \;>\; 0 \;>\; 1 \;>\; \oplus.$$

To conclude termination it is enough to check that for each rule $l \rightsquigarrow r$ in $\mathcal{R}_{ss}$, it holds that $l >_{lpo} r$. Note that this check can be easily executed by a computer program – we used [28] – but to give a concrete grasp to the reader, we report here the computations.

We start with the rule $x_1 \otimes 1 \rightsquigarrow x_1$. We have to check that $x_1 \otimes 1 >_{lpo} x_1$. But this trivially holds by (LPO1). The same argument works for $1 \otimes x_1 \rightsquigarrow x_1$, $x_1 \oplus 0 \rightsquigarrow x_1$ and $0 \oplus x_1 \rightsquigarrow x_1$.

Consider now the rule $x_1 \otimes 0 \rightsquigarrow 0$. We have to check that $x_1 \otimes 0 >_{lpo} 0$. This holds by (LPO2b): indeed $\otimes > 0$ and since 0 is a constant (namely n in the definition above is 0), there is nothing else to check. The same can be said for $0 \otimes x_1 \rightsquigarrow 0$.

The computations for the remaining four rules are more involved: for the rule $(x_1 \otimes x_2) \otimes x_3 \rightsquigarrow x_1 \otimes (x_2 \otimes x_3)$, we have to check that

$$(x_1 \otimes x_2) \otimes x_3 >_{lpo} x_1 \otimes (x_2 \otimes x_3).$$

Since $\otimes = \otimes$, we are in the case (LPO2c) but we need to additionally check that

$$(x_1 \otimes x_2) \otimes x_3 >_{lpo} x_1, \qquad (x_1 \otimes x_2) \otimes x_3 >_{lpo} x_2 \otimes x_3 \quad \text{and} \quad x_1 \otimes x_2 >_{lpo} x_1.$$

Both the leftmost and the rightmost inequations hold by (LPO1). For the central one, since $\otimes = \otimes$, we are again in (LPO2c) and we have to check that

$$(x_1 \otimes x_2) \otimes x_2 >_{lpo} x_2, \qquad (x_1 \otimes x_2) \otimes x_3 >_{lpo} x_3 \quad \text{and} \quad x_1 \otimes x_2 >_{lpo} x_2.$$

The three inequations hold for (LPO1).

The computations for $(x_1 \oplus x_2) \oplus x_3 \rightsquigarrow x_1 \oplus (x_2 \oplus x_3)$ are the same as above.

For the rule of right distributor $(x_1 \oplus x_2) \otimes x_3 \rightsquigarrow (x_1 \otimes x_3) \oplus (x_2 \otimes x_3)$, we have to check that

$$(x_1 \oplus x_2) \otimes x_3 >_{lpo} (x_1 \otimes x_3) \oplus (x_2 \otimes x_3).$$

Since $\otimes > \oplus$, we are in the case (LPO2b) but we need to check that

$$(x_1 \oplus x_2) \otimes x_3 >_{lpo} (x_1 \otimes x_3) \qquad \text{and} \qquad (x_1 \oplus x_2) \otimes x_3 >_{lpo} (x_2 \otimes x_3).$$

We illustrate the computations for the leftmost, the one for the rightmost are identical. For the leftmost inequation, since $\otimes = \otimes$, we are in the case (LPO2c), but we still need to check that

$$(x_1 \oplus x_2) \otimes x_3 >_{lpo} x_1, \qquad (x_1 \oplus x_2) \otimes x_3 >_{lpo} x_3 \quad \text{and} \quad x_1 \oplus x_2 >_{lpo} x_1.$$

The three inequations hold for (LPO1).

We are now left to check the rule for partial left distributivity: $A \otimes (x_1 \oplus x_2) \rightsquigarrow (A \otimes x_1) \oplus (A \otimes x_2)$. We need to check that

$$A \otimes (x_1 \oplus x_2) >_{lpo} (A \otimes x_1) \oplus (A \otimes x_2).$$

Since $\otimes > \oplus$, we are in the case (LPO2b) but we need to check that

$$A \otimes (x_1 \oplus x_2) >_{lpo} A \otimes x_1 \qquad \text{and} \qquad A \otimes (x_1 \oplus x_2) >_{lpo} A \otimes x_2.$$

We illustrate the computations for the leftmost, the one for the rightmost are identical. For the leftmost inequation, since $\otimes = \otimes$, we are in the case (LPO2c), but we still need to check that

$$A \otimes (x_1 \oplus x_2) >_{lpo} A, \qquad A \otimes (x_1 \oplus x_2) >_{lpo} x_1 \quad \text{and} \quad x_1 \oplus x_2 >_{lpo} x_1.$$

Observe that the rightmost pair is necessary, because the first argument of $\otimes$ in $A \otimes (x_1 \oplus x_2)$ and $A \otimes x_1$ is the same: the term A. The rightmost and the central inequations hold for (LPO1). The leftmost one holds for (LPO2b): indeed $\otimes > A$ and since A is a constant, there is nothing else to check.