7. Laplace transform

Vincenzo Recupero
Department of Mathematical Sciences, Politecnico di Torino
vincenzo.recupero@polito.it

Version: June 5, 2013
Revision: June 8, 2016

Analysis Lecture Notes
04LSI Mathematical Methods

Disclaimer: These notes are meant to be outlines, summarizing and sometimes supplementing the lectures. They are not as good as the real thing: classroom lectures.

1 Laplace transform of functions

Definition 1.1. If $f : [0, +\infty[\longrightarrow \mathbb{C}$ is locally summable, we set

$$\Omega_f := \{s \in \mathbb{C} \ : \ f(t)e^{-st} \text{ is summable in the variable } t \in [0, +\infty[\} \subseteq \mathbb{C}, \quad (1.1)$$

and we call *Laplace transform of f* the function $\mathcal{L}(f) : \Omega_f \longrightarrow \mathbb{C}$ defined by

$$\mathcal{L}(f)(s) := \int_0^{+\infty} f(t)e^{-st}\,\mathrm{d}t, \qquad s \in \Omega_f. \quad (1.2)$$

The Laplace transform $\mathcal{L}(f)$ is often defined for functions f whose domain is the whole real line $\mathbb{R}$ and that are null on $]-\infty, 0[$: $\mathcal{L}(f)(s) := \int_{-\infty}^{+\infty} f(t)e^{-st}\,\mathrm{d}t$. In order to avoid any ambiguity in this case we will write

$$\mathcal{L}(f)(s) = \int_{-\infty}^{+\infty} H(t)f(t)e^{-st}\,\mathrm{d}t.$$

Let us observe that it can happen that integral in (1.2) is convergent even $f(t)e^{-st}$ is not summable (in t).

It is not hard to prove the following

Proposition 1.1. *If $f : [0, +\infty[\longrightarrow \mathbb{C}$ is locally summable and we set*

$$\lambda_f := \inf \left\{ \operatorname{Re} s \ : \ f(t)e^{-st} \text{ is summable in the variable } t \in [0, +\infty[\right\} \tag{1.3}$$

(we agree to define $\inf \varnothing := +\infty$*), then only one of the following cases can occurr*

(i) $\Omega_f = \{s \in \mathbb{C} \ : \ \operatorname{Re} s > \lambda_f\}$ *(if λ_f is finite)*

(ii) $\Omega_f = \{s \in \mathbb{C} \ : \ \operatorname{Re} s \geqslant \lambda_f\}$ *(if λ_f is finite)*

(iii) $\Omega_f = \mathbb{C}$

(iv) $\Omega_f = \varnothing$.

Definition 1.2. A locally summable function $f : [0, +\infty[\longrightarrow \mathbb{C}$ is said to be $\mathcal{L}$-*transformable* (or *Laplace (absolutely) transformable*) if $\Omega \neq \varnothing$. The number λ_f is called *abscissa of (absolute) convergence.* The set Ω_f is called *set of (absolute) convergence.*

The previous proposition implies that the set of convergence is always a half plane whose boundary is a vertical line (when it is not empty or the whole $\mathbb{C}$).

Example 1.1. Let us compute the Laplace transform of $H(t)$. We have

$$\mathcal{L}(H(t))(s) = \int_0^{+\infty} e^{-st}\, \mathrm{d}t = \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=+\infty}.$$

Now $|e^{-st}| = e^{-(\operatorname{Re} s)t}$, hence $\lim_{t\to+\infty} e^{-st}$ exists (in $\mathbb{C}$) if and only if $\operatorname{Re} s > 0$. In this case this limit is zero, therefore $\lambda_H = 0$ and Ω_H is the half plane $\{x > 0\}$. We have

$$\boxed{\mathcal{L}(H(t))(s) = \frac{1}{s}, \qquad \operatorname{Re} s > 0.} \tag{1.4}$$

♡

In the next proposition it is convenient to adopt the following notations:

$$\Omega_f + s_0 := \{s_1 + s_0 \ : \ s_1 \in \Omega_f\},$$
$$a\Omega_f := \{as_1 \ : \ s_1 \in \Omega_f\},$$

holding for $s_0 \in \mathbb{C}$ and $a > 0$. In the first case Ω_f is translated from 0 to s_0, Ω_f being a halfplane; in the second case we are dilating (or shrinking) Ω_f by a real factor a.

Proposition 1.2. *If $f, g : [0, \infty[\longrightarrow \mathbb{C}$ are $\mathcal{L}$-transformable and $\lambda, \mu \in \mathbb{C}$, $s_0 \in \mathbb{C}$, $t_0, a > 0$, then:*

(i) $\mathcal{L}(\lambda f + \mu g) = \lambda\mathcal{L}(f) + \mu\mathcal{L}(g) \qquad$ *in* $\Omega_f \cap \Omega_g$

(ii) $\mathcal{L}(e^{s_0 t} f(t))(s) = \mathcal{L}(f(t))(s - s_0) \qquad$ *for every* $s \in \Omega_f + s_0$

(iii) $\mathcal{L}(f(t - t_0)H(t - t_0))(s) = e^{-t_0 s}\mathcal{L}(f(t))(s) \qquad$ *for every* $s \in \Omega_f$

(iv) $\mathcal{L}(f(at))(s) = \dfrac{1}{a}\mathcal{L}(f(t))\left(\dfrac{s}{a}\right) \qquad$ *for every* $s \in a\Omega_f$

(v) $\exists[\mathcal{L}(f(t))]'(s) = -\mathcal{L}(tf(t))(s) \qquad$ *for every s in the interior of* Ω_f

(vi) *if $f \in C([0, +\infty[)$ is differentiable in $]0, +\infty[$ and f is $\mathcal{L}$-transformable, then f is $\mathcal{L}$-transformable and*

$\mathcal{L}(f'(t))(s) = s\mathcal{L}(f(t))(s) - f(0+)$ *for every* $s \in \Omega_{f'}$.

It is important to note that statement (v) of the previous Proposition 1.2 implies that $\mathcal{L}(f)(s)$ is a holomorphic function in the interior of Ω_f.

Example 1.2. Let us compute $\mathcal{L}(tH(t))$ by means of property (v) of Proposition 1.2. We find

$$\mathcal{L}(tH(t))(s) = -[\mathcal{L}(H(t))]'(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}$$

for $\operatorname{Re} s > 0$. In the same fashion, for $\operatorname{Re} s > 0$, we have

$$\mathcal{L}(t^2 H(t))(s) = \mathcal{L}(ttH(t))(s) = -[\mathcal{L}(tH(t))]'(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3},$$

and, iterating the procedure, we get

$$\boxed{\mathcal{L}(t^k H(t))(s) = \frac{k!}{s^{k+1}}, \qquad \operatorname{Re} s > 0.} \tag{1.5}$$

♡

More generally one can iterate the formula in Proposition 1.2(v): first we get $[\mathcal{L}(f(t))]''(s) = [-\mathcal{L}(tf(t))]'(s) = +\mathcal{L}(t^2 f(t))(s)$, in general

$$\boxed{[\mathcal{L}(f(t))]^{(k)}(s) = (-1)^k \mathcal{L}(t^k f(t))(s), \qquad k \in \mathbb{N}.} \tag{1.6}$$

Example 1.3. Compute $\mathcal{L}(e^{s_0 t}H(t))$ for $s_0 \in \mathbb{C}$.
This transform can be easily computed by means of the definition. However we use Proposition 1.2(ii). We have $\mathcal{L} = (e^{s_0 t}H(t))(s) = \mathcal{L}(H(t))(s - s_0) = \frac{1}{s - s_0}$ per $\operatorname{Re} s > \operatorname{Re} s_0$:

$$\boxed{\mathcal{L}(e^{s_0 t}H(t))(s) = \frac{1}{s - s_0}, \qquad \operatorname{Re} s > \operatorname{Re} s_0.} \tag{1.7}$$

♡

Example 1.4. Compute $\mathcal{L}(\cos(\omega t)H(t))$, where $\omega \in \mathbb{R}$.

Thanks to (i) and (ii) of Proposition 1.2

$$\begin{aligned}\mathcal{L}(\cos(\omega t)H(t))(s) &= \mathcal{L}\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}H(t)\right)(s)\\ &= \frac{1}{2}\mathcal{L}(e^{i\omega t}H(t))(s)+\frac{1}{2}\mathcal{L}(e^{-i\omega t}H(t))(s)\\ &= \frac{1}{2}\mathcal{L}(H(t))(s-i\omega)+\frac{1}{2}\mathcal{L}(H(t))(s+i\omega)\\ &= \frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right)=\frac{s}{s^2+\omega^2}\end{aligned}$$

where last equality holds for $\operatorname{Re} s > \operatorname{Re} i\omega = 0$ and $\operatorname{Re} s > \operatorname{Re} -i\omega = 0$. Summarizing

$$\boxed{\mathcal{L}(\cos(\omega t)H(t))(s) = \frac{s}{s^2+\omega^2}, \qquad \operatorname{Re} s > 0.} \tag{1.8}$$

Check as an exercise that

$$\boxed{\mathcal{L}(\sin(\omega t)H(t))(s) = \frac{\omega}{s^2+\omega^2}, \qquad \operatorname{Re} s > 0.} \tag{1.9}$$

♡

Example 1.5. Compute $\mathcal{L}(\sinh(\omega t)H(t))$, where $\omega \in \mathbb{R}$.
We have that

$$\begin{aligned}\mathcal{L}(\sinh(\omega t)H(t))(s) &= \mathcal{L}\left(\frac{e^{\omega t}-e^{\omega t}}{2}H(t)\right)(s)\\ &= \frac{1}{2}\mathcal{L}(e^{\omega t}H(t))(s)-\frac{1}{2}\mathcal{L}(e^{-\omega t}H(t))(s)\\ &= \frac{1}{2}\mathcal{L}(H(t))(s-\omega)-\frac{1}{2}\mathcal{L}(H(t))(s+\omega)\\ &= \frac{1}{2}\left(\frac{1}{s-\omega}-\frac{1}{s+\omega}\right)=\frac{\omega}{s^2+\omega^2}\end{aligned}$$

for $\operatorname{Re} s > \omega$ and $\operatorname{Re} s > -\omega$. Therefore

$$\boxed{\mathcal{L}(\sinh(\omega t)H(t))(s) = \frac{\omega}{s^2-\omega^2}, \qquad \operatorname{Re} s > |\omega|.} \tag{1.10}$$

We leave to the reader the verification of

$$\boxed{\mathcal{L}(\cosh(\omega t)H(t))(s) = \frac{s}{s^2-\omega^2}, \qquad \operatorname{Re} s > |\omega|.} \tag{1.11}$$

♡

From Proposition 1.2 one can deduce the following properties.

Proposition 1.3. *If f is $\mathcal{L}$-transformable, then*

(i) *($\mathcal{L}$-transform of the antiderivative) Every antiderivative of f is $\mathcal{L}$-transformable and*

$$\mathcal{L}\left(\int_0^t f(\tau)\,\mathrm{d}\tau\right)(s) = \frac{\mathcal{L}(f(t))(s)}{s}, \quad s \in \Omega_f,\ \operatorname{Re} s > 0.$$

(ii) *If $f(t)/t$ is $\mathcal{L}$-transformable, then*

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^{+\infty} \mathcal{L}(f)(\sigma)\,\mathrm{d}\sigma, \quad s \in \Omega_f \cap \mathbb{R},\ s > 0.$$

(iii) *If f is T-periodic, then*

$$\mathcal{L}(f(t))(s) = \frac{1}{1-e^{-Ts}} \int_0^T f(t)e^{-st}\,\mathrm{d}t.$$

Now we present some other properties (without proofs).

Proposition 1.4. *If f is $\mathcal{L}$-transformable, then*

(i) $\lim_{\operatorname{Re} s \to +\infty} \mathcal{L}(f)(s) = 0$.

(ii) *(Initial value theorem) If $f(0+)$ exists and is finite, then*

$$f(0+) = \lim_{\operatorname{Re} s \to +\infty} s\mathcal{L}(f)(s).$$

(iii) *(Final value theorem) If $f(+\infty) := \lim_{t\to+\infty} f(t)$ exists and is finite, then*

$$f(+\infty) = \lim_{s\to 0+} s\mathcal{L}(f)(s).$$

Example 1.6. Compute $\mathcal{L}(H(t-2))$.
We have

$$\mathcal{L}(H(t-2))(s) = \mathcal{L}(H(t-2)H(t-2))(s) = e^{-2s}\mathcal{L}(H(t))(s) = e^{-2s}\frac{1}{s}$$

for $\operatorname{Re} s > 0$. ♡

Example 1.7. Compute $\mathcal{L}(p_1(t-2))$.
We have $p_1(t-2) = H(t-3/2) - H(t-5/2)$, hence

$$\begin{aligned}
\mathcal{L}(p_1(t-2))(s) &= \mathcal{L}(H(t-3/2))(s) - \mathcal{L}(H(t-5/2))(s) \\
&= \mathcal{L}(H(t-3/2)H(t-3/2))(s) - \mathcal{L}(H(t-5/2)H(t-5/2))(s) \\
&= e^{-3s/2}\mathcal{L}(H(t))(s) - e^{-5s/2}\mathcal{L}(H(t))(s) \\
&= e^{-3s/2}\frac{1}{s} - e^{-5s/2}\frac{1}{s} = \frac{e^{-3s/2} - e^{-5s/2}}{s}
\end{aligned}$$

for $\operatorname{Re} s > 0$. ♡

Example 1.8. Compute $\mathcal{L}(f)$, where

$$f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ 2-t & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}.$$

We have $f(t) = [H(t)-H(t-1)]+(2-t)[H(t-1)-H(t-2)] = H(t)-(t-1)H(t-1)+(t-2)H(t-2)$, therefore

$$\begin{aligned} \mathcal{L}(f(t))(s) &= \mathcal{L}(H(t))(s) - \mathcal{L}((t-1)H(t-1))(s) + \mathcal{L}((t-2)H(t-2))(s) \\ &= \mathcal{L}(H(t))(s) - e^{-s}\mathcal{L}(tH(t))(s) + e^{-2s}\mathcal{L}(tH(t))(s) \\ &= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \frac{s - e^{-s} + e^{-2s}}{s^2} \end{aligned}$$

for $\operatorname{Re} s > 0$. ♡

Example 1.9. Compute $\mathcal{L}(tp_1(t-1/2))$.
We have

$$\begin{aligned} \mathcal{L}(tp_1(t-1/2))(s) &= -[\mathcal{L}(p_1(t-1/2))]'(s) = -[\mathcal{L}(H(t)-H(t-1))]'(s) \\ &= -\mathcal{L}(H(t))'(s) + \mathcal{L}(H(t-1))'(s) \\ &= -\mathcal{L}(H(t))'(s) + \mathcal{L}(H(t-1)H(t-1))'(s) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{e^{-s}}{s} = \frac{1 - e^{-s} - se^{-s}}{s^2} \end{aligned}$$

for $\operatorname{Re} s > 0$. ♡

Example 1.10. Compute $\mathcal{L}(e^{-it}t^7H(t))$.
We have

$$\mathcal{L}(e^{-it}t^7H(t))(s) = \mathcal{L}(t^7H(t))(s+i) = \frac{7!}{(s+i)^8} \tag{1.12}$$

for $\operatorname{Re} s > \operatorname{Re}(-i) = 0$. ♡

Example 1.11. Compute $\mathcal{L}((t-4)H(t-5))$.
Since $(t-4)H(t-5) = (t-5)H(t-5) + H(t-5)$, we have

$$\begin{aligned} \mathcal{L}((t-4)H(t-5))(s) &= \mathcal{L}((t-5)H(t-5))(s) + \mathcal{L}(H(t-5))(s) \\ &= e^{-5s}\mathcal{L}(tH(t))(s) + e^{-5s}\mathcal{L}(H(t))(s) \\ &= e^{-5s}\left[\frac{1}{s^2} + \frac{1}{s}\right]. \end{aligned}$$

♡

2 Inversion formula

Let us assume that $f : [0, +\infty[\longrightarrow \mathbb{C}$ is continuous and $\mathcal{L}$-transformable and that

$$\mathcal{L}(f)(s_1 + is_2) \text{ is summable in the variable } s_2 \in \mathbb{R} \text{ for some fixed } s_1 > \lambda_f. \tag{2.1}$$

We extend the domain of f to $]-\infty, 0[$ by setting $f(t) = 0$ for every $t < 0$. We have that

$$\begin{aligned}\mathcal{L}(f)(s_1 + is_2) &= \int_{-\infty}^{+\infty} H(t)f(t)e^{-s_1 t}e^{-2\pi i(s_2/2\pi)t}\,\mathrm{d}t \\ &= \mathcal{F}(H(t)f(t)e^{-s_1 t})\left(\frac{s_2}{2\pi}\right)\end{aligned}$$

for every $s_2 \in \mathbb{R}$. Hence this formula also holds with s_2 replaced by $2\pi s_2$, therefore

$$\mathcal{L}(f)(s_1 + i2\pi s_2) = \mathcal{F}(H(t)f(t)e^{-s_1 t})(s_2) \qquad \forall s_2 \in \mathbb{R}. \tag{2.2}$$

The assumption (2.1) allows us to apply the pointwise Fourier inversion formula (Corollary 4.1 of the lecture notes on the Fourier transform and the following remark[1]) to both sides of this equation as functions of s_2. We get that for every $t > 0$

$$\begin{aligned}f(t)e^{-s_1 t} &= \mathcal{F}^{-1}(\mathcal{L}(f)(s_1 + 2\pi i s_2))(t) \\ &= \int_{-\infty}^{+\infty} \mathcal{L}(f)(s_1 + 2\pi i s_2)e^{2\pi i t s_2}\,\mathrm{d}s_2 \quad (\tau = 2\pi s_2,\ \mathrm{d}\tau = 2\pi\,\mathrm{d}s_2) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{L}(f)(s_1 + i\tau)e^{it\tau}\,\mathrm{d}\tau.\end{aligned}$$

Hence

$$f(t) = \lim_{R\to+\infty} \frac{1}{2\pi}\int_{-R}^{R} \mathcal{L}(f)(s_1 + i\tau)e^{(s_1+i\tau)t}\,\mathrm{d}\tau \qquad \forall t > 0. \tag{2.3}$$

If we recall the definition of complex line integral, we consider the curve $\gamma_{R,s_1} : [-R, R] \longrightarrow \mathbb{C}$ defined by

$$\gamma_{R,s_1}(\tau) = s_1 + i\tau, \qquad \tau \in [-R, R],$$

parametrizing the vertical segment from $s_1 - iR$ to $s_1 + iR$. Since $\gamma'_{R,s_1}(\tau) = i$ we find that

$$f(t) = \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{\gamma_{R,s_1}} \mathcal{L}(f)(s)e^{st}\,\mathrm{d}s \qquad \forall t > 0. \tag{2.4}$$

Taking the limit as $R \to +\infty$ we have a sort of improper integral on the vertical line passing through s_1. Usually the notation

$$\int_{s_1-i\infty}^{s_1+i\infty} \mathcal{L}(f)(s)e^{st}\,\mathrm{d}s := \lim_{R\to+\infty}\int_{\gamma_{R,s_1}} \mathcal{L}(f)(s)e^{st}\,\mathrm{d}s, \tag{2.5}$$

is used, hence (2.4) reads

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{s_1-i\infty}^{s_1+i\infty} \mathcal{L}(f)(s)e^{st}\,\mathrm{d}s \qquad \forall t > 0,} \tag{2.6}$$

which is called *Riemann-Fourier formula*. Summarizing we have proved the following

[1] $H(t)f(t)e^{-s_1 t}$ is not necessarily continuous at $t = 0$

Theorem 2.1. *Assume that $f : [0, +\infty[\longrightarrow \mathbb{C}$ is continuous and $\mathcal{L}$-transformable and that*

$$\mathcal{L}(f)(s_1 + is_2) \text{ is summable in } s_2 \in \mathbb{R} \text{ for some } s_1 > \lambda_f. \tag{2.7}$$

Then the Riemann-Fourier (2.6) *holds for every $t > 0$.*

Remark 2.1. By means of advanced tools from integration theory, one could prove that the assumption that f *is continuous* is not really necessary: if we do not require the continuity, it is possible to prove that f (more precisely $H(t)f(t)$) agrees with a continuous function eveywhere except on a set that is "negligible", in the sense that the integral does not change. We can say that f is essentially continuous, more precisely $H(t)f(t)$ is essentially continuous, and we have that $f(0) = 0$. This fact limits the range of application of Theorem 2.1, nevertheless it allows to prove the following fundamental Theorem 2.2. ♢

The previous theorem requires a strong assumtpion, the summability of $\mathcal{L}(f)(s_1 + is_2)$ in the variable s_2, which is not satisfied by very simple functions like $f(t) = H(t)$ for which $F(s) = \mathcal{L}(f)(s) = 1/s$. Nevertheless it has an important consequence: the invertibility of the Laplace transform. Indeed we have the following

Theorem 2.2. *The Laplace transform is injective on the set of continuous $\mathcal{L}$-transformable functions, i.e. if $\mathcal{L}(f) = \mathcal{L}(g)$ then $f = g$ for every pair of continuous $\mathcal{L}$-transformable functions f, g.*

Proof. If $\mathcal{L}(f) = \mathcal{L}(g)$ then $\mathcal{L}(f - g) = \mathcal{L}(f) - \mathcal{L}(g) = 0$, therefore $\mathcal{L}(f - g)$ is summable. Therefore we can apply the Riemann-Fourier formula and we find

$$f(t) - g(t) = (2\pi i)^{-1} \int_{s_1 - i\infty}^{s_1 + i\infty} \mathcal{L}(f - g)(s) e^{st} \, \mathrm{d}s = 0$$

for every $t > 0$. □

Remark 2.2. Taking into account Remark 2.1, one can see that in some way the continuity of f and g is not necessary also in Theorem 2.2. ♢

Theorem 2.2 states that $\mathcal{L}$ is invertible, therefore if a given $F(s)$ is the Laplace transform of some function (continuous on $[0, +\infty[$), then this function is unique, i.e. there exists a unique $f(t)$ such that $\mathcal{L}(f(t))(s) = F(s)$. Such function f is called *inverse Laplace transform of F* and is denoted by $\mathcal{L}^{-1}(F)$.

Example 2.1. Compute the inverse $\mathcal{L}$-transform of $F(s) = \dfrac{s}{4s^2 + 9}$.
We have

$$F(s) = \frac{s}{4s^2 + 9} = \frac{1}{4} \frac{s}{s^2 + 9/4}$$

therefore from (1.8) we get

$$\mathcal{L}^{-1}(F(s))(t) = \frac{1}{4} \cos(3t/2) H(t).$$

♡

Example 2.2. Let us compute the inverse $\mathcal{L}$-transform of $F(s) = \dfrac{e^{-3s}s}{4s^2+9}$.
Since $\mathcal{L}(f(t-t_0)H(t-t_0))(s) = e^{-t_0 s}\mathcal{L}(f(t))(s)$ we have $\mathcal{L}^{-1}(e^{-t_0 s}\mathcal{L}(f(t))(s)) = f(t-t_0)H(t-t_0)$, in other words if $G(s) = \mathcal{L}(f(t))(s)$ then $\mathcal{L}^{-1}(e^{-t_0 s}G(s)) = \mathcal{L}^{-1}(G(s))(t-t_0)$. Hence, exploiting the result of the previous exercise we get (here $G(s) = \dfrac{s}{4s^2+9}$ and $t_0 = 3$)

$$\mathcal{L}^{-1}(F(s))(t) = \mathcal{L}^{-1}\left(\frac{s}{4s^2+9}\right)(t-3) = \frac{1}{4}\cos\left(\frac{3}{2}(t-3)\right)H(t-3)$$

$\heartsuit$

Example 2.3. Let us compute the inverse $\mathcal{L}$-transform of $F(s) = \dfrac{1}{(s-3)^2}$.
Since $\mathcal{L}(e^{s_0 t}f(t))(s) = \mathcal{L}(f(t))(s-s_0)$ we have

$$\mathcal{L}^{-1}(F(s))(t) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)(t) = e^{3t}tH(t)$$

$\heartsuit$

Example 2.4. Find the inverse Laplace transform of $F(s) = \dfrac{1}{(2s-1)^2}$.

Since $F(s) = \dfrac{1}{4(s-1/2)^2}$, we have

$$\mathcal{L}^{-1}(F(s)) = \frac{1}{4}e^{t/2}tH(t).$$

$\heartsuit$

Example 2.5. Let us compute the inverse Laplace transform of $F(s) = \dfrac{2}{(s+1)^2+4}$.
We have

$$\mathcal{L}^{-1}(F(s)) = e^{-t}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = e^{-t}\sin(2t)H(t).$$

$\heartsuit$

Now we see a sufficient condition for $F(s)$ to be the Laplace transform and such that its inverse transform is provided by the Riemann-Fourier formula. We omit the proof.

Theorem 2.3. *Let $F(s)$ be holomorphic in the set $\{s \in \mathbb{C} \;:\; \operatorname{Re} s > \lambda_0\}$ for some $\lambda_0 \in \mathbb{R}$. Assume moreover that there exists $M > 0$ such that $|F(s)| \leqslant M/|s|^2$ for every $s \in \mathbb{C}$ such that $\operatorname{Re} s > \lambda_0$. Then F is the $\mathcal{L}$-transform $f(t) = \mathcal{L}^{-1}(F)(t)$ given by the Riemann-Fourier formula* (2.6) *(with $s_1 > \lambda_0$) for $t > 0$.*

Now we prove the following Theorem 2.4. You can skip the proof and go directly to the statement of the theorem.

Theorem 2.3 applies for instance to functions like $F(s) = \frac{1}{s^2+\omega^2}$, indeed if $\operatorname{Re} s > \omega > 0$, we have (recall that $||z| - |w|| \leqslant |z-w|$)

$$\left|\frac{1}{s^2+\omega^2}\right| \leqslant \frac{1}{||s|^2-\omega^2|} = \frac{1}{|s|^2-\omega^2} \sim \frac{1}{|s|^2} \quad \text{as } |s| \to +\infty.$$ [2]

[2] hence there exists $A > 0$ such that $|F(s)| \leqslant 2/|s|^2$ if $|s| > A$; if instead $|s| \leqslant A$ by the Weierstrass theorem there exists $C > 0$ such that $|F(s)| \leqslant C \leqslant CA^2/|s|^2$. We can therefore take $M = \max\{2, CA^2\}$.

Therefore the theorem also applies to positive powers of $\frac{1}{s^2+\omega^2}$. In a similar way one can see that the assumptions are satisfied by functions like $1/(s-s_0)^m$ with $m \geqslant 2$. Instead the theorem is useless for $F(s) = 1/s$ (or $F(s) = 1/(s-s_0)$). For such simple, but important, functions, the validity of the Riemann-Fourier formula can be checked by a direct computation involving the residue theorem. Collecting together these remarks we infer that the inverse transform of

$$F(s) = \frac{P(s)}{Q(s)}, \qquad \text{with } P, Q \text{ polynomial}, \quad \text{degree}(P) < \text{degree}(Q)$$

is provided by the Riemann-Fourier formula for t positive, i.e.

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) = \frac{1}{2\pi i}\int_{s_1-i\infty}^{s_1+i\infty} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s = \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{s_1-iR}^{s_1+iR} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s. \tag{2.8}$$

Let us assume now that P and Q have no common roots (otherwise we divide P by Q). If we choose s_1 to the right of all the poles $z_1, \ldots, z_N$ of $F = P/Q$ and if we consider R large enough such that all these poles are contained in the rectangle C_R having vertices $s_1 - iR,\ s_1 + iR,\ -R + iR,\ -R - iR$, then

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) &= \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{s_1-iR}^{s_1+iR} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s \\
&= \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{C_R} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s - \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{D_R} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s
\end{aligned}$$

where D_R is the polygonal path connecting the points $s_1 + iR,\ -R + iR,\ -R - iR, s_1 - iR$. It is possible to prove that $\lim_{R\to+\infty} \frac{1}{2\pi i}\int_{D_R} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s = 0$, hence, by the residue theorem we have that

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) = \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{C_R} \frac{P(s)}{Q(s)} e^{st}\,\mathrm{d}s = \sum_{k=1}^{N} \mathrm{Res}_{\frac{P(s)}{Q(s)}e^{ts}}(z_k). \tag{2.9}$$

for $t > 0$. If the poles are simple we can write for $t > 0$,

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) = \sum_{k=1}^{N} \mathrm{Res}_{\frac{P(s)}{Q(s)}e^{ts}}(z_k) = \sum_{k=1}^{N} \frac{P(z_k)}{Q'(z_k)} e^{z_k t} \tag{2.10}$$

which is called *Heaviside formula*. Summarizing:

Theorem 2.4. *Let $P(s), Q(s)$ be two polynomials with* degree(P) < degree(Q) *(with no common roots), and let $z_1, \ldots, z_N$ be the pole of $F(s) := P(s)/Q(s)$. Then*

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) = H(t)\sum_{k=1}^{N} \mathrm{Res}_{F(s)e^{ts}}(z_k) = H(t)\sum_{k=1}^{N} \mathrm{Res}_{\frac{P(s)}{Q(s)}e^{ts}}(z_k). \tag{2.11}$$

If the poles of F are simple then the following Heaviside formula *holds:*

$$\mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right)(t) = H(t)\sum_{k=1}^{N} \frac{P(z_k)}{Q'(z_k)} e^{z_k t} \tag{2.12}$$

Example 2.6. Compute the Laplace inverse transform of $F(s) = \dfrac{1}{(s^2+3)(s-2)}$.
If we decompose F in partial fraction we get

$$F(s) = \frac{1}{7}\left(\frac{-s-2}{s^2+3} + \frac{1}{s-2}\right),$$

(we used the real decomposition, but you can use the complex one), therefore by linearity

$$\begin{aligned}
\mathcal{L}^{-1}(F(s))(t) &= -\frac{1}{7}\mathcal{L}^{-1}\left(\frac{s}{s^2+3}\right)(t) - \frac{2}{7}\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)(t) + \frac{1}{7}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)(t)\\
&= -\frac{1}{7}\mathcal{L}^{-1}\left(\frac{s}{s^2+3}\right)(t) - \frac{2}{7\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)(t) + \frac{1}{7}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)(t)\\
&= -\frac{1}{7}\cos(\sqrt{3}t)H(t) - \frac{2}{7\sqrt{3}}\sin(\sqrt{3}t)H(t) + \frac{1}{7}e^{2t}H(t).
\end{aligned}$$

This exercise can be solved by applying directly Theorem 2.4. Since all the poles are simple we can use (2.12) and obtain

$$\mathcal{L}^{-1}(F(s))(t) = H(t)\left[\frac{e^{i\sqrt{3}t}}{2i\sqrt{3}(i\sqrt{3}-2)} + \frac{e^{-i\sqrt{3}t}}{-2i\sqrt{3}(i\sqrt{3}+2)} + \frac{e^{2t}}{7}\right]. \tag{2.13}$$

Check, by the Euler formulas, that this formula provides the same result. ♡

Example 2.7. Compute the inverse $\mathcal{L}$-transform of $F(s) = \dfrac{e^{-s}}{s^2(s^2+2)}$.
Let us start computing the inverse transform of $G(s) = \dfrac{1}{s^2(s^2+2)}$, the final result will be then obtained by a translation. If we want to apply Theorem 2.4, we have to use formula 2.11, because there is a pole that is not simple. We find

$$\begin{aligned}
\mathcal{L}^{-1}(G(s))(t) &= H(t)\left[\mathrm{Res}_{G(s)e^{st}}(0) + \mathrm{Res}_{G(s)e^{st}}(i\sqrt{2}) + \mathrm{Res}_{G(s)e^{st}}(-i\sqrt{2})\right]\\
&= H(t)\left[\frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^{ts}}{s^2+2}\right)\Big|_{s=0} + \left(\frac{e^{ts}}{s^2(s+i\sqrt{2})}\right)\Big|_{s=i\sqrt{2}} + \left(\frac{e^{ts}}{s^2(s-i\sqrt{2})}\right)\Big|_{s=-i\sqrt{2}}\right]
\end{aligned}$$

(observe that $e^{ts} \neq 0$ for $s=0$ and $s = \pm i\sqrt{2}$ (indeed for every s!)). Proceeding with the calculations we find

$$\mathcal{L}^{-1}(G(s))(t) = \left[\frac{1}{2}t - \frac{1}{4\sqrt{2}i}(e^{i\sqrt{2}t} - e^{-i\sqrt{2}i})\right]H(t) = \frac{1}{2}tH(t) - \frac{1}{2\sqrt{2}}\sin(\sqrt{2}t)H(t)$$

where in the last equality we used the Euler formulas. Recalling that $\mathcal{L}(f(t-t_0)H(t-t_0))(s) = e^{-t_0 s}\mathcal{L}(f(t))(s)$, we obtain

$$\mathcal{L}^{-1}(F(s))(t) = \frac{1}{2}(t-1)H(t-1) - \frac{1}{2\sqrt{2}}\sin(\sqrt{2}(t-1))H(t-1).$$

Of course we can solve the exercise by using the partial fraction decomposition $G(s)$:

$$G(s) = \frac{2}{2s^2(s^2+2)} = \frac{2+s^2-s^2}{2s^2(s^2+2)} = \frac{1}{2s^2} - \frac{1}{2(s^2+2)}$$

(there are other ways to decompose G). ♡

3 $\mathcal{L}$-transform of distributions

In order to extend the Laplace transform to distributions, let us first consider the classical case of functions. If $f(t)$ is $\mathcal{L}$-transformable, after we extend its definition to $]-\infty, 0[$ by setting $f(t) = 0$ for $t < 0$, we could be tempted to write

$$\mathcal{L}(f)(s) = \int_0^{+\infty} f(t)e^{-st}\,\mathrm{d}t = \langle T_{f(t)}, e^{-st}\rangle.$$

But the function $\varphi(t) = e^{-st} \in C^\infty$ has not compact support, thus the right hand side makes sense if the support of f is compact, i.e. if $\operatorname{supp}(T_f)$ is compact. In this case the equation holds for every $s \in \mathbb{C}$. Hence we start with the following

Definition 3.1. If $T \in \mathscr{D}'(\mathbb{R})$ has compact support contained in $[0, +\infty[$, we define the function $\mathcal{L}(T) : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$\mathcal{L}(T)(s) := \langle T(t), e^{-st} \rangle, \qquad s \in \mathbb{C}. \tag{3.1}$$

Observe that if $\operatorname{Re} s > 0$ then $\varphi(t) = e^{-st}$ has not compact support, but it is rapidly decreasing as $t \to +\infty$[3] as $t \to +\infty$. Instead, if $\operatorname{Re} s > 0$, the modulus of the test function $\varphi(t) = e^{-st}$ approaches $+\infty$ as $t \to -\infty$. Nevertheless we are assuming that $\operatorname{supp}(T) \subseteq [0, +\infty[$, hence we can imagine that the behaviour of the test functions for $t < 0$ is immaterial. Therefore the first definition of distributional Laplace transform can be essentially extended to the tempered distributions. In order to do this in a proper way we need to introduce the auxiliary function $\xi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\xi \in C^\infty(\mathbb{R}) \ : \ \begin{cases} \xi(t) = 0 & \text{if } t \leqslant -1 \\ \xi(t) \in [0,1] & \text{if } -1 < t < 0 \ ^{4} \\ \xi(t) = 1 & \text{if } t \leqslant 0 \end{cases} \tag{3.2}$$

Replacing e^{-st} with $\xi(t)e^{-st} \in \mathscr{S}(\mathbb{R})$ we can therefore give the following rigorous definition:

Definition 3.2. Let $T \in \mathscr{S}'(\mathbb{R})$ be such that $\operatorname{supp}(T) \subseteq [0, +\infty[$ and set $\Omega := \{s \in \mathbb{C} \ : \ \operatorname{Re} s > 0\}$. We define the function $\mathcal{L}(T) : \Omega \longrightarrow \mathbb{C}$ by setting

$$\mathcal{L}(T)(s) := \langle T(t), \xi(t)e^{-st} \rangle, \qquad s \in \mathbb{C}, \quad \operatorname{Re} s > 0. \tag{3.3}$$

Note that $\psi(t) := \xi(t)e^{-st} \in \mathscr{S}(\mathbb{R})$, therefore the "bracket" in (3.3) makes sense because now T is tempered. It is possible to prove that Definition 3.2 is independent of the choice of ξ satisfying (3.2). It is also possible to check that if T has compact support, then this definition is consistent with the previous Definition 3.1

Definition 3.2 is not completely satisfactory because it cannot be used, for instance, for a distribution like T_{e^t}, but e^t is $\mathcal{L}$-transformable by means of the classical Definition 1.1. The idea for a further extension to the whole $\mathscr{D}'(\mathbb{R})$ is to mimic the classical definition by replacing the statement "$e^{-st}f(t)$ is summable" with the statement "$e^{-st}T(t)$ is tempered". Let us see how to do this. We start with the following

[3]we mean that $\lim_{t\to+\infty} t^p\varphi(t) = 0$ and $\lim_{t\to+\infty} t^p\varphi^{(q)}(t) = 0$ for every $p, q \in \mathbb{N}$.

[4]it is reasonable to guess that such function exists, if you want to prove this you can define it starting from a "bell shaped" function $\rho \in C^\infty$ such that $\rho \geqslant 0$, $\operatorname{supp}(\rho) = [-1, 0]$ and $\int_{\mathbb{R}} \rho(t)\, \mathrm{d}t = 1$. Hence we can take $\xi(t) := \int_{-\infty}^{t} \rho(\tau)\, \mathrm{d}\tau$.

Definition 3.3. Let $T \in \mathscr{D}'(\mathbb{R})$ be such that $\operatorname{supp}(T) \subseteq [0,+\infty[$. We say that T *is $\mathcal{L}$-transformble* if there exists $s_0 \in \mathbb{C}$ such that $e^{-s_0 t}T(t) \in \mathscr{S}'(\mathbb{R})$. Hence we set

$$\lambda_T := \inf\{\operatorname{Re} s \ : \ s \in \mathbb{C},\ e^{-st}T(t) \in \mathscr{S}'(\mathbb{R})\}, \tag{3.4}$$
$$\Omega_T := \{s \in \mathbb{C} \ : \ \operatorname{Re} s > \lambda_T\}. \tag{3.5}$$

It is not hard to prove that, under the assumptions of the previous definition,

$$\operatorname{Re} s > \lambda_T \quad \Longrightarrow \quad e^{-st}T(t) \in \mathscr{S}'(\mathbb{R}), \tag{3.6}$$

hence we can finally define the $\mathcal{L}$-transform for general distributions:

Definition 3.4. Let $T \in \mathscr{D}'(\mathbb{R})$ be $\mathcal{L}$-transformable. We call *Laplace transform of T* the function $\mathcal{L}(T) : \Omega_T \longrightarrow \mathbb{C}$ defined by

$$\mathcal{L}(T)(s) := \langle e^{-\lambda_0 t}T(t), \xi(t)e^{-(s-\lambda_0)t}\rangle, \qquad s \in \Omega_T, \tag{3.7}$$

where $\lambda_0 \in\,]\lambda_T, \operatorname{Re} s[$ and $\xi(t)$ is a function satisfying (3.2).

Observe that if $\lambda_T < \lambda_0 < \operatorname{Re} s$ then $\operatorname{Re}(s-\lambda_0) < 0$, therefore $\xi(t)e^{-(s-\lambda_0)t} \in \mathscr{S}$. Moreover, thanks to (3.6) $e^{-\lambda_0 t}T(t) \in \mathscr{S}'$, hence the "bracket" in (3.7) makes sense. Again one can prove that Definition 3.4 does not depend on the choice of $\lambda_0 \in\,]\lambda_T, \operatorname{Re} s[$ and ξ satisfying (3.2). Of course if $T \in \mathscr{S}'$, this definition is consistent with the previous Definitions 3.1 and 3.2. In order to understand better Definition 3.4 one should remember that $\operatorname{supp}(T) \subseteq [0,+\infty[$ and develop the following non rigorous computation: $\langle e^{-\lambda_0 t}T(t), \xi(t)e^{-(s-\lambda_0)t}\rangle = \langle T(t), e^{-\lambda_0 t}\xi(t)e^{-(s-\lambda_0)t}\rangle = \langle T(t), \xi(t)e^{-st}\rangle = \langle T(t), e^{-st}\rangle$ (why are these inequalities not correct?).

One can prove the following important

Proposition 3.1. *If $f : [0,+\infty[\longrightarrow \mathbb{C}$ is $\mathcal{L}$-transformable and we extend f to $]-\infty,0[$ by setting $f(t) = 0$ for $t < 0$, we have*

$$\mathcal{L}(f)(s) = \mathcal{L}(T_f)(s) \qquad \forall s \in \mathbb{C}, \quad \operatorname{Re} s > \lambda_f.$$

Therefore the previous definitions are actually a generalization of the Laplace transform of functions.

Let us observe that in general if f is $\mathcal{L}$-transformable only the inequality $\lambda_{T_f} \leqslant \lambda_f$ holds true, and it can happen that $\lambda_{T_f} < \lambda_f$.

Example 3.1. Let us compute $\mathcal{L}(\delta_{x_0})$, $x_0 \in \mathbb{R}$.
Since δ_{x_0} has compact support we can use the first Definition 3.1:

$$\mathcal{L}(\delta_{x_0})(s) = \langle \delta_{x_0}(t), e^{-st}\rangle = e^{-sx_0}.$$

Hence

$$\boxed{\mathcal{L}(\delta_{x_0})(s) = e^{-sx_0}, \qquad s \in \mathbb{C}.} \tag{3.8}$$

In particular if $x_0 = 0$

$$\boxed{\mathcal{L}(\delta_0)(s) = 1, \qquad s \in \mathbb{C}.} \tag{3.9}$$

♡

The following properties are analogous to those in Proposition 1.2.

Proposition 3.2. *Let $T, S \in \mathscr{D}'(\mathbb{R})$ be $\mathcal{L}$-transformable and $\lambda, \mu \in \mathbb{C}$, $s_0 \in \mathbb{C}$, $t_0, a > 0$. Then*

(i) $\mathcal{L}(\lambda T + \mu S) = \lambda \mathcal{L}(T) + \mu \mathcal{L}(S)$ *in* $\Omega_T \cap \Omega_S$

(ii) $\mathcal{L}(e^{s_0 t} T(t))(s) = \mathcal{L}(T(t))(s - s_0)$ *for every* $s \in \Omega_T + s_0$

(iii) $\mathcal{L}(T(t - t_0))(s) = e^{-t_0 s} \mathcal{L}(T(t))(s)$ *for every* $s \in \Omega_T$

(iv) $\mathcal{L}(T(at))(s) = \dfrac{1}{a} \mathcal{L}(T(t)) \left(\dfrac{s}{a}\right)$ *for every* $s \in a\Omega_T$

(v) $\exists [\mathcal{L}(T(t))]'(s) = -\mathcal{L}(tT(t))(s)$ *for every* $s \in \Omega_T$

(vi) $\mathcal{L}(T'(t))(s) = s\mathcal{L}(T(t))(s)$ *for every* $s \in \Omega_T$.

The derivative in (vi) *is meant in the sense of distributions.*

We omit the proofs of the previous properties and we only observe that from (v) it follows that $\mathcal{L}(T)(s)$ is holomorphic in Ω_T. However it is important to understand that statement (vi) does not contradict the analogous (vi) of Proposition 1.2 (observe the difference between these formulas). This difference is due to the fact that when we consider the $\mathcal{L}$-transform of T_f, we have to extend the definition of f to $]-\infty, 0[$, hence the resulting function may have a jump at the point $t = 0$, so that we will get a Dirac delta when differentiating in the sense of ditributions. Let us perform the precise computation. If $f \in C([0, +\infty[)$ is differentiable in $]0, +\infty[$, and f' is $\mathcal{L}$-transformable, then we extend f to $\mathbb{R}$ by setting $f(t) = 0$ for every $t < 0$. Therefore we have $T_f' = T_{f'} + f(0+)\delta_0$ and

$$\begin{aligned} \mathcal{L}(T_f')(s) &= \mathcal{L}(T_{f'} + f(0+)\delta_0)(s) = \mathcal{L}(T_{f'})(s) + f(0+)\mathcal{L}(\delta_0)(s) \\ &= \mathcal{L}(f')(s) + f(0+) = [s\mathcal{L}(f)(s) - f(0+)] + f(0+) \\ &= s\mathcal{L}(f)(s) = s\mathcal{L}(T)(s). \end{aligned}$$

Example 3.2. Let us compute $\mathcal{L}(\delta'_{x_0})$, $x_0 \in \mathbb{R}$.
Thanks to Proposition 3.2(vi) we find

$$\mathcal{L}(\delta'_{x_0})(s) = s\mathcal{L}(\delta_{x_0})(s) = se^{-sx_0}.$$

Iterating this formula we get

$$\boxed{\mathcal{L}(\delta_{x_0}^{(p)})(s) = s^p e^{-sx_0}, \qquad s \in \mathbb{C}.} \tag{3.10}$$

In particular for $x_0 = 0$

$$\boxed{\mathcal{L}(\delta_0^{(p)})(s) = s^p, \qquad s \in \mathbb{C}.} \tag{3.11}$$

♡

Example 3.3. Let us compute the $\mathcal{L}$-transform of the so called *impulse train for positive times* $T = \sum_{k=0}^{\infty} \delta_k$, the tempered distribution defined by

$$\left\langle \sum_{k=0}^{\infty} \delta_k, \varphi \right\rangle := \sum_{k=0}^{\infty} \varphi(k), \qquad \varphi \in \mathscr{S}.$$

Using the second Definition 3.2 we get

$$\mathcal{L}(T)(s) = \left\langle \sum_{k=0}^{\infty} \delta_k, \xi(t) e^{-st} \right\rangle = \sum_{k=0}^{\infty} \xi(k) e^{-sk} = \sum_{k=0}^{\infty} e^{-sk} = \sum_{k=0}^{\infty} (e^{-s})^k = \frac{1}{1-e^{-s}}.$$

♡

Now we provide some hints about the important relationship between convolution and Laplace transform. First of all let us observe that for any pair of $\mathcal{L}$-transformable functions f, g, their convolution can be written in the following alternative way:

$$(f * g)(t) = \int_0^t f(t-s) g(s)\, \mathrm{d}s \qquad \forall t \in \mathbb{R}, \tag{3.12}$$

indeed $f(s) = g(s) = 0$ for every $s < 0$, hence if $t \in \mathbb{R}$ is fixed we have $f(t-s) = 0$ if $s > t$, so

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s) g(s)\, \mathrm{d}s = \int_0^t f(t-s) g(s)\, \mathrm{d}s.$$

In particular $(f * g)(t) = 0$ for every $t < 0$ and it can be proved that $f * g$ is $\mathcal{L}$-transformable and that

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g), \qquad s \in \Omega_f \cap \Omega_g. \tag{3.13}$$

Concerning the convolution of two distributions, one can prove that if $T, S \in \mathscr{D}'(\mathbb{R})$ are $\mathcal{L}$-transformable then $T * S$ is $\mathcal{L}$-transformable and

$$\mathcal{L}(T * S) = \mathcal{L}(T)\mathcal{L}(S), \qquad s \in \Omega_T \cap \Omega_S. \tag{3.14}$$

Even if we defined the convolution $T * S$ when at least of the two distributions T and S has compact support, it can be proved that when both the supports of T and S are contained in $[0, +\infty[$, the formula defining $T * S$ makes sense.

Let us conclude by mentioning the fact that the Laplace transform is invertible also in the framework of distributions, i.e. if $F(s)$ is the transform of a distribution $T \in \mathscr{D}'(\mathbb{R})$, then T is also the only distribution such that $\mathcal{L}(T)(s) = F(s)$. Hence the inverse transform $\mathcal{L}^{-1}(F)$ is well defined for a suitable class of function $F(s)$. In particular it is possible to find the inverse transform of any polynomial and, thanks to formula (3.11) and to the linearity we get

$$\boxed{\mathcal{L}^{-1}(a_n s^n + \cdots + a_1 s + a_0) = a_n \delta_0^{(n)} + \cdots + a_1 \delta_0' + a_0 \delta_0, \qquad s \in \mathbb{C}.} \tag{3.15}$$

for every $a_1, a_2, \dots, a_n \in \mathbb{C}$.

Example 3.4. Compute the inverse Laplace transform of $F(s) = \dfrac{s^2 - 3s - 3}{s-4}$.
The degree of the numerator is *grater or equal* than the degree of the denominator. Hence, after dividing the two polynomials, we find $s^2 - 3s - 3 = (s+1)(s-4) + 1$, so that

$$F(s) = s + 1 + \frac{1}{s-4}.$$

Therefore

$$\mathcal{L}^{-1}(F(s))(t) = \mathcal{L}^{-1}(s)(t) + \mathcal{L}^{-1}(1)(t) + \mathcal{L}^{-1}\left(\frac{1}{s-4}\right)(t) = \delta_0' + \delta_0 + e^{4t} H(t).$$

♡

4 Exercise

a) Compute the Laplace transform of the following functions

1. $f(t) = p_a(t - t_0)$ where $a, t_0 > 0$ e $a/2 < t_0$.
2. $f(t) = \mathbb{1}_{[a,b]}(t)$ where $0 \leqslant a < b$.
3. $f(t) = p_4(t - 1)H(t)$
4. $f(t) = \mathbb{1}_{[-1,2]}(t)H(t)$
5. $$f(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant 1 \\ 2 - t & \text{if } 1 < t < 2 \\ t - 2 & \text{if } t \geqslant 2 \\ 0 & \text{if } t < 0 \end{cases}$$
6. $$f(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ -2 & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$
7. $f(t) = H(t)e^{t-1} + H(t - 2)\cos t$
8. $f(t) = e^{3t} \int_0^t \frac{\sin(2x)}{x}\,\mathrm{d}x$ (compute $\mathcal{L}(f)(s)$ for real s only. Hint: use Proposition 1.3).
9. $f(t) = e^{s_0 t} t^k H(t)$, where $s_0 \in \mathbb{C}$ and $k \in \mathbb{N}$.
10. $f(t) = \cos^2 t H(t)$
11. $f(t) = (t-3)H(t-2)e^{t+1}$ (hint: $(t-3)H(t-2)e^{t+1} = [(t-2)-1]H(t-2)e^{t-2}e^3$)

b) Compute the inverse Laplace transform of the following functions.

1. $F(s) = \dfrac{1}{(s+5)^3}$
2. $F(s) = \dfrac{2e^{-3s}\cosh s}{s^2 + s^3}$ (hint: $\cosh s = (e^s + e^{-s})/2$)
3. $F(s) = \dfrac{(s-2)e^{-\pi s}}{s^2 + 2s + 1}$
4. $F(s) = \dfrac{s^3 + 2s^2 + s - 1}{s^2 + 2s + 1}$
5. $F(s) = \dfrac{s^2 + 3}{(s+1)(s^2 + 6s + 9)}$
6. $F(s) = \dfrac{s^4 + 3s^3 + 2s^2 + 4s + 4}{(s+3)(s^2+1)}$

Answers

a)

1. $\dfrac{e^{-t_0 s} 2\sinh(as/2)}{s}$ for $\operatorname{Re} s > 0$.
2. $\dfrac{e^{-as} - e^{-bs}}{s}$ for $\operatorname{Re} s > 0$.
3. $\dfrac{1 - e^{-3s}}{s}$ for $\operatorname{Re} s > 0$.
4. $\dfrac{1 - e^{-2s}}{s}$ for $\operatorname{Re} s > 0$.
5. $\dfrac{1 - 2e^{-s} + 2e^{-2s}}{s^2}$ for $\operatorname{Re} s > 0$.
6. $\dfrac{1 - 3e^{-s} + 2e^{-2s}}{s}$ for $\operatorname{Re} s > 0$.
7. $\dfrac{e^{-1}}{s-1} + \dfrac{e^{2(i-s)}}{2(s-i)} + \dfrac{e^{-2(i+s)}}{2(s+i)}$ for $\operatorname{Re} s > 1$.
8. $\dfrac{1}{s-3}\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{s-3}{2}\right)\right]$ for $s > 3$.
9. $\dfrac{k!}{(s-s_0)^{k+1}}$ for $\operatorname{Re} s > \operatorname{Re} s_0$.
10. $\dfrac{4s^2 + 8}{4s^3 + 16s}$ for $\operatorname{Re} s > 0$.
11. $e^{3-2s}\left[\dfrac{1}{(s-1)^2} - \dfrac{1}{s-1}\right]$ for $\operatorname{Re} s > 0$.

b)
1. $\dfrac{e^{-5t}}{2} t^2 H(t)$
2. $(t-2)H(t-2) - H(t-2) - e^{-(t-2)}H(t-2) + (t-4)H(t-4) - H(t-4) - e^{-(t-4)}H(t-4)$
3. $e^{-(t-\pi)}(1 - 3(t-\pi))H(t-\pi)$
4. $\delta_0' - te^{-t}H(t)$
5. $e^{-t}H(t) - 6te^{-3t}H(t)$
6. $\delta_0' + e^{-3t}H(t) + \sin t H(t)$

5 Applications to ordinary differential equations

The Laplace transform is a powerful tool for solving linear ordinary differential equations with constant coefficients, in a sense it is the natural tool for such equations, since it transform them into algebraic equations. Let us see how it works in some simple but fundamental examples.

5.1 Simple harmonic oscillator with sinusoidal driving force.

Assume that $\omega > 0$, $\beta > 0$, $F > 0$, and $\omega \neq \beta$. We look for the C^2 function $y : [0, +\infty[\longrightarrow \mathbb{R}$ satisfying the Cauchy problem

$$\begin{cases} y''(t) + \omega^2 y(t) = F\cos(\beta t) & \forall t \geqslant 0, \\ y(0) = y_0, \\ y'(0) = y_1. \end{cases} \tag{5.1}$$

The first equation is the *simple harmonic oscillator equation with a sinusoidal driving force*: notice that we are dealing with the case $\beta \neq \omega$, Let us take the $\mathcal{L}$-transform of both sides of the equation. If we set $Y(s) := \mathcal{L}(y)(s)$, from Proposition 1.2-(vi) we get

$$s^2 Y(s) - sy(0+) - y'(0+) + \omega^2 Y(s) = F\mathcal{L}(H(t)\cos(\beta t))(s),$$

thus using the initial conditions and the fact that $y \in C^1([0, +\infty[)$ we infer that

$$s^2 Y(s) - sy_0 - y_1 + \omega^2 Y(s) = \frac{Fs}{s^2+\beta^2},$$

i.e.

$$Y(s) = \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} + \frac{Fs}{(s^2+\beta^2)(s^2+\omega^2)}. \tag{5.2}$$

If we decompose the last term of the previous equation in partial fractions (exercise), we get that

$$\frac{Fs}{(s^2+\beta^2)(s^2+\omega^2)} = \frac{Fs}{(\omega^2-\beta^2)(s^2+\beta^2)} - \frac{Fs}{(\omega^2-\beta^2)(s^2+\omega^2)},$$

therefore

$$\mathcal{L}^1\left(\frac{Fs}{(s^2+\beta^2)(s^2+\omega^2)}\right)(t) = \frac{F}{\omega^2-\beta^2}H(t)\cos(\beta t) - \frac{F}{\omega^2-\beta^2}H(t)\cos(\omega t),$$

hence, applying $\mathcal{L}^{-1}$ to equation (5.2) we deduce the solution of (5.1)

$$y(t) = y_0 H(t)\cos(\omega t) + \frac{y_1 H(t)}{\omega}\sin(\omega t) + \frac{F}{\omega^2-\beta^2}H(t)\cos(\beta t) - \frac{F}{\omega^2-\beta^2}H(t)\cos(\omega t)$$

in other terms

$$y(t) = \left(y_0 - \frac{F}{\omega^2-\beta^2}\right)\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{F}{\omega^2-\beta^2}\cos(\beta t) \qquad \forall t \geqslant 0.$$

5.2 Simple harmonic oscillator with arbitrary driving force

More generally we can consider the Cauchy problem

$$\begin{cases} y''(t) + \omega^2 y(t) = f(t) & \forall t \geqslant 0, \\ y(0) = y_0, \\ y'(0) = y_1, \end{cases} \tag{5.3}$$

where we are given $\omega > 0$ and $f \in C([0, +\infty[)$, an arbitrary driving force for the harmonic oscillator. In order to find the solution $y \in C^1([0, +\infty[)$ we follow the same procedure and we take $\mathcal{L}$ to both sides of the differential equation. Using the initial conditions we find

$$\begin{aligned} Y(s) &= \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} + \frac{\mathcal{L}(f(t))(s)}{(s^2+\omega^2)} \\ &= \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} + \frac{1}{\omega}\mathcal{L}(f(t))(s)\mathcal{L}(\sin(\omega t))(s) \\ &= \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} + \frac{1}{\omega}\mathcal{L}(f(t) * \sin(\omega t))(s), \end{aligned}$$

thus taking $\mathcal{L}^{-1}$ we deduce that

$$\begin{aligned} y(t) &= y_0 \cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{1}{\omega} f(t) * \sin(\omega t) \\ &= y_0 \cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{1}{\omega}\int_0^t f(x)\sin(\omega(t-x))\,\mathrm{d}x \quad \forall t \geqslant 0. \end{aligned}$$

If we consider the particular case $f(t) = F\cos(\omega t)$ with $F > 0$, we have that

$$\begin{aligned} \int_0^t f(x)\sin(\omega(t-x))\,\mathrm{d}x &= F\int_0^t \cos(\omega x)\sin(\omega(t-x))\,\mathrm{d}x \\ &= \frac{F}{2}\int_0^t (\sin(\omega t) + \sin(\omega(2x-t))\,\mathrm{d}x \\ &= \frac{F}{2}\left[x\sin(\omega t) - \frac{\cos(\omega(2x-t))}{2\omega}\right]_{x=0}^{x=t} \\ &= \frac{F}{2} t\sin(\omega t), \end{aligned}$$

therefore we find

$$y(t) = y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{F}{2\omega} t\sin(\omega t) \qquad \forall t \geqslant 0.$$

Notice the resonance phenomenon due to the fact that the frequency of the driving force is exactly ω.

Remark 5.1. The particular case $f(t) = F\cos(\omega t)$ with $F > 0$ can be treated with no use of the convolution, indeed we can exploit the complex partial fraction decomposition and write

$$\begin{aligned} \frac{\mathcal{L}(f(t))(s)}{(s^2+\omega^2)} = \frac{Fs}{(s^2+\omega^2)^2} &= \frac{Fs}{(s-i\omega)^2(s+i\omega)^2} \\ &= -\frac{Fi}{4\omega(s-i\omega)^2} + \frac{Fi}{4\omega(s+i\omega)^2} \end{aligned}$$

so that

$$\begin{aligned}
Y(s) &= \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} + \frac{\mathcal{L}(f(t))(s)}{(s^2+\omega^2)} \\
&= \frac{sy_0}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2} - \frac{Fi}{4\omega(s-i\omega)^2} + \frac{Fi}{4\omega(s+i\omega)^2}
\end{aligned}$$

and taking $\mathcal{L}^{-1}$

$$\begin{aligned}
y(t) &= y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) - \frac{Fi}{4\omega}te^{i\omega t} + \frac{Fi}{4\omega}te^{-i\omega t} \\
&= y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{F}{2\omega}\frac{1}{2i}te^{i\omega t} - \frac{F}{2\omega}\frac{1}{2i}te^{-i\omega t} \\
&= y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) + \frac{F}{2\omega}t\sin(\omega t) \qquad \forall t \geqslant 0.
\end{aligned}$$

$\diamondsuit$

5.3 Simple harmonic oscillator with an impulsive driving force

Fix $\omega > 0$. We look for a function $y : [0,+\infty[\longrightarrow \mathbb{R}$ satisfying the harmonic oscillator equation with a forcing term F given by a unitary impulse at the time, say, $t = 1$, i.e. $F = \delta_1$, and with prescribed initial conditions for y and y'. We need to set the problem in the framework of distributions $\mathscr{D}'(\mathbb{R})$, thus we extend the domain of the searched solution y to $\mathbb{R}$ and we look for $y : \mathbb{R} \longrightarrow \mathbb{R}$ such that $y(t) := 0$ for $t < 0$ (equivalently $\operatorname{supp}(y) \subseteq [0,+\infty[$) such that

$$T_{y''} + \omega^2 T_y = \delta_1 \qquad \text{in the sense of distributions} \tag{5.4}$$

with

$$y(0+) = y_0, \tag{5.5}$$
$$y'(0+) = y_1. \tag{5.6}$$

Equation (5.4) leads to a distributional equation of the form $T'' + \omega^2 T = G$ where $T \in \mathscr{D}'(\mathbb{R})$ is the unknow and $G \in \mathscr{D}'(\mathbb{R})$ is given, and it makes no sense to prescribe initial conditions for T, since a priori it is not a function. So the initial conditions (5.5)–(5.6) need to be inserted *into* the equation, more precisely in the right hand side G in the following way: $T_y'' = T_{y''} + y'(0+)\delta_0 + y(0+)\delta_0'$, thus (5.4) reads

$$T_y'' + \omega^2 T_y = \delta_1 + y'(0+)\delta_0 + y(0+)\delta_0'$$

Thus the distributional Cauchy problem we need to study is the following:

$$T'' + \omega^2 T = \delta_1 + y_0\delta_0' + y_1\delta_0 \qquad \text{in } \mathscr{D}'(\mathbb{R}). \tag{5.7}$$

Let us apply $\mathcal{L}$ to both sides of this equation. If $Y := \mathcal{L}(T)$, an application of Proposition 3.2-(vi) leads to

$$s^2Y(s) + \omega^2 Y(s) = e^{-s} + y_0 s + y_1$$

hence

$$Y(s) = \frac{e^{-s}}{s^2+\omega^2} + \frac{y_0 s}{s^2+\omega^2} + \frac{y_1}{s^2+\omega^2}.$$

Coming back with $\mathcal{L}^{-1}$ we find

$$y(t) = \frac{1}{\omega}H(t-1)\sin(\omega(t-1)) + y_0 H(t)\cos(\omega t) + \frac{y_1}{\omega}H(t)\sin(\omega t)$$

i.e.

$$y(t) = \frac{1}{\omega}H(t-1)\sin(\omega(t-1)) + y_0\cos(\omega t) + \frac{y_1}{\omega}\sin(\omega t) \qquad \forall t \geqslant 0.$$

Notice that for $0 < t < 1$ we have that $y(t)$ is equal to the solution of the homogeneous equation $y'' + \omega^2 y = 0$ satisftying (5.5)–(5.6); at the time $t = 1$ the impulsive force δ_1 is applied to the system and for $t > 1$ the solution $y(t)$ has the additional term $\frac{1}{\omega}H(t-1)\sin(\omega(t-1))$.

Changes from revision June 10, 2015 to revision June 8, 3016 :

1. Example 2.6: A missing $H(t)$ has been added.
2. Hints added to Exercises a)-11 and b)-2.
3. An appendix on applications to differential equations has been added. This part is *not* required for the exam