The Pigeonhole Principle and It's Applications

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Abstract

The pigeonhole principle asserts that there is no injective mapping from m pigeons to n pigeonholes as long as m>n. It is a simple but a powerful idea which expresses one of the most basic primitives in mathematics and is the most extensively studied combinatorial principle. It's applications are developed in the corollaries of Ramsay's Theorem along with proof projection in Fermat's Theorem and Dirichlet's Theorem.

1. The Pigeonhole Principle

Historic Context
The principle was constructed by Johann Dirichlet, a German mathematician, in the early 1800's who specialized in the field of analysis and combinatorics.

Theorem 1.1: Let S be a finite set where $|S|$=n. Let s_1,s_2,...,s_k be a partition of S into k subsets. Then at least one subset s_i of S contains at least $\lceil$n/k$\rceil$ elements, where $\lceil . \rceil$ is the ceiling function.

*__Ceiling Function__
If $\forall$ x $\in \mathcal{R}$:$\lceil$x$\rceil$=inf(m$\in\mathcal{Z}$: m $\geq$ x) and $\lceil$x$\rceil$ is the smallest integer $\geq$ to x.

Corollary
If a set of n distinct objects is partitioned into k subsets, where 0<k<n, then at least one subset must contain at least two elements.

Proof By Contradiction
Suppose this was not the case, and no subset s_i of S has as many as $\lceil$n/k$\rceil$ elements. Then at most, the number of elements of any s_i would be $\lceil$n/k$\rceil$-1. So the total number of elements of S would be no more than k($\lceil$n/k$\rceil$-1) = $\lceil$n/k$\rceil$-k.

There are two cases:
1. n is divisible by k
2. n is not divisible by k

Case 1: k/n
Then $\lceil$n/k$\rceil$=n/k is an integer and $\lceil$n/k$\rceil$-k=n-k. Then $\sum_{i=0}^{k}|s_i|\leq$n-k<n.

This contradicts our assumption that no subset s_i of S has as many as $\lceil n/k \rceil$ elements.

Case 2: n|k
Then $k\lceil n/k \rceil - k < k(n+k)/k - k = n$.

This again contradicts our assumption that no subset s_i of S has as many as $\lceil n/k \rceil$ elements.

Either way, there has to be at least $\lceil n/k \rceil$ elements in $s_i \subseteq S$.

2. Applications

Application to Corollaries of Ramsay's Theorem

1. Erdos-Szekers Theorem

Theorem 2.1: Suppose $m,n \geq 1$ and $a_1, a_2, ..., a_{mn+1}$ is a sequence of distinct real numbers. Then either there exists an increasing sequence of length m+1 or decreasing sequence of length n+1.

Proof By Contradiction
For each $1 \leq i \leq mn+1$, let $f(i)=(r_i, s_i)$ where r_i and s_i are the longest increasing and decreasing sequence starting with a_i. If the theorem fails for this sequence, we must have that $f(i) \leq [m]*[n]$. On the other hand, we can see that f is injective. For $i \neq j$, if $a_i < a_j$ then $r_i \geq r_{j+1}$ and if $a_i > a_j$ then $s_i \geq s_{j+1}$.

This is a contradiction to the Pigeonhole Principle.

2. Graph Theory Problem

Theorem 2.2: Show that every finite simple graph G with more than one vertex has at least two vertices with the same degree.

Predefinitions
1. Graph: Denoted by G=(V,E), consists of non-empty set of vertices V and a set of edges E.
2. Degree: Number of edges incident with a vertex.
3. Connected: Any two vertices are connected by a path.

Proof
Let G be any finite simple graph with more than one vertex and $|V_G|=n$. We notice that at most the degree of any vertex in G is less than or equal to n-1. Also, if our graph is not connected then the maximal degree is strictly less than n-1. So we have two cases.

Case 1: Graph is connected.
We cannot have a vertex of degree 0 in G, so the set of vertex degrees is a subset of S=$\{1, 2, ..., n-1\}$. Since the graph has n vertices(pigeons) and n-1 vertex degree(pigeonholes), by pigeonhole principle we can find two vertices of the same degree G.

Case 2: Graph is not connected.
We cannot have a vertex of degree n-1, so the set of vertex degree is a subset of S$^{'}$=$\{0, 1, 2, ..., n-2\}$. Since the graph has n-1 vertices(pigeons) and n-2 vertex degree(pigeonholes), by pigeonhole principle we can find two vertices of the same degree G.

Application to Approximation of Real Numbers

1. Dirichlet's Theorem

Theorem 2.3: Let α be an irrational number, and let N be a positive integer. Then there is a rational p/q such that the denominator q is between 1 and N, such that $|\alpha - p/q|$≤1/q^{N}.

Proof
qα is within 1/N of an integer(choose p as the integer). Divide the interval [0,1) into N sub-intervals, each of length 1/N: [[0,1/N),[1/N,2/N),...,[N-1/N,1)]. These are our pigeonholes.

Use x(fractional part after the decimal) as $\{x\}$. Pigeons will be the N+1 numbers i.e 0,$\{\alpha\}$,...,$\{N\alpha\}$.

Pigeonhole principle tells that $\{r\alpha\}$ and $\{s\alpha\}$ with r<s lie within the same subinterval. Then (s-r)α lie within 1/N of an integer. Now define q=s-r. Then certainly 1≤q≤N and qα is within 1/N of an integer.

Application to Fermat's Theorem of Gaussian Primes

1. Fermat's Theorem

Theorem 2.4: Show that every prime number of the form p=1(mod 4) can be written as a sum of squares of two integers.

Proof By Contradiction
Let p be such a prime, $\exists$ x such that x^2 = -1(mod p). It suffices to show that $\exists$ integer pairs (u,v), such that u+vx= 0(mod p) and $|u|$,$|v|$≤$\sqrt{p}$. The desired result would follow: u^2+v^2=0(mod 0) and 0< u^2+v^2 < 2p.Suppose on the contrary that such a pair does not exist. Consider all pairs (u,v) $\neq$) =(0,0) where

-$\sqrt{p}$ $\leq$ u,v $\leq$ $\sqrt{p}$. There are at least $(2[p]+1)^2$ - 1 $\geq$ 4p such pairs. Consider all numbers of the form u+vx(pigeons). Since there are p-1 possible residues mode p(pigeonholes) by the pigeonhole principle there must exist pairs (u,v) $\neq$ (u$^{'}$,v$^{'}$) such that u+vx= u$^{'}$ + v$^{'}$x(mod p). But ($|u-u^{'}|$,$|v-v^{'}|$) is a pair satisfying the desired condition.

This is a contradiction.

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