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MASTERS THESIS

Frames and Equilogical Spaces

Alessio Santamaria

Supervisor
Prof. Giuseppe Rosolini

Co-supervisor
Dott. Ruggero Pagnan

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Introduction

The category $\mathbb{F}\text{rames}$ is a prominent category because it suggests an algebraic counterpart to the category of topological spaces; particularly relevant in this respect is the so-called *Stone duality*, which consists in a pair of adjoint functors

$$\begin{tikzcd} \mathbb{T}\text{op} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbb{L}\text{oc} \arrow[l, bend left] \end{tikzcd}$$

between the category of topological spaces and the category of Locales, which is the opposite category $\mathbb{F}\text{rames}^{\text{op}}$ of frames. This adjunction determines in a canonical way two subcategories, one of $\mathbb{T}\text{op}$, the other of $\mathbb{L}\text{oc}$, which are equivalent. One is the category of sober spaces (i.e. T_0-spaces where open sets generate the points of the space) and the category of spatial locales, i.e. subframes of Boolean algebras of sets. The study of $\mathbb{F}\text{rames}$ is therefore particularly interesting because it allows us to deepen our knowledge about topological spaces, which are heavily used in every branch of mathematics.

In the thesis we try to give an abstract motivation for the intuition that $\mathbb{F}\text{rames}$ are the algebras of topological spaces, similarly to the fact that Boolean algebras are the algebras of finite sets. In order to do this, we consider the category $\mathbb{E}\text{qu}$ of equilogical spaces, which embeds $\mathbb{T}\text{op}_0$ as a full and faithful subcategory, and is cartesian closed. On that category we study the double-exponential monad

$$T = \Sigma^{(\Sigma^{(-)})}$$

for Σ the Sierpinski space (considered as equilogical space). Then we compare to $\mathbb{F}\text{rames}$ the category $\mathbb{E}\text{qu}^T$ of Eilenberg-Moore algebras for T. We start considering the global sections functor

$$\begin{tikzcd} \mathbb{E}\text{qu} \arrow[r, "\Gamma'"] & \mathbb{S}\text{et} \\ A \arrow[r, mapsto] & \mathbb{E}\text{qu}(\mathbb{1}, A) \end{tikzcd}$$

and determine a "natural" frame structure on the set $\Gamma'(A)$ when A is a $\Sigma^{(\Sigma^{(-)})}$-algebra, thus inducing from the global section functor another functor $\Gamma\colon \mathbb{E}\text{qu}^T \to \mathbb{F}\text{rames}$. In fact, we show out that every T-algebra A inherits—in a functorial way—every algebraic operation on Σ, meaning that every $f\colon \Sigma^I \to \Sigma^J$ in $\mathbb{E}\text{qu}$ gives rise to a $f_A\colon A^I \to A^J$. Using this fact, we can transfer the frame structure of Σ onto $\Gamma(A)$.

We get close to show that Γ is an equivalence of categories, but at the time of writing the thesis the problem is still unsolved.

With the collaboration of Prof. Rosolini and Dott. Frosoni, I spent several months trying to find the complete answer. Now, through testing various different attempts, we have found the candidate functor $\mathcal{L}\colon \mathbb{F}\text{rames} \to \mathbb{E}\text{qu}^T$ as it acts as a right "inverse" for Γ, i.e. it makes Γ essentially surjective on objects. Here we show how we obtained this interesting result, yet partial.

Very recently, we proved that $\mathcal{L}$ is a left adjoint of Γ, but for timing reasons I am not able to write it down in this thesis.

The thesis structured in three chapters. In chapter 1, we study cartesian closed categories in total generality, introducing the concept of internal language for cartesian closed categories, that is a typed-λ calculus that, roughly speaking, allows to treat every cartesian closed category as if it were the category $\mathbb{S}\mathrm{et}$. We use the internal language to study the double-exponential monad $X^{(X^{(-)})}$ in the setting of an arbitrary cartesian closed category, in particular we focus on the properties of algebras for such a monad.

In chapter 2 we start the study of the particular problem, introducing the category $\mathbb{E}\mathrm{qu}$ and using the results of chapter 1 to work with the monad $\Sigma^{(\Sigma^{(-)})}$, and to construct the functor Γ.

Finally, in chapter 3 we pursue the search of an inverse functor. First we present an approach that is inconclusive, but interesting by itself because it shows several remarkable properties of the algebras for the double-exponential monad; next we focus on how to define a T-algebra, given an arbitrary frame, defining finally the functor $\mathcal{L}$.

Chapter 1

Cartesian closed categories

The thesis will be focused mainly on *cartesian closed* categories, a particular class of categories, but extremely relevant in many respects. Our concerns will be on categorical logic and topology.

Rather than introducing directly the definition of cartesian closure, we begin with the *internal language* which will fit the notion of cartesian closed category so tightly as to make them look very much like the category $\mathbb{S}\mathrm{et}$ of sets and functions, see [Cro93].

1.1 Internal Language

We need several definitions before we can introduce the concept of $\lambda\times$-theory needed to present the internal language of cartesian closed categories.

Definition 1.1. A $\lambda\times$-*signature*, Sg, is given by the following data:

- A collection of *ground types*. A *type* is defined inductively as follows:
  - *unit* is a type;
  - any ground type is a type;
  - if β and γ are types, then $\beta \times \gamma$ is a type;
  - if β and γ are types, then γ^β is a type.

  We call $\beta \times \gamma$ a *binary product type* and γ^β a *function type*.
- A collection of *function symbols* each of which has an *arity* which is a natural number (when it is 0 we refer to such a symbol as a *constant*).
- A *sorting* for each function symbol f, which is a non-empty list $[\alpha_1, \dots, \alpha_a, \alpha]$ of $a+1$ types, where a is the arity of f. We shall write $f \colon \alpha_1, \dots, \alpha_a \to \alpha$ to denote the sorting of f. In the case that k is a function symbol of arity 0 we shall denote the sorting by $k : \alpha$ and the function symbol k will be referred to as a *constant* of type α.

We can now define the *raw terms* generated by a $\lambda\times$-signature Sg, assuming that we are given a countably infinite stock of *variables*, say $Var = \{x, y, \dots\}$. A raw term is defined inductively as follows:

- a variable x is a raw term;

- a constant k is a raw term;
- if f is any function symbol of non-zero arity a and $N_1, \ldots, N_a$ are raw terms, then $f(N_1, \ldots, N_a)$ is a raw term;
- $()$ is a raw term;
- if N_1 and N_2 are raw terms, then (N_1, N_2) is a raw term;
- if N is a raw term, then $\mathsf{Fst}(N)$ and $\mathsf{Snd}(N)$ are raw terms;
- if α is a type and N a raw term, then $\lambda x : \alpha.\, N$ is a raw term;
- if M and N are raw terms, then $M\,N$ is a raw term.

Informally, think of the raw terms in the following ways:

- x is an arbitrary element;
- k is a given constant;
- $f(N_1, \ldots, N_a)$ is an expression which is the application of f to a arguments;
- $()$ can be thought of as "a unique element of a one point set";
- $(-, -)$ takes a pair of arguments N_1 and N_2 and returns the pair (N_1, N_2);
- Fst takes a pair P and returns the first argument $\mathsf{Fst}(P)$ and similarly Snd takes a pair P and returns the second argument $\mathsf{Snd}(P)$;
- $\lambda x : \alpha.\, N$ is a function which acts on the generic argument x producing N;
- $M\,N$ is the result of the application of a function M to an argument N.

Remark 1.2. We shall often write syntax in an *informal* fashion. When expressions are complicated, we might write parentheses for $M\,N$ in order to improve readbility. We will use brackets "(" and ")" in an informal fashion to indicate sufficient structure to reconstruct the formal syntax.

We will need to define the concept of free and bound variables of raw terms. In order to do so we shall need some auxiliary definitions. We define the relation "R is a *raw subterm* of M", written $R \prec M$, through the following clauses:

- $M \prec M$;
- given M_i for $1 \leq i \leq n$, if $R \prec M_i$ for some i, then $R \prec f(M_1, \ldots, M_n)$;
- if $R \prec M$ or $R \prec N$, then $R \prec (M, N)$;
- if $R \prec P$ then $R \prec \mathsf{Fst}(P)$;
- if $R \prec P$ then $R \prec \mathsf{Snd}(P)$;
- if $R = x$ or $R \prec M$, then $R \prec \lambda x : \alpha.\, M$;
- if $R \prec M$ or $R \prec N$, then $R \prec MN$.

Definition 1.3. Let M be a raw term and x a variable. If x occurs in M, then x is *bound* if it occurs in a subterm of M of the form $\lambda x : \alpha. N$. If x occurs in M and is not bound, it is *free in* M. If x has at least one free occurrence in M, then x is said to be a *free variable* of M. We denote $fv(M)$ the set of all the free variables of the raw term M.

Example 1.4. Let

$$M \stackrel{\text{def}}{=} \lambda x : \alpha. \, yx(\lambda x : \beta. \, y(\lambda y : \delta. \, z)x).$$

The first and second occurrences of y in M, from the left hand side, are free, but the rightmost y is bound. All occurrences of x are bound.

We are now ready to define the concept of *substitution* of the raw term N for the variable x in the raw term M (written $M[N/x]$). We do so via structural induction on M, through the following clauses:

- $x[N/x] \stackrel{\text{def}}{=} N$;
- $y[N/x] \stackrel{\text{def}}{=} y$ where y is a variable distinct from x;
- $k[N/x] \stackrel{\text{def}}{=} k$ where k is a constant function symbol;
- $f(M_1, \ldots, M_a)[N/x] \stackrel{\text{def}}{=} f(M_1[N/x], \ldots, M_a[N/x])$ where f is a function symbol of non-zero arity a;
- $()[N/x] \stackrel{\text{def}}{=} ()$;
- $(M_1, M_2)[N/x] \stackrel{\text{def}}{=} (M_1[N/x], M_2[N/x])$;
- $\mathsf{Fst}(M)[N/x] \stackrel{\text{def}}{=} \mathsf{Fst}(M[N/x])$ and $\mathsf{Snd}(M)[N/x] \stackrel{\text{def}}{=} \mathsf{Snd}(M[N/x])$;
- $(M_1 M_2)[N/x] \stackrel{\text{def}}{=} (M_1[N/x])(M_2[N/x])$.
- $(\lambda x : \alpha. \, M)[N/x] \stackrel{\text{def}}{=} \lambda x : \alpha. \, M$;
- $(\lambda y : \alpha. \, M)[N/x] \stackrel{\text{def}}{=} \lambda z : \alpha. \, (M[z/y][N/x])$ where $z \notin fv(M) \cup fv(N)$ and z is different from x and y;

Note that the last clause amounts to a simple renaming of the variable y to ensure that occurrences of y in N are not captured by $\lambda y : \alpha$ when N is substituted for x. For this reason, substitution is only well defined up to renaming of bound variables.

Definition 1.5. A *context* is a finite list of (variable, type) pairs, written as $\Gamma = [x_1 : \alpha_1, \ldots, x_n : \alpha_n]$, where the variables are all distinct. A *term-in-context* is a judgement of the form $\Gamma \vdash M : \alpha$, where Γ is a context, M is a raw term and α a type. We shall now define a certain class of judgements of the form $Sg \, \rhd \, \Gamma \vdash M : \alpha$ where $\Gamma \vdash M : \alpha$ is a term-in-context, we refer to $Sg \, \rhd \, \Gamma \vdash M : \alpha$ as a proved term. Formally, the *proved terms* are generated by the following rules:

- **Variables**

$$\frac{}{Sg \, \rhd \, \Gamma, x : \alpha, \Gamma' \vdash x : \alpha}$$

- **Unit Term**

$$\frac{}{Sg \;\rhd\; \Gamma \vdash () : \mathit{unit}}$$

- **Function Symbols**

$$\frac{}{Sg \;\rhd\; \Gamma \vdash k : \alpha}\,(k : \alpha)$$

$$\frac{Sg \;\rhd\; \Gamma \vdash M_1 : \alpha_1 \qquad \dots \qquad Sg \;\rhd\; \Gamma \vdash M_a : \alpha_a}{Sg \;\rhd\; \Gamma \vdash f(M_1, \dots, M_a) : \beta}\,(f \colon \alpha_1, \dots, \alpha_a \to \beta)$$

- **Binary Product Terms**

$$\frac{Sg \;\rhd\; \Gamma \vdash M : \alpha \qquad Sg \;\rhd\; \Gamma \vdash N : \beta}{Sg \;\rhd\; \Gamma \vdash (M, N) : \alpha \times \beta}$$

$$\frac{Sg \;\rhd\; \Gamma \vdash P : \alpha \times \beta}{Sg \;\rhd\; \Gamma \vdash \mathsf{Fst}(P) : \alpha} \qquad \frac{Sg \;\rhd\; \Gamma \vdash P : \alpha \times \beta}{Sg \;\rhd\; \Gamma \vdash \mathsf{Snd}(P) : \beta}$$

- **Function Terms**

$$\frac{Sg \;\rhd\; \Gamma, x : \alpha \vdash F : \beta}{Sg \;\rhd\; \Gamma \vdash \lambda x : \alpha .\, F : \beta^\alpha} \qquad \frac{Sg \;\rhd\; \Gamma \vdash M : \beta^\alpha \qquad Sg \;\rhd\; \Gamma \vdash N : \alpha}{Sg \;\rhd\; \Gamma \vdash MN : \beta}$$

Remark 1.6. It is assumed that both the hypotheses and conclusion of each rule are well formed. For example, in the rule which introduces a function type, it is implicit that x does not appear in Γ, because Γ is a well formed context in both the hypothesis and conclusion of the rule.

Informally, think of the raw term M in the term-in-context $\Gamma \vdash M : \alpha$ as a program, and the variables which appear in the context Γ as an environment for M, and think of the raw term M in the proved term $Sg \;\rhd\; \Gamma \vdash M : \alpha$ as a well typed program.

Remark 1.7. The terms-in-context which appear as part of the judgement $Sg \;\rhd\; \Gamma \vdash M : \alpha$ form a sub-class of all the terms-in-context; the formal symbol $Sg \rhd$ is indicating that $\Gamma \vdash M : \alpha$ is in this sub-class, and reminds us that we work with respect to the signature $Sg \rhd$. We shall just refer to a proved term $\Gamma \vdash M : \alpha$, when it is clear to which signature we refer.

Definition 1.8. A $\lambda\times$-*theory*, Th, is a pair (Sg, Ax) where Sg is a $\lambda\times$-signature and Ax is a collection of equations-in-context. An *equation-in-context* is a judgement of the form $\Gamma \vdash M = M' : \alpha$ where $\Gamma \vdash M : \alpha$ and $\Gamma \vdash M' : \alpha$ are proved terms. The equations-in-context in Ax are called the *axioms* of the theory. We indicate this by writing $Ax \;\rhd\; \Gamma \vdash M = M' : \alpha$. The *theorems* of Th consist of the judgements of the form $Th \;\rhd\; \Gamma \vdash M = M' : \alpha$ (where $Sg \;\rhd\; \Gamma \vdash M : \alpha$ and $Sg \;\rhd\; \Gamma \vdash M' : \alpha$) generated by the following rules.

- **Axioms**

$$\frac{Ax \;\rhd\; \Gamma \vdash M = M' : \alpha}{Th \;\rhd\; \Gamma \vdash M = M' : \alpha}$$

- **Equational Reasoning**

$$\frac{Sg \rhd \Gamma \vdash M : \alpha}{Th \rhd \Gamma \vdash M = M : \alpha} \qquad \frac{Th \rhd \Gamma \vdash M = M' : \alpha}{Th \rhd \Gamma \vdash M' = M : \alpha}$$

$$\frac{Th \rhd \Gamma \vdash M = M' : \alpha \qquad Th \rhd \Gamma \vdash M' = M'' : \alpha}{Th \rhd \Gamma \vdash M = M'' : \alpha}$$

- **Permutation**

$$\frac{Th \rhd \Gamma \vdash M = M' : \alpha}{Th \rhd \pi\Gamma \vdash M = M' : \alpha} \text{ (where } \pi \text{ is a permutation)}$$

- **Weakening**

$$\frac{Th \rhd \Gamma \vdash M = M' : \alpha}{Th \rhd \Gamma' \vdash M = M' : \alpha} \text{ (where } \Gamma \subseteq \Gamma')$$

- **Substitution**

$$\frac{Th \rhd \Gamma, x : \alpha \vdash N = N' : \beta \qquad Th \rhd \Gamma \vdash M = M' : \alpha}{Th \rhd \Gamma \vdash N[M/x] = N'[M'/x] : \beta}$$

- **Unit Equations**

$$\frac{Sg \rhd \Gamma \vdash M : \mathit{unit}}{Th \rhd \Gamma \vdash M = () : \mathit{unit}}$$

- **Binary Product Equations**

$$\frac{Sg \rhd \Gamma \vdash M : \alpha \qquad Sg \rhd \Gamma \vdash N : \beta}{Th \rhd \Gamma \vdash \mathsf{Fst}((M, N)) = M : \alpha} \qquad \frac{Sg \rhd \Gamma \vdash M : \alpha \qquad Sg \rhd \Gamma \vdash N : \beta}{Th \rhd \Gamma \vdash \mathsf{Snd}((M, N)) = N : \beta}$$

$$\frac{Sg \rhd \Gamma \vdash P : \alpha \times \beta}{Th \rhd \Gamma \vdash (\mathsf{Fst}(P), \mathsf{Snd}(P)) = P : \alpha \times \beta}$$

- **Function Equations**

$$\frac{Sg \rhd \Gamma, x : \alpha \vdash F : \beta \qquad Sg \rhd \Gamma \vdash M : \alpha}{Th \rhd \Gamma \vdash (\lambda x : \alpha.\, F)M = F[M/x] : \beta}$$

$$\frac{Sg \rhd \Gamma \vdash M : \beta^\alpha}{Th \rhd \Gamma \vdash \lambda x : \alpha.\, (Mx) = M : \beta^\alpha} \text{ (provided } x \notin fv(M))$$

$$\frac{Th \rhd \Gamma, x : \alpha \vdash F = F' : \beta}{Th \rhd \Gamma \vdash \lambda x : \alpha.\, F = \lambda x : \alpha.\, F' : \beta^\alpha}$$

In order to give a semantics to $\lambda\times$-theories, we shall use cartesian closed categories, which we define next.

1.2 Cartesian closed categories

We shall refer to [Mac98] for the standard elementary notions about categories.

Definition 1.9. A category with finite products $\mathbb{C}$ is said *cartesian closed* if for all $A, B \in \mathrm{Ob}(\mathbb{C})$ there is an object B^A, called *exponential object*, and an *evaluation* morphism *eval*: $B^A \times A \to B$ such that for every Z object in $\mathbb{C}$ and for every morphism $f\colon Z \times A \to B$ there is a unique morphism $\lambda f\colon Z \to B^A$ forming a commutative diagram

$$\begin{tikzcd} B^A \times A \arrow[r, "eval"] & B \\ Z \times A \arrow[u, "\lambda f \times \mathrm{id}_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

The semantics we want to give to a $\lambda\times$-theory is a *categorical* semantics: types will be modelled by objects in a cartesian closed category, while the proved terms will be interpreted by morphisms. In particular, the *unit* type will correspond to a terminal object (which we shall denote $\mathbb{1}$), the product type to the product in the category, and the function type to the exponential object.

Definition 1.10. Let $\mathbb{C}$ be a cartesian closed category and *Sg* a $\lambda\times$-signature. Then a *structure* $\mathbf{M}$ for *Sg* in $\mathbb{C}$ is specified by giving:

- for every ground type γ of *Sg* an object $[\![\gamma]\!]$ of $\mathbb{C}$;
- for every constant function symbol $k : \alpha$, a global element $[\![k]\!]$ of $[\![\alpha]\!]$;
- for every function symbol $f\colon \alpha_1, \ldots, \alpha_n \to \beta$ of *Sg* with non-zero arity, a morphism

$$[\![f]\!]\colon [\![\alpha_1]\!] \times \cdots \times [\![\alpha_n]\!] \to [\![\beta]\!]$$

The object $[\![\alpha]\!]$ is defined via structural induction on the type α, setting $[\![unit]\!] \stackrel{\text{def}}{=} \mathbb{1}$, $[\![\alpha \times \beta]\!] \stackrel{\text{def}}{=} [\![\alpha]\!] \times [\![\beta]\!]$ and $[\![\beta^\alpha]\!] \stackrel{\text{def}}{=} [\![\beta]\!]^{[\![\alpha]\!]}$.

Given a context $\Gamma = [x_1 : \alpha_1, \ldots, x_n : \alpha_n]$ we set $[\![\Gamma]\!] \stackrel{\text{def}}{=} [\![\alpha_1]\!] \times \cdots \times [\![\alpha_n]\!]$. Then for every proved term $\Gamma \vdash M : \alpha$ we shall use the structure $\mathbf{M}$ to specify a morphism

$$[\![\Gamma \vdash M : \alpha]\!] : [\![\Gamma]\!] \to [\![\alpha]\!]$$

in $\mathbb{C}$. The semantics of proved terms is given by the following rules:

- **Variables**

$$\frac{}{[\![\Gamma, x : \alpha, \Gamma' \vdash x : \alpha]\!] \stackrel{\text{def}}{=} \pi\colon [\![\Gamma]\!] \times [\![\alpha]\!] \times [\![\Gamma']\!] \to [\![\alpha]\!]}$$

- **Unit Term**

$$\frac{}{[\![\Gamma \vdash () : unit]\!] \stackrel{\text{def}}{=} !\colon [\![\Gamma]\!] \to \mathbb{1}}$$

- **Function Symbols**

$$\frac{}{[\![\Gamma \vdash k : \alpha]\!] \stackrel{\text{def}}{=} [\![k]\!] \circ ! \colon [\![\Gamma]\!] \to \mathbb{1} \to [\![\alpha]\!]}\ (k : \alpha)$$

$$\frac{[\![\Gamma \vdash k : \alpha]\!] = m_1 \colon [\![\Gamma]\!] \to [\![\alpha_1]\!] \qquad \dots \qquad [\![\Gamma \vdash M_n : \alpha_n]\!] = m_n \colon [\![\Gamma]\!] \to [\![\alpha_n]\!]}{\left[\!\!\left[\Gamma \vdash f(\vec{M}) : \beta\right]\!\!\right] = [\![f]\!] \circ \langle m_1, \dots, m_n \rangle \colon [\![\Gamma]\!] \to [\![\alpha_1]\!] \times \cdots \times [\![\alpha_n]\!] \to \beta}$$
$$(f \colon \alpha_1, \dots, \alpha_n \to \beta)$$

- **Binary Product Terms**

$$\frac{[\![\Gamma \vdash M : \alpha]\!] = m \colon [\![\Gamma]\!] \to [\![\alpha]\!] \qquad [\![\Gamma \vdash N : \beta]\!] = n \colon [\![\Gamma]\!] \to [\![\beta]\!]}{[\![\Gamma \vdash (M, N) : \alpha \times \beta]\!] = \langle m, n \rangle \colon [\![\Gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!]}$$

$$\frac{[\![\Gamma \vdash P : \alpha \times \beta]\!] = p \colon [\![\Gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!]}{[\![\Gamma \vdash \mathsf{Fst}(P) : \alpha]\!] = \pi_1 \circ p \colon [\![\Gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!] \to [\![\alpha]\!]}$$

$$\frac{[\![\Gamma \vdash P : \alpha \times \beta]\!] = p \colon [\![\Gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!]}{[\![\Gamma \vdash \mathsf{Snd}(P) : \beta]\!] = \pi_2 \circ p \colon [\![\Gamma]\!] \to [\![\alpha]\!] \times [\![\beta]\!] \to [\![\beta]\!]}$$

- **Function Terms**

$$\frac{[\![\Gamma, x : \alpha \vdash F : \beta]\!] = f \colon [\![\Gamma]\!] \times [\![\alpha]\!] \to [\![\beta]\!]}{[\![\Gamma \vdash \lambda x : \alpha.\, F : \beta^\alpha]\!] = \lambda f \colon [\![\Gamma]\!] \to [\![\beta]\!]^{[\![\alpha]\!]}}$$

$$\frac{[\![\Gamma \vdash M : \beta^\alpha]\!] = m \colon [\![\Gamma]\!] \to [\![\beta]\!]^{[\![\alpha]\!]} \qquad [\![\Gamma \vdash N : \alpha]\!] = n \colon [\![\Gamma]\!] \to [\![\alpha]\!]}{[\![\Gamma \vdash MN : \beta]\!] \stackrel{\text{def}}{=} \mathit{eval} \circ \langle m, n \rangle \colon [\![\Gamma]\!] \to [\![\beta]\!]^{[\![\alpha]\!]} \times [\![\alpha]\!] \to [\![\beta]\!]}$$

The fundamental result about categorical semantics (in particular, for that of a $\lambda\times$-theory) is that substitution of a term for a variable in a term is interpreted by composition of morphisms. More precisely, we have the following Theorem.

Theorem 1.11. *Let $\Gamma' \vdash N : \beta$ be a proved term where $\Gamma' = [x_1 : \alpha_1, \dots, x_n : \alpha_n]$ and let $\Gamma \vdash M_i : \alpha_i$ be proved terms for $i \in \{1, \dots, n\}$. Then $\Gamma \vdash N[\vec{M}/\vec{x}] : \beta$ is a proved term and*

$$\left[\!\!\left[\Gamma \vdash N[\vec{M}/\vec{x}] : \beta\right]\!\!\right] = [\![\Gamma' \vdash N : \beta]\!] \circ \langle [\![\Gamma \vdash M_1 : \alpha_1]\!], \dots, [\![\Gamma \vdash M_n : \alpha_n]\!] \rangle$$

where $N[\vec{M}/\vec{x}]$ denotes simultaneous substitution.

Definition 1.12. Let $\mathbf{M}$ be a structure for a $\lambda\times$-signature in a cartesian closed category $\mathbb{C}$. Given an equation-in-context $\Gamma \vdash M = M' : \alpha$ we say that $\mathbf{M}$ *satisfies* the equation-in-context if $[\![\Gamma \vdash M : \alpha]\!]$ and $[\![\Gamma \vdash M' : \alpha]\!]$ are equal morphisms in $\mathbb{C}$. We say that $\mathbf{M}$ is a *model* of a $\lambda\times$-theory $\mathit{Th} = (\mathit{Sg}, \mathit{Ax})$ if $\mathbf{M}$ satisfies all of the equations-in-context in Ax.

The following theorem shows that for a model $\mathbf{M}$ in order to satisfy all of the theorems of a $\lambda\times$-signature it suffices to satisfy only the axioms, i.e. to be a model of the theory.

Theorem 1.13 (Soundness)**.** *Let $\mathbb{C}$ be a cartesian closed category, Th a $\lambda\times$-theory and $\mathbf{M}$ a model of Th in $\mathbb{C}$. Then $\mathbf{M}$ satisfies any equation-in-context which is a theorem of Th.*

Theorem 1.14. *Let $\mathbb{C}$ be a cartesian closed category. Then $\mathbb{C}$ generates a $\lambda\times$-theory $Th(\mathbb{C}) = (Sg(\mathbb{C}), Ax(\mathbb{C}))$ that has a canonical model in $\mathbb{C}$.*

Proof. The $\lambda\times$-signature $Sg(\mathbb{C})$ has ground types which are copies of the objects of $\mathbb{C}$ and function symbols which are copies of the morphisms of $\mathbb{C}$, together with some distinguished function symbols which will witness certain isomorphisms. More precisely, for each object A of $\mathbb{C}$ there is a ground type A. The types of $Sg(\mathbb{C})$ are therefore of the form *unit*, or A, or $\alpha\ulcorner\times\urcorner\beta$, or $\ulcorner\beta^{\alpha}\urcorner$ (with α and β types of less "complexity"). The notation $\ulcorner\times\urcorner$ simply distinguishes formal binary products of $Sg(\mathbb{C})$ from binary products $\times$ in $\mathbb{C}$ (and similarly for the exponential).
For each morphism of the form $k\colon \mathbb{1} \to A$ in $\mathbb{C}$ there is a constant function symbol $k : A$, and for each morphism of the form $f\colon A_1 \times \cdots \times A_n \to B$ in $\mathbb{C}$ there is a function symbol $f\colon A_1, \ldots, A_n \to B$. There are also function symbols, each of arity 1, with the following sortings:

- $I_\alpha\colon [\![\alpha]\!] \to \alpha$;
- $J_\alpha\colon \alpha \to [\![\alpha]\!]$ where $[\![\alpha]\!]$ is defined below. (Here, α runs over all types of $Sg(\mathbb{C})$).

The canonical structure $\mathbf{M}$ for $Sg(\mathbb{C})$ in $\mathbb{C}$ is defined by setting $[\![A]\!] \stackrel{\text{def}}{=} A$ for each ground type of $Sg(\mathbb{C})$, $[\![k]\!] \stackrel{\text{def}}{=} k$, $[\![f]\!] \stackrel{\text{def}}{=} f$ and $[\![I_\alpha]\!] = [\![J_\alpha]\!] \stackrel{\text{def}}{=} \mathrm{id}_{[\![\alpha]\!]}$. Note that *by definition* of structure we have $[\![\alpha\ulcorner\times\urcorner\beta]\!] = [\![\alpha]\!] \times [\![\beta]\!]$ and $[\![\ulcorner\beta^{\alpha}\urcorner]\!] = [\![\beta]\!]^{[\![\alpha]\!]}$.

The collection of axioms $Ax(\mathbb{C})$ consists of those equations-in-context generated from $Sg(\mathbb{C})$ which are satisfied by $\mathbf{M}$. This means that $\mathbf{M}$ is indeed a model of the internal language in $\mathbb{C}$. □

The *internal language* of a cartesian closed category $\mathbb{C}$ is the theory $Th(\mathbb{C})$. This language is extremely useful because it allows us to reason about the category $\mathbb{C}$ as it were the category $\mathbb{S}\mathrm{et}$ of sets and functions. For example, if $f\colon A \to B$ is a morphism of $\mathbb{C}$, then there is a proved term $x : A \vdash f(x) : B$ in $Th(\mathbb{C})$. We can think of x as an "element" of the "set" A and $f(x)$ as the action of the "function" f on x. The soundness theorem allows us to prove facts about $\mathbb{C}$ using the internal language. For example, if we wish to prove that $h = gf$ in $\mathbb{C}$ it is enough to show that $Th \rhd x : A \vdash g(f(x)) = h(x) : B$, because the soundness theorem gives

$$h = [\![x : A \vdash h(x) : B]\!] = [\![x : A \vdash g(f(x)) : B]\!] = gf.$$

It is often easier to perform calculations in the internal language than argue category-theoretically. For example, suppose we wish to prove that for any morphism $f\colon A \times B \to C$ and $g\colon C \to D$ in $\mathbb{C}$ one has $\lambda(gf) = \lambda(g \circ \mathit{eval})\lambda f$. We can prove this directly using the universal property of exponential: we have

$$\begin{aligned}
\mathit{eval} \circ \Big(\lambda(g \circ \mathit{eval})(\lambda f) \times \mathrm{id}\Big) &= \mathit{eval} \circ \Big(\lambda(g \circ \mathit{eval}) \times \mathrm{id}\Big) \circ (\lambda f \times \mathrm{id}) \\
&= g \circ \mathit{eval} \circ (\lambda f \times \mathrm{id}) \\
&= gf
\end{aligned}$$

and hence the required equality follows from the said universal property. However, we can proceed more directly using the internal language: we write down proved terms of $Th(\mathbb{C})$ which make use of the *function symbols* $f\colon A, B \to C$ and $g\colon C \to D$, namely

$$Sg \rhd x : A \vdash \lambda y : B.\, f(x, y) : C^B$$

(that corresponds to λf) and

$$Sg \rhd z : C^B \vdash \lambda w : B.\, g(zw) : D^B$$

(that corresponds to $\lambda(g \circ \mathit{eval})$). We have then

$$Sg \ \rhd\ x : A \vdash \lambda w : B.\, g\Big(\big(\lambda y : B.\, f(x,y)\big)w\Big) : D^B$$

and this latter proved terms corresponds to the composition of $\lambda(g \circ \mathit{eval})$ and $\lambda(f)$. But

$$Th \ \rhd\ x : A \vdash \lambda w : B.\, g\Big(\big(\lambda y : B.\, f(x,y)\big)w\Big) = \lambda w : B.\, g\big(f(x,w)\big) : D^B$$

thus writing $[\![-]\!]$ for the canonical model of $Th(\mathbb{C})$ in $\mathbb{C}$ we have

$$\begin{aligned}
\lambda(gf) &= [\![x : A \vdash \lambda w : B.\, g\big(f(x,w)\big) : D^B]\!] \\
&= \Big[\!\!\Big[x : A \vdash \lambda w : B.\, g\Big(\big(\lambda y : B.\, f(x,y)\big)w\Big)\Big]\!\!\Big] \\
&= \lambda(g \circ \mathit{eval})\lambda f.
\end{aligned}$$

Remark 1.15. The crucial step in the above proof using the internal lanuage is the *derivation* of the given theorem *from* the proved terms, which is often easier than the corresponding calculation involving the category morphisms f and g.

We shall make heavy use of the internal language in the rest of the thesis: it will allow us to discuss the properties of the double-exponential monad in total generality, yet in a set-theoretic way, as if we worked in $\mathbb{S}\mathrm{et}$.

We shall find useful also the following notation: when a diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "g_1"'] & B \arrow[d, "f_2"] \\
C \arrow[r, "g_2"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "g"] & B \\
C \arrow[u, "f"] \arrow[ur, "h"'] &
\end{tikzcd}$$

commutes, we write a symbol $=$ inside it, expressing the equality between the composites of the maps which border it—*e.g.* $f_2 \circ f_1 = g_2 \circ g_1$ and $h = g \circ f$—, like so:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "="] & B \arrow[d, "f_2"] \\
C \arrow[r, "g_2"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "g"] \arrow[r, phantom, "=", shift right=3] & B \\
C \arrow[u, "f"] \arrow[ur, "h"'] &
\end{tikzcd}$$

1.3 The double-exponential monad

Recall from [Mac98] the notion of a monad.

Definition 1.16. Let $\mathbb{C}$ be a category. A *monad* over $\mathbb{C}$ is a triple (T, η, μ), where

- $T \colon \mathbb{C} \to \mathbb{C}$ is an endofunctor,
- $\eta\colon \mathrm{id}_{\mathbb{C}} \to T \colon \mathbb{C} \to \mathbb{C}$ is a natural transformation (the *unit*),
- $\mu\colon T^2 \to T \colon \mathbb{C} \to \mathbb{C}$ is a natural transformation (the *multiplication*),

such that

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] \arrow[dr, phantom, "="] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] \arrow[dl, phantom, "="] \arrow[dr, phantom, "="] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

Definition 1.17. Let (T, η, μ) be a monad over a category $\mathbb{C}$. A *T-algebra*, or *Eilenberg-Moore algebra* for T, is a pair (A, α), where

- A is an object of $\mathbb{C}$, the *underlying object of* (A, α),
- $\alpha\colon T(A) \to \mathbb{C}$ is a morphism of $\mathbb{C}$, the *algebraic structure of* (A, α),

such that

$$\begin{tikzcd} T^2(A) \arrow[r, "\mu_A"] \arrow[d, "T(\alpha)"'] \arrow[dr, phantom, "="] & T(A) \arrow[d, "\alpha"] \\ T(A) \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & T(A) \arrow[d, "\alpha"] \arrow[dl, phantom, "="] \\ & A \end{tikzcd}$$

The category $\mathbb{C}^T$ of T-algebras consists of:

- T-algebras as objects,
- for every (A, α) and (B, β) T-algebras, a morphism $f\colon (A, \alpha) \to (B, \beta)$ is a morphism $f\colon A \to B$ in $\mathbb{C}$ such that

$$\begin{tikzcd} T(A) \arrow[r, "\alpha"] \arrow[d, "T(f)"'] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ T(B) \arrow[r, "\beta"] & B \end{tikzcd}$$

For the rest of the chapter, $\mathbb{C}$ is a cartesian closed category, and X a fixed object in $\mathbb{C}$.

Proposition 1.18. *The assignment*

$$\begin{tikzcd} X^{(-)}\colon \mathbb{C}^{op} \arrow[r] & \mathbb{C} \\ A \arrow[d, "f"] \arrow[r, mapsto] & X^A \\ {} & X^f \arrow[u] \\ B \arrow[r, mapsto] & X^B \end{tikzcd} \qquad f \longmapsto X^f$$

where $X^f \coloneqq [\![F : X^B \vdash \lambda a : A.\, F(f(a)) : X^A]\!]$, *is a functor.*

Proof. First we prove that $X^{(-)}$ preserves identities and compositions. Let $A \in \mathrm{Ob}(\mathbb{C})$: we have

$$\begin{aligned} X^{\mathrm{id}_A} &= [\![F : X^A \vdash \lambda a : A.\, F(a) : X^A]\!] \\ &= [\![F : X^A \vdash F : X^A]\!] \\ &= \mathrm{id}_{X^A}\,. \end{aligned}$$

Consider $A \xrightarrow{f} B \xrightarrow{g} C$ two comparable morphisms in $\mathbb{C}$. Then

$$\begin{aligned} X^f &= \llbracket F : X^B \vdash \lambda a : A.\, F(f(a)) : X^A \rrbracket \\ X^g &= \llbracket G : X^C \vdash \lambda b : B.\, G(g(b)) : X^B \rrbracket \end{aligned}$$

and

$$\begin{aligned} X^f \circ X^g &= \llbracket G : X^C \vdash \lambda a : A.\, \big(\lambda b : B.\, G\big(g(b)\big)\big)\big(f(a)\big) : X^A \rrbracket \\ &= \llbracket G : X^C \vdash \lambda a : A.\, G\big(g\big(f(a)\big)\big) : X^A \rrbracket \\ &= \llbracket G : X^C \vdash \lambda a : A.\, G\big((g \circ f)(a)\big) : X^A \rrbracket \\ &= X^{g \circ f}. \end{aligned}$$

□

The universal property of the exponentitation gives rise to a monad (T, η, μ) over $\mathbb{C}$ on the endofunctor

$$T = X^{(X^{(-)})} \colon \big(\mathbb{C}^{\mathrm{op}}\big)^{\mathrm{op}} = \mathbb{C} \longrightarrow \mathbb{C}$$

According to 1.16, we have to have two natural transformations η and μ and appropriate commutative diagrams. First, we write the action of $X^{(X^{(-)})}$ on a morphism $A \xrightarrow{f} B$, because it will be used several times in the following:

$$\begin{aligned} X^{(X^f)} &= \left\llbracket F : X^{(X^A)} \vdash \lambda G : X^B.\, F\big(X^f(G)\big) : X^{(X^B)} \right\rrbracket \\ &= \left\llbracket F : X^{(X^A)} \vdash \lambda G : X^B.\, F\big(\lambda a : A.\, G\big(f(a)\big)\big) : X^{(X^B)} \right\rrbracket. \end{aligned}$$

Also, we may at times write an application of the functor X^A simply as $X(A)$ in order to turn towers of exponential notation such as $X^{(X^{(X^{(X^A)})})}$ into a more readable $X^4(A)$.

For every $A \in \mathrm{Ob}(\mathbb{C})$, we define

$$\begin{aligned} \eta_A &= \left\llbracket a : A \vdash \lambda F : X(A).\, F(a) : X^{(X^A)} \right\rrbracket, \\ \mu_A &= \left\llbracket F : X^{(X^{(X^{(X^A)})})} \vdash \lambda G : X(A).\, F\big(\eta_{X(A)}(G)\big) : X^{(X^A)} \right\rrbracket. \end{aligned}$$

Lemma 1.19. *The families η: $id_{\mathbb{C}} \longrightarrow X^2$ and μ: $X^4 \longrightarrow X^2$ are natural transformations.*

Proof. Let $A \xrightarrow{f} B$ in $\mathbb{C}$. We must prove that the following squares commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & X^2(A) \arrow[d, "X^2(f)"] \\ B \arrow[r, "\eta_B"] & X^2(B) \end{tikzcd} \qquad \begin{tikzcd} X^4(A) \arrow[r, "\mu_A"] \arrow[d, "X^4(f)"'] & X^2(A) \arrow[d, "X^2(f)"] \\ X^4(B) \arrow[r, "\mu_B"] & X^2(B) \end{tikzcd}$$

We do that comparing the expressions of the two compositions in internal language:

$$\begin{aligned} X^2(f) \circ \eta_A &= \llbracket a : A \vdash \lambda G : X(B).\, \eta_A(a)(\lambda a' : A.\, G\,(f(a'))\rrbracket \\ &= \llbracket a : A \vdash \lambda G : X(B).\, G\,(f(a))\rrbracket \\ &= \eta_B \circ f. \end{aligned}$$

So η is natural. Now,

$$\begin{aligned} X^2(f)\circ\mu_A &= \Big[\!\!\Big[F : X^4(A) \vdash \lambda G : X(B).\, \mu_A(F)\big(X(f)(G)\big)\Big]\!\!\Big] \\ &= \Big[\!\!\Big[F : X^4(A) \vdash \lambda G : X(B).\, F\Big(\eta_{X(A)}\big(X(f)(G)\big)\Big)\Big]\!\!\Big]. \end{aligned}$$

Since $X^4(f) = \Big[\!\!\Big[F : X^4(A) \vdash \lambda G : X^3(B).\, F\Big(X^3(f)(G)\Big) : X^4(B)\Big]\!\!\Big]$, we have

$$\mu_B \circ X^4(f) = \Big[\!\!\Big[F : X^4(A) \vdash \lambda G : X(B).\, F\Big(X^3(f)\big(\eta_{X(B)}(G)\big)\Big) : X^2(B)\Big]\!\!\Big].$$

We are left to prove that $G : X(B) \vdash \eta_{X(A)}\big(X(f)(G)\big) = X^3(f)\big(\eta_{X(B)}(G)\big)$ is a theorem in $Th(\mathbb{C})$, but

$$\begin{tikzcd} X(B) \arrow[r, "\eta_{X(B)}"] \arrow[d, "X(f)"'] & X^3(B) \arrow[d, "X^3(f)"] \\ X(A) \arrow[r, "\eta_{X(A)}"] & X^3(A) \end{tikzcd}$$

commutes by naturality of η on $X(f)$. So the proof is complete. □

We can now show that the functor X^2 is a monad.

Theorem 1.20. *For $T = X^2$ and η and μ defined as on p.11, (T, η, μ) is a monad over $\mathbb{C}$.*

Proof. We must show that the two axioms for the unit and multiplication of a monad are satisfied. We start with the first one, considering the following diagram:

$$\begin{tikzcd} X^2 \arrow[r, "\eta X^2"] \arrow[dr, "\mathrm{id}_{X^2}"'] & X^4 \arrow[d, "\mu"] & X^2 \arrow[l, "X^2\eta"'] \arrow[dl, "\mathrm{id}_{X^2}"] \\ & X^2 & \end{tikzcd}$$

Let $A \in \mathrm{Ob}(\mathbb{C})$. We have

$$\Big(\eta X^2\Big)_A = \eta_{X^2(A)} = \Big[\!\!\Big[F : X^2(A) \vdash \lambda G : X^3(A).\, G(F) : X^4(A)\Big]\!\!\Big],$$

so

$$\begin{aligned} \mu_A \circ \eta_{X^2(A)} &= \Big[\!\!\Big[F : X^2(A) \vdash \lambda G : X(A).\, \eta_{X(A)}(G)(F) : X^2(A)\Big]\!\!\Big] \\ &= \Big[\!\!\Big[F : X^2(A) \vdash \lambda G : X(A).\, F(G) : X^2(A)\Big]\!\!\Big] \\ &= \Big[\!\!\Big[F : X^2(A) \vdash F : X^2(A)\Big]\!\!\Big] \\ &= \mathrm{id}_{X^2(A)}\,. \end{aligned}$$

As for the other triangle,

$$\left(X^2\eta\right)_A = X^2(\eta_A) = \left[\!\!\left[F : X^2(A) \vdash \lambda G : X^3(A).\, F\big(X(\eta_A)(G)\big) : X^4(A) \right]\!\!\right]$$

so

$$\mu_A \circ X^2(\eta_A) = \left[\!\!\left[F : X^2(A) \vdash \lambda G : X(A).\, F\Big(X(\eta_A)\big(\eta_{X(A)}(G)\big)\Big) : X^2(A) \right]\!\!\right].$$

If we show that $X(\eta_A)\big(\eta_{X(A)}(G)\big) = G$ for $G : X(A)$, we are done. Since $\eta_{X(A)}(G) = \lambda H : X^2(A).\, H(G)$,

$$\begin{aligned} X(\eta_A)\big(\eta_{X(A)}(G)\big) &= \lambda a : A.\, \eta_{X(A)}(G)\big(\eta_A(a)\big) \\ &= \lambda a : A.\, \eta_{X(A)}(G)\big(\lambda H : X(A).\, H(a)\big) \\ &= \lambda a : A.\, G(a) \\ &= G. \end{aligned}$$

Next we show that the following square commutes:

$$\begin{array}{ccc} X(X^5) & \xrightarrow{\mu X^2} & X^4 \\ {\scriptstyle X^2\mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ X^4 & \xrightarrow{\mu} & X^2 \end{array}$$

We have

$$\left(\mu X^2\right)_A = \mu_{X^2(A)} = \left[\!\!\left[F : X(X^5(A)) \vdash \lambda G : X^3(A).\, F\Big(\eta_{X^3(A)}(G)\Big) : X^4(A) \right]\!\!\right]$$

$$\left(X^2\mu\right)_A = X^2(\mu_A) = \left[\!\!\left[F : X(X^5(A)) \vdash \lambda G : X^3(A).\, F\big(X(\mu_A)(G)\big) : X^4(A) \right]\!\!\right]$$

whence

$$\mu_A \circ \mu_{X^2(A)} = \left[\!\!\left[F : X(X^5(A)) \vdash \lambda G : X(A).\, F\Big(\eta_{X^3(A)}\big(\eta_{X(A)}(G)\big)\Big) : X^2(A) \right]\!\!\right]$$

$$\mu_A \circ X^2(\mu_A) = \left[\!\!\left[F : X(X^5(A)) \vdash \lambda G : X(A).\, F\Big(X(\mu_A)\big(\eta_{X(A)}(G)\big)\Big) : X^2(A) \right]\!\!\right].$$

To see that $\eta_{X^3(A)}\big(\eta_{X(A)}(G)\big) = X(\mu_A)\big(\eta_{X(A)}(G)\big)$ for $G : X(A)$, compute

$$\eta_{X^3(A)}\big(\eta_{X(A)}(G)\big) = \lambda H : X^4(A).\, H\big(\eta_{X(A)}(G)\big)$$

while

$$\begin{aligned} X(\mu_A)\big(\eta_{X(A)}(G)\big) &= \lambda H : X^4(A).\, \eta_{X(A)}(G)\big(\mu_A(H)\big) \\ &= \lambda H : X^4(A).\, \mu_A(H)(G) \\ &= \lambda H : X^4(A).\, H\big(\eta_{X(A)}(G)\big). \end{aligned}$$

□

1.4 Properties of $X^{(X^{(-)})}$-algebras

For the rest of this chapter, $T = X^2$ is the double-exponential monad as defined on p.11. We first show that X itself is a T-algebra, defining $\sigma\colon X^2(X) \to X$ as

$$\sigma = \left[\!\!\left[F : X^2(X) \vdash F(\lambda x : X.\, x) : X \right]\!\!\right].$$

Proposition 1.21. *(X, σ) is a X^2-algebra.*

Proof. We must show that these two diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & X^2(X) \arrow[d, "\sigma"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X^4(X) \arrow[r, "\mu_X"] \arrow[d, "X^2(\sigma)"'] & X^2(X) \arrow[d, "\sigma"] \\ X^2(X) \arrow[r, "\sigma"] & X \end{tikzcd}$$

We have

$$\sigma \circ \eta_X = [\![x : X \vdash \mathrm{id}_X(x) : X]\!] = \mathrm{id}_X$$

so the triangle commutes; moreover

$$\begin{aligned} \sigma \circ \mu_X &= \left[\!\!\left[F : X^4(X) \vdash F\big(\eta_{X(X)}(\mathrm{id}_X)\big) : X \right]\!\!\right] \\ \sigma \circ X^2(\sigma) &= \left[\!\!\left[F : X^4(X) \vdash F\big(X(\sigma)(\mathrm{id}_X)\big) : X \right]\!\!\right]. \end{aligned}$$

where

$$\begin{aligned} X(\sigma)(\mathrm{id}_X) &= \lambda G : X^2(X).\, \mathrm{id}_X\big(\sigma(G)\big) \\ &= \lambda G : X^2(X).\, \sigma(G) \\ &= \lambda G : X^2(X).\, G(\mathrm{id}_X) \\ &= \eta_{X(X)}(\mathrm{id}_X) \end{aligned}$$

thus the proof is complete. □

Proposition 1.22. *Let $I \in \mathrm{Ob}(\mathbb{C})$. The following assignment*

$$\begin{tikzcd} (-)^I : \mathbb{C} \arrow[r] & \mathbb{C} \\ A \arrow[r, mapsto] \arrow[d, "f"] & A^I \arrow[d] \\ B \arrow[r, mapsto] & B^I \end{tikzcd} \qquad f \mapsto f^I$$

defines a functor, where $f^I = \left[\!\!\left[F : A^I \vdash \lambda i : I.\, f\big(F(i)\big) : B^I \right]\!\!\right]$.

Proof. The action of $(-)^I$ on id_A is

$$(\mathrm{id}_A)^I = \left[\!\!\left[F : A^I \vdash \lambda i : I.\, F(i) : A^I \right]\!\!\right] = \mathrm{id}_{A^I}$$

so $(-)^I$ preserves identities. Let now $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{C}$. Then

$$\begin{aligned}
(g \circ f)^I &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, (g \circ f)\big(F(i)\big) : C^I \right]\!\!\right] \\
&= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, g\Big(f\big(F(i)\big)\Big) : C^I \right]\!\!\right] \\
&= g^I \circ f^I
\end{aligned}$$

□

For every $A \in \mathrm{Ob}(\mathbb{C})$ we define the morphism $\omega_A : T(A^I) \to (TA)^I$ as:

$$\omega_A = \left[\!\!\left[F : X^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\Big(\lambda f : A^I.\, g\big(f(i)\big)\Big)\Big) : \Big(X^2(A)\Big)^I \right]\!\!\right].$$

Lemma 1.23. *$\omega : T \circ (-)^I \to (-)^I \circ T : \mathbb{C} \to \mathbb{C}$ is a natural transformation.*

Proof. Let $\alpha : A \to B$ be a morphism in $\mathbb{C}$. We must show that the following is a commutative square:

$$\begin{tikzcd}
T(A^I) \arrow[r, "\omega_A"] \arrow[d, "T(\alpha^I)"'] & (TA)^I \arrow[d, "(T\alpha)^I"] \\
T(B^I) \arrow[r, "\omega_B"] & (TB)^I
\end{tikzcd}$$

First we compute the action of the two compositions of functors on α:

$$\begin{aligned}
(T\alpha)^I &= \left[\!\!\left[\beta : \Big(X^2(A)\Big)^I \vdash \lambda i : I.\, (T\alpha)\big(\beta(i)\big) \right]\!\!\right] \\
&= \left[\!\!\left[\beta : \Big(X^2(A)\Big)^I \vdash \lambda i : I.\, \Big(\lambda G : X(B).\, \beta(i)\big(X(\alpha)(G)\big)\Big) : \Big(X^2(B)\Big)^I \right]\!\!\right],
\end{aligned}$$

$$\begin{aligned}
T(\alpha^I) &= \left[\!\!\left[F : X^2(A^I) \vdash \lambda R : X(B^I).\, F\Big(X(\alpha^I)(R)\Big) : X^2(B^I) \right]\!\!\right] \\
&= \left[\!\!\left[F : X^2(A^I) \vdash \lambda R : X(B^I).\, F\Big(\lambda f : A^I.\, R\big(\alpha^I(f)\big)\Big) : X^2(B^I) \right]\!\!\right].
\end{aligned}$$

Now we focus on the previous square:

$$\begin{aligned}
(T\alpha)^I \circ \omega_A &= \left[\!\!\left[F : X^2(A^I) \vdash \lambda i : I.\, \Big(\lambda G : X(B).\, F\Big(\lambda f : A^I.\, \overbrace{X(\alpha)(G)\big(f(i)\big)}^{G\left(\alpha\left(f(i)\right)\right)}\Big)\Big) : \Big(X^2(B)\Big)^I \right]\!\!\right] \\
\omega_B \circ T(\alpha^I) &= \left[\!\!\left[F : X^2(A^I) \vdash \lambda i : I.\, \Big(\lambda G : X(B).\, F\Big(\lambda f : A^I.\, G\big(\underbrace{\alpha^I(f)(i)}_{\alpha\left(f(i)\right)}\big)\Big)\Big) : \Big(X^2(B)\Big)^I \right]\!\!\right]
\end{aligned}$$

and the proof is complete. □

Another property of T is that T-algebras are closed under the action of the functor $(-)^I$ in the following sense:

Theorem 1.24. *If* $\big(A, T(A) \xrightarrow{\alpha} A\big)$ *is a* T*-algebra, then* $\big(A^I, T(A^I) \xrightarrow{\beta} A^I\big)$*, where*

$$\beta = \alpha^I \circ \omega_A = \left[\!\!\left[F : X^2(A^I) \vdash \lambda i : I.\, \alpha\Big(\lambda g : X(A).\, F\Big(\lambda f : A^I.\, g\big(f(i)\big)\Big)\Big) : A^I \right]\!\!\right],$$

is a T*-algebra for every* $I \in \mathrm{Ob}(\mathbb{C})$*.*

Proof. We first show that the following triangle is commutative:

$$\begin{tikzcd}
A^I \arrow[r, "\eta_{A^I}"] \arrow[dr, "\mathrm{id}_{A^I}"'] & T(A^I) \arrow[d, "\beta"] \\
& A^I
\end{tikzcd}$$

We have

$$\beta \circ \eta_{A^I} = \left[\!\!\left[F : A^I \vdash \lambda i : I.\, \alpha\Big(\lambda f : X(A).\, f\big(F(i)\big)\Big) : A^I \right]\!\!\right],$$

but $\lambda f : X(A).\, f\big(F(i)\big) = \eta_A\big(F(i)\big)$ and since $\alpha \circ \eta_A = \mathrm{id}_A$ (remember that A is a T-algebra by hypothesis), we have

$$\begin{aligned}
\beta \circ \eta_{A^I} &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, \alpha\Big(\eta_A\big(F(I)\big)\Big) : A^I \right]\!\!\right] \\
&= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, F(i) : A^I \right]\!\!\right] \\
&= \mathrm{id}_{A^I}
\end{aligned}$$

In order to prove that the second axiom for T-algebras is satisfied, we have to prove that the external square of the following diagram commutes:

$$\begin{tikzcd}
T^2(A^I) \arrow[rrrr, "\mu_{A^I}"] \arrow[dr, "T\omega_A"] \arrow[ddd, "T\beta"'] & & & & T(A^I) \arrow[dl, "\omega_A"'] \arrow[ddd, "\beta"] \\
& T((TA)^I) \arrow[r, "\omega_{TA}"] \arrow[ddl, "T(\alpha^I)"'] & (T^2A)^I \arrow[r, "\mu_A^I"] \arrow[d, "(T\alpha)^I"] \arrow[u, phantom, "1"] & (TA)^I \arrow[ddr, "\alpha^I"'] & \\
& & (TA)^I \arrow[drr, "\alpha^I"] \arrow[l, phantom, "2"] \arrow[r, phantom, "3"] & & \\
T(A^I) \arrow[urr, "\omega_A"] \arrow[rrrr, "\beta"] & & & & A^I
\end{tikzcd}$$

We already know that diagrams 2 and 3 commute (the former by naturality of ω on α while the latter because $(-)^I$ is a functor and (A, α) is a T-algebra). So, if we show that diagram 1 is commutative, we are done.

By definition,

$$\mu_{A^I} = \left[\!\!\left[F : T^2(A^I) \vdash \lambda G : X(A^I).\, F(\eta_{X(A^I)}(G)) : X^2(A^I) \right]\!\!\right],$$

then

$$\begin{aligned}\omega_A \circ \mu_{A^I} &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, \mu_{A^I}(F)\big(\lambda f : A^I.\, g(f(i))\big)\Big) : \big(X^2(A)\big)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\Big(\eta_{X(A^I)}\big(\lambda f : A^I.\, g(f(i))\big)\Big)\Big) : \big(X^2(A)\big)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\Big(\lambda H : T(A^I).\, H\big(\lambda f : A^I.\, g(f(i))\big)\Big)\Big) : \big(X^2(A)\big)^I \right]\!\!\right].\end{aligned}$$

Again by definition,

$$\begin{aligned}T(\omega_A) &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda G : X((TA)^I).\, F\big(X(\omega_A)(G)\big) \right]\!\!\right] \\ \omega_{TA} &= \left[\!\!\left[H : T\big((TA)^I\big) \vdash \lambda i : I.\, \Big(\lambda g : X(TA).\, H\big(\lambda f : (TA)^I.\, g(f(i))\big)\Big) : (T^2A)^I \right]\!\!\right] \\ (\mu_A)^I &= \left[\!\!\left[F : (T^2A)^I \vdash \lambda i : I \mu_A\big(F(i)\big) : (TA)^I \right]\!\!\right],\end{aligned}$$

so

$$\begin{aligned}&\omega_{TA} \circ T\omega_A = \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(TA).\, F\Big(X(\omega_A)\big(\lambda f : (TA)^I.\, g(f(i))\big)\Big)\Big) : (T^2A)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(TA).\, F\big(\lambda H : T(A^I).\, g(\omega_A(H)(i))\big)\Big) : (T^2A)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(TA).\, F\Big(\lambda H : T(A^I).\, g\Big(\lambda h : X(A).\, H\big(\lambda f : A^I.\, h(f(i))\big)\Big)\Big)\Big) : (T^2A)^I. \right]\!\!\right]\end{aligned}$$

Hence

$$\begin{aligned}&(\mu_A)^I \circ \omega_{TA} \circ T\omega_A = \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \mu_A\Big(\lambda G : X(TA).\, \Big(\lambda H : T(A^I).\, G\Big(\lambda h : X(A).\, H\big(\lambda f : A^I.\, h(f(i))\big)\Big)\Big)\Big) : (TA)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\big(\lambda H : T(A^I).\, \eta_{X(A)}(g)\big(\omega_A(H)(i)\big)\big)\Big) : (TA)^I \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\Big(\lambda H : T(A^I).\, \eta_{X(A)}(g)\Big(\lambda h : X(A).\, H\big(\lambda f : A^I.\, h(f(i))\big)\Big)\Big)\Big) \right]\!\!\right] \\ &= \left[\!\!\left[F : T^2(A^I) \vdash \lambda i : I.\, \Big(\lambda g : X(A).\, F\Big(\lambda H : T(A^I).\, H\big(\lambda f : A^I.\, g(f(i))\big)\Big)\Big) : (TA)^I \right]\!\!\right] \\ &= \omega_A \circ \mu_{A^I}.\end{aligned}$$

□

Remark 1.25. Since $(X, \sigma\colon T(X) \to X)$ is a T-algebra, we have that for every $J \in \mathrm{Ob}(\mathbb{C})$ $(X(J), \beta)$ is a T-algebra, where

$$\begin{aligned}
\beta = \sigma^I \circ \omega_X &= \Big[\!\!\Big[F : T(X(I)) \vdash \lambda i : I.\, \sigma\Big(\lambda g : X(X).\, F\Big(\lambda f : X(I).\, g\big(f(i)\big)\Big)\Big)\Big]\!\!\Big] \\
&= \Big[\!\!\Big[F : T(X(I)) \vdash \lambda i : I.\, F\underbrace{\big(\lambda f : X(I).\, f(i)\big)}_{\eta_I(i)}\Big]\!\!\Big] \\
&= X(\eta_I).
\end{aligned}$$

Consider now $J = X(I)$. It follows that $(T(I), X(\eta_{X(I)})\colon T^2(I) \to T(I))$ is a T-algebra and $(T(I), \mu_I\colon T^2(I) \to T(I))$ is a T-algebra too because μ_I satisfies the two axioms of T-algebras thanks to the axioms of multiplication of the monad T. Indeed, $X(\eta_{X(I)}) = \mu_I$ as one can immediately see from the definition using the internal language of $\mathbb{C}$.

1.5 Algebraic Theories on X

Definition 1.26. The category $\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$ of *algebraic theory of* X is defined as follows:

- objects are those of $\mathbb{C}$;
- morphisms from I to J are morphisms in $\mathbb{C}$ from X^I to X^J, that is
$$\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}(I, J) = \mathbb{C}\Big(X^I, X^J\Big);$$
- composition is that of $\mathbb{C}$.

Remark 1.27. Consider the Kleisli category $\mathbb{C}_T$ associated to a monad (T, η, μ) over a category $\mathbb{C}$, whose objects are the objects of $\mathbb{C}$ and given $I, J \in \mathrm{Ob}(\mathbb{C})$,

$$\mathbb{C}_T(I, J) = \mathbb{C}(I, TJ).$$

The identity morphism for an object I is given by η_I and if $I \xrightarrow{f} J \xrightarrow{g} K$ are in $\mathbb{C}_T$, $g \circ_T f$ is defined as $\mu_K \circ Tg \circ f$:

$$\begin{tikzcd}
I \arrow[r, dotted, "g \circ_T f"] \arrow[d, "f"'] & TK \\
TJ \arrow[r, "Tg"] & T^2K \arrow[u, "\mu_K"']
\end{tikzcd}$$

In the case in which $\mathbb{C}$ is cartesian closed and $T = X^{(X^{(-)})}$, the categories $\mathbb{C}_T$ and $(\mathbb{A}\mathrm{lg}_X^{\mathbb{C}})^{\mathrm{op}}$ are isomorphic via the functors

$$\begin{tikzcd}
(\mathbb{A}\mathrm{lg}_X^{\mathbb{C}})^{\mathrm{op}} \arrow[r] & \mathbb{C}_T \\
A \arrow[r, mapsto] \arrow[d, "f"'] & A \\
B \arrow[r, mapsto] & B \arrow[u, "X(f) \circ \eta_B"']
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathbb{C}_T \arrow[r] & (\mathbb{A}\mathrm{lg}_X^{\mathbb{C}})^{\mathrm{op}} \\
A \arrow[r, mapsto] & A \arrow[d, dash] \\
g \arrow[r, mapsto] & X(g) \circ \eta_{X(A)} \arrow[d] \\
B \arrow[r, mapsto] \arrow[uu] & B
\end{tikzcd}
$$

We shall show that every T-algebra (A, α) gives rise to a functor $A^{(-)} \colon \mathbb{A}\mathrm{lg}_X^{\mathbb{C}} \to \mathbb{C}$ that will be crucial in the following. In order to define it, consider $o \colon X^I \to X$ a morphism in $\mathbb{C}$. We regard o as an I-ary operation on X and we want to show that o induces an I-ary operation on A, that is a morphism $o_A \colon A^I \to A$, that satisfies a particular property.

In order to define o_A it suffices to know o_{TA} (TA is a T-algebra by μ_A), defining o_A as

$$o_A = \; A^I \xrightarrow{\eta_A^I} (TA)^I \xrightarrow{o_{TA}} TA \xrightarrow{\alpha} A.$$

Similarly, we define o_{TA} using the multiplication of the T-algebra TA as follows:

$$o_{TA} = \; (TA)^I \xrightarrow{\eta_{TA}^I} (T^2A)^I \xrightarrow{o_{T^2A}} T^2A \xrightarrow{\mu_A} TA$$

where we define $o_{T^2A} = \left[\!\left[F : (T^2A)^I \vdash \lambda G : X^3(A).\, o\big(\lambda i : I.\, F(i)(G)\big) : T^2A \right]\!\right]$. It follows that

$$
\begin{aligned}
o_{TA} &= \mu_A \circ \left[\!\left[F : (TA)^I \vdash o_{T^2A}\Big(\lambda i : I.\, \eta_{TA}\big(F(i)\big)\Big) : T^2A \right]\!\right] \\
&= \mu_A \circ \left[\!\left[F : (TA)^I \vdash o_{T^2A}\bigg(\lambda i : I.\, \Big(\lambda G : X^3(A).\, G\big(F(i)\big)\Big)\bigg) : T^2A \right]\!\right] \\
&= \mu_A \circ \left[\!\left[F : (TA)^I \vdash \lambda G : X^3(A).\, o\Big(\lambda i : I.\, G\big(F(i)\big)\Big) : T^2A \right]\!\right] \\
&= \left[\!\left[F : (TA)^I \vdash \mu_A\bigg(\lambda G : X^3(A).\, o\Big(\lambda i : I.\, G\big(F(i)\big)\Big)\bigg) : TA \right]\!\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
o_A &= \alpha \circ \left[\!\left[F : A^I \vdash \mu_A\bigg(\lambda G : X^3(A).\, o\Big(\lambda i : I.\, G\big(\eta_A^I(F)(i)\big)\Big)\bigg) : TA \right]\!\right] \\
&= \alpha \circ \left[\!\left[F : A^I \vdash \mu_A\bigg(\lambda G : X^3(A).\, o\Big(\lambda i : I.\, G\big(\eta_A(F(i))\big)\Big)\bigg) : TA \right]\!\right] \\
&= \alpha \circ \left[\!\left[F : A^I \vdash \lambda G : X(A).\, o\Big(\lambda i : I.\, \eta_A\big(F(i)\big)(G)\Big) : TA \right]\!\right] \\
&= \left[\!\left[F : A^I \vdash \alpha\bigg(\lambda G : X(A).\, o\Big(\lambda i : I.\, G\big(F(i)\big)\Big)\bigg) : A \right]\!\right].
\end{aligned}
$$

Proposition 1.28. *Let $(A, \alpha \colon TA \to A)$ be a T-algebra, o_A, o_{TA} as previously defined. Then the following square*

$$
\begin{tikzcd}
(TA)^I \arrow[r, "o_{TA}"] \arrow[d, "\alpha^I"'] & TA \arrow[d, "\alpha"] \\
A^I \arrow[r, "o_A"] & A
\end{tikzcd}
$$

commutes.

Proof. We have

$$o_A \circ \alpha^I = \left[\!\!\left[F : (TA)^I \vdash \alpha\Big(\lambda G : X(A).\, o\Big(\lambda i : I.\, G\big(\alpha(F(i))\big)\Big)\Big) : A \right]\!\!\right]$$

and we are to show that given $F : (TA)^I$

$$\underbrace{\alpha\Big(\mu_A\Big(\lambda G : X^3(A).\, o\big(\lambda i : I.\, G(F(i))\big)\Big)\Big)}_{(\alpha\circ o_{TA})(F)} = \underbrace{\alpha\Big(\lambda G : X(A).\, o\Big(\lambda i : I.\, G\big(\alpha(F(i))\big)\Big)\Big)}_{(o_A\circ\alpha^I)(F)}.$$

Since (A, α) is a T-algebra, we have that

$$\alpha\Big(\mu_A\Big(\lambda G : X^3(A).\, o\big(\lambda i : I.\, G(F(i))\big)\Big)\Big) = \alpha\Big(T\alpha\Big(\lambda G : X^3(A).\, o\big(\lambda i : I.\, G(F(i))\big)\Big)\Big),$$

but

$$T\alpha\Big(\lambda G : X^3(A).\, o\big(\lambda i : I.\, G(F(i))\big)\Big) = \lambda G : X(A).\, o\Big(\lambda i : I.\, G\big(\alpha(F(i))\big)\Big)$$

and this completes the proof. $\square$

Now we want to generalize this argument saying that given A a T-algebra, every J-family of I-ary operations on X, that is a morphism $f\colon X^I \to X^J$, induces a J-family of I-ary operations on A, that is a morphism $f_A\colon A^I \to A^J$.
Following the one-dimensional case, we define f_A using only f_{T^2A} as follows:

$$\begin{tikzcd}
A^I \arrow[r, "\eta_A^I"] & (TA)^I \arrow[r, "f_{TA}"] \arrow[d, "\eta_{TA}^I"'] \arrow[dr, phantom, "/\!/"] & (TA)^J \arrow[r, "\alpha^J"] & A^J \\
& (T^2A)^I \arrow[r, "f_{T^2A}"] & (T^2A)^J \arrow[u, "\mu_A^J"'] &
\end{tikzcd}$$

If $\mathbb{C}$ is $\mathbb{S}\mathrm{et}$, then, for every $j \in J$, $f\colon X^I \to X^J$ would determine a function $f_j\colon X^I \to X$. In this situation we can consider $f_{j,A}\colon A^I \to A$ for every $j \in J$ and define for every $F \in A^I$ $f_A(F) = (J \ni j \mapsto f_{j,A}(F)) \in A^J$.

In general that construction would make no sense, but we can do a similar thing noting that our morphism f is equal to $\lambda(\overline{f})$ where

$$\overline{f}\colon X^I \times J \to X = \left[\!\!\left[F : X^I, j : J \vdash f(F)(j) : X \right]\!\!\right].$$

If for $o : X^I \to X$ we defined $o_{T^2A} = \left[\!\!\left[F : (T^2A)^I \vdash \lambda G : X^3(A).\, o\big(\lambda i : I.\, F(i)(G)\big) : T^2A \right]\!\!\right]$, we put

$$\begin{aligned}
f_{T^2A} &= \left[\!\!\left[F : (T^2A)^I \vdash \lambda j : J.\, \Big(\lambda G : X^3(A).\, \overline{f}\big(\lambda i : I.\, F(i)(G), j\big)\Big) : (T^2A)^J \right]\!\!\right] \\
&= \left[\!\!\left[F : (T^2A)^I \vdash \lambda j : J.\, \Big(\lambda G : X^3(A).\, \Big[f\big(\lambda i : I.\, F(i)(G)\big)\Big](j)\Big) : (T^2A)^J \right]\!\!\right]
\end{aligned}$$

With the same calculations as in the one-dimensional case, one obtains

$$f_{TA} = \left[\!\!\left[F : (TA)^I \vdash \lambda j : J.\, \mu_A\Big(\lambda G : X^3(A).\, \big[f\big(\lambda i : I.\, G\big(F(i)\big)\big)\big](j)\Big) : (T^2A)^J \right]\!\!\right]$$
$$f_A = \left[\!\!\left[F : A^I \vdash \lambda j : J.\, \alpha\Big(\lambda G : X(A).\, \big[f\big(\lambda i : I.\, G\big(F(i)\big)\big)\big](j)\Big) : A^J \right]\!\!\right].$$

Lemma 1.29. *With the above notations, the following square*

$$\begin{tikzcd} (TA)^I \arrow[r, "f_{TA}"] \arrow[d, "\alpha^I"'] & (TA)^J \arrow[d, "\alpha^J"] \\ A^I \arrow[r, "f_A"] & A^J \end{tikzcd}$$

commutes.

Proof. Similar to that of the Proposition 1.28. □

Now we can show that this construction indeed defines a functor.

Theorem 1.30. *Let $(A, \alpha\colon TA \to A)$ be a T-algebra. Then the following assignment*

$$\begin{tikzcd} A^{(-)} : \mathbb{A}\mathrm{lg}^{\mathbb{C}}_X \arrow[r] & \mathbb{C} \\ I \arrow[r, mapsto] \arrow[d, "f"] & A^I \arrow[d] \\ J \arrow[r, mapsto] & A^J \end{tikzcd} \qquad f \longmapsto f_A$$

is a functor.

Proof. $A^{(-)}$ preserves the identities, since

$$\begin{aligned} A^{(-)}(\mathrm{id}_I) &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, \alpha\Big(\lambda G : X(A).\, \big[\lambda j : I.\, G\big(F(j)\big)\big](i)\Big) : A^I \right]\!\!\right] \\ &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, \alpha\Big(\lambda G : X(A).\, G\big(F(i)\big)\Big) : A^I \right]\!\!\right] \\ &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, \alpha\Big(\eta_A\big(F(i)\big)\Big) : A^I \right]\!\!\right] \\ &= \left[\!\!\left[F : A^I \vdash \lambda i : I.\, F(i) : A^I \right]\!\!\right] \\ &= \mathrm{id}_{A^I}\,. \end{aligned}$$

Furthermore, $A^{(-)}$ preserves compositions: let $X^I \xrightarrow{f} X^J \xrightarrow{g} X^K$ two morphisms of $\mathbb{C}$ that give rise to $A^I \xrightarrow{f_A} A^J \xrightarrow{g_A} A^K$. We have that

$$\begin{aligned} f_A &= \alpha^J \circ f_{TA} \circ \eta_A^I \\ g_A &= \alpha^K \circ g_{TA} \circ \eta_A^J \\ (gf)_A &= \alpha^K \circ (gf)_{TA} \circ \eta_A^I \end{aligned}$$

and for Lemma 1.29 the following diagram commutes

$$\begin{tikzcd}
(TA)^I \arrow[r, "f_{TA}"] \arrow[d, "\alpha^I"'] \arrow[dr, phantom, "/\!/"] & (TA)^J \arrow[r, "g_{TA}"] \arrow[d, "\alpha^J"] \arrow[dr, phantom, "/\!/"] & (TA)^K \arrow[d, "\alpha^K"] \\
A^I \arrow[r, "f_A"] & A^J \arrow[r, "g_A"] & A^K
\end{tikzcd}$$

whence

$$\begin{aligned}
g_A \circ f_A &= \underbrace{\alpha^K \circ g_{TA}}_{g_A \circ \alpha^J} \circ \eta_A^J \circ \underbrace{\alpha^J \circ f_{TA}}_{f_A \circ \alpha^I} \circ \eta_A^I \\
&= g_A \circ \underbrace{\alpha^J \circ \eta_A^J}_{\mathrm{id}_{A^J}} \circ f_A \circ \alpha^I \circ \eta_A^I \\
&= \alpha^K \circ g_{TA} \circ f_{TA} \circ \eta_A^I.
\end{aligned}$$

We prove then that $g_{TA} \circ f_{TA} = (gf)_{TA}$.

$$\begin{aligned}
g_{TA} \circ f_{TA} &= \mu_A^K \circ g_{T^2A} \circ \eta_{TA}^J \circ \mu_A^J \circ f_{T^2A} \circ \eta_{TA}^I \\
&= \mu_A^K \circ g_{T^2A} \circ f_{T^2A} \circ \eta_{TA}^I,
\end{aligned}$$

while

$$(gf)_{TA} = \mu_A^K \circ (gf)_{T^2A} \circ \eta_{TA}^I.$$

Now,

$$\begin{aligned}
&g_{T^2A} \circ f_{T^2A} = \\
&= \left[\!\left[F : (T^2A)^I \vdash g_{T^2A}\big(f_{T^2A}(F)\big) : (T^2A)^K \right]\!\right] \\
&= \left[\!\left[F : (T^2A)^I \vdash \lambda k : K. \left(\lambda G : X^3(A). \left[g\big(\lambda j : J. f_{T^2A}(F)(j)(G)\big)\right](k)\right) : (T^2A)^K \right]\!\right] \\
&= \left[\!\left[F : (T^2A)^I \vdash \lambda k : K. \left(\lambda G : X^3(A). \left[g\left(\lambda j : J. \left(f\big(\lambda i : I. F(i)(G)\big)(j)\right)\right)\right](k)\right) : (T^2A)^K \right]\!\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\mu_A^K \circ g_{T^2A} \circ f_{T^2A} = \\
&\left[\!\left[F : (T^2A)^I \vdash \lambda k : K. \mu_A \left(\lambda G : X^3(A) \left[g\left(\lambda j : J. \left(f\big(\lambda i : I. F(i)(G)\big)\right)(j)\right)\right](k)\right) : (TA)^K \right]\!\right].
\end{aligned}$$

Since $(\eta_{TA})^I(F) = \lambda i : I. \eta_{TA}(F(i)) = \lambda i : I. \left(\lambda G : X^3(A). G\big(F(i)\big)\right)$, we have

$$\begin{aligned}
&g_{TA} \circ f_{TA} = \\
&= \left[\!\left[F : (TA)^I \vdash \lambda k : K. \mu_A \left(\lambda G : X^3(A) \left[g\left(\lambda j : J. \left(f\big(\lambda i : I. G(F(i))\big)\right)(j)\right)\right](k)\right) : (TA)^K \right]\!\right] \\
&= \left[\!\left[F : (TA)^I \vdash \lambda k : K. \left(\lambda G : X(A). \left[g \underbrace{\left(\lambda j : J. \left[f\big(\lambda i : I. F(i)(G)\big)\right](j)\right)}_{f\left(\lambda i : I. F(i)(G)\right)}\right](k)\right) : (TA)^K \right]\!\right]
\end{aligned}$$

while

$$(gf)_{T^2A} \circ (\eta_{TA})^I = \left[\!\!\left[F : (TA)^I \vdash \lambda k : K. \left(\lambda G : X^3(A). \left[gf\Big(\lambda i : I. G\big(F(i)\big)\Big)\right](k) \right) : (T^2A)^K \right]\!\!\right]$$

hence finally

$$\begin{aligned}(gf)_{TA} &= \left[\!\!\left[F : (TA)^I \vdash \lambda k : K. \mu_A \left(\lambda G : X^3(A). \left[gf\Big(\lambda i : I. G\big(F(i)\big)\Big)\right](k) \right) : (TA)^K \right]\!\!\right] \\ &= \left[\!\!\left[F : (TA)^K \vdash \lambda k : K. \Big(\lambda G : X(A). \big[\lambda i : I. F(i)(G)\big](k) \Big) : (TA)^K \right]\!\!\right]. \end{aligned}$$ □

We want now to prove that the functor $A^{(-)} \colon \mathbb{Alg}_X^{\mathbb{C}} \to \mathbb{C}$ preserves a particular kind of products. First we will show a general property of powers.

Lemma 1.31. *Let $\mathbb{C}$ be a cartesian closed category, $(A_i)_{i\in I}$ a family of objects of $\mathbb{C}$. Suppose the coproduct $(\coprod_i A_i,\, u_i \colon A_i \to \coprod_i A_i)$ exists in $\mathbb{C}$. Then $(X(\coprod_i A_i),\, X(u_i) \colon X(\coprod_i A_i) \to X(A_i))$ is a product of the family $(X(A_i))$ in $\mathbb{C}$ for every $X \in \mathrm{Ob}(\mathbb{C})$.*

Proof. Let B be an object of $\mathbb{C}$ and $(f_i)_i$ a family of morphisms from B to $X(A_i)$.

$$\begin{tikzcd} X((\coprod A_i)) \arrow[d, "X(u_i)"'] & B \arrow[dl, "f_i"] \\ X(A_i) & \end{tikzcd}$$

We have that $f_i = \lambda(\overline{f_i})$ where $\overline{f_i} = \left[\!\!\left[F : X(B),\, a_i : A_i \vdash f_i(F)(a_i) : X \right]\!\!\right]$.

In turn, $\overline{f_i}$ induces $\widehat{f_i} \colon A_i \to X(B)$ where

$$\begin{aligned}\widehat{f_i} &= \left[\!\!\left[a_i : A_i \vdash \lambda b : B. \overline{f_i}(b, a_i) : X \right]\!\!\right] \\ &= \left[\!\!\left[a_i : A_i \vdash \lambda b : B. f_i(b)(a_i) : X \right]\!\!\right]. \end{aligned}$$

But now, for the universal property of $\coprod A_i$, we have that there exists a unique $\hat{f} \colon \coprod A_i \to X(B)$ such that $\widehat{f_i} = \hat{f} \circ u_i$.

$$\begin{tikzcd} \coprod A_i \arrow[r, dotted, "\hat{f}"] & X(B) \\ A_i \arrow[u, "u_i"] \arrow[ur, "\widehat{f_i}"'] \arrow[u, phantom, shift left=6, "="] & \end{tikzcd}$$

Analogously, we have that $\hat{f} = \lambda(\bar{f})$ where $\bar{f} = \left[\!\!\left[F : \coprod A_i,\, b : B \vdash \hat{f}(F)(b) : X \right]\!\!\right]$ and $\hat{f}$ induces $f = \left[\!\!\left[b : B \vdash \lambda F : \coprod A_i. \hat{f}(F)(b) : X(\coprod A_i) \right]\!\!\right]$. We prove now that this f is the solution to the universal problem.

The following triangle indeed commutes

$$\begin{tikzcd} X(\coprod A_i) \arrow[d, "X(u_i)"'] & B \arrow[l, "f"'] \arrow[dl, "f_i"] \\ X(A_i) & \end{tikzcd}$$

since

$$\begin{aligned} X(u_i) \circ f &= \left[\!\left[b : B \vdash \lambda a_i : A_i.\, \hat{f}\big(u_i(a)\big)(b) : X \right]\!\right] \\ &= \left[\!\left[b : B \vdash \lambda a_i : A_i.\, \widehat{f_i}(a)(b) : X \right]\!\right] \\ &= \left[\!\left[b : B \vdash \lambda a_i : A_i.\, f_i(b)(a_i) : X \right]\!\right] \\ &= \lambda(\overline{f_i}) = f_i. \end{aligned}$$

We are to show that f is the unique solution to the universal problem. Consider then $g\colon B \to X(\coprod A_i)$ such that $X(u_i) \circ g = f_i$. Again, we have $g = \lambda(\bar{g})$ where $\bar{g} = \left[\!\left[b : B,\, F : \coprod A_i \vdash g(b)(F) : X \right]\!\right]$ and $\hat{g} = \left[\!\left[F : \coprod A_i \vdash \lambda b : B.\, g(b)(F) : X(B) \right]\!\right]$.

We observe that

$$\hat{g} \circ u_i = \left[\!\left[a_i : A_i \vdash \hat{g}\big(u_i(a_i)\big) : X(B) \right]\!\right] = \left[\!\left[a_i : A_i \vdash \lambda b : B.\, g(b)\big(u_i(a_i)\big) : X(B) \right]\!\right]$$

while $f_i = X(u_i) \circ g$, so

$$\widehat{f_i} = \left[\!\left[a_i : A_i \vdash \lambda b : B.\, (X(u_i \circ g)(b)(a_i) : X(B) \right]\!\right] = \left[\!\left[a_i : A_i \vdash \lambda b : B.\, g(b)\big(u_i(a_i)\big) : X(B) \right]\!\right],$$

then $\hat{g} \circ u_i = \widehat{f_i}$, whence $\hat{g} = \hat{f}$. This means that

$$\hat{f} = \left[\!\left[F : \coprod A_i \vdash \lambda b : B.\, g(b)(F) : X(B) \right]\!\right] = \hat{g}$$

and follows that

$$f = \left[\!\left[b : B \vdash \lambda F : \coprod A_i.\, g(b)(F) : X(\coprod A_i) \right]\!\right] = \left[\!\left[b : B \vdash g(b) : X(\coprod A_i) \right]\!\right] = g. \qquad \square$$

Remark 1.32. In particular, $(\coprod_{i \in I} A_i, X(u_i))$ is a product in $\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$.

Theorem 1.33. *Let (A, α) be a X^2-algebra. Then the functor $A^{(-)}\colon \mathbb{A}\mathrm{lg}_X^{\mathbb{C}} \to \mathbb{C}$ preserves all products in $\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$ of the form $\coprod_{i \in I} A_i$ for some I.*

Proof. Consider a family $(A_i)_{i \in I}$ of objects of $\mathbb{C}$ such that there exists its coproduct $(\coprod_i A_i,\, u_i\colon A_i \to \coprod_i A_i)$ in $\mathbb{C}$. By the previous lemma, considering as X the object A, we know that $(A(\coprod A_i),\, A(u_i))$ is a product in $\mathbb{C}$: we just need to show then that $(X(u_i))_A = A(u_i)$.

$$\begin{aligned}
(X(u_i))_A &= \left[\!\!\left[F : A(\textstyle\coprod A_i) \vdash \lambda a_i : A_i.\, \alpha\left(\lambda G : X(A).\, \left[X(u_i)\Big(\lambda b : \textstyle\coprod A_i.\, G\big(F(b)\big)\Big)\right](a_i)\right) : A(A_i) \right]\!\!\right] \\
&= \left[\!\!\left[F : A(\textstyle\coprod A_i) \vdash \lambda a_i : A_i.\, \alpha\Bigg(\underbrace{\lambda G : X(A).\, G\Big(F\big(u_i(a_i)\big)\Big)}_{\eta_A\left(F\left(u_i(a_i)\right)\right)}\Bigg) : A(A_i) \right]\!\!\right] \\
&= \left[\!\!\left[F : A(\textstyle\coprod A_i) \vdash \lambda a_i : A_i.\, F\big(u_i(a_i)\big) : A(A_i) \right]\!\!\right] \\
&= A(u_i)
\end{aligned}$$

$\square$

Chapter 2

Equilogical Spaces

In this chapter we present the category $\mathbb{E}\text{qu}$ of Equilogical Spaces, introduced by Dana S. Scott in [Sco76]. It is easy to show that $\mathbb{E}\text{qu}$ embeds the category $\mathbb{T}\text{op}_0$ of T_0 topological spaces as a full and faithful subcategory, but what is not so obvious it that $\mathbb{E}\text{qu}$ is indeed *cartesian closed*, as $\mathbb{T}\text{op}_0$ is not. In order to prove cartesian closedness for $\mathbb{E}\text{qu}$ we use results in Domain Theory (the so-called *E-theorems*) about algebraic lattices. For that reason, we start discussing those, following [Sco96] and [TS01].

2.1 Complete and algebraic lattices

We denote a topological space as a pair $\mathcal{T} = (T, \tau)$ where T is the set of points of the space, and τ is the set of open sets of $\mathcal{T}$. We often write $|\mathcal{T}|$ for the set T.

Definition 2.1. Let $\mathcal{T} = (T, \tau)$ be a topological space, $x \in T$. The *neighbourhood filter* of x is defined as

$$\tau(x) := \{ U \in \tau \mid x \in U \}.$$

Definition 2.2. A topological space $\mathcal{T} = (T, \tau)$ is T_0 if for every pair of distinct points there is an open set that contains one but not the other. In other words, for all $x, y \in T$, the condition $\tau(x) = \tau(y)$ ensures that $x = y$. The category of all such spaces and continuous mappings between them is denoted by $\mathbb{T}\text{op}_0$.

Definition 2.3. The *specialization ordering* of a topological space $\mathcal{T}$ is defined by

$$x \leq_{\mathcal{T}} y \iff \tau(x) \subseteq \tau(y)$$

for all $x, y \in |\mathcal{T}|$.

Remark 2.4. The T_0-condition on a space $\mathcal{T}$ is equivalent to the fact that the specialization ordering is a partial ordering of the set $|\mathcal{T}|$ of points.

In a partial order $(P, \leq)$, a subset $S \subseteq P$ is *upward closed* if for any $x \in S$ and $x \leq y$ it is also $y \in S$. It is easy to show that the upward closed subsets of a partial order $(P, \leq)$ form a topology on P; furthermore the specialization ordering induced by such a topology is precisely the same ordering as the one we started out with. As we shall see, there is a similar interplay between complete lattices and T_0-spaces.

Definition 2.5. A *lattice* is a partial ordered set $\mathcal{L} = (L, \leq)$ such that every pair of elements x, y in L has a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$. A lattice $\mathcal{L}$ is called *complete* if every subset S of $|\mathcal{L}|$ has a least upper bound $\bigvee S$ (equivalently every subset has a greatest lower bound $\bigwedge S$).

Definition 2.6. Let $\mathcal{L}$ be a complete lattice. The *Scott-topology* on $\mathcal{L}$ is defined as the collection of all upward closed subsets $U \subseteq |\mathcal{L}|$ such that, whenever $S \subseteq |\mathcal{L}|$ and $\bigvee S \in U$, there is a finite subset $S_0 \subseteq S$ such that $\bigvee S_0 \in U$. The collection of all such subsets is denoted by $\sigma_{\mathcal{L}}$.

When S is a finite subset of a set L, we write $S \subseteq_{\mathit{fin}} L$. The following theorem gives the connection between complete lattices and T_0-spaces we mentioned before.

Theorem 2.7. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ be a complete lattice. Then $(L, \sigma_{\mathcal{L}})$ is a T_0-space whose specialization ordering is exactly $\leq_{\mathcal{L}}$.*

Proof. We first prove that $\sigma_{\mathcal{L}}$ is a topology. Obviously, both L and $\varnothing$ satisfy the definition. Let $\{ U_i \}_{i \in I}$ be a family of open sets, and $U = \bigcup_{i \in I} U_i$: U is still upward closed and if $S \subseteq L$ is such that $\bigvee S \in U$ then there exists an $i \in I$ such that $\bigvee S \in U_i$, but U_i is open and then there is a $S_0 \subseteq_{\mathit{fin}} S$ such that $\bigvee S_0 \in U_i \subseteq U$, so U is open as well. Next, let $U, V \in \Sigma_{\mathcal{L}}$ and $S \subseteq L$ such that $\bigvee S \in U \cap V$. Then there exist $S_0, S_1 \subseteq_{\mathit{fin}} S$ such that $\bigvee S_0 \in U$ and $\bigvee S_1 \in V$. It follows that $S_0 \cup S_1 \subseteq_{\mathit{fin}} S$ and $\bigvee (S_0 \cup S_1)$ is greater than both $\bigvee S_0$ and $\bigvee S_1$: since S_0 and S_1 are upward closed, $\bigvee (S_0 \cup S_1) \in U \cap V$, as desired. Moreover, $U \cap V$ is upward closed, hence is open in $\sigma_{\mathcal{L}}$.

To check the T_0-property, we observe that if $x \in L$, then

$$\bar{x} = \{ y \in L \mid y \not\leq x \}$$

is open: it is upward closed, since if $y \not\leq x$ and $y \leq z$, then $z \not\leq x$ because otherwise we should have that $y \leq x$, which is a contradiction. Moreover, assume that $\bigvee S \not\leq x$ for a subset $S \subseteq L$. Then there exists an $s_0 \in S$ such that $s_0 \not\leq x$, since if this did not happen then x would be an upper bound for S so that $\bigvee S \leq x$. This means that $\{s_0\} \subseteq_{\mathit{fin}} \bar{x}$, so $\bar{x}$ is indeed open. Given now two distinct points $x, y \in L$, there are two cases: either x and y are incomparable, or one is less than the other. If they are incomparable, then we have that $x \notin \bar{x}$ but $y \in \bar{x}$. Otherwise, if $x \leq y$ (the case $y \leq x$ is similar), then we must have that $y \not\leq x$ since $\leq$ is antisymmetric and x and y are distinct. This means that $y \in \bar{x}$, whereas $x \notin \bar{x}$.

Finally, we show that the specialization ordering and $\leq_{\mathcal{L}}$ coincide. If $x \leq_{\mathcal{L}} y$ and $U \in \sigma_{\mathcal{L}}(x)$ (that is U is an open neighbourhood of x in the Scott-topology), then $y \in U$ because U is upward closed, so $\sigma_{\mathcal{L}}(x) \subseteq \sigma_{\mathcal{L}}(y)$. Viceversa, assume that $\sigma_{\mathcal{L}}(x) \subseteq \sigma_{\mathcal{L}}(y)$. If $x \not\leq y$, then $x \in \bar{y}$: this means that $\bar{y}$ is an open neighbourhood of x and $y \in \bar{y}$, which is a contradiction. $\square$

The theorem states that every complete lattice can be canonically equipped with a topology that makes it a T_0-space. We have not considered yet what the closed sets in $(L, \sigma_{\mathcal{L}})$ are. An upcoming characterization deals with this, and it is useful to introduce some further terminology.

Definition 2.8. A *directed* subset of a partially ordered set $(P, \leq)$ is a subset $D \subseteq P$ such that every finite subset of D has an upper bound in D. We say that a subset $A \subseteq P$ is *closed under directed sups* if for any directed set $D \subseteq A$, we have that $\bigvee D$ exists and $\bigvee D \in A$. If D is a directed set, we shall denote its supremum as $\bigvee^{\uparrow} D$.

Theorem 2.9. *Let $\mathcal{L} = (L, \leq)$ be a complete lattice. A subset $A \subseteq L$ is closed in the Scott-topology if and only if it is downward closed and closed under directed sups.*

Proof. Let A be a closed set in the Scott-topology (so that $\complement A$ is open). If $x \in A$ and $z \leq x$, then we must have that $z \in A$, for if $z \in \complement A$, then the upward closedness of $\complement A$ would imply $x \in \complement A$, which is a contradiction. Therefore, A is downward closed. Next, assume that $D \subseteq A$ is a directed set, and consider the element $\bigvee^{\uparrow} D$. If $\bigvee^{\uparrow} D \in \complement A$, then there would exist $D_0 \subseteq_{fin} D$ with $\bigvee D_0 \in \complement A$ because $\complement A$ is open. But since D is directed, we have that $\bigvee^{\uparrow} D_0 \in D \subseteq A$, which is a contradiction. This means that $\bigvee^{\uparrow} D \in A$, so A is closed under directed sups.

Conversely, assume that A is downwards closed and closed under directed sups. We are to show that $\complement A$ is open. It follows by an easy argument dual to the one above that $\complement A$ is upward closed. Furthermore, let $S \subseteq L$ be such that $\bigvee S \in \complement A$, and assume that for all $S_0 \subseteq_{fin} S$ we have that $\bigvee S_0 \in A$. Consider the set

$$D = \left\{ \bigvee S_0 \mid S_0 \subseteq_{fin} S \right\}.$$

We claim that D is directed. If $s_1, \ldots, s_n \in D$, then there are finite sets $S_1, \ldots, S_n$ such that $s_i = \bigvee S_i$ for each $i \in \{ 1, \ldots, n \}$. Then the set $S_0 = S_1 \cup \cdots \cup S_n$ is finite and $s_0 = \bigvee S_0 \in D$ is an upper bound for $s_1, \ldots, s_n$. By assumption, we have that $D \subseteq A$, and since A is closed under directed sups, we get that $\bigvee^{\uparrow} D \in A$. But

$$\bigvee^{\uparrow} D = \bigvee^{\uparrow} \left\{ \bigvee S_0 \mid S_0 \subseteq_{fin} S \right\} = \bigvee S \notin A$$

which is a contradiction. □

We note now that continuity in the Scott-topology implies monotonicity.

Corollary 2.10. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices, and assume that $f \colon \mathcal{L} \to \mathcal{M}$ is continuous. Then f is monotone.*

Proof. Let $x \leq_{\mathcal{L}} y$: in order to prove that $f(x) \leq_{\mathcal{M}} f(y)$ it suffices to show that $\sigma_{\mathcal{M}}(f(x)) \subseteq \sigma_{\mathcal{M}}(f(y))$, that is all open sets that contain $f(x)$ also contain $f(y)$. Let then $U \subseteq M$ be an open set with $f(x) \in U$. This means that $x \in f^{-1}(U)$ that is open in L because f is continuous; in particular, it is upward closed: it follows that $y \in f^{-1}(U)$, that is $f(y) \in U$. □

Definition 2.11. The category $\mathbb{CL}$att is defined as follows:

- objects are complete lattices;
- morphisms are function between lattices that are continuous with respect to the Scott-topology;
- composition is functional composition.

The final goal of this section is to prove that $\mathbb{CL}$att is cartesian closed. First we need a characterization of continuous functions between complete lattices.

Theorem 2.12. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{M})$ be complete lattices, and let $f \colon L \to M$ be a function. Then $f \colon (L, \sigma_{\mathcal{L}}) \to (M, \sigma_{\mathcal{M}})$ is continuous if and only if it preserves directed sups, that is for all directed subsets $D \subseteq L$*

$$f(\bigvee^{\uparrow} D) = \bigvee^{\uparrow} f(D).$$

Proof. Suppose that $f(\bigvee^{\uparrow} D) = \bigvee^{\uparrow} f(D)$ for all directed substs $D \subseteq L$. We first show that f is monotone when considered as a function $f\colon \mathcal{L} \to \mathcal{M}$. Assume therefore that $x \leq_{\mathcal{L}} y$: this makes $\{x, y\}$ a directed subset of L. By assumption, we therefore have that

$$f(y) = f\left(\bigvee\nolimits^{\uparrow}\{x, y\}\right) = \bigvee\nolimits^{\uparrow} f(\{x, y\}) = \bigvee\nolimits^{\uparrow}\{f(x), f(y)\} = f(x) \vee_{\mathcal{M}} f(y).$$

This means that $f(x) \leq_{\mathcal{M}} f(y)$, so f is monotone. To show that it is continuous, we show that the inverse image of any closed subset in M is closed in L. So let $A \subseteq M$ be a closed subset of M, and consider $f^{-1}(A)$. We need to show that it is downward closed and closed under directed sups. Let $x \in f^{-1}(A)$ and $z \leq_{\mathcal{L}} x$. This means that $f(x) \in A$, and since f is monotone, we get that $f(z) \leq_{\mathcal{M}} f(x)$, which, since A is downward closed, in turn means that $f(z) \in A$, that is $z \in f^{-1}(A)$, so $f^{-1}(A)$ is downward closed. For the other condition, assume that $D \subseteq f^{-1}(A)$ is a directed subset of L (this means that $f(D) \subseteq A$). We are to show that $\bigvee^{\uparrow} D \in f^{-1}(A)$, or equivalently, that $f\left(\bigvee^{\uparrow} D\right) \in A$. By our assumption about f, we have that

$$f\left(\bigvee\nolimits^{\uparrow} D\right) = \bigvee\nolimits^{\uparrow} f(D).$$

We have that $f(D) \subseteq M$ is directed since f is monotone. As A is closed under directed sups, we get that $f\left(\bigvee^{\uparrow} D\right) \in A$, as requested.

Viceversa, assume that $f\colon (L, \sigma_{\mathcal{L}}) \to (M, \sigma_{\mathcal{M}})$ is continuous, and let $D \subseteq L$ be a directed set. Since f is monotone, we have that $f\left(\bigvee^{\uparrow} D\right) \geq_{\mathcal{M}} \bigvee^{\uparrow} f(D)$. We are to show that $f\left(\bigvee^{\uparrow} D\right)$ is the least among the upper bounds of $f(D)$. So let $y \geq_{\mathcal{M}} f(x)$ for all $x \in D$. Showing that $y \geq_{\mathcal{M}} f(\bigvee^{\uparrow} D)$ amounts to show that $\sigma_{\mathcal{M}}(f(\bigvee^{\uparrow} D)) \subseteq \sigma_{\mathcal{M}}(y)$: let $A \in \sigma_{\mathcal{M}}$ be such that $f(\bigvee^{\uparrow} D) \in A$. Then $\bigvee^{\uparrow} D \in f^{-1}(A)$ that is open because f is continuous. It follows that there exists $D_0 \subseteq_{fin} D$ such that $\bigvee^{\uparrow} D_0 \in f^{-1}(A)$. Now, D is directed, so there exists $d \in D$ such that $d \geq_{\mathcal{L}} \bigvee^{\uparrow} D_0$ and since $f^{-1}(A)$ is upward closed, we have that $d \in f^{-1}(A)$. This means that $f(d) \in A$ and $y \geq_{\mathcal{M}} f(d)$ (because $d \in D$): by the upward closedness of A, we get $y \in A$, and this completes the proof. $\square$

We can now start proving that $\mathbb{CL}\mathrm{att}$ is cartesian closed. To do so, we consider the set of all continuous functions between two complete lattices, and show that it is a complete lattice.

Definition 2.13. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices, and consider the set $(\mathcal{L} \to \mathcal{M})$ of continuous functions from $\mathcal{L}$ to $\mathcal{M}$. We define the *point-wise ordering* on this set by

$$f \leq_{(\mathcal{L}\to\mathcal{M})} g :\iff \forall x \in L \quad f(x) \leq_{\mathcal{M}} g(x).$$

Theorem 2.14. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be complete lattices. Then the function space $(\mathcal{L} \to \mathcal{M})$ is a complete lattice under $\leq_{(\mathcal{L}\to\mathcal{M})}$.*

Proof. Let $F \subseteq (\mathcal{L} \to \mathcal{M})$ be a family of continuous functions from $\mathcal{L}$ to $\mathcal{M}$. We define the function

$$\begin{aligned} \bigvee F\colon\ L &\longrightarrow M \\ x &\longmapsto \textstyle\bigvee_{\mathcal{M}} \{\, f(x) \mid f \in F \,\} \end{aligned}$$

and we claim that $\bigvee F$ is in fact the least upper bound for F in $((\mathcal{L} \to \mathcal{M}), \leq_{(\mathcal{L}\to\mathcal{M})})$.

First of all, we need to show that $\bigvee F \in (\mathcal{L} \to \mathcal{M})$, that is it is continuous. Let then $D \subseteq L$ be a directed set. We have

$$\begin{aligned} \bigvee F\left(\bigvee\nolimits^{\uparrow} D\right) &= \bigvee\nolimits_{\mathcal{M}} \left\{\, f\left(\bigvee\nolimits^{\uparrow} D\right) \mid f \in F \,\right\} \\ &= \bigvee\nolimits_{\mathcal{M}} \left\{\, \bigvee\nolimits^{\uparrow} f(D) \mid f \in F \,\right\} \\ &= \bigvee\nolimits^{\uparrow} \left(\bigvee F\right)(D) \end{aligned}$$

because every $f \in F$ is continuous. Next, $\bigvee F$ is obviously an upper bound for F in $(\mathcal{L} \to \mathcal{M})$ by its definition. Finally, if $g \geq_{(\mathcal{L}\to\mathcal{M})} f$ for all $f \in F$, then given $x \in L$ we have

$$g(x) \geq_{\mathcal{M}} \bigvee\nolimits_{\mathcal{M}} \{ f(x) \mid f \in F \} = \bigvee F(x)$$

and then $\bigvee F$ is the least upper bound of F. $\square$

In order to prove that $\mathbb{C}\mathbb{L}\mathrm{att}$ is cartesian closed, we start by establishing that it has terminal object and binary products (which implies that it has all finite products). Now, the terminal object is just the lattice $\{ * \}$, while given $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ complete lattices, their product $\mathcal{L} \times \mathcal{M}$ has as underlying set the cartesian product $L \times M$ with the pointwise ordering so that sups and infs are computed pointwise.

Theorem 2.15. $\mathbb{C}\mathbb{L}\mathrm{att}$ *is a cartesian closed category.*

Proof. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$, $\mathcal{M} = (M, \leq_{\mathcal{M}})$ and $\mathcal{N} = (N, \leq_{\mathcal{N}})$ be complete lattices. We define the *exponential lattice*

$$\mathcal{N}^{\mathcal{M}} := ((\mathcal{M} \to \mathcal{N}), \leq_{(\mathcal{M}\to\mathcal{N})})$$

and the *evaluation map*

$$\begin{aligned} \mathit{eval}\colon\ \mathcal{N}^{\mathcal{M}} \times \mathcal{M} &\longrightarrow \mathcal{N} \\ (f, y) &\longmapsto f(y) \end{aligned}$$

Recall that $(\mathcal{M} \to \mathcal{N})$ is the set of all *continuous* function from M to N regarded as complete lattices. First of all, we need to prove that *eval* is indeed continuous. Let $D \subseteq N^M \times M$ be a non-empty directed subset. Then $\pi_1(D)$ and $\pi_2(D)$ are both directed: if $D_0 \subseteq_{\mathit{fin}} \pi_1(D)$ and $d \in \pi_2(D)$ is any element of $\pi_2(D)$, then $D_0 \times \{d\} \subseteq_{\mathit{fin}} D$. It follows that there exists $(e_1, e_2) \in D$ that is an upper bound for $D_0 \times \{d\}$: we have so that $e_1 \in \pi_1(D)$ and it is an upper bound for D_0. Similarly, $\pi_2(D)$ is directed. We have

$$\bigvee\nolimits^{\uparrow} D = \left(\bigvee\nolimits^{\uparrow} \pi_1(D), \bigvee\nolimits^{\uparrow} \pi_2(D)\right).$$

Thus

$$\mathit{eval}\left(\bigvee\nolimits^{\uparrow} D\right) = \left(\bigvee\nolimits^{\uparrow} \pi_1(D)\right)\left(\bigvee\nolimits^{\uparrow} \pi_2(D)\right) = \bigvee^{\uparrow}_{f\in\pi_1(D)} f\left(\bigvee\nolimits^{\uparrow} \pi_2(D)\right)$$

by definition of least upper bound of a family of function as $\pi_1(D)$ is. Now, every $f \in \pi_1(D)$ is continuous and $\pi_2(D)$ is directed, so we obtain

$$\mathit{eval}\left(\bigvee\nolimits^{\uparrow} D\right) = \bigvee^{\uparrow}_{f\in\pi_1(D)} \bigvee\nolimits^{\uparrow} f(\pi_2(D)) = \bigvee^{\uparrow}_{f\in\pi_1(D)} \bigvee^{\uparrow}_{y\in D} f(y) = \bigvee^{\uparrow}_{(f,y)\in D} \mathit{eval}(f, y) = \bigvee\nolimits^{\uparrow} \mathit{eval}(D)$$

that means that *eval* is continuous.

Let now $\varphi\colon \mathcal{L} \times \mathcal{M} \to \mathcal{N}$ be a continuous function. For $x \in L$, consider the function

$$\begin{aligned} \varphi_x\colon\ M &\longrightarrow N \\ y &\longmapsto \varphi(x, y) \end{aligned}$$

We have that φ_x is continuous: let $D \subseteq M$ be a directed subset. We call $D_x = \{ (x, y) \mid y \in D \} \subseteq L \times M$. Then D_x is directed and $\bigvee^{\uparrow} D_x = (x, \bigvee^{\uparrow} D)$, thus we have:

$$\varphi_x\left(\bigvee\nolimits^{\uparrow} D\right) = \varphi\left(x, \bigvee\nolimits^{\uparrow} D\right) = \varphi\left(\bigvee\nolimits^{\uparrow} D_x\right) = \bigvee\nolimits^{\uparrow} \varphi\left(D_x\right) = \bigvee^{\uparrow}_{y\in D} \varphi(x, y) = \bigvee\nolimits^{\uparrow} \varphi_x(D).$$

We consider then

$$\begin{aligned} \lambda\varphi\colon\ L &\longrightarrow (\mathcal{M} \to \mathcal{N}) \\ x &\longmapsto \varphi_x \end{aligned}$$

and we claim that it is a continuous function. So let $D \subseteq L$ directed: we have to prove that $\lambda\varphi\left(\bigvee^{\uparrow} D\right)$ and $\bigvee^{\uparrow}_{x\in D} \varphi_x$ coincide as functions from M to N. So, given $y \in M$, we compute

$$\lambda\varphi\left(\bigvee^{\uparrow} D\right)(y) = \varphi\left(\bigvee^{\uparrow} D, y\right) = \varphi\left(\bigvee^{\uparrow} D_y\right) = \bigvee^{\uparrow} \varphi\left(D_y\right) = \bigvee^{\uparrow}_{x\in D} \varphi(x,y)$$

as $D_y = \{\, (x,y) \mid x \in D \,\}$ is directed and $\bigvee^{\uparrow} D_y = (\bigvee^{\uparrow} D, x)$. Also

$$\left(\bigvee^{\uparrow}_{x\in D} \varphi_x\right)(y) = \bigvee^{\uparrow}_{x\in D} \left(\varphi_x(y)\right) = \bigvee^{\uparrow}_{x\in D} \varphi(x,y)$$

so $\lambda\varphi$ is indeed continuous. Finally, $\lambda\varphi$ is clearly the unique morphism of $\mathbb{C}\mathbb{L}$att that makes the following diagram commute:

$$\begin{tikzcd} N^M \times M \arrow[r, "eval"] & N \\ L \times M \arrow[u, "\lambda\varphi \times \mathrm{id}"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

□

Now we consider a full subcategory of $\mathbb{C}\mathbb{L}$att, namely that on the *algebraic lattices*. It will have a crucial role when we introduce equilogical spaces. An algebraic lattice is a complete lattice that satisfies an additional condition. Before we formulate that condition however, we need to introduce an instrumental order-theoretic notion.

Definition 2.16. Let $L = (L, \leq_{\mathcal{L}})$ be a complete lattice. A *compact* element e of $\mathcal{L}$ is an element of L such that if $e \leq \bigvee S$ for a subset $S \subseteq L$, then there exists $S_0 \subseteq_{\mathit{fin}} S$ such that $e \leq \bigvee S_0$. The set of compact elements in the lattice $\mathcal{L}$ is denoted $\mathcal{K}(\mathcal{L})$.

It is easy to show that the bottom element $\bot_{\mathcal{L}}$ of a complete lattice $\mathcal{L}$ is compact and that the least upper bound of a finite collection of compact elements is also compact.

Definition 2.17. A complete lattice $\mathcal{L} = (L, \leq_{\mathcal{L}})$ is called *algebraic* if every element is determined by the compact elements that precede it. In other words if

$$x, y \in L \text{ and } (\forall e \in \mathcal{K}(\mathcal{L})\ e \leq_{\mathcal{L}} x \iff e \leq_{\mathcal{L}} y) \text{ always imply } x = y.$$

Proposition 2.18. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ be a complete lattice. Then $\mathcal{L}$ is algebraic if and only if for every $x \in L$*

$$x = \bigvee \{\, e \in \mathcal{K}(\mathcal{L}) \mid e \leq x \,\}.$$

Proof. Let $x \in L$ and $A = \{\, e \in \mathcal{K}(\mathcal{L}) \mid e \leq x \,\}$: we want to prove that $x = \bigvee^{\uparrow} A$. We write $y = \bigvee^{\uparrow} A$: x is obviously an upper bound, so $y \leq x$ and then given $e \in \mathcal{K}(L)$, if $e \leq y$ then $e \leq x$. On the other hand, if $e \leq x$ then $e \in A$ and so $e \leq y$: we have shown that for every e compact element of $\mathcal{L}$, e precedes x if and only if it precedes y. By definition of algebraic lattice, it follows that $x = y$.

Viceversa, let $x, y \in L$ and suppose that given any e compact element, saying that $e \leq x$ is equivalent to say that $e \leq y$. We know that $x = \bigvee^{\uparrow} A$ and $y = \bigvee^{\uparrow} B$ where A, B are the sets of all the compact elements that precede x and y respectively. Then $A = B$ thanks to our assumption, that means that $x = y$. □

Definition 2.19. The category $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$ of *algebraic lattices* is the full subcategory of $\mathbb{C}\mathbb{L}\mathrm{att}$ on the algebraic lattices.

We aim to show that $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$ is a sub-cartesian closed category of $\mathbb{C}\mathbb{L}\mathrm{att}$.

For two algebraic lattices $\mathcal{L}$ and $\mathcal{M}$, we already know that the function space $(\mathcal{L} \to \mathcal{M})$ is a complete lattice (since $\mathcal{L}$ and $\mathcal{M}$ are complete). In order to show that it is indeed algebraic, we will characterize the compact elements in $(\mathcal{L} \to \mathcal{M})$ and show that there are enough of them to distinguish all element of the space. The argument is based on the step functions, which we define next.

Definition 2.20. Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices and let $a \in \mathcal{K}(\mathcal{L})$, $y \in M$. We define the *step function*

$$[a, y] \colon L \longrightarrow M$$

$$x \longmapsto \begin{cases} y & \text{if } a \leq x \\ \bot_{\mathcal{M}} & \text{otherwise} \end{cases}$$

First we show that step functions $[a, y] \colon \mathcal{L} \to \mathcal{M}$ are in fact morphism in $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$, hence they are elements of the function space $(\mathcal{L} \to \mathcal{M})$.

Proposition 2.21. *Let $[a, y] \colon \mathcal{L} \to \mathcal{M}$ be a step function between algebraic lattices as in the previous definition. Then $[a, y]$ is continuous.*

Proof. Let $V \subseteq M$ be a proper open set in $\mathcal{M}$, that is $\bot_{\mathcal{M}} \notin V$ (every open set is upward closed). If $y \notin V$, then $[a, y]^{-1}(V) = \varnothing$. If $y \in V$, then

$$[a, y]^{-1}(V) = \{\, x \in L \mid x \geq a \,\}.$$

We write $U = [a, y]^{-1}(V)$. We have that U is obviously upward closed and given $S \subseteq L$ such that $\bigvee S \in U$, we get that $a \leq \bigvee S$. But a is compact, thus there exists $S_0 \subseteq_{\mathit{fin}} S$ such that $a \leq \bigvee S_0$, that is $\bigvee S_0 \in U$. This means that U is indeed open in the Scott-topology. $\square$

Proposition 2.22. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices, $a \in \mathcal{K}(\mathcal{L})$, $y \in M$, $f \colon \mathcal{L} \to \mathcal{M}$ a continuous function. Then $[a, y] \leq f$ if and only if $y \leq f(a)$.*

Proof. Assume that for every $x \in L$ $[a, y](x) \leq f(x)$. This means that in particular $y \leq f(a)$, since $a \leq a$. Conversely, suppose that $y \leq f(a)$ and let $x \in L$. If $x \ngeq a$, then $[a, y](x) = \bot_{\mathcal{M}}$; thus it obviously precedes $f(x)$. If $x \geq a$, then by monotonicity of f we have $f(x) \geq f(a) \geq y$, as requested. $\square$

Proposition 2.23. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices and let $a \in \mathcal{K}(\mathcal{L})$, $b \in \mathcal{K}(\mathcal{M})$. Then the step function $[a, b]$ is compact in $(\mathcal{L} \to \mathcal{M})$.*

Proof. Let $S \subseteq (\mathcal{L} \to \mathcal{M})$ be such that $[a, b] \leq \bigvee S$. As we know, $\bigvee S$ is a continuous function from $\mathcal{L}$ to $\mathcal{M}$, so by Proposition 2.22 we have that

$$b \leq \bigvee S(a) = \bigvee_{f \in S} f(a).$$

Since b is compact, there exists $S_0 \subseteq_{\mathit{fin}} S$ such that $b \leq \bigvee_{f \in S_0} f(a) = \bigvee S_0(a)$, thus $[a, b] \leq \bigvee S_0$. $\square$

Remark 2.24. There is also a kind of converse to 2.23, but we limit ourselves just to mention.

Theorem 2.25. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ and $\mathcal{M} = (M, \leq_{\mathcal{M}})$ be algebraic lattices. Then the function space $(\mathcal{L} \to \mathcal{M})$ is an algebraic lattice.*

Proof. Let $f\colon L \to M$ be a continuous function: we want to prove that f is the least upper bound of all the compact elements that precede it.

We first observe that

$$f = \bigvee^{\uparrow}_{a \in \mathcal{K}(\mathcal{L})} [a, f(a)].$$

In fact, for every $x \in \mathcal{L}$,

$$\left(\bigvee^{\uparrow}_{a \in \mathcal{K}(\mathcal{L})} [a, f(a)] \right)(x) = \bigvee^{\uparrow}_{a \in \mathcal{K}(\mathcal{L})} [a, f(a)](x) = \bigvee^{\uparrow}_{\substack{a \in \mathcal{K}(\mathcal{L}) \\ a \leq x}} f(a) = f\left(\bigvee^{\uparrow}_{\substack{a \in \mathcal{K}(\mathcal{L}) \\ a \leq x}} a \right) = f(x)$$

since f is continuous and the set $\{ a \in \mathcal{K}(\mathcal{L}) \mid a \leq x \}$ is directed. Since $\mathcal{M}$ is algebraic, for a in L,

$$f(a) = \bigvee^{\uparrow}_{\substack{b \in \mathcal{K}(\mathcal{M}) \\ b \leq f(a)}} b.$$

Hence

$$f = \bigvee^{\uparrow}_{\substack{a \in \mathcal{K}(\mathcal{L}) \\ b \in \mathcal{K}(\mathcal{M}) \\ b \leq f(a)}} [a, b].$$

We have then written f as the least upper bound of a set of compact elements, the step functions $[a, b]$ (with b compact), that precede f (Proposition 2.22). This means that

$$f = \bigvee^{\uparrow}_{\substack{a \in \mathcal{K}(\mathcal{L}) \\ b \in \mathcal{K}(\mathcal{M}) \\ b \leq f(a)}} [a, b] \leq \bigvee^{\uparrow}_{\substack{e \in \mathcal{K}(\mathcal{L} \to \mathcal{M}) \\ e \leq f}} e \leq f$$

so $(\mathcal{L} \to \mathcal{M})$ is indeed algebraic. □

All we still have to do in order to prove that $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$ is cartesian closed, is that it has terminal object and binary products. Regarding the first, it suffices to note that $\{*\}$ is an algebraic lattice; moreover it is easy to see that in a product lattice $\mathcal{L} \times \mathcal{M}$ of two complete lattices, an element (x, y) is compact if and only if both x and y are compact in $\mathcal{L}$ and $\mathcal{M}$, respectively: this means that $\mathcal{L} \times \mathcal{M}$ is algebraic if and only if $\mathcal{L}$ and $\mathcal{M}$ are. Since $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$ is a full subcategory of $\mathbb{C}\mathbb{L}\mathrm{att}$, extending the proof ot Theorem 2.15 we immediately obtain the following result.

Theorem 2.26. $\mathbb{A}\mathrm{lg}\mathbb{L}\mathrm{att}$ *is a cartesian closed category.*

We conclude this section with a theorem that will come in handy in the following.

Theorem 2.27. *Let $\mathcal{L}, \mathcal{M}$ be algebraic lattices. Then every monotone function $f\colon \mathcal{K}(\mathcal{L}) \to \mathcal{M}$ can be uniquely extended to a continuous function $\bar{f}\colon \mathcal{L} \to \mathcal{M}$ defining*

$$\bar{f}(x) := \bigvee^{\uparrow}_{\substack{c \in \mathcal{K}(\mathcal{L}) \\ c \leq x}} f(c).$$

2.2 The E-Theorems

In this section we prove two well-known theorems, the Embedding and the Extension Theorems, that are going to come in handy when we treat the equilogical spaces. Before doing this, though, we need a preliminary result about the Scott-topology for an algebraic lattice.

Theorem 2.28. *Let $\mathcal{L} = (L, \leq_{\mathcal{L}})$ be an algebraic lattice. For an element $a \in L$, define the set $a^{\leq} \subseteq L$ as*

$$a^{\leq} = \{\, x \in L \mid a \leq x \,\}\,.$$

Then $\left\{\, e^{\leq} \mid e \in \mathcal{K}(\mathcal{L}) \,\right\}$ is a basis for $\sigma_{\mathcal{L}}$.

Proof. The sets $e^{\leq}$, with e compact, are open as one can check in the same way as in 2.21. To show that $\left\{\, e^{\leq} \mid e \in \mathcal{K}(\mathcal{L}) \,\right\}$ is indeed a basis, let W be an open set. It is enough to show that

$$W = \bigcup_{e \in W \cap \mathcal{K}(\mathcal{L})} e^{\leq}.$$

Consider a compact $e \in W$ and assume that $x \in e^{\leq}$: it means that $e \leq x$, thus $x \in W$ since W is upward closed. Conversely, let an $x \in W$ be given. By 2.18, we have that

$$x = \bigvee \{\, e \in \mathcal{K}(\mathcal{L}) \mid e \leq x \,\} \in W$$

and since W is open, there exists a finite collection of compact elements $e_1 \leq x, \ldots, e_n \leq x$ such that $e := e_1 \vee \ldots \vee e_n \in W$. But e is again compact and $e \leq x$: this means that

$$x \in e^{\leq} \subseteq \bigcup_{e \in W \cap \mathcal{K}(\mathcal{L})} e^{\leq} \qquad \square$$

Recall that a topological embedding of a topological space into another is a function that is continuous, open and injective. The Embedding Theorem says that any T_0-space can be embedded into an algebraic lattice, indeed a powerset where the compact elements are the finite subsets.

Theorem 2.29 (The Embedding Theorem)**.** *Let $\mathcal{T} = (T, \tau)$ be a T_0-space. Then*

$$\begin{aligned} \varphi\colon\ (T,\tau) &\longrightarrow (\mathscr{P}(\tau), \sigma_{\mathscr{P}(\tau)}) \\ x &\longmapsto \tau(x) \end{aligned}$$

is a topological embedding.

Proof. Since $\mathcal{T}$ is a T_0-space, φ is clearly an injection. To show that φ is continuous, it suffices to show that the inverse image of an element of the basis of $\sigma_{\mathscr{P}(\tau)}$ is open. By Theorem 2.28, such an element is of the form

$$\mathscr{A}^{\subseteq} = \{\, \mathscr{B} \in \mathscr{P}(\tau) \mid \mathscr{A} \subseteq \mathscr{B} \,\}$$

with $\mathscr{A} = \{U_1, \ldots, U_n\}$ a finite subset of τ. We have

$$\begin{aligned} \varphi^{-1}(\mathscr{A}^{\subseteq}) &= \left\{\, x \in T \mid \tau(x) \in \mathscr{A}^{\subseteq} \,\right\} \\ &= \{\, x \in T \mid \mathscr{A} \subseteq \tau(x) \,\} \\ &= \{\, x \in T \mid \forall i \in \{1, \ldots, n\}\ x \in U_i \,\} \\ &= U_1 \cap \cdots \cap U_n \end{aligned}$$

which is open. Finally, let $U \in \tau$: we are to show that $\varphi(U)$ is open in the Scott-topology of $\mathscr{P}(\tau)$. In fact

$$\begin{aligned}\varphi(U) &= \{\, \tau(x) \mid x \in U \,\} \\ &= \{\, \tau(x) \mid x \in T \text{ and } U \in \tau(x) \,\} \\ &= \varphi(T) \cap \{\, \mathscr{B} \in \mathscr{P}(\tau) \mid \{U\} \subseteq \mathscr{B} \,\} \\ &= \varphi(T) \cap \{U\}^{\subseteq}\end{aligned}$$

which is open since $\{U\}$ is compact. □

Theorem 2.30 (The Extension Theorem)**.** *Let $\mathcal{L} = (L, \leq)$ be an algebraic lattice, and let $\mathcal{X} = (X, \tau_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \tau_{\mathcal{Y}})$ be two topological spaces such that $\mathcal{X}$ is a subspace of $\mathcal{Y}$. If a function $f \colon X \to L$ is continuous with respect to the Scott-topology on L, then f has a continuous extension to all of Y.*

Proof. Let $f \colon X \to L$ be a continuous function. We define $\bar{f} \colon Y \to L$ as

$$\bar{f}(y) := \bigvee_{U \in \tau_{\mathcal{Y}}(y)} \bigwedge_{x \in U \cap X} f(x)$$

for all $y \in Y$. We claim that $\bar{f}$ is the desired extension, so we must show that $\bar{f}$ extends f and is continuous.

For the first claim, let $a \in X$. For any open $U \in \tau_{\mathcal{X}}(a)$ we have that

$$\bigwedge_{x \in U \cap X} f(x) \leq f(a)$$

so, by definition of $\bar{f}$, we obtain $\bar{f}(a) \leq f(a)$. Since $\mathcal{L}$ is algebraic, it now suffices to show that for any compact element $e \in L$ we have that $e \leq f(a)$ implies $e \leq \bar{f}(a)$, since this would mean that every compact element that precedes $f(a)$ precedes $\bar{f}(a)$ too and viceversa, thus $f(a) = \bar{f}(a)$. Therefore, let $e \leq f(a)$ be a compact element. By Theorem 2.28, the set $e^{\leq}$ is open, so $f^{-1}(e^{\leq})$ is open in $\mathcal{X}$ by continuity of f and $a \in f^{-1}(e^{\leq})$. Also note that if $x' \in f^{-1}(e^{\leq})$, then $e \leq f(x')$. This means that

$$e \leq \bigwedge_{x' \in X \cap f^{-1}(e^{\leq})} f(x') \leq \bar{f}(a),$$

as required.

To show that $\bar{f}$ is continuous, we show that the inverse image of a basic open set $e^{\leq}$ is open. Let a $y \in Y$ be such that $\bar{f}(y) \in e^{\leq}$ for a compact element $e \in L$. This means that

$$\bigvee_{U \in \tau_{\mathcal{Y}}(y)} \bigwedge_{x \in U \cap X} f(x) \in e^{\leq}.$$

Since $e^{\leq}$ is open, we can find $U_1, \ldots, U_n \in \tau_{\mathcal{Y}}(y)$ such that

$$\bigvee_{i=1}^{n} \bigwedge_{x \in U_i \cap X} f(x) \in e^{\leq}$$

that is

$$e \leq \bigvee_{i=1}^{n} \bigwedge_{x \in U_i \cap X} f(x).$$

Let the open set $U_0 \in \tau_{\mathcal{Y}}$ be defined as $U_0 = U_1 \cap \cdots \cap U_n$ and note that $y \in U_0$. We are to show that $U_0 \subseteq \bar{f}^{-1}(e^{\leq})$, proving so that $\bar{f}^{-1}(e^{\leq})$ is open in $\mathcal{X}$. Since $U_0 \subseteq U_i$ for all $i \in \{1, \ldots, n\}$, we have that

$$\bigwedge_{x \in U_i \cap X} f(x) \leq \bigwedge_{x \in U_0 \cap X} f(x),$$

hence

$$\begin{aligned} e &\leq \bigvee_{i=1}^{n} \bigwedge_{x \in U_i \cap X} f(x) \\ &\leq \bigvee_{i=1}^{n} \bigwedge_{x \in U_0 \cap X} f(x) \\ &= \bigwedge_{x \in U_0 \cap X} f(x). \end{aligned}$$

Now, if $y' \in U_0$, then $U_0 \in \tau_{\mathcal{Y}}(y')$, so

$$e \leq \bigwedge_{x \in U_0 \cap X} f(x) \leq \bigvee_{U \in \tau_{\mathcal{Y}}(y')} \bigwedge_{x \in U \cap X} f(x) = \bar{f}(y').$$

This means that $\bar{f}(y') \in e^{\leq}$, i.e. $y' \in \bar{f}^{-1}(e^{\leq})$. □

2.3 The category $\mathbb{E}$qu

We are now ready to define the category $\mathbb{E}$qu. In order to prove that $\mathbb{E}$qu is cartesian closed, we introduce another category, $\mathbb{PE}$qu, and show that it is cartesian closed. Next, we shall prove that $\mathbb{PE}$qu is equivalent to $\mathbb{E}$qu, whence its cartesian closedness.

Definition 2.31. The category $\mathbb{E}$qu of *equilogical spaces* is defined as consisting of

- objects $E = (|E|, \tau_E, \equiv_E)$, where $(|E|, \tau_E)$ is a T_0 topological space and $\equiv_E$ is an equivalence relation on the set $|E|$. These structures are called equilogical spaces;
- a morphism from the equilogical space $E = (|E|, \tau_e, \equiv_E)$ to $F = (|F|, \tau_F, \equiv_F)$ is an equivalence class modulo $\equiv_{E \to F}$ of continuous functions between the topological spaces that respect the equivalence relations—such a map will be thus deemed *equivariant*. The equivalence relation $\equiv_{E \to F}$ on maps is defined by

$$f \equiv_{E \to F} g \iff \forall x, y \in |E| \big(x \equiv_E y \implies f(x) \equiv_F g(y)\big).$$

Remark 2.32. There is a natural embedding of $\mathbb{T}\text{op}_0$ into $\mathbb{E}$qu which considers the identity relation as equivalence relation for a T_0 topological space.

Before the definition of $\mathbb{PE}$qu, recall that a *partial* equivalence relation $\equiv$ on a set A is a relation on A that is symmetric and transitive, which will be reflexive on its (possibly empty) domain contained in A. If $x, y \in A$ are such that $x \equiv y$, then by symmetry and transitivity of $\equiv$ we have that $x \equiv x$ and $y \equiv y$. Therefore, $\equiv$ splits A up partially into "equivalence classes", which will be pairwise disjoint, but there may be elements that are not in any of these classes.

Definition 2.33. The category $\mathbb{PE}$qu of *partial equilogical spaces* is defined as consisting of

- objects $A = (|A|, \equiv_A)$, where $|A|$ is an algebraic lattice equipped with the Scott-topology and $\equiv_A$ is a partial equivalence relation on $|A|$;
- a morphism from the partial equilogical space A to the partial equilogical space B is an equivalence class modulo $\equiv_{A\to B}$ of continuous equivariant maps. The relation $\equiv_{A\to B}$ is defined just like in the case of equilogical spaces.

First of all, we show that $\mathbb{P}\mathbb{E}$qu has finite products. The partial equilogical space

$$\mathbb{1} = \big(\{\,*\,\}, =\big),$$

is a terminal object, and it has binary products since we can just take the structure with the cartesian product of the sets, the product topology, and the relation given componentwise on pairs which is immediate to see that it is a partial equivalence relation.

In order to show that $\mathbb{P}\mathbb{E}$qu is cartesian closed, we need to determine function spaces in a suitable way. We know that $\mathbb{A}$lg$\mathbb{L}$att is cartesian closed; so, given $A = (|A|, \equiv_A)$ and $B = (|B|, \equiv_B)$ two partial equilogical spaces, we use the algebraic lattice $|B^A| := |B|^{|A|}$ as "base" of the function space B^A while, as a partial equivalence relation on it, we use $\equiv_{A\to B}$. We make this explicit in the following theorem.

Theorem 2.34. $\mathbb{P}\mathbb{E}$qu *is a cartesian closed category.*

Proof. For simplicity we shall omit the bars $|*|$ on underlying sets of partial equilogical spaces. Let so $(A, \equiv_A)$ and $(B, \equiv_B)$ be two partial equilogical spaces: in $\mathbb{A}$lg$\mathbb{L}$att we have the evaluation map

$$\begin{aligned} B^A \times A &\xrightarrow{eval} B \\ (f,\, a) &\longmapsto f(a) \end{aligned}$$

which we know to be continuous: we want to prove that it is equivariant. If $(f,\, a_1) \equiv_{B^A\times A} (g, a_2)$ then $f \equiv_{B^A} g$ and $a_1 \equiv_A a_2$: it follows that $f(a_1) \equiv_B g(a_2)$, so *eval* is indeed equivariant. This means that we can consider $[eval]\colon (B^A, \equiv_{B^A}) \times (A, \equiv_A) \to (B, \equiv_B)$ as the evaluation morphism in $\mathbb{P}\mathbb{E}$qu.

Let $[\varphi]\colon (Z, \equiv_Z) \times (A, \equiv_A) \to (B, \equiv_B)$ be a morphism in $\mathbb{P}\mathbb{E}$qu. We want to prove that there exists a unique $\lambda[\varphi]\colon (Z, \equiv_Z) \to (B^A, \equiv_{B^A})$ such that the following triangle commutes

$$\begin{tikzcd}
(B^A \times A, \equiv_{B^A} \times \equiv_A) \arrow[d, equal] & \\
(B^A, \equiv_{B^A}) \times (A, \equiv_A) \arrow[r, "{[eval]}"] & (B, \equiv_B) \\
(Z, \equiv_Z) \times (A, \equiv_A) \arrow[u, dashed, "\lambda[\varphi] \times \mathrm{id}"] \arrow[ur, "{[\varphi]}"'] \arrow[d, equal] & \\
(Z \times A, \equiv_Z \times \equiv_A) &
\end{tikzcd}$$

Having $[\varphi]\colon (Z, \equiv_Z) \times (A, \equiv_A) \to (B, \equiv_B)$ means having $\varphi\colon Z \times A \to B$ in $\mathbb{A}$lg$\mathbb{L}$att, thus there exists a unique $\lambda\varphi\colon Z \to B^A$ continuous such that

$$\begin{tikzcd} B^A \times A \arrow[r, "eval"] & B \\ Z \times A \arrow[u, dashed, "\lambda\varphi \times \mathrm{id}"] \arrow[ur, "\varphi"'] \arrow[u, phantom, shift left=4ex, "="] \end{tikzcd}$$

$\lambda\varphi$ is in fact defined by $\lambda\varphi(z) := (A \ni a \mapsto \varphi(z,a))$. $\lambda\varphi$ is equivariant too: given $z_1 \equiv_Z z_2$, we have that $\lambda\varphi(z_1) \equiv_{B^A} \lambda\varphi(z_2)$ if and only if whenever $a_1, a_2 \in A$ are such that $a_1 \equiv_A a_2$ one has $\lambda\varphi(z_1)(a_1) \equiv_B \lambda\varphi(z_2)(a_2)$, that is $\varphi(z_1,a_1) \equiv_B \varphi(z_2,a_2)$ which is true because φ is equivariant. Moreover, if $\psi\colon Z \times A \to B$ is another representative of $[\varphi]$, then $[\lambda\varphi] = [\lambda\psi]$ because given $z_1 \equiv_Z z_2$ and $a_1 \equiv_A a_2$ we have that $\varphi(z_1,a_1) \equiv_B \psi(z_2,a_2)$ inasmuch as $(z_1,a_1) \equiv_{Z\times A} (z_2,a_2)$ and $\varphi \equiv_{Z\times A \to B} \psi$ and so $\lambda\varphi(z_1) \equiv_{B^A} \lambda\psi(z_2)$.

We can thus define

$$\lambda[\varphi] := [\lambda\varphi]$$

and since the projection maps in $\mathbb{PE}$qu are the equivalence class of the projections maps in $\mathbb{A}$lg$\mathbb{L}$att, we have that

$$[\varphi] = [eval \circ (\lambda\varphi \times \mathrm{id})] = [eval] \circ [\lambda\varphi \times \mathrm{id}] = [eval] \circ [\lambda\varphi] \times \mathrm{id}$$

that is

$$\begin{tikzcd} (B^A, \equiv_{B^A}) \times (A, \equiv_A) \arrow[r, "{[eval]}"] & (B, \equiv_B) \\ (Z, \equiv_Z) \times (A, \equiv_A) \arrow[u, dashed, "\lambda[\varphi] \times \mathrm{id}"] \arrow[ur, "{[\varphi]}"'] \end{tikzcd}$$

Finally, if $[\alpha]\colon (Z, \equiv_Z) \to (B^A, \equiv_{B^A})$ is such that $[\varphi] = [eval] \circ ([\alpha] \times \mathrm{id}) = [eval \circ (\alpha \times \mathrm{id})]$, then given $z_1 \equiv_Z z_2$ we have $\alpha(z_1) \equiv_{B^A} \lambda\varphi(z_2)$ because for every $a_1 \equiv_A a_2$ we have

$$\alpha(z_1)(a_1) \equiv_B \varphi(z_2,a_2) = \lambda\varphi(z_2)(a_2)$$

by hypothesis, so $[\lambda\varphi]$ is the unique morphism in $\mathbb{PE}$qu that make the triangle commute. □

Theorem 2.35. $\mathbb{E}$qu *and* $\mathbb{PE}$qu *are equivalent categories.*

Proof. We produce a functor $\mathcal{F}\colon \mathbb{PE}\mathrm{qu} \to \mathbb{E}\mathrm{qu}$ which is full, faithful and essentially surjective on objects. Within the scope of the present proof we find convenient, given a set A endowed with a partial equivalence relation $\equiv$, to denote as A_{tot} the domain $\{\, x \in A \mid x \equiv x \,\} \subseteq A$ of the relation $\equiv$. Recall that $\equiv$ is reflexive when restricted to this subset—thus an equivalence relation. For a partial equilogical space $A = (|A|, \equiv_A)$, set

$$\mathcal{F}(A) := \left(|A|_{\mathrm{tot}}, \tau_{|A|_{\mathrm{tot}}}, \equiv_{|A|_{\mathrm{tot}}}\right)$$

where $\tau_{|A|_{\mathrm{tot}}}$ is the topology induced on $|A|_{\mathrm{tot}}$ by the Scott-topology on $|A|$ and $\equiv_{|A|_{\mathrm{tot}}}$ is the restriction of $\equiv_A$ to $|A|_{\mathrm{tot}}$. Given $[f]\colon A \to B$ in $\mathbb{PE}$qu, define

$$\mathcal{F}([f]) := \left[f{\restriction}_{|A|_{\mathrm{tot}}}\right]$$

$\mathcal{F}$ is a functor. Since the Scott-topology separates points in the algebraic lattice $|A|$, the space $(|A|_{\mathrm{tot}}, \tau_{|A|_{\mathrm{tot}}})$ is still T_0. So $\mathcal{F}(A)$ is an equilogical space. Consider now a map $[f]\colon A \to B$ in $\mathbb{PE}$qu. The codomain of $f\restriction_{|A|_{\mathrm{tot}}}$ is indeed $|B|_{\mathrm{tot}}$, as, for $a \equiv_A a$, it is $f(a) \equiv_B f(a)$. It respects the equivalence relation $\equiv_{|A|_{\mathrm{tot}}}$, since f respects $\equiv_A$, and it is continuous as a restriction of a continuous function. It is clear that $\mathcal{F}$ preserves identities and compositions, and therefore is a functor.

$\mathcal{F}$ is faithful. Let two partial equilogical spaces $A = (|A|, \equiv_A)$ and $B = (|B|, \equiv_B)$ be given, and let $[f], [g]\colon A \to B$ be two morphisms in $\mathbb{PE}$qu such that $\mathcal{F}([f]) = \mathcal{F}([g])$. We need to show that $f \equiv_{A\to B} g$, so let $x, y \in |A|$ satisfy $x \equiv_A y$. This means that $x \equiv_A x$ and $y \equiv_A y$, and therefore

$$f(x) = f\restriction_{|A|_{\mathrm{tot}}}(x) \equiv_{|B|_{\mathrm{tot}}} g\restriction_{|A|_{\mathrm{tot}}}(y) = g(y).$$

Since $\equiv_{|B|_{\mathrm{tot}}}$ is a restriction of $\equiv_B$, this means that $f(x) \equiv_B g(y)$, so $\mathcal{F}$ is faithful.

$\mathcal{F}$ is full. For $A = (|A|, \equiv_A)$ and $B = (|B|, \equiv_B)$ partial equilogical spaces, let $[f]\colon \mathcal{F}(A) \to \mathcal{F}(B)$ be a morphisms between the two images in $\mathbb{E}$qu: we have to show that there is a valid map $[\varphi]\colon A \to B$ in $\mathbb{PE}$qu such that $\mathcal{F}([\varphi]) = [f]$. Note that f is a continuous map $f\colon (|A|_{\mathrm{tot}}, \tau_{|A|_{\mathrm{tot}}}) \to (B, \tau_B)$. Since (B, τ_B) is the Scott-topology of an algebraic lattice, the Extension Theorem tells us that f has a continuous extension $\bar{f}\colon A \to B$. $\bar{f}$ preserves the partial equivalence relation: let $x, y \in |A|$ be such that $x \equiv_A y$. Again, this means that $x, y \in |A|_{\mathrm{tot}}$, so since f was equivariant and $\bar{f}$ extends f, we have $\bar{f}(x) \equiv_B \bar{f}(b)$. This means that $\bar{f}$ is equivariant: defining $\varphi = \bar{f}$, we have that $\mathcal{F}([\varphi]) = [f]$ since φ is the extension of f from the set $|A|_{\mathrm{tot}}$.

$\mathcal{F}$ is essentially surjective on objects. Let $E = (|E|, \tau_E, \equiv_E)$ be an equilogical space. We have to construct a partial equilogical space A such that $\mathcal{F}(A)$ is isomorphic to E. For this we shall use the Embedding Theorem.

As the underlying set of A we take the algebraic lattice $\mathscr{P}(\tau_E)$. For two elements $\mathscr{A}, \mathscr{B} \in \mathscr{P}(\tau_E)$, define $\mathscr{A} \equiv_{\mathscr{P}(\tau_E)} \mathscr{B}$ as follows:

$$\mathscr{A} \equiv_{\mathscr{P}(\tau_E)} \mathscr{B} \iff \exists x, y \in |E|.\, x \equiv_E y \text{ and } \mathscr{A} = \tau_E(x) \text{ and } \mathscr{B} = \tau_E(y).$$

It is obviously symmetric on $\mathscr{P}(\tau_E)$. Assume now that $\mathscr{A} \equiv_{\mathscr{P}(\tau_E)} \mathscr{B}$ and $\mathscr{B} \equiv_{\mathscr{P}(\tau_E)} \mathscr{C}$ for three elements $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathscr{P}(\tau_E)$. So there are $x, y, z, w \in |E|$ with

$$x \equiv_E y, \quad \mathscr{A} = \tau_E(x), \quad , \mathscr{B} = \tau_E(y), \quad z \equiv_E w, \quad B = \tau_E(z), \quad \mathscr{C} = \tau_E(w).$$

Since (E, τ_E) is a T_0 space, the conditions $\mathscr{B} = \tau_E(y)$ and $\mathscr{B} = \tau_E(z)$ imply that $y = z$. Thus

$$x \equiv_E w \text{ and } \mathscr{A} = \tau_E(x) \text{ and } \mathscr{C} = \tau_E(w).$$

Hence $\mathscr{A} \equiv_{\mathscr{P}(\tau_E)} \mathscr{C}$. This proves that is $\equiv_{\mathscr{P}(\tau_E)}$ is transitive. Finally we observe that for an element $\mathscr{A} \in \mathscr{P}(\tau_E)$,

$$\begin{aligned}
\mathscr{A} \equiv_{\mathscr{P}(\tau_E)} \mathscr{A} &\iff \exists x, y \in |E|.\, x \equiv_E y \text{ and } \mathscr{A} = \tau_E(x) \text{ and } \mathscr{A} = \tau_E(y) \\
&\iff \exists x \in |E|.\, x \equiv_E x \text{ and } \mathscr{A} = \tau_E(x) \\
&\iff \exists x \in |E|.\, \mathscr{A} = \tau_E(x)
\end{aligned}$$

because τ_E is T_0 and $\equiv_E$ is reflexive. Therefore, we obtain that the underlying set of $\mathcal{F}(A) \in \mathrm{Ob}(\mathbb{E}\mathrm{qu})$ is $\{\, \tau_E(x) \mid x \in |E| \,\}$, and the topology $\tau_{\mathcal{F}(A)}$ and $\equiv_{\mathcal{F}(A)}$ are those induced from it. The Embedding Theorem states that the map

$$\begin{aligned} \varphi_E : (E, \tau_E) &\longrightarrow \big(\{\, \tau_E(x) \mid x \in |E| \,\}, \tau_{\mathcal{F}(A)}\big) \\ x &\longmapsto \tau_E(x) \end{aligned}$$

is an isomorphism of topological spaces. All that is left to show is that φ_E and

$$\begin{aligned} (\varphi_E)^{-1} : \big(\{\, \tau_E(x) \mid x \in |E| \,\}, \tau_{\mathcal{F}(A)}\big) &\longrightarrow (E, \tau_E) \\ \tau_E(x) &\longmapsto x \end{aligned}$$

are equivariant. If $x, y \in |E|$ are equivalent, it is clear that $\varphi(x) = \tau_E(x) \equiv_{\mathscr{P}(\tau_E)} \tau_E(y) = \varphi(y)$. Conversely, given $x, y \in |E|$, $\tau_E(x) \equiv_{\mathscr{P}(\tau_E)} \tau_E(y)$ implies that $x \equiv_E y$, so φ^{-1} does indeed respect the equivalence relations. □

Corollary 2.36. $\mathbb{E}$qu *is a cartesian closed category.*

Proof. It follows immediately from 2.35 and 2.34. □

In order to study the Eilenberg-Moore algebras for a specific double-exponential monad over $\mathbb{E}$qu, it is useful to study the cartesian closed structure of $\mathbb{E}$qu directly, as inherited from the structure of $\mathbb{PE}$qu. Consider thus the equivalence of categories

$$\begin{tikzcd} \mathbb{PE}\mathrm{qu} \arrow[r, bend left, "\mathcal{F}"] & \mathbb{E}\mathrm{qu} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

where $\mathcal{G}$ is defined on objects as in the last part of the proof of Theorem 2.35. Given A and B two equilogical spaces, the exponential B^A and the evaluation map are defined up to isomorphism in terms of the respective exponential and evaluation map in $\mathbb{PE}$qu as follows:

$$B^A = \mathcal{F}\Big(\mathcal{G}(B)^{\mathcal{G}(A)}\Big)$$

and for $E = (|E|, \tau_E, \equiv_E)$ an equilogical space consider the isomorphism $[\varphi_E] : E \to \mathcal{F}(\mathcal{G}(E))$ provided by Theorem 2.29, where

$$\begin{aligned} \varphi_E : \ |E| &\longrightarrow |\mathcal{F}(\mathcal{G}(E))| \\ x &\longmapsto \tau_E(x) \end{aligned}$$

We have then an isomorphism

$$\mathrm{id}_\times \, \varphi_A : \mathcal{F}\left(\mathcal{G}(B)^{\mathcal{G}(A)}\right) \times A \to \mathcal{F}\left(\mathcal{G}(B)^{\mathcal{G}(A)}\right) \times \mathcal{F}(\mathcal{G}(A)).$$

Since $\mathcal{F}$ preserves products on the nose, we have

$$\mathcal{F}\left(\mathcal{G}(B)^{\mathcal{G}(A)}\right) \times \mathcal{F}(\mathcal{G}(A)) = \mathcal{F}\left(\mathcal{G}(B)^{\mathcal{G}(A)} \times \mathcal{G}(A)\right).$$

Applying the image under $\mathcal{F}$ of the evaluation map in $\mathbb{P}\mathbb{E}$qu and then $(\varphi_A)^{-1}$ we obtain

$$\begin{aligned} eval\colon\ B^A \times A &\longrightarrow B \\ \Big[(f,a) &\longmapsto \varphi_A^{-1}\big(f(\tau_A(a))\big)\Big]. \end{aligned}$$

Remark 2.37. If $P \in \mathrm{Ob}(\mathbb{P}\mathbb{E}\mathrm{qu})$ is such that $\mathcal{F}(P) \cong E$ with $E \in \mathrm{Ob}(\mathbb{E}\mathrm{qu})$, then $\mathcal{G}(E) \cong P$ in $\mathbb{P}\mathbb{E}$qu because $\mathcal{F}\big(\mathcal{G}(E)\big) \cong E \cong \mathcal{F}(P)$ and $\mathcal{F}$ reflects isomorphisms. This means that, when we compute B^A in $\mathbb{E}$qu, we can replace $\mathcal{G}(B)$ (or $\mathcal{G}(A)$) with any partial equilogical space whose image under $\mathcal{F}$ is still isomorphic to B (or A).

For simplicity, from now on we shall avoid the use of the bars $|-|$ for the underlying set of a (partial) equilogical space if no confusion arises.

Example. Consider the Sierpinski Space

$$\Sigma = \big(\{\perp, \top\}, \{\varnothing, \{\top\}, \{\perp, \top\}\}, =\big)$$

in $\mathbb{E}$qu. This triple is indeed a partial equilogical space, since Σ is an algebraic lattice defining $\perp \leq \top$: in the notation of the previous remark, if we take both P and E as Σ we have that $\mathcal{G}(\Sigma) \cong \Sigma$ in $\mathbb{P}\mathbb{E}$qu. Since a map $f\colon \Sigma \to \Sigma$ is continuous if and only if it is monotone we obtain that

$$\begin{aligned} \Sigma^\Sigma &= \{ f\colon \Sigma \to \Sigma \text{ continuous} \} \\ &= \{ const_\perp, \mathrm{id}_\Sigma, const_\top \} \end{aligned}$$

equipped with the Scott topology and the identity relation. We observe that open sets are just the upward closed sets since Σ^Σ is finite.

If F is an arbitrary generic element of Σ^Σ, we will use the notation

$$F = \begin{pmatrix} \perp \mapsto F(\perp) \\ \top \mapsto F(\top) \end{pmatrix}$$

or even

$$F = \begin{pmatrix} F(\perp) \\ F(\top) \end{pmatrix}. \qquad \square$$

Example. An element $F \in \Sigma^{(\Sigma^\Sigma)}$ is of the form

$$\begin{pmatrix} const_\perp \mapsto F(const_\perp) \\ \mathrm{id} \mapsto F(\mathrm{id}) \\ const_\top \mapsto F(const_\top) \end{pmatrix} = \begin{pmatrix} F(const_\perp) \\ F(\mathrm{id}) \\ F(const_\top) \end{pmatrix}$$

where necessarily $F(const_\perp) \leq F(\mathrm{id}) \leq F(const_\top)$ since F is monotone. It follows that

$$\Sigma^{(\Sigma^\Sigma)} = \left\{ \begin{pmatrix} \perp \\ \perp \\ \perp \end{pmatrix}, \begin{pmatrix} \perp \\ \perp \\ \top \end{pmatrix}, \begin{pmatrix} \perp \\ \top \\ \top \end{pmatrix}, \begin{pmatrix} \top \\ \top \\ \top \end{pmatrix} \right\}$$

with the Scott topology and the identity relation. $\square$

Example. Consider the terminal object

$$\mathbb{1} = \big(\{\, * \,\}\,, \mathrm{dscrt}, =\big)$$

and consider the coproduct in $\mathbb{E}\mathrm{qu}$ and $\mathbb{PE}\mathrm{qu}$

$$2 = \mathbb{1} + \mathbb{1} = \big(\{\, 0,\, 1 \,\}\,, \mathrm{dscrt}, =\big).$$

Then we have

$$\begin{aligned}
\Sigma^2 &= \{\, f \colon 2 \to \Sigma \,\} \\
&= \left\{\, const_\bot,\, \begin{pmatrix} 0 \mapsto \bot \\ 1 \mapsto \top \end{pmatrix}, \begin{pmatrix} 0 \mapsto \top \\ 1 \mapsto \bot \end{pmatrix}, const_\top \,\right\} \\
&\cong \{\, (\bot, \bot),\, (\bot, \top),\, (\top, \bot),\, (\top, \top) \,\}
\end{aligned}$$

with the Scott topology and the identity relation. Here open sets are upward-closed sets with respect to the point-wise order. □

Example. Let A be an equilogical space. We have that

$$A^\Sigma = \{\, f \colon \Sigma \to \mathscr{P}\,(\tau_A) \mid f \text{ continuous, } f \equiv_{A^\Sigma} f \,\}\,.$$

By definition, $f \equiv_{A^\Sigma} f$ if and only if $f(\bot) \equiv_{\mathscr{P}(\tau_A)} f(\bot)$ and $f(\top) \equiv_{\mathscr{P}(\tau_A)} f(\top)$, and this is equivalent to say that for every $\sigma \in \Sigma$ there exist a_σ, b_σ in A such that $a_\sigma \equiv_A b_\sigma$ and $f(\sigma) = \tau_A(a_\sigma) = \tau_A(b_\sigma)$. But A is a T_0 space, so

$$A^\Sigma = \{\, f \colon \Sigma \to \mathscr{P}\,(\tau_A) \mid \forall \sigma \in \Sigma \;\, \exists!\, a_\sigma \in A.\, f(\sigma) = \tau_A(a_\sigma) \,\}$$

equipped with the Scott topology and the restriction of the usual partial equivalence relation $\equiv_{A^\Sigma}$.

Conversely,

$$\Sigma^A = \{\, g \colon \mathscr{P}\,(\tau_A) \to \Sigma \mid g \text{ continuous, } g \equiv_{\Sigma^A} g \,\}$$

where $g \equiv_{\Sigma^A} g$ if and only if for every $\mathscr{A}, \mathscr{B} \in \mathscr{P}\,(\tau_A)$ the condition $\mathscr{A} \equiv_{\mathscr{P}(\tau_A)} \mathscr{B}$ implies that $g(\mathscr{A}) = g(\mathscr{B})$. But $\mathscr{A} \equiv_{\mathscr{P}(\tau_A)} \mathscr{B}$ means that there exist $a, b \in A$ such that $a \equiv_A b$, $\mathscr{A} = \tau_A(a)$ and $\mathscr{B} = \tau_A(b)$, so

$$\Sigma^A = \{\, g \colon \mathscr{P}\,(\tau_A) \to \Sigma \mid g \text{ continuous, } \forall a, b \in A \quad a \equiv_A b \implies g\,(\tau_A(a)) = g\,(\tau_A(b)) \,\}\,. \quad \square$$

2.4 The double-exponential monad on Σ

We are now ready to study the algebras for the double-exponential monad $T = \Sigma^{(\Sigma^{(-)})}$ in $\mathbb{E}\mathrm{qu}$. By the general definition of T in a cartesian closed category $\mathbb{C}$, see p. 11, and by definition of the evaluation morphism in $\mathbb{E}\mathrm{qu}$ on p. 42, we have that for every A equilogical space

$$\begin{aligned}
\eta_A &= \big[A \ni a \mapsto \big(\Sigma^A \ni F \mapsto F(\tau_A(a))\big)\big]\,, \\
\mu_A &= \left[\Sigma^{(\Sigma^{(\Sigma^{(\Sigma^A)})})} \ni F \mapsto \big(\Sigma^A \ni G \mapsto F\,(\eta_{\Sigma^A}(G))\big)\right].
\end{aligned}$$

We aim to prove that every T-algebra A gives rise to a frame in a functorial way, that is we want to define a functor $\mathbb{E}\mathrm{qu}^T \to \mathbb{F}\mathrm{rames}$.

Definition 2.38. A *frame* F is a complete lattice satisfying the distributive law:

$$x \wedge \bigvee_{y \in S} y = \bigvee_{y \in S} (x \wedge y)$$

for any $x \in F$ and $S \subseteq F$. The category $\mathbb{F}\mathrm{rames}$ has frames as objects and the maps are functions that preserve finite infs and arbitrary sups.

Toward the definition of the functor $\Gamma \colon \mathbb{E}\mathrm{qu} \to \mathbb{S}\mathrm{et}$, consider the following assignment:

$$\begin{tikzcd}
\Gamma' \colon \mathbb{E}\mathrm{qu} \arrow[r] & \mathbb{S}\mathrm{et} \\
A \arrow[r, mapsto] \arrow[d] & \mathbb{E}\mathrm{qu}(\mathbb{1}, A) \arrow[d] \\
B \arrow[r, mapsto] & \mathbb{E}\mathrm{qu}(\mathbb{1}, B)
\end{tikzcd} \qquad [f] \mapsto f \circ (-)$$

Γ' is obviously a functor, and we observe that

$$\begin{aligned}
\mathbb{E}\mathrm{qu}(\mathbb{1}, A) &= \{\, [f] \mid f \colon \mathbb{1} \to |A| \,\} \\
&= \{\, \{\, g \colon \mathbb{1} \to |A| \mid g(0) \equiv_A a \,\} \mid a \in |A| \,\} \\
&= \{\, \{\, b \in |A| \mid a \equiv_A b \,\} \mid a \in |A| \,\} \\
&= \{\, \bar{a} \mid a \in |A| \,\} \\
&= |A| /_{\equiv_A}.
\end{aligned}$$

So it follows easily that Γ' is isomorphic to the functor

$$\begin{tikzcd}
\Gamma \colon \mathbb{E}\mathrm{qu} \arrow[r] & \mathbb{S}\mathrm{et} \\
A \arrow[r, mapsto] \arrow[d] & |A|/_{\equiv_A} \arrow[d] \\
B \arrow[r, mapsto] & |B|/_{\equiv_B}
\end{tikzcd} \qquad [f] \longmapsto \bar{f}$$

where $\bar{f}(\bar{a}) = \overline{f(a)}$.

Remark 2.39. Γ is clearly faithful.

We aim to prove that, when A is a T-algebra, the set $\Gamma(A)$ is a frame. First we need to show a crucial, though straightforward, property of Γ.

Proposition 2.40. Γ *preserves all products.*

Proof. Let $\left(\prod_{i \in I} A_i, p_i\right)$ be a product of the family $(A_i)_{i \in I}$ in $\mathbb{E}\mathrm{qu}$: it is given by the product of the underlying topological spaces with the product relation. This means that

$$\Gamma\Big(\prod A_i\Big) = \Big(\prod |A_i|\Big)/_{\equiv_{(\prod A_i)}} \underset{\mathbb{S}\mathrm{et}}{\cong} \prod \Big(|A_i|/_{\equiv_{A_i}}\Big) = \prod \big(\Gamma(A_i)\big) \qquad \square$$

In order to prove that $\Gamma(A)$ is a frame, we shall "translate" the frame structure of Σ into $\Gamma(A)$ using the functors $A^{(-)}$ and Γ. First of all, Σ is a partially ordered set (with $\bot \leq \top$) that has two operations of meet and join

$$\wedge\colon \Sigma^2 \to \Sigma \qquad \vee\colon \Sigma^2 \to \Sigma.$$

Both $\wedge$ and $\vee$ are monotone, hence continuous because Σ is finite. The functor $A^{(-)}$ gives rise to two operations on A

$$\wedge_A\colon A^2 \to A \qquad \vee_A\colon A^2 \to A$$

such that for every pair $(a,b) \in A^2$

$$a \wedge_A b = \alpha\left(\Sigma^A \ni g \mapsto g\left(\tau_A(a)\right) \wedge g\left(\tau_A(b)\right)\right),$$
$$a \vee_A b = \alpha\left(\Sigma^A \ni g \mapsto g\left(\tau_A(a)\right) \vee g\left(\tau_A(b)\right)\right).$$

Recall that

$$\Sigma^A = \left\{ g \in \Sigma^{\mathscr{P}(\tau_A)} \mid \forall a, b \in |A| \quad a \equiv_A b \implies g\left(\tau_A(a)\right) = g\left(\tau_A(b)\right) \right\}.$$

Since the functor Γ preserves products, we obtain two binary operations $\wedge_{\Gamma(A)} := \Gamma(\wedge_A)$ and $\vee_{\Gamma(A)} := \Gamma(\vee_A)$ such that for every pair $(\bar{a}, \bar{b}) \in \Gamma(A)^2$

$$\bar{a} \wedge_{\Gamma(A)} \bar{b} = \overline{\alpha\left(\Sigma^A \ni g \mapsto g\left(\tau_A(a)\right) \wedge g\left(\tau_A(b)\right)\right)},$$
$$\bar{a} \vee_{\Gamma(A)} \bar{b} = \overline{\alpha\left(\Sigma^A \ni g \mapsto g\left(\tau_A(a)\right) \vee g\left(\tau_A(b)\right)\right)}.$$

We recall a general proposition about lattices.

Proposition 2.41. *Let X be a set and $\wedge\colon X^2 \to X$, $\vee\colon X^2 \to X$ be two binary operations such that, for every $x, y, z \in X$,*

commutativity *$x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;*

associativity *$(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$;*

absorption laws *$(x \vee y) \wedge x = (x \wedge y) \vee x = x$.*

Then, for every $x, y \in X$,

(i) *$x \wedge x = x$ and $x \vee x = x$;*

(ii) *$(x \wedge y = x) \iff (x \vee y = y)$.*

Moreover, the binary relation on X defined by $[x \leq y \iff x \wedge y = x]$ is a partial order on X and X is a lattice where $\inf\{x, y\} = x \wedge y$ and $\sup\{x, y\} = x \vee y$.

We show that $\wedge_{\Gamma(A)}$ and $\vee_{\Gamma(A)}$ are both associative and commutative and respect the absorption laws, and to do that we express these properties for $\wedge$ and $\vee$ on Σ using commutative diagrams which involve only products and thus are preserved by the functors $A^{(-)}$ and Γ.

Theorem 2.42. *$(\Gamma(A), \wedge_{\Gamma(A)}, \vee_{\Gamma(A)})$ is a lattice.*

Proof. The commutative property of $\wedge$ and $\vee$ on Σ is equivalent to the following commutativities

$$\begin{tikzcd} \Sigma^2 \arrow[rr, "\langle \pi_2, \pi_1 \rangle"] \arrow[dr, "\wedge"'] & \arrow[d, phantom, "/\!/"] & \Sigma^2 \arrow[dl, "\wedge"] \\ & \Sigma & \end{tikzcd} \qquad \begin{tikzcd} \Sigma^2 \arrow[rr, "\langle \pi_2, \pi_1 \rangle"] \arrow[dr, "\vee"'] & \arrow[d, phantom, "/\!/"] & \Sigma^2 \arrow[dl, "\vee"] \\ & \Sigma & \end{tikzcd}$$

while the associative law is equivalent to the commutativity of

$$\begin{tikzcd} \Sigma^3 \arrow[r, "\langle \wedge\langle \pi_1, \pi_2 \rangle, \pi_3 \rangle"] \arrow[d, "\langle \pi_1, \wedge\langle \pi_2, \pi_3 \rangle\rangle"'] \arrow[dr, phantom, "/\!/"] & \Sigma^2 \arrow[d, "\wedge"] \\ \Sigma^2 \arrow[r, "\wedge"] & \Sigma \end{tikzcd} \qquad \begin{tikzcd} \Sigma^3 \arrow[r, "\langle \vee\langle \pi_1, \pi_2 \rangle, \pi_3 \rangle"] \arrow[d, "\langle \pi_1, \vee\langle \pi_2, \pi_3 \rangle\rangle"'] \arrow[dr, phantom, "/\!/"] & \Sigma^2 \arrow[d, "\vee"] \\ \Sigma^2 \arrow[r, "\vee"] & \Sigma \end{tikzcd}$$

Finally, the absorption laws are expressed by

$$\begin{tikzcd} \Sigma^2 \arrow[r, "\langle \vee, \pi_1 \rangle"] \arrow[d, "\langle \wedge, \pi_1 \rangle"'] \arrow[dr, "\pi_1"] & \Sigma^2 \arrow[d, "\wedge"] \\ \Sigma^2 \arrow[r, "\vee"] & \Sigma \end{tikzcd}$$

Applying the functors $A^{(-)}$ and Γ, which preserve products, we obtain again commutative diagrams that are equivalent to the corresponding properties for $\wedge_{\Gamma(A)}$ and $\vee_{\Gamma(A)}$, so the proof is complete. $\square$

Now we want to study the partial order on $\Gamma(A)$ given by

$$\bar{a} \leq \bar{b} \iff \bar{a} \wedge_{\Gamma(A)} \bar{b} = \bar{a}.$$

It turns out, in fact, that this order can be usefully expressed in another way: consider $f\colon 2 \to \Sigma$ in $\mathbb{E}\mathrm{qu}$ such that

$$f = \begin{pmatrix} 0 \mapsto \perp \\ 1 \mapsto \top \end{pmatrix}.$$

Applying the functor $\Sigma^{(-)}\colon \mathbb{E}\mathrm{qu}^{\mathrm{op}} \to \mathbb{E}\mathrm{qu}$ we have $\Sigma^f\colon \Sigma^\Sigma \to \Sigma^2$ where for every $F \in |\Sigma^\Sigma|$

$$\Sigma^f(F) = \big(F(\perp), F(\top)\big).$$

In the following we write φ as an abbreviation for Σ^f.

Proposition 2.43. *φ is a split monomorphism.*

Proof. Consider the function $\psi\colon \Sigma^2 \to \Sigma^\Sigma$ such that

$$\psi(a, b) = \begin{pmatrix} a \wedge b \\ b \end{pmatrix}$$

First of all, $\psi(a,b) \in \Sigma^\Sigma$ because it is monotone and then continuous as a function from Σ to Σ. ψ is in turn continuous since monotone; moreover, ψ preserves the equivalence relations since we have the identity relation on both domain and codomain, thus ψ is a morphism in $\mathbb{E}\mathrm{qu}$.

Finally, we have for every $F \in \Sigma^\Sigma$

$$(\psi \circ \varphi)(F) = \psi(F(\bot), F(\top)) = \binom{F(\bot)}{F(\top)} = F$$

because F is monotone and then $F(\bot) \wedge F(\top) = F(\bot)$. $\square$

In particular, φ is a monomorphism. This means that φ determines a binary relation on Σ. And, its image

$$\operatorname{Im}\varphi = \{\, (\bot,\bot), (\bot,\top), (\top,\top) \,\}$$

coincides with the natural order on Σ given by $\bot \leq \top$.

Consider now the composition $\varphi\psi\colon \Sigma^2 \to \Sigma^2$: we have that for every pair $(a,b) \in \Sigma^2$

$$\varphi\psi(a,b) = (a \wedge b, b).$$

Proposition 2.44. *The fixed points of $\varphi\psi$ are exactly the elements of $\operatorname{Im}\varphi$. Moreover, $\varphi\psi$ is an idempotent operator, that is $\varphi\psi \circ \varphi\psi = \varphi\psi$.*

Proof. Immediate. $\square$

The fact that φ is a split mono is crucial, because it means that also $\varphi_A\colon A^\Sigma \to A^2$ is split mono as every functor preserves split monos. So $\Gamma(\varphi_A)\colon \Gamma(A^\Sigma) \to \Gamma(A^2)$ is split mono as well.

Since $\Gamma(A^2) \cong \Gamma(A)^2$ in $\mathbb{S}\mathrm{et}$, we obtained a binary relation, which we temporarily write $\prec$, on $\Gamma(A)$

$$\bar{a} \prec \bar{b} \iff \big(\bar{a}, \bar{b}\big) \in \operatorname{Im}\Gamma(\varphi_A)$$

where

$$\begin{aligned} \varphi_A\colon\ & A^\Sigma \longrightarrow A^2 \\ & \Big[f \longmapsto \Big(\underbrace{\alpha\Big(\Sigma^A \ni G \mapsto G\big(f(\bot)\big)\Big)}_{f(\bot)}, \underbrace{\alpha\Big(\Sigma^A \ni G \mapsto G\big(f(\top)\big)\Big)}_{f(\top)} \Big) \Big]. \end{aligned}$$

Then

$$\begin{aligned} \Gamma(\varphi_A)\colon\ \Gamma(A^\Sigma) &\longrightarrow \Gamma(A)^2 \\ \overline{f} &\longmapsto \big(\overline{f(\bot)}, \overline{f(\top)}\big) \end{aligned}$$

Necessarily, the fixed points of $\Gamma(\varphi_A) \circ \Gamma(\psi_A) = \Gamma((\varphi\psi)_A)$ are exactly the elements of $\operatorname{Im}\Gamma(\varphi_A)$ since $\Gamma(\varphi_A) \circ \Gamma(\psi_A) \circ \Gamma(\varphi_A) = \Gamma((\varphi\psi\varphi)_A) = \Gamma(\varphi_A)$. Thus every element of $\operatorname{Im}\Gamma(\varphi_A)$ is a fixed point for $\Gamma((\varphi\psi)_A)$ and every fixed point of $\Gamma(\varphi_A) \circ \Gamma(\psi_A)$ is an element of $\operatorname{Im}\Gamma(\varphi_A)$.

We aim to prove that the partial order $\leq$ on $\Gamma(A)$ coincides with the relation $\prec$. We start with a lemma.

Lemma 2.45. *Let $a, b \in A$. Then:*

i. $\bar{a} \wedge_{\Gamma(A)} \bar{b} \prec \bar{a}$;

ii. $\bar{a} \wedge_{\Gamma(A)} \bar{b} \prec \bar{b}$.

Proof. We prove that the pairs $(\bar{a} \wedge_{\Gamma(A)} \bar{b}, \bar{a})$ and $(\bar{a} \wedge_{\Gamma(A)} \bar{b}, \bar{b})$ are fixed point for $\Gamma((\varphi\psi)_A)$. We know that the following diagram commutes:

$$\begin{tikzcd}
\Sigma^2 \arrow[r, "{\langle \wedge, \pi_1 \rangle}"] \arrow[dr, "{\langle \wedge, \pi_1 \rangle}"'] \arrow[dr, phantom, "=", shift left=3] & \Sigma^2 \arrow[d, "\varphi\psi"] & \Sigma^2 \arrow[l, "{\langle \wedge, \pi_2 \rangle}"'] \arrow[dl, "{\langle \wedge, \pi_2 \rangle}"] \arrow[dl, phantom, "=", shift right=3] \\
& \Sigma^2 &
\end{tikzcd}$$

since its commutativity is equivalent to the fact that for every pair (a, b) we have $a \wedge b \leq a$ and $a \wedge b \leq b$. Applying $A^{(-)}$ and Γ we obtain

$$\begin{tikzcd}
\Gamma(A)^2 \arrow[r, "{\langle \wedge_{\Gamma(A)}, \pi_1 \rangle}"] \arrow[dr, "{\langle \wedge_{\Gamma(A)}, \pi_1 \rangle}"'] \arrow[dr, phantom, "=", shift left=3] & \Sigma^2 \arrow[d, "\Gamma((\varphi\psi)_A)"] & \Sigma^2 \arrow[l, "{\langle \wedge_{\Gamma(A)}, \pi_2 \rangle}"'] \arrow[dl, "{\langle \wedge_{\Gamma(A)}, \pi_2 \rangle}"] \arrow[dl, phantom, "=", shift right=3] \\
& \Sigma^2, &
\end{tikzcd}$$

thus the claim. ☐

Theorem 2.46. *Let $\bar{a}, \bar{b} \in \Gamma(A)$. It is $\bar{a} \leq \bar{b}$ if and only if $\bar{a} \prec \bar{b}$.*

Proof. If $\bar{a} \prec \bar{b}$ then by definition $\bar{a} \wedge_{\Gamma(A)} \bar{b} = \bar{a}$ and then for the previous lemma we have $\bar{a} \prec \bar{b}$. Viceversa, we have to prove that $\Gamma((\varphi\psi)_A)(\bar{a}, \bar{b}) = (\bar{a}, \bar{b})$ whenever $\bar{a}$ and $\bar{b}$ are such that $\bar{a} \wedge_{\Gamma(A)} \bar{b} = \bar{a}$. Since $\varphi\psi = \langle \wedge, \pi_2 \rangle$, we have that $\Gamma((\varphi\psi)_A) = \langle \wedge_{\Gamma(A)}, \pi_2 \rangle$ and then

$$\Gamma((\varphi\psi)_A)(\bar{a}, \bar{b}) = (\bar{a} \wedge_{\Gamma(A)} \bar{b}, \bar{b}).$$

Now it is immediate to see why if $\bar{a} \wedge_{\Gamma(A)} \bar{b} = \bar{a}$ then $(\bar{a}, \bar{b})$ is a fixed point for $\Gamma((\varphi\psi)_A)$. ☐

We want now to introduce the definition of *arbitrary* sups on $\Gamma(A)$. The cleanest way is to define

$$\bigvee_I : \Sigma^I \longrightarrow \Sigma$$

on Σ where I is a discrete topological space, and for every $(a_i)_{i \in I} \in \Sigma^I$

$$\bigvee_I ((a_i)_{i \in I}) = \bigvee_{i \in I} a_i.$$

Notice that $I = \coprod_{i \in I} \mathbb{1}$, thus $\Sigma^I = \prod_{i \in I} \Sigma$ (this is why its elements are families of elements of Σ) by Lemma 1.31. We have that

$$\bigvee_I^{-1} \{\top\} = \bigcup_{i \in I} \pi_i^{-1} \{\top\}$$

that is open in the product topology, hence $\bigvee_I$ is continuous. Applying the functors $A^{(-)}$ and Γ, we have that $\Gamma(A^I) \cong \Gamma(A)^I$ since $I = \coprod_{i \in I} \mathbb{1}$. We obtain then an I-ary operation on $\Gamma(A)$, which we still call $\bigvee_I$, such that for every $(\bar{a}_i)_{i \in I} \in \Gamma(A)^I$

$$\bigvee_I ((\bar{a}_i)_{i \in I}) = \overline{\alpha \left(\Sigma^A \ni g \mapsto \bigvee_{i \in I} g(\tau_A(a_i)) \right)}.$$

Instead of $\bigvee_I((\bar{a}_i)_{i\in I})$ we shall simply write $\bigvee_{i\in I}\bar{a}_i$ since, by the following theorem, what we have obtained is indeed the least upper bound of the family $(\bar{a}_i)_{i\in I}$.

Theorem 2.47. *Let $(\bar{a}_i)_{i\in I}$ be a family in $\Gamma(A)$. Then $\bigvee_{i\in I}\bar{a}_i = \sup\{\,\bar{a}_i \mid i\in I\,\}$.*

Proof. We start proving that $\bigvee_{i\in I}\bar{a}_i \geq \bar{a}_i$ for every $i\in I$. This can be achieved by showing that the following triangle commutes

$$\begin{tikzcd}
\Gamma(A)^I \arrow[r, "\langle \pi_i, \bigvee_I\rangle"] \arrow[dr, "\langle \pi_i, \bigvee_I\rangle"'] & \Gamma(A)^2 \arrow[d, "\Gamma((\varphi\psi)_A)"] \\
& \Gamma(A)^2
\end{tikzcd}$$

for every $i\in I$. But we know that for every $i\in I$

$$\begin{tikzcd}
\Sigma^I \arrow[r, "\langle \pi_i, \bigvee_I\rangle"] \arrow[dr, "\langle \pi_i, \bigvee_I\rangle"'] \arrow[dr, phantom, shift left=3, "/\!/"] & \Sigma^2 \arrow[d, "\varphi\psi"] \\
& \Sigma^2
\end{tikzcd}$$

So the first diagram commutes too.

In order to prove that $\bigvee_{i\in I}\bar{a}_i$ is the least among the upper bounds of the set $\{\,\bar{a}_i \mid i\in I\,\}\subseteq \Gamma(A)$, we prove a stronger condition, that is

$$\left[\forall i\in I \quad \bar{b}_i \leq \bar{a}_i\right] \implies \bigvee_{i\in I}\bar{b}_i \leq \bigvee_{i\in I}\bar{a}_i. \tag{2.1}$$

As usual, we show that the corresponding statement for Σ holds because equivalent to the commutativity of a diagram. We have to consider

$$\Sigma^{2\times I} \cong \left(\Sigma^2\right)^I \cong \left(\Sigma^I\right)^2$$

using each time the appropriate isomorphic version of $\Sigma^{2\times I}$. The required condition is given by

$$\begin{tikzcd}
\Sigma^{2\times I} \arrow[r, "(\varphi\psi)^I"] & \Sigma^{2\times I} \arrow[r, "\bigvee_I\times\bigvee_I"] \arrow[dr, "\bigvee_I\times\bigvee_I"'] \arrow[dr, phantom, shift left=3, "/\!/"] & \Sigma^2 \arrow[d, "\varphi\psi"] \\
& & \Sigma^2
\end{tikzcd}$$

where

$$(\varphi\psi)^I\left((b_i,a_i)_{i\in I}\right) = (b_i\wedge a_i, a_i)_{i\in I}.$$

Now, the condition (2.1) implies that if for every $i \in I$ one has that $\bar{b}_i \leq \bar{a}$, then $\bigvee_{i \in I} \bar{b}_i \leq \bar{a}$, since

$$\begin{tikzcd} \Sigma \arrow[r, "\langle \mathrm{id}_\Sigma \rangle_{i \in I}"] \arrow[dr, "\mathrm{id}_\Sigma"'] & \Sigma^I \arrow[d, "\bigvee_I"] \arrow[dl, phantom, "/\!/"] \\ & \Sigma. \end{tikzcd}$$

Hence

$$\begin{tikzcd} \Gamma(A) \arrow[r, "\langle \mathrm{id}_{\Gamma(A)} \rangle_{i \in I}"] \arrow[dr, "\mathrm{id}_{\Gamma(A)}"'] & \Gamma(A)^I \arrow[d, "\bigvee_I"] \arrow[dl, phantom, "/\!/"] \\ & \Gamma(A) \end{tikzcd}$$

□

Corollary 2.48. $\Gamma(A)$ *is a complete lattice.*

Proof. Given $S \subseteq \Gamma(A)$, we consider the family $(\bar{s})_{\bar{s} \in S}$ ($I = S$ in the above notations) and then we simply have thanks to the previous theorem

$$\sup S = \bigvee_{\bar{s} \in S} \bar{s} = \overline{\alpha \left(\Sigma^A \ni g \mapsto \bigvee_{\bar{s} \in S} g \left(\tau_A(s) \right) \right)}$$

□

Finally, we are ready to prove our goal.

Theorem 2.49. $\Gamma(A)$ *is a frame.*

Proof. We have to show that the distributive law holds:

$$\bar{a} \wedge \left(\bigvee_{i \in I} \bar{b}_i \right) = \bigvee_{i \in I} \left(\bar{a} \wedge \bar{b}_i \right).$$

As usual, we express this law for Σ using a commutative diagram. In order to define a map $\Sigma \times \Sigma^I \to \Sigma^I$ that given $(a, (b_i)_{i \in I})$ create the family $(a \wedge b_i)_{i \in I}$, we have to consider

$$\beta := \lambda \left(\Sigma \times \Sigma^I \times I \xrightarrow{\langle \pi_1, eval \langle \pi_2, \pi_3 \rangle \rangle} \Sigma \times \Sigma \xrightarrow{\wedge} \Sigma \right)$$

and then we have

$$\begin{tikzcd} \Sigma \times \Sigma^I \arrow[r, "\mathrm{id} \times \bigvee_I"] \arrow[d, "\beta"'] \arrow[dr, phantom, "/\!/"] & \Sigma \times \Sigma \arrow[d, "\wedge"] \\ \Sigma^I \arrow[r, "\bigvee_I"] & \Sigma \end{tikzcd}$$

It follows that

$$
\begin{tikzcd}
\Gamma(A) \times \Gamma(A)^I \arrow[r, "\mathrm{id} \times \bigvee_I"] \arrow[d, "\Gamma(\beta_A)"'] \arrow[dr, phantom, "/\!/"] & \Gamma(A) \times \Gamma(A) \arrow[d, "\wedge_{\Gamma(A)}"] \\
\Gamma(A)^I \arrow[r, "\bigvee_I"] & \Gamma(A)
\end{tikzcd}
$$

We are left to show that $\Gamma(\beta_A)(\bar{a}, (\bar{b}_i)_{i\in I}) = (\bar{a} \wedge_{\Gamma(A)} \bar{b}_i)_{i\in I}$. By definition of Γ,

$$\Gamma(\beta_A)(\bar{a}, (\bar{b}_i)_{i\in I}) = \overline{\beta_A\,(a, (b_i)_{i\in I})}$$

while by definition of $A^{(-)}$,

$$
\begin{aligned}
\beta_A &= \left[\!\left[(a, F) : A \times A^I \vdash \lambda i : I.\, \alpha\left(\lambda G : \Sigma^A.\, [\lambda j : I.\, G(a) \wedge G(F(j))]\,(i)\right) : A^I\right]\!\right] \\
&= \left[\!\left[(a, F) : A \times A^I \vdash \lambda i : I.\, \alpha\left(\lambda G : \Sigma^A.\, G(a) \wedge G(F(i))\right) : A^I\right]\!\right] \\
&= \left[\!\left[(a, F) : A \times A^I \vdash \lambda i : I.\, a \wedge_A F(i) : A^I\right]\!\right].
\end{aligned}
$$

Then

$$\beta_A(a, (b_i)_{i\in I}) = (a \wedge_A b_i)_{i\in I}$$

and

$$
\begin{aligned}
\overline{\beta_A\,(a, (b_i)_{i\in I})} &= \overline{(a \wedge_A b_i)_{i\in I}} \\
&= (\overline{a \wedge_A b_i})_{i\in I} \\
&= (\bar{a} \wedge_{\Gamma(A)} \bar{b}_i)_{i\in I}
\end{aligned}
$$

as required. □

Chapter 3

The inverse functor

We have proved that, if we restrict the functor $\Gamma\colon \mathbb{E}\mathrm{qu} \to \mathbb{S}\mathrm{et}$ to the category of $\Sigma^{(\Sigma^{(-)})}$-algebras $\mathbb{E}\mathrm{qu}_{\Sigma^{(\Sigma^{(-)})}}$ then it can be extended to produce a functor into the category of $\mathbb{F}\mathrm{rames}$. We now intend to address the question if Γ is an equivalence of categories

$$\begin{tikzcd} \mathbb{E}\mathrm{qu}^{\Sigma^{(\Sigma^{(-)})}} \arrow[rr, bend left, "\Gamma"] & & \mathbb{F}\mathrm{rames} \arrow[ll, bend left, "?"] \end{tikzcd}$$

At the moment we can only offer a partial answer to the question. We shall thus limit ourselves to propose where the study has gone so far, in order to study the open problem.

3.1 Enriched functors and $\Sigma^{(\Sigma^{-})}$-algebras

In this section we show a category-theoretical approach that highlights several properties of $X^{(X^{(-)})}$-algebras in an arbitrary cartesian closed category $\mathbb{C}$ and applies usefully to the case at hand.

We first need a preliminary definition of category enriched over a category with finite products $\mathbb{C}$. This definition can be generalized considering $\mathbb{C}$ as a monoidal category, but that generalization is not relevant for our purposes.

Definition 3.1. Let $\mathbb{C}$ be a category with finite products. A category $\mathbb{D}$ *enriched over* $\mathbb{C}$, or $\mathbb{C}$-*category*, consists of

- a class of objects $\mathrm{Ob}(\mathbb{D})$;
- a *hom-object* $[A, B] \in \mathrm{Ob}(\mathbb{C})$ for every pair $(A, B) \in \mathrm{Ob}(\mathbb{D}) \times \mathrm{Ob}(\mathbb{D})$;
- a morphism $u_A\colon \mathbb{1} \to [A, A]$ in $\mathbb{C}$ for every $A \in \mathrm{Ob}(\mathbb{D})$;
- a morphism
$$c_{A,B,C}\colon [A, B] \times [B, C] \to [A, C]$$
in $\mathbb{C}$ for every triple $(A, B, C) \in \mathrm{Ob}(\mathbb{D}) \times \mathrm{Ob}(\mathbb{D}) \times \mathrm{Ob}(\mathbb{D})$;

which satisfy the following axioms

associativity: for every $A, B, C, D \in \mathrm{Ob}(\mathbb{D})$

$$\begin{tikzcd}
{[A,B] \times [B,C] \times [C,D]} \arrow[r, "\langle c_{A,B,C}\langle \pi_1, \pi_2\rangle, \pi_3\rangle"] \arrow[d, "\langle \pi_1, c_{B,C,D}\langle \pi_2, \pi_3\rangle\rangle"'] \arrow[dr, phantom, "/\!/"] & {[A,C] \times [C,D]} \arrow[d, "c_{A,C,D}"] \\
{[A,B] \times [B,D]} \arrow[r, "c_{A,B,D}"] & {[A,D]}
\end{tikzcd}$$

unit: for every $A, B \in \mathrm{Ob}(\mathbb{D})$

$$\begin{tikzcd}
\mathbb{1} \times [A,B] \arrow[r, "u_A \times \mathrm{id}"] \arrow[dr, "\pi_2"'] & {[A,A] \times [A,B]} \arrow[d, "c_{A,A,B}"] \\
& {[A,B]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A,B] \times \mathbb{1}} \arrow[r, "\mathrm{id} \times u_B"] \arrow[dr, "\pi_1"'] & {[A,B] \times [B,B]} \arrow[d, "c_{A,B,B}"] \\
& {[A,B]}
\end{tikzcd}$$

Remark 3.2. A $\mathbb{S}\mathrm{et}$-enriched category is just a category.

Remark 3.3. Given a functor $F : \mathbb{C} \to \mathbb{C}'$ which preserves finite products and a $\mathbb{C}$-enriched category $\mathbb{D}$, it is easy to obtain a $\mathbb{C}'$-enriched category $\mathbb{D}'$ by applying F to the data of $\mathbb{D}$.

Definition 3.4. Let $\mathbb{D}$ be a $\mathbb{C}$-category. The *underlying category* of $\mathbb{D}$ is the category $\mathbb{D}_0$ obtained by applying the product preserving functor $U = \mathbb{C}(\mathbb{1}, -) : \mathbb{C} \to \mathbb{S}\mathrm{et}$:

- $\mathrm{Ob}(\mathbb{D}_0) = \mathrm{Ob}(\mathbb{D})$,
- $\mathbb{D}'(A, B) = U([A, B])$ for every $A, B \in \mathrm{Ob}(\mathbb{D}_0)$,
- the identity morphism of A in $\mathbb{D}_0$ is u_A for every $A \in \mathrm{Ob}(\mathbb{D}_0)$,
- for every $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{D}'$, the composition of f with g is given by

$$g \circ_{\mathbb{D}_0} f := c_{A,B,C} \circ_{\mathbb{C}} \langle f, g\rangle : \mathbb{1} \xrightarrow[\mathbb{C}]{} [A, C].$$

Proposition 3.5. *Let $\mathbb{C}$ be a cartesian closed category. Then $\mathbb{C}$ is enriched over itself. Moreover, the* $\mathbb{S}\mathrm{et}$*-enriched category obtained from it by the action of U is isomorphic to the category $\mathbb{C}$.*

Proof. We define for every $A, B \in \mathrm{Ob}(\mathbb{C})$

$$[A, B] := B^A.$$

Now we need to define a morphism $c_{A,B,C} : B^A \times C^B \to C^A$ in $\mathbb{C}$ that has to play the role of "composition": the natural definition is

$$\begin{aligned} c_{A,B,C} &= [\![f : B^A,\, g : C^B \vdash \lambda a : A.\, g\,(f(a)) : C^A]\!] \\ &= [\![f : B^A,\, g : C^B \vdash g \circ f : C^A]\!] \end{aligned}$$

where $g \circ f$ is only a syntactic abbreviation for $\lambda a : A.\, g\,(f(a))$. Note that f and g are *not* morphisms of $\mathbb{C}$ and $g \circ f$ is meaningless by itself. Yet when we write in internal language, and focus only on what is on the right-hand side of the turnstyle $\vdash$, f and g are not morphisms of $\mathbb{C}$ in any case: we treat $a : A$, $b : B$, $c : C$ as elements of sets and f, g as if they were set functions from A to B and from B to C respectively, so "$\lambda a : A.\, g\,(f(a))$" does express the composition of f with g as functions. In this spirit, we define the identity element $u_A \colon \mathbb{1} \to A^A$ as

$$\begin{aligned} u_A &= \llbracket () : \mathit{unit} \vdash \lambda a : A.\, a : A^A \rrbracket \\ &= \llbracket () : \mathit{unit} \vdash \mathrm{id}_A : A^A \rrbracket \end{aligned}$$

Again, we have to be careful: here "id_A" is not the identity morphism $\mathrm{id}_A \colon A \to A$ in $\mathbb{C}$, but only an abbreviation of "$\lambda a : A.\, a$" that would express the identity *function* on the *set* A.

Now, the associativity axiom requires that

$$\llbracket f : B^A,\, g : C^B,\, h : D^C \vdash h \circ (g \circ f) : D^A \rrbracket = \llbracket f : B^A,\, g : C^B,\, h : D^C \vdash (h \circ g) \circ f : D^A \rrbracket$$

which is true because application is sequential, while the unit axioms requires that

$$\llbracket () : \mathit{unit}, f : B^A \vdash f \circ \mathrm{id}_A : B^A \rrbracket = \llbracket () : \mathit{unit}, f : B^A \vdash f : B^A \rrbracket$$

and

$$\llbracket f : B^A, () : \mathit{unit} \vdash \mathrm{id}_B \circ f : B^A \rrbracket = \llbracket f : B^A, () : \mathit{unit} \vdash f : B^A \rrbracket$$

and these equations hold because in $\mathbb{S}\mathrm{et}$ the identity function is the unit of composition (or, more precisely, thanks to the axioms and rules of the internal language).

Given $\mathbb{D}$ the category enriched over $\mathbb{C}$ so defined, we want to show that its underlying category $\mathbb{D}_0$ is isomorphic to $\mathbb{C}$. By definition, $\mathrm{Ob}(\mathbb{D}_0) = \mathrm{Ob}(\mathbb{D}) = \mathrm{Ob}(\mathbb{C})$, while for every $A, B \in \mathrm{Ob}(\mathbb{D}_0)$

$$\mathbb{D}_0(A, B) = U\left([A, B]\right) = U\left(B^A\right) = \mathbb{C}(\mathbb{1}, B^A) \underset{\mathbb{S}\mathrm{et}}{\cong} \mathbb{C}(A, B)$$

where the last bijection is so defined: if $f \colon \mathbb{1} \to B^A$ in $\mathbb{C}$, then

$$\hat{f} = \llbracket a : A \vdash f(())(a) : B \rrbracket : A \to B.$$

We define then

$$\begin{tikzcd} (\hat{-}) \colon \mathbb{D}_0 \arrow[r] & \mathbb{C} \\ A \arrow[d, "f"'] \arrow[r, mapsto] & A \arrow[d, "\hat{f}"] \\ B \arrow[r, mapsto] & B \end{tikzcd} \qquad f \longmapsto \hat{f}$$

that is a functor $\widehat{u_A} = \llbracket a : A \vdash u(())(a) : A \rrbracket = \llbracket a : A \vdash a : A \rrbracket = \mathrm{id}_A$. Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbb{D}_0$ it is

$$\hat{g} \circ \hat{f} = \llbracket a : A \vdash g(())\,(f(())(a)) : C \rrbracket$$

So

$$\begin{aligned} \widehat{g \circ f} &= \llbracket a : A \vdash (c_{A,B,C} \circ (f \times g))\,(())(a) : C \rrbracket \\ &= \llbracket a : A \vdash c_{A,B,C}\,(f \times g(()))\,(a) : C \rrbracket \\ &= \llbracket a : A \vdash c_{A,B,C}\,(f(()), g(()))\,(a) : C \rrbracket \\ &= \llbracket a : A \vdash g(())\,(f(())(a)) : C \rrbracket \\ &= \hat{g} \circ \hat{f}. \end{aligned}$$

Analogously, the inverse functor is given by

$$\begin{array}{rcl} \lambda\colon \mathbb{C} & \longrightarrow & \mathbb{D}_0 \\ A & \longmapsto & A \\ \Big\downarrow f & \mapsto \lambda(f) & \Big\downarrow \\ B & \longmapsto & B \end{array}$$

(where $\lambda(f) = [\![() : \mathit{unit} \vdash \lambda a : A.\, f(a) : B^A]\!]$). $\square$

Theorem 3.6. *Let $\mathbb{C}$ be a cartesian closed category, $X \in \mathrm{Ob}(\mathbb{C})$. Then $\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$ is enriched over $\mathbb{C}$.*

Proof. Consider $\mathbb{D}$ defined as follows:

- $\mathrm{Ob}(\mathbb{D}) = \mathrm{Ob}(\mathbb{C})$,
- for every $J, K \in \mathrm{Ob}(\mathbb{D})$ $\quad [J, K] = \left(X^K\right)^{\left(X^J\right)}$,
- for every $J, K, L \in \mathrm{Ob}(\mathbb{D})$

$$c_{J,K,L} = \left[\!\!\left[F : \left(X^K\right)^{\left(X^J\right)}, G : \left(X^L\right)^{\left(X^K\right)} \vdash \lambda f : X^J.\, G(F(f)) : \left(X^L\right)^{\left(X^J\right)}\right]\!\!\right]$$

- for every $J \in \mathrm{Ob}(\mathbb{D})$

$$u_J = \left[\!\!\left[() : \mathit{unit} \vdash \lambda f : X^J.\, F : \left(X^J\right)^{\left(X^J\right)}\right]\!\!\right].$$

Then $\mathbb{D}$ is a $\mathbb{C}$-category with an argument similar to that of the previous theorem. Moreover, the underlying category $\mathbb{D}_0$ is such that

- $\mathrm{Ob}(\mathbb{D}_0) = \mathrm{Ob}(\mathbb{D}) = \mathrm{Ob}(\mathbb{C}) = \mathrm{Ob}(\mathbb{A}\mathrm{lg}_{\mathrm{X}}^{\mathbb{C}})$,
- for every $J, K \in \mathrm{Ob}(\mathbb{D}_0)$

$$\mathbb{D}_0(J, K) = U\left(\left(X^K\right)^{\left(X^J\right)}\right) = \mathbb{C}\left(\mathbb{1}, \left(X^K\right)^{\left(X^J\right)}\right) \underset{\mathbb{S}\mathrm{et}}{\cong} \mathbb{C}\left(X^J, X^K\right) = \mathbb{A}\mathrm{lg}_X^{\mathbb{C}}(J, K)$$

and thus $\mathbb{D}_0 \cong \mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$ because the above bijection is functorial. $\square$

Definition 3.7. Let $\mathbb{C}$ be a cartesian closed category, $\mathbb{D}$ a category enriched over $\mathbb{C}$, $K \in \mathrm{Ob}(\mathbb{D})$ and $P \in \mathrm{Ob}(\mathbb{C})$. We define the *power of K by P* as an object $P \pitchfork K$ of $\mathbb{D}$ such that, for every $J \in \mathrm{Ob}(\mathbb{D})$, there is a natural isomorphism in $\mathbb{C}$ $[J, K]^P \longrightarrow [J, P \pitchfork K]$.

Proposition 3.8. *Let $\mathbb{C}$ be a cartesian closed category. Then $\mathbb{C}$ has all powers.*

Proof. Given $P, K \in \mathrm{Ob}(\mathbb{C})$, the power of K by P is simply $P \pitchfork K := K^P$, since we have that for every $J \in \mathrm{Ob}(\mathbb{C})$

$$[J, K]^P = \left(K^J\right)^P \cong K^{J \times P} \cong K^{P \times J} \cong \left(K^P\right)^J = [J, K^P].$$ $\square$

Proposition 3.9. *Let $\mathbb{C}$ be a cartesian closed category, $X \in \mathrm{Ob}(\mathbb{C})$. Then $\mathbb{A}\mathrm{lg}_X^{\mathbb{C}}$ has all powers.*

Proof. We know that $\mathrm{Ob}(\mathbb{A}\mathrm{lg}^{\mathbb{C}}_{\mathrm{X}}) = \mathrm{Ob}(\mathbb{C})$: given then P and K in $\mathbb{C}$, we define $P \pitchfork K := K \times P$ and we have

$$[J,K]^P = \left(\left(X^K\right)^{\left(X^J\right)}\right)^P \cong \left(X^K\right)^{\left(X^J \times P\right)} \cong \left(\left(X^K\right)^P\right)^{\left(X^J\right)} \cong \left(X^{(K\times P)}\right)^{\left(X^J\right)} = [J, K\times P]. \ \square$$

Definition 3.10. Let $\mathbb{D}$ and $\mathbb{E}$ be two enriched category over a category $\mathbb{C}$ with finite products. An *enriched functor* $T\colon \mathbb{D} \to \mathbb{E}$ consists of

- a function $T\colon \mathrm{Ob}(\mathbb{D}) \to \mathrm{Ob}(\mathbb{E})$,
- a $\mathrm{Ob}(\mathbb{D}) \times \mathrm{Ob}(\mathbb{D})$-indexed collection of morphisms in $\mathbb{C}$

$$T_{A,B}\colon\ [A,B]_{\mathbb{D}} \xrightarrow[\mathbb{C}]{} [T(A),T(B)]_{\mathbb{E}}$$

such that the following diagrams commute for every $A, B, C \in \mathrm{Ob}(\mathbb{D})$:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "u^{\mathbb{D}}_A"] \arrow[dr, "u^{\mathbb{E}}_{T(A)}"'] & {[A,A]_{\mathbb{D}}} \arrow[d, "T_{A,A}"] \\
 & {[T(A),T(A)]_{\mathbb{E}}}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[A,B]_{\mathbb{D}} \times [B,C]_{\mathbb{D}}} \arrow[r, "c^{\mathbb{D}}_{A,B,C}"] \arrow[d, "T_{A,B}\circ\pi_1 \times T_{B,C}\circ\pi_2"'] & {[A,C]_{\mathbb{D}}} \arrow[d, "T_{A,C}"] \\
{[T(A),T(B)]_{\mathbb{E}} \times [T(B),T(C)]_{\mathbb{E}}} \arrow[r, "c^{\mathbb{E}}_{T(A),T(B),T(C)}"] & {[T(A),T(C)]_{\mathbb{E}}}
\end{tikzcd}$$

The *underlying functor* $T_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ between the underlying ordinary categories is defined:

- on objects by $T\colon \mathrm{Ob}(\mathbb{D}) \to \mathrm{Ob}(\mathbb{E})$,
- on morphisms by $\mathbb{C}(\mathbb{1}, -)$: for every $A, B \in \mathrm{Ob}(\mathbb{D})$

$$\begin{array}{ccccccc}
\mathbb{D}_0(A,B) & = & \mathbb{C}(\mathbb{1},[A.B]) & \longrightarrow & \mathbb{E}_0(T(A),T(B)) & = & \mathbb{C}(\mathbb{1},[T(A),T(B)]) \\
 & & f & \longmapsto & T_{A,B}\circ f & &
\end{array}$$

Proposition 3.11. *Let $\mathbb{C}$ be a cartesian closed category, $X \in \mathrm{Ob}(\mathbb{C})$, (A,α) a $X^{(X^{(-)})}$-algebra. Then $A^{(-)}\colon \mathbb{A}\mathrm{lg}^{\mathbb{C}}_X \to \mathbb{C}$ is an enriched functor.*

Proof. We have to define for every $J, K \in \mathrm{Ob}(\mathbb{C})$ a morphism in $\mathbb{C}$ $A^{(-)}_{J,K}\colon \left(X^K\right)^{\left(X^J\right)} \to \left(A^K\right)^{\left(A^J\right)}$. We know that the functor $A^{(-)}$ transforms a morphism $f\colon X^J \to X^K$ in $f_A\colon A^J \to A^K$ where

$$f_A = \left[\!\!\left[F : A^J \vdash \lambda k : K.\, \alpha\Big(\lambda G : X^A.\, \Big[f\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big](k)\Big) : A^K \right]\!\!\right].$$

We define then

$$A^{(-)}_{J,K} = \\ = \left[\!\!\left[\mathscr{F} : \left(X^K\right)^{\left(X^J\right)} \vdash \lambda F : A^J.\, \bigg(\lambda k : K.\, \alpha\Big(\lambda G : X^A.\, \Big[\mathscr{F}\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big](k)\Big)\bigg) : \left(A^K\right)^{\left(A^J\right)} \right]\!\!\right]$$

where we treat the pseudo-element $\mathscr{F}$ as it were a morphism in $\mathbb{C}$ $\mathscr{F}\colon X^J \to X^K$ and then available to be considered by the action of $A^{(-)}$. If we abbreviate

$$\lambda k : K.\, \alpha\Big(\lambda G : X^A.\, \Big[\mathscr{F}\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big](k)\Big)$$

as $\mathscr{F}_A(F)$, we then write

$$A^{(-)}_{J,K} = \Big[\!\!\Big[\mathscr{F} : \big(X^K\big)^{\left(X^J\right)} \vdash \lambda F : A^J.\, \mathscr{F}_A(F) : \big(A^K\big)^{\left(A^J\right)}\Big]\!\!\Big]$$

and even more

$$A^{(-)}_{J,K} = \Big[\!\!\Big[\mathscr{F} : \big(X^K\big)^{\left(X^J\right)} \vdash \mathscr{F}_A : \big(A^K\big)^{\left(A^J\right)}\Big]\!\!\Big]\,.$$

We have now to prove that our morphism $A^{(-)}_{J,K}$ preserves units and compositions of the $\mathbb{C}$-categories $\mathbb{A}\mathrm{lg}^{\mathbb{C}}_X$ and $\mathbb{C}$ itself, but we claim that *we already did* when we proved that $A^{(-)}$ is a functor, because in that proof we used only properties within the internal language: for $f\colon X^J \to X^K$ and $g\colon X^K \to X^L$ are two morphisms in $\mathbb{C}$, we proved the following derived rules in the $\lambda\times$-theory.

- Preservation of identities:

$$\frac{Sg \;\rhd\; a : X^J \vdash a : X^J}{Th \;\rhd\; F : A^J \vdash \lambda j : J.\, \alpha\Big(\lambda G : X^A.\, \Big[\lambda k : J.\, G\big(F(k)\big)\Big](j)\Big) = F}$$

  that is,

$$\frac{Sg \;\rhd\; a : X^J \vdash a : X^J}{Th \;\rhd\; F : A^J \vdash \left(\mathrm{id}_{X^J}\right)_A(F) = \mathrm{id}_{A^J}(F)}$$

  This rule proves the first axiom for enriched functors, since

$$\begin{aligned} A^{(-)}_{J,J} \circ u_J &= \Big[\!\!\Big[() : \mathit{unit} \vdash \lambda F : A^J.\, \lambda j : J.\, \alpha\Big(\lambda G : X^A.\, \Big[\lambda k : J.\, G\big(F(k)\big)\Big](j)\Big) : \big(A^J\big)^{\left(A^J\right)}\Big]\!\!\Big] \\ &= \Big[\!\!\Big[() : \mathit{unit} \vdash \lambda F : A^J.\, F : \big(A^J\big)^{\left(A^J\right)}\Big]\!\!\Big] \\ &= u_{A^J}. \end{aligned}$$

- Preservation of compositions:

$$\frac{Sg \;\rhd\; a : X^J \vdash f(a) : X^K \qquad Sg \;\rhd\; b : X^K \vdash g(b) : X^L}{Th \;\rhd\; F : A^J \vdash (gf)_A(F) = (g_A \circ f_A)(F)}$$

  where the last line is an abbreviation of

$$\begin{aligned} Th \;\rhd\; F : A^J \vdash \lambda l : L.\, \alpha\Big(\lambda G : X^A.\, \Big[g\Big(f\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big)\Big](l)\Big) &= \\ = \lambda l : L.\, \alpha\Big(\lambda G : X^A.\, \Big[g\Big(\lambda j : J.\, G\big(f_A(F)(j)\big)\Big)\Big](l)\Big) \end{aligned}$$

As we can see, we expressed the fact that $\mathrm{id}_{X^J} : X^J \to X^J$ is a morphism of $\mathbb{C}$ by assuming in hypothesis that $a : X^J \vdash a : X^J$ is a proved term (and it is because it is an axiom of the theory, see **Variables**, p. 3), while the fact that f and g are morphisms of $\mathbb{C}$ assures that $a : X^J \vdash f(a) : X^K$ and $b : X^K \vdash g(b) : X^L$ are proved terms by definition of internal language.

Now, the following are proved terms thanks to **Function Symbols** rule on p. 4:

$$\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)}, a : X^J \vdash \mathscr{F}(a) : X^K$$
$$\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)}, b : X^K \vdash \mathscr{G}(b) : X^L.$$

Then, by the **Weakening** rule of p. 5), we obtain

$$\mathit{Th} \rhd \mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)}, F : A^J \vdash \left(\mathscr{G}\mathscr{F}\right)_A(F) = \mathscr{G}_A \circ \mathscr{F}_A(F),$$

where we use the following abbreviations

$$\left(\mathscr{G}\mathscr{F}\right)_A(F) = \lambda l : L.\, \alpha\Big(\lambda G : X^A.\, \Big[\mathscr{G}\Big(\mathscr{F}\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big)\Big](l)\Big)$$
$$\mathscr{G}_A \circ \mathscr{F}_A(F) = \mathscr{G}_A\left(\mathscr{F}_A(F)\right).$$

This is exactly what we need to prove that $A^{(-)}_{J,K}$ preserves compositions: indeed

$$\begin{aligned}
&A^{(-)}_{J,L} \circ c_{X^J,X^K,X^L} = \\
&= \Big[\!\Big[\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)} \vdash \\
&\qquad \lambda F : A^J.\, \Big(\lambda l : L.\, \alpha\Big(\lambda G : X^A.\, \Big[\mathscr{G}\Big(\mathscr{F}\Big(\lambda j : J.\, G\big(F(j)\big)\Big)\Big)\Big](l)\Big)\Big) : \left(A^L\right)^{\left(A^J\right)}\Big]\!\Big] \\
&= \Big[\!\Big[\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)} \vdash \lambda F : A^J.\, \mathscr{G}_A\left(\mathscr{F}_A(F)\right) : \left(A^L\right)^{\left(A^J\right)}\Big]\!\Big] \\
&= \Big[\!\Big[\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)} \vdash c_{A^J,A^K,A^L}\left(\mathscr{F}_A, \mathscr{G}_A\right) : \left(A^L\right)^{\left(A^J\right)}\Big]\!\Big] \\
&= \Big[\!\Big[\mathscr{F} : \left(X^K\right)^{\left(X^J\right)}, \mathscr{G} : \left(X^L\right)^{\left(X^K\right)} \vdash c_{A^J,A^K,A^L}\left(A^{(-)}_{J,K}(\mathscr{F}), A^{(-)}_{K,L}(\mathscr{G})\right) : \left(A^L\right)^{\left(A^J\right)}\Big]\!\Big] \\
&= c_{A^J,A^K,A^L} \circ \left(A^{(-)}_{J,K}\pi_1 \times A^{(-)}_{K,L}\pi_2\right).
\end{aligned}$$

Finally, applying the functor $\mathbb{C}(\mathbb{1}, -)$ to $A^{(-)}_{J,K}$ we obtain

$$\begin{tikzcd}
f \arrow[ddd, mapsto, bend right] & \mathbb{C}\left(X^J, X^K\right) \arrow[d, phantom, "\cong"] & & \mathbb{C}\left(A^J, A^K\right) \arrow[d, phantom, "\cong"] & f_A \\
& C\left(\mathbb{1}, \left(X^K\right)^{\left(X^J\right)}\right) \arrow[rr] & & \mathbb{C}\left(\mathbb{1}, \left(A^K\right)^{\left(A^J\right)}\right) & \\
& & & & \\
\Big[\!\Big[() : \mathit{unit} \vdash \lambda a : X^J.\, f(a) : \left(X^K\right)^{\left(X^J\right)}\Big]\!\Big] \arrow[rrr, mapsto] & & & \Big[\!\Big[() : \mathit{unit} \vdash \lambda F : A^J.\, f_A(F) : \left(A^K\right)^{\left(A^J\right)}\Big]\!\Big] \arrow[ruuu, bend right] &
\end{tikzcd}$$

Thus the underlying functor is indeed $A^{(-)} \colon \mathbb{A}\mathrm{lg}^{\mathbb{C}}_X \to \mathbb{C}$. □

Proposition 3.12. *Let $\mathbb{C}$ be a cartesian closed category, $X \in \mathrm{Ob}(\mathbb{C})$, (A, α) a $X^{(X^{(-)})}$-algebra. Then $A^{(-)} \colon \mathbb{A}\mathrm{lg}_X^{\mathbb{C}} \to \mathbb{C}$ preserves all powers.*

Proof. Let $K, P \in \mathrm{Ob}(\mathbb{C})$. Then

$$A^{(-)}(P \pitchfork K) = A^{(-)}(K \times P) = A^{K \times P} \cong \left(A^K\right)^P = \left(A^{(-)}(K)\right)^P = P \pitchfork A^{(-)}(K). \qquad \square$$

We have seen then that every $X^{(X^{(-)})}$-algebra (A, α) gives rise to an enriched functor that preserves all powers. We have tried to analyze if the converse is true: given $F \colon \mathbb{A}\mathrm{lg}_X^{\mathbb{C}} \to \mathbb{C}$ an enriched functor preserving all powers, is it of the form $F \cong A^{(-)}$ for some A $X^{(X^{(-)})}$-algebra? If the answer is positive, as every frame would give rise to such a functor, we would have found a way to define the inverse of $\Gamma \colon \mathbb{E}\mathrm{qu}_{\Sigma^{(\Sigma^{(-)})}} \longrightarrow \mathbb{F}\mathrm{rames}$.

First of all we have that, for every $K, P \in \mathrm{Ob}(\mathbb{C})$,

$$F(P)^K \cong F(P \times K) \cong F(K \times P) \cong F(K)^P.$$

In particular, this means that for every $K \in \mathrm{Ob}(\mathbb{C})$

$$F(K) \cong F(\mathbb{1} \times K) \cong F(\mathbb{1})^K.$$

We call φ_K the isomorphism $F(\mathbb{1})^K \cong F(K)$. This fact tells us that F is of the form $A^{(-)}$ with $A = F(\mathbb{1})$. Maybe $F(\mathbb{1})$ is the $X^{(X^{(-)})}$-algebra that we are looking for? Notice that given A a $X^{(X^{(-)})}$-algebra, $A^{(-)}(\mathbb{1})$ is indeed A.

Since F is enriched, we have for every $J, K \in \mathrm{Ob}(\mathbb{C})$ a morphism

$$\left(X^K\right)^{(X^J)} \xrightarrow{F_{J,K}} F(K)^{F(J)}\ .$$

If $K = \mathbb{1}$ and $J = F(\mathbb{1})$ then we have:

$$X^{(X^{F(\mathbb{1})})} \quad \cong \quad \left(X^{\mathbb{1}}\right)^{(X^{F(\mathbb{1})})} \xrightarrow{F_{F(\mathbb{1}),\mathbb{1}}} F(\mathbb{1})^{F\left(F(\mathbb{1})\right)} \xrightarrow{F(\mathbb{1})^{\varphi_{F(\mathbb{1})}}} F(\mathbb{1})^{(F(\mathbb{1})^{F(\mathbb{1})})} \xrightarrow{\quad\beta\quad} F(\mathbb{1})$$

where

$$\beta := \left[\!\left[\mathscr{F} : F(\mathbb{1})^{(F(\mathbb{1})^{F(\mathbb{1})})} \vdash \mathscr{F}\big(\lambda a : F(\mathbb{1}).\, a\big) : F(\mathbb{1}) \right]\!\right].$$

Defining then $A := F(\mathbb{1})$ and $\alpha = \beta \circ A^{\varphi_A} \circ F_{A,\mathbb{1}}$, we want to prove that the first axiom for $X^{(X^{(-)})}$-algebras is satisfied, that is the commutativity of

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & X^{(X^A)} \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

3.2 A functor $\mathcal{L} \colon \mathbb{F}\mathrm{rames} \to \mathbb{E}\mathrm{qu}^{\Sigma^{(\Sigma^{(-)})}}$

In this section we take a different, more direct approach to the problem. Given a frame, we shall define a $\Sigma^{(\Sigma^{(-)})}$-algebra whose image under the action of the functor Γ is isomorphic to the initial frame.

We start considering a subclass of the class of frames, namely on *algebraic* frames—i.e. frames that are algebraic lattices. One reason for this choice is that the equilogical space $\mathbf{F} = (F, \sigma_F, =)$, with F algebraic frame, is a partial equilogical space too: this means that it is very easy to compute $\Sigma^{(\Sigma^{\mathbf{F}})}$. Moreover, we can make use of compact elements. It turns out that we can define a simple map $\alpha\colon \Sigma^{(\Sigma^{\mathbf{F}})} \to \mathbf{F}$ that indeed makes $\mathbf{F}$ a $\Sigma^{(\Sigma^{(-)})}$-algebra.

Theorem 3.13. *Let F be an algebraic frame. Then the equilogical space $\mathbf{F} = (F, \sigma_F, =)$ is a $\Sigma^{(\Sigma^{(-)})}$-algebra.*

Proof. Since F is algebraic, we can compute $\Sigma^{(\Sigma^{\mathbf{F}})}$ in $\mathbb{PE}$qu and then considering the sub-lattice of all those elements that are equivalent to themselves. We first observe that since both $\mathbf{F}$ and Σ have the identity relation, any continuous map $f\colon F \to \Sigma$ is equivariant (i.e. preserves the relation) and its equivalence class consists only of f itself, that is $[f] = \{f\}$. This means that

$$\Sigma^{\mathbf{F}} = (\Sigma^F, \sigma_{\Sigma^F}, =).$$

For the same reason, we have that

$$\Sigma^{(\Sigma^{\mathbf{F}})} = \left(\Sigma^{\Sigma^F}, \sigma_{\Sigma^{(\Sigma^F)}}, =\right).$$

In order to give a structure map for F, we need to define a continuous map $\alpha\colon \Sigma^{(\Sigma^F)} \to F$. First we note that for any compact element $c \in F$, the map

$$\begin{aligned} c^{\leq}\colon\ F &\longrightarrow \Sigma \\ x &\longmapsto [\![c \leq x]\!] \end{aligned}$$

where $[\![c \leq x]\!] = \top$ if and only if $c \leq x$, is continuous since $\{\, x \in F \mid c \leq x \,\}$ is an open set in F (see proof of Proposition 2.21, p. 33), thus is in Σ^F. We define for every $G \in \Sigma^{(\Sigma^F)}$

$$\alpha(G) = \bigvee_{\substack{c \in \mathcal{K}(F) \\ G(c^{\leq}) = \top}} c.$$

We show that $\alpha\colon \Sigma^{(\Sigma^F)} \to F$ preserves arbitrary sups, thus it is continuous. Let $S \subseteq \Sigma^{(\Sigma^F)}$.

$$\alpha\left(\bigvee S\right) = \bigvee_{\substack{c \in \mathcal{K}(F) \\ \bigvee_{G \in S} G(c^{\leq}) = \top}} c = \bigvee_{\substack{c \in \mathcal{K}(F) \\ \exists G \in S. G(c^{\leq}) = \top}} c = \bigvee_{\substack{c \in \mathcal{K}(F) \\ c^{\leq} \in \bigcup_{G \in S} G^{-1}\{\top\}}} c = \bigvee_{G \in S} \bigvee_{\substack{c \in \mathcal{K}(F) \\ G(c^{\leq}) = \top}} c = \bigvee_{G \in S} \alpha(G).$$

Next, we have that the following triangle commutes

$$\begin{tikzcd} F \arrow[r, "\eta_F"] \arrow[dr, "\mathrm{id}_F"'] & \Sigma^{(\Sigma^F)} \arrow[d, "\alpha"] \\ & F \end{tikzcd}$$

where $\eta_F(a) = (f \mapsto f(a))$ (see p.43), since

$$\alpha(\eta_F(a)) = \alpha(f \mapsto f(a)) = \bigvee_{\substack{c \in \mathcal{K}(F) \\ c \leq a}} c = a$$

Finally we show the commutativity of the square

$$\begin{tikzcd}
\Sigma^{(\Sigma^{(\Sigma^{(\Sigma^F)})})} \arrow[r, "\mu_F"] \arrow[d, "\Sigma^{(\Sigma^\alpha)}"'] & \Sigma^{(\Sigma^F)} \arrow[d, "\alpha"] \\
\Sigma^{(\Sigma^F)} \arrow[r, "\alpha"] & F
\end{tikzcd}$$

We compute

$$\alpha(\mu_F(\beta)) = \alpha\Big(f \mapsto \beta\big(G \mapsto G(f)\big)\Big) = \bigvee_{\substack{c\in\mathcal{K}(F)\\ \beta\left(G\mapsto G(c^{\leq})\right)=\top}} c$$

and

$$\alpha\left(\Sigma^{(\Sigma^\alpha)}(\beta)\right) = \alpha\Bigg(f \mapsto \beta\Big(G \mapsto f\big(\alpha(G)\big)\Big)\Bigg) = \bigvee_{\substack{c\in\mathcal{K}(F)\\ \beta\left(G\mapsto c^{\leq}\left(\alpha(G)\right)\right)=\top}} c$$

and we prove that

$$G(c^{\leq}) = c^{\leq}\big(\alpha(G)\big)$$

for every $G \in \Sigma^{(\Sigma^F)}$ and $c \in \mathcal{K}(F)$. By definition of $c^{\leq}$,

$$c^{\leq}\big(\alpha(G)\big) = \left[\!\!\left[c \leq \bigvee_{\substack{k\in\mathcal{K}(F)\\ G\left(k^{\leq}\right)=\top}} k \right]\!\!\right].$$

It follows that if $G(c^{\leq}) = \top$ then $c \leq \bigvee_{\substack{k\in\mathcal{K}(F)\\ G\left(k^{\leq}\right)=\top}} k$, thus $c^{\leq}\big(\alpha(G)\big) = \top$. Conversely, if $c^{\leq}\big(\alpha(G)\big) = \top$, then there is a $k \in \mathcal{K}(F)$ such that $G(k^{\leq}) = \top$ and $k \geq c$ (because c is compact). This implies that $k^{\leq} \leq c^{\leq}$ and by monotonicity of G we obtain

$$\top = G(k^{\leq}) \leq G(c^{\leq}),$$

hence $\alpha\colon \Sigma^{(\Sigma^{\mathbf{F}})} \to \mathbf{F}$ is indeed a structure map for $\mathbf{F}$ in $\mathbb{E}$qu. □

Obviously, this result does not apply to a generic frame, because we took advantage of the compact elements in F. Thus we introduce the concept of *free algebraic lattice* on an order, and we shall prove that given F frame, the equilogical space given by the free algebraic lattice on the order F equipped with the Scott-topology and a natural equivalence relation is indeed a $\Sigma^{(\Sigma^{(-)})}$-algebra.

Let $\mathbf{P}$ be a poset and consider the category $[\mathbf{P}^{\mathrm{op}}, \Sigma]$ on the functors from $\mathbf{P}^{\mathrm{op}}$ to Σ, i.e. the order-reversing functions from $\mathbf{P}$ to Σ, and natural transformations between them. Note that given functors $x, y\colon \mathbf{P}^{\mathrm{op}} \to \Sigma$, there is at most one natural transformation $\eta\colon x \to y$ and this happens if and only if $x(a) \leq y(a)$ for every $a \in \mathbf{P}$. This means that $[\mathbf{P}^{\mathrm{op}}, \Sigma]$ is a poset. Moreover, $\mathbf{P}$ embeds via the Yoneda embedding (in its poset version)

$$\begin{array}{rcl}
\mathbf{P} & \xrightarrow{\;Y\;} & [\mathbf{P}^{\mathrm{op}}, \Sigma] \\
a & \longmapsto & a^{\geq} := \lambda p.\, [\![a \geq p]\!]
\end{array}$$

Remark 3.14. If we consider as elements of $[\mathbf{P}^{\mathrm{op}}, \Sigma]$ the inverse image of $\{\top\}$ of the order-reversing functions from $\mathbf{P}$ to Σ, we get all the downward closed subsets of $\mathbf{P}$. We can make unions and intersections with them, obtaining again downward closed sets, thus $[\mathbf{P}^{\mathrm{op}}, \Sigma]$ is a complete lattice. It is indeed the free complete lattice on the order $\mathbf{P}$: intuitively, because we added arbitrary sups to $\mathbf{P}$, since every downward closed subset $A \subseteq \mathbf{P}$ is such that $A = \bigcup_{a\in A} a^{\geq}$ (here $a^{\geq} = \{\, x \in \mathbf{P} \mid a \geq x \,\}$, i.e. the inverse image of the element $a^{\geq}$ of the image of Y).

We shall consider the subposet $\mathscr{I}(\mathbf{P})$ of $[\mathbf{P}^{\mathrm{op}}, \Sigma]$: it consists of those $x\colon \mathbf{P}^{\mathrm{op}} \to \Sigma$ which satisfy:

inhabitation: $\top = \bigvee_{a\in\mathbf{P}} x(a)$;

directedness: for every $a, b \in \mathbf{P}$, $x(a) \wedge x(b) = \bigvee_{a,b\leq c} x(c)$.

Remark 3.15. If $\mathbf{P}$ is a bounded join-semilattice, then

$$\begin{aligned} x \text{ inhabited} &\iff x(\bot) = \top \\ x \text{ directed} &\iff \forall a, b \in \mathbf{P} \quad x(a \vee b) = x(a) \wedge x(b) \end{aligned}$$

Remark 3.16. $Y\colon \mathbf{P} \to [\mathbf{P}^{\mathrm{op}}, \Sigma]$ maps into $\mathscr{I}(\mathbf{P})$.

Theorem 3.17. *Let $\mathbf{P}$ be a join-semilattice. Then $\mathscr{I}(\mathbf{P})$ is the free algebraic lattice on the order $\mathbf{P}$ in the sense that, for every monotone map $f\colon \mathbf{P} \to \mathscr{A}$ from $\mathbf{P}$ to an algebraic lattice $\mathscr{A}$, there is a unique continuous map $f_c\colon \mathscr{I}(\mathbf{P}) \to \mathscr{A}$ such that the diagram*

$$\begin{tikzcd} \mathbf{P} \arrow[r, "Y"] \arrow[dr, "f"'] & \mathscr{I}(\mathbf{P}) \arrow[d, "f_c"] \\ & \mathscr{A} \end{tikzcd}$$

commute.

Proof. In $\mathscr{I}(\mathbf{P})$ arbitrary meets are computed pointwise as well as directed joins, so it is a complete lattice. We observe that for every $a \in \mathbf{P}$ and $x \in \mathscr{I}(\mathbf{P})$

$$Y(a) \leq x \iff x(a) = \top$$

in fact if $Y(a) \leq x$ then $\top = Y(a)(a) \leq x(a)$, and if $x(a) = \top$ then for every $p \in \mathbf{P}$ such that $Y(a)(p) = \top$ (i.e. $p \leq a$) one has $x(p) \geq x(a) = \top$ since x is order reversing. This means that the elements of the form $Y(a)$ are compact: let $S \subseteq \mathscr{I}(\mathbf{P})$ be such that $Y(a) \leq \bigvee S$. Then

$$\top = \bigvee S(a) = \left[\bigvee^{\uparrow}_{E\subseteq_{\mathit{fin}} S} \left(\bigvee E\right)\right](a) = \bigvee^{\uparrow}_{E\subseteq_{\mathit{fin}} S} \left[\bigvee E(a)\right]$$

(because directed sups are computed pointwise), thus there is $E \subseteq_{\mathit{fin}} S$ such that $\bigvee E(a) = \top$ that amounts to saying that $Y(a) \leq \bigvee E$.

It also follows that

$$x = \bigvee^{\uparrow}_{x(a)=\top} Y(a)$$

where the join is directed by definition of $\mathscr{I}(\mathbf{P})$. So $\mathscr{I}(\mathbf{P})$ is algebraic, and the compact elements are the finite joins of elements of the form $Y(a)$.

Now, let $g\colon \mathscr{I}(\mathbf{P}) \to \mathscr{A}$ be a continuous function such that $g \circ Y = f$. Then for every $x \in \mathscr{I}(\mathbf{P})$

$$g(x) = g\left(\bigvee^{\uparrow}_{x(a)=\top} Y(a)\right) = \bigvee^{\uparrow}_{x(a)=\top} g(Y(a)) = \bigvee^{\uparrow}_{x(a)=\top} f(a),$$

so we have proved the uniqueness part of the statement. For $x \in \mathscr{I}(\mathbf{P})$, let so

$$f_c(x) = \bigvee^{\uparrow}_{x(a)=\top} f(a).$$

Then f_c makes the diagram commute and to see that f_c is continuous, let $D \subseteq \mathscr{I}(\mathbf{P})$ be a directed subset. We have

$$f_c\left(\bigvee^{\uparrow} D\right) = \bigvee^{\uparrow}_{\bigvee^{\uparrow} D(a)=\top} f(a) = \bigvee^{\uparrow}_{\exists x\in D. x(a)=\top} f(a) = \bigvee^{\uparrow}_{a\in\bigcup_{x\in D} x^{-1}\{\top\}} f(a) = \bigvee^{\uparrow}_{x\in D} \bigvee^{\uparrow}_{x(a)=\top} f(a) = \bigvee^{\uparrow}_{x\in D} f_c(x)$$

as required. □

Theorem 3.18. *The functor $Y\colon \mathbf{P} \to \mathscr{I}(\mathbf{P})$ preserves arbitrary meets which exist in $\mathbf{P}$ as well as finite joins which exist in $\mathbf{P}$.*

Proof. Suppose $a = \bigwedge_{i\in I} a_i$ in $\mathbf{P}$. Then, for any $p \in \mathbf{P}$, it is

$$\Big(\bigwedge_{i\in I} Y(a_i)\Big)(p) = \bigwedge_{i\in I}\big(Y(a_i)(p)\big) = \bigwedge_{i\in I}[\![a_i \geq p]\!] = [\![\forall i \in I a_i \geq p]\!] = \Big[\!\!\Big[\Big(\bigwedge_{i\in I} a_i\Big) \geq p\Big]\!\!\Big] = Y\Big(\bigwedge_{i\in I} a_i\Big)(p).$$

Suppose $c = a \vee b$ in $\mathbf{P}$. Then $Y(a), Y(b) \leq Y(c)$. Suppose $x \in \mathscr{I}(\mathbf{P})$ is such that $Y(a), Y(b) \leq x$: then $x(a) = \top = x(b)$ and since x is directed, there is d in $\mathbf{P}$ such that $a, b \leq d$ and $x(d) = \top$. Then $c = a \vee b \leq d$ in $\mathbf{P}$ and $x(c) = \top$, so $Y(c) \leq x$. □

Remark 3.19. If $\mathbf{P}$ is a join-semilattice, the compact elements of $\mathscr{I}(\mathbf{P})$ are the functors in $[\mathbf{P}^{\mathrm{op}}, \Sigma]$ of the form $Y(a)$ for some a in $\mathbf{P}$ because, in $\mathscr{I}(\mathbf{P})$, it is $Y(a) \vee Y(b) = Y(a \vee b)$.

Remark 3.20. An algebraic lattice is a frame if and only if it is distributive.

Theorem 3.21. *Let $\mathbf{L}$ be a distributive lattice. Then $\mathscr{I}(\mathbf{L})$ is a frame.*

Proof. We first note that $x \in \mathscr{I}(\mathbf{L})$ if and only if $x^{-1}\{\top\}$ is an *ideal* of $\mathbf{L}$, i.e. a non-empty downward closed subset or $\mathbf{L}$ which is closed under finite sups. In this proof we shall consider then the elements of $\mathscr{I}(\mathbf{L})$ as ideals of $\mathbf{L}$.

Since $\mathscr{I}(\mathbf{L})$ is algebraic, in order to prove that it is a frame it suffices to show that it is distributive. Intersection of ideals is again an ideal, but unions (even finite) are not in general. Given $A \subseteq \mathbf{L}$, the ideal $\langle A\rangle$ generated by A consists of those $x \in \mathbf{L}$ such that there are finitely many $a_1, \ldots, a_n \in A$ and $x \leq a_1 \vee \ldots \vee a_n$. Let I, J and K be ideals, and consider $i \in I \cap \langle J \cup K\rangle$. So there are $j \in J$ and $k \in K$ such that $i \leq j \vee k$. By distributivity,

$$i = (i \wedge j) \vee (i \wedge k) \in \langle (I \cap J) \cup (I \cap K)\rangle.$$

Vivecersa, let $x \in \langle (I \cap J) \cup (I \cap K)\rangle$. Then there are $y \in I \cap J$ and $z \in I \cap K$ such that $x \leq y \vee z$. It follows immediately that $x \in \langle J \cup K\rangle$; moreover, since $x, y \in I$ and I is closed under finite sups, $x \vee y \in I$ and then $x \in I$. □

When $\mathbf{P}$ has joins of directed sets (i.e. is a complete lattice), we write $\overline{x} := \bigvee^{\uparrow}_{xa=\top} a$ for $x \in \mathscr{I}(\mathbf{P})$.

Suppose $\mathbf{F}$ be a frame. Consider the equilogical space

$$\mathcal{L}(\mathbf{F}) := \big(\mathscr{I}(\mathbf{F}), \sigma, \equiv\big)$$

where σ is the Scott topology on the algebraic lattice $\mathscr{I}(\mathbf{F})$ and, for $x, y \in \mathscr{I}(\mathbf{F})$, it is $x \equiv y$ when

$$\overline{x} = \overline{y}.$$

Theorem 3.22. *The exponential $\Sigma^{\mathcal{L}(\mathbf{F})}$ is isomorphic to the subspace*

$$\mathrm{C}(\mathbf{F}, \Sigma) \hookrightarrow [\mathbf{F}, \Sigma]$$

on the directed-join preserving functors from $\mathbf{F}$ to Σ, i.e. the space of Scott-continuous functions from $\mathbf{F}$ to Σ with the topology τ induced from the algebraic lattice of monotone functions from $\mathbf{F}$ to Σ.

Proof. Since we must compute exponentials, we shall turn the three equilogical spaces $\mathcal{L}(\mathbf{F})$, Σ, and $\mathrm{C}(\mathbf{F}, \Sigma)$ into partial equilogical spaces which are mapped by the equivalence $\mathcal{F}\colon \mathbb{PE}\mathrm{qu} \to \mathbb{E}\mathrm{qu}$ to each, see Theorem 2.35. The underlying topological space of the first is an algebraic lattice, so we shall just use that; the second is an algebraic lattice. So we compute the exponential $\Sigma^{\mathcal{L}(\mathbf{F})}$ in $\mathbb{PE}\mathrm{qu}$ as a partial equivalence relation on $\Sigma^{(\mathscr{I}(\mathbf{F}),\sigma)}$: it consists of a subreflexive partial equivalence relation where, for a Scott-continuous $f\colon \mathscr{I}(\mathbf{F}) \to \Sigma$, it is $f \sim f$ if and only if

$$\forall x, y \in \mathscr{I}(\mathbf{F}) \quad \big(x \equiv y \implies f(x) = f(y)\big).$$

We shall show that the isomorphism of algebraic lattices

$$\Sigma^{(\mathscr{I}(\mathbf{F}),\sigma)} \xrightarrow{\ \restriction_{\mathbf{F}}\ } [\mathbf{F}, \Sigma]$$

given by Theorem 2.27 maps $\{\, f \mid f \sim f \,\}$ onto $\mathrm{C}(\mathbf{F}, \Sigma)$. So it induces an isomorphism in $\mathbb{PE}\mathrm{qu}$ from $\Sigma^{\mathcal{L}(\mathbf{F})}$ and the subreflexive partial equivalence relation $\big([\mathbf{F}, \Sigma], \sigma, =_{\mathrm{C}(\mathbf{F},\Sigma)}\big)$, where

$$=_{\mathrm{C}(\mathbf{F},\Sigma)} := \{\, (f, f) \mid f \in \mathrm{C}(\mathbf{F}, \Sigma) \,\}.$$

Therefore, there is an isomorphism in $\mathbb{E}\mathrm{qu}$ between $\Sigma^{\mathcal{L}(\mathbf{F})}$ and $\big(\mathrm{C}(\mathbf{F}, \Sigma), \tau, =\big)$.

Suppose $f \sim f$ and consider a directed family $(a_i)_{i \in I}$ in $\mathbf{F}$. Then

$$f\left(\bigvee^{\uparrow}_{i\in I} Y(a_i)\right) = \bigvee^{\uparrow}_{i\in I} f\left(Y\left(a_i\right)\right)$$

because f is Scott-continuous. But also

$$f\left(\bigvee^{\uparrow}_{i\in I} Y(a_i)\right) = f\left(Y\left(\bigvee^{\uparrow}_{i\in I} a_i\right)\right)$$

because $\overline{\bigvee^{\uparrow}_{i\in I} Y(a_i)} = \overline{Y\big(\bigvee^{\uparrow}_{i\in I} a_i\big)}$. In fact, for $a \in \mathbf{F}$,

$$\bigvee^{\uparrow}_{i\in I} Y(a_i)(a) = \top \iff \exists i \in I.\, a_i \geq a$$

then

$$\overline{\bigvee^{\uparrow}_{i\in I} Y(a_i)} = \bigvee^{\uparrow}_{\exists i\in I. a_i \geq a} a = \bigvee^{\uparrow}_{i\in I} a_i = \bigvee^{\uparrow}_{a \leq \bigvee^{\uparrow}_{i\in I} a_i} a = \overline{Y\big(\bigvee^{\uparrow}_{i\in I} a_i\big)}.$$

So f restricts to a Scott-continuous functor from $\mathbf{F}$ to Σ, as requested. $\square$

Theorem 3.23. *The equilogical space* $\Sigma^{(\Sigma^{\mathcal{L}(\mathbf{F})})}$ *is*

$$\left(\Sigma^{[F,\Sigma]}, \sigma, \sim\right)$$

where the equivalence relation $\sim$ *is given by functional equality on the subset* $\mathrm{C}(\mathbf{F},\Sigma) \rightarrowtail [\mathbf{F},\Sigma]$*:*

$$\mathfrak{a} \sim \mathfrak{b} \iff \mathfrak{a} \restriction_{\mathrm{C}(\mathbf{F},\Sigma)} = \mathfrak{b} \restriction_{\mathrm{C}(\mathbf{F},\Sigma)}$$

Proof. With a similar argument of the previous proof, we compute $\Sigma^{(\Sigma^{\mathcal{L}(\mathbf{F})})}$ in $\mathbb{PE}$qu. We have that $\Sigma^{\mathcal{L}(\mathbf{F})} \cong \left([\mathbf{F},\Sigma], \sigma, =_{\mathrm{C}(\mathbf{F},\Sigma)}\right)$, so applying $\Sigma^{(-)}$ in $\mathbb{PE}$qu we obtain the equivalence relation $\left(\Sigma^{[F,\Sigma]}, \sigma, \sim\right)$ that is an equilogical space too. □

Remark 3.24. The functors of the form $a^{\not\geq} := \lambda x. [\![a \not\geq x]\!] \colon \mathbf{F} \to \Sigma$ are Scott-continuous, hence in $\mathrm{C}(\mathbf{F},\Sigma)$.

To determine a $\Sigma^{(\Sigma^{(-)})}$-algebra structure

$$\alpha \colon \Sigma^{(\Sigma^{\mathcal{L}(\mathbf{F})})} \to \mathcal{L}(\mathbf{F})$$

on $\mathcal{L}(\mathbf{F})$, one must give a Scott-continuous function $\Sigma^{[\mathbf{F},\Sigma]} \to \mathscr{I}(\mathbf{F})$ which preserves the equivalence relations. We start considering, as a representative for α, the following function

$$\begin{aligned} \Sigma^{[\mathbf{F},\Sigma]} &\longrightarrow \mathscr{I}(\mathbf{F}) \\ \mathfrak{a} &\longmapsto Y\left(\textstyle\bigwedge_{\mathfrak{a}(a^{\not\geq})=\perp} a\right) \end{aligned} \tag{†}$$

Indeed, it may not be the right function just because it need not to be Scott-continuous. But it does preserve the equivalence relation, so (†) can be taken on just the compact elements and extended by continuity to all $\Sigma^{[\mathbf{F},\Sigma]}$. We shall try this in the following, but first we need to determine the compact elements of $\Sigma^{[\mathbf{F},\Sigma]}$.

It is easy to show that the elements $a^{\leq}$ are compact in $[\mathbf{F},\Sigma]$. Since every element of $[\mathbf{F},\Sigma]$ is a join of elements of the form $Y(a)$, the compact elements of $[\mathbf{F},\Sigma]$ are exactly finite joins of elements of that form.

Recall that for A and B algebraic lattices, $a \in \mathcal{K}(A)$ and $y \in B$, the step function $[a,y] \colon A \to B$ is defined as

$$x \mapsto \begin{cases} y & \text{if } a \leq x \\ \perp_{\mathcal{M}} & \text{otherwise} \end{cases}$$

The compact elements of $\mathscr{I}(\mathbf{F})$ are exactly the elements of the form $Y(a)$ with $a \in \mathbf{F}$. So, in $\Sigma^{\mathscr{I}(\mathbf{F})}$, the step functions $[Y(a),\top]$ are compact (see Proposition 2.23). Indeed $[Y(a),\top]$ is mapped to $a^{\leq}$ via the retriction isomorphism

$$\Sigma^{\mathscr{I}(\mathbf{F})} \xrightarrow{\restriction_{\mathbf{F}}} [\mathbf{F},\Sigma].$$

The general compact element of $\mathscr{I}(\mathbf{F})$ is therefore of the form

$$Y(a_1) \vee \ldots \vee Y(a_n).$$

Consider now $\Sigma^{[\mathbf{F},\Sigma]}$. Since all step functions $[c,\perp]$ are equal to the constant $\perp$ function, we consider only step functions of the form $[f,\top]$.

Fist of all, note that

$$[Y(a),\top](f) = \begin{cases} \top & \text{if } Y(a) \leq f \\ \bot & \text{otherwise} \end{cases}$$
$$= \begin{cases} \top & \text{if } \top \leq f(a) \\ \bot & \text{otherwise} \end{cases}$$
$$= \widehat{a}(f)$$

where for a in $\mathbf{F}$,

$$\begin{aligned} \widehat{a}\colon\ [\mathbf{F},\Sigma] &\longrightarrow \Sigma \\ f &\longmapsto f(a) \end{aligned}$$

Next

$$[Y(a)\vee Y(b),\top](f) = \begin{cases} \top & \text{if } Y(a)\vee Y(b) \leq f \\ \bot & \text{otherwise} \end{cases}$$
$$= \begin{cases} \top & \text{if } Y(a) \leq f \text{ and } Y(b) \leq f \\ \bot & \text{otherwise} \end{cases}$$
$$= \left(\widehat{a}\wedge\widehat{b}\right)(f)$$

Analogously, the arbitrary compact element $\Cup_i[\Cup_j Y(a_{ij}),\top]$ (denoting with $\Cup$ a finite sup) is equal to $\Cup_i(\Cap_j \widehat{a_{ij}})$.

We evaluate (†), first on compact elements, then on arbitrary elements.

Since $\widehat{a}\left(c^{\not\geq}\right) = \bot \iff (\llbracket c \not\geq a\rrbracket = \bot) \iff c \geq a$, it is

$$\bigwedge_{\widehat{a}(c^{\not\geq})=\bot} c = \bigwedge_{c\geq a} c = a.$$

Since $\left(\widehat{a}\wedge\widehat{b}\right)\left(c^{\not\geq}\right) = \bot \iff \left(\widehat{a}\left(c^{\not\geq}\right) = \bot \ \vee\ \widehat{b}\left(c^{\not\geq}\right) = \bot\right) \iff (c \geq a \ \vee\ c \geq b)$, it is

$$\bigwedge_{\left(\widehat{a}\wedge\widehat{b}\right)(c^{\not\geq})=\bot} c = \bigwedge_{c\geq a\ \vee\ c\geq b} c = a\vee b.$$

For the general case, since the lattice is distributive it is easier to handle the case of finite meets of finite joins:

$$\begin{aligned} \left[\left(\widehat{a}\vee\widehat{a'}\right)\wedge\left(\widehat{b}\vee\widehat{b'}\right)\right]\left(c^{\not\geq}\right) = \bot &\iff \Big[(c\geq a \ \wedge\ c\geq a') \ \vee\ (c\geq b \ \wedge\ c \geq b')\Big] \\ &\iff \Big[(c\geq a\vee a') \ \vee\ (c \geq b\vee b')\Big]. \end{aligned}$$

Hence it is

$$\bigwedge_{\left[\left(\widehat{a}\vee\widehat{a'}\right)\wedge\left(\widehat{b}\vee\widehat{b'}\right)\right](c^{\not\geq})=\bot} c = \bigwedge_{(c\geq a\vee a')\ \vee\ (c\geq b\vee b')} c = (a\vee a')\wedge(b\vee b').$$

Consider now an arbitrary $\mathfrak{a}\in\Sigma^{[\mathbf{F},\Sigma]}$, it is the directed join of all the compact elements below itself:

$$\mathfrak{a} = \bigvee_h^{\uparrow}\left(\Cup_i(\Cap_j \widehat{a_{hij}})\right).$$

Again it follows that

$$\begin{aligned}
\mathfrak{a}\left(c^{\not\geq}\right) = \bot &\iff \forall h \big(\mathbb{W}_i(\mathbb{A}_j\, \widehat{a_{hij}})\big)\left(c^{\not\geq}\right) = \bot \\
&\iff \forall hc \geq \mathbb{W}(\mathbb{A}\, a_{hij}) \\
&\iff c \geq \bigvee_h \big(\mathbb{W}_i(\mathbb{A}_j\, a_{hij})\big)
\end{aligned}$$

and we just extend (†) by continuity defining

$$\begin{aligned}
\Sigma^{[\mathbf{F},\Sigma]} &\xrightarrow{\qquad\alpha\qquad} \mathscr{I}(\mathbf{F}) \\
\bigvee_h \big(\mathbb{W}_i(\mathbb{A}_j\, \widehat{a_{hij}})\big) &\longmapsto \bigvee_h Y\big(\mathbb{W}_i(\mathbb{A}_j\, a_{hij})\big)
\end{aligned}$$

Note that if $\mathfrak{a} \restriction_{\mathrm{C}(\mathbf{F},\Sigma)} = \mathfrak{b} \restriction_{\mathrm{C}(\mathbf{F},\Sigma)}$, then $\overline{\alpha(\mathfrak{a})} = \overline{\alpha(\mathfrak{b})}$, because if $\mathfrak{a} = \bigvee_h \big(\mathbb{W}_i(\mathbb{A}_j\, \widehat{a_{hij}})\big)$ and $\mathfrak{b} = \bigvee_h \big(\mathbb{W}_i(\mathbb{A}_j\, \widehat{b_{hij}})\big)$, then

$$\overline{\alpha(\mathfrak{a})} = \bigvee_h \mathbb{W}_i(\mathbb{A}_j\, a_{hij})$$

and

$$\overline{\alpha(\mathfrak{b})} = \bigvee_h \mathbb{W}_i(\mathbb{A}_j\, b_{hij}).$$

Thus if $c := \bigvee_h \mathbb{W}_i(\mathbb{A}_j\, b_{hij})$, we have

$$\begin{aligned}
c \geq \bigvee_h \mathbb{W}_i(\mathbb{A}_j\, a_{hij}) &\iff \mathfrak{a}\left(c^{\not\geq}\right) = \bot \\
&\iff \mathfrak{b}\left(c^{\not\geq}\right) = \bot \\
&\iff \bigvee_h \big(\mathbb{W}_i(\mathbb{A}_j\, b_{hij})\big) \geq c
\end{aligned}$$

and analogously the other inequality.

In order to prove that α is a structure map, we prove that α *coincides* with the structure map on the algebraic frame $\mathscr{I}(\mathbf{F})$ given by Theorem 3.13, therefore the triangle in $\mathbb{E}\mathrm{qu}$

$$\begin{tikzcd}
\mathcal{L}(\mathbf{F}) \arrow[r, "\eta_{\mathcal{L}(\mathbf{F})}"] \arrow[dr, "\mathrm{id}_{\mathcal{L}(\mathbf{F})}"'] & \Sigma^{(\Sigma^{\mathcal{L}(\mathbf{F})})} \arrow[d, "\alpha"] \\
& \mathcal{L}(\mathbf{F})
\end{tikzcd}$$

commutes since the corresponding triangle of continuous functions

$$\begin{tikzcd}
\mathscr{I}(\mathbf{F}) \arrow[r, "\eta_{\mathscr{I}(\mathbf{F})}"] \arrow[dr, "\mathrm{id}_{\mathscr{I}(\mathbf{F})}"'] & \Sigma^{[\mathbf{F},\Sigma]} \arrow[d, "\alpha"] \\
& \mathscr{I}(\mathbf{F})
\end{tikzcd}$$

does. And similarly for the square.

We start recalling the definition of the structure map on an algebraic frame $\mathbf{G}$:

$$\begin{aligned} \Sigma^{(\Sigma^{\mathbf{G}})} &\xrightarrow{\beta} \mathbf{G} \\ G &\longmapsto \bigvee_{\substack{c\in\mathcal{K}(\mathbf{G}) \\ G(c^{\leq})=\top}} c. \end{aligned}$$

Since the compact elements of $\mathscr{I}(\mathbf{F})$ are exactly the elements of the form $a^{\geq}$ for some $a \in \mathbf{F}$, we obtain in the case $\mathbf{G} = \mathscr{I}(\mathbf{F})$

$$\begin{aligned} \Sigma^{(\Sigma^{\mathscr{I}(\mathbf{F})})} &\xrightarrow{\beta} \mathscr{I}(\mathbf{F}) \\ G &\longmapsto \bigvee_{\substack{c\in\mathbf{F} \\ G\left((a^{\geq})^{\leq}\right)=\top}} c^{\geq}. \end{aligned}$$

but for $x \in \mathscr{I}(\mathbf{F})$

$$\begin{aligned} \left(c^{\geq}\right)^{\leq}(x) &= \begin{cases} \top & \text{if } x \geq c^{\geq} \\ \bot & \text{otherwise} \end{cases} \\ &= \begin{cases} \top & \text{if } x(c) = \top \\ \bot & \text{otherwise} \end{cases} \\ &= \widehat{c}(x) \end{aligned}$$

therefore

$$\begin{aligned} \Sigma^{(\Sigma^{\mathscr{I}(\mathbf{F})})} &\xrightarrow{\beta} \mathscr{I}(\mathbf{F}) \\ G &\longmapsto \bigvee_{\substack{c\in\mathbf{F} \\ G(\widehat{c})=\top}} c^{\geq}. \end{aligned}$$

In order to confront α and β, we need to consider the isomorphism $\varphi\colon \Sigma^{[\mathbf{F},\Sigma]} \to \Sigma^{(\Sigma^{\mathscr{I}(\mathbf{F})})}$ such that for every continuous $\mathfrak{a}\colon [\mathbf{F},\Sigma] \to \Sigma$

$$\begin{aligned} \varphi(\mathfrak{a})\colon\ \Sigma^{\mathscr{I}(\mathbf{F})} &\longrightarrow \Sigma \\ f &\longmapsto \mathfrak{a}(f \circ Y) \end{aligned}$$

thus

$$\beta(\varphi(\mathfrak{a})) = \bigvee_{\substack{c\in\mathbf{F} \\ \varphi(\mathfrak{a})(\widehat{c})=\top}} c^{\geq} = \bigvee_{\substack{c\in\mathbf{F} \\ \mathfrak{a}(\widehat{c}\circ Y)=\top}} c^{\geq}$$

We consider then

$$\begin{aligned} \beta'\colon\ \Sigma^{[\mathbf{F},\Sigma]} &\longrightarrow \mathscr{I}(\mathbf{F}) \\ \mathfrak{a} &\longmapsto \bigvee_{\substack{c\in\mathbf{F} \\ \mathfrak{a}(\widehat{c}\circ Y)=\top}} c^{\geq} \end{aligned}$$

Both α and β' are continuous, so if they coincide on compact elements of $\Sigma^{[\mathbf{F},\Sigma]}$, they are equal. But with simple calculations, one can prove that they coincide on the compact elements of the form $\widehat{a}$, then on finite infs of them, then on finite sups of finite infs of them, so indeed α makes $\mathscr{I}(\mathbf{F})$ a $\Sigma^{(\Sigma^{(-)})}$-algebra.

Consider now $f\colon \mathbf{F} \to \mathbf{G}$ a morphism in $\mathbb{F}$rames. In order to define a continuous function $\bar{f}\colon \mathscr{I}(\mathbf{F}) \to \mathscr{I}(\mathbf{G})$, it suffices to give a monotone function $\varphi\colon \mathbf{F} \to \mathscr{I}(\mathbf{G})$ thanks to Theorem 3.17. Since f preserves (arbitrary) sups, it is monotone, thus $\varphi := Y_{\mathbf{G}} \circ f\colon \mathbf{F} \to \mathscr{I}(\mathbf{G})$ is again a monotone function, therefore there is a unique $\varphi_c\colon \mathscr{I}(\mathbf{F}) \to \mathscr{I}(\mathbf{G})$ that is continuous and such that

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "Y_{\mathbf{F}}"] \arrow[dr, "\varphi"'] & \mathscr{I}(\mathbf{F}) \arrow[d, "\varphi_c"] \arrow[dl, phantom, "/\!/"] \\
 & \mathscr{I}(\mathbf{G})
\end{tikzcd}$$

Let $x, y \in \mathscr{I}(\mathbf{F})$ be such that $\bar{x} = \bar{y}$, this means that

$$\bigvee^{\uparrow}_{x(a)=\top} a = \bigvee^{\uparrow}_{y(a)=\top} a.$$

We are to prove that $\bigvee^{\uparrow}_{\varphi_c(x)(a)=\top} a = \bigvee^{\uparrow}_{\varphi_c(y)(a)=\top} a$. By definition,

$$\varphi_c(x) = \bigvee^{\uparrow}_{x(a)=\top} \varphi(a),$$

but since φ preserves all sups, we have

$$\bigvee^{\uparrow}_{x(a)=\top} \varphi(a) = \varphi\left(\bigvee^{\uparrow}_{x(a)=\top} a\right) = \varphi\left(\bigvee^{\uparrow}_{y(a)=\top} a\right) = \varphi_c(y)$$

therefore φ_c preserves indeed the relations, i.e. is a map in $\mathbb{E}$qu. Calling $\mathcal{L}(f) := \varphi_c$, it is easy to prove that $\mathcal{L}(\mathrm{id}_{\mathbf{F}}) = \mathrm{id}_{\mathcal{L}(\mathbf{F})}$ and that $\mathcal{L}(gf) = \mathcal{L}(g) \circ \mathcal{L}(f)$. Thus we have defined a functor

$$\mathcal{L}\colon \mathbb{F}\text{rames} \to \mathbb{E}\text{qu}.$$

In order to prove that φ_c is a homomorphism of $\Sigma^{(\Sigma^{(-)})}$-algebras, we have to show that the following diagram

$$\begin{tikzcd}
\Sigma^{[\mathbf{F},\Sigma]} \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & \mathscr{I}(\mathbf{F}) \arrow[ddd, "\varphi_c"] \\
\Sigma^{(\Sigma^{\mathscr{I}(\mathbf{F})})} \arrow[d, "\Sigma^{(\Sigma^{\varphi_c})}"'] & \\
\Sigma^{(\Sigma^{\mathscr{I}(\mathbf{G})})} \arrow[d, "\varphi^{-1}"'] & \\
\Sigma^{[\mathbf{G},\Sigma]} \arrow[r, "\beta"] & \mathscr{I}(\mathbf{G})
\end{tikzcd}$$

commutes, where

$$\begin{tikzcd}
\Sigma^{[\mathbf{F},\Sigma]} \arrow[r, "\alpha"] & \mathscr{I}(\mathbf{F}) \\
\bigvee^{\uparrow}_h \big(\Cup_i (\Cap_j \widehat{a_{hij}})\big) \arrow[r, mapsto] & \bigvee^{\uparrow}_h Y\big(\Cup_i (\Cap_j a_{hij})\big)
\end{tikzcd}$$

and similarly β. The result can be achieved after several computations showing that the diagram commutes when applied to the compact elements of $\Sigma^{[\mathbf{F},\Sigma]}$, so

$$\mathcal{L}\colon \mathbb{F}\mathrm{rames} \to \mathbb{E}\mathrm{qu}^{\Sigma^{(\Sigma^{(-)})}}.$$

As a last result, we prove that the functor $\mathcal{L}$ acts as a right "inverse" functor for Γ.

Theorem 3.25. *Let* $\mathbf{F}$ *be a frame. Then* $\Gamma(\mathcal{L}(\mathbf{F})) \cong \mathbf{F}$ *in* $\mathbb{F}$rames.

Proof. We have a bijection of sets

$$\begin{array}{ccc} \mathscr{I}(\mathbf{F})/_{\equiv} & \xrightarrow{\varphi} & \mathbf{F} \\ [x] & \longmapsto & \bar{x} \\ [Y(a)] & \longleftarrow & a \end{array}$$

since $[x] = [(\bar{x})^{\geq}]$ and $\bigvee_{b \leq a} b = a$. We first prove that φ and φ^{-1} are monotone.

We know that $\mathscr{I}(\mathbf{F})/_{\equiv}$ is a frame with respect to the order

$$[x] \leq [y] \iff \exists f \in \mathscr{I}(\mathbf{F})^{\Sigma}.\, [f(\bot)] = [x] \text{ and } [f(\top)] = [y]$$

(see p.47). First, we observe that this order coincides with another order, given by

$$[x] \preceq [y] \iff \bar{x} \leq \bar{y}$$

because if $[x] \leq [y]$ then there is a continuous $f\colon \Sigma \to \mathscr{I}(\mathbf{F})$ such that $f(\bot) \equiv x$ and $f(\top) \equiv y$, so

$$\bigvee_{f(\bot)(a)=\top} a = \bar{x}$$
$$\bigvee_{f(\top)(a)=\top} a = \bar{y}$$

Since f is monotone, it is $(f(\bot))^{-1}\{\top\} \subseteq (f(\top))^{-1}\{\top\}$, therefore the first sup is less or equal than the second, i.e. $\bar{x} \leq \bar{y}$. Viceversa, if $\bar{x} \leq \bar{y}$, then consider the function

$$\begin{array}{rccc} f\colon & \Sigma & \longrightarrow & \mathscr{I}(\mathbf{F}) \\ & \bot & \longmapsto & (\bar{x})^{\geq} \\ & \top & \longmapsto & (\bar{y})^{\geq} \end{array}$$

that is monotone (hence continuous since Σ is finite) and as we already observed $[x] = [(\bar{x})^{\geq}]$. So the two orders $\leq$ and $\preceq$ are equal, hence φ and φ^{-1} are monotone. In order to prove that they preserve arbitrary sups and finite infs, it suffices to note that

$$\bigvee_{i \in I} [x_i] = \left[\Big(\bigvee_{i \in I} \bar{x}_i\Big)^{\geq}\right]$$

and

$$[x] \wedge [y] = \left[(\bar{x} \wedge \bar{y})^{\geq}\right]$$

and with simple computations we obtain that both φ and φ^{-1} are morphisms in $\mathbb{F}$rames. $\square$

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