6. Fourier transform

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Analysis Lecture Notes
04LSI Mathematical Methods

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1 Fourier transform: first properties

Definition 1.1. If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable, we call *Fourier transform of g* the function $\mathcal{F}(g) : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{F}(g)(\nu) := \int_{-\infty}^{+\infty} g(t)e^{-2\pi i\nu t}\,\mathrm{d}t, \qquad \nu \in \mathbb{R}. \tag{1.1}$$

The notation $\hat{g} := \mathcal{F}(g)$ is also used.

This definition makes sense since if ν is fixed, then $|g(t)e^{-2\pi i\nu t}| = |g(t)|$ for every t, thus $g(t)e^{-2\pi i\nu t}$ is integrable by the comparison criterion for improper integrals. It is worth noting that the integral in (1.1) can be convergent even if g is not summable, therefore the Fourier transform can be defined also for some non summable functions. We do not consider such cases, since we will set them in the wider framework of the theory of distributions. We will also use the notation $\int_{\mathbb{R}}$ rather that $\int_{-\infty}^{+\infty}$. Let us start by studying some properties of the Fourier transform.

Proposition 1.1. *If $g, h : \mathbb{R} \longrightarrow \mathbb{C}$ are summable, $\lambda, \mu \in \mathbb{C}$, $\nu_0, t_0, a \in \mathbb{R}$, $a \neq 0$, then*

(i) $\mathcal{F}(\lambda g + \mu h) = \lambda \mathcal{F}(f) + \mu \mathcal{F}(h)$.

(ii) $\mathcal{F}(e^{2\pi i \nu_0 t} g(t))(\nu) = \mathcal{F}(g(t))(\nu - \nu_0)$ *for every* $\nu \in \mathbb{R}$.

(iii) $\mathcal{F}(g(t - t_0))(\nu) = e^{-2\pi i t_0 \nu} \mathcal{F}(g(t))(\nu)$ *for every* $\nu \in \mathbb{R}$.

(iv) $\mathcal{F}(g(at))(\nu) = \dfrac{1}{|a|} \mathcal{F}(g(t)) \left(\dfrac{\nu}{a}\right)$ *for every* $\nu \in \mathbb{R}$.

(v) $\mathcal{F}(g(-t))(\nu) = \mathcal{F}(g(t))(-\nu)$ *for every* $\nu \in \mathbb{R}$.

(vi) $\mathcal{F}(\overline{g(t)})(\nu) = \overline{\mathcal{F}(g(t))(-\nu)}$ *for every* $\nu \in \mathbb{R}$.

Proof.

(i) It is a consequence of the linearity of the integral.

(ii) $\mathcal{F}(e^{2\pi i \nu_0 t} g(t))(\nu) = \int_{\mathbb{R}} e^{2\pi i \nu_0 t} g(t) e^{-2\pi i \nu t}\, \mathrm{d}t = \int_{\mathbb{R}} g(t) e^{-2\pi i (\nu - \nu_0) t}\, \mathrm{d}t = \mathcal{F}(g(t))(\nu - \nu_0)$.

(iii) By the change of variable $s = t - t_0$ one has $\mathcal{F}(g(t - t_0))(\nu) = \int_{\mathbb{R}} g(t - t_0) e^{-2\pi i \nu t}\, \mathrm{d}t = \int_{\mathbb{R}} g(s) e^{-2\pi i \nu (s + t_0)}\, \mathrm{d}s = e^{-2\pi i \nu t_0} \int_{\mathbb{R}} g(s) e^{-2\pi i \nu s}\, \mathrm{d}s = e^{-2\pi i \nu t_0} \mathcal{F}(g)(\nu)$.

(iv) We set $s = at$ and we infer that

$$\mathcal{F}(g(at))(\nu) = \int_{-\infty}^{+\infty} g(at) e^{-2\pi i \nu t}\, \mathrm{d}t = \begin{cases} \int_{-\infty}^{+\infty} g(s) e^{-(2\pi i \nu s)/a} \frac{1}{a}\, \mathrm{d}s & \text{se } a > 0 \\ \int_{-\infty}^{+\infty} g(s) e^{-(2\pi i \nu s)/a} \frac{1}{-a}\, \mathrm{d}s & \text{se } a < 0 \end{cases}$$

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} g(s) e^{-[2\pi i (\nu/a) s]}\, \mathrm{d}s = \frac{1}{|a|} \mathcal{F}(g(s)) \left(\frac{\nu}{a}\right).$$

(v) It follows from from (iv) with $a = -1$.

(vi) $\mathcal{F}(\overline{g(t)})(\nu) = \int_{\mathbb{R}} \overline{g(t)} e^{-2\pi i \nu t}\, \mathrm{d}t = \int_{\mathbb{R}} \overline{g(t) e^{2\pi i \nu t}}\, \mathrm{d}t = \overline{\int_{\mathbb{R}} g(t) e^{2\pi i \nu t}\, \mathrm{d}t} = \overline{\mathcal{F}(g(t))(-\nu)}$.

□

Let us observe that $e^{-2\pi i \nu t} = \cos(-2\pi \nu t) + i \sin(-2\pi \nu t) = \cos(2\pi \nu t) - i \sin(2\pi \nu t)$, since cos is even and sin is odd. Thus

$$\boxed{\mathcal{F}(g)(\nu) = \int_{\mathbb{R}} g(t) \cos(2\pi \nu t)\, \mathrm{d}t - i \int_{\mathbb{R}} g(t) \sin(2\pi \nu t)\, \mathrm{d}t.} \tag{1.2}$$

The first integral is an even function of ν, the second is odd. If g is even, we have that $g(t) \sin(2\pi \nu t)$ is odd (in t), so the second integral null. If g is odd, then $g(t) \cos(2\pi \nu t)$ is odd (in t), and the first integral is zero. Therefore we have proved the following

Proposition 1.2. *Let $g : \mathbb{R} \longrightarrow \mathbb{C}$ be summable. Then*

(i) g *even* $\quad\Longrightarrow\quad \mathcal{F}(g)(\nu) = \int_{\mathbb{R}} g(t) \cos(2\pi \nu t)\, \mathrm{d}t$ *and* $\mathcal{F}(g)$ *is even.*

(ii) g *odd* $\quad\Longrightarrow\quad \mathcal{F}(g)(\nu) = -i \int_{\mathbb{R}} g(t) \sin(2\pi \nu t)\, \mathrm{d}t$ *and* $\mathcal{F}(g)$ *is odd.*

Example 1.1. Let us consider $a > 0$ and $g(t) := H(t)e^{-at}$, $t \in \mathbb{R}$. The function g is summable on $\mathbb{R}$ and

$$\hat{g}(\nu) = \int_0^{+\infty} e^{-(a+2\pi i\nu)t}\,\mathrm{d}t = \left[\frac{e^{-(a+2\pi i\nu)t}}{-(a+2\pi i\nu)}\right]_{t=0}^{t\to+\infty}.$$

Since $a > 0$, we have

$$|e^{-(a+2\pi\nu)t}| = e^{\mathrm{Re}(-a-2\pi i\nu)t} = e^{-at} \to 0 \qquad \text{as } t \to +\infty,$$

therefore

$$\boxed{\mathcal{F}(H(t)e^{-at})(\nu) = \frac{1}{a+2\pi i\nu} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{1.3}$$

♡

Example 1.2. If $a > 0$ and $g(t) := e^{-a|t|}$, $t \in \mathbb{R}$, the function g is summable in $\mathbb{R}$ and we have

$$e^{-a|t|} = H(t)e^{-at} + H(-t)e^{at}$$

hence, exploiting (i) and (v) of Proposition 1.1, we find

$$\begin{aligned}
\mathcal{F}(g)(\nu) &= \mathcal{F}(H(t)e^{-at})(\nu) + \mathcal{F}(H(-t)e^{at})(\nu)\\
&= \mathcal{F}(H(t)e^{-at})(\nu) + \mathcal{F}(H(t)e^{-at})(-\nu)\\
&= \frac{1}{a+2\pi i\nu} + \frac{1}{a-2\pi i\nu} = \frac{2a}{a^2+4\pi^2\nu^2}.
\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}(e^{-a|t|})(\nu) = \frac{2a}{a^2+4\pi^2\nu^2} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{1.4}$$

♡

Example 1.3. If $a > 0$ and $g(t) := p_a(t)$, $t \in \mathbb{R}$, we have

$$\hat{g}(\nu) = \int_{-a/2}^{a/2} e^{-2\pi i\nu t}\,\mathrm{d}t = \left[\frac{e^{-2\pi i\nu t}}{-2\pi i\nu}\right]_{t=-a/2}^{t=a/2} = \frac{e^{\pi i\nu a} - e^{-\pi i\nu a}}{2\pi i\nu} = \frac{\sin(a\pi\nu)}{\pi\nu}.$$

So

$$\boxed{\mathcal{F}(p_a)(\nu) = \frac{\sin(a\pi\nu)}{\pi\nu} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{1.5}$$

♡

Example 1.4. If $a > 0$ and $g(t) := (a-|t|)p_{2a}(t)$, $t \in \mathbb{R}$, then g is even, thus thanks to Proposition 1.2 we have

$$\hat{g}(\nu) = \int_{-a}^{a} (a-|t|)\cos(2\pi\nu t)\,\mathrm{d}t = 2\int_0^a (a-t)\cos(2\pi\nu t)\,\mathrm{d}t,$$

where in the second equality we exploited the fact that the integrand function is even in t and the integration interval is symmetric with respect to $t = 0$. Therefore, thanks also to an integration by parts,

$$\begin{aligned}
\hat{g}(\nu) &= 2a\int_0^a \cos(2\pi\nu t)\,\mathrm{d}t - 2\int_0^a t\cos(2\pi\nu t)\,\mathrm{d}t\\
&= \frac{\sin(2\pi\nu a)}{\pi\nu} - 2\left(\left[\frac{t\sin(2\pi\nu t)}{2\pi\nu}\right]_{t=0}^{t=a} - \int_0^a \frac{\sin(2\pi\nu t)}{2\pi\nu}\,\mathrm{d}t\right)\\
&= 2\left[-\frac{\cos(2\pi\nu t)}{(2\pi\nu)^2}\right]_{t=0}^{t=a} = \frac{1}{2\pi^2\nu^2}(1-\cos(2\pi\nu a)) = \frac{\sin^2(\pi\nu a)}{\pi^2\nu^2}
\end{aligned} \tag{1.6}$$

Hence

$$\boxed{\mathcal{F}\big(p_{2a}(t)(a-|t|)\big)(\nu) = \frac{\sin^2(a\pi\nu)}{\pi^2\nu^2} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{1.7}$$

♡

Example 1.5. If $a > 0$ and $g(t) := e^{-at^2}$, $t \in \mathbb{R}$, using methods from complex analysis we computed the following integral (see the additional file)

$$\boxed{\mathcal{F}(e^{-at^2})(\nu) = \sqrt{\tfrac{\pi}{a}}\; e^{-\frac{\pi^2\nu^2}{a}} \qquad \forall \nu \in \mathbb{R}, \quad a > 0.} \tag{1.8}$$

In the particular case $a = \pi$ we find

$$\boxed{\mathcal{F}(e^{-at^2})(\nu) = e^{-\pi\nu^2} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{1.9}$$

♡

Proposition 1.3. *If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable, then*

(i) $\hat{g} \in C(\mathbb{R})$.

(ii) $\|\hat{g}\|_\infty \leqslant \|g\|_1 := \int_{\mathbb{R}} |g(t)|\,\mathrm{d}t$.

Proof. STOP

(i) For $\nu_0 \in \mathbb{R}$ arbitrarily fixed, we show that $\hat{g}$ is continuous at ν_0, equivalently that for every sequence $\nu_n \to \nu_0$ we have $\lim_{n\to\infty} g(\nu_n) = g(\nu_0)$. Thanks to the Dominated convergence Theorem 8.1 in the Appendix we can pass to the limit under the integral sign and we get $\lim_{n\to\infty} \hat{g}(\nu_n) = \lim_{n\to\infty} \int_{\mathbb{R}} g(t) e^{-2\pi i \nu_n t}\,\mathrm{d}t = \int_{\mathbb{R}} \lim_{n\to\infty} g(t) e^{-2\pi i \nu_n t}\,\mathrm{d}t$ $= \int_{\mathbb{R}} g(t) e^{-2\pi i \nu_0 t}\,\mathrm{d}t = \hat{g}(\nu_0)$.

(ii) We have

$$\|\hat{g}\|_\infty := \sup_{\nu\in\mathbb{R}} \left| \int_{\mathbb{R}} g(t) e^{-2\pi i \nu t}\,\mathrm{d}t \right| \leqslant \sup_{\nu\in\mathbb{R}} \int_{\mathbb{R}} |g(t)||e^{-2\pi i \nu t}|\,\mathrm{d}t = \int_{\mathbb{R}} |g(t)|\,\mathrm{d}t.$$

□

It is also possible to prove the so called *Riemann-Lebesgue lemma*, stating that

$$\lim_{\nu\to\pm\infty} \hat{g}(\nu) = 0.$$

2 Fourier transform of rapidly decreasing functions

The Fourier transform is a useful tool for the analysis of non-periodic signals, but its range of application is limited by the fact that the integral in (1.1) does not converge for non-summable signals like $g(t) = H(t)$ or $g(t) = \sin t$. In order to overcome this problem we could try to consider the distributional framework, giving a suitable definition of Fourier transform that preserves the good properties of (1.1). As usual, we first consider the regular case and we study the action of $T_{\hat{g}}$ on test functions, where g is a summable function. Let us observe that, thanks to Fubini theorem 8.3 (cf. Appendix) about iterated integrals, we have

$$\begin{aligned}\langle T_{\hat{g}}, \varphi\rangle &= \int_{\mathbb{R}} \hat{g}(\nu)\varphi(\nu)\,\mathrm{d}\nu = \int_{\mathbb{R}}\int_{\mathbb{R}} g(t)e^{-2\pi i\nu t}\,\mathrm{d}t\varphi(\nu)\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}} g(t)\int_{\mathbb{R}} \varphi(\nu)e^{-2\pi i\nu t}\,\mathrm{d}\nu\,\mathrm{d}t = \int_{\mathbb{R}} g(t)\hat{\varphi}(t)\,\mathrm{d}t = \langle T_g, \hat{\varphi}\rangle,\end{aligned}$$

hence a natural attempt is to define the Fourier transform of T setting $\langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle$. Unfortunately $\mathcal{F}(\varphi)$ is not with compact support, hence this definition is meaningless, unless $\operatorname{supp}(T)$ is compact: indeed it can be proved that $\mathcal{F}(\varphi) \in C^\infty(\mathbb{R})$ and we previously defined the action of a compact supported distributions on a C^∞ function. Thus

$$\langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle, \quad \varphi \in \mathscr{D}'(\mathbb{R}), \qquad \text{if } T \in \mathscr{D}'(\mathbb{R}), \operatorname{supp}(T) \text{ compact.} \tag{2.1}$$

For instance, for the Dirac delta δ_{x_0}, for every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$\begin{aligned} \langle \mathcal{F}(\delta_{x_0}), \varphi \rangle &:= \langle \delta_{x_0}, \mathcal{F}(\varphi) \rangle = \langle \delta_{x_0}, \int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t} \, \mathrm{d}t \rangle \\ &= \int_{\mathbb{R}} \varphi(t) e^{-2\pi i x_0 t} \, \mathrm{d}t = \langle T_{e^{-2\pi i x_0 t}}, \varphi(t) \rangle, \end{aligned}$$

and, denoting $T_{e^{-2\pi i x_0 \nu}}$ simply by $e^{-2\pi i x_0 \nu}$, we have

$$\mathcal{F}(\delta_{x_0})(\nu) = e^{-2\pi i x_0 \nu} \qquad \forall \nu \in \mathbb{R}.$$

But regular distributions like $H(t)$ (T_H) and $\sin t$ ($T_{\sin}$) are not with compact support, therefore we search for a new kind of distributions acting on a new space of test functions $\mathscr{S}$ that is invariant with respect to the Fourier transform, i.e. such that

$$\varphi \in \mathscr{S} \quad \Longrightarrow \quad \mathcal{F}(\varphi) \in \mathscr{S}.$$

In this way formula $\langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle$ would make sense. Let us now describe this new space of test functions.

Definition 2.1. A function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be *rapidly decreasing (at infinity)* if $\varphi \in C^\infty(\mathbb{R})$ and

$$\lim_{t \to \pm\infty} t^p \varphi^{(q)}(t) = 0 \qquad \forall p, q \in \mathbb{N}.$$

The set of such functions is denoted by $\mathscr{S}(\mathbb{R})$, or simply by $\mathscr{S}$.
If $\varphi_n, \varphi \in \mathscr{S}(\mathbb{R})$ are given we say that $\varphi_n \to \varphi$ *in* $\mathscr{S}(\mathbb{R})$ if

$$t^p \varphi_n^{(q)}(t) \to t^p \varphi^{(q)}(t) \qquad \text{uniformly in } \mathbb{R} \quad \forall p, q \in \mathbb{N}.$$

A typical example of rapidly decreasing function is $\varphi(t) = e^{-t^2}$. Another example is provided by the C^∞ functions with compact support, hence

$$\mathscr{D}(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R}).$$

It is not hard to check that $\mathscr{S}(\mathbb{R})$ is a vector space:

$$\varphi, \psi \in \mathscr{S}(\mathbb{R}), \quad \lambda, \mu \in \mathbb{C} \quad \Longrightarrow \quad \lambda\varphi + \mu\psi \in \mathscr{S}(\mathbb{R}).$$

Moreover $\varphi \in \mathscr{S}(\mathbb{R})$ is summable over $\mathbb{R}$, indeed taking $p = 2$ and $q = 0$, we have $\lim_{t \to \pm\infty} t^2 \varphi(t) = 0$, therefore there exists $M > 0$ such that $|t^2 \varphi(t)| \leqslant 1$ whenever $|t| \geqslant M$. Whence

$$|\varphi(t)| \leqslant \frac{1}{t^2} \qquad \text{if } |t| \geqslant M$$

and we apply the comparison theorem for improper integrals.
Now we see that $\mathscr{S}$ is invariant with respect to the Fourier transform. In order to see this, we need some useful properties relating the Fourier transform and the differentation.

Proposition 2.1. *If $\varphi \in \mathscr{S}(\mathbb{R})$ then*

(i) $[\mathcal{F}(\varphi)]^{(p)}(\nu) = (-2\pi i)^p \mathcal{F}(t^p \varphi(t))(\nu)$ *for every* $p \in \mathbb{N}$.

(ii) $\mathcal{F}(\varphi^{(p)})(\nu) = (2\pi i)^p \nu^p \mathcal{F}(\varphi)(\nu)$ *for every* $p \in \mathbb{N}$.

(iii) $\mathcal{F}(\varphi) \in \mathscr{S}(\mathbb{R})$ *(i.e. $\mathscr{S}$ is invariant with respect to $\mathcal{F}$).*

(iv) $\varphi_n \to \varphi$ *in* $\mathscr{S}(\mathbb{R}) \quad \Longrightarrow \quad \mathcal{F}(\varphi_n) \to \mathcal{F}(\varphi) \qquad$ *as* $n \to \infty$.

Proof.

(i) We can differentiate under the integral sign (cf. Theorem 8.2), hence

$$\begin{aligned}[\mathcal{F}(\varphi)]'(\nu) &= \frac{\mathrm{d}}{\mathrm{d}\nu} \int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t}\,\mathrm{d}t = \int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t}(-2\pi i t)\,\mathrm{d}t \\ &= (-2\pi i) \int_{\mathbb{R}} t\varphi(t) e^{-2\pi i \nu t}\,\mathrm{d}t = \mathcal{F}(t\varphi(t))(\nu),\end{aligned}$$

therefore we have proved (i) for $p = 1$. The case $p > 1$ is obtained by iterating the previous formula.

(ii) Integrating by parts

$$[\mathcal{F}(\varphi')](\nu) = \int_{\mathbb{R}} \varphi'(t) e^{-2\pi i \nu t}\,\mathrm{d}t = \left[\varphi(t) e^{-2\pi i \nu t}\right]_{t=-\infty}^{t=+\infty} - \int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t}(-2\pi i \nu)\,\mathrm{d}t.$$

Observe that

$$\lim_{t \to \pm\infty} \varphi(t) e^{-2\pi i \nu t} = 0$$

since $\varphi(t) \to 0$ and $e^{-2\pi i \nu t}$ is bounded. Hence

$$[\mathcal{F}(\varphi')](\nu) = -\int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t}(-2\pi i \nu)\,\mathrm{d}t = (2\pi i \nu)\mathcal{F}(\varphi)(\nu)$$

and (ii) is proved for $p = 1$. The case $p > 1$ is obtained by iterating the previous formula.

(iii) Thanks to (i) we have $\mathcal{F}(\varphi) \in C^\infty(\mathbb{R})$. For every $p, q \in \mathbb{N}$ we also have

$$\begin{aligned}|\nu^p [\mathcal{F}(\varphi)]^{(q)}(\nu)| &= |\nu^p (-2\pi i)^q \mathcal{F}(t^q \varphi(t))(\nu)| && \text{(by (i))} \\ &= |(-2\pi i)^q \nu^p \mathcal{F}(t^q \varphi(t))(\nu)| \\ &= |(-2\pi i)^{q-p} [\mathcal{F}((t^q \varphi(t))^{(p)})](\nu)| && \text{(by (ii))} \\ &= |2\pi|^{q-p} \left| \int_{\mathbb{R}} (t^q \varphi(t))^{(p)} e^{-2\pi i \nu t}\,\mathrm{d}t \right| \\ &\leqslant |2\pi|^{q-p} \int_{\mathbb{R}} |(t^q \varphi(t))^{(p)}|\,\mathrm{d}t := C_{p,q}.\end{aligned}$$

Therefore

$$|\nu^{p+1} [\mathcal{F}(\varphi)]^{(q)}(\nu)| \leqslant C_{p+1,q}$$

and this implies that

$$|\nu^p[\mathcal{F}(\varphi)]^{(q)}(\nu)| \leqslant \frac{C_{p+1,q}}{|\nu|} \to 0 \qquad \text{as } \nu \to \pm\infty.$$

(iv) Assume that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R})$, i.e. $x^p(\varphi_n^{(q)} - \varphi^{(q)}) \to 0$ uniformly for every $p, q \in \mathbb{N}$. We want to prove that $\nu^p\left([\mathcal{F}(\varphi_n)]^{(q)}(\nu) - [\mathcal{F}(\varphi)]^{(q)}(\nu)\right) \to 0$ uniformly for every $p, q \in \mathbb{N}$. Observe that $[\mathcal{F}(\varphi_n)]^{(q)}(\nu) - [\mathcal{F}(\varphi)]^{(q)}(\nu) = [\mathcal{F}(\varphi_n) - \mathcal{F}(\varphi)]^{(q)}(\nu) = [\mathcal{F}(\varphi_n - \varphi)]^{(q)}(\nu)$, hence setting $\psi_n := \varphi_n - \varphi$, we need to prove that $\nu^p[\mathcal{F}(\psi_n)]^{(q)} \to 0$ uniformly. Thanks to (ii) and (i)

$$\begin{aligned}
\left|\nu^p[\mathcal{F}(\psi_n)]^{(q)}\right| &= |\nu^p(-2\pi i)^q \mathcal{F}(t^q(\psi_n(t)))(\nu)| && \text{(by (i))}\\
&= |(-2\pi i)^{q-p}[\mathcal{F}((t^q(\psi_n(t)))^{(p)})](\nu)| && \text{(by (ii))}\\
&\leqslant (2\pi)^{q-p}\int_{\mathbb{R}} |(t^q\psi_n(t))^{(p)}|\,\mathrm{d}t\\
&= (2\pi)^{q-p}\int_{\mathbb{R}} (1+t^2)|(t^q\psi_n(t))^{(q)}|\frac{1}{1+t^2}\,\mathrm{d}t\\
&\leqslant (2\pi)^{q-p}\|(1+t^2)(t^q\psi_n(t))^{(q)}\|_\infty \int_{\mathbb{R}} \frac{1}{1+t^2}\,\mathrm{d}t.
\end{aligned}$$

Now the last term goes to zero as $n \to \infty$, since $\psi_n = \varphi_n - \varphi \to 0$ in $\mathscr{S}(\mathbb{R})$, so $\|(1+t^2)(t^q\psi_n(t))^{(q)}\|_\infty \leqslant \|(t^q\psi_n(t))^{(q)}\|_\infty + \|t^2(t^q\psi_n(t))^{(q)}\|_\infty \to 0$. Therefore

$$\|\nu^p[\mathcal{F}(\psi_n)]^{(q)}\|_\infty \leqslant (2\pi)^{q-p}\|(1+t^2)(t^q\psi_n(t))^{(q)}\|_\infty \int_{\mathbb{R}} \frac{1}{1+t^2}\,\mathrm{d}t \to 0$$

as $n \to \infty$ and we are done.

□

Properties (i) and (ii) hold also for more general functions, but the precise assumptions are rather tecnical and hard to remember. It is simpler to state this properties in the more general framework of distributions. Now we introduce the important notion of inverse Fourier transform.

Definition 2.2. If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable, we call *inverse Fourier transform of* g the function $\check{g} : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\check{g}(\nu) := \int_{\mathbb{R}} g(t)e^{2\pi i\nu t}\,\mathrm{d}t, \qquad \nu \in \mathbb{R}. \tag{2.2}$$

In other words

$$\check{g}(\nu) := \hat{g}(-\nu), \qquad \nu \in \mathbb{R}. \tag{2.3}$$

The notation $\mathcal{F}^{-1}(g) := \check{g}$ is also used, since we will see that in $\mathscr{S}$ it is actually the inverse transformation of $\mathcal{F}$. Again $\mathcal{F}^{-1}(g)(\nu) := \mathcal{F}(g)(-\nu)$.

The following proposition can be proved in the same way we proved the analogous properties of the Fourier transform.

Proposition 2.2. *If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable, then*

(i) $\check{g} \in C(\mathbb{R})$.

(ii) $\|\check{g}\|_\infty \leqslant \|g\|_1$.

Also in this case we have $\lim_{\nu \to \pm\infty} \check{g}(\nu) = 0$.

In the next theorem we show that in the space $\mathscr{S}(\mathbb{R})$ the transformation $\mathcal{F}^{-1}$ is actually the inverse of $\mathcal{F}$. In general this is not true.

Theorem 2.1 (Inversion formula in $\mathscr{S}$)**.** *If $\varphi \in \mathscr{S}(\mathbb{R})$ then $\check{\hat{\varphi}} = \varphi = \hat{\check{\varphi}}$, that is*

$$\mathcal{F}^{-1}(\mathcal{F})(\varphi) = \varphi = \mathcal{F}(\mathcal{F}^{-1})(\varphi). \tag{2.4}$$

Using (2.3) we can rewrite the inversion formula in the following way

$$\mathcal{F}(\mathcal{F}(\varphi))(t) = \varphi(-t) \qquad \forall t \in \mathbb{R}. \tag{2.5}$$

Proof. If we could apply the Fubini theorem (cf. Theorem 8.3) we would infer that:

$$\begin{aligned}
\check{\hat{\varphi}}(\nu) &= \int_{\mathbb{R}} \hat{\varphi}(t) e^{2\pi i \nu t} \,\mathrm{d}t = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \varphi(s) e^{-2\pi i t s} \,\mathrm{d}s \right] e^{2\pi i \nu t} \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \varphi(s) \left[\int_{\mathbb{R}} e^{-2\pi i (s-\nu) t} \,\mathrm{d}t \right] \mathrm{d}s \\
&= \int_{\mathbb{R}} \varphi(s) \hat{1}(s-\nu) \,\mathrm{d}s = (\varphi * \hat{1})(\nu).
\end{aligned}$$

But this computations are not correct because the function $\varphi(s) e^{-2\pi i (s-\nu) t}$ is not integrable over $\mathbb{R}^2$. Therefore we approximate this function by multiplying it by a rapidly decreasing function of t:

$$f_\varepsilon(t) := e^{-\pi(\varepsilon t)^2}, \qquad t \in \mathbb{R},$$

where $\varepsilon > 0$ is small. Let us recall that

$$f_\varepsilon(t) = f(\varepsilon t) \qquad \text{dove } f(t) := e^{-\pi t^2}, \quad t \in \mathbb{R} \tag{2.6}$$

$$\int_{\mathbb{R}} f(t) \,\mathrm{d}t = 1, \tag{2.7}$$

$$\hat{f} = f, \tag{2.8}$$

therefore

$$\hat{f}_\varepsilon(t) = \int_{\mathbb{R}} f(\varepsilon s) e^{-2\pi i t s} \,\mathrm{d}s = \frac{1}{\varepsilon} \int_{\mathbb{R}} f(\sigma) e^{-2\pi i \frac{t}{\varepsilon} \sigma} \,\mathrm{d}s = \frac{1}{\varepsilon} \hat{f}\left(\frac{t}{\varepsilon}\right) = \frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon}\right).$$

Now we can apply the Fubini theorem and infer that

$$\begin{aligned}
\int_{\mathbb{R}} \hat{\varphi}(t) f_\varepsilon(t) e^{2\pi i\nu t}\,\mathrm{d}t &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \varphi(s) e^{-2\pi i t s}\,\mathrm{d}s\right] f_\varepsilon(t) e^{2\pi i\nu t}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \varphi(s) \left[\int_{\mathbb{R}} f_\varepsilon(t) e^{-2\pi i(s-\nu)t}\,\mathrm{d}t\right] \mathrm{d}s \\
&= \int_{\mathbb{R}} \varphi(s) \hat{f}_\varepsilon(s-\nu)\,\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}} \varphi(s) f\left(\frac{s-\nu}{\varepsilon}\right) \mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}} \varphi(s) e^{-\pi(s-\nu)^2/\varepsilon^2}\,\mathrm{d}s \\
&= \int_{\mathbb{R}} \varphi(\nu+\varepsilon\sigma) e^{-\pi\sigma^2}\,\mathrm{d}\sigma \qquad \left[\sigma = \frac{(s-\nu)}{\varepsilon}\right].
\end{aligned}$$

So, since $\lim_{\varepsilon\to 0+} f_\varepsilon(t) = 1$ for every $t \in \mathbb{R}$, we can pass to the limit under the integral sign (Theorem 8.1) and we obtain that

$$\begin{aligned}
\check{\hat{\varphi}}(\nu) &= \int_{\mathbb{R}} \hat{\varphi}(t) e^{2\pi i\nu t}\,\mathrm{d}t = \int_{\mathbb{R}} \lim_{\varepsilon\to 0} \hat{\varphi}(t) f_\varepsilon(t) e^{2\pi i\nu t}\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} \hat{\varphi}(t) f_\varepsilon(t) e^{2\pi i\nu t}\,\mathrm{d}t \\
&= \lim_{\varepsilon\to 0+}\int_{\mathbb{R}} \varphi(\nu+\varepsilon\sigma) e^{-\pi\sigma^2}\,\mathrm{d}\sigma = \int_{\mathbb{R}} \lim_{\varepsilon\to 0+} \varphi(\nu+\varepsilon\sigma) e^{-\pi\sigma^2}\,\mathrm{d}\sigma = \varphi(\nu)\int_{\mathbb{R}} e^{-\pi\sigma^2}\,\mathrm{d}\sigma = \varphi(\nu).
\end{aligned}$$

Thus we have proved that $\mathcal{F}^{-1}(\mathcal{F}(\varphi)) = \varphi$. The other identity follows by virtue of Proposition 1.1-(v), since

$$\mathcal{F}(\mathcal{F}^{-1}(\varphi)(t))(\nu) = \mathcal{F}(\mathcal{F}(\varphi)(-t))(\nu) = \mathcal{F}(\mathcal{F}(\varphi)(t))(-\nu) = \mathcal{F}^{-1}(\mathcal{F}(\varphi)(t))(\nu) = \varphi(\nu).$$

□

3 Tempered distributions

In this section we introduce the class of distributions that will be Fourier transformable.

Definition 3.1. A functional $T : \mathscr{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is called *tempered distribution* if it is linear and *continuous in* $\mathscr{S}$:

$$\langle T, \lambda\varphi + \mu\psi\rangle = \lambda\langle T, \varphi\rangle + \mu\langle T, \psi\rangle, \qquad \forall \varphi, \psi \in \mathscr{S}(\mathbb{R}),\ \lambda, \mu \in \mathbb{C}, \tag{3.1}$$

$$\varphi_n \to \varphi \text{ in } \mathscr{S}(\mathbb{R}) \quad \Longrightarrow \quad \langle T, \varphi_n\rangle \to \langle T, \varphi\rangle. \tag{3.2}$$

The set of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R})$.

Sometimes it is useful to write

$$\langle T(t), \varphi(t)\rangle$$

(rather than $\langle T, \varphi\rangle$), even if this notation is not correct (T is not a function of t). There is no danger of confusion and in this way it is clear what is the independent variable of φ.
It is not so hard to prove that $\mathscr{S}'(\mathbb{R})$ is a vector space, that is

$$T, S \in \mathscr{S}'(\mathbb{R}), \quad \lambda, \mu \in \mathbb{C} \quad \Longrightarrow \quad \lambda T + \mu S \in \mathscr{S}'(\mathbb{R}).$$

As in the case of distributions in $\mathscr{D}'$, the continuity (3.2) is a technical requirement.

Proposition 3.1. *If $T : \mathscr{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is linear, then T is continuous if and only if the following implication holds:*

$$\varphi_n \to 0 \text{ in } \mathscr{S}(\mathbb{R}) \quad \Longrightarrow \quad \langle T, \varphi_n \rangle \to 0.$$

Proof. Assume that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R})$. Then $\psi_n := \varphi_n - \varphi \to 0$ in $\mathscr{S}(\mathbb{R})$, indeed $\sup_{t\in\mathbb{R}} |t^p \psi_n^{(q)}(t)| = \sup_{t\in\mathbb{R}} |t^p(\varphi_n^{(q)}(t) - \varphi_n^{(q)}(t))| = \sup_{t\in\mathbb{R}} |t^p \varphi_n^{(q)}(t) - t^p \varphi_n^{(q)}(t)| \to 0$ as $n \to \infty$. Thus $|\langle T, \varphi_n \rangle - \langle T, \varphi \rangle| = |\langle T, \varphi_n - \varphi \rangle| \to 0$ that is $\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle$. □

Let us consider a tempered distribution $T \in \mathscr{S}'$, i.e. a linear and continuous functional $T : \mathscr{S} \longrightarrow \mathbb{C}$. As $\mathscr{D} \subseteq \mathscr{S}$, we can consider the restriction of T on $\mathscr{D}$, that is $T : \mathscr{D} \longrightarrow \mathbb{C}$ (a common notation for the restriction is $T|_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathbb{C}$). Clearly T is linear. It is less obvious that T is continuous in the sense of $\mathscr{D}'$: indeed if $\varphi_n \to 0$ in $\mathscr{D}$, then $\lim_{n\to\infty} \|\varphi_n^{(q)}\|_\infty = \lim_{n\to\infty} \sup_{t\in\mathbb{R}} |\varphi_n^{(q)}(t)| = 0$ for every $q \in \mathbb{N}$, and there exists $R \geqslant 0$ such that $\operatorname{supp} \varphi_n \subseteq [-R, R]$ for every $n \in \mathbb{N}$, therefore $\|t^p \varphi_n^{(q)}(t)\|_\infty = \sup_{t\in\mathbb{R}} |t^p \varphi_n^{(q)}(t)| \leqslant \sup_{t\in\mathbb{R}} |R|^p |\varphi_n^{(q)}(t)| \to 0$ as $n \to \infty$, i.e. $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R})$. So we have shown that a functional $T \in \mathscr{S}'$ can be considered as a functional of $\mathscr{D}'$, if we consider its restriction to $\mathscr{D}$: $T|_{\mathscr{D}} : \mathscr{D} \longrightarrow \mathbb{C}$ (or simply $T : \mathscr{D} \longrightarrow \mathbb{C}$) The following proposition also holds true and shows that the association

$$\mathscr{S}'(\mathbb{R}) \ni T \longmapsto T|_{\mathscr{D}} \in \mathscr{D}'(\mathbb{R})$$

is injective (one-to-one).

Proposition 3.2. *If $T_1, T_2 \in \mathscr{S}'(\mathbb{R})$ and $T_1 \neq T_2$, then T_1 and T_2 restricted to $\mathscr{D}(\mathbb{R})$ are two different distributions of $\mathscr{D}'(\mathbb{R})$.*

Proof. We already checked that T_k, $k = 1, 2$, restricted to $\mathscr{D}(\mathbb{R})$ are distributions. We are left to prove that T_1 and T_2 are not the same element of $\mathscr{D}'(\mathbb{R})$. Since they differ in $\mathscr{S}'$, there exists $\varphi \in \mathscr{S}$ such that $\langle T_1, \varphi \rangle \neq \langle T_2, \varphi \rangle$. Assume by contradiction that $T_1 = T_2$ as elements of $\mathscr{D}'$ and let us approximate φ with a sequence of test functions $\varphi_n \in \mathscr{D}$ such that $\varphi_n \to \varphi$ in $\mathscr{S}$ (this is possibile by virtue of the following technical Lemma 3.1). Thus $\langle T_1, \varphi_n \rangle = \langle T_2, \varphi_n \rangle$ for every n, and taking the limit as $n \to \infty$ we get $\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle$, a contradiction. □

Lemma 3.1. *If $\varphi \in \mathscr{S}(\mathbb{R})$ then there exists $\varphi_n \in \mathscr{D}(\mathbb{R})$ such that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R})$.*

Proof. The strategy of the proof consists in multiplying φ, at every step n, by a function $\psi_n \in \mathscr{D}$ such that $\psi_n(t) = 1$ if $|t| \leqslant n$. So let us take $\psi_1 \in \mathscr{D}$ equal to 1 on $[-1, 1]$ and equal to zero outside $[-2, 2]$, and we set $\psi_n(t) := \psi_1(t/n)$. Let us consider the sequence $\varphi_n := \varphi\psi_n$, satisfying the conditions $\varphi_n \in \mathscr{D}$, $\operatorname{supp}(\varphi_n) \subseteq [-2n, 2n]$ and $\varphi_n(t) = \varphi(t)$ if

$t \in [-n, n]$. We have that $\varphi_n \to \varphi$ uniformly on $\mathbb{R}$, indeed

$$\begin{aligned}\|\varphi - \varphi_n\|_\infty &= \sup_{t\in\mathbb{R}} |\varphi(t) - \varphi_n(t)| = \sup_{t\in\mathbb{R}} |\varphi(t) - \varphi(t)\psi_n(t)| \\ &= \sup_{t\in\mathbb{R}} |\varphi(t)||1 - \psi_n(t)| = \sup_{|t|\geqslant n} |\varphi(t)||1 - \psi_n(t)| \leqslant \sup_{|t|\geqslant n} |\varphi(t)| \to 0\end{aligned}$$

as $n \to \infty$ since $\varphi(t) \to 0$ as $t \to \pm\infty$. Now if $q \in \mathbb{N}$, $q \geqslant 1$, thanks to the Leibniz formula for the derivative of a product, we have

$$\begin{aligned}|\varphi_n^{(q)}(t) - \varphi^{(q)}(t)| &= |[\varphi(t)(\psi_n(t) - 1)]^{(q)}| = |[\varphi(t)(\psi_1(t/n) - 1)]^{(q)}| \\ &\leqslant \sum_{k=0}^{q} \binom{q}{k} |\varphi^{(q-k)}(t)||(\psi_1(t/n) - 1)^{(k)}| \\ &= |\varphi^{(q)}(t)||\psi_1(t/n) - 1| + \sum_{k=1}^{q} \binom{q}{k} |\varphi^{(q-k)}(t)||\psi_1^{(k)}(t/n)|1/n^k\end{aligned}$$

hence

$$\begin{aligned}\|\varphi_n^{(q)} - \varphi\|_\infty &\leqslant \sup_{|t|\geqslant n} |\varphi^{(q)}(t)||\psi_1(t/n) - 1| + \sum_{k=1}^{q} \binom{q}{k} \|\varphi^{(q-k)}\|_\infty \|\psi_1^{(k)}\|_\infty 1/n^k \\ &\leqslant \sup_{|t|\geqslant n} |\varphi^{(q)}(t)| + \sum_{k=1}^{q} \binom{q}{k} \|\varphi^{(q-k)}\|_\infty \|\psi_1^{(k)}\|_\infty 1/n^k \to 0 \qquad (3.3)\end{aligned}$$

as $n \to \infty$. Therefore if $p \in \mathbb{N}$

$$\|t^p\varphi_n^{(q)}(t) - t^p\varphi_n^{(q)}(t)\|_\infty \leqslant \sup_{|t|\geqslant n} |t^p||\varphi^{(q)}(t)| + \sum_{k=1}^{q} \binom{q}{k} \|t^p\varphi^{(q-k)}(t)\|_\infty \|\psi_1^{(k)}\|_\infty 1/n^k \qquad (3.4)$$

which goes to zero as $n \to \infty$ since $\lim_{t\to\pm\infty} |t^p||\varphi^{(q)}(t)| = 0$. The lemma is proved. □

Thanks to the previous lemmas we can identify the space of tempered distributions $\mathscr{S}'$ with a subspace of $\mathscr{D}'$, hence we can write

$$\boxed{\mathscr{S}'(\mathbb{R}) \subseteq \mathscr{D}'(\mathbb{R}).}$$

It is useful to have some criterions in order to establish if a given ditribution $T \in \mathscr{D}'(\mathbb{R})$ is tempered, i.e. if it belongs to $\mathscr{S}'(\mathbb{R})$. The next three important examples provide such criterions.

Example 3.1. If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is summable then T_f is tempered.

STOP It is clear that T_f is linear. In order to check the continuity let us consider $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R})$. Then

$$\langle T_f, \varphi_n\rangle = \int_{\mathbb{R}} f(t)\varphi_n(t)\,\mathrm{d}t = \|\varphi_n\|_\infty \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t \to 0. \qquad (3.5)$$

♡

Example 3.2. If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is locally summable and f is *slowly increasing*, i.e.

$$\exists A > 0,\ p \in \mathbb{N}\ :\ |f(t)| \leqslant A(1 + |t|)^p \quad \forall t \in \mathbb{R}, \qquad (3.6)$$

then $T_f \in \mathscr{S}'(\mathbb{R})$.
Condition (3.6) means that the modulus of f is smaller that the modulus of a polynomial. Particular cases of slowly increasing functions are polynomials and bounded functions, hence

(i) f polynomial $\implies$ f slowly increasing $\implies$ $T_f \in \mathscr{S}'(\mathbb{R})$.
(ii) f bounded $\implies$ f slowly increasing $\implies$ $T_f \in \mathscr{S}'(\mathbb{R})$.

Let us see the proof that a slowly increasing function is (generates) a tempered distribution. If $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R})$ then

$$\begin{aligned}|\langle T_f, \varphi_n\rangle| &= \left|\int_{\mathbb{R}} f(t)\varphi_n(t)\,\mathrm{d}t\right| \leqslant A\int_{\mathbb{R}}(1+|t|)^p|\varphi_n|\,\mathrm{d}t\\ &= A\int_{\mathbb{R}}(1+|t|)^{p+2}|\varphi_n(t)|\frac{(1+|t|)^p}{(1+|t|)^{p+2}}\,\mathrm{d}x\\ &\leqslant A\|(1+|t|)^{p+2}\varphi_n(t)\|_\infty \int_{\mathbb{R}}\frac{(1+|t|)^p}{(1+|t|)^{p+2}}\,\mathrm{d}t \to 0.\end{aligned}$$

♡

Example 3.3. If $T \in \mathscr{D}'(\mathbb{R})$ has compact support then $T \in \mathscr{S}'(\mathbb{R})$.
In order to see this we consider $\varphi_n \to 0$ in $\mathscr{S}(\mathbb{R})$ and we assume that $\mathrm{supp}(T) \subseteq [-R, R]$ for some $R > 0$. If $\psi \in \mathscr{D}(\mathbb{R})$ is equal to 1 on $[-R, R]$ then

$$\langle T, \varphi_n\rangle = \langle T, \psi\varphi_n\rangle \to 0 \tag{3.7}$$

since $\psi\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$, indeed $\mathrm{supp}(\psi\varphi_n) \subseteq [-R, R]$ and thanks to the Leibniz formula

$$\|D^p(\psi\varphi_n)\|_\infty \leqslant \sum_{q=0}^{p}\binom{p}{q}\|\psi^{(q)}\|_\infty\|\varphi_n^{(p-q)}\|_\infty \to 0.$$

♡

Another example is provided by the *impulse train* (or *Dirac comb*)

$$T = \sum_{k=-\infty}^{\infty}\delta_k,$$

which is defined by

$$\left\langle \sum_{k=-\infty}^{\infty}\delta_k, \varphi\right\rangle := \sum_{k=-\infty}^{\infty}\varphi(k), \qquad \varphi \in \mathscr{S}(\mathbb{R}).$$

Indeed

Proposition 3.3. *The impulse train is a tempered distribution.*

Proof. First of all note that $\sum_{k=-\infty}^{\infty}\varphi(k)$ is convergent, since $|\varphi(k)| \leqslant 1/|k|^2$ if k is large enough, φ being rapidly decreasing. The linearity of T is easy to check. In order to prove that T is continuous let $\varphi_n \to 0$ in $\mathscr{S}$. Hence we have that $|\varphi_n(k)| = |(1+k^2)\varphi_n(k)|/(1+k^2)$ for every k, thus

$$\begin{aligned}|\langle T, \varphi_n\rangle| &= \left|\sum_{k=-\infty}^{\infty}\varphi_n(k)\right| \leqslant \sum_{k=-\infty}^{\infty}|\varphi_n(k)| = \sum_{k=-\infty}^{\infty}\frac{|(1+k^2)\varphi_n(k)|}{1+k^2}\\ &\leqslant \|(1+x^2)\varphi_n(x)\|_\infty\left(\sum_{k=-\infty}^{\infty}\frac{1}{1+k^2}\right) \to 0\end{aligned} \tag{3.8}$$

as $n \to \infty$ since $\varphi_n \to 0$ in $\mathscr{S}$. □

4 Fourier transform of tempered distributions

Definition 4.1. If $T \in \mathscr{S}'(\mathbb{R})$ we call *Fourier transform of T* the functional $\mathcal{F} : \mathscr{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle \mathcal{F}(T), \varphi \rangle := \langle T, \mathcal{F}(\varphi) \rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}). \tag{4.1}$$

We also use the notation $\widehat{T} := \mathcal{F}(T)$, hence $\langle \widehat{T}, \varphi \rangle := \langle T, \widehat{\varphi} \rangle$ for every $\varphi \in \mathscr{S}$.

Proposition 4.1. *If $T \in \mathscr{S}'(\mathbb{R})$ then $\mathcal{F}(T) \in \mathscr{S}'(\mathbb{R})$.*

Proof. Linearity is obvious. If $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R})$, then thanks to Proposition 2.1-(iv) we have that $\mathcal{F}(\varphi_n) \to \mathcal{F}(\varphi)$ as $n \to \infty$, therefore

$$\langle \mathcal{F}(T), \varphi_n \rangle = \langle T, \mathcal{F}(\varphi_n) \rangle \to \langle T, \mathcal{F}(\varphi) \rangle = \langle \mathcal{F}(T), \varphi \rangle$$

as $n \to \infty$, and we infer that T is continuous. □

Proposition 4.2. *If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable then*

$$\mathcal{F}(T_g) = T_{\mathcal{F}(g)},$$

in other words $\mathcal{F}(T_g)$ is regular and it is associated to the function

$$\mathcal{F}(g)(\nu) = \int_{\mathbb{R}} g(t) e^{-2\pi i \nu t} \, \mathrm{d}t, \qquad \nu \in \mathbb{R}.$$

Proof. Thenks to Theorem 1.3(i) we have that $\hat{g}$ is continuous, hence it is locally summable. Therefore for every $\varphi \in \mathscr{S}$ we have, by the Fubini theorem,

$$\begin{aligned} \langle \mathcal{F}(T_g), \varphi \rangle = \langle T_g, \mathcal{F}(\varphi) \rangle &= \int_{\mathbb{R}} g(\nu) \mathcal{F}(\varphi)(\nu) \, \mathrm{d}\nu = \int_{\mathbb{R}} g(\nu) \int_{\mathbb{R}} \varphi(t) e^{-2\pi i \nu t} \, \mathrm{d}t \, \mathrm{d}\nu \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} g(\nu) e^{-2\pi i \nu t} \, \mathrm{d}\nu \varphi(t) \, \mathrm{d}t = \int_{\mathbb{R}} \mathcal{F}(g)(t) \varphi(t) \, \mathrm{d}t = \langle \mathcal{F}(g), \varphi \rangle. \end{aligned}$$

□

Proposition 4.3. *If $T \in \mathscr{S}'(\mathbb{R})$ then*

(i) $[\mathcal{F}(T(t))]^{(p)} = (-2\pi i)^p \mathcal{F}(t^p T(t))$ *for every $p \in \mathbb{N}$.*

(ii) $\mathcal{F}(T^{(p)})(\nu) = (2\pi i)^p \nu^p \mathcal{F}(T)(\nu)$ *for every $p \in \mathbb{N}$.*

Every derivative here is meant in the sense of distributions.

Proof.

(i) For every $\varphi \in \mathscr{D}(\mathbb{R})$ we have that

$$\begin{aligned}
\langle [\mathcal{F}(T)]^{(p)}, \varphi\rangle &= (-1)^p \langle \mathcal{F}(T), \varphi^{(p)}\rangle && \text{(distributional derivative)}\\
&= (-1)^p \langle T, \mathcal{F}(\varphi^{(p)})\rangle && (\mathcal{F}\text{-transform in } \mathscr{S}')\\
&= (-1)^p \langle T, (2\pi i)^p \nu^p \mathcal{F}(\varphi)\rangle && \text{(Proposition 2.1-(ii))}\\
&= (-1)^p (2\pi i)^p \langle T, \nu^p \mathcal{F}(\varphi)\rangle && \text{(linearity of } T)\\
&= (-1)^p (2\pi i)^p \langle \nu^p T, \mathcal{F}(\varphi)\rangle && \text{(multiplication by a } C^\infty \text{ function)}\\
&= (-1)^p (2\pi i)^p \langle \mathcal{F}(\nu^p T), \varphi\rangle && (\mathcal{F}\text{-transform in } \mathscr{S}')\\
&= \langle (-2\pi i)^p \mathcal{F}(\nu^p T), \varphi\rangle
\end{aligned}$$

(ii)

$$\begin{aligned}
\langle [\mathcal{F}(T^{(p)})], \varphi\rangle &= \langle T^{(p)}, \mathcal{F}(\varphi)\rangle && (\mathcal{F}\text{-transform in } \mathscr{S}')\\
&= (-1)^p \langle T, (\mathcal{F}(\varphi))^{(p)}\rangle && \text{(distributional derivative)}\\
&= (-1)^p \langle T, (-2\pi i)^p \mathcal{F}(t^p \varphi)\rangle && \text{(Proposition 2.1-(i))}\\
&= (-1)^p (-2\pi i)^p \langle T, \mathcal{F}(t^p \varphi)\rangle && \text{(linearity of } T)\\
&= (2\pi i)^p \langle \mathcal{F}(T), t^p \varphi\rangle && (\mathcal{F}\text{-transform in } \mathscr{S}')\\
&= (2\pi i)^p \langle t^p \mathcal{F}(T), \varphi\rangle && \text{(multiplication by a } C^\infty \text{ function)}\\
&= \langle (2\pi i)^p t^p \mathcal{F}(T), \varphi\rangle
\end{aligned}$$

□

In a similar way, i.e. letting $\mathcal{F}$ act on the test function and exploiting the analogous properties in $\mathscr{S}$, we prove the following

Proposition 4.4. *If $T, S \in \mathscr{S}'(\mathbb{R})$, $\lambda, \mu \in \mathbb{C}$, $\nu_0, t_0, a \in \mathbb{R}$, $a \neq 0$, then*

(i) $\mathcal{F}(\lambda T + \mu S) = \lambda \mathcal{F}(T) + \mu \mathcal{F}(S)$.

(ii) $\mathcal{F}(e^{2\pi i \nu_0 t} T(t))(\nu) = \mathcal{F}(T(t))(\nu - \nu_0)$.

(iii) $\mathcal{F}(T(t - t_0))(\nu) = e^{-2\pi i t_0 \nu} \mathcal{F}(T(t))(\nu)$.

(iv) $\mathcal{F}(T(at))(\nu) = \dfrac{1}{|a|} \mathcal{F}(T(t)) \left(\dfrac{\nu}{a}\right)$.

(v) $\mathcal{F}(T(-t))(\nu) = \mathcal{F}(T(t))(-\nu)$.

Now we can define the inverse transform for distributions.

Definition 4.2. If $T \in \mathscr{S}'(\mathbb{R})$ we call *Fourier inverse transform of T* the tempered distribution $\check{T} \in \mathscr{S}'(\mathbb{R})$ defined by

$$\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}). \tag{4.2}$$

The notation $\mathcal{F}^{-1}(T) := \check{T}$ is also used: $\langle \mathcal{F}^{-1}(T), \varphi\rangle := \langle T, \mathcal{F}^{-1}(\varphi)\rangle$ for every $\varphi \in \mathscr{S}(\mathbb{R})$. We will see that $\mathcal{F}^{-1}$ in $\mathscr{S}'(\mathbb{R})$ is in fact the inverse transformation of $\mathcal{F}$.

Let us observe that if $\varphi \in \mathscr{S}$ then

$$\begin{aligned}
\langle \check{T}, \varphi \rangle &= \langle T, \check{\varphi} \rangle && \text{(definition of } \check{T}) \\
&= \langle T(t), \check{\varphi}(t) \rangle && \text{(we insert a fictitious independent variable)} \\
&= \langle T(t), \hat{\varphi}(-t) \rangle && \text{(definition of } \check{\varphi}) \\
&= \langle T(-t), \hat{\varphi}(t) \rangle && \text{(definition of } T(-t)) \\
&= \langle T(-t), \hat{\varphi} \rangle && \\
&= \langle \widehat{T(-t)}, \varphi \rangle && \text{(definition of } \widehat{T(-t)}) \\
&= \langle \widehat{T(-t)}(\nu), \varphi(\nu) \rangle && \\
&= \langle \widehat{T}(-\nu), \varphi(\nu) \rangle && \text{(Proposition 4.4(v)).}
\end{aligned} \tag{4.3}$$

Hence we have that

$$\boxed{\check{T}(\nu) = \hat{T}(-\nu),} \tag{4.4}$$

i.e., using the $\mathcal{F}$ notation,

$$\boxed{\mathcal{F}^{-1}(T)(\nu) = \mathcal{F}(T)(-\nu).} \tag{4.5}$$

Now we see that the symbol $\mathcal{F}^{-1}$ is justified by the fact that the inverse Fourier transform is actually the inverse transformation of $\mathcal{F}$ in $\mathscr{S}'$.

Theorem 4.1 (Inversion formula in $\mathscr{S}'$)**.** *The Fourier transform $\mathcal{F} : \mathscr{S}' \longrightarrow \mathscr{S}'$ is invertible and its inverse transformation is $\mathcal{F}^{-1}$, i.e. $\check{\hat{T}} = T = \hat{\check{T}}$, or in other words*

$$\mathcal{F}^{-1}(\mathcal{F}(T)) = T = \mathcal{F}^{-1}(\mathcal{F}(T)) \qquad \forall T \in \mathscr{S}'(\mathbb{R}). \tag{4.6}$$

Using (4.5) *we can write such formula as*

$$\mathcal{F}(\mathcal{F}(T))(t) = T(-t). \tag{4.7}$$

Proof. For every φ one has

$$\begin{aligned}
\langle \check{\hat{T}}, \varphi \rangle &= \langle \hat{T}, \check{\varphi} \rangle && \text{(inverse } \mathcal{F}\text{-transform in } \mathscr{S}') \\
&= \langle T, \hat{\check{\varphi}} \rangle && (\mathcal{F}\text{-transform in } \mathscr{S}') \\
&= \langle T, \varphi \rangle && \text{(inversion formula in } \mathscr{S}),
\end{aligned}$$

hence $\mathcal{F}^{-1}$ is actually the inverse of $\mathcal{F}$. The equality $T = \hat{\check{T}}$ can be proved in the same way. □

As a corollary we obtain a pointwise inversion formula, showing an elegant analogy with the Fourier expansion of a periodic funciton.

Corollary 4.1. *If $g \in C(\mathbb{R})$ is summable and if $\hat{g}$ is summable, then*

$$g(t) = \int_{\mathbb{R}} \hat{g}(\nu) e^{2\pi i t \nu} \, \mathrm{d}\nu \qquad \forall t \in \mathbb{R}. \tag{4.8}$$

Proof. By the inversion formula in $\mathscr{S}'$ we infer that $T_g = \check{\hat{T}}_g$, hence, exploiting also Fubini theorem, we deduce that

$$\begin{aligned}\langle T_g, \varphi\rangle = \langle \check{\hat{T}}_g, \varphi\rangle = \langle \hat{T}_g, \check{\varphi}\rangle = \langle T_{\hat{g}}, \check{\varphi}\rangle &= \int_{\mathbb{R}} \hat{g}(\nu)\check{\varphi}(\nu)\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}} \hat{g}(\nu) \int_{\mathbb{R}} \varphi(t)e^{2\pi i\nu t}\,\mathrm{d}t\,\mathrm{d}\nu \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu\varphi(t)\,\mathrm{d}t = \left\langle T_{\int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu}, \varphi(t)\right\rangle.\end{aligned}$$

Therefore $T_g = T_{\int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu}$, but thanks to Proposition 2.2(i) the function (of t) $f(t) := \int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu$ is continuous, hence $g = \int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi i\nu t}\,\mathrm{d}\nu$, which is what we wanted to prove (recall that if $f_1, f_2 \in C(\mathbb{R})$ and $T_{f_1} = T_{f_2}$, then $f_1 = f_2$). □

Corollary 4.1 could be interpreted by saying that the Fourier transform is a "continuous" version of Fourier series, where the summation is replaced by the integral, and the Fourier coefficients are replaced by $\hat{g}(\nu)$.
There are other versions of the pointwise inversion formula (4.8). For instance it is possible to prove the following implication :

$$\begin{cases} g : \mathbb{R} \longrightarrow \mathbb{C} \text{ summable, piecewise } C^0 \\ \int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi it\nu}\,\mathrm{d}\nu \text{ convergent} \end{cases} \implies \frac{g(t+) + g(t-)}{2} = \int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi it\nu}\,\mathrm{d}\nu. \tag{4.9}$$

Let us note that the summability of $\hat{g}$ is not required here, but we only need that the improper integral $\int_{\mathbb{R}} \hat{g}(\nu)e^{2\pi it\nu}\,\mathrm{d}\nu$ is convergent. If the convergence of such integral is dropped, then we need an additional regularity of g, but we obtain a weaker result, indeed one can prove that if g is summable, piecewise continuous and with piecewise continuous derivative, then

$$\frac{g(t+) + g(t-)}{2} = \lim_{R\to+\infty} \int_{-R}^{R} \hat{g}(\nu)e^{2\pi it\nu}\,\mathrm{d}\nu \tag{4.10}$$

(recall from Analysis 1 that the limit in the right hand side of the previous equation can exist even if the improper integral is not convergent).

Let us conclude this section with some hints about convolution. It is possible to prove that

$$T, S \in \mathscr{S}'(\mathbb{R}), \quad \mathrm{supp}(T) \text{ compact} \quad \implies \begin{cases} \mathcal{F}(T) \in C^\infty(\mathbb{R}) \\ \mathcal{F}(T * S) = \mathcal{F}(T)\mathcal{F}(S) \end{cases} \tag{4.11}$$

The formula $\mathcal{F}(T) \in C^\infty(\mathbb{R})$ means that $\mathcal{F}(T)$ is a regular distribution associated to a C^∞ function. Therefore it makes sense to consider the product $\mathcal{F}(T)\mathcal{F}(S)$. Concerning the convolution of two functions f and g, it is possible to prove that $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ whenever both sides make sense.

5 Calculation of Fourier transforms

In this section we compute some important Fourier transforms, and we solve some exercises.

Example 5.1. We already computed the Fourier transform of the Dirac δ_{x_0}. If $x_0 \in \mathbb{R}$, then for every $\varphi \in \mathscr{S}$ we have

$$\begin{aligned}\langle \mathcal{F}(\delta_{x_0}), \varphi\rangle &= \langle \delta_{x_0}(\nu), \mathcal{F}(\varphi)(\nu)\rangle = \left\langle \delta_{x_0}(\nu), \int_{\mathbb{R}} \varphi(t) e^{-2\pi i\nu t}\,\mathrm{d}t \right\rangle \\ &= \int_{\mathbb{R}} \varphi(t) e^{-2\pi i x_0 t}\,\mathrm{d}t = \langle T_{e^{-2\pi i x_0 t}}, \varphi(t)\rangle = \langle T_{e^{-2\pi i x_0 \nu}}, \varphi(\nu)\rangle,\end{aligned}$$

therefore

$$\boxed{\mathcal{F}(\delta_{x_0})(\nu) = e^{-2\pi i x_0 \nu} \qquad \forall \nu \in \mathbb{R}.} \tag{5.1}$$

Taking the Fourier transform of both sides of (5.1) and using the inversion formula (4.7), we get

$$\mathcal{F}(e^{-2\pi i \nu x_0})(t) = \mathcal{F}(\mathcal{F}(\delta_{x_0}))(t) = \delta_{x_0}(-t) = \delta_{-x_0}(t),$$

i.e.

$$\boxed{\mathcal{F}(e^{2\pi i x_0 t}) = \delta_{x_0}} \tag{5.2}$$

where $x_0 \in \mathbb{R}$. If $x_0 = 0$ we obtain formulas

$$\boxed{\mathcal{F}(\delta_0)(\nu) = 1, \qquad \mathcal{F}(1) = \delta_0.} \tag{5.3}$$

♡

Example 5.2. Let us compute the Fourier transform of $g(t) = \dfrac{1}{a^2+t^2}$, $a > 0$.
Since g is summable, it is also a tempered distribution (more precisely T_g is a tempered distribution). We know that if $b > 0$ then

$$\mathcal{F}(e^{-b|t|})(\nu) = \frac{2b}{b^2 + 4\pi^2\nu^2} \qquad \forall \nu \in \mathbb{R}.$$

Thus taking $\mathcal{F}$ of both side and using the inversion formula we infer that

$$\mathcal{F}\left(\frac{2b}{b^2+4\pi^2\nu^2}\right)(t) = \mathcal{F}(\mathcal{F}(e^{-b|t|}))(t) = e^{-b|-t|} = e^{-b|t|},$$

therefore, exchanging t and ν,

$$\mathcal{F}\left(\frac{2b}{b^2+4\pi^2 t^2}\right)(\nu) = e^{-b|\nu|} \qquad \forall \nu \in \mathbb{R} \tag{5.4}$$

Then, in order to obtain the tranform of $g(t)$ we write

$$\mathcal{F}\left(\frac{1}{a^2+t^2}\right)(\nu) = \mathcal{F}\left(\frac{\pi}{a}\frac{2(2\pi a)}{(2\pi a)^2 + 4\pi^2 t^2}\right)(\nu) = \frac{\pi}{a}\mathcal{F}\left(\frac{2(2\pi a)}{(2\pi a)^2+4\pi^2 t^2}\right)(\nu) = \frac{\pi}{a} e^{-2\pi a|\nu|}$$

where we used formula (5.4) with $b = 2\pi a$. Summarizing

$$\boxed{\mathcal{F}\left(\frac{1}{a^2+t^2}\right)(\nu) = \frac{\pi}{a} e^{-2\pi a|\nu|} \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{5.5}$$

♡

Example 5.3. Let us compute the Fourier transform of $g(t) = \dfrac{\sin(at)}{t}$, $a > 0$.
The function g is not summable, nevertheless it is bounded, hence it can be considered as a tempered distribution. If $b > 0$ we know that

$$\mathcal{F}(p_b(t))(\nu) = \frac{\sin(b\pi\nu)}{\pi\nu} \qquad \forall \nu \in \mathbb{R}.$$

Using the inversion formula we get

$$\mathcal{F}\left(\frac{\sin(b\pi\nu)}{\pi\nu}\right)(t) = \mathcal{F}(\mathcal{F}(p_b(t)))(t) = p_b(-t) = p_b(t),$$

and, changing t and ν,

$$\mathcal{F}\left(\frac{\sin(b\pi t)}{\pi t}\right)(\nu) = p_b(\nu) \qquad \forall \nu \in \mathbb{R}. \tag{5.6}$$

Now we compute the transform of $g(t)$ by writing

$$\mathcal{F}\left(\frac{\sin(at)}{t}\right)(\nu) = \mathcal{F}\left(\pi\frac{\sin(\frac{a}{\pi}\pi t)}{\pi t}\right)(\nu) = \pi\mathcal{F}\left(\frac{\sin(\frac{a}{\pi}\pi t)}{\pi t}\right)(\nu) = \pi p_{a/\pi}(\nu)$$

where we used formula (5.4) with $b = a/\pi$. Therefore

$$\boxed{\mathcal{F}\left(\frac{\sin(at)}{t}\right)(\nu) = \pi p_{\frac{a}{\pi}}(\nu) \qquad \forall \nu \in \mathbb{R} \quad (a > 0).} \tag{5.7}$$

♡

Example 5.4. Compute $\mathcal{F}(t^n)$ for every $n \in \mathbb{N}$.
The polynomial t^n is slowly increasing, and we can compute its Fourier transform in the sense of tempered distribution

$$\mathcal{F}(t^n)(\nu) = \mathcal{F}(t^n 1)(\nu) = \left(\frac{1}{-2\pi i}\right)^n [\mathcal{F}(1)]^{(n)}(\nu) = \left(\frac{1}{-2\pi i}\right)^n \delta_0^{(n)}.$$

♡

Example 5.5. Compute $\mathcal{F}(\sin t)$.
The function $\sin t$ is bounded, therefore it is a tempered distribution and we have

$$\begin{aligned}\mathcal{F}(\sin t)(\nu) &= \mathcal{F}\left(\frac{e^{it}-e^{-it}}{2i}\right)(\nu) = \frac{1}{2i}\left[\mathcal{F}(e^{it})(\nu) - \mathcal{F}(e^{-it})(\nu)\right] \\ &= \frac{1}{2i}\left[\mathcal{F}\left(e^{2\pi i(1/2\pi)t}\right)(\nu) - \mathcal{F}\left(e^{2\pi i(-1/2\pi)t}\right)(\nu)\right] = \frac{1}{2i}(\delta_{1/2\pi} - \delta_{-1/2\pi}).\end{aligned}$$

♡

Example 5.6. If $g(t) = te^{-t}H(t)$, $t \in \mathbb{R}$, compute $\mathcal{F}(g(t))$.
We have that g is summable.Thanks to Proposition 4.3(i) we get

$$\begin{aligned}\mathcal{F}(tH(t)e^{-t})(\nu) &= \left(-\frac{1}{2\pi i}\right)\left(\mathcal{F}(H(t)e^{-t})\right)'(\nu) = \left(-\frac{1}{2\pi i}\right)\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{1+2\pi i\nu}\right) \\ &= \left(-\frac{1}{2\pi i}\right)\frac{-2\pi i}{(1+2\pi i\nu)^2} = \frac{1}{(1+2\pi i\nu)^2}.\end{aligned}$$

♡

Example 5.7. Compute $\mathcal{F}\left(\frac{1}{t^2+t+1}\right)$.
Note that $g(t) = 1/(t^2+t+1)$ is summable. Thanks to Proposition (4.4)(iii) and to formula (5.7) we infer that

$$\begin{aligned}\mathcal{F}\left(\frac{1}{t^2+t+1}\right)(\nu) &= \mathcal{F}\left(\frac{1}{(t+1/2)^2+3/4}\right)(\nu) = e^{-2\pi i\nu(-1/2)}\mathcal{F}\left(\frac{1}{t^2+3/4}\right)(\nu) \\ &= e^{\pi i\nu}\frac{2\pi}{\sqrt{3}}e^{-2\pi(\sqrt{3}/2)|\nu|} = \frac{2\pi}{\sqrt{3}}e^{\pi(i\nu-\sqrt{3}|\nu|)}.\end{aligned}$$

♡

Example 5.8. Compute $\mathcal{F}(e^{-|2t|}\operatorname{sgn}(t))$.
We heve that $e^{-|2t|}\operatorname{sgn}(t) = -H(-t)e^{2t} + H(t)e^{-2t}$, hence

$$\begin{aligned}\mathcal{F}(e^{-|2t|}\operatorname{sgn}(t))(\nu) &= -\mathcal{F}(H(-t)e^{2t})(\nu) + \mathcal{F}(H(t)e^{-2t})(\nu) \\ &= -\mathcal{F}(H(t)e^{-2t})(-\nu) + \mathcal{F}(H(t)e^{-2t})(\nu) \\ &= -\frac{1}{2+2\pi i(-\nu)} + \frac{1}{2+2\pi i\nu} \\ &= -\frac{1}{2-2\pi i\nu} + \frac{1}{2+2\pi i\nu} = \frac{-4\pi i\nu}{4+4\pi^2\nu^2} = \frac{-\pi i\nu}{1+\pi^2\nu^2}.\end{aligned}$$

♡

Example 5.9. Compute $\mathcal{F}\left(\dfrac{t}{(9+4t^2)^2}\right)$.
The function $t/(9+4t^2)^2$ is summable. We can reduce to well known transforms by observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(9+4t^2)} = -\frac{8t}{(9+4t^2)^2},$$

hence, using Proposition 4.3(ii), Proposition 4.4(iv), and formula (5.7), we get

$$\begin{aligned}\mathcal{F}\left(\frac{t}{(9+4t^2)^2}\right)(\nu) &= \mathcal{F}\left(-\frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{(9+4t^2)}\right)(\nu) = -\frac{1}{8}2\pi i\nu\mathcal{F}\left(\frac{1}{(9+4t^2)}\right)(\nu)\\ &= -\frac{\pi i\nu}{4}\mathcal{F}\left(\frac{1}{(9+(2t)^2)}\right)(\nu) = -\frac{\pi i\nu}{4}\frac{1}{2}\mathcal{F}\left(\frac{1}{(9+t^2)}\right)\left(\frac{\nu}{2}\right)\\ &= -\frac{\pi i\nu}{8}\frac{\pi}{3}e^{-2\pi 3|\nu/2|} = \frac{\pi^2\nu}{24i}e^{-3\pi|\nu|}.\end{aligned}$$

♡

In order to compute the next transforms we need the following

Definition 5.1. A distribution $T \in \mathscr{D}'(\mathbb{R})$ is said to be *even* if $T(-t) = T(t)$.
A distribution $T \in \mathscr{D}'(\mathbb{R})$ is said to be *odd* if $T(-t) = -T(t)$.

Example 5.10.

a. If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is even then T_f is even.
b. If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is odd then T_f is odd.
c. δ_0 is even, indeed if $\varphi \in \mathscr{D}$ we have $\langle\delta_0(-t), \varphi(t)\rangle = \langle\delta_0(t), \varphi(-t)\rangle = \varphi(0) = \langle\delta_0(t), \varphi(t)\rangle$.
d. p.v. $\frac{1}{t}$ is odd: by the change of variable $s = -t$ we get that for every φ

$$\begin{aligned}\langle \text{p.v.}\,\frac{1}{t}(-t), \varphi(t)\rangle = \langle \text{p.v.}\,\frac{1}{t}(t), \varphi(-t)\rangle &= \lim_{\varepsilon\to 0+}\int_{-\infty}^{-\varepsilon}\frac{\varphi(-t)}{t}\,\mathrm{d}t + \int_{\varepsilon}^{+\infty}\frac{\varphi(-t)}{t}\,\mathrm{d}t\\ &= \lim_{\varepsilon\to 0+} -\int_{\varepsilon}^{+\infty}\frac{\varphi(s)}{s}\,\mathrm{d}s - \int_{-\infty}^{-\varepsilon}\frac{\varphi(s)}{s}\,\mathrm{d}s = -\langle \text{p.v.}\,\frac{1}{t}(t), \varphi(t)\rangle.\end{aligned}$$

e. If T is odd, then $\mathcal{F}(T)$ is odd. This follows from Proposition 4.4(v).
f. If T is even, then $\mathcal{F}(T)$ is even. This follows from Proposition 4.4(v).

♡

The following property holds:

Proposition 5.1.

(i) *If $T \in \mathscr{D}'(\mathbb{R})$ is odd, then T' is even.*

(ii) *If $T \in \mathscr{D}'(\mathbb{R})$ is even, then T' is odd.*

Proof. We prove (i) and we leave (ii) as an exercise. For every test function φ we have

$$\begin{aligned}\langle T'(-t), \varphi(t)\rangle &= \langle T'(t), \varphi(-t)\rangle = -\langle T(t), -\varphi'(-t)\rangle = \langle T(t), \varphi'(-t)\rangle\\ &= \langle T(-t), \varphi'(t)\rangle = \langle T(t), \varphi'(t)\rangle = -\langle T(t), \varphi(t)\rangle.\end{aligned}$$

□

Now we compute the Fourier transform of p.v. $\frac{1}{t}$. First we observe that, as $t(\frac{1}{t}) = 1$, it is reasonable to conjecture that

$$t\left(\text{p.v.}\,\frac{1}{t}\right) = 1. \tag{5.8}$$

The conjecture is true: indeed for every test function φ we have

$$\left\langle t\left(\text{p.v.}\,\frac{1}{t}\right)\ ,\ \varphi(t)\right\rangle = \left\langle \text{p.v.}\,\frac{1}{t}\ ,\ t\varphi(t)\right\rangle = \lim_{\varepsilon\to 0+}\int_{-\infty}^{-\varepsilon}\frac{t\varphi(t)}{t}\,\mathrm{d}t + \int_{\varepsilon}^{+\infty}\frac{t\varphi(t)}{t}\,\mathrm{d}t$$
$$= \int_{\mathbb{R}}\varphi(t)\,\mathrm{d}t = \langle 1,\varphi\rangle,$$

where 1 means the regular distribution associated to the constant function 1. Taking the Fourier transform of both sides of (5.8) we have

$$\mathcal{F}\left(t\left(\text{p.v.}\,\frac{1}{t}\right)\right) = \delta_0,$$

thus by Proposition 4.3(i) we get

$$\left(-\frac{1}{2\pi i}\right)\left[\mathcal{F}\left(\text{p.v.}\,\frac{1}{t}\right)\right]' = \delta_0$$

i.e., since $2\delta_0 = \text{sgn}'$,

$$\left[\pi i\,\text{sgn} + \mathcal{F}\left(\text{p.v.}\,\frac{1}{t}\right)\right]' = 0$$

and therefore there is a constant C such that

$$\pi i\,\text{sgn} + \mathcal{F}\left(\text{p.v.}\,\frac{1}{t}\right) = C.$$

As p.v. $\frac{1}{t}$ is odd, its transform is odd. The sgn function is also odd, hence the distribution on the left hand side of the previous equation is odd, and this is possible if and only if $C = 0$. Summarizing we have that

$$\boxed{\mathcal{F}\left(\text{p.v.}\,\frac{1}{t}\right)(\nu) = -\pi i\,\text{sgn}(\nu).} \tag{5.9}$$

We apply the inversion formula:

$$-\pi i\mathcal{F}(\text{sgn}(\nu))(t) = \left(\text{p.v.}\,\frac{1}{t}\right)(-t) = -\left(\text{p.v.}\,\frac{1}{t}\right)(t)$$

or, exchanging ν with t,

$$\boxed{\mathcal{F}(\text{sgn}(t))(\nu) = \frac{1}{\pi i}\,\text{p.v.}\,\frac{1}{\nu}.} \tag{5.10}$$

Now, as $H(t) = \frac{1}{2}\,\text{sgn}(t) + \frac{1}{2}$ we also get

$$\boxed{\mathcal{F}(H(t))(\nu) = \frac{1}{2\pi i}\left(\text{p.v.}\,\frac{1}{\nu}\right) + \frac{1}{2}\delta_0.} \tag{5.11}$$

Now we transform the impulse train. To this aim we use the so called *Poisson summation formula.*

Theorem 5.1. *For every $\varphi \in \mathscr{S}(\mathbb{R})$ one has*

$$\sum_{n=-\infty}^{\infty} \varphi(n) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) \tag{5.12}$$

Proof. If $f(t) := \sum_{n=-\infty}^{\infty} \varphi(t-n)$, since φ is rapidly decreasing, it is not hard to see that $f(t)$ is convergent for every t. Moreover f is C^∞ and periodic with period 1. Therefore we can apply the theory of Fourier series and write

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt}$$

where the Fourier coefficients c_k are

$$\begin{aligned} c_k &= \int_{-1/2}^{1/2} f(s)e^{-2\pi iks}\,\mathrm{d}s = \sum_{n=-\infty}^{\infty} \int_{-1/2}^{1/2} \varphi(s-n)e^{-2\pi iks}\,\mathrm{d}s \\ &= \sum_{n=-\infty}^{\infty} \int_{-1/2-n}^{1/2-n} \varphi(\sigma)e^{-2\pi ik\sigma}\,\mathrm{d}\sigma = \int_{-\infty}^{+\infty} \varphi(\sigma)e^{-2\pi ik\sigma}\,\mathrm{d}\sigma = \hat{\varphi}(k), \end{aligned} \tag{5.13}$$

hence

$$\sum_{n=-\infty}^{\infty} \varphi(t-n) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)e^{2\pi ikt}$$

and we conclude taking $t = 0$. ☐

Let us consider therefore the impulse train $T = \sum_{n=-\infty}^{\infty} \delta_n$ and prove that

$$\boxed{\mathcal{F}\left(\sum_{n=-\infty}^{\infty} \delta_n\right) = \sum_{n=-\infty}^{\infty} \delta_n.} \tag{5.14}$$

Indeed thanks to the Poisson summation formula, for every test functions φ we have

$$\langle T, \varphi\rangle = \sum_{n=-\infty}^{\infty} \varphi(n) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) = \sum_{n=-\infty}^{\infty} \langle \delta_n, \hat{\varphi}\rangle = \sum_{n=-\infty}^{\infty} \langle \hat{\delta}_n, \varphi\rangle = \sum_{n=-\infty}^{\infty} \langle e^{2\pi in\nu}, \varphi\rangle$$

i.e.

$$T = \sum_{n=-\infty}^{\infty} e^{2\pi in\nu}.$$

Taking the Fourier transform of both sides:

$$\mathcal{F}(T) = \sum_{n=-\infty}^{\infty} \mathcal{F}(e^{2\pi in\nu}) = \sum_{n=-\infty}^{\infty} \delta_n.$$

Let us observe that we used the continuity of $\mathcal{F}$ in $\mathscr{S}'$, which is proved in the following lemma, whose proof is a simple exercise.

Lemma 5.1. *Assume that $T_n, T \in \mathscr{S}'(\mathbb{R})$ and that $T_n \to T$ in $\mathscr{S}'$, i.e.*

$$\lim_{n\to\infty} \langle T_n, \varphi\rangle = \langle T, \varphi\rangle \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}).$$

Then $\mathcal{F}(T_n) \to \mathcal{F}(T)$, i.e. $\lim_{n\to\infty}\langle \mathcal{F}(T_n), \varphi\rangle = \langle \mathcal{F}(T), \varphi\rangle$ for every φ.

Let us conclude with some words about the so called *quadratic theory of the Fourier transform.* A locally summable function $g : \mathbb{R} \longrightarrow \mathbb{C}$ is called *square-summable (over $\mathbb{R}$)* if $|g|^2 : \mathbb{R} \longrightarrow \mathbb{R}$ è summable. We denote by $L^2(\mathbb{R})$ the square-summable functions. It is possible to prove the following result. If $g : \mathbb{R} \longrightarrow \mathbb{C}$ is summable and square-summable, then $\hat{g}$ is square-summable and

$$\|g\|_2 = \|\hat{g}\|_2$$

where $\|f\|_2 := \left(\int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}t\right)^{1/2}$. This equation is called *Parseval identity.*

It is also possible to prove that every $g \in L^2$ (not necessarily summable) is a tempered distribution (more precisely $T_g \in \mathscr{S}'$ if $g \in L^2$). Therefore it makes sense to consider its Fourier transform in the sense of distributions. Using an extension of the Riemann integral, the *Lebesgue integral*, one can prove that $\mathcal{F}(g)$ is a actually a function satisfying the Parseval identity and the inversion formula in a suitable generalized sense.

6 Tables

Properties

$\mathcal{F}(e^{2\pi i\nu_0 t}T(t))(\nu) = \mathcal{F}(T(t))(\nu - \nu_0)$	$(\nu_0 \in \mathbb{R})$
$\mathcal{F}(T(t - t_0))(\nu) = e^{-2\pi i t_0 \nu}\mathcal{F}(T(t))(\nu)$	$(t_0 \in \mathbb{R})$
$\mathcal{F}(T(at))(\nu) = \dfrac{1}{\lvert a\rvert}\mathcal{F}(T(t))\left(\dfrac{\nu}{a}\right)$	$(a \in \mathbb{R}\setminus\{0\})$
$\mathcal{F}(T(-t))(\nu) = \mathcal{F}(T(t))(-\nu)$	
$[\mathcal{F}(T(t))]^{(p)} = (-2\pi i)^p\mathcal{F}(t^p T(t))$	$(p \in \mathbb{N})$
$\mathcal{F}(T^{(p)})(\nu) = (2\pi i)^p\nu^p\mathcal{F}(T)(\nu)$	$(p \in \mathbb{N})$

Transforms $(a > 0)$

$T(t)$	$\mathcal{F}(T(t))(\nu)$
$H(t)e^{-at}$	$\dfrac{1}{a + 2\pi i\nu}$
$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2 + 4\pi^2\nu^2}$
$p_a(t)$	$\dfrac{\sin(a\pi\nu)}{\pi\nu}$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\pi^2\nu^2/a}$
$\dfrac{1}{a^2 + t^2}$	$\dfrac{\pi}{a}e^{-2\pi a\lvert\nu\rvert}$

$T(t)$	$\mathcal{F}(T(t))(\nu)$
$\dfrac{\sin(at)}{t}$	$\pi p_{\frac{a}{\pi}}(\nu)$
δ_{x_0}	$e^{-2\pi i x_0\nu}$
$e^{2\pi i x_0\nu}$	δ_{x_0}
$\text{p.v.}\,\frac{1}{t}$	$-\pi i\,\mathrm{sgn}(\nu)$
$\mathrm{sgn}(t)$	$\dfrac{1}{\pi i}\,\text{p.v.}\,\frac{1}{\nu}$
$H(t)$	$\dfrac{1}{2\pi i}\,\text{p.v.}\,\frac{1}{\nu} + \frac{\delta_0}{2}$

7 Esercises

Compute the Fourier transform of the following functions and distributions.

1. $g(t) = te^{-t^2}$
2. $g(t) = e^{-2t^2+4t}$ (hint: complete the square)
3. $g(t) = \cos t$
4. $g(t) = \cos(2t + 1)$
5. $g(t) = \cos t \; e^{-3t} H(t)$
6. $g(t) = \dfrac{t^2}{1 + t^2}$
7. $g(t) = |t|e^{-|t|}$
8. $g(t) = p_T(t - t_0)$ (where $t_0 \in \mathbb{R}$ and $T > 0$)
9. $g(t) = \mathbb{1}_{[a,b]}$ (where $a, b \in \mathbb{R}$ and $a < b$)
10. $g(t) = \begin{cases} 2 & \text{if } -1 < t < 0 \\ -1 & \text{if } 0 \leqslant t < 2 \\ 0 & \text{otherwise} \end{cases}$
    (hint: write g as a sum of suitable gate functions).
11. $g(t) = te^{-|t+2|/2}$
12. $g(t) = \dfrac{t \sin(2t)}{(t^2 + 4)^2}$ (hint: reason as in the Example 5.9)
13. $g(t) = tH(t)$
14. $g(t) = |t|$ (hint: $|t| = t \operatorname{sgn} t$)
15. $g(t) = H(t - 2)e^{-2t}$
16. $g(t) = p_{2\pi}(t) \sin t$
17. $g(t) = e^{-5t} \sin t H(t)$
18. $g(t) = p_\pi(t) \cos t$
19. $g(t) = \mathbb{1}_{[1,2]}(t) \sin t$ (you can use exercise 9)

20.** $g(t) = \arctan t$

21.** $g(t) = \log(t^2 + 1) - 2 \log |t|$

Answers:

1. $-i\pi\sqrt{\pi}\nu e^{-\pi^2\nu^2}$

2. $\sqrt{\dfrac{\pi}{2}}e^2e^{-\pi\nu(2i+\pi\nu/2)}$

3. $\dfrac{1}{2}(\delta_{1/2\pi}+\delta_{-1/2\pi})$

4. $\dfrac{1}{2}(e^i\delta_{1/\pi}+e^{-i}\delta_{-1/\pi})$

5. $\dfrac{3+2\pi i\nu}{(3+2\pi i\nu)^2+1}$

6. $-\pi e^{-2\pi|\nu|}+\delta_0$

7. $\dfrac{2-8\pi^2\nu^2}{(1+4\pi^2\nu^2)^2}$

8. $e^{-2\pi i\nu t_0}\frac{\sin(\pi\nu T)}{\pi\nu}$

9. $e^{-\pi i\nu(a+b)}\frac{\sin(\pi\nu(b-a))}{\pi\nu}$

10. $\dfrac{1}{\pi\nu}(2e^{\pi i\nu}\sin(\pi\nu)-e^{-2\pi i\nu}\sin(2\pi\nu))$

11. $8ie^{4\pi i\nu}\dfrac{i(1+16\pi^2\nu^2)-8\pi\nu}{(1+16\pi^2\nu^2)^2}$

12. $\dfrac{\pi}{4}\left[(1-\pi\nu)e^{-4\pi|\nu-1/\pi|}+(1+\pi\nu)e^{-4\pi|\nu+1/\pi|}\right].$

13. $\dfrac{1}{4\pi}\left[\dfrac{1}{\pi}\left(\text{p.v.}\,\frac{1}{\nu}\right)'+i\delta_0'\right]$

14. $\dfrac{1}{2\pi^2}\left(\text{p.v.}\,\frac{1}{\nu}\right)'$

15. $\dfrac{e^{-4(1+\pi i\nu)}}{2+2\pi i\nu}$

16. $\dfrac{1}{2i}\left[\dfrac{\sin(2\pi^2(\nu-1/2\pi))}{\pi(\nu-1/2\pi)}-\dfrac{\sin(2\pi^2(\nu+1/2\pi))}{\pi(\nu+1/2\pi)}\right]$

17. $\dfrac{1}{2i}\left[\dfrac{1}{5+2\pi i(\nu-1/2\pi)}-\dfrac{1}{5+2\pi i(\nu+1/2\pi)}\right]$

18. $\dfrac{1}{2}\left[\dfrac{\sin(\pi^2(\nu-1/2\pi))}{\pi(\nu-1/2\pi)}+\dfrac{\sin(\pi^2(\nu+1/2\pi))}{\pi(\nu+1/2\pi)}\right]$

19. $\dfrac{1}{2i}\left[e^{-3\pi i(\nu-1/2\pi)}\dfrac{\sin(\pi(\nu-1/2\pi))}{\pi(\nu-1/2\pi)}-e^{-3\pi i(\nu+1/2\pi)}\dfrac{\sin(\pi(\nu+1/2\pi))}{\pi(\nu+1/2\pi)}\right]$

20. Since $g'(t)=\frac{1}{1+t^2}$ we have $\widehat{g'}(\nu)=\mathcal{F}(1/(1+t^2))(\nu)=\pi e^{-2\pi|\nu|}$. But $\widehat{g'}(\nu)=2\pi i\nu\hat{g}(\nu)$, hence $2i\nu\hat{g}(\nu)=e^{-2\pi|\nu|}$. We cannot divide by ν because $\hat{g}$ is a distribution, hence recalling that $\nu\,\text{p.v.}\,\frac{1}{\nu}=1$, we write $2i\nu\hat{g}(\nu)=e^{-2\pi|\nu|}\nu\,\text{p.v.}\,\frac{1}{\nu}$,

i.e. $\nu(2i\hat{g}(\nu) - e^{-2\pi|\nu|}\,\text{p.v.}\,\frac{1}{\nu}) = 0$. Let us recall that $\nu T(\nu) = 0$ implies that $T(\nu) = c\delta_0$ for some constant c. Therefore we find $2i\hat{g}(\nu) - e^{-2\pi|\nu|}\,\text{p.v.}\,\frac{1}{\nu} = c\delta_0$. Now $e^{-2\pi|\nu|}$ is even and $\text{p.v.}\,\frac{1}{\nu}$ is odd, hence $e^{-2\pi|\nu|}\,\text{p.v.}\,\frac{1}{\nu}$ is odd (exercise). Moreover $g(t)$ is odd, thus$\hat{g}(\nu)$ is odd. It follows $2i\hat{g}(\nu) - e^{-2\pi|\nu|}\,\text{p.v.}\,\frac{1}{\nu}$ is odd, but it is equal to $c\delta_0$ which is even, hence $c = 0$. Summarizing we have $\hat{g}(\nu) = \frac{1}{2i}e^{-2\pi|\nu|}\,\text{p.v.}\,\frac{1}{\nu}$.

21. The function $g(t)$ is summable on $\mathbb{R}$, indeed as $t \to 0$ $g(t) \sim 2\log|t|$, and as $t \to \pm\infty$ $g(t) = \log(1 + 1/t^2) \sim 1/t^2$. We have $g'(t) = \frac{2t}{1+t^2} - 2\,\text{p.v.}(\frac{1}{t})$, hence transforming the equation $\widehat{g'}(\nu) = 2\left(-\frac{1}{2\pi i}\right)\left[\mathcal{F}(\frac{1}{1+t^2})\right]'(\nu) + 2\pi i\,\text{sgn}(\nu) = \frac{i}{\pi}\frac{\mathrm{d}}{\mathrm{d}\nu}(e^{-2\pi|\nu|}) + 2\pi i\,\text{sgn}(\nu) = \frac{i}{\pi}(-2\pi)e^{-2\pi|\nu|}\,\text{sgn}(\nu) + 2\pi i\,\text{sgn}(\nu) = 2\pi i\,\text{sgn}(\nu)(1 - \frac{e^{-2\pi|\nu|}}{\pi})$. On the other hand $\widehat{g'}(\nu) = 2\pi i\nu\hat{g}(\nu)$, therefore $2\pi i\nu\hat{g}(\nu) = 2\pi i\,\text{sgn}(\nu)(1 - \frac{e^{-2\pi|\nu|}}{\pi})$, and we infer that $\nu\hat{g}(\nu) = \text{sgn}(\nu)(1 - \frac{e^{-2\pi|\nu|}}{\pi})$, thus $\hat{g}(\nu) = \frac{\text{sgn}(\nu)}{\nu}(1 - \frac{e^{-2\pi|\nu|}}{\pi}) = \frac{1}{|\nu|}(1 - \frac{e^{-2\pi|\nu|}}{\pi})$, indeed, g being summable, we have that $\hat{g} \in C(\mathbb{R})$, i.e. $\hat{g}$ is a function).

8 Appendix: some theorems about integration

In this appendix we state, without proofs, some theorems about improper integrals.

Theorem 8.1 (Dominated convergence). *If $n \in \mathbb{N}$, let $f_n : \mathbb{R} \longrightarrow \mathbb{C}$ be a summable function such that for every $t \in \mathbb{R}$ there exists $f(t) := \lim_{n\to\infty} f_n(t) \in \mathbb{C}$. Assume that f is summable and that there exists $g : \mathbb{R} \longrightarrow [0,\infty]$ summable such that $|f_n(t)| \leqslant g(t)$ for every t. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(t)\,\mathrm{d}t = \int_{\mathbb{R}}\lim_{n\to\infty} f_n(t)\,\mathrm{d}t = \int_{\mathbb{R}} f(t)\,\mathrm{d}t.$$

Theorem 8.2. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a C^1 function such that for every $x \in \mathbb{R}$ the integral*

$$F(x) := \int_{\mathbb{R}} f(x,t)\,\mathrm{d}t$$

is convergent. If there is a summable function $g : \mathbb{R} \longrightarrow [0,\infty]$ such that $|\frac{\mathrm{d}}{\mathrm{d}t}f(x,t)| \leqslant g(t)$, then F is differentiable and

$$F'(x) = \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x}f(x,t)\,\mathrm{d}t.$$

Theorem 8.3 (Fubini)**.** *If $F : \mathbb{R}^2 \longrightarrow \mathbb{C}$ is integrable over $\mathbb{R}^2$, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} F(x, y) \, \mathrm{d}y \, \mathrm{d}x, \tag{8.1}$$

where we assume that both sides are convergent.

Concerning Fubini theorem, it is possible to prove that it is not necessary that both sides of (8.1) are convergent: the integrability of F over $\mathbb{R}^2$ implies the convergence of the two iterated integrals, provided the definition of F is modified in a suitable "negligible" set that does not change the values of these integrals.

Changes from revision June 9, 2015 to revision June 01, 2016:

1. The computation in the Example 5.8 have been rewritten.
2. Hint added to Exercise 2.
3. Everywhere: v.p. $\rightsquigarrow$ p.v.