Distributions

01URO - Mathematical methods for Computer Science
2023/2024 course

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Disclaimer: This is a translation of already existing notes written in Italian. **These notes cannot substitute classroom lectures, but only integrate them, it is heavily advised to go to the lectures**.

Contents

1 Distributional derivative

1.1 Theory

Let $\varphi(x) \in \mathbb{D}$ be a test function and $T \in \mathbb{D}'$ be a distribution. The definition of **distributional derivative T'** is the following:

$$\langle T', \varphi(x)\rangle = -\langle T, \varphi'(x)\rangle \tag{1}$$

It has the following proprieties:

- **Linearity** : $(\lambda T_1 + \mu T_2)' = \lambda T_1' + \mu T_2'$ with $\lambda, \mu \in \mathbb{R}$
- $(T(ax+b))' = aT'(ax+b)$ with $a, b \in \mathbb{R}$ and $a \neq 0$
- $((g(x)T(x))' = g'(x)T(x) + g(x)T'(x)$ with $g(x) \in C^\infty(\mathbb{R})$

The distributional derivative of a regular distribution (induced by a function $f(\mathrm{x}) \in L^1_{loc}(\mathbb{R})$) is differentiable everywhere except for a finite number of points $x_1, ..., x_k$ where, each point represents a **jump discontinuity**.
The derivative can be defined as:

$$T'_{f(x)} = T_{f'(x)} + \sum_{i=1}^{k} [f(x_i^+) - f(x_i^-)] \cdot \delta_{x_i} \qquad with \quad f(x_i^\pm) := \lim_{x \to x_i^\pm} f(x) \tag{2}$$

where $T_{f'(x)}$ is the distributional derivative of the regular distribution where $f(x)$ is differentiable, so $\mathbb{R}$ / $\{x_1, ..., x_k\}$. This theorem shows that:

- the points $x_0 \in \mathbb{R}$ where the function $f(x)$ has, either, an angle point or a removable discontinuity do not give any contribution to the distributional derivative.
  If we consider them in the right hand side of the formula 2, they would give no contribution whatsoever because $f(x_i^+) = f(x_i^-) \implies f(x_i^+) - f(x_i^-) = 0$.
- the value of the function $f(x)$ in the point x_i ($f(x_i)$) is meaningless since in 2 we just need to evaluate the limits of $f(x)$ for $x \to x_i^\pm$.

1.2 Exercises

Exercise 1: Evaluate the distributional derivative of the regular distribution induced by:

$$f(x) = sign(x) + 2x \tag{3}$$

using the following definition for sign(x):

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Solution

To correctly solve the exercise the distributional derivative formula 2 must be used since $f(x)$ cannot be derived for $\forall\ x \in \mathbb{R}$ since it has a jump discontinuity for $x = 0$.

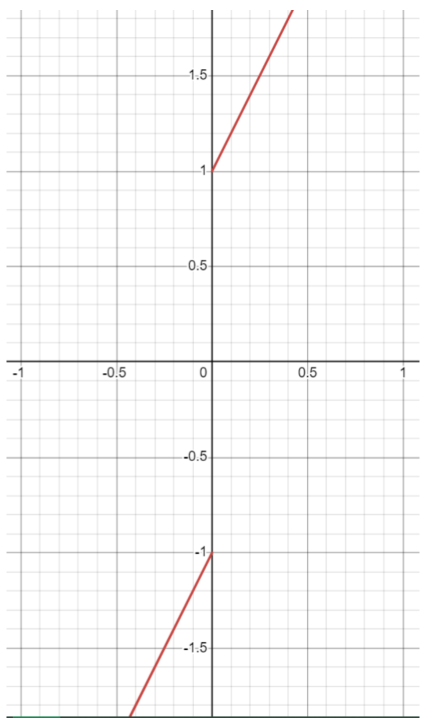

Figure 1: function 3 near $x = 0$

The function can be differentiated (for $\forall x \in \mathbb{R}/\{0\}$) as it follows :

$$f'(x) = \frac{d}{dx} f(x) = \begin{cases} \frac{d}{dx}(2x-1) & \text{if } x < 0 \\ \frac{d}{dx}(2x+1) & \text{if } x > 0 \end{cases} \qquad \rightarrow \qquad f'(x) = 2, \quad x \neq 0$$

Instead, for x = 0 we have to study the two limits, $x \to 0^{\pm}$.

$$f(0^-) = \lim_{x \to 0-} f(x) = -1 \qquad\qquad f(0^+) = \lim_{x \to 0+} f(x) = 1$$

to correctly finish the exercise, we must now apply the formula for distributional derivatives 2:

$$T'_{sign(x)+2x} = T_{[sign(x)+2x]'} + [f(0^+) - f(0^-)] \cdot \delta_0 = T_2 + 2\delta_0$$

where the first part of the formula highlights the regular part, whilst, the second part of the formula highlights the presence of a jump.
As discussed previously, we can see that the value of the function 3 in the point $x_0 = 0$ does not play any role when it regards calculating the final result.

Exercise 2: Evaluate the distributional derivative of the following distribution:

$$T = e^{x^2} \cdot \delta_{-1} + T_{-3 \cdot sign(x)} \tag{4}$$

using the following definition for sign(x):

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Solution
To correctly solve the exercise we must use the linearity propriety, we can write:

$$T' = (e^{x^2} \cdot \delta_{-1})' + T'_{-3 \cdot sign(x)}$$

and we can study the two pieces separately.

The first piece is really easy to evaluate, since, when a function that is $C^{\infty}(\mathbb{R})$ is multiplied by Dirac's delta we can say that, by the virtue of the sampling propriety of Dirac's delta:

$$e^{x^2} \cdot \delta_{-1} = e^{(-1)^2} \cdot \delta_{-1} = e \cdot \delta_{-1}$$

Done this, since e^{x^2} is now constant, because, when you have a function multiplied by Dirac's delta you have to substitute x with the value of the center of the delta, we can write that:

$$(e^{x^2} \cdot \delta_{-1})' = e \cdot \delta'_{-1}$$

The second part of the original distribution 4 can be studied as done in the previous exercise.
The first thing to do is to evaluate the intervals where the function that induces the distribution is continuous and differentiable, so that we can use the formula

2.
The function 3sign(-x) can be written as:

$$3{\cdot}\text{sign}(-x) = \begin{cases} -3 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ 3 & \text{if } x < 0 \end{cases}$$

as it was in the previous exercise, we found that the only point of non differ-

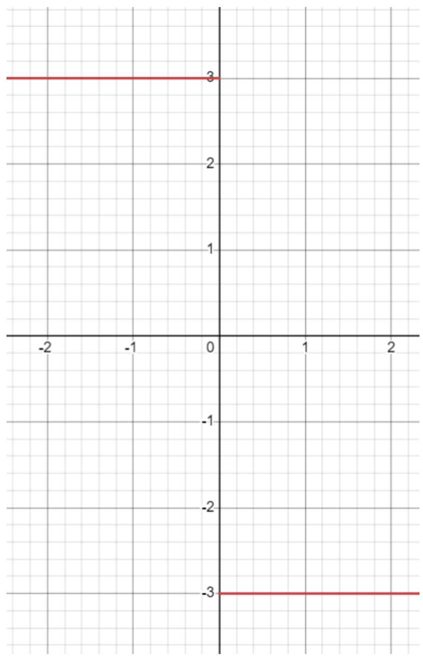

Figure 2: function 4 near $x = 0$

entiability is $x_0 = 0$ where the function has a jump discontinuity.
Evaluating the two limits and applying the final formula 2 we can write that

$$T' = e \cdot \delta'_{-1} + [f(0^+) - f(0^-)] \cdot \delta_0 = e \cdot \delta'_{-1} - 6\delta_0$$

Exercise 3: Let $\varphi(x) \in \mathbb{D}$ be a test function such that $\varphi'(0) = -2$ evaluate:

$$\lambda = \langle (\sin x) \cdot \delta''_0, \varphi(x) \rangle \tag{5}$$

Solution
Since $f(x) := \sin x \in C^\infty(\mathbb{R})$, the product property can be used to solve this exercise:

$$\lambda = \langle (\sin x) \cdot \delta''_0, \varphi(x) \rangle = \langle \delta''_0, \varphi(x) \cdot (\sin x) \rangle$$

Since we are asked to take the second derivative of 5 we can use 1 twice:

$$\lambda = \langle \delta_0'', \varphi(x) \cdot (\sin x) \rangle \stackrel{1}{=} -\langle \delta_0', [\sin(x) \cdot \varphi(x)]' \rangle =$$

$$= -\langle \delta_0', \cos(x) \cdot \varphi(x) + \sin(x) \cdot \varphi'(x) \rangle \stackrel{1}{=} \langle \delta_0, [\cos(x) \cdot \varphi(x) + \sin(x) \cdot \varphi'(x)]' \rangle =$$

$$= \langle \delta_0, -\sin(x) \cdot \varphi(x) + \cos(x) \cdot \varphi'(x) + \cos(x) \cdot \varphi'(x) + \sin(x) \cdot \varphi''(x) \rangle =$$

$$= -\sin(0) \cdot \varphi(0) + 2 \cdot \cos(0) \cdot \varphi'(0) + \sin(0) \cdot \varphi''(0) = -4$$

Exercise 4: Find every distribution $T \in \mathbb{D}'$ for which the following equation is true:

$$T' = 2 \cdot \delta_{-2} - 6 \cdot \delta_1 + \delta_4'$$

Solution

To solve this exercise we must apply the distributional derivative formula 2 from right to left.

The first thing that we may note is the fact that the function that gives the given equation is created by a function that when derived will surely be zero for $\forall x \in \mathbb{R}$.

The first two terms are created after differentiating something that creates a jump discontinuity multiplied by an opportune constant value, whilst, the last term is created by taking the first derivative of an already existing delta.

The first delta can be obtained by either differentiating a shifted Heaviside function $H(t+2)$ multiplied by 2 or a shifted sign function $sign(t+2)$.

Following a similar way of reasoning, the second delta can be found by differentiating a shifted Heaviside function $H(t-1)$ multiplied by -6 or a shifted sign function $sign(t-1)$ multiplied by -3.

The third delta is instead obtained by differentiating an already existing delta in the form δ_4.

Since we are asked to find every possible distribution we must also consider the existence of a constant term (c) that when differentiated will be zero.

To conclude a possible solution (using Heaviside function) is:

$$T = 2 \cdot T_{H(t+2)} - 6 \cdot T_{H(t-1)} + \delta_4 + c$$

2 Sequence of Distributions

2.1 Theory

Convergence can be defined in the set of distributions $\mathbb{D}'$. Given a sequence of distributions $T_n \in \mathbb{D}'$, it converges (in the sense of distribution) to a certain $T \in \mathbb{D}'$ if:

$$\lim_{n\to+\infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle \qquad \forall \varphi \in \mathbb{D} \tag{6}$$

and it can be written that $T_n \to T$.
Moreover, given two convergent sequences $T_n \to T$ and $S_n \to S$, it can be written that:

$$\lambda T_n + \mu S_n \to \lambda T + \mu S, \qquad \forall \lambda, \mu \in \mathbb{R} \tag{7}$$

When it comes to regular distributions, given a function sequence $f_n : \mathbb{R} \to \mathbb{R}$ that evenly converges to $f : \mathbb{R} \to \mathbb{R}$ in a compact support, we can conclude that $T_{f_n} \to T_f$.

2.2 Exercises

Exercise 1: Let $T_n = n^n \cdot \delta_n$ be a sequence of distributions, determine whether or not the sequence converges, and if so, determine the distribution T such that $T_n \to T$.

Solution
To study whether the sequence converges or not we need to evaluate $\langle T_n, \varphi \rangle$ for $n \to +\infty$ as stated in 6:

$$\lim_{n\to+\infty} \langle T_n, \varphi \rangle = \lim_{n\to+\infty} \langle n^n \cdot \delta_n, \varphi \rangle$$

Since $n^n \in \mathbb{N}$, we can use the linearity propriety of distributions to write:

$$\lim_{n\to+\infty} \langle n^n \cdot \delta_n, \varphi \rangle = \lim_{n\to+\infty} n^n \cdot \langle \delta_n, \varphi \rangle = \lim_{n\to+\infty} n^n \cdot \varphi(n)$$

The function $\varphi \in \mathbb{D}$ is a test function, so it has a compact support (and it is also $\varphi \in C^{\infty}(\mathbb{R})$).
If we use the definition of test function we can conclude that

$$\exists M \in \mathbb{N} : n > M \implies \varphi(x) = 0$$

We could conclude that $lim_{n\to+\infty}\varphi(n) = 0$ and it would be correct, but, inaccurate, since, not only the test function goes to 0 for $n \to +\infty$ but <u>is 0</u>, which is an even stronger conclusion.
So what could be, incorrectly, interpreted as an indeterminate form '$0 \cdot \infty$' is just a multiplication by 0:

$$\lim_{n\to+\infty} n^n \cdot \varphi = 0$$

so,

$$\langle n^n \cdot \delta_n, \varphi \rangle \to \langle 0, \varphi \rangle \implies n^n \cdot \delta_n \to 0$$

Exercise 2: Let $T_n = n \cdot \left(\delta_{\frac{1}{n}} + \delta_0 \right)$ be a sequence of distributions, show that the sequence does not converge in $\mathbb{D}'$:

Solution
To prove that the sequence of the given distribution does not converge, we must use a test function φ.
Moreover, we can write that:

$$\langle T_n, \varphi \rangle = \langle n \cdot \left(\delta_{\frac{1}{n}} + \delta_0 \right), \varphi \rangle = \langle n \cdot \delta_{\frac{1}{n}} + n \cdot \delta_0, \varphi \rangle$$

By using the linearity property alongside Dirac's delta property, we can write:

$$\langle n \cdot \delta_{\frac{1}{n}} + n \cdot \delta_0, \varphi \rangle = n \cdot \varphi \left(\frac{1}{n} \right) + n \cdot \varphi(0)$$

Now, depending on the value of $\varphi(0)$ we can have different cases.
If the value of $\varphi(0) \neq 0$ (for example, $\varphi(0) = 1$), we can conclude that $n \cdot \varphi\left(\frac{1}{n}\right) + n \cdot \varphi(0) \to +\infty$ as $n \to +\infty$, thus showing that the given sequence is not convergent.

Exercise 3: Find the limit (in the sense of distribution) for which T_{f_n} converges to:

$$f_n = n \cdot P_{\left[-\frac{2}{n};\frac{2}{n}\right]}(x)$$

with $P_{\left[-\frac{2}{n};\frac{2}{n}\right]}(x)$ as the door function 8 with center 0 and amplitude $\frac{4}{n}$.
The door function can be written as:

$$P_{\left[-\frac{2}{n};\frac{2}{n}\right]}(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{2}{n} \\ 0 & \text{if } |x| > \frac{2}{n} \end{cases} \tag{8}$$

Solution
With the given definition of the door function, we can write the regular distribution induced by f_n as:

$$\langle T_{f_n}, \varphi \rangle = \int_{-\infty}^{+\infty} f_n(x) \cdot \varphi(x)\, dx = \int_{-\infty}^{+\infty} n \cdot P_{\left[-\frac{2}{n};\frac{2}{n}\right]}(x) \cdot \varphi(x)\, dx =^{8}$$

$$= n \cdot \int_{-\frac{2}{n}}^{\frac{2}{n}} \varphi(x)\, dx$$

Now, we can use the fundamental theorem of calculus to evaluate last integral:

$$n \cdot \int_{-\frac{2}{n}}^{\frac{2}{n}} \varphi(x)\, dx = n \cdot \left[\Phi\left(\frac{2}{n}\right) - \Phi\left(-\frac{2}{n}\right)\right]$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a primitive of φ. Now, we may apply the following formula $\varphi(N(n)) + \nu((n)) - \varphi(N(n))$ to then evaluate the difference quotient:

$$n \cdot \left[\Phi\left(\frac{2}{n}\right) - \Phi\left(-\frac{2}{n}\right)\right] = n \cdot \frac{4}{n} \cdot \frac{\Phi\left(-\frac{2}{n} + \frac{4}{n}\right) - \Phi\left(-\frac{2}{n}\right)}{\frac{4}{n}}$$

Since we want to evaluate the convergence of T_{f_n} we must study the limit for $n \to +\infty$ of the previous expression, doing so the expression becomes the difference quotient:

$$\lim_{n \to +\infty} \langle T_{f_n}, \varphi \rangle = \lim_{n \to +\infty} n \cdot \frac{4}{n} \cdot \frac{\Phi\left(-\frac{2}{n} + \frac{4}{n}\right) - \Phi\left(-\frac{2}{n}\right)}{\frac{4}{n}} = \lim_{n \to +\infty} 4 \cdot \Phi'\left(\frac{-2}{n}\right) = 4 \cdot \varphi(0)$$

So, T_{f_n} converges to a distribution which can be expressed as $\varphi(0)$.
The obvious and correct answer is that the distribution is Dirac's delta with $x_0 = 0$ as its center and multiplied by 4, we can conclude that:

$$T_{f_n} \to 4 \cdot \delta_0$$

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