Deconstructing the Calculus of Relations with Tape Diagrams

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Rig categories with finite biproducts are categories with two monoidal products, where one is a biproduct and the other distributes over it. In this work we present tape diagrams, a sound and complete diagrammatic language for these categories, that can be intuitively thought as string diagrams of string diagrams. We test the effectiveness of our approach against the positive fragment of Tarski's calculus of relations.

CCS Concepts: • **Theory of computation** → **Logic**; **Categorical semantics**.

Additional Key Words and Phrases: calculus of relations, rig categories, string diagrams

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1 INTRODUCTION

Diagrammatic notations have been used in computer science since its early stages. A famous example is the proof of the structured program theorem [Böhm and Jacopini 1979] by Böhm and Jacopini: they rely on a syntax of flow diagrams and, by means of several transformations preserving the semantics, prove the existence of a normal form. In physics, diagrams by Feynman [Kaiser 2009] and Penrose [Penrose 1971] became essential linguistic tools: on the one hand they provide intuitive visualisations for otherwise arcane formulas; on the other, they allow a critical simplification of calculations, pretty much like adding two Hindu-Arabic numerals is far easier than two Roman ones [Fibonacci 2020].

In general, well-behaved diagrammatic languages share some desirable features: (a) diagrams can be composed, like one composes formulas in mathematics, programs in a programming language or sentences in English; (b) they are equipped with a compositional semantics: the meaning of a compound diagram is given by the meaning of its components; (c) some basic laws allow us to transform diagrams into semantically equivalent ones, like the laws of arithmetic allow safe manipulation of expressions.

Motivated by the interest in dealing with diagrammatic languages that enjoy such features, a growing number of works [Baez and Erbele 2015; Bonchi et al. 2022b, 2019a, 2015; Coecke and Duncan 2011; Fong et al. 2016; Fong and Spivak 2020; Ghica and Jung 2016; Muroya et al. 2018; Piedeleu and Zanasi 2021] exploits *string diagrams* [Joyal and Street 1991; Selinger 2010], a graphical notation that emerged in the field of category theory. Formally, string diagrams are arrows of a

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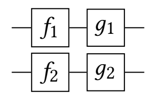
strict symmetric monoidal category freely generated by a *monoidal signature*. A symbol s in the signature is represented as a box $-\boxed{s}-$, and arbitrary string diagrams are depicted by composing horizontally (;) and vertically ($\otimes$) such symbols (plus some wiring technology that we are going to ignore in this introduction). The following is our first example of a string diagram:



Observe that it can be regarded as both $(f_1; g_1) \otimes (f_2; g_2)$ and $(f_1 \otimes f_2); (g_1 \otimes g_2)$. This ambiguity is not an issue, since in any monoidal category the law $(f_1; g_1) \otimes (f_2; g_2) = (f_1 \otimes f_2); (g_1 \otimes g_2)$ holds for all arrows f_1, f_2, g_1, g_2. More generally, the main result in [Joyal and Street 1991] states that the diagrammatic representation identifies exactly *all and only* the laws of strict monoidal categories. This is the key feature of string diagrams. Indeed, by virtue of this fact, one can safely exploit diagrams to make proofs, which in this way often amount to suggestive manipulations of diagrams.

By carefully crafting the monoidal signature, hereafter denoted as Σ, one obtains the syntaxes of several languages specifying a large variety of systems: quantum processes [Coecke and Kissinger 2017], linear dynamical systems [Bonchi et al. 2015], Petri nets [Bonchi et al. 2019a], concurrent connectors [Bruni et al. 2006], digital circuits [Ghica and Jung 2016], automata [Piedeleu and Zanasi 2021], or conjunctive queries [Bonchi et al. 2018]. In these approaches, the semantics are defined by monoidal functors

$$\mathbf{C}_\Sigma \xrightarrow{[\![\cdot]\!]} \mathbf{D} \tag{1}$$

going from the category of string diagrams $\mathbf{C}_\Sigma$ to some monoidal category $\mathbf{D}$ representing the semantic domain. Since $[\![\cdot]\!]$ is a monoidal functor, it preserves ; and $\otimes$, and thus the semantics is guaranteed to be *compositional*. Typically, the languages come with a set of axioms, namely equalities or inequalities between string diagrams, that are sound with respect to the semantics interpretation. Interestingly, the *same* algebraic structures seem to appear in many different contexts, e.g. commutative monoids and comonoids, Frobenius algebras, bialgebras, etc.

However, in several occasions the string diagrammatic syntax seems to be too restrictive. For instance, in the context of the ZX-calculus [Coecke and Duncan 2008; Coecke and Kissinger 2017], a well known quantum diagrammatic language, several works [Stollenwerk and Hadfield 2022; Toumi et al. 2021; Zhao and Gao 2021] make use of a mixture of diagrammatic and algebraic syntax to represent, for example, addition of diagrams. Similarly, in [Boisseau and Piedeleu 2022] $\sqcup$-props have been introduced in order to enrich string diagrams with a join operation. Sometimes, the structure of monoidal category is not enough and one needs to depict arrows of *rig categories* [Johnson and Yau 2022; Laplaza 1972], roughly categories equipped with two monoidal products: $\otimes$ and $\oplus$. In these cases, the authors often introduce novel kinds of diagrams [Duncan 2009; James and Sabry 2012; Staton 2015] to convey the intuition to the reader, but without a soundness and completeness theorem like in [Joyal and Street 1991] for string diagrams. The main challenge in depicting arrows of a rig category is given by the possibility of composing them not only with ; (horizontally) and $\otimes$ (vertically), but also with the novel monoidal product $\oplus$. The natural solution consists in exploiting 3 dimensions. This is the approach taken by *sheet diagrams* [Comfort et al. 2020], certain topological manifolds that, modulo a notion of isotopy, capture exactly the laws of rig categories.

In this paper, we introduce *tape diagrams* as a way to depict arrows not of arbitrary rig categories but only of those where $\oplus$ is a *biproduct* [Coecke et al. 2018; Mac Lane 1978]. The payoff of this restriction in expressiveness is a better usability: tape diagrams are two dimensional pictures and for this reason they are, in our opinion, more intuitive and more easily drawable than three dimensional diagrams. A second important novelty is that we do not need to define ad-hoc topological structures and transformations, since tape diagrams are, literally, *string diagrams of string diagrams*.

Our main result, Theorem 5.11, is analogous to the one of [Joyal and Street 1991]: it states that the category of tape diagrams, hereafter referred to as $\mathbf{T}_\Sigma$, is the rig category with finite biproducts freely generated by a *monoidal* signature. Another result, Theorem 4.9, states that for finite biproduct rig categories considering only monoidal signatures, rather than the more general *rig* signatures, does not affect the expressivity, in the sense that for every rig signature Σ one can find a monoidal signature Σ_M such that the finite biproduct rig category generated by Σ is isomorphic to the one generated by Σ_M. So, Theorems 5.11 and 4.9 together allow us to state that tape diagrams form a universal diagrammatic language for rig categories with finite biproducts (see Remark 5.16).

A useful consequence of Theorem 5.11 is Corollary 5.12, which states that whenever the semantic domain $\mathbf{D}$ of a string diagrammatic language as in (1) carries the structure of a finite biproduct rig category, the semantics map $[\![\cdot]\!]$ can be extended to tape diagrams as follows.

$$\begin{tikzcd} \mathbf{C}_\Sigma \arrow[r, "{[\![\cdot]\!]}"] \arrow[d, hook] & \mathbf{D} \\ \mathbf{T}_\Sigma \arrow[ur, "{[\![\cdot]\!]^\sharp}"'] & \end{tikzcd}$$

In Example 5.13 we quickly show that applying the above construction to the ZX-calculus [Coecke and Duncan 2008], one can easily express through a tape diagram a quantum Controlled Unitary gate. In Example 6.13 we show, with the help of four "adjointness" axioms (in Figure 2), that $\sqcup$-props from [Boisseau and Piedeleu 2022] can be comfortably translated into tape diagrams so to obtain a purely graphical calculus that avoids the use of algebraic operators. Finally, by taping the calculus of graphical conjunctive queries from [Bonchi et al. 2018], one obtains a complete axiomatisation for the calculus of relations by Tarski [Tarski 1941]. This is our main application, which we will illustrate in the next section.

Structure of the Paper. In Section 3 we recall string diagrams and monoidal categories. In particular, we show that **Rel**, the category of sets and relations, carries two monoidal structures satisfying distinct algebraic properties: $(\mathbf{Rel}, \oplus, 0)$ is a finite biproduct (fb) category, while $(\mathbf{Rel}, \otimes, 1)$ is a cartesian bicategory (cb). In Section 4 we recall rig categories and we introduce a novel (to the best of our knowledge) notion of strictness that is useful to simplify the presentation. We also illustrate rig signatures, finite biproduct rig categories and Theorem 4.9.

In Section 5 we introduce tape diagrams and we prove Theorem 5.11. The key step of its proof consists in showing that tapes form a rig category (Theorem 5.10): this is carefully done by inductively defining left and right whiskerings (Definition 5.7) that enjoy beautiful algebraic properties (Table 5). Differently from $\oplus$, the representation of the product $\otimes$ of two tapes involves some computations. However, a further result, Theorem 5.15, allows us to avoid this issue: in diagrammatic proofs one can safely forget about $\otimes$ of tapes.

In Section 6 we investigate the matrix calculus for $\mathbf{T}_\Sigma$ that is provided by its finite biproduct structure. Corollary 6.6 characterises tape diagrams as matrices of multisets of string diagrams. Interestingly enough $\oplus$ of tapes corresponds to direct sum of matrices, while $\otimes$ to their Kronecker product. Such correspondence is then extended to a poset enriched setting: the category of tapes resulting from the four adjointness axioms of Figure 2 is isomorphic to matrices of downsets of string diagrams (Theorem 6.9). From this follows a characterisation of the induced poset (Corollary 6.10) that is fundamental for the completeness result in Section 7: Theorem 7.5.

This result is analogous to Theorem 2.2 in [Selinger 2012] stating that finite dimensional Hilbert spaces are complete for dagger compact closed categories. Theorem 7.5 shows that **Rel** is complete for *fb-cb rig categories* (Definition 7.1), namely rig categories where $\otimes$ forms a cartesian bicategory and $\oplus$ a finite biproduct category enjoying the aforementioned adjointness axioms. From the completeness theorem, one can easily show (Corollary 7.8) that the laws of fb-cb categories provide a sound and complete axiomatisation for the positive fragment of Tarski's calculus of relations.

This paper only contains sketches of proofs; an extended version of this article, containing fully detailed proofs and additional examples, can be found in [Bonchi et al. 2022a].

2 LEADING EXAMPLE: THE CALCULUS OF RELATIONS

In order to provide a preliminary intuition about tape diagrams and, meanwhile, explain their main application investigated in this paper, we recall now the positive fragment of the calculus of binary relations by Tarski [Tarski 1941]. Its syntax is specified by the following context free grammar

$$E \quad ::= \quad R \mid 1 \mid E;E \mid \bot \mid E \cup E \mid \top \mid E \cap E \mid E^{\dagger} \qquad (\mathrm{CR}_\Sigma)$$

where R is taken from a given set Σ of generating symbols. The expression 1 denotes the identity relation, ; relational composition, $\cdot^{\dagger}$ the opposite relation and the remaining expressions are the usual set-theoretic union $\cup$ and intersection $\cap$, together with their units $\bot$ and $\top$ being, respectively, the empty relation and the total relation. Formally, the semantics of CR_Σ is defined with respect to a *relational interpretation* $\mathcal{I}$, i.e. a set X together with a binary relation $\rho(R) \subseteq X \times X$ for each $R \in \Sigma$.

$$\begin{array}{rclrcl}
\langle\!\langle R \rangle\!\rangle_{\mathcal{I}} &=& \rho(R) & \langle\!\langle E^{\dagger} \rangle\!\rangle_{\mathcal{I}} &=& \{(y,x) \mid (x,y) \in \langle\!\langle E \rangle\!\rangle_{\mathcal{I}}\} \\
\langle\!\langle \bot \rangle\!\rangle_{\mathcal{I}} &=& \{\,\} & \langle\!\langle E_1 \cup E_2 \rangle\!\rangle_{\mathcal{I}} &=& \langle\!\langle E_1 \rangle\!\rangle_{\mathcal{I}} \cup \langle\!\langle E_2 \rangle\!\rangle_{\mathcal{I}} \\
\langle\!\langle \top \rangle\!\rangle_{\mathcal{I}} &=& \{(x,y) \mid x,y \in X\} & \langle\!\langle E_1 \cap E_2 \rangle\!\rangle_{\mathcal{I}} &=& \langle\!\langle E_1 \rangle\!\rangle_{\mathcal{I}} \cap \langle\!\langle E_2 \rangle\!\rangle_{\mathcal{I}} \\
\langle\!\langle 1 \rangle\!\rangle_{\mathcal{I}} &=& id_X = \{(x,x) \mid x \in X\} & \langle\!\langle E_1;E_2 \rangle\!\rangle_{\mathcal{I}} &=& \{(x,z) \mid \exists y.\ (x,y) \in \langle\!\langle E_1 \rangle\!\rangle_{\mathcal{I}} \wedge (y,z) \in \langle\!\langle E_2 \rangle\!\rangle_{\mathcal{I}}\}
\end{array} \qquad (2)$$

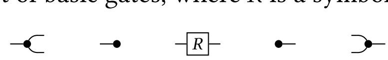
Two expressions E_1, E_2 are said to be *equivalent*, written $E_1 \equiv_{\mathrm{CR}} E_2$, if and only if $\langle\!\langle E_1 \rangle\!\rangle_{\mathcal{I}} = \langle\!\langle E_2 \rangle\!\rangle_{\mathcal{I}}$ for all interpretations $\mathcal{I}$. Inclusion, denoted by $\leq_{\mathrm{CR}}$, is defined analogously by replacing = with $\subseteq$. For instance, the following two laws assert that ; distributes over $\cup$ but only laxly over $\cap$.

$$R;(S \cup T) \equiv_{\mathrm{CR}} (R;S) \cup (R;T) \qquad R;(S \cap T) \leq_{\mathrm{CR}} (R;S) \cap (R;T) \qquad (3)$$

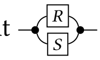
The question left open by Tarski is whether or not $\equiv_{\mathrm{CR}}$ can be axiomatised: is there a basic set of laws from which one can prove all the valid equivalences? Unfortunately, many negative results show that there are no finite complete axiomatisations for the whole calculus [Monk 1964], for the positive fragment [Hodkinson and Mikulás 2000] and several other fragments, e.g. [Bollig et al. 2020; Freyd and Scedrov 1990; Redko 1964]. See [Pous 2018] for a recent overview.

In this paper we propose a solution to the same problem, but from a radically different perspective: we encode the calculus of relations into a novel calculus that is based on, in our opinion, more primitive linguistic constructions. Our language, named $\mathbf{T}_{\mathbf{CB}_\Sigma}$, is strictly more expressive than CR_Σ but allows to obtain a complete axiomatisation of equivalence and inclusion.

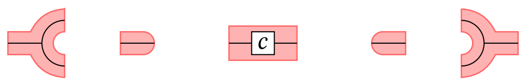
The syntax of $\mathbf{T}_{\mathbf{CB}_\Sigma}$ is based on *circuits* and *tapes*. Circuits are obtained by composing horizontally and vertically the following set of basic gates, where R is a symbol in Σ.



To abbreviate, we will denote , , and respectively as $\blacktriangleleft$, $!$, $¡$ and $\blacktriangleright$. Intuitively, $\blacktriangleleft$ acts as a *copier*: it receives a signal (i.e. a value from some set X) on its left wire and sends it to *both* its wires on the right. Instead, ! is a *discharger*: it throws away the signal coming from its only wire on the left. Formally, $\blacktriangleleft$ is the pairing function $\langle id_X, id_X \rangle$ going from X to $X \times X$, namely the cartesian product of X with itself, while ! is the only function going from a set X to a singleton set 1. The gates $\blacktriangleright$ and ¡ are interpreted as the opposite relations of $\blacktriangleleft$ and !, respectively. Finally $-\boxed{R}-$ simply denotes an arbitrary binary relation R on X.

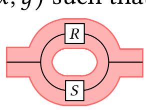
It is worth emphasising here that signal flows through circuits as a *wave*, i.e. it passes through *all* the vertical components at the same time. For instance, the circuit  denotes the relation $R \cap S$, i.e. the set of all pairs of signals (x, y) such that, at the same time, $x\,R\,y$ and $x\,S\,y$.

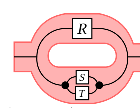
Tapes are obtained by composing horizontally and vertically the following generators



where the middle one represents a circuit c wrapped inside a tape. To abbreviate, we will denote the first two generators on the left as $\lhd$ and ⫯ and the last two on the right as ⫰ and $\rhd$. Intuitively, $\rhd$ lets pass to the right the signal coming from either the top *or* the bottom branch on the left, while ⫰ simply closes a branch. Formally, $\rhd$ is the coparing function $[id_X, id_X]$ going from $X + X$, i.e. the disjoint union of X with itself, to X, while ⫰ is the only function going from the the empty set 0 to X. The generators $\lhd$ and ⫯ are interpreted as the opposite relations of $\rhd$ and ⫰, respectively.

Differently from circuits, a signal flows through a tape as a *particle*, i.e. it passes through *only one* of the vertical components at a time. For instance, the tape in (4) denotes the relation $R \cup S$, i.e. the set of all pairs (x, y) such that either $x\,R\,y$ or $x\,S\,y$.


(4)

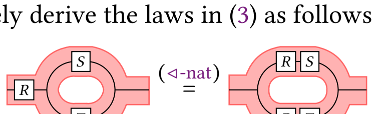

(5)

Since circuits can be nested inside tapes, an expression such as $R \cup (S \cap T)$ can be represented as the tape in (5). A formal semantics of $\mathbf{T}_{\mathbf{CB}_\Sigma}$, as well as an encoding of CR_Σ into $\mathbf{T}_{\mathbf{CB}_\Sigma}$, will be given in Section 7. Moreover, Theorem 7.5 will state that the axioms in Figures 1, 2 and 3 are complete. Figure 1 features the axioms for "plain" tape diagrams and Figure 2 for their partial order enrichment (discussed in Sections 6.2 and 7); Figure 3 lists axioms to be used specifically for $\mathbf{T}_{\mathbf{CB}_\Sigma}$.

Our axiomatisation is far from those proposed in more traditional approaches to CR, but it elegantly features some well-known algebraic structures that occur frequently in various fields [Baez and Erbele 2015; Bonchi et al. 2022b, 2019a,b, 2015; Bruni et al. 2006; Coecke and Duncan 2011; Fritz 2009; Lafont 2003]. The second group of axioms in Figures 1 and 3 state that ($\lhd$, ⫯) and ($\blacktriangleleft$, $!$) are cocommutative comonoids while ($\rhd$, ⫰) and ($\blacktriangleright$, ¡) are commutative monoids. The third group expresses the facts that ($\lhd$, ⫯, $\rhd$, ⫰) form a bialgebra and ($\blacktriangleleft$, $!$, $\blacktriangleright$, ¡) a special Frobenius bimonoid. The axioms in Figure 2 assert that the monoid ($\rhd$, ⫰) is *left* adjoint to the comonoid ($\lhd$, ⫯), while those in the last group in Figure 3 state the the monoid ($\blacktriangleright$, ¡) is *right* adjoint to the monoid ($\blacktriangleleft$, $!$).

The formal justification to these axioms will be clarified through the paper but it is worth establishing now a preliminary intuition: consider the axioms ($\rhd\lhd$) and ($\blacktriangleright\blacktriangleleft$). In the leftmost tape of ($\rhd\lhd$), the signal flows either in the top branch or in the bottom one, while in the rightmost tape, the signal coming from any of the two branches on the left may go through any of the branches on the right. In the two tapes of ($\blacktriangleright\blacktriangleleft$) the signal flows, at the same time, through the top and bottom wires: in the leftmost such signals must be equal, in the rightmost they may be different.

The remaining axioms are those in the first and fourth groups in Figures 1 and 3. The laws in the first group assert that crossings of tapes and wires behave like symmetries. The axioms in the fourth group force naturality for $\lhd$, ⫯, $\rhd$ and ⫰, while for $\blacktriangleleft$ and $!$ naturality only holds laxly (and from these one can easily derive that $\blacktriangleright$ and ¡ are colax natural). With these naturality axioms one can immediately derive the laws in (3) as follows:



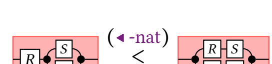


Table 1. Axioms for C_Σ

Objects ($A \in \mathcal{S}$)	Arrows ($A \in \mathcal{S}, s \in \Sigma$)
$X \; ::= \; A \mid I \mid X \odot X$	$f \; ::= \; id_A \mid id_I \mid s \mid f;f \mid f \odot f \mid \sigma^{\odot}_{A,B}$
$(X \odot Y) \odot Z = X \odot (Y \odot Z)$ $X \odot I = X$ $I \odot X = X$	$(f;g);h = f;(g;h)$ $\quad id_X;f = f = f;id_Y$ $(f_1 \odot f_2);(g_1 \odot g_2) = (f_1;g_1) \odot (f_2;g_2)$ $id_I \odot f = f = f \odot id_I$ $\quad (f \odot g) \odot h = f \odot (g \odot h)$ $\sigma^{\odot}_{A,B};\sigma^{\odot}_{B,A} = id_{A \odot B}$ $\quad (s \odot id_Z);\sigma^{\odot}_{Y,Z} = \sigma^{\odot}_{X,Z};(id_Z \odot s)$

Typing rules					
$id_A : A \to A$	$id_I : I \to I$	$\sigma^{\odot}_{A,B} : A \odot B \to B \odot A$	$\dfrac{s : ar(s) \to coar(s) \in \Sigma}{s : ar(s) \to coar(s)}$	$\dfrac{f : X_1 \to Y_1 \quad g : X_2 \to Y_2}{f \odot g : X_1 \odot X_2 \to Y_1 \odot Y_2}$	$\dfrac{f : X \to Y \quad g : Y \to Z}{f;g : X \to Z}$

By means of simple graphical manipulations prescribed by the laws in Figures 1, 2 and 3, one can prove all the valid equivalences in CR. We conclude this preliminary section by remarking that the discussed axiomatisation has some redundancies, e.g. ( ) evidently follows from (bo). We have however chosen this presentation to emphasise the various elegant dualities.

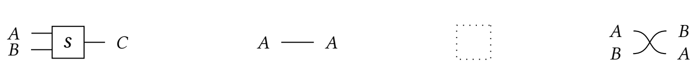
3 STRING DIAGRAMS AND MONOIDAL CATEGORIES

We begin our exposition by regarding string diagrams [Joyal and Street 1991; Selinger 2010] as terms of a typed language. Given a set $\mathcal{S}$ of basic *sorts*, hereafter denoted by $A, B \dots$, types are elements of $\mathcal{S}^\star$, i.e. words over $\mathcal{S}$. Terms are defined by the following context free grammar

$$f \quad ::= \quad id_A \mid id_I \mid s \mid \sigma^{\odot}_{A,B} \mid f;f \mid f \odot f \tag{6}$$

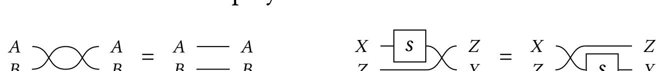
where s belongs to a fixed set Σ of *generators* and I is the empty word. Each $s \in \Sigma$ comes with two types: arity $ar(s)$ and coarity $coar(s)$. The tuple $(\mathcal{S}, \Sigma, ar, coar)$, Σ for short, forms a *monoidal signature*. Amongst the terms generated by (6) we consider only those that can be typed according to the rules in Table 1. String diagrams are such terms modulo the axioms in Table 1 where, for any $X, Y \in \mathcal{S}^\star$, id_X and $\sigma^{\odot}_{X,Y}$ can be easily built using id_I, id_A, $\sigma^{\odot}_{A,B}$, $\odot$ and ; (see e.g. [Zanasi 2015]).

String diagrams enjoy an elegant graphical representation: a generator s in Σ with arity X and coarity Y is depicted as a *box* having *labelled wires* on the left and on the right representing, respectively, the words X and Y. For instance $s : AB \to C$ in Σ is depicted as the leftmost diagram below. Moreover, id_A is displayed as one wire, id_I as the empty diagram and $\sigma^{\odot}_{A,B}$ as a crossing:



Finally, composition $f;g$ is represented by connecting the right wires of f with the left wires of g when their labels match, while the monoidal product $f \odot g$ is depicted by stacking the corresponding diagrams on top of each other:

The first three rows of axioms for arrows in Table 1 are implicit in the graphical representation while the axioms in the last row are displayed as



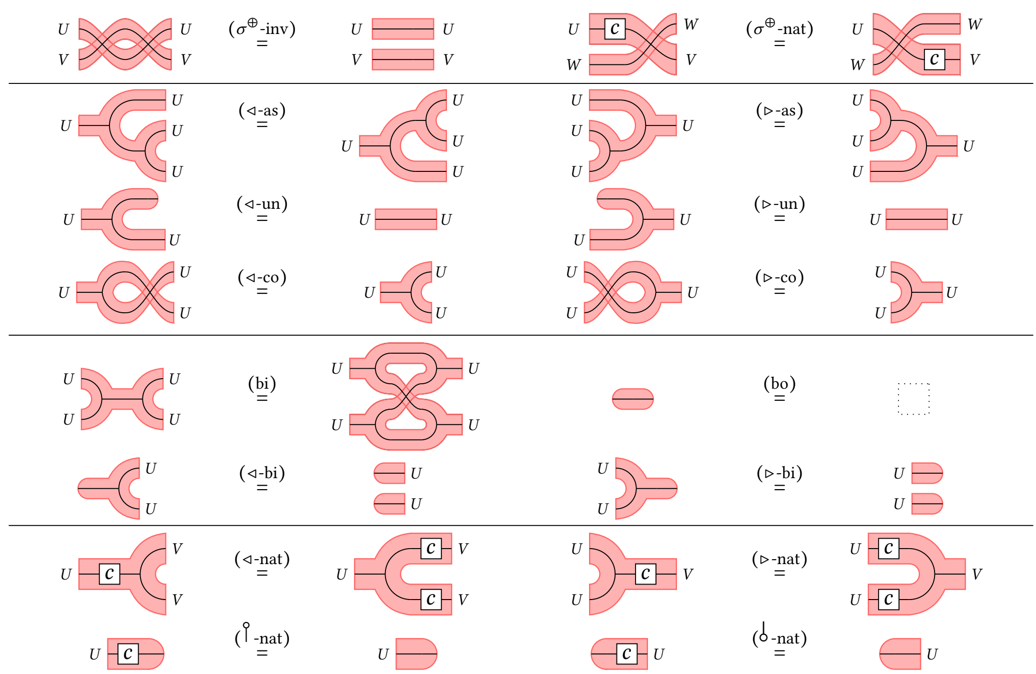


Fig. 1. Axioms for tape diagrams

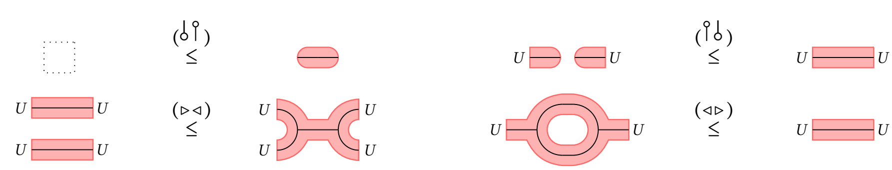


Fig. 2. Axioms for ⫰ ⊣ ⫯ and $\rhd \dashv \lhd$

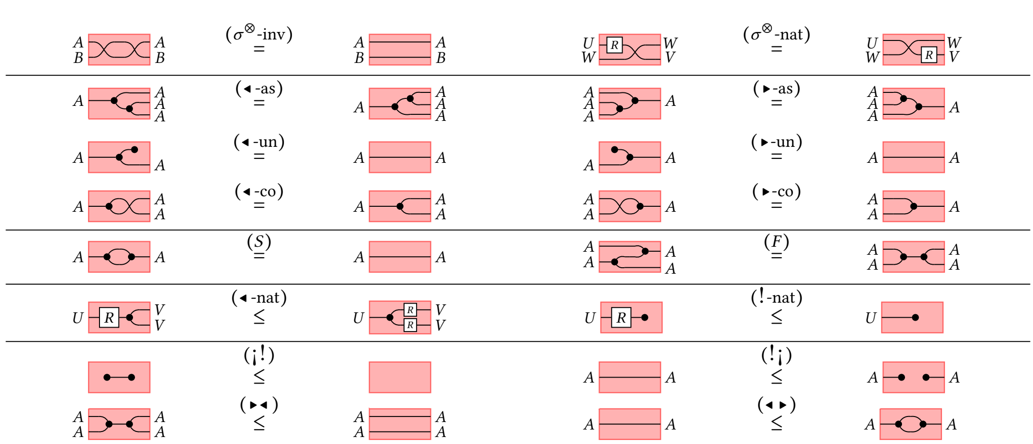


Fig. 3. Axioms of cartesian bicategories

Hereafter, we call $\mathbf{C}_\Sigma$ the category having as objects words in $\mathcal{S}^\star$ and as arrows string diagrams. Theorem 2.3 in [Joyal and Street 1991] states that $\mathbf{C}_\Sigma$ is a *symmetric strict monoidal category freely generated* by Σ.

Definition 3.1. A *symmetric monoidal category* consists of a category $\mathbf{C}$, a bifunctor $\odot\colon \mathbf{C}\times\mathbf{C}\to\mathbf{C}$, an object I and natural isomorphisms

$$\alpha_{X,Y,Z}\colon (X\odot Y)\odot Z \to X\odot(Y\odot Z) \qquad \lambda_X\colon I\odot X\to X \qquad \rho_X\colon X\odot I\to X \qquad \sigma^{\odot}_{X,Y}\colon X\odot Y\to Y\odot X$$

satisfying the usual coherence axioms [Mac Lane 1978]. $\mathbf{C}$ is said to be *strict* when α, λ and ρ are identity natural isomorphisms. A *strict symmetric monoidal functor* is a functor $F\colon\mathbf{C}\to\mathbf{D}$ preserving $\odot$, I and $\sigma^{\odot}$.

Remark 3.2. In *strict* symmetric monoidal (ssm) categories the symmetry σ is not forced to be the identity, since this would raise some problems: for instance, $(f_1;g_1)\odot(f_2;g_2) = (f_1;g_2)\odot(f_2;g_1)$ for all $f_1,f_2\colon A\to B$ and $g_1,g_2\colon B\to C$. As we will see in Section 4, this fact will make the issue of strictness for rig categories rather subtle.

To illustrate in which sense $\mathbf{C}_\Sigma$ is freely generated, it is convenient to introduce *interpretations* in a fashion similar to [Selinger 2010]: an interpretation $\mathcal{I}$ of Σ into an ssm category $\mathbf{D}$ consists of two functions $\alpha_{\mathcal{S}}\colon\mathcal{S}\to Ob(\mathbf{D})$ and $\alpha_\Sigma\colon\Sigma\to Ar(\mathbf{D})$ such that, for all $s\in\Sigma$, $\alpha_\Sigma(s)$ is an arrow having as domain $\alpha_{\mathcal{S}}^\sharp(ar(s))$ and codomain $\alpha_{\mathcal{S}}^\sharp(coar(s))$, for $\alpha_{\mathcal{S}}^\sharp\colon\mathcal{S}^\star\to Ob(\mathbf{D})$ the inductive extension of $\alpha_{\mathcal{S}}$. $\mathbf{C}_\Sigma$ is freely generated by Σ in the sense that, for all ssm categories $\mathbf{D}$ and all interpretations $\mathcal{I}$ of Σ in $\mathbf{D}$, there exists a unique ssm functor $[\![-]\!]_{\mathcal{I}}\colon\mathbf{C}_\Sigma\to\mathbf{D}$ extending $\mathcal{I}$ (i.e. $[\![s]\!]_{\mathcal{I}} = \alpha_\Sigma(s)$ for all $s\in\Sigma$).

One can easily extend the notion of interpretation of Σ into a symmetric monoidal category $\mathbf{D}$ that is not necessarily strict. In this case we set $\alpha_{\mathcal{S}}^\sharp\colon\mathcal{S}^\star\to Ob(\mathbf{D})$ to be the *right bracketing* of the inductive extension of $\alpha_{\mathcal{S}}$. For instance, $\alpha_{\mathcal{S}}^\sharp(ABC) = \alpha_{\mathcal{S}}(A)\odot(\alpha_{\mathcal{S}}(B)\odot\alpha_{\mathcal{S}}(C))$.

Example 3.3. The set Σ in CR_Σ can be regarded as a monoidal signature. The set of sorts $\mathcal{S}$ is the singleton set $\{A\}$, while the set of generators is exactly Σ. Each generator $R\in\Sigma$ has both arity and coarity A. Now, interpretations $\mathcal{I}=(\alpha_{\mathcal{S}},\alpha_\Sigma)$ of this monoidal signature into $\mathbf{Rel}$, the category of sets and relations, are exactly relational interpretations as intended for CR: a set $\alpha_{\mathcal{S}}(A)$ (named X in CR) together with, for all $R\in\Sigma$, a relation $\alpha_\Sigma(R)\colon\alpha_{\mathcal{S}}(A)\to\alpha_{\mathcal{S}}(A)$ (named $\rho(R)$ in CR).

The Two Monoidal Structures of **Rel**. It is often the case that the same category carries more than one monoidal product. An example relevant to this work is $\mathbf{Rel}$ which exhibits two monoidal structures: $(\mathbf{Rel},\otimes,1)$ and $(\mathbf{Rel},\oplus,0)$. In the former, $\otimes$ is given by the cartesian product, i.e. $R\otimes S \stackrel{\text{def}}{=} \{(\,(x_1,x_2),\,(y_1,y_2)\,) \mid (x_1,y_1)\in R \text{ and } (x_2,y_2)\in S\}$ for all relations R,S, and the monoidal unit is the singleton set $1=\{\bullet\}$. In the latter, $\oplus$ is given by disjoint union, i.e. $R\oplus S \stackrel{\text{def}}{=} \{(\,(x,0),\,(y,0)\,) \mid (x,y)\in R\} \cup \{(\,(x,1),\,(y,1)\,) \mid (x,y)\in S\}$, and the monoidal unit 0 is the empty set. It is worth recalling that in $\mathbf{Rel}$ the empty set is both an initial and final object, i.e. a zero object, and that the disjoint union is both a coproduct and product, in fact a biproduct. Indeed $(\mathbf{Rel},\oplus,0)$ is our first example of a finite biproduct category.

Definition 3.4. A *finite biproduct (fb) category* is a symmetric monoidal category $(\mathbf{C},\odot,I)$ where for every object X there are morphisms $\triangleright_X\colon X\odot X\to X$, $\mathrel{\overset{\circ}{\shortmid}}_X\colon I\to X$, $\triangleleft_X\colon X\to X\odot X$, $\mathrel{\underset{\circ}{\shortmid}}_X\colon X\to I$ s.t.

(1) $(\triangleright_X,\mathrel{\overset{\circ}{\shortmid}}_X)$ is a commutative monoid and $(\triangleleft_X,\mathrel{\underset{\circ}{\shortmid}}_X)$ is a cocommutative comonoid, satisfying the coherence axioms in [Selinger 2010, Table 4.7] and their duals (see also [Fox 1976]),
(2) every arrow $f\colon X\to Y$ is both a monoid and a comonoid homomorphism, i.e.

$$(f\odot f);\triangleright_Y=\triangleright_X;f \qquad \mathrel{\overset{\circ}{\shortmid}}_X;f=\mathrel{\overset{\circ}{\shortmid}}_Y \qquad f;\triangleleft_Y=\triangleleft_X;(f\odot f) \qquad f;\mathrel{\underset{\circ}{\shortmid}}_Y=\mathrel{\underset{\circ}{\shortmid}}_X.$$

A *morphism of finite biproduct categories* is a symmetric monoidal functor preserving $\triangleright_X, \multimap_X, \triangleleft_X, \multimap_X$.

Observe that (2) above simply amounts to naturality of monoids and comonoids. Monoids and comonoids in $(\mathbf{Rel}, \oplus, 0)$ are illustrated in the first column below:

$$
\begin{aligned}
\triangleright_X &\stackrel{\text{def}}{=} \{((x,0),\ x) \mid x \in X\} \cup \{((x,1),\ x) \mid x \in X\} & \blacktriangleright_X &\stackrel{\text{def}}{=} \blacktriangleleft_X^\dagger \\
\multimap_X &\stackrel{\text{def}}{=} \{\} & ¡_X &\stackrel{\text{def}}{=} !_X^\dagger \\
\triangleleft_X &\stackrel{\text{def}}{=} \triangleright_X^\dagger & \blacktriangleleft_X &\stackrel{\text{def}}{=} \{(x,\ (x,x)) \mid x \in X\} \\
\multimap_X &\stackrel{\text{def}}{=} \multimap_X^\dagger & !_X &\stackrel{\text{def}}{=} \{(x,\bullet) \mid x \in X\} \subseteq X \times 1
\end{aligned}
\tag{7}
$$

$(\mathbf{Rel}, \otimes, 1)$ has monoids and comonoids too, shown in the second column above. However, they fail to be natural, thus $(\mathbf{Rel}, \otimes, 1)$ is not a fb category. It is instead the archetypical cartesian bicategory.

Definition 3.5. A *cartesian bicategory*, in the sense of [Carboni and Walters 1987], is a symmetric monoidal category $(\mathbf{C}, \odot, I)$ enriched over the category of posets where for every object X there are morphisms $\blacktriangleright_X : X \odot X \to X$, $¡_X : I \to X$, $\blacktriangleleft_X : X \to X \odot X$, $!_X : X \to I$ such that

(1) $(\blacktriangleright_X, ¡_X)$ is a commutative monoid and $(\blacktriangleleft_X, !_X)$ is a cocommutative comonoid, satisfying the coherence axioms as in Definition 3.4.1,
(2) every arrow $f : X \to Y$ is a lax comonoid homomorphism, i.e.

$$f ; \blacktriangleleft_Y \leq \blacktriangleleft_X ; (f \odot f) \qquad f ; !_Y \leq !_X,$$

(3) monoids and comonoids form special Frobenius bimonoids (see e.g. [Lack 2004]),
(4) the comonoid $(\blacktriangleleft_X, !_X)$ is left adjoint to the monoid $(\blacktriangleright_X, ¡_X)$, i.e.

$$¡_X ; !_X \leq id_I \qquad \blacktriangleright_X ; \blacktriangleleft_X \leq id_X \odot id_X \qquad id_X \leq !_X ; ¡_X \qquad id_X \leq \blacktriangleleft_X ; \blacktriangleright_X .$$

A *morphism of cartesian bicategories* is a poset enriched symmetric monoidal functor preserving monoids and comonoids.

In [Bonchi et al. 2018] the authors introduced a string diagrammatic language, named $\mathbf{CB}_\Sigma$, expressing the cartesian bicategory structure of $(\mathbf{Rel}, \otimes, 1)$. One can similarly define a language for $(\mathbf{Rel}, \oplus, 0)$, however combining them would require a diagrammatic language able to handle two monoidal products at once. The appropriate categorical structures for this are rig categories.

4 RIG CATEGORIES

Rig categories [Johnson and Yau 2022; Laplaza 1972], also known as *bimonoidal categories*, involve two monoidal structures where one distributes over the other. The structures introduced below are sometimes referred to as *symmetric* rig categories, but we will just call them rig categories.

Definition 4.1. A *rig category* is a category $\mathbf{C}$ with two symmetric monoidal structures $(\mathbf{C}, \otimes, 1, \sigma^\otimes)$ and $(\mathbf{C}, \oplus, 0, \sigma^\oplus)$ and natural isomorphisms

$$\delta^l_{X,Y,Z} : X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z) \qquad \lambda^\bullet_X : 0 \otimes X \to 0$$

$$\delta^r_{X,Y,Z} : (X \oplus Y) \otimes Z \to (X \otimes Z) \oplus (Y \otimes Z) \qquad \rho^\bullet_X : X \otimes 0 \to 0$$

satisfying the coherence axioms in [Laplaza 1972]. A rig category is said to be *right* (*left*) *strict* when both its monoidal structures are strict and $\lambda^\bullet, \rho^\bullet$ and δ^r (δ^l) are identity natural isomorphisms.

The natural isomorphism δ^r (δ^l) is called *right* (*left*) *distributor*. The reader may wonder why only one of the two distributors is forced to be the identity within a strict rig category. This can

Table 2. Equations for the objects of a free sesquistrict rig category

$X \ ::= \ A \mid 1 \mid 0 \mid X \otimes X \mid X \oplus X$		
$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$	$1 \otimes X = X$	$X \otimes 1 = X$
$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$	$0 \oplus X = X$	$X \oplus 0 = X$
$(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z)$	$0 \otimes X = 0$	$X \otimes 0 = 0$
$A \otimes (Y \oplus Z) = (A \otimes Y) \oplus (A \otimes Z)$		

(a)

n-ary sums and products		
$\bigoplus_{i=1}^{0} X_i = 0$	$\bigoplus_{i=1}^{1} X_i = X_1$	$\bigoplus_{i=1}^{n+1} X_i = X_1 \oplus (\bigoplus_{i=1}^{n} X_{i+1})$
$\bigotimes_{i=1}^{0} X_i = 1$	$\bigotimes_{i=1}^{1} X_i = X_1$	$\bigotimes_{i=1}^{n+1} X_i = X_1 \otimes (\bigotimes_{i=1}^{n} X_{i+1})$

(b)

intuitively be explained as follows: imagine requiring that both distributors are identities. This would imply that both equations below should hold for all objects X, Y, Z of any such strict category:

$$(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z) \quad \text{and} \quad X \otimes (Y \oplus Z) = (X \otimes Y) \oplus (X \otimes Z).$$

The coexistence of the above laws would raise the same problems of strictification of symmetries (see Remark 3.2). Indeed it would hold at once that

$$(A \oplus B) \otimes (C \oplus D) = ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) = ((A \otimes C) \oplus (B \otimes C)) \oplus ((A \otimes D) \oplus (B \otimes D))$$

and

$$(A \oplus B) \otimes (C \oplus D) = (A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D)) = ((A \otimes C) \oplus (A \otimes D)) \oplus ((B \otimes C) \oplus (B \otimes D))$$

Note that $(B \otimes C)$ and $(A \otimes D)$ are in the opposite order in the two terms.

The traditional approach to strictness is however unsatisfactory when studying freely generated categories. To illustrate our concerns, consider a right strict rig category freely generated by a signature Σ with sorts $\mathcal{S}$. The objects of this category are terms generated by the grammar in Table 2a modulo the equations in the first three rows of the same table. These equivalence classes of terms do not come with a very handy form, unlike, for instance, the objects of a strict monoidal category, which are words. For this reason several authors, like [Comfort et al. 2020; Johnson and Yau 2022], prefer to take as objects polynomials in $\mathcal{S}$ at the price of working with a category that is not freely generated but only equivalent to a freely generated one. This fact forces one to consider functors that are not necessarily strict, thus most of the constructions need to properly deal with the tedious natural isomorphisms.

Here we propose a solution that might look a bit technical but, in our opinion, is rewarding. We focus on freely generated rig categories that we call *sesquistrict*, i.e. right strict but only partially left strict: namely the left distributor $\delta^l_{X,Y,Z}: X \otimes (Y \oplus Z) \to (X \otimes Y) \oplus (X \otimes Z)$ is the identity only when X is a basic sort $A \in \mathcal{S}$. In terms of the equations to impose on objects, this amounts to the one in the fourth row in Table 2a for each $A \in \mathcal{S}$. It is useful to observe that the addition of these equations avoids the problem of using left and right strictness at the same time. Indeed $(A \oplus B) \otimes (C \oplus D)$ turns out to be equal to $(A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D)$ but not to $(A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D)$. Moreover, by orienting all the equations in Table 2a from left to right, one obtains a rewriting system that is confluent and terminating and, most importantly, the unique normal forms are exactly polynomials.

Definition 4.2. A term X of the grammar in Table 2a is said to be in *polynomial* form if there exist n, m_i and $A_{i,j} \in \mathcal{S}$ for $i = 1 \dots n$ and $j = 1 \dots m_i$ such that $X = \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{m_i} A_{i,j}$ (for n-ary sums and products as in Table 2b).

We will always refer to terms in polynomial form as *polynomials* and, for a polynomial like X above, we will call *monomials* of X the n terms $\bigotimes_{j=1}^{m_i} A_{i,j}$. For instance the monomials of $(A \otimes B) \oplus 1$ are $A \otimes B$ and 1. Note that, differently from the polynomials we are used to dealing with, here neither $\oplus$ nor $\otimes$ is commutative so, for instance, $(A \otimes B) \oplus 1$ is different from both $1 \oplus (A \otimes B)$ and

$(B \otimes A) \oplus 1$. Note that non-commutative polynomials are in one to one correspondence with *words of words* over $\mathcal{S}$, while monomials are words over $\mathcal{S}$.

Notation. Through the whole paper, we will denote by $A, B, C \dots$ the sorts in $\mathcal{S}$, by $U, V, W \dots$ the words in $\mathcal{S}^\star$ and by $P, Q, R, S \dots$ the words of words in $(\mathcal{S}^\star)^\star$. Given two words $U, V \in \mathcal{S}^\star$, we will write UV for their concatenation and 1 for the empty word. Given two words of words $P, Q \in (\mathcal{S}^\star)^\star$, we will write $P \oplus Q$ for their concatenation and 0 for the empty word of words. Given a word of words P, we will write πP for the corresponding term in polynomial form, for instance $\pi(A \oplus BCD \oplus 1)$ is the term $A \oplus ((B \otimes (C \otimes D)) \oplus 1)$. Throughout this paper we often omit π, thus we implicitly identify words of words with polynomials.

Beyond concatenation ($\oplus$), one can define on $(\mathcal{S}^\star)^\star$ a product operation $\otimes$ by taking the unique normal form of $\pi(P) \otimes \pi(Q)$ for any $P, Q \in (\mathcal{S}^\star)^\star$. More explicitly for $P = \bigoplus_i U_i$ and $Q = \bigoplus_j V_j$,

$$P \otimes Q \stackrel{\text{def}}{=} \bigoplus_i \bigoplus_j U_i V_j. \tag{8}$$

Observe that, if both P and Q are monomials, namely $P = U$ and $Q = V$ for some $U, V \in \mathcal{S}^\star$, then $P \otimes Q = UV$. We can thus safely write PQ in place of $P \otimes Q$ without the risk of any confusion.

Words of words are also exploited in the traditional *strictification* theorem for rig categories (see e.g. Theorem 5.4.6 in [Johnson and Yau 2022]): given a rig category $\mathbf{C}$, one can build a right strict rig category $\overline{\mathbf{C}}$ that is equivalent to it. The objects of the strictification $\overline{\mathbf{C}}$ are words of words of objects of $\mathbf{C}$, while $\overline{\mathbf{C}}[P, Q] = \mathbf{C}[\pi P, \pi Q]$. It is easy to see that $\mathbf{C}$ embeds into $\overline{\mathbf{C}}$: a morphism $f: X \to Y$ in $\mathbf{C}$ can be seen as a morphism in $\overline{\mathbf{C}}[X, Y]$, where X and Y are considered as unary words made of a single unary word. This embedding forms an equivalence of rig categories $\mathbf{C} \simeq \overline{\mathbf{C}}$. See [Johnson and Yau 2022] for further details.

It turns out that $\overline{\mathbf{C}}$ does actually satisfy a partial left distribution, with $Ob(\mathbf{C})$ playing the role of $\mathcal{S}$, as it were. To make this statement precise, we introduce the following notion.

Definition 4.3. A *sesquistrict rig category* is a functor $H: \mathbf{S} \to \mathbf{C}$, where $\mathbf{S}$ is a discrete category and $\mathbf{C}$ is a strict rig category, such that for all $A \in \mathbf{S}$

$$\delta^l_{H(A),X,Y}: H(A) \otimes (X \oplus Y) \to (H(A) \otimes X) \oplus (H(A) \otimes Y)$$

is an identity morphism. We will also say, in this case, that $\mathbf{C}$ is a $\mathbf{S}$-sesquistrict rig category.

Given $H: \mathbf{S} \to \mathbf{C}$ and $H': \mathbf{S}' \to \mathbf{C}'$ two sesquistrict rig categories, a *sesquistrict rig functor* from H to H' is a pair $(\alpha: \mathbf{S} \to \mathbf{S}', \beta: \mathbf{C} \to \mathbf{C}')$, with α a functor and β a strict rig functor, such that $\alpha; H' = H; \beta$.

Theorem 4.4. *Let $\mathbf{C}$ be a rig category. Then its strictification $\overline{\mathbf{C}}$ is a $Ob(\mathbf{C})$-sesquistrict rig category.*

Proof. We recall from [Johnson and Yau 2022] some facts about $\overline{\mathbf{C}}$. If P and Q are in $\overline{\mathbf{C}}$, then $P \otimes Q$ is given as in (8) and it is immediate to see that $A \otimes (P \oplus Q) = (A \otimes P) \oplus (A \otimes Q)$ if $A \in Ob(\mathbf{C})$. To prove that $id_{A \otimes (P \oplus Q)} = \delta^l_{A,P,Q}$ in $\overline{\mathbf{C}}$, one needs to expand the definitions of δ^l and $\sigma^\otimes$ in $\overline{\mathbf{C}}$, which involve structural isomorphisms of $\mathbf{C}$, and use the coherence for rig categories, as stated in [Johnson and Yau 2022, Thm. 3.9.1]. For this it is enough to observe that $\pi(A \otimes (P \oplus Q))$ is *regular* in the sense of [Johnson and Yau 2022, Def. 3.1.25]. □

Corollary 4.5 (Sesquistrictification)**.** *Any rig category is rig equivalent to a sesquistrict rig category.*

In light of the corollary above, from now on we will only consider strictified rig categories. Moreover, whenever we will be considering ssm functors $F: (\mathbf{C}, \otimes 1) \to (\mathbf{D}, \otimes, 1)$ where $\mathbf{D}$ is a rig category, we will actually mean that $F: \mathbf{C} \to \overline{\mathbf{D}}$ and we will assume that F sends objects of $\mathbf{C}$ to objects of $\mathbf{D}$ (embedded in $\overline{\mathbf{D}}$). This assumption is not restrictive, as the following lemma shows.

Lemma 4.6. *Let* $\mathbf{D}$ *be a rig category,* $(\mathbf{C}, \otimes, 1)$ *a strict symmetric monoidal category and* $F\colon \mathbf{C} \to (\overline{\mathbf{D}}, \otimes, 1)$ *a strict symmetric monoidal functor. Then there exists a ssm functor* $F'\colon \mathbf{C} \to \overline{\mathbf{D}}$*, monoidally-naturally isomorphic to* F*, such that* $F(Ob(\mathbf{C})) \subseteq Ob(\mathbf{D})$*.*

Finite Biproduct Rig Categories. On many occasions, one is interested in rig categories where $\oplus$ has some additional structure. For instance, distributive monoidal categories are rig categories where $\oplus$ is a coproduct. In this paper we will focus on rig categories where $\oplus$ is a biproduct.

Definition 4.7. A *finite biproduct rig category* is a rig category $(\mathbf{C}, \oplus, 0, \otimes, 1)$ such that $(\mathbf{C}, \oplus, 0)$ is a finite biproduct category.

Rel, with the two monoidal structures defined as in Section 3, is a fb rig category. Another example relevant to this paper is $\mathbf{Mat}(\mathbb{C})$, the category of complex matrices, with $\otimes$ and $\oplus$ being, respectively, the Kronecker product and the direct sum of matrices. Beyond these examples, our interest in fb rig categories is motivated by Theorem 4.9, which we are going to illustrate now.

Given $\mathcal{S}$ a set of sorts, a *rig signature* is a tuple $(\mathcal{S}, \Sigma, ar, coar)$ where ar and $coar$ assign to each $s \in \Sigma$ an arity and a coarity respectively, which are terms in the grammar specified in Table 2a modulo the equations underneath it. (Notice that any monoidal signature is a rig signature.) To define the notion of free sesquistrict fb rig category, we need to extend interpretations of monoidal signatures to the fb rig case. An *interpretation* of a rig signature $(\mathcal{S}, \Sigma, ar, coar)$ in a sesquistrict fb rig category $H\colon \mathbf{M} \to \mathbf{D}$ is a pair of functions $(\alpha_{\mathcal{S}}\colon \mathcal{S} \to Ob(\mathbf{M}), \alpha_\Sigma\colon \Sigma \to Ar(\mathbf{D}))$ such that, for all $s \in \Sigma$, $\alpha_\Sigma(s)$ is an arrow having as domain and codomain $(\alpha_{\mathcal{S}}; H)^\sharp(ar(s))$ and $(\alpha_{\mathcal{S}}; H)^\sharp(coar(s))$.

Definition 4.8. Let $(\mathcal{S}, \Sigma, ar, coar)$ (simply Σ for short) be a rig signature. A sesquistrict fb rig category $H\colon \mathbf{M} \to \mathbf{D}$ is said to be *freely generated* by Σ if there is an interpretation $(\alpha_S, \alpha_\Sigma)$ of Σ in H such that for every sesquistrict rig category $H'\colon \mathbf{M}' \to \mathbf{D}'$ and every interpretation $(\alpha'_{\mathcal{S}}\colon \mathcal{S} \to Ob(\mathbf{M}'), \alpha'_\Sigma\colon \Sigma \to Ar(\mathbf{D}'))$ there exists a unique sesquistrict rig functor $(\alpha\colon \mathbf{M} \to \mathbf{M}', \beta\colon \mathbf{D} \to \mathbf{D}')$ such that $\alpha_{\mathcal{S}}; \alpha = \alpha'_{\mathcal{S}}$ and $\alpha_\Sigma; \beta = \alpha'_\Sigma$.

Sesquistrict fb rig categories generated by a given signature are isomorphic to each other, hence we will refer to “the” free sesquistrict rig category generated by a signature.

Theorem 4.9. *For every rig signature* $(\mathcal{S}, \Sigma)$ *there exists a monoidal signature* $(\mathcal{S}, \Sigma_M)$ *such that the free sesquistrict fb rig categories generated by* $(\mathcal{S}, \Sigma)$ *and by* $(\mathcal{S}, \Sigma_M)$ *are isomorphic.*

The proof of the above theorem follows the same pattern as the proof of the well known fact that one can reduce monoidal signatures to standard cartesian signatures when generating the free finite product category. In particular, Σ_M is obtained from Σ by replacing each generator $s\colon \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_j$ in Σ with new, formal symbols $s_{j,i}$ of type $U_i \to V_j$.

5 TAPE DIAGRAMS

In Section 3 we saw that string diagrams provide a convenient graphical language for strict monoidal categories. Here we introduce tape diagrams to depict arrows of sesquistrict rig categories.

To show our main idea, we recall the following functors amongst the categories **Cat** of categories and functors, **SMC**, of ssm categories and functors, and **FBC**, of strict fb categories and functors:

$$\begin{tikzcd} \mathbf{SMC} \arrow[r, "U_1"] & \mathbf{Cat} \arrow[r, bend left, "F_2"] \arrow[r, phantom, "\perp"] & \mathbf{FBC} \arrow[l, bend left, "U_2"] \end{tikzcd} \tag{9}$$

U_1 and U_2 are the forgetful functors, while F_2, the left adjoint to U_2, can be described as follows.

Table 3. Additional axioms for $F_2(\mathbf{C})$. Above, $c\colon A \to B$ is an arbitrary arrow of $\mathbf{C}$

$\lhd_A ; (id_A \odot \lhd_A) = \lhd_A ; (\lhd_A \odot id_A)$		$\lhd_A ; (\text{⫯}_A \odot id_A) = id_A$	$\lhd_A ; \sigma_{A,A} = \lhd_A$
$(id_A \odot \rhd_A) ; \rhd_A = (\rhd_A \odot id_A) ; \rhd_A$		$(\text{⫰}_A \odot id_A) ; \rhd_A = id_A$	$\sigma_{A,A} ; \rhd_A = \rhd_A$
$\rhd_A ; \lhd_A = \lhd_{A \odot A} ; (\rhd_A \odot \rhd_A)$	$\text{⫰}_A ; \text{⫯}_A = id_I$	$\text{⫰}_A ; \lhd_A = \text{⫰}_{A \odot A}$	$\lhd_A ; \text{⫯}_A = \text{⫯}_{A \odot A}$
$\overline{\underline{c}} ; \text{⫯}_B = \text{⫯}_A$	$\overline{\underline{c}} ; \lhd_B = \lhd_A ; (\overline{\underline{c}} \odot \overline{\underline{c}})$	$\text{⫰}_A ; \overline{\underline{c}} = \text{⫰}_B$	$\rhd_A ; \overline{\underline{c}} = (\overline{\underline{c}} \odot \overline{\underline{c}}) ; \rhd_B$
$\overline{\underline{id_A}} = id_A$		$\overline{\underline{c ; d}} = \overline{\underline{c}} ; \overline{\underline{d}}$	

Definition 5.1. Let $\mathbf{C}$ be a category. The strict fb category freely generated by $\mathbf{C}$, hereafter denoted by $F_2(\mathbf{C})$, has as objects words of objects of $\mathbf{C}$. Arrows are terms inductively generated by the following grammar, where A, B and c range over arbitrary objects and arrows of $\mathbf{C}$:

$$f \quad ::= \quad id_A \mid id_I \mid \overline{\underline{c}} \mid \sigma^{\odot}_{A,B} \mid f ; f \mid f \odot f \mid \text{⫯}_A \mid \lhd_A \mid \text{⫰}_A \mid \rhd_A \tag{10}$$

modulo the axioms in Tables 1 and 3. Notice in particular the last two from Table 3:

$$\overline{\underline{id_A}} = id_A \qquad \overline{\underline{c ; d}} = \overline{\underline{c}} ; \overline{\underline{d}} \tag{Tape}$$

The assignment $\mathbf{C} \mapsto F_2(\mathbf{C})$ easily extends to functors $H\colon \mathbf{C} \to \mathbf{D}$. The unit of the adjunction $\eta\colon Id_{\mathbf{Cat}} \Rightarrow F_2 U_2$ is defined for each category $\mathbf{C}$ as the functor $\overline{\underline{\cdot}}\colon \mathbf{C} \to U_2 F_2(\mathbf{C})$ which is the identity on objects and maps every arrow c of $\mathbf{C}$ into the arrow $\overline{\underline{c}}$ of $U_2 F_2(\mathbf{C})$. Observe that $\overline{\underline{\cdot}}$ is indeed functorial thanks to the axioms (Tape). We will refer hereafter to $\overline{\underline{\cdot}}$ as the *taping functor*.

Lemma 5.2. *$F_2\colon \mathbf{Cat} \to \mathbf{FBC}$ is left adjoint to $U_2\colon \mathbf{FBC} \to \mathbf{Cat}$.*

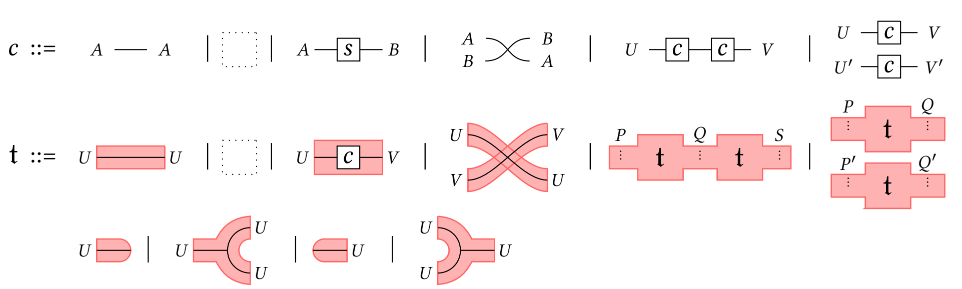
The main result of this section (Theorem 5.11) states that the sesquistrict fb rig category freely generated by a monoidal signature Σ is $F_2 U_1(\mathbf{C}_\Sigma)$, hereafter referred to as $\mathbf{T}_\Sigma$. This is somehow the fb rig category analogue to Theorem 2.3 in [Joyal and Street 1991] (cf. Section 3). Let us see why.

Recall that the set of objects of $\mathbf{C}_\Sigma$ is $\mathcal{S}^\star$, i.e. words of sorts in $\mathcal{S}$. The set of objects of $\mathbf{T}_\Sigma$ is thus $(\mathcal{S}^\star)^\star$, namely words of words of sorts in $\mathcal{S}$. For arrows, consider the following two-layer grammar where $s \in \Sigma$, $A, B \in \mathcal{S}$ and $U, V \in \mathcal{S}^\star$.

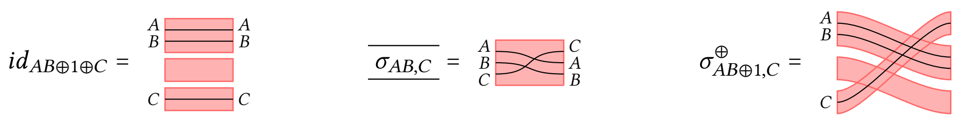
$$\begin{array}{rcl} c & ::= & id_A \mid id_1 \mid s \mid \sigma_{A,B} \mid c ; c \mid c \otimes c \\ \mathfrak{t} & ::= & id_U \mid id_0 \mid \overline{\underline{c}} \mid \sigma^{\oplus}_{U,V} \mid \mathfrak{t} ; \mathfrak{t} \mid \mathfrak{t} \oplus \mathfrak{t} \mid \text{⫯}_U \mid \lhd_U \mid \text{⫰}_U \mid \rhd_U \end{array} \tag{11}$$

The terms of the first row, denoted by c, are called *circuits*. Modulo the axioms in Table 1 (after replacing $\odot$ with $\otimes$), these are exactly the arrows of $\mathbf{C}_\Sigma$. The terms of the second row, denoted by $\mathfrak{t}$, are called *tapes*. Modulo the axioms in Tables 1 and 3 (after replacing $\odot$ with $\oplus$ and A, B with U, V), these are exactly the arrows of $F_2 U_1(\mathbf{C}_\Sigma)$, i.e. $\mathbf{T}_\Sigma$.

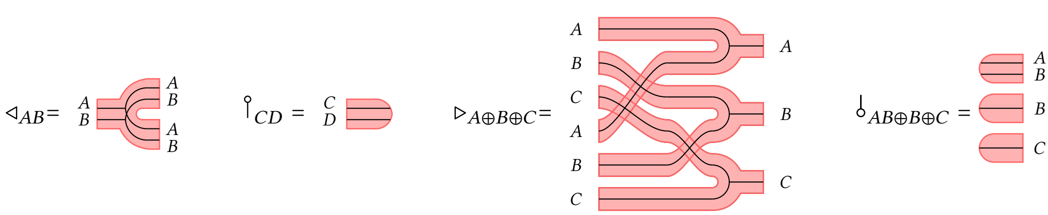
Since circuits are arrows of $\mathbf{C}_\Sigma$, these can be graphically represented as string diagrams. Also tapes can be represented as string diagrams, since they satisfy all of the axioms of ssm category. Note however that *inside* tapes there are string diagrams: this justifies the motto *tape diagrams are string diagrams of string diagrams*. We can thus render graphically the grammar in (11):



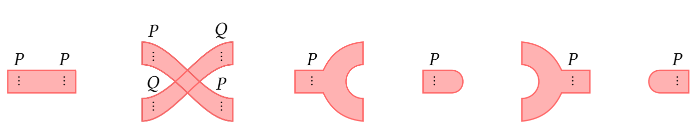
The identity id_0 is rendered as the empty tape, while id_1 is : a tape filled with the empty circuit. For a monomial $U = A_1 \dots A_n$, id_U is depicted as a tape containing n wires labelled by A_i. For instance, id_{AB} is rendered as . When clear from the context, we will simply represent it as a single wire with the appropriate label. Similarly, for a polynomial $P = \bigoplus_{i=1}^{n} U_i$, id_P is obtained as a vertical composition of tapes, as illustrated below on the left.



We can render graphically the symmetries $\overline{\underline{\sigma_{U,V}}} : UV \to VU$ and $\sigma^{\oplus}_{P,Q} : P \oplus Q \to Q \oplus P$ as crossings of wires and crossings of tapes, see the two rightmost diagrams above. The diagonal $\lhd_U : U \to U \oplus U$ is represented as a splitting of tapes, while the bang $!_U : U \to 0$ is a tape closed on its right boundary. Codiagonals and cobangs are represented in the same way but mirrored along the y-axis. Exploiting the coherence axioms of fb categories, we can construct (co)diagonals and (co)bangs for arbitrary polynomials P. For example, $\lhd_{AB}$, $!_{CD}$, $\rhd_{A\oplus B\oplus C}$ and $¡_{AB\oplus B\oplus C}$ are depicted as:



When the structure inside a tape is not relevant the graphical language can be "compressed" in order to simplify the diagrammatic reasoning. For example, for arbitrary polynomials P, Q we represent $id_P, \sigma^{\oplus}_{P,Q}, \lhd_P, !_P, \rhd_P, ¡_P$ as follows:



Moreover, for an arbitrary tape diagram $\mathfrak{t} : P \to Q$ we write .

It is important to observe that the graphical representation takes care of the two axioms in (Tape): both sides of the leftmost axiom are depicted as while both sides of the rightmost axiom as . The axioms of monoidal categories are also implicit in the graphical representation, while those for symmetries and the fb-structure (in Table 3) have to be depicted explicitly as in Figure 1. In particular, the diagrams in the first row express the inverse law and naturality of $\sigma^{\oplus}$. In the second group there are the (co)monoid axioms and in the third group the bialgebra ones. Finally, the last group depicts naturality of the (co)diagonals and (co)bangs.

5.1 The Finite Biproduct Rig Structure of Tapes

By definition $\mathbf{T}_\Sigma$ is equipped with a monoidal product $\oplus$ that is a biproduct. We now show that $\mathbf{T}_\Sigma$ carries a second monoidal product $\otimes$ which makes it a sesquistrict rig category.

On objects, $\otimes$ is defined as in (8). For instance, $(A \oplus B) \otimes (C \oplus D)$ is $AC \oplus AD \oplus BC \oplus BD$ and not $AC \oplus BC \oplus AD \oplus BD$. This is justified by the fact that we want $\mathbf{T}_\Sigma$ to be a sesquistrict rig category and thus $\otimes$ distributes over $\oplus$ on the right (and only partially on the left).

Table 4. Inductive definition of δ^l and $\sigma^{\otimes}$ in $\mathbf{T}_{\Sigma}$

$\delta^l_{P,Q,R} \colon P \otimes (Q \oplus R) \to (P \otimes Q) \oplus (P \otimes R)$
$\delta^l_{0,Q,R} \stackrel{\text{def}}{=} id_0$
$\delta^l_{U \oplus P',Q,R} \stackrel{\text{def}}{=} (id_{U \otimes (Q \oplus R)} \oplus \delta^l_{P',Q,R}) ; (id_{U \otimes Q} \oplus \sigma^{\oplus}_{U \otimes R, P' \otimes Q} \oplus id_{P' \otimes R})$

(a)

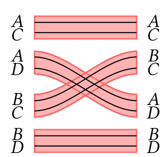
$\sigma^{\otimes}_{P,Q} \colon P \otimes Q \to Q \otimes P$, with $P = \bigoplus_i U_i$
$\sigma^{\otimes}_{P,0} \stackrel{\text{def}}{=} id_0$
$\sigma^{\otimes}_{P,V \oplus Q'} \stackrel{\text{def}}{=} \delta^l_{P,V,Q'} ; (\bigoplus_i \overline{\underline{\sigma_{U_i,V}}} \oplus \sigma^{\otimes}_{P,Q'})$

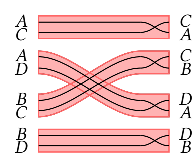
(b)

Remark 5.3. Observe that left distributivity holds not only for $A \in \mathcal{S}$, but for all monomials U: $U \otimes (P \oplus Q) = (U \otimes P) \oplus (U \otimes Q)$ for all polynomials P, Q.

In general, distributivity on the left is possible, but has to be made explicit through left distributors built just from identities and $\oplus$-symmetries, as shown in Table 4a. Similarly, arbitrary $\otimes$-symmetries can be built, as shown in Table 4b, from left distributors and symmetries within tapes $\overline{\underline{\sigma_{U,V}}}$.

Example 5.4. By means of the inductive definition in Table 4, $\delta^l_{A \oplus B,C,D} \colon (A \oplus B)(C \oplus D) \to (A \oplus B)C \oplus (A \oplus B)D$ and $\sigma^{\otimes}_{A \oplus B, C \oplus D} \colon (A \oplus B)(C \oplus D) \to (C \oplus D)(A \oplus B)$ are depicted as follows:

$\delta^l_{A \oplus B,C,D} =$ 

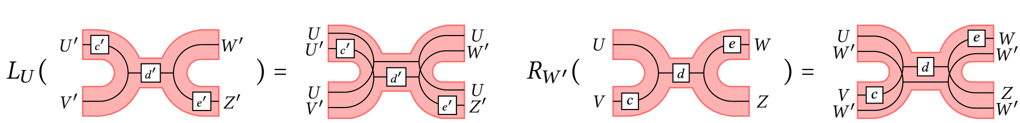
$\sigma^{\otimes}_{A \oplus B, C \oplus D} =$ 

We can now introduce left and right whiskering for those objects U that are monomials.

Definition 5.5. Let U be a monomial. The *left* and *right whiskering* (with respect to U) are two functors $L_U, R_U \colon \mathbf{T}_{\Sigma} \to \mathbf{T}_{\Sigma}$ defined on objects as $L_U(P) \stackrel{\text{def}}{=} U \otimes P$, $R_U(P) \stackrel{\text{def}}{=} P \otimes U$ and on arrows as:

$$
\begin{array}{llll}
L_U(id_0) \stackrel{\text{def}}{=} id_0 & & R_U(id_0) \stackrel{\text{def}}{=} id_0 & \\
L_U(\overline{\underline{c}}) \stackrel{\text{def}}{=} \overline{\underline{id_U \otimes c}} & L_U(\sigma^{\oplus}_{V,W}) \stackrel{\text{def}}{=} \sigma^{\oplus}_{UV,UW} & R_U(\overline{\underline{c}}) \stackrel{\text{def}}{=} \overline{\underline{c \otimes id_U}} & R_U(\sigma^{\oplus}_{V,W}) \stackrel{\text{def}}{=} \sigma^{\oplus}_{VU,WU} \\
L_U(\triangleleft_V) \stackrel{\text{def}}{=} \triangleleft_{UV} & L_U(\text{⫯}_V) \stackrel{\text{def}}{=} \text{⫯}_{UV} & R_U(\triangleleft_V) \stackrel{\text{def}}{=} \triangleleft_{VU} & R_U(\text{⫯}_V) \stackrel{\text{def}}{=} \text{⫯}_{VU} \\
L_U(\triangleright_V) \stackrel{\text{def}}{=} \triangleright_{UV} & L_U(\text{⫰}_V) \stackrel{\text{def}}{=} \text{⫰}_{UV} & R_U(\triangleright_V) \stackrel{\text{def}}{=} \triangleright_{VU} & R_U(\text{⫰}_V) \stackrel{\text{def}}{=} \text{⫰}_{VU} \\
L_U(\mathfrak{t}_1 ; \mathfrak{t}_2) \stackrel{\text{def}}{=} L_U(\mathfrak{t}_1) ; L_U(\mathfrak{t}_2) & & R_U(\mathfrak{t}_1 ; \mathfrak{t}_2) \stackrel{\text{def}}{=} R_U(\mathfrak{t}_1) ; R_U(\mathfrak{t}_2) & \\
L_U(\mathfrak{t}_1 \oplus \mathfrak{t}_2) \stackrel{\text{def}}{=} L_U(\mathfrak{t}_1) \oplus L_U(\mathfrak{t}_2) & & R_U(\mathfrak{t}_1 \oplus \mathfrak{t}_2) \stackrel{\text{def}}{=} R_U(\mathfrak{t}_1) \oplus R_U(\mathfrak{t}_2) &
\end{array}
$$

Example 5.6. The meaning of the monomial whiskering is quite immediate in graphical terms: below we draw the left whiskering of a tape $\mathfrak{s}$ and the right whiskering of a tape $\mathfrak{t}$. $L_U(\mathfrak{s})$ *thickens* the tapes which $\mathfrak{s}$ is made of, by stacking the wires of id_U inside them; the right whiskering works analogously except that the additional wires are stacked at the bottom of the single tapes:



We can extend Definition 5.5 to arbitrary polynomials S as follows.

Table 5. The algebra of whiskerings

$L_S(id_P) = id_{SP}$		$R_S(id_P) = id_{PS}$	(W1)
$L_S(\mathfrak{t};\mathfrak{s}) = L_S(\mathfrak{t});L_S(\mathfrak{s})$		$R_S(\mathfrak{t};\mathfrak{s}) = R_S(\mathfrak{t});R_S(\mathfrak{s})$	(W2)
$L_1(\mathfrak{t}) = \mathfrak{t}$		$R_1(\mathfrak{t}) = \mathfrak{t}$	(W3)
$L_0(\mathfrak{t}) = id_0$		$R_0(\mathfrak{t}) = id_0$	(W4)
$L_S(\mathfrak{t}_1 \oplus \mathfrak{t}_2) = \delta^l_{S,P_1,P_2};(L_S(\mathfrak{t}_1) \oplus L_S(\mathfrak{t}_2));\delta^{-l}_{S,Q_1,Q_2}$		$R_S(\mathfrak{t}_1 \oplus \mathfrak{t}_2) = R_S(\mathfrak{t}_1) \oplus R_S(\mathfrak{t}_2)$	(W5)
$L_{S\oplus T}(\mathfrak{t}) = L_S(\mathfrak{t}) \oplus L_T(\mathfrak{t})$		$R_{S\oplus T}(\mathfrak{t}) = \delta^l_{P,S,T};(R_S(\mathfrak{t}) \oplus R_T(\mathfrak{t}));\delta^{-l}_{Q,S,T}$	(W6)
$L_{P_1}(\mathfrak{t}_2);R_{Q_2}(\mathfrak{t}_1) = R_{P_2}(\mathfrak{t}_1);L_{Q_1}(\mathfrak{t}_2)$			(W7)
$R_S(\lhd_U) = \lhd_{US}$ $\quad R_S(\rhd_U) = \rhd_{US}$	(W8)	$R_S(\mathring{\mid}_U) = \mathring{\mid}_{US}$ $\quad R_S(\underset{\circ}{\mid}_U) = \underset{\circ}{\mid}_{US}$	(W9)
$R_S(\sigma^{\oplus}_{P,Q}) = \sigma^{\oplus}_{PS,QS}$	(W10)	$\sigma^{\otimes}_{PQ,S} = L_P(\sigma^{\otimes}_{Q,S});R_Q(\sigma^{\otimes}_{P,S})$	(W11)
$R_S(\mathfrak{t});\sigma^{\otimes}_{Q,S} = \sigma^{\otimes}_{P,S};L_S(\mathfrak{t})$	(W12)	$L_S(R_T(\mathfrak{t})) = R_T(L_S(\mathfrak{t}))$	(W13)
$L_{ST}(\mathfrak{t}) = L_S(L_T(\mathfrak{t}))$	(W14)	$R_{TS}(\mathfrak{t}) = R_S(R_T(\mathfrak{t}))$	(W15)
$R_S(\delta^l_{P,Q,R}) = \delta^l_{P,QS,RS}$	(W16)	$L_S(\delta^l_{P,Q,R}) = \delta^l_{SP,Q,R};\delta^{-l}_{S,PQ,PR}$	(W17)

Definition 5.7. Let S be a polynomial. $L_S, R_S \colon \mathbf{T}_\Sigma \to \mathbf{T}_\Sigma$ are defined on objects as $L_S(P) \stackrel{\text{def}}{=} S \otimes P$ and $R_S(P) \stackrel{\text{def}}{=} P \otimes S$ and on arrows $\mathfrak{t} \colon P \to Q$ by induction on S:

$$L_0(\mathfrak{t}) \stackrel{\text{def}}{=} id_0 \qquad\qquad R_0(\mathfrak{t}) \stackrel{\text{def}}{=} id_0$$

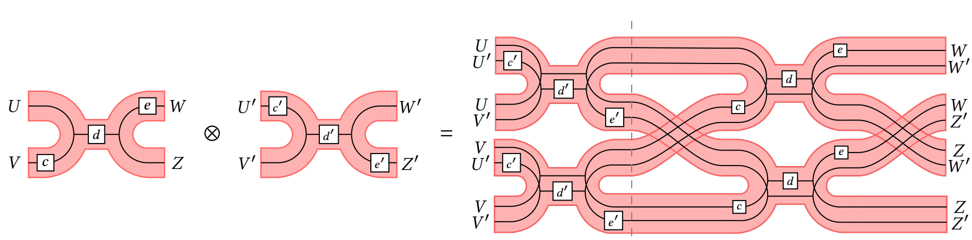
$$L_{W\oplus S'}(\mathfrak{t}) \stackrel{\text{def}}{=} L_W(\mathfrak{t}) \oplus L_{S'}(\mathfrak{t}) \qquad\qquad R_{W\oplus S'}(\mathfrak{t}) \stackrel{\text{def}}{=} \delta^l_{P,W,S'};(R_W(\mathfrak{t}) \oplus R_{S'}(\mathfrak{t}));\delta^{-l}_{Q,W,S'}$$

Observe that there is an asymmetry in the definition of left and right whiskerings for polynomials: again, this is justified by the fact that $\otimes$ distributes over $\oplus$ on the right. The whiskering of a tape $\mathfrak{t} \colon A \oplus B \to A' \oplus B'$ on the right with the polynomial $C \oplus D$, that is $R_{C\oplus D}(\mathfrak{t})$, should go from $(A \oplus B) \otimes (C \oplus D) = AC \oplus AD \oplus BC \oplus BD$ to $(A' \oplus B') \otimes (C \oplus D) = A'C \oplus A'D \oplus B'C \oplus B'D$, whereas $R_C(\mathfrak{t}) \oplus R_D(\mathfrak{t})$ goes from $AC \oplus BC \oplus AD \oplus BD$ to $A'C \oplus B'C \oplus A'D \oplus B'D$.

With Definition 5.7 we can finally introduce $\otimes$ on arrows $\mathfrak{t}_1 \colon P_1 \to Q_1, \mathfrak{t}_2 \colon P_2 \to Q_2$ as

$$\mathfrak{t}_1 \otimes \mathfrak{t}_2 \stackrel{\text{def}}{=} L_{P_1}(\mathfrak{t}_2);R_{Q_2}(\mathfrak{t}_1). \tag{12}$$

Example 5.8. Consider $\mathfrak{t} \colon U \oplus V \to W \oplus Z$ and $\mathfrak{s} \colon U' \oplus V' \to W' \oplus Z'$ from Example 5.6, then $\mathfrak{t} \otimes \mathfrak{s}$ is simply the sequential composition of $L_{U\oplus V}(\mathfrak{s})$ and $R_{W'\oplus Z'}(\mathfrak{t})$:



The dashed line highlights the boundary between left and right polynomial whiskerings: $L_{U\oplus V}(\mathfrak{s})$, on the left, is simply the vertical composition of the monomial whiskerings $L_U(\mathfrak{s})$ and $L_V(\mathfrak{s})$ while, on the right, $R_{W'\oplus Z'}(\mathfrak{t})$ is rendered as the vertical composition of $R_{W'}(\mathfrak{t})$ and $R_{Z'}(\mathfrak{t})$, precomposed and postcomposed with left distributors.

Lemma 5.9. *The laws in Table 5 hold for any* $\mathfrak{t} \colon P \to Q, \mathfrak{s} \colon Q \to R, \mathfrak{t}_1 \colon P_1 \to Q_1, \mathfrak{t}_2 \colon P_2 \to Q_2$.

The laws in Table 5 are useful in several occasions. In particular, they make it possible to easily prove that $\mathbf{T}_\Sigma$ is a fb rig category. For instance, functoriality of $\otimes$ immediately follows from (W2) and (W7): for all $t_1 : P \to Q, t_2 : Q \to S, t_3 : P' \to Q', t_4 : Q' \to S'$,

$$\begin{aligned}
(t_1; t_2) \otimes (t_3; t_4) &= L_P(t_3; t_4); R_{S'}(t_1; t_2) && \text{(Def. } \otimes) \\
&= L_P(t_3); L_P(t_4); R_{S'}(t_1); R_{S'}(t_2) && \text{(W2)} \\
&= L_P(t_3); R_{Q'}(t_1); L_Q(t_4); R_{S'}(t_2) && \text{(W7)} \\
&= (t_1 \otimes t_3); (t_2 \otimes t_4) && \text{(Def. } \otimes)
\end{aligned}$$

Theorem 5.10. $\mathbf{T}_\Sigma$ *is a $\mathcal{S}$-sesquistrict finite biproduct rig category.*

In the future we will refer to the sesquistrict fb rig category $\mathcal{S} \to \mathbf{T}_\Sigma$ simply as $\mathbf{T}_\Sigma$. We can now state the main result of this section.

Theorem 5.11. $\mathbf{T}_\Sigma$ *is the free sesquistrict fb rig category generated by the monoidal signature* $(\mathcal{S}, \Sigma)$.

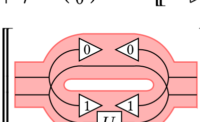
Proof. We have a trivial interpretation of $(\mathcal{S}, \Sigma)$ in $\mathcal{S} \to \mathbf{T}_\Sigma$, given by $(id_{\mathcal{S}}, \underline{\overline{\cdot}})$. Suppose now that $H : \mathbf{M} \to \mathbf{D}$ is a sesquistrict fb rig category with an interpretation $\mathcal{I} = (\alpha_{\mathcal{S}} : \mathcal{S} \to Ob(\mathbf{M}), \alpha_\Sigma : \Sigma \to Ar(\mathbf{D}))$. We aim to find a sesquistrict fb rig functor $(\alpha : \mathcal{S} \to \mathbf{M}, \beta : \mathbf{T}_\Sigma \to \mathbf{D})$ such that $id_{\mathcal{S}}; \alpha = \alpha_{\mathcal{S}}$ and $\underline{\overline{\cdot}}; \beta = \alpha_\Sigma$. Since $(\mathcal{S}, \Sigma)$ is a monoidal signature, $\mathcal{I}$ is in fact a monoidal interpretation of $(\mathcal{S}, \Sigma)$ into the ssm category $(\mathbf{D}, \otimes, 1)$, hence by freeness of $\mathbf{C}_\Sigma$ there exists a unique ssm functor $[\![-]\!]_{\mathcal{I}} : \mathbf{C}_\Sigma \to \mathbf{D}$ that extends $\mathcal{I}$. Now, because $\mathbf{T}_\Sigma = F_2(\mathbf{C}_\Sigma)$ is by definition the free fb category generated by $\mathbf{C}_\Sigma$ and $\mathbf{D}$ is also a fb category, we have that there is a unique fb functor $\beta : \mathbf{T}_\Sigma \to \mathbf{D}$ that extends $[\![-]\!]_{\mathcal{I}}$. The fact that β preserves the rest of the rig structure can be proved using the inductive definitions of the symmetries, distributors and the tensor of $\mathbf{T}_\Sigma$. □

Corollary 5.12. *Let $\mathbf{D}$ be a fb rig category and $F : \mathbf{C}_\Sigma \to (\mathbf{D}, \otimes, 1)$ be a strict symmetric monoidal functor. Then there exists a unique sesquistrict rig functor $F^\sharp : \mathbf{T}_\Sigma \to \mathbf{D}$ such that $F = F^\sharp \circ \underline{\overline{\cdot}}$.*

The above corollary is rather effective: take $\mathbf{C}_\Sigma$ to be the syntax of a string diagrammatic language and the functor $F : \mathbf{C}_\Sigma \to \mathbf{D}$ to be its semantics. Whenever $\mathbf{D}$ carries the structure of an fb rig category, then one can extend the semantics F to the language of tape diagrams $\mathbf{T}_\Sigma$.

Example 5.13 (ZX-calculus)**.** Consider the ZX-calculus with scalars and let **ZX** be the smc of ZX diagrams and $[\![-]\!] : \mathbf{ZX} \to \mathbf{Mat}(\mathbb{C})$ the semantic functor given in [Backens 2015, Definition 3.1.1]. Since $\mathbf{Mat}(\mathbb{C})$ is a fb rig category, one can easily derive the semantics $[\![-]\!]^\sharp : \mathbf{T}_{\mathbf{ZX}} \to \mathbf{Mat}(\mathbb{C})$ of ZX *tapes* (i.e. tape diagrams of ZX diagrams) as suggested in Corollary 5.12 above.

We show that this construction allows us to obtain an immediate graphical representation of control. Let $-\boxed{U}-$ be a ZX diagram for a unitary U and $\lhd\!\!0-$, $\lhd\!\!1-$, $-0\!\!\rhd$, $-1\!\!\rhd$ be shorthands for the computational basis states and effects given in [Backens 2015, Equation 3.72]. For instance, $[\![\lhd\!\!0-]\!] = |0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $[\![-0\!\!\rhd\lhd\!\!0-]\!] = |0\rangle\langle 0| = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right)$. Consider the following diagram:



$$[\![\ \cdot\ ]\!]^\sharp = |00\rangle\langle 00| + |01\rangle\langle 01| + (|1\rangle \otimes U|0\rangle)\langle 10| + (|1\rangle \otimes U|1\rangle)\langle 11|$$

As witnessed by the semantics the first wire acts as a control qubit on the unitary U and thus the whole diagram can be regarded as a rendering of the controlled unitary CU. The correspondence between the diagrammatic notation and the sum of matrices $(+)$ will be clear in Section 6.

5.2 Diagrammatic Reasoning with Tapes

String diagrams allow graphical reasoning on arrows of (symmetric) monoidal categories. In particular, graphical proofs are significantly simpler, since the diagrammatic representation implicitly embodies the laws of strict monoidal categories (Table 1). In this section, we show that tape diagrams allow the same kind of graphical reasoning as string diagrams. However, this fact is not completely obvious because of the peculiar role played by $\otimes$ in tape diagrams.

The usual way of reasoning through string diagrams is based on monoidal theories, namely a signature plus a set of axioms: either equations or inequations. Similarly a *tape theory* is a pair $(\Sigma, \mathbb{I})$ where Σ is a monoidal signature (or by Theorem 4.9 even a rig signature) and $\mathbb{I}$ is a set of axioms, namely a set of pairs of tapes with same domain and codomain. Hereafter, we think of each pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ as an inequation $\mathfrak{t}_1 \leq \mathfrak{t}_2$, but the results that we develop in this section trivially hold also for equations: it is enough to add in $\mathbb{I}$ a pair $(\mathfrak{t}_2, \mathfrak{t}_1)$ for each $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathbb{I}$.

In the following, we write $\mathfrak{t}_1 \mathbb{I} \mathfrak{t}_2$ for $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathbb{I}$ and $\leq_{\mathbb{I}}$ for the smallest precongruence (w.r.t. $\oplus$, $\otimes$ and ;) generated by $\mathbb{I}$, i.e. the relation inductively generated as

$$\frac{\mathfrak{t}_1 \mathbb{I} \mathfrak{t}_2}{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2}\,(\mathbb{I}) \qquad \frac{-}{\mathfrak{t} \leq_{\mathbb{I}} \mathfrak{t}}\,(r) \qquad \frac{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \quad \mathfrak{t}_2 \leq_{\mathbb{I}} \mathfrak{t}_3}{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_3}\,(t)$$

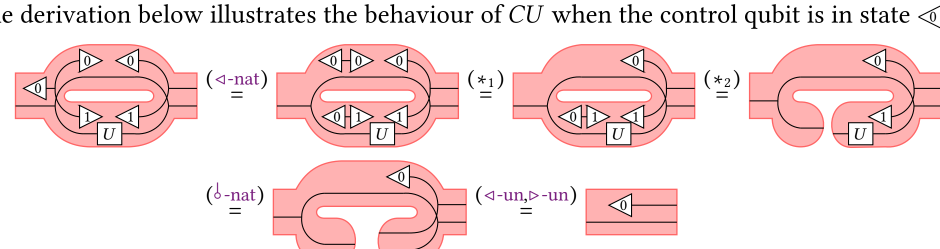
$$\frac{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{s}_2}{\mathfrak{t}_1 \oplus \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \oplus \mathfrak{s}_2}\,(\oplus) \qquad \frac{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{s}_2}{\mathfrak{t}_1 \otimes \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \otimes \mathfrak{s}_2}\,(\otimes) \qquad \frac{\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{s}_2}{\mathfrak{t}_1 ; \mathfrak{s}_1 \leq_{\mathbb{I}} \mathfrak{t}_2 ; \mathfrak{s}_2}\,(;)$$

By enriching $\mathbf{T}_\Sigma$ with $\leq_{\mathbb{I}}$, we obtain a *preorder enriched rig category* (namely, $\oplus$, $\otimes$ and ; are monotone) that we denote hereafter by $\mathbf{T}_{\Sigma,\mathbb{I}}$. For an arbitrary category $\mathbf{C}$ enriched over a preorder $\leq$, one can define a corresponding poset enriched category $\mathbf{C}^{\sim}$ by quotienting the homsets of $\mathbf{C}$ by the equivalence relation $\sim$ defined as $\sim \stackrel{\text{def}}{=} \leq \cap \geq$. Moreover, if $\mathbf{C}$ is a preorder enriched rig category, then $\mathbf{C}^{\sim}$ is a *poset enriched rig category*. Particularly relevant for this paper will be $\mathbf{T}^{\sim}_{\Sigma,\mathbb{I}}$. Similarly, for a monoidal theory $(\Sigma, \mathbb{I})$, one can construct preorder and poset enriched monoidal categories, hereafter denoted by $\mathbf{C}_{\Sigma,\mathbb{I}}$ and $\mathbf{C}^{\sim}_{\Sigma,\mathbb{I}}$, respectively.

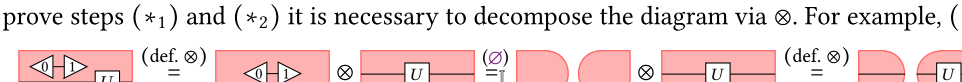
Now, given two tape diagrams $\mathfrak{s}$ and $\mathfrak{t}$, one would like to prove that $\mathfrak{s} \leq_{\mathbb{I}} \mathfrak{t}$ through some graphical manipulation involving the axioms in $\mathbb{I}$ and those in Figure 1. Unfortunately, this is not completely obvious with tapes, as illustrated by the following example.

Example 5.14. Consider the ZX-calculus mentioned in Example 5.13 and let $\mathbb{I}$ be the set consisting of the following axioms (which are part of a larger set of axioms stating that ◁0— and ◁1— form an orthonormal basis, see [Coecke and Kissinger 2017, Theorem 5.32]):

The derivation below illustrates the behaviour of CU when the control qubit is in state ◁0—.


(13)

To prove steps $(\star_1)$ and $(\star_2)$ it is necessary to decompose the diagram via $\otimes$. For example, $(\star_2)$ is



The proof above is not entirely graphical because of the decomposition via $\otimes$. In the following we show that one can easily avoid this inconvenience by taking the right whiskering, for all monomials U, of each of the axioms in $\mathbb{I}$. In other words, rather than $\mathbb{I}$, we consider the following set of axioms

$$\hat{\mathbb{I}} = \{(R_U(\mathfrak{t}_1), R_U(\mathfrak{t}_2)) \mid (\mathfrak{t}_1, \mathfrak{t}_2) \in \mathbb{I} \text{ and } U \in \mathcal{S}^\star\} \tag{14}$$

and we write $\lesssim_{\hat{\mathbb{I}}}$ for the smallest precongruence (w.r.t. $\oplus$ and ;) generated by $\hat{\mathbb{I}}$, i.e. the relation inductively defined as

$$\frac{\mathfrak{t}_1 \hat{\mathbb{I}} \mathfrak{t}_2}{\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2} (\hat{\mathbb{I}}) \qquad \frac{-}{\mathfrak{t} \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}} (r) \qquad \frac{\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 \quad \mathfrak{t}_2 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_3}{\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_3} (t) \qquad \frac{\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{s}_2}{\mathfrak{t}_1 \oplus \mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 \oplus \mathfrak{s}_2} (\oplus) \qquad \frac{\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 \quad \mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{s}_2}{\mathfrak{t}_1 ; \mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 ; \mathfrak{s}_2} (;)$$

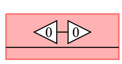
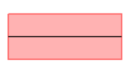
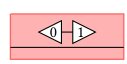
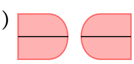
Observe that in the above definition we do not close $\lesssim_{\hat{\mathbb{I}}}$ by $\otimes$. Yet, as stated by the following theorem, $\lesssim_{\hat{\mathbb{I}}}$ coincides with $\leq_{\mathbb{I}}$.

Theorem 5.15. *For all tapes* $\mathfrak{t}_1, \mathfrak{t}_2$, $\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2$ *if and only if* $\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2$.

Proof. To prove that $\leq_{\mathbb{I}} \subseteq \lesssim_{\hat{\mathbb{I}}}$, first observe that $\mathbb{I} \subseteq \hat{\mathbb{I}}$ by (W3). Thus, to conclude it is enough to show that $\lesssim_{\hat{\mathbb{I}}}$ is closed under $\otimes$, i.e. that if $\mathfrak{t}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2$ and $\mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{s}_2$, then $\mathfrak{t}_1 \otimes \mathfrak{s}_1 \lesssim_{\hat{\mathbb{I}}} \mathfrak{t}_2 \otimes \mathfrak{s}_2$. But this is easy using the definition of $\otimes$ and the algebra of whiskering.

To prove that $\lesssim_{\hat{\mathbb{I}}} \subseteq \leq_{\mathbb{I}}$, we show that $\hat{\mathbb{I}} \subseteq \leq_{\mathbb{I}}$: if $\mathfrak{t}_1 \hat{\mathbb{I}} \mathfrak{t}_2$, there exists $(\mathfrak{t}_1', \mathfrak{t}_2') \in \mathbb{I}$ such that $R_U(\mathfrak{t}_i') = \mathfrak{t}_i$ for some monomial U. It is easy to see that $R_U(\mathfrak{t}_i) = \mathfrak{t}_i \otimes id_U$, thus by rules (r) and $(\otimes)$, $\mathfrak{t}_1 \leq_{\mathbb{I}} \mathfrak{t}_2$. □

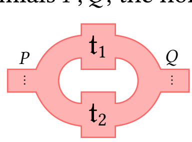
The proof in Example 13 can then be carried out completely diagrammatically by virtue of Theorem 5.15. Indeed, steps $(*_1)$ and $(*_2)$ become trivial by means of the whiskered axioms

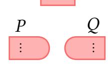
 $\stackrel{(\text{Wid})}{=}$   $\stackrel{(\text{W}\varnothing)}{=}$ 

Remark 5.16. Tape diagrams for finite coproduct rig categories, namely rig categories where $\oplus$ is a coproduct and 0 is an initial object, can be obtained by discarding from the definition of $\mathbf{T}_\Sigma$ the comonoids $(\triangleleft_U, {}^{\dagger}_U)$. Similarly, by ignoring the monoids $(\triangleright_U, \downarrow_U)$, one obtains tape diagrams for finite product rig categories. All the results in the current section hold verbatim for these constructions. However, our focus on finite biproduct categories is justified by the fact that, thanks to Theorem 4.9 which only holds when $\oplus$ is a biproduct, tape diagrams provide a universal language for fb rig categories: all *rig* signatures generate fb rig categories whose arrows are tapes. In the next section, we illustrate several results that rely on the finite biproduct structure of $\mathbf{T}_\Sigma$.

6 TAPES AS MATRICES

Like any category with finite biproducts, $\mathbf{T}_\Sigma$ is enriched over **CMon**, the category of commutative monoids. For all polynomials P, Q, the homset $\mathbf{T}_\Sigma[P, Q]$ carries a commutative monoid defined as

$\mathfrak{t}_1 + \mathfrak{t}_2 \stackrel{\text{def}}{=}$  (i.e. $\mathfrak{t}_1 + \mathfrak{t}_2 \stackrel{\text{def}}{=} \triangleleft_P ; (\mathfrak{t}_1 \oplus \mathfrak{t}_2) ; \triangleright_Q$) (15)

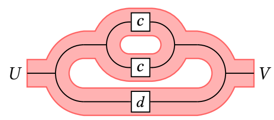
$\mathfrak{o}_{P,Q} \stackrel{\text{def}}{=}$  (i.e. $\mathfrak{o}_{P,Q} \stackrel{\text{def}}{=} {}^{\dagger}_P ; \downarrow_Q$)

for all $\mathfrak{t}_1, \mathfrak{t}_2 : P \to Q$. This structure distributes not only over ; but also with respect to $\otimes$.

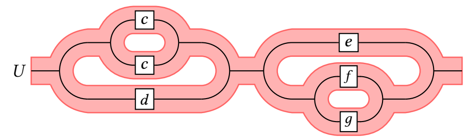
Proposition 6.1. Let $\mathfrak{t}_1, \mathfrak{t}_2 : P \to Q$ and $\mathfrak{s}$ of the appropriate type. It holds that

(1) $(\mathfrak{t}_1 + \mathfrak{t}_2) ; \mathfrak{s} = (\mathfrak{t}_1 ; \mathfrak{s}) + (\mathfrak{t}_2 ; \mathfrak{s})$
(2) $\mathfrak{s} ; (\mathfrak{t}_1 + \mathfrak{t}_2) = (\mathfrak{s} ; \mathfrak{t}_1) + (\mathfrak{s} ; \mathfrak{t}_2)$
(3) $\mathfrak{o} ; \mathfrak{s} = \mathfrak{o} = \mathfrak{s} ; \mathfrak{o}$
(4) $(\mathfrak{t}_1 + \mathfrak{t}_2) \otimes \mathfrak{s} = (\mathfrak{t}_1 \otimes \mathfrak{s}) + (\mathfrak{t}_2 \otimes \mathfrak{s})$
(5) $\mathfrak{s} \otimes (\mathfrak{t}_1 + \mathfrak{t}_2) = (\mathfrak{s} \otimes \mathfrak{t}_1) + (\mathfrak{s} \otimes \mathfrak{t}_2)$
(6) $\mathfrak{o} \otimes \mathfrak{s} = \mathfrak{o} = \mathfrak{s} \otimes \mathfrak{o}$

In this section, we illustrate how such enrichment can be exploited to define a matrix calculus of tapes. First of all we need to identify the entries of these matrices. Let **Mnm** be the full subcategory of $\mathbf{T}_\Sigma$ whose objects are just monomials (i.e. unary sums). It is immediate to see that **Mnm** is a ssm category (w.r.t. $\otimes$), because if U and V are two monomials, then so is $U \otimes V$ by definition of $\otimes$. It is also clearly enriched over **CMon**. Here are two examples of morphisms in **Mnm**.



(16)



(17)

Notice that (16) is in fact $\overline{c} + \overline{c} + \overline{d}$, with $c, d \in \mathbf{C}_\Sigma$. Now, by definition, a morphism in **Mnm** is a tape of $\mathbf{T}_\Sigma$ with only one 'input' and one 'output', but in between these two it can be arbitrarily complicated, like in (17) above. However, it turns out that every tape in **Mnm** can be written as the diagram in (16), that is, as a local sum $\sum_i \overline{c_i}$. This is a consequence of the fact that, as we will see in Corollary 6.7, **Mnm** is isomorphic to $\mathbf{C}_\Sigma^+$: the free **CMon**-enriched category generated by $\mathbf{C}_\Sigma$.

Definition 6.2. Let $\mathbf{C}$ be any category. The *free* **CMon**-*enriched category generated by* $\mathbf{C}$, denoted as $\mathbf{C}^+$, is the category whose objects are those of $\mathbf{C}$, while $\mathbf{C}^+[A, B]$ is the free commutative monoid generated by $\mathbf{C}[A, B]$: a morphism in $\mathbf{C}^+[A, B]$ is a finite multiset of morphisms in $\mathbf{C}[A, B]$. We write multisets as $\{\!|\, a_1, \dots, a_n \,|\!\}$ where the a_i are not necessarily distinct. The identity for A in $\mathbf{C}^+$ is $\{\!|\, id_A^{\mathbf{C}} \,|\!\}$, while if $f : A \to B$ and $g : B \to C$ are morphisms of $\mathbf{C}^+$, then $f ; g \stackrel{\text{def}}{=} \{\!|\, a ; b \mid a \in f,\ b \in g \,|\!\}$ ($a ; b$ has multiplicity equal to the product of the multiplicities of a in the multiset f and b in the multiset g). Addition of multisets is given by union, with the empty multiset being the neutral element.

The isomorphism between **Mnm** and $\mathbf{C}_\Sigma^+$ assigns to each arrow $\mathfrak{m} : U \to V$ in **Mnm**, hereafter referred to as *monomial tapes*, a multiset whose elements are the morphisms of $\mathbf{C}_\Sigma$ appearing in a path in the diagrammatic representation of $\mathfrak{m}$. For instance, the monomial tape in (16) corresponds to the multiset $\{\!|\, c, c, d \,|\!\}$, while the one in (17) to $\{\!|\, c;e,\ c;f,\ c;g,\ c;e,\ c;f,\ c;g,\ d;e,\ d;f,\ d;g \,|\!\}$. Vice versa, every multiset $\{\!|\, c_1, \dots, c_n \,|\!\}$ of arrows in $\mathbf{C}_\Sigma$ corresponds to the monomial tape $\mathfrak{m} = \sum_{i=1}^n \overline{c_i}$.

We can now consider an arbitrary tape $\mathfrak{t} : \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_j$. One can represent $\mathfrak{t}$ as a $m \times n$ matrix $M(\mathfrak{t})$ whose (j, i) entry (row j, column i), is the monomial tape

$$\mathfrak{t}_{ji} \stackrel{\text{def}}{=} \left(U_i \xrightarrow{\mu_i} \bigoplus_{k=1}^n U_k \xrightarrow{\mathfrak{t}} \bigoplus_{k=1}^m V_k \xrightarrow{\pi_j} V_j \right) \tag{18}$$

where $\mu_i \stackrel{\text{def}}{=} \bigoplus_{k=1}^{i-1} \text{¡}_{U_k} \oplus id_{U_i} \oplus \bigoplus_{k=i+1}^n \text{¡}_{U_k}$ and π_j is defined dually. Hence we can associate to $\mathfrak{t}$ a $m \times n$ matrix $\mathcal{F}(\mathfrak{t})$ whose entries are the multisets corresponding to the monomial tapes $\mathfrak{t}_{ji}$.

Example 6.3. Consider again $\mathfrak{t}$ and $\mathfrak{s}$ from Example 5.6, then

$$\mathcal{F}(\mathfrak{t}) = \begin{array}{c} \\ W \\ Z \end{array} \!\!\begin{array}{c} \begin{array}{cc} U & V \end{array} \\ \begin{pmatrix} \{\!|\, \boxed{d}\boxed{e} \,|\!\} & \{\!|\, \boxed{c}\boxed{d}\boxed{e} \,|\!\} \\ \{\!|\, \boxed{d} \,|\!\} & \{\!|\, \boxed{c}\boxed{d} \,|\!\} \end{pmatrix} \end{array} \quad \text{and} \quad \mathcal{F}(\mathfrak{s}) = \begin{array}{c} \\ W' \\ Z' \end{array} \!\!\begin{array}{c} \begin{array}{cc} U' & V' \end{array} \\ \begin{pmatrix} \{\!|\, \boxed{c'}\boxed{d'} \,|\!\} & \{\!|\, \boxed{d'} \,|\!\} \\ \{\!|\, \boxed{c'}\boxed{d'}\boxed{e'} \,|\!\} & \{\!|\, \boxed{d'}\boxed{e'} \,|\!\} \end{pmatrix} \end{array}$$

This correspondence forms an isomorphism that we now illustrate in detail. Given a **CMon**-enriched category $\mathbf{S}$, one can form the *biproduct completion* [Coecke et al. 2018; Mac Lane 1978] of $\mathbf{S}$, denoted as $\mathbf{Mat}(\mathbf{S})$. Its objects are formal $\oplus$'s of objects of $\mathbf{S}$, while a morphism $M : \bigoplus_{k=1}^n A_k \to \bigoplus_{k=1}^m B_k$ is a $m \times n$ matrix where $M_{ji} \in \mathbf{S}[A_i, B_j]$. Composition is given by matrix multiplication, with the addition being the plus operation on the homsets (provided by the enrichment) and multiplication being composition. The identity morphism of $\bigoplus_{k=1}^n A_k$ is given by the $n \times n$ matrix (δ_{ji}), where $\delta_{ji} = id_{A_j}$ if $i = j$, while if $i \neq j$, then δ_{ji} is the zero morphism of $\mathbf{S}[A_i, A_j]$.

Theorem 6.4. *Let* $\mathbf{C}$ *be any category. Then* $F_2(\mathbf{C}) \cong \mathbf{Mat}(\mathbf{C}^+)$ *as fb categories.*

Proof. By [Mac Lane 1978, Exercises VIII.2.5-6], we have a pair of adjunctions

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathbf{CMonCat} \arrow[l, bend left] \arrow[r, bend left, "\mathbf{Mat}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{FBC} \arrow[l, bend left] \end{tikzcd}$$

where the right adjoints are forgetful functors. Since adjunctions compose, the composite functor $\mathbf{Mat}((-)^+)$ is left adjoint to the forgetful functor $U\colon \mathbf{FBC} \to \mathbf{Cat}$, as is F_2: hence they are naturally isomorphic. This makes $F_2(\mathbf{C})$ and $\mathbf{Mat}(\mathbf{C}^+)$ isomorphic as fb categories. Explicitly, the functor $\mathcal{F}\colon F_2(\mathbf{C}) \to \mathbf{Mat}(\mathbf{C}^+)$ is the identity on objects and it is defined on morphisms by induction:

- $\mathcal{F}(\underline{\overline{c}}) = \begin{pmatrix} \{\!\mid c \mid\!\} \end{pmatrix}$ (the 1×1 matrix of the multiset consisting of one copy of c)
- $\mathcal{F}(id) = id$, $\mathcal{F}(\mathfrak{t}_2 \circ \mathfrak{t}_1) = \mathcal{F}(\mathfrak{t}_2) \circ \mathcal{F}(\mathfrak{t}_1)$
- Given $A, B \in \mathbf{C}$, $\mathcal{F}(\sigma_{A,B}) = \begin{pmatrix} \varnothing & \{\!\mid id_B \mid\!\} \\ \{\!\mid id_A \mid\!\} & \varnothing \end{pmatrix}$ of size 2×2,

$$\mathcal{F}(\lhd_A) = \begin{pmatrix} \{\!\mid id_A \mid\!\} \\ \{\!\mid id_A \mid\!\} \end{pmatrix} \qquad \mathcal{F}(\rhd_A) = \begin{pmatrix} \{\!\mid id_A \mid\!\} & \{\!\mid id_A \mid\!\} \end{pmatrix}$$

$$\mathcal{F}(\,\rotatebox[origin=c]{180}{$\multimap$}_A) = \text{empty matrix of size } 0 \times 1 \qquad \mathcal{F}(\multimap_A) = \text{empty matrix of size } 1 \times 0$$

- For $f_1\colon \bigoplus_{i=1}^{n} A_i \to \bigoplus_{i=1}^{n'} A'_i$ and $f_2\colon \bigoplus_{i=1}^{m} B_j \to \bigoplus_{i=1}^{m'} B'_j$,

$$\mathcal{F}(f_1 \oplus f_2) = \begin{pmatrix} \mathcal{F}(f_1) & \varnothing_{n' \times m} \\ \varnothing_{m' \times n} & \mathcal{F}(f_2) \end{pmatrix} \qquad \square$$

6.1 Kronecker Product in $\mathbf{Mat}(\mathbf{C}_\Sigma^+)$

If $(\mathbf{C}, \otimes_{\mathbf{C}}, 1_{\mathbf{C}})$ is a ssm category, then we can define a monoidal product on $\mathbf{C}^+$:

$$f \otimes_{\mathbf{C}^+} g \stackrel{\text{def}}{=} \{\!\mid a \otimes_{\mathbf{C}} b \mid a \in f,\ b \in g \mid\!\} : A \otimes_{\mathbf{C}} B \to A' \otimes_{\mathbf{C}} B'. \tag{19}$$

In turn, this allows us to define a monoidal product in $\mathbf{Mat}(\mathbf{C}^+)$ *à la* Kronecker. We will denote it as $\circledast$. On objects it is given as $\left(\bigoplus_{i=1}^n A_i\right) \circledast \left(\bigoplus_{j=1}^m B_j\right) \stackrel{\text{def}}{=} \bigoplus_{i=1}^n \bigoplus_{j=1}^m A_i B_j$. If $M\colon \bigoplus_{i=1}^n A_i \to \bigoplus_{i'=1}^{n'} A'_{i'}$ and $N\colon \bigoplus_{j=1}^m B_j \to \bigoplus_{j'=1}^{m'} B'_{j'}$, then $M \circledast N$ is the matrix of size $n'm' \times nm$ defined as in the usual Kronecker product of matrices (with $\otimes_{\mathbf{C}^+}$ playing the role of multiplication).

Theorem 6.5. *Let* $\mathbf{C}$ *be an ssm category. Then* $\mathbf{Mat}(\mathbf{C}^+)$ *is a fb rig category, with tensor given by* $\circledast$*, and* $F_2(\mathbf{C}) \cong \mathbf{Mat}(\mathbf{C}^+)$ *as fb rig categories.*

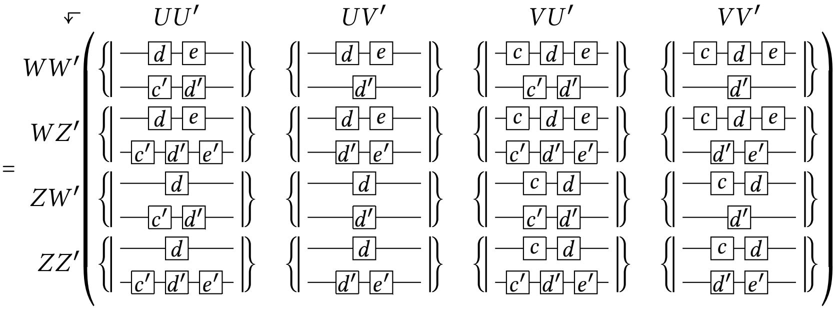
Proof. One can define on $F_2(\mathbf{C})$ a monoidal product $\otimes$ distributing on $\oplus$ in a completely analogous way of $\mathbf{T}_\Sigma$, making it a fb rig category. Thus $\mathbf{Mat}(\mathbf{C}^+)$ inherits the structure of a strict fb rig category from $F_2(\mathbf{C})$ via the isomorphism $\mathcal{F}$ of Theorem 6.4, by simply setting $M \otimes N \stackrel{\text{def}}{=} \mathcal{F}(\mathcal{F}^{-1}(M) \otimes_{\mathbf{T}_\Sigma} \mathcal{F}^{-1}(N))$ and similarly for the rest of the rig structure. This makes $\mathcal{F}$ an isomorphism of fb rig categories. Moreover, one can prove that $\otimes = \circledast$ with direct calculations. $\square$

We conclude by applying our general result to $\mathbf{C} = \mathbf{C}_\Sigma$.

Corollary 6.6. $\mathbf{T}_\Sigma \cong \mathbf{Mat}(\mathbf{C}_\Sigma^+)$ *as fb rig categories.*

Corollary 6.7. $\mathbf{Mnm} \cong \mathbf{C}_\Sigma^+$ *as ssm categories.*

Example 6.8. Recall from Example 5.8 the tape diagram $\mathfrak{t} \otimes \mathfrak{s}$. The corresponding matrix, which can be computed as the Kronecker product of $\mathcal{F}(\mathfrak{t})$ and $\mathcal{F}(\mathfrak{s})$ in Example 6.3, is illustrated below.

$\mathcal{F}(\mathfrak{t} \otimes \mathfrak{s}) =$ 

6.2 Poset Enrichment

We saw that $\mathbf{T}_\Sigma$ is isomorphic to a category of matrices in Corollary 6.6. Here we want to extend the isomorphism to the case where the category of tapes is equipped with a poset generated by a tape theory. We shall focus on a particular case of tape theory, namely the one arising from a monoidal theory $(\Sigma, \mathbb{I})$ by simply adding the four tape axioms in Figure 2. These axioms force $\rhd_U$ to be left adjoint to $\lhd_U$ and $⫰_U$ to $⫯_U$. Moreover, they make $+$, as defined in (15), idempotent, thus a join $\sqcup$, with $\mathfrak{0}$ being its bottom element $\perp$.

For a monoidal theory $(\Sigma, \mathbb{I})$, we call *the generated tape theory* $(\Sigma, \tilde{\mathbb{I}})$ the tape theory where $\tilde{\mathbb{I}}$ consists of the four inequalities in Figure 2 together with all those pairs $(\overline{c}, \overline{d})$ whenever $(c, d) \in \mathbb{I}$. We can then form the category $\mathbf{T}^{\sim}_{\Sigma,\tilde{\mathbb{I}}}$ as explained in Section 5.2. It turns out that $\mathbf{T}^{\sim}_{\Sigma,\tilde{\mathbb{I}}}$ is isomorphic to a certain category of matrices that we are going to illustrate next.

Let $(X, \leq)$ be a preordered set and $S \subseteq X$. The *downward closure* of S is defined to be the set $S\downarrow = \{x \in X \mid \exists s \in S.\, x \leq s\}$. S is said to be *downward closed* if $S = S\downarrow$; a downward closed set $A \subseteq X$ is said to be *finitely generated* if there exists a finite set $B \subseteq X$ such that $A = B\downarrow$. Consider now an arbitrary category $\mathbf{C}$ enriched over a preorder $\leq_{\mathbf{C}}$. We can form a new category $\mathbf{C}^{\downarrow}$ whose objects are $Ob(\mathbf{C})$ while a morphism of type $X \to Y$ in $\mathbf{C}^{\downarrow}$ is a finitely generated downward closed subset of $\mathbf{C}[X, Y]$. $\mathbf{C}^{\downarrow}[X, Y]$ is partially ordered by inclusion, has all finite joins given by unions, composition is defined using the composition in $\mathbf{C}$ and this makes $\mathbf{C}^{\downarrow}$ a finite join-semilattice enriched category. For a monoidal theory $(\Sigma, \mathbb{I})$, one can construct the poset enriched monoidal category $\mathbf{C}^{\sim}_{\Sigma,\mathbb{I}}$ as in Section 5.2 and, since ${\mathbf{C}^{\sim}_{\Sigma,\mathbb{I}}}^{\downarrow}$ is enriched over finite join semilattices (and thus commutative monoids), one can take its biproduct completion.

Theorem 6.9. $\mathbf{T}^{\sim}_{\Sigma,\tilde{\mathbb{I}}} \cong \mathbf{Mat}({\mathbf{C}^{\sim}_{\Sigma,\mathbb{I}}}^{\downarrow})$ *as poset-enriched fb rig categories.*

Proof. Let $\mathbf{C}$ be preorder-enriched: then we can equip $\mathbf{C}^{+}[X, Y]$ with the Egli-Milner preorder

$$\{\!|\, c_1, \dots, c_n \,|\!\} \leq^{EM} \{\!|\, d_1, \dots, d_m \,|\!\} \iff \forall i.\, \exists j.\, c_i \leq_{\mathbf{C}} d_j,$$

making $\mathbf{C}^{+}$, and in turn $\mathbf{Mat}(\mathbf{C}^{+})$, preorder-enriched. We can also enrich $F_2(\mathbf{C})$ in a way that the isomorphism $\mathcal{F}$ of Theorem 6.4 becomes preorder-enriched. To do so we add on the homsets of $F_2(\mathbf{C})$ the precongruence, with respect to composition, $\oplus$ and $\otimes$, generated by the following axioms:

$$f \leq_{\mathbf{C}} g \Rightarrow \underline{\overline{f}} \leq \underline{\overline{g}} \qquad \lhd_A ; \rhd_A \leq id_A \qquad id_{A \oplus A} \leq \rhd_A ; \lhd_A \qquad ⫯_A ; ⫰_A \leq id_A \qquad id_0 \leq ⫰_A ; ⫯_A$$

(The last four axioms are the generalised version of Figure 2 for $F_2(\mathbf{C})$.) One can then prove that $\mathcal{F}$ preserves and reflects the inequalities of $F_2(\mathbf{C})$. Taking now $F_2(\mathbf{C})^{\sim}$ and $\mathbf{Mat}(\mathbf{C}^{+})^{\sim}$, we obtain a *poset*-enriched isomorphism. Observe that $\mathbf{Mat}(\mathbf{C}^+)^\sim \cong \mathbf{Mat}((\mathbf{C}^+)^\sim)$, because the order on matrices is entry by entry, and that $(\mathbf{C}^+)^\sim \cong \mathbf{C}^\downarrow$, because for all $S, T \in \mathbf{C}^+[X, Y]$, $S \leq^{EM} T$ iff $S \downarrow \subseteq T \downarrow$.

If we apply this to $\mathbf{C} = \mathbf{C}_{\Sigma,\mathbb{I}}$, then we have that $F_2(\mathbf{C})^\sim = \mathbf{T}^\sim_{\Sigma,\tilde{\mathbb{I}}}$ and we obtain $\mathbf{T}^\sim_{\Sigma,\tilde{\mathbb{I}}} \cong \mathbf{Mat}(\mathbf{C}_{\Sigma,\mathbb{I}}{}^\downarrow)$. We conclude by simply observing that $\mathbf{C}_{\Sigma,\mathbb{I}}{}^\downarrow \cong \mathbf{C}^\sim_{\Sigma,\mathbb{I}}{}^\downarrow$. □

A consequence of Theorem 6.9 is that we can characterise the partial order of $\mathbf{T}^\sim_{\Sigma,\tilde{\mathbb{I}}}[P, Q]$ as follows: for arbitrary monomials $\sum_{h=1}^{n} \underline{\overline{c_h}}$ and $\sum_{k=1}^{m} \underline{\overline{d_k}}$ we define $\leq^{EM}_{\mathbb{I}}$ as

$$\sum_{h=1}^{n} \underline{\overline{c_h}} \leq^{EM}_{\mathbb{I}} \sum_{k=1}^{m} \underline{\overline{d_k}} \iff \forall h.\, \exists k.\, c_h \leq_{\mathbb{I}} d_k$$

and we extend it to arbitrary tapes $\mathfrak{t}, \mathfrak{s} \colon \bigoplus_i U_i \to \bigoplus_j V_j$ as

$$\mathfrak{t} \leq^{EM}_{\mathbb{I}} \mathfrak{s} \iff \forall i, j.\, \mathfrak{t}_{ji} \leq^{EM}_{\mathbb{I}} \mathfrak{s}_{ji}$$

where $\mathfrak{t}_{ji}$ and $\mathfrak{s}_{ji}$ are the monomials in (18).

Corollary 6.10. $\mathfrak{t} \leq_{\tilde{\mathbb{I}}} \mathfrak{s}$ *iff* $\mathfrak{t} \leq^{EM}_{\mathbb{I}} \mathfrak{s}$.

Remark 6.11. Consider the case of $\mathbb{I}$ being 0, the empty set. Then Theorem 6.9 asserts that $\mathbf{T}^\sim_{\Sigma,\tilde{0}}$, which is simply the poset enriched version of $\mathbf{T}_\Sigma$ generated by the inequalities of Figure 2, is isomorphic to the biproduct completion of the free *join-semilattice enriched* category generated by $\mathbf{C}_\Sigma$. The latter is the same as $\mathbf{C}^+_\Sigma$ except that its homsets are sets, rather than multisets, of arrows of $\mathbf{C}_\Sigma$. Indeed, since the preorder on $\mathbf{C}_{\Sigma,0}[X, Y]$ is discrete, we have that the finitely generated downward closed subsets of $\mathbf{C}_{\Sigma,0}[X, Y]$ are exactly finite subsets of $\mathbf{C}_{\Sigma,0}[X, Y]$.

Example 6.12 ($\mathbf{T}_{\mathbf{CB}_\Sigma}$)**.** We recall from [Bonchi et al. 2018] $\mathbf{CB}_\Sigma$: the cartesian bicategory freely generated by a single sorted, i.e. $\mathcal{S} = \{A\}$, monoidal signature Σ. In a nutshell, $\mathbf{CB}_\Sigma$ can be described as $\mathbf{C}^\sim_{\Gamma,\mathbb{I}}$ where the monoidal signature Γ is $\Sigma \cup \{ \blacktriangleright_A \colon A \odot A \to A,\ ¡_A \colon I \to A,\ \blacktriangleleft_A \colon A \to A \odot A,\ !_A \colon A \to I \}$ and the set of axioms $\mathbb{I}$ consists of those of cartesian bicategories, see Figure 3. There we draw, for all $U \in \mathcal{S}^{\star}$[1], $\blacktriangleright_U$, $¡_U$, $\blacktriangleleft_U$, $!_U$ as

In the generated tape theory $(\Sigma, \tilde{\mathbb{I}})$, the set of axioms $\tilde{\mathbb{I}}$ consists of those in Figures 2 and 3. In the next section we will show that $\mathbf{T}^\sim_{\Gamma,\tilde{\mathbb{I}}}$ provides a complete calculus for relations. An essential ingredient is Corollary 6.10 above. Hereafter we will write $\mathbf{T}_{\mathbf{CB}_\Sigma}$ for $\mathbf{T}^\sim_{\Gamma,\tilde{\mathbb{I}}}$ and denote the poset of $\mathbf{CB}_\Sigma$ as $\leq_{\mathbf{CB}_\Sigma}$ while the one in $\mathbf{T}_{\mathbf{CB}_\Sigma}$ as $\leq_T$.

Example 6.13 ($\sqcup$-props)**.** By exploiting a mixture of algebraic and diagrammatic syntax, the authors of [Boisseau and Piedeleu 2022] introduced $\sqcup$-props to model piecewise linear systems (e.g. diodes). Tape diagrams, instead, provide a purely graphical calculus for $\sqcup$-props.

Consider a single sorted, i.e. $\mathcal{S} = \{A\}$, monoidal signature Σ and take $\mathbb{I}$ to be the empty set 0 (no axioms). The objects of $\mathbf{T}^\sim_{\Sigma,\tilde{0}}$ can easily be seen to be in one-to-one correspondence with words of natural numbers: $\bigoplus_i^n A^{m_i} \mapsto m_1 m_2 \dots m_n$. In particular monomials are natural numbers and $\otimes$ on monomials is just addition. Thus, by taking $\mathbf{Mnm}$, the full subcategory of $\mathbf{T}^\sim_{\Sigma,\tilde{0}}$ where objects are monomials, one obtains a *prop* [Lack 2004; Mac Lane 1965]. It is immediate to see that $\mathbf{Mnm}$ is a $\sqcup$-*prop*: every homset carries a join semilattice with bottom (as in (15)), which is preserved by composition and monoidal product (Proposition 6.1). Most importantly, $\mathbf{Mnm}$ is the $\sqcup$-prop freely generated by Σ, as defined in [Boisseau and Piedeleu 2022]. One can readily see this by means of

[1]The coherence axioms of (co)monoids provide a recipe to define inductively, $\blacktriangleright_U$, $¡_U$, $\blacktriangleleft_U$, $!_U$, for all $U \in \mathcal{S}^\star$.

Theorem 6.9 and Remark 6.11. Therefore, whenever one is interested in the $\sqcup$-prop freely generated by some monoidal signature Σ, one can rather embed it into $\mathbf{T}^{\sim}_{\Sigma,\tilde{0}}$ and exploit the graphical calculus of tapes. Notice that there is a little mismatch with the definition in [Boisseau and Piedeleu 2022], where the freely generated $\sqcup$-prop has as arrows only *non-empty* finite sets. The empty set is instead denoted in **Mnm** by $\mathfrak{o}$: the presence of $\mathfrak{o}$ seems however a feature rather than an issue as illustrated, for instance, by the axiom ($\varnothing$).

7 BACK TO RELATIONS

The tape axioms introduced in Section 6.2 forcing $(\triangleright_X, \text{⫰}_X)$ to be left adjoint to $(\triangleleft_X, \text{⫯}_X)$ (Figure 2) suggest a general categorical notion reconciling the two monoidal structures of **Rel** (Section 3).

Definition 7.1. A poset enriched rig category $\mathbf{C}$ is said to be a *fb-cb rig category* (or simply fb-cb category for short) if

(1) $(\mathbf{C}, \oplus, 0)$ is a finite biproduct category;
(2) $(\mathbf{C}, \otimes, 1)$ is a cartesian bicategory;
(3) the monoid $(\triangleright_X, \text{⫰}_X)$ of $\oplus$ is left adjoint to the comonoid $(\triangleleft_X, \text{⫯}_X)$, i.e.

$$id_0 \leq \text{⫰}_X; \text{⫯}_X \qquad id_X \oplus id_X \leq \triangleright_X; \triangleleft_X \qquad \text{⫯}_X; \text{⫰}_X \leq id_X \qquad \triangleleft_X; \triangleright_X \leq id_X$$

(4) the (co)monoids of both $(\mathbf{C}, \oplus, 0)$ and $(\mathbf{C}, \otimes, 1)$ satisfy the following coherence axioms:

$$\begin{aligned} \blacktriangleleft_{X\oplus Y} &= (\blacktriangleleft_X \oplus \text{⫰}_{XY} \oplus \text{⫰}_{YX} \oplus \blacktriangleleft_Y); (\delta^{-l}_{X,X,Y} \oplus \delta^{-l}_{Y,X,Y}); \delta^{-r}_{X,Y,X\oplus Y} \qquad & !_{X\oplus Y} &= (!_X \oplus !_Y); \triangleright_1 \\ \blacktriangleright_{X\oplus Y} &= \delta^{r}_{X,Y,X\oplus Y}; (\delta^{l}_{X,X,Y} \oplus \delta^{l}_{Y,X,Y}); (\blacktriangleright_X \oplus \text{⫯}_{XY} \oplus \text{⫯}_{YX} \oplus \blacktriangleright_Y) & ¡_{X\oplus Y} &= \triangleleft_1; (¡_X \oplus ¡_Y) \end{aligned}$$

A *morphism of fb-cb categories* is a poset enriched rig functor that is both a morphism of finite biproduct categories and a morphism of cartesian bicategories.

We have already seen in Section 3 that $(\mathbf{Rel}, \oplus, 0)$ is an fb-category and $(\mathbf{Rel}, \otimes, 1)$ is a cartesian bicategory. To conclude that **Rel** is an fb-cb category is enough to check that conditions (3) and (4) above are satisfied: this is trivial by using the definitions of the two (co)monoids of **Rel** in (7).

Remark 7.2. By exploiting the structure of fb-cb category, one can express finite unions and intersections in **Rel**. The reader can easily check that for all relations $R, S \colon X \to Y$, it holds that

$$R \cup S = \triangleleft_X; (R \oplus S); \triangleright_Y \qquad \{\} = \text{⫯}_X; \text{⫰}_Y \qquad R \cap S = \blacktriangleleft_X; (R \otimes S); \blacktriangleright_Y \qquad X \otimes Y = !_X; ¡_Y$$

Another example of fb-cb rig category is $\mathbf{T}_{\mathbf{CB}_\Sigma}$ from Example 6.12.

Theorem 7.3. $\mathbf{T}_{\mathbf{CB}_\Sigma}$ *is an fb-cb category.*

Proof. By construction $\mathbf{T}_{\mathbf{CB}_\Sigma}$ is a poset enriched fb rig category with $(\triangleright_X, \text{⫰}_X)$ left adjoint to $(\triangleleft_X, \text{⫯}_X)$. To prove that $(\mathbf{T}_{\mathbf{CB}_\Sigma}, \otimes, 1)$ is a cartesian bicategory, we need to define $\blacktriangleleft_P \colon P \to P \otimes P$, $!_P \colon P \to 1$, $\blacktriangleright_P \colon P \to P \otimes P$ and $¡_P \colon 1 \to P$ for all polynomials P. The coherence axioms in Definition 7.1 provide us the recipe:

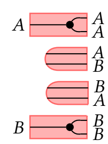
$$\begin{aligned} \blacktriangleleft_0 &\stackrel{\text{def}}{=} id_0 & !_0 &\stackrel{\text{def}}{=} \text{⫰}_1 \\ \blacktriangleleft_{U\oplus P'} &\stackrel{\text{def}}{=} \blacktriangleleft_U \oplus \text{⫰}_{UP'} \oplus ((\text{⫰}_{P'U} \oplus \blacktriangleleft_{P'}); \delta^{-l}_{P',U,P'}) \qquad & !_{U\oplus P'} &\stackrel{\text{def}}{=} (!_U \oplus !_{P'}); \triangleright_1 \end{aligned} \tag{20}$$

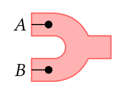
$$\begin{aligned} \blacktriangleright_0 &\stackrel{\text{def}}{=} id_0 & ¡_0 &\stackrel{\text{def}}{=} \text{⫯}_1 \\ \blacktriangleright_{U\oplus P'} &\stackrel{\text{def}}{=} \blacktriangleright_U \oplus \text{⫯}_{UP'} \oplus (\delta^{l}_{P',U,P'}; (\text{⫯}_{P'U} \oplus \blacktriangleright_{P'})) \qquad & ¡_{U\oplus P'} &\stackrel{\text{def}}{=} \triangleleft_1; (¡_U \oplus ¡_{P'}) \end{aligned} \tag{21}$$

Table 6. The morphisms $[\![\cdot]\!]_{\mathcal{I}} \colon \mathbf{CB}_\Sigma \to \mathbf{Rel}$ and $[\![\cdot]\!]^\sharp_{\mathcal{I}} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ extending an interpretation $\mathcal{I} = (\alpha_{\mathcal{S}}, \alpha_\Sigma)$ of a monoidal signature Σ in $\mathbf{Rel}$. We write X, W, W' for $\alpha_{\mathcal{S}}(A)$, $\alpha^\sharp_{\mathcal{S}}(U)$ and $\alpha^\sharp_{\mathcal{S}}(U')$ respectively, and $\blacktriangleright_X, ¡_X, \blacktriangleleft_X, !_X, \rhd_W, ^{\downarrow}_W, \lhd_W, ^{\uparrow}_W$ for the (co)monoids in $\mathbf{Rel}$ defined as in (7).

$[\![U \boxed{R} V]\!]_{\mathcal{I}} = \alpha_\Sigma(R)$	$[\![A \blacktriangleleft \begin{smallmatrix} A \\ A \end{smallmatrix}]\!]_{\mathcal{I}} = \blacktriangleleft_X$	$[\![A \multimap]\!]_{\mathcal{I}} = !_X$	$[\![\begin{smallmatrix} A \\ A \end{smallmatrix} \blacktriangleright A]\!]_{\mathcal{I}} = \blacktriangleright_X$	$[\![\bullet\!\!- A]\!]_{\mathcal{I}} = ¡_X$
$[\![id_A]\!]_{\mathcal{I}} = id_X$	$[\![id_1]\!]_{\mathcal{I}} = id_1$	$[\![\sigma^{\otimes}_{A,A}]\!]_{\mathcal{I}} = \sigma^{\otimes}_{X,X}$	$[\![c_1 ; c_2]\!]_{\mathcal{I}} = [\![c_1]\!]_{\mathcal{I}} ; [\![c_2]\!]_{\mathcal{I}}$	$[\![c_1 \otimes c_2]\!]_{\mathcal{I}} = [\![c_1]\!]_{\mathcal{I}} \otimes [\![c_2]\!]_{\mathcal{I}}$
$[\![U \boxed{c} V]\!]^\sharp_{\mathcal{I}} = [\![c]\!]_{\mathcal{I}}$	$[\![U \lhd \begin{smallmatrix} U \\ U \end{smallmatrix}]\!]^\sharp_{\mathcal{I}} = \lhd_W$	$[\![U \supset]\!]^\sharp_{\mathcal{I}} = {}^{\uparrow}_W$	$[\![\begin{smallmatrix} U \\ U \end{smallmatrix} \rhd U]\!]^\sharp_{\mathcal{I}} = \rhd_W$	$[\![\subset U]\!]^\sharp_{\mathcal{I}} = {}^{\downarrow}_W$
$[\![id_U]\!]^\sharp_{\mathcal{I}} = id_W$	$[\![id_0]\!]^\sharp_{\mathcal{I}} = id_0$	$[\![\sigma^{\oplus}_{U,U'}]\!]^\sharp_{\mathcal{I}} = \sigma^{\oplus}_{W,W'}$	$[\![\mathfrak{s} ; \mathfrak{t}]\!]^\sharp_{\mathcal{I}} = [\![\mathfrak{s}]\!]^\sharp_{\mathcal{I}} ; [\![\mathfrak{t}]\!]^\sharp_{\mathcal{I}}$	$[\![\mathfrak{s} \oplus \mathfrak{t}]\!]^\sharp_{\mathcal{I}} = [\![\mathfrak{s}]\!]^\sharp_{\mathcal{I}} \oplus [\![\mathfrak{t}]\!]^\sharp_{\mathcal{I}}$

For instance, $\blacktriangleleft_{A \oplus B} \colon A \oplus B \to (A \oplus B) \otimes (A \oplus B) = AA \oplus AB \oplus BA \oplus BB$ and $!_{A \oplus B} \colon A \oplus B \to 1$ are

$\blacktriangleleft_{A \oplus B} =$ 

$!_{A \oplus B} =$ 

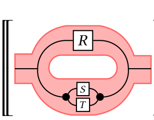
□

We now focus on fb-cb morphisms from $\mathbf{T}_{\mathbf{CB}_\Sigma}$ to $\mathbf{Rel}$. In [Bonchi et al. 2018] it is shown that every interpretation $\mathcal{I}$ of the monoidal signature Σ in $\mathbf{Rel}$ gives rise to a morphism of cartesian bicategories $[\![\cdot]\!]_{\mathcal{I}} \colon \mathbf{CB}_\Sigma \to \mathbf{Rel}$. This extends in turn to a morphism of fb-cb categories $[\![\cdot]\!]^\sharp_{\mathcal{I}} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ as illustrated in Table 6. More generally, the following holds.

Proposition 7.4. The following are (pairwise) in bijective correspondence:

(1) interpretations of the monoidal signature Σ in $\mathbf{Rel}$;
(2) morphisms of cartesian bicategories $\mathbf{CB}_\Sigma \to \mathbf{Rel}$;
(3) morphisms of sesquistrict fb-cb categories $\mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$.

Arrows in $\mathbf{T}_{\mathbf{CB}_\Sigma}$ can be thought of as expressions and $[\![\cdot]\!]^\sharp_{\mathcal{I}} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ as a semantic map assigning to each expression $\mathfrak{t}$ the denoted relation. For instance by using the definition in Table 6 and Remark 7.2, the reader can easily check that

$[\![$  $]\!]^\sharp_{\mathcal{I}} = \alpha_\Sigma(R) \cup (\alpha_\Sigma(S) \cap \alpha_\Sigma(T)).$ (22)

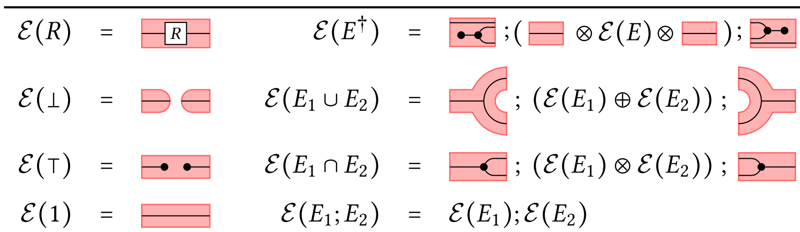
It turns out that if $[\![\mathfrak{s}]\!]^\sharp_{\mathcal{I}} \subseteq [\![\mathfrak{t}]\!]^\sharp_{\mathcal{I}}$ for all interpretations $\mathcal{I}$ of Σ in $\mathbf{Rel}$, then one can derive that $\mathfrak{s} \leq_{\mathbb{T}} \mathfrak{t}$ in $\mathbf{T}_{\mathbf{CB}_\Sigma}$. In other words, the axioms of $\mathbf{T}_{\mathbf{CB}_\Sigma}$ are *complete* with respect to relational inclusion.

Theorem 7.5 (Completeness). *Let* $\mathfrak{s}, \mathfrak{t}$ *be arrows in* $\mathbf{T}_{\mathbf{CB}_\Sigma}$. *Then* $\mathfrak{s} \leq_{\mathbb{T}} \mathfrak{t}$ *if and only if* $\mathcal{M}(\mathfrak{s}) \leq \mathcal{M}(\mathfrak{t})$ *for all morphisms of sesquistrict fb-cb categories* $\mathcal{M} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$.

Proof. Soundness is trivial: since the $\mathcal{M}$'s by definition preserve the ordering, then $\mathfrak{s} \leq_{\mathbb{T}} \mathfrak{t}$ entails $\mathcal{M}(\mathfrak{s}) \leq \mathcal{M}(\mathfrak{t})$ for all $\mathcal{M}$. For completeness we will make use of the following facts, where every $\mathcal{M}$ appearing below is intended to be a morphism of fb-cb categories:

(1) For all $\mathcal{M} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ and $c, d \in \mathbf{CB}_\Sigma[U, V]$, $\mathcal{M}(\overline{c} + \overline{d}) = \mathcal{M}(\overline{c}) \cup \mathcal{M}(\overline{d})$.
(2) Let $c, d_1, \ldots, d_m \in \mathbf{CB}_\Sigma[U, V]$. If for all $\mathcal{M} \colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ we have $\mathcal{M}(\overline{c}) \leq \mathcal{M}(\sum_{j=1}^{m} \overline{d_j})$, then there exists a j such that $c \leq_{\mathbf{CB}_\Sigma} d_j$.

Table 7. The encoding $\mathcal{E}(-)\colon \mathsf{CR}_\Sigma \to \mathbf{T}_{\mathbf{CB}_\Sigma}[A,A]$



(3) Let $c_1,\dots c_n, d_1,\dots,d_m \in \mathbf{CB}_\Sigma[U,V]$. If for all $\mathcal{M}\colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ we have $\mathcal{M}(\sum_{i=1}^n \overline{\underline{c_i}}) \leq \mathcal{M}(\sum_{j=1}^m \overline{\underline{d_j}})$, then for all i there exists a j such that $c_i \leq_{\mathbf{CB}_\Sigma} d_j$.

(4) Let $\mathfrak{s},\mathfrak{t}$ be arrows in **Mnm**. If for all $\mathcal{M}\colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$ we have $\mathcal{M}(\mathfrak{s}) \leq \mathcal{M}(\mathfrak{t})$, then $\mathfrak{s} \leq_T \mathfrak{t}$.

(5) Let $\mathbf{C}$ be a fb category monoidally enriched over a preorder $\leq$. Let $f,g\colon \bigoplus_{i=1}^n A_i \to \bigoplus_{j=1}^m B_j$ be arrows in $\mathbf{C}$. Let f_{ji} and g_{ji} be, respectively, $\mu_i;f;\pi_j$ and $\mu_i;g;\pi_j$. Then $f \leq g$ if and only if $f_{ji} \leq g_{ji}$ for all i,j.

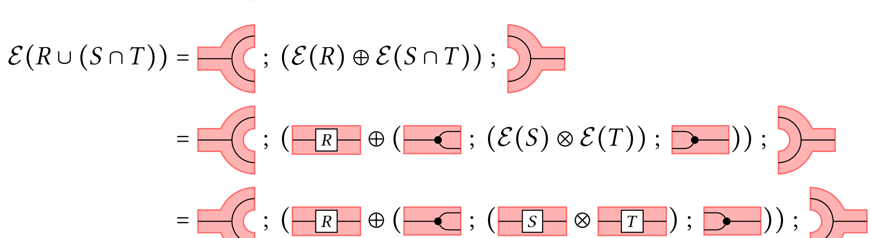
(1) and (5) are easy to prove. (2) follows from the proof of Theorem 17 in [Bonchi et al. 2018]: for all $c \in \mathbf{CB}_\Sigma[U,V]$, there exist a morphism $\mathcal{U}_c\colon \mathbf{CB}_\Sigma \to \mathbf{Rel}$ and an element $(\iota,\omega) \in \mathcal{U}_c(c)$ such that for all $d \in \mathbf{CB}_\Sigma[U,V]$, if $(\iota,\omega) \in \mathcal{U}_c(d)$ then $c \leq_{\mathbf{CB}_\Sigma} d$. (3) follows from (1) and (2). (4) follows from (3) and Corollary 6.10.

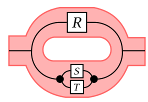
By recalling that, for arbitrary tape diagrams $\mathfrak{s},\mathfrak{t}\colon \bigoplus U_i \to \bigoplus V_j$, $\mathfrak{s}_{ji}$ and $\mathfrak{t}_{ji}$ are in **Mnm**, we can easily extend (4): if $\mathcal{M}(\mathfrak{s}) \leq \mathcal{M}(\mathfrak{t})$ for all morphisms of fb-cb categories $\mathcal{M}\colon \mathbf{T}_{\mathbf{CB}_\Sigma} \to \mathbf{Rel}$, then, by applying (5) to **Rel**, $\mathcal{M}(\mathfrak{s})_{ji} \leq \mathcal{M}(\mathfrak{t})_{ji}$. Hence $\mathcal{M}(\mathfrak{s}_{ji}) = \mathcal{M}(\mathfrak{s})_{ji} \leq \mathcal{M}(\mathfrak{t})_{ji} = \mathcal{M}(\mathfrak{t}_{ji})$ for all fb-cb morphisms $\mathcal{M}$. By (4) we have $\mathfrak{s}_{ji} \leq_T \mathfrak{t}_{ji}$ and thus, again by (5), $\mathfrak{s} \leq_T \mathfrak{t}$. □

Remark 7.6. Point (2) in the above proof already appears in different flavours in the literature: for instance it closely resembles Lemma 13 in [Chandra and Merlin 1977] and Lemma 3 in [Andréka and Bredikhin 1995]. In this sense, the proof above is analogous to the one in [Sagiv and Yannakakis 1980] that provides an algorithm for checking inclusion of disjunctive-conjunctive queries by relying on the algorithm in [Chandra and Merlin 1977] for conjunctive queries.

Deconstructing the Calculus of Relations with Tapes. We conclude by showing how tapes can help in dealing with $\leq_{\mathsf{CR}}$, i.e. the semantic inclusion of the calculus of binary relations (Section 2).

Recall from Example 3.3 that the set Σ in CR_Σ can be regarded as a monoidal signature Σ with set of sorts $\mathcal{S} = \{A\}$. From this signature one constructs $\mathbf{T}_{\mathbf{CB}_\Sigma}$ as prescribed in Example 6.12 and encodes expressions of CR_Σ into tapes of $\mathbf{T}_{\mathbf{CB}_\Sigma}$. The encoding $\mathcal{E}(\cdot)$ is illustrated in Table 7 where, to make the notation lighter, we avoided labelling all the wires with A. It is easy to see that all expressions are mapped into tapes of type $A \to A$. For instance,



= 

Observe that $\llbracket \mathcal{E}(R \cup (S \cap T)) \rrbracket^\sharp$, illustrated in (22), coincides with $\langle\!\langle R \cup (S \cap T) \rangle\!\rangle_{\mathcal{I}}$ defined in (2). More generally, a simple inductive argument confirms that $\mathcal{E}(\cdot)$ preserves the semantics.

Proposition 7.7. For all expressions $E \in \mathsf{CR}_\Sigma$ and interpretations $\mathcal{I}$, $\langle\!\langle E \rangle\!\rangle_{\mathcal{I}} = \llbracket \mathcal{E}(E) \rrbracket^\sharp_{\mathcal{I}}$.

By Propositions 7.4, 7.7 and the completeness theorem the next result follows immediately.

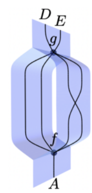
Corollary 7.8. *For all $E_1, E_2 \in \mathsf{CR}_\Sigma$, $E_1 \leq_{\mathsf{CR}} E_2$ if and only if $\mathcal{E}(E_1) \leq_{\mathcal{T}} \mathcal{E}(E_2)$.*

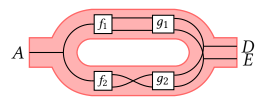
Remark 7.9. The reader may wonder whether it is possible to encode tape diagrams into the calculus of relations. This is not the case, even when considering only tapes in $\mathbf{T}_{\mathbf{CB}_\Sigma}[A, A]$. To prove this it suffices to observe that a tape of type $A \to A$ can express the graph (8) in [Pous 2018] which is known to be not expressible in CR_Σ. More generally, $\mathbf{T}_{\mathbf{CB}_\Sigma}$ can express coherent logic, i.e. the fragment of first order logic consisting of $\exists$, $\wedge$, $\top$, $\vee$ and $\bot$, while the positive fragment of the calculus of relations expresses the restriction of coherent logic to formulae using at most three variables: indeed one needs four variables to express the graph (8) in [Pous 2018]. The reader can try to extend the encoding in Figure 4 of [Bonchi et al. 2018] with tapes expressing $\vee$ and $\bot$.

8 CONCLUSION

Related Work. The idea of inserting string diagrams into superstructures, like we do when passing from $\mathbf{C}_\Sigma$ to $\mathbf{T}_\Sigma$, is not new. Here we mention a few examples, but this list is by no means exhaustive. Functorial boxes [Melliès 2006], for instance, are a graphical expedient that allows one to draw in a simple, intuitive manner monoidal functors $F\colon \mathbf{C} \to \mathbf{D}$: the image along F of a string diagram c in $\mathbf{C}$ can be depicted as a box, labelled by F, drawn around c. A clear analogy to our tape diagrams, however, as a diagrammatic calculus for finite biproduct rig categories, does not seem to be evident. A more apparent resemblance with our drawings are *twisting polygraphs* used in [Acclavio 2019] to give a formal syntax for the representation of proof nets in Multiplicative Linear Logic. In there the author develops a string diagrammatic calculus, where diagrams look like tree-shaped "tapes", whose leaves are the axioms appearing in the proof. However, there is no mention of rig categories in [Acclavio 2019]. Also in [Bartlett et al. 2015] the authors use a diagrammatic language consisting of certain surfaces that host internal string diagrams within them but these are, like sheet diagrams in [Comfort et al. 2020], inherently three dimensional.

All in all, sheet diagrams are the closest related structures to tape diagrams. Similarities include the fact that the objects are polynomials in both cases and indeed also the category of sheet diagrams turns out to be a sesquistrict rig category. Moreover, the definition of $\otimes$ in [Comfort et al. 2020] exploits a notion of whiskering given in terms of diagrammatic manipulations, while in our approach it is defined inductively. The key difference between the two languages is that sheet diagrams may have nodes (morphisms) in the intersection of two different surfaces. In the presence of a biproduct, by virtue of our Theorem 4.9, we can reduce a rig signature into a monoidal one. In this case, all the generators will appear as nodes in a single sheet and never in an intersection of two or more. The only morphisms that will appear in an intersection are (co)diagonals whose existence is guaranteed by the biproduct structure of $\oplus$.





For example, consider the sheet diagram above (on the left) as a representation of a **Vect** morphism. Since **Vect** has biproducts the generator $f\colon A \to BC \oplus BC$ is decomposed as $\triangleleft_A ; (f_1 \oplus f_2)$ where $f_1, f_2\colon A \to BC$ are now generators of the novel monoidal signature. Applying the same procedure to $g\colon BC \oplus CB \to DE$ yields the tape diagram on the right. When $\oplus$ is not a biproduct, Theorem 4.9 does not hold and thus the procedure described above may fail. One example is **Set**, where $+$ is not a biproduct and thus a function $f\colon A \times B \to X + Y$ cannot be canonically decomposed in two functions $f_1\colon A \times B \to X$ and $f_2\colon A \times B \to Y$. Therefore $f\colon A \times B \to X + Y$ can be drawn using sheet diagrams, crucially in 3 dimensions, but not using tape diagrams.

Conclusion and Future Work. Like string diagrams provide, by Theorem 2.3 in [Joyal and Street 1991], a graphical language for symmetric monoidal categories, tape diagrams are, by Theorems 4.9 and 5.11, a universal language for rig categories with finite biproducts. In particular, by Corollary 5.12, whenever the semantic domain of a string diagrammatic language carries the structure of a finite biproduct rig category (or even a finite (co)product, see Remark 5.16), one can wrap string diagrams into tapes and obtain a meaningful language. By applying this approach to the ZX-calculus, we can easily specify a primitive form of quantum control (Examples 5.13 and 5.14). Other relevant instances of tape diagrams are $\sqcup$-props from [Boisseau and Piedeleu 2022] (Example 6.13).

The leading example investigated in this paper is CR, the positive fragment of the calculus of relations [Tarski 1941]. The tape diagrams axioms in Figures 1, 2 and 3 allow us to prove all and only the valid equivalences between expressions of CR (Corollary 7.8). This result follows easily from the completeness theorem (Theorem 7.5) that is close in spirit to those in [Hasegawa et al. 2008] and [Selinger 2012]. Note that Corollary 7.8 does not contradict the impossibility of a finite axiomatisation proved in [Hodkinson and Mikulás 2000], as our syntax is a radical departure from CR. However, the fact that a simple axiomatisation—consisting of several well known algebraic structures—is possible with tape diagrams seems to suggest that the tape notation is more suitable than the traditional one.

The obvious next step consists in considering the extension of CR with Kleene star (reflexive and transitive closure), see e.g. [Brunet and Pous 2015; Pous 2018]. In terms of tapes this can be obtained by adding a trace to the monoidal structure given by $\oplus$. Note that $\mathbf{T}_{\mathbf{CB}_\Sigma}$ is actually traced on $\otimes$, but this trace corresponds to feedbacks and not to Kleene star (iteration). The relationship between the two traced monoidal structures of **Rel** has been studied in [Selinger 1998], rephrasing an early work by Bainbridge [Bainbridge 1976], who was the first to observe the duality between *data flow* and *control flow* provided by $(\mathbf{Rel}, \otimes, 1)$ and $(\mathbf{Rel}, \oplus, 0)$. Indeed, the language resulting from adding a trace to $\mathbf{T}_{\mathbf{CB}_\Sigma}$ would represent data flow at the level of circuits and control flow at the level of tapes. Such language would be similar in spirit to a diagrammatic version of Hoare Logic [Hoare 1969] consisting of imperative programs, predicates on them and, in place of a proof system, diagrammatic laws. The ubiquity of our axioms gives us the hope that this language could be easily adjusted to deal with concurrent and quantum computations. Preliminary inspirations come, respectively, from [Baldan and Gadducci 2019; Hoare et al. 2011; Kappé et al. 2018] and [Harding 2008].

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