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Extreme Value Theory and Application

A mission improbable

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Statistical Risk Aggregation Methodology

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SECTION 1

Theory in a nutshell

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Extreme Values

Definition

t-years event: given a random variable *X* with distribution function *F*, for *t* sufficiently large, is the event corresponding to a value

$$x_t = F^{\leftarrow}(1-t^{-1})$$

where $F^{\leftarrow}$ is the generalized inverse of *F*.

Goal

Calculation of the *upper-quantile q* = 1 - t^{-1} of the df *F*, where *t* is larger than the lenght of the time-series of the available empirical observations (eg. *t* = 100-10'000 years), i.e. the level that with probability *q* will not be superceeded.

Applications

- Engineering, hydrology (1953 North Sea flood, 1953-98 Delta Works, 2004-2011 tsunami)
- Geology, meteorology, environmental analysis
- Insurance (fire & storm claims, reinsurance, …)
- **Finance (market & operational risk, stress testing)**

Historical Background

*"… In the case of the dyke problem, what were the available historical data? The basic data consisted of observations on maximum annual heights. Moreover, the previous record surge occurred in 1570 with a recorded value of (NAP+4)m, where NAP stands for normal Amsterdam level. The 1953 flooding corresponded to a (NAP+3.85)m surge. The statistical analysis in the Delta report led to an **estimated (1 - t^{-1})-quantile of (NAP+5.14)m** for the yearly maximum, based on a total of 166 historical observations.(…) Clearly, the task of estimating such a 10000year event leads to estimating well beyond the range of the data."*

Out-of-sample Estimation

Estimation well beyond the range of the data: the solution requires making extra assumptions on the underlying model.

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Useful Tools

Empirical distribution function

$$F_n(x) = \frac{\#\{i : 1 \le i \le n \quad and \quad X_i \le x\}}{n}, \quad x \in \Re$$

Order Statistics

$$X_{1,n} = \max(X_1,\ldots,X_n) \ge X_{2,n} \ge \ldots \ge X_{n,n} = \min(X_1,\ldots,X_n)$$

p-quantile

$$x_p = F^{\leftarrow}(p) = \inf\{x \in \Re : F(x) \ge p\}$$

$$x_{p,n} = F_n^{\leftarrow}(p) = X_{k,n} \qquad 1 - \frac{k}{n} < p \le 1 - \frac{k-1}{n}$$

Shortfall distribution

$$P(X - u_L \le -x \mid X < u_L) = \vartheta(x \mid u_L)$$

Excess distribution *F_u*

$$F_u(x) = P(X - u \le x \mid X > u), \quad x \ge 0$$

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Graphical Exploratory Analysis

Mean Excess Function (MEF): the mean value of the excesses above a certain threshold *u*. It is useful to classify the tail of the distribution (short vs long-tailed).

Theoretical MEF: $$e(u) = E(X - u \mid X > u), \quad u \geq 0$$

Empirical MEF: $$e(u) = \frac{\sum_{i=1}^{n} (X_i - u)^+}{\sum_{i=1}^{n} I_{\{X_i > u\}}}$$

QQ-Plot: used to test the goodness-of-fit of a parametric model $F(.,\theta)$ compared to the empirical distribution of the values X_i. For appropriate plotting sequence $p_{k,n}$ (in particular $X_{k,n}$):

$$\left\{ \left(X_{k,n}, \hat{F}^{\leftarrow}(p_{k,n}) \right) : k = 1, \ldots, n \right\}$$

Mean Excess Function: Graphical Example

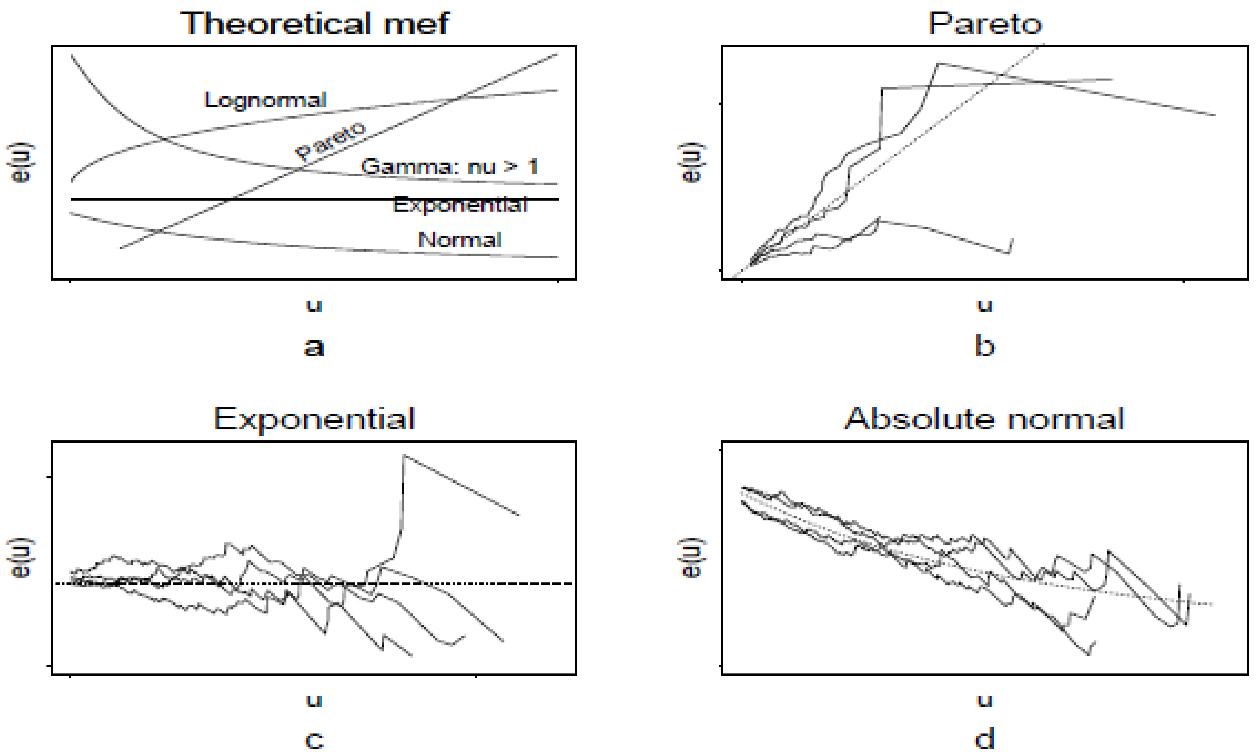

Figure 5: *Theoretical mefs (Figure a) and some examples of empirical mefs for simulated data (n=200). Normal in Figure a stands for the absolute value of a normally distributed rv. In Figure b the empirical mef of 4 simulated Pareto*(1.5) *distributed data-sets are plotted, Figure c corresponds to Exp*(1)*, and d to the absolute value of a normally distributed rv. The dotted line shows the theoretical mef of the underlying distribution.*

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Sum Statistics vs Maximum Statistics

Suppose $X_1,\ldots,X_n$ are iid with df F.

Define the **partial sum** $\boldsymbol{S_n = X_1 + \ldots + X_n}$

If there exist constants $a_n>0$ and b_n real so that:

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} Y \quad , \quad n \to \infty$$

where Y is non-degenerate with df G.

Special Case: distributions with finite variance

then $a_n = \sqrt{n}\sigma$, $b_n = n\mu$ and $G = N(0,1)$

All distributions with finite second moment are "attracted" to the standard normal.

This is the **Central Limit Theorem (CLT)**

Suppose $X_1,\ldots,X_n$ are iid with df F.

Define the **maximum** $\boldsymbol{M_n = X_{1,n} = \max(X_1,\ldots,X_n)}$.

If there exist constants $a_n>0$ and b_n real so that:

$$\frac{M_n - b_n}{a_n} \xrightarrow{d} Y \quad , \quad n \to \infty$$

where Y is non-degenerate with df G, then G is one of the following types:

1. Gumbel

$$\Lambda(x) = \exp\left\{-e^{-x}\right\}$$

2. Fréchet

$$\Phi_\alpha(x) = \begin{cases} 0 & if \quad x \le 0 \\ \exp\left\{-x^{-\alpha}\right\} & if \quad x > 0 \end{cases} \quad for \quad \alpha > 0$$

3. Weibull

$$\Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^{\alpha}\right\} & if \quad x < 0 \\ 1 & if \quad x \ge 0 \end{cases} \quad for \quad \alpha > 0$$

This is the **Fisher-Tippett Theorem (FT)**

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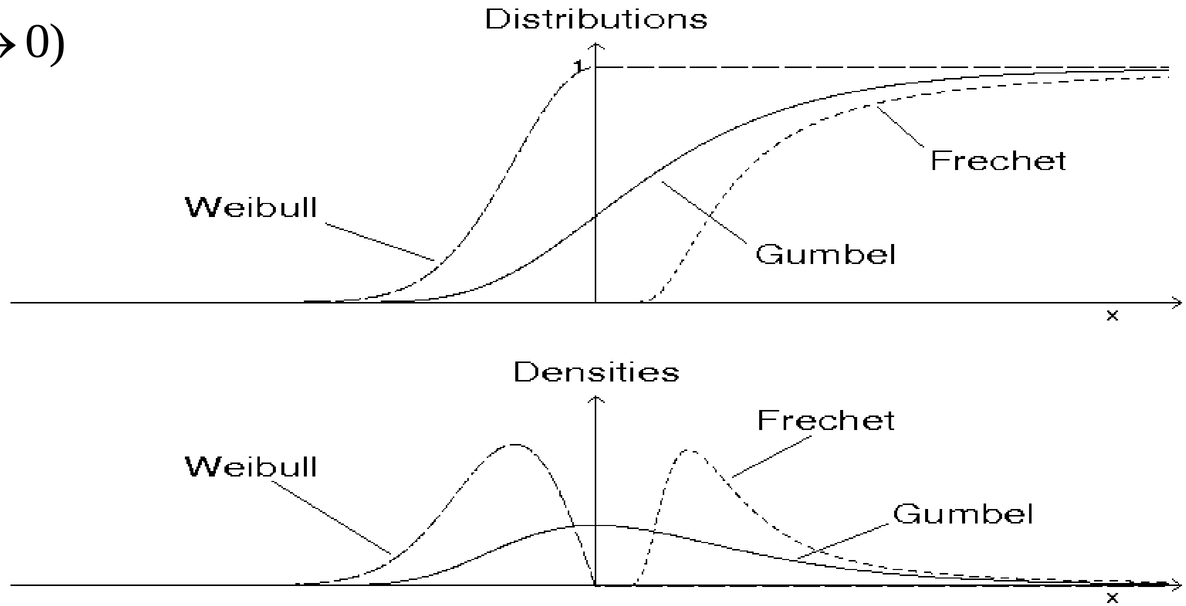
Generalized Extreme Value Distributions (GEV)

These three classes of dfs can be reparametrised using a GEV df characterized by the parameters: $\xi \in \Re$ *(shape),* $\beta > 0$ *(scale),* $\mu \in \Re$ *(location):*

$$H_{\xi,\beta,\mu}(x) = \exp\left\{-\left(1+\xi\frac{x-\mu}{\beta}\right)^{-\frac{1}{\xi}}\right\}, \quad 1+\xi\frac{x-\mu}{\beta} \geq 0$$

Particular values of ξ correspond to the case:

- Gumbel: $"\xi = 0" \quad (\xi \to 0)$
- Fréchet: $\xi > 0$
- Weibull: $\xi < 0$



Block Maxima Method

In these terms the problem of the ***maximum domain of attraction*** **(MDA)** is formalized:

the solution is given by dfs *F* such that, given H_ξ, FT theorem holds for appropriate sequences a_n and b_n. In this case we can say $\boldsymbol{F \in MDA(H_\xi)}$: *F* belongs to the MDA of H_ξ with norming constants a_n, b_n.

Suppose $X_1,\ldots,X_n$ are iid with df $F \in MDA(H_\xi)$, then:

$$P(a_n^{-1}(M_n - b_n) \le x) = P(M_n \le a_n x + b_n) = F^n(a_n x + b_n) \;\; \to \;\; H_\xi(x), \qquad n \to \infty$$

where $u = a_n x + b_n$. Using Taylor expansion one can obtain an *upper-tail* estimator:

$$\hat{\overline{F}}(u) = \frac{1}{n}\left(1 + \hat{\xi}\,\frac{u - \hat{b}_n}{\hat{a}_n}\right)^{-\frac{1}{\xi}}$$

And then an estimator for the quantile x_p is available:

$$\boxed{\hat{x}_p = \hat{b}_n + \frac{\hat{a}_n}{\hat{\xi}}\left((n(1-p))^{-\hat{\xi}} - 1\right)}$$

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Tail Index Estimators

Hill estimator:

$$\hat{\xi}^{H}_{n,k} = \frac{1}{k}\sum_{j=1}^{k} \ln X_{j,n} - \ln X_{k,n}$$

Pickands estimator:

$$\hat{\xi}^{P}_{n,k} = \frac{1}{\ln 2} \ln \frac{X_{k,n} - X_{2k,n}}{X_{2k,n} - X_{4k,n}}$$

Dekkers – Einmahl - de Haan estimator:

$$\hat{\xi}^{D}_{n} = 1 + H_n^{(1)} + \frac{1}{2}\left(\frac{(H_n^{(1)})^2}{H_n^{(2)}} - 1 \right)^{-1}$$

$$H_n^{(1)} = \frac{1}{k}\sum_{j=1}^{k} (\ln X_{j,n} - \ln X_{k+1,n})$$

$$H_n^{(2)} = \frac{1}{k}\sum_{j=1}^{k} (\ln X_{j,n} - \ln X_{k+1,n})^2$$

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Gnedenko – Pickands – Balkema - de Haan Theorem

Suppose *F* is a df with excess distribution F_u, $u\geq 0$. Then, for $\xi\in\Re$, $F\in MDA(H_\xi)$ if and only if there exists a positive measurable function β(u) so that

$$\lim_{u\to x_F}\ \sup_{0\leq x\leq x_F-u}\left|\overline{F}_u(x)-\overline{G}_{\xi,\beta(u)}(x)\right|=0$$

Where *G* is a **Generalized Pareto Distribution** with parameters β>0, $\xi\in\Re$ so defined:

$$G_{\xi,\beta}(x)=\begin{cases}1-\left(1+\xi\dfrac{x}{\beta}\right)^{-\frac{1}{\xi}}, & \xi\neq 0\\[2ex] 1-\exp\left\{-\dfrac{x}{\beta}\right\}, & \xi=0\end{cases}$$

Shape parameter

Scale parameter

GPD Properties:

- **ξ > 0** : Pareto distribution (heavy-tailed)
- **ξ = 0** : Exponential distribution
- **ξ < 0** : Pareto type II (short-tailed)

$$\boxed{G_{\xi,\beta}(x)\in MDA(H_\xi)\qquad \forall\xi\in\Re}$$

$$\boxed{\lim_{\xi\to 0} G_{\xi,\beta}(x)=G_{0,\beta}(x)}$$

$$\boxed{if\quad \xi>0,\quad E[X^k]=\infty\quad\Leftrightarrow\quad k\geq\frac{1}{\xi}\qquad\Rightarrow\qquad \exists\quad \mu=E[X]=\frac{\beta}{1-\xi}\quad\Leftrightarrow \xi<1}$$

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Peaks-Over-Threshold Method

As a consequence, it is possible to use an *upper-tail estimator* for $x,u \geq 0$ (in particular for $u \to \infty$):

$$\overline{F}(u+x) = \overline{F}_u(x)\overline{F}(u) \approx \overline{G}_{\hat{\xi},\hat{\beta}}(x)\frac{N_u}{n}$$

Where the GPD parameters can be obtained through estimation (eg. MLE), while N_u:

$$N_u = \#\{i : 1 \leq i \leq n, X_i > u\}$$

Finally it is possible to evaluate the quantile x_p:

$$\hat{x}_p = u + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{n}{N_u}(1-p)\right)^{-\hat{\xi}} - 1\right) \qquad ES_p = \frac{VaR_p}{1-\xi} + \frac{\beta - \xi u}{1-\xi}$$

SECTION 2

ETV Applications: Operational Risk Modeling

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Operational risk: regulatory definition

From Solvency II

Operational risk *is the risk of a change in value caused by the fact that actual losses, incurred for inadequate or failed internal processes, people and systems, or from external events (including legal risk), differ from the expected losses.*

From the technical point of view:

- (1) The operational-risk measure is a VaR at confidence level 99.9 % with a holding period of one year.
- (2) The measurement approach must capture potentially severe tail loss events.

Operational risk: examples

Basel categories and taxonomies

#	Event Type	Examples
1	Internal Fraud	misappropriation of assets, tax evasion, intentional mismarking of positions, bribery
2	External Fraud	theft of information, hacking damage, third-party theft and forgery
3	Employment Practices and Workplace Safety	discrimination, workers compensation, employee health and safety
4	Clients, Products, and Business Practice	market manipulation, antitrust, improper trade, product defects, fiduciary breaches, account churning
5	Damage to Physical Assets	natural disasters, terrorism, vandalism
6	Business Disruption and Systems Failures	utility disruptions, software failures, hardware failures
7	Execution, Delivery, and Process Management	data entry errors, accounting errors, failed mandatory reporting, negligent loss of client assets

Operational Risk Capital calculation

Methods for determining the minimum capital requirement

Less

Basic Indicator Approach (BIA)
Average gross Income of the last 3 years → x 15% → Minimum Capital Requirement

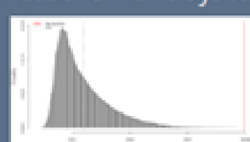
Standardised Approach (SA)
Retail Banking
Retail Brokerage
Asset Management
(Gross Income) → x 12%

Commercial Banking Agency Services
(Gross Income) → x 15%

Corporate Finance
Trading and Sales Payment & Settlement
(Gross Income) → x 18%

→ Minimum Capital Requirement

Advanced Measurement Approach (AMA)
Operational Risk Measurement System



→ Internal Model* → Minimum Capital Requirement

Qualitative Qualifying Standards

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AMA Capital calculation

Building Blocks of an Operational Risk Quantification Approach

Internal Losses	External Losses	Scenario Analysis	Business Environment and Internal Control Factors (BEICF)

According to the BCBS Supervisory Guidelines, an AMA framework must include the use of four data elements: (i) Internal loss data (ILD), (ii) External data (ED), (iii) Scenario analysis (SBA), and (iv) Business environment and internal control factors (BEICFs).

The banking industry has developed different types of models to estimate operational risk that include the four key elements in different ways.

We will focus on the category of models defined by the regulator as **LDA**:

> *Under the Loss Distribution Approach, the bank estimates, for each business line/risk type cell, the probability distribution functions of the single event impact and the event frequency for the next (one) year using its internal data, and computes the probability distribution function of the cumulative operational loss.*

LDA: Loss Distribution Approach

Model component	Description	Modelling choice
Units of Measure (UoM) = Risk Category	External Fraud, Market Conduct, etc.	• UoM taxonomy to satisfy homogeneity, not necessarily corresponding to Basel categories
Event Frequency (per UoM)	N_j: Random number of losses per year for UoM j	• Usually Poisson or Negative Binomial distribution
Event Severity (per UoM)	X_i: Random size of a loss event i	• Body: Empirical distribution or Continuous version of the empirical distribution • Tail: parametric "heavy tailed" distribution (Lognormal, Truncated lognormal, Pareto, GPD, Weibull, Burr, etc.)

Cumulative loss per year (per UoM)	$Y = X_1 + X_2 + \cdots + X_{N\_j}$
	
	Convolution of frequency and severity distributions
		• Calculation of cumulative loss distribution (CLD) per UoM via:
		    • Deterministic algorithms: FFT, Panjer, SLA
		    • Monte Carlo algorithms

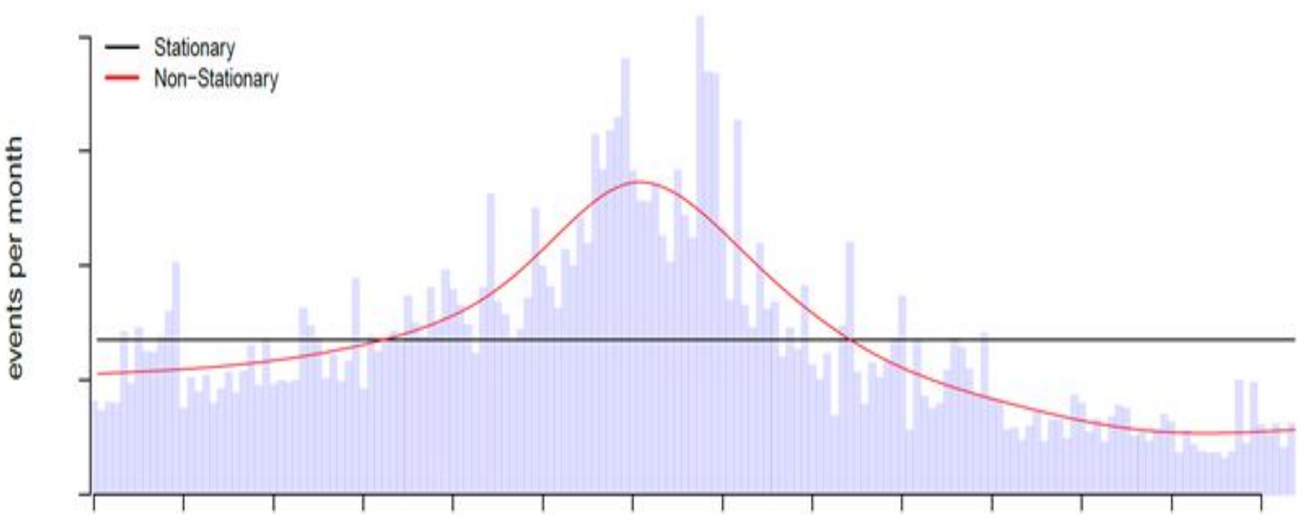
Diversification/Dependence between UoM cumulative loss distributions	To aggregate the cumulative losses per UoM to the total loss, a model for the dependence structure between cumulative losses per UoM is needed	• Independence is usually not conservative • Copula (Gaussian, t, asymmetrical) • "perfect dependence": comonotonicity
Capital calculation	Calculation of the normative capital requirement as the quantile (99.9%) of the total loss distribution	• Aggregation of UoM CLD's via copula using Monte Carlo simulation. To control Monte Carlo simulation error, capital calculation can be repeated. • Calculation of 99.9% quantile of the total yearly losses

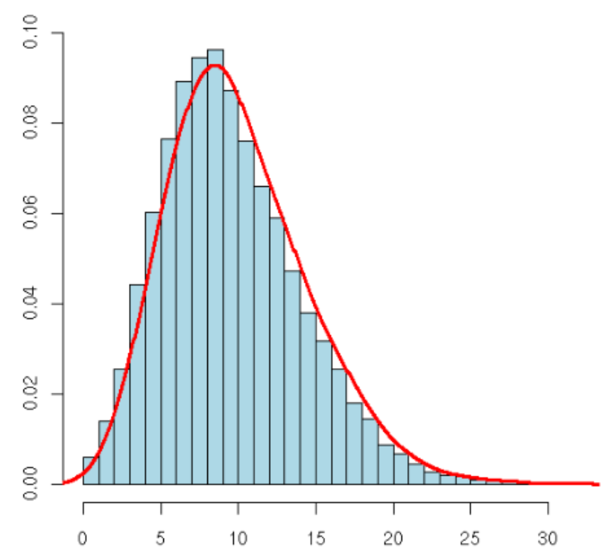
Frequency Estimation

Distribution of the number of events per UoM per year

Key elements to consider:

- Independence between frequency and severity distribution is an assumption
- Presence of temporary evolution (stationary vs non-stationary models)
- Presence of under/overdispersion (Poisson, negBin, etc.)





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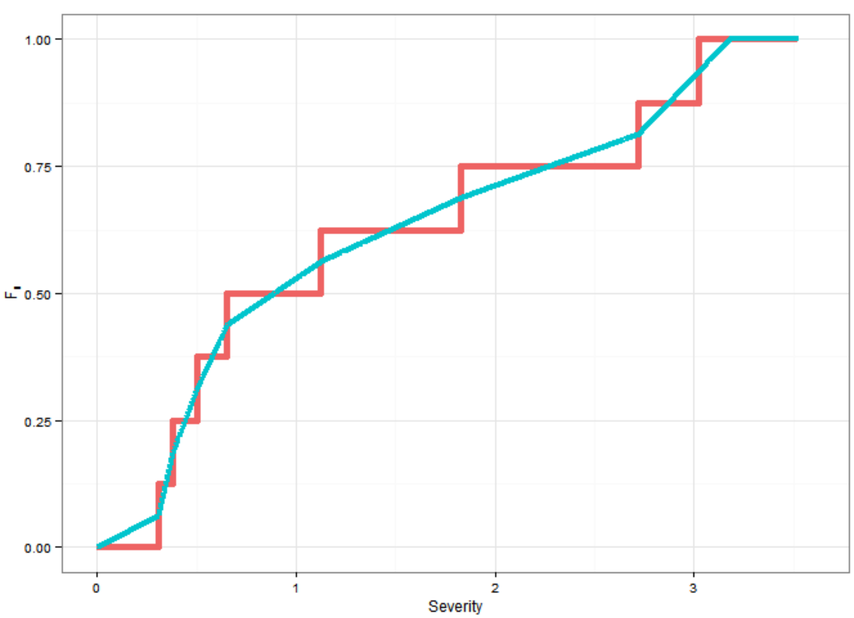
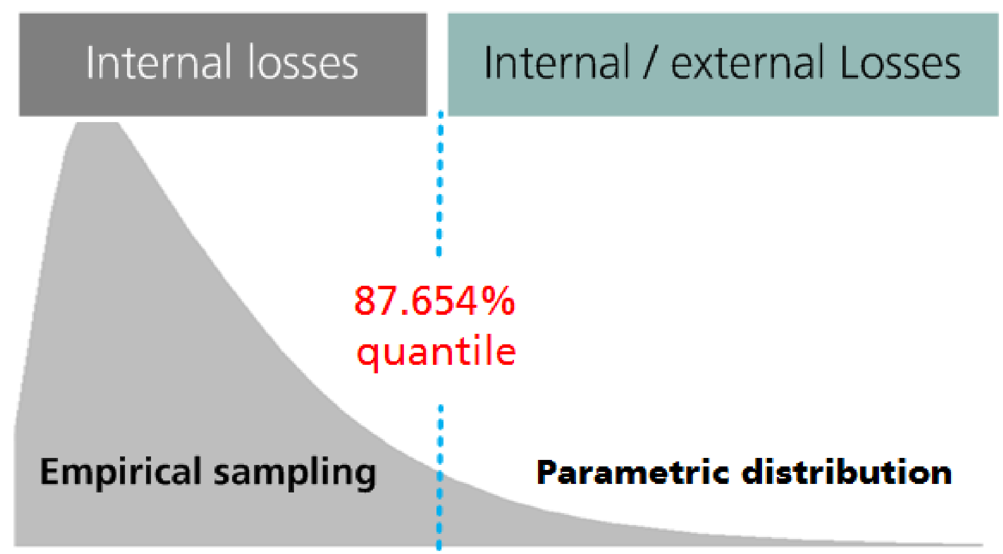
Severity Estimation

Distribution of the size of the single events per UoM

Key elements to consider:

- Independence between the size of different events in the severity distribution is an assumption.
- Use of a mixed distribution (body + tail) to better represent the empirical properties.
- The parametric piece with heavy tailed distr. to account for (potentially Inf.) events not historically experienced is a regulatory requirement.
- EVT distributions are a common choice to fit the parametric tail.





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SLA: Single Loss Approximation

For rv X_i with subexponential distribution F:

$$\lim_{x\to\infty} \frac{P(X_1 + \cdots + X_n > x)}{P(\max(X_1, \ldots, X_n) > x)} = 1 \quad \text{for some (all) } n \geq 2.$$

Operational VaR: a Closed-Form Approximation

Klaus Böcker Claudia Klüppelberg

Abstract

Klaus Böcker and Claudia Klüppelberg investigate a simple loss distribution model for operational risk. They show that, when loss data are heavy-tailed (which in practice they are), a simple closed-form approximation for the OpVaR can be obtained. They apply this approximation in particular to the Pareto severity model, for which they obtain also a simple time scaling rule for the operational VaR.

Under mild conditions for the frequency distribution – satisfied by Poisson, negBin – it can be proved that for the tail of the aggregate loss distribution G_t:

$$\overline{G}_t(x) \sim EN(t)\overline{F}(x) , \quad x \to \infty , \tag{2}$$

where EN is the expected frequency.

It follows an analytical solution of the convolution problem for asymptotic quantiles:

Theorem 2.4 (Analytical OpVaR). *Consider the Standard LDA model for fixed $t > 0$ and a subexponential severity with distribution function F. Assume, moreover, that the tail estimate (2) holds. Then, the $VaR_t(\kappa)$ satisfies the approximation*

$$VaR_t(\kappa) = F^{\leftarrow}\left(1 - \frac{1-\kappa}{EN(t)}(1 + o(1))\right) , \quad \kappa \to 1 . \tag{3}$$

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Thanks for your attention!

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