5. Theory of distributions

Vincenzo Recupero
Department of Mathematical Sciences, Politecnico di Torino
vincenzo.recupero@polito.it

Version: June 13, 2013
Revision: June 1, 2016

Analysis Lecture Notes
04LSI Mathematical Methods

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1 Preliminaries

1.1 Topological notions in $\mathbb{R}$

The topological notions defined in $\mathbb{C}$ can be analogously defined in $\mathbb{R}$ if we replace the open ball $B_r(z_0)$ by the open interval $(x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\}$, $r > 0$. For such particular intervals we can use the term *neighborhood of x with radius r* as well. For instance, if $S \subseteq \mathbb{R}$ then:

(i) we say that S is *open* if every point $x_0 \in S$ has a neighborhood $(x_0 - r, x_0 + r)$, $r > 0$, which is entirely contained in S

(ii) we say that $x_0 \in \mathbb{R}$ is a *boundary point of S* if for every $r > 0$ and every neighborhood $(x_0 - r, x_0 + r)$ of x_0 there exist $x \in S$ and $y \in \mathbb{R} \smallsetminus S$ such that $x \in (x_0 - r, x_0 + r)$ and $y \in (x_0 - r, x_0 + r)$. The set of boundary points of S is denoted by ∂S and is called *the boundary of S*

(iii) the set $\overline{S} := S \cup \partial S$ is called *closure of S*

(iv) we say that S *is closed* if $\overline{S} = S$

(v) the set $\mathring{S} := S \smallsetminus \partial S$ is called *interior of S*

Note that we could use the notation $B_r(x_0) := \{x \ : \ |x - x_0| < r\}$ in $\mathbb{R}$, in $\mathbb{C}$ and in $\mathbb{R}^n$ and give only one definition of the previous topological notions: in every particular case we have to interpret the symbol $|x - x_0|$ as the absolute value in $\mathbb{R}$, the modulus in $\mathbb{C}$, or the norm in $\mathbb{R}^n$.

One can prove the following property

Proposition 1.1. *If S is given, the closure $\overline{S}$ is the smallest closed set containing S.*

Proof. . □

The following terminology is frequently used

Definition 1.1. We say that a set S is *compact* if it is closed and bounded.

Example 1.1. For the following subsets S of $\mathbb{R}$ we have:

(a) $S = \left]0,1\right[$, $\partial S = \{0,1\}$, $\overline{S} = [0,1]$, $\mathring{S} = S$, S is open, S is not closed.

(b) $S = \left[2,+\infty\right[$, $\partial S = \{2\}$, $\overline{S} = S$, $\mathring{S} = \left]2,+\infty\right[$, S is not open, S is closed.

(c) $S = \{1/n \ : \ n \in \mathbb{N},\ n > 1\}$, $\partial S = S \cup \{0\}$, $\overline{S} = S \cup \{0\}$, $\mathring{S} = \varnothing$, S is not open, S is not closed.

(d) $S = \mathbb{N}$, $\partial S = \mathbb{N}$, $\overline{S} = \mathbb{N}$, $\mathring{S} = \varnothing$, S is not open, S is closed.

(e) $S = \mathbb{R}$, $\partial S = \varnothing$, $\overline{S} = \mathbb{R}$, $\mathring{S} = \mathbb{R}$, S is open, S is closed.

(f) $S = \varnothing$, $\partial S = \varnothing$, $\overline{S} = \varnothing$, $\mathring{S} = \varnothing$, S is open, S is closed.

♡

1.2 The support of a function

Definition 1.2. The *support* of a function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is the set

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R} \ : \ f(x) \neq 0\}},$$

i.e. the closure of the set where f is different from zero.

Example 1.2.

(a) If

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1 \end{cases}$$

then $\operatorname{supp}(f) = \overline{\left]-1,1\right[} = [-1,1]$.

(b) If $f(x) = x^2$ then $\operatorname{supp}(f) = \overline{\mathbb{R} \smallsetminus \{0\}} = \mathbb{R}$.

(c) If $f(x) = \sin x$ then $\operatorname{supp}(f) = \overline{\mathbb{R} \smallsetminus \{k\pi \ : \ k \in \mathbb{Z}\}} = \mathbb{R}$.

(d) If $f(x) = 0$ then $\operatorname{supp}(f) = \overline{\varnothing} = \varnothing$.

♡

Remark 1.1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function. We have that $\operatorname{supp}(f)$ is compact (i.e. is closed and bounded in $\mathbb{R}$) if and only if

$$\exists a, b \in \mathbb{R}, \quad a \leqslant b \quad : \quad \varphi(x) = 0 \quad \forall x \notin [a,b]\,.$$

◇

Remark 1.2. Assume that $f : \mathbb{R} \longrightarrow \mathbb{C}$ is differentiable and has compact support, i.e. $\operatorname{supp}(f)$ is closed and bounded. Then its derivative f' has compact support as well, indeed there exist $a, b \in \mathbb{R}$, $a < b$, such that

$$f(x) = 0 \quad \forall x \notin [a, b]\,,$$

hence f is identically equal to zero outside $[a, b]$, thus

$$f'(x) = 0 \quad \forall x \notin [a, b]\,.$$

$\diamondsuit$

1.3 Locally summable functions

Definition 1.3. Let $I \subseteq \mathbb{R}$ be an interval in $\mathbb{R}$ having endpoints a and b ($-\infty \leqslant a < b \leqslant +\infty$) and let $f : I \longrightarrow \mathbb{C}$ be a given function. We say that f is *summable on I* if the integrals

$$\int_a^b f(x)\,\mathrm{d}x, \qquad \int_a^b |f(x)|\,\mathrm{d}x$$

(meant as improper integrals if necessary) are finite (or convergent).

Definition 1.4. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a given function. We say that f is *locally summable (in $\mathbb{R}$)* if

$$\int_a^b f(x)\,\mathrm{d}x, \quad \int_a^b |f(x)|\,\mathrm{d}x \quad \text{are finite for every } a, b \in \mathbb{R},\ -\infty < a < b < +\infty.$$

(meant as improper integrals if necessary).

Example 1.3.

a) If $f : \mathbb{R} \longrightarrow \mathbb{R} : f(x) := x^2$, then for every $a, b \in \mathbb{R}$ we have that $\int_a^b f(x)\,\mathrm{d}x = \int_a^b |f(x)|\,\mathrm{d}x = \int_a^b x^2\,\mathrm{d}x$ is finite because f is continuous on $[a, b]$.

b) More generally any continuous function $f \in C(\mathbb{R})$ is locally summable.

c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) := 1/\sqrt{|x|}$ for $x \neq 0$ ($f(0)$ arbitrarily defined). Let $a, b \in \mathbb{R}$. If $0 \notin [a, b]$ then $\int_a^b f(x)\,\mathrm{d}x = \int_a^b |f(x)|\,\mathrm{d}x = \int_a^b 1/\sqrt{|x|}\,\mathrm{d}x$ is finite because f is continuous on $[a, b]$. If $0 \in [a, b]$ then the improper integral $\int_a^b 1/\sqrt{|x|}\,\mathrm{d}x$ is convergent. Thus f is locally summable.

d) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) := 1/x$ for $x \neq 0$ ($f(0)$ arbitrarily defined). The integral $\int_0^1 1/x\,\mathrm{d}x$ is not convergent, therefore f is not locally summable.

$\heartsuit$

Sometimes we will use the symbol $\int_{\mathbb{R}} f(x)\,\mathrm{d}x := \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ for the integral of a function over the whole real line, but we warn the reader that the notation $\int_{-\infty}^{+\infty}$ is easier to handle if a change of variable is needed.

In the following example we define some simple but important locally summable functions that we will use in the sequel of the course.

Example 1.4.

(a) If S is a given subset, the *indicator function of the set S* is the function $\mathbb{1}_S : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathbb{1}_S(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

Some authors use the symbol χ_S and the name *characteristic function of the set S*.

(b) The *sign function* $\operatorname{sign} : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\operatorname{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(c) The *Heaviside function* is the function $H : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$H(x) := \mathbb{1}_{[0,+\infty[}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

(c) If $a > 0$, the *gate function having full-width a* (or *rectangular function*) is the function $p_a : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$p_a(x) := \mathbb{1}_{[-a/2,a/2]}(x) = \begin{cases} 1 & \text{if } |x| \leqslant a/2 \\ 0 & \text{if } x > a/2 \end{cases}$$

♡

1.4 Uniform convergence

Definition 1.5. Let $D \subseteq \mathbb{R}$ and let $f : D \longrightarrow \mathbb{C}$ be given. The *supremum norm of f on D* (or the *∞-norm of f on D*) is defined by

$$\|f\|_{\infty,D} := \sup_{x \in D} |f(x)| \in [0, \infty] .$$

We will also use the symbol $\|f\|_\infty$ if the definition set D is fixed and no confusion may arise.

It is not hard to prove that

$$\begin{aligned} &\|f\|_\infty = 0 \quad \Longleftrightarrow \quad f = 0, \\ &\|\lambda f\|_\infty = |\lambda| \|f\|_\infty, \\ &\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty. \end{aligned}$$

Definition 1.6. Let $D \subseteq \mathbb{R}$ and let $f_n, f : I \longrightarrow \mathbb{C}$ be given for every $n \in \mathbb{N}$. We say that f_n *converges uniformly to f on I* (or $f_n \to f$ *uniformly on I*) if

$$\forall \varepsilon > 0 \qquad \exists n_\varepsilon \in \mathbb{N} \quad : \quad [\ n > n_\varepsilon, \quad x \in D \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon\ ]$$

i.e. if and only if

$$\forall \varepsilon > 0 \qquad \exists n_\varepsilon \in \mathbb{N} \quad : \quad \left[\ n > n_\varepsilon \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \varepsilon\ \right]$$

i.e. if and only if

$$\forall \varepsilon > 0 \qquad \exists n_\varepsilon \in \mathbb{N} \quad : \quad [\ n > n_\varepsilon \quad \Longrightarrow \quad \|f_n - f\|_{\infty, D} < \varepsilon\ ]$$

i.e. if and only if

$$\lim_{n \to \infty} \|f_n - f\|_{\infty, D} = 0.$$

2 Test functions

Definition 2.1. We say that $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is a *test function* if

(i) $\varphi \in C^\infty(\mathbb{R})$

(ii) $\mathrm{supp}(\varphi)$ is compact in $\mathbb{R}$, i.e. $\mathrm{supp}(\varphi)$ is closed and bounded in $\mathbb{R}$.

The set of test functions is denoted by $\mathscr{D}(\mathbb{R})$ (or simply by $\mathscr{D}$)

Remark 2.1. We recall that the fact that $\mathrm{supp}(\varphi)$ is closed and bounded in $\mathbb{R}$ means that

$$\exists a, b \in \mathbb{R}, \quad a \leqslant b \quad : \quad \varphi(x) = 0 \quad \forall x \notin [a, b]\,.$$

$\diamondsuit$

Proposition 2.1. *The set $\mathscr{D}(\mathbb{R})$ is a vector space, i.e. if $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ and $\lambda, \mu \in \mathbb{C}$, then $\lambda\varphi + \mu\psi \in \mathscr{D}(\mathbb{R})$.*

Proof. It is easy. $\square$

Definition 2.2. Assume that $\varphi_n, \varphi \in \mathscr{D}(\mathbb{R})$ for every $n \in \mathbb{N}$. We say that the sequence φ_n *converges to φ in $\mathscr{D}(\mathbb{R})$* (or $\varphi_n \to \varphi$ in $\mathscr{D}(\mathbb{R})$ as $n \to \infty$) if

(i) $\exists a, b \in \mathbb{R}, \quad a < b, \quad : \quad \mathrm{supp}(\varphi_n) \subseteq [a, b] \quad \forall n \in \mathbb{N}$.

(ii) $\varphi_n^{(p)} \to \varphi^{(p)}$ uniformly on $\mathbb{R}$ for every $p \in \mathbb{N}$.

Remark 2.2. In the previous Definition 2.2 condition (i) means that all the supports of the functions φ_n are contained in a unique closed bounded interval:

there exist $a, b \in \mathbb{R}$ such that $\varphi_n(x) = 0$ for every $x \notin [a, b]$ and for every $n \in \mathbb{N}$.

The second condition (ii) means that the sequence φ_n and the sequence $\varphi_n^{(p)}$ of the derivatives of any order $p \in \mathbb{N}$ converge uniformly on $\mathbb{R}$ (in fact on $[a, b]$) to φ and to $\varphi^{(p)}$. $\diamondsuit$

Before proceeding with the theory we should check that the set of test functions $\mathscr{D}(\mathbb{R})$ contains at least one nonzero function. This is the content of the following example.

Example 2.1. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\varphi(x) := \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqslant 1 \end{cases} \tag{2.1}$$

Let us study the function φ. Clearly $\operatorname{supp}(\varphi) = [-1, 1]$. Moreover $\lim_{x\to\pm1} \varphi(x) = 0$ thus φ is continuous. We have

$$\varphi'(x) = \begin{cases} -e^{\frac{1}{x^2-1}} \dfrac{2x}{(x^2-1)^2} & \forall x \in \,]-1, 1[\\ \\ 0 \quad \forall x \in \mathbb{R} \setminus [-1, 1] \end{cases}$$

and

$$\lim_{x\to 1-} \varphi'(x) = \lim_{x\to -1+} \varphi'(x) = 0.$$

thus f is differentiable at $x = \pm 1$ and $\varphi'(1) = \varphi'(-1) = 0$. By induction one can prove that $\varphi \in C^\infty(\mathbb{R})$, thus $\varphi \in \mathscr{D}(\mathbb{R})$: indeed

$$\begin{aligned} \varphi''(x) &= e^{\frac{1}{x^2-1}} \left(\frac{2x}{(x^2-1)^2} \right)^2 - e^{\frac{1}{x^2-1}} \frac{2(x^2-1)[(x^2-1) - 4x^2]}{(x^2-1)^4} \\ &= e^{\frac{1}{x^2-1}} \left[\frac{4x^2 - 2(x^2-1)[(x^2-1) - 4x^2]}{(x^2-1)^4} \right] \qquad \forall x \in \,]-1, 1[\,, \end{aligned}$$

and if $p > 1$

$$\varphi^{(p)}(x) = e^{\frac{1}{x^2-1}} \frac{P(x)}{(x^2-1)^{2p}} \qquad \forall x \in \,]-1, 1[\tag{2.2}$$

where $P(x)$ is a polynomial, indeed if we assume (2.2) as induction hypothesis then

$$\varphi^{(p+1)}(x) = -e^{\frac{1}{x^2-1}} \frac{2x}{(x^2-1)^2} \frac{P(x)}{(x^2-1)^{2p}} + e^{\frac{1}{x^2-1}} \frac{P'(x)(x^2-1)^{2p} - P(x)4p(x^2-1)^{2p-1}2x}{(x^2-1)^{4p}}.$$

Therefore

$$\lim_{x\to 1-} \varphi^{(p)}(x) = \lim_{x\to -1+} \varphi^{(p)}(x) = 0.$$

$\heartsuit$

Example 2.2. Let $\varphi \in \mathscr{D}$ be the bell shaped function of Example 2.1. For every $n \in \mathbb{N}$, $n > 1$, let $\rho_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\rho_n(x) := \frac{\varphi(nx)}{\int_{\mathbb{R}} \varphi(nt)\,\mathrm{d}t}, \qquad x \in \mathbb{R}.$$

Then $\operatorname{supp}(\rho_n) = [-1/n, 1/n]$ and $\rho_n \in C^\infty(\mathbb{R})$, hence $\rho_n \in \mathscr{D}(\mathbb{R})$. Moreover observe that $\int_{\mathbb{R}} \rho_n(x)\,\mathrm{d}x = 1$ for every $n \in \mathbb{N}$, $n > 1$, indeed

$$\int_{-\infty}^{\infty} \rho_n(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\varphi(nx)}{\int_{\mathbb{R}} \varphi(nt)\,\mathrm{d}t}\,\mathrm{d}x = \frac{1}{\int_{\mathbb{R}} \varphi(nt)\,\mathrm{d}t} \int_{-\infty}^{\infty} \varphi(nx)\,\mathrm{d}x = 1.$$

. $\heartsuit$

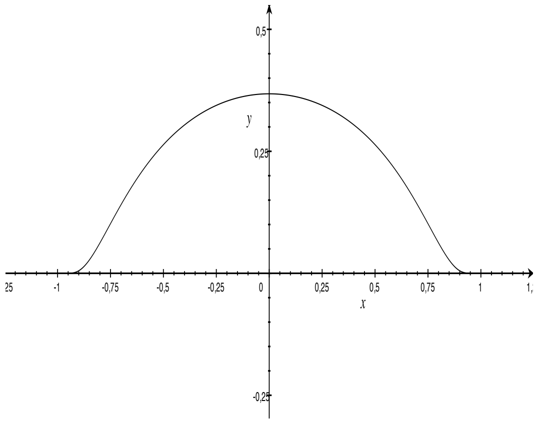

Figure 1: The bell shaped function of Example 2.1

3 Distributions (generalized functions)

In the sequel of this chapter we will frequently deal with mappings (functions) whose domain is $\mathscr{D}(\mathbb{R})$ and whose codomain is $\mathbb{C}$ (or $\mathbb{R}$). These are mappings $T : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ associating a number $T(\varphi) \in \mathbb{C}$ to every function $\varphi \in \mathscr{D}(\mathbb{R})$. In order to stress the fact that the domain is a set which consists of functions, the mapping T will be called *functional*. Moreover when T is linear we will use the notation

$$\langle T, \varphi \rangle := T(\varphi), \qquad \varphi \in \mathscr{D}(\mathbb{R})$$

which is widely used. Summarizing for T linear:

$$\begin{array}{cccc} T : & \mathscr{D}(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ & \varphi & \longmapsto & \langle T, \varphi \rangle \end{array}$$

Observe that the independent variable x of a test function $\varphi(x)$ does not appear in the notation $\langle T, \varphi \rangle$, indeed it would be unnecessary. Nevertheless there will be occasions when the not so proper notation

$$\langle T, \varphi(x) \rangle := \langle T, \varphi \rangle$$

will be very useful. It could even be convenient to write the completely incorrect

$$\langle T(x), \varphi(x) \rangle$$

even if x is not the variable of the mapping T. In other words the notation $T(x)$ does not makes sense, but it can be useful to write $\langle T(x), \varphi(x) \rangle$ if we remember that its meaning is simply $\langle T, \varphi \rangle$.

Definition 3.1. A functional $T : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ is called a *distribution* (or *generalized function*) if

(i) T is linear, i.e. if for every $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ and every $\lambda, \mu \in \mathbb{C}$

$$\langle T, \lambda\varphi + \mu\psi\rangle = \lambda\langle T, \varphi\rangle + \mu\langle T, \psi\rangle$$

(ii) T is *continuous* in the following sense: if $\varphi_n, \varphi \in \mathscr{D}(\mathbb{R})$,

$$\varphi_n \to \varphi \text{ in } \mathscr{D}(\mathbb{R}) \quad \Longrightarrow \quad \langle T, \varphi_n\rangle \to \langle T, \varphi\rangle \text{ in } \mathbb{C}$$

The set of distributions is denoted by $\mathscr{D}'(\mathbb{R})$ (or simply by $\mathscr{D}'$).

The condition (ii) of the previous definition is essentially technical in nature and allows a consistent development of the theory.

Proposition 3.1. *The set $\mathscr{D}'(\mathbb{R})$ is a vector space, i.e. if $T_1, T_2 \in \mathscr{D}'(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 T_1 + \lambda_2 T_2 \in \mathscr{D}'(\mathbb{R})$, where we set*

$$\begin{aligned}\langle \lambda T, \varphi\rangle &:= \lambda\langle T, \varphi\rangle, \qquad T \in \mathscr{D}'(\mathbb{R}),\ \lambda \in \mathbb{C},\\ \langle T + S, \varphi\rangle &:= \langle T, \varphi\rangle + \langle S, \varphi\rangle \qquad T, S \in \mathscr{D}'(\mathbb{R}).\end{aligned}$$

Proof. Easy. $\square$

The following lemma shows a typical property of <u>linear</u> functionals: it is enough to check continuity at $\varphi = 0$.

Lemma 3.1. *If $T : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ is <u>linear</u>, then in order to check the continuity of T it is sufficient to prove that*

$$\varphi_n \to 0 \text{ in } \mathscr{D}(\mathbb{R}) \quad \Longrightarrow \quad \langle T, \varphi_n\rangle \to 0.$$

Proof. . Assume that $\varphi_n \to \varphi$ in $\mathscr{D}(\mathbb{R})$, i.e. $\lim_{n\to\infty} \|\varphi_n^{(p)} - \varphi^{(p)}\|_\infty = 0$ for every $p \in \mathbb{N}$ and there exist $a, b \in \mathbb{R}$ such that $\varphi_n(x) = 0$ for every $x \notin [a, b]$ and for every $n \in \mathbb{N}$. In particular $\varphi(x) = 0$ whenever $x \notin [a, b]$. Therefore $\psi_n := \varphi_n - \varphi \to 0$ in $\mathscr{D}(\mathbb{R})$ hence $|\langle T, \varphi_n\rangle - \langle T, \varphi\rangle| = |\langle T, \varphi_n - \varphi\rangle| \to 0$, that is $\langle T, \varphi_n\rangle \to \langle T, \varphi\rangle$. $\square$

Definition 3.2 (Regular distributions)**.** If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is locally summable, we define the distribution $T_f \in \mathscr{D}'(\mathbb{R})$ by setting

$$\langle T_f, \varphi\rangle := \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{3.1}$$

The distribution T_f is also called the *regular distribution associated to f*.

We have to check that T_f is actually a distribution. Let us start by verifying that the integral in (3.1) makes sense. If $\varphi \in \mathscr{D}(\mathbb{R})$ then its support is contained in a bounded interval $[a,b]$ thus $\varphi(x) = 0$ if $x \notin [a,b]$. Moreover by the Weierstrass theorem

$$\|\varphi\|_\infty = \sup_{x\in\mathbb{R}} |\varphi(x)| = \sup_{x\in[a,b]} |\varphi(x)| = \max_{x\in[a,b]} |\varphi(x)| < \infty,$$

thus

$$\begin{aligned} \left|\int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x\right| &= \left|\int_a^b f(x)\varphi(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)||\varphi(x)|\,\mathrm{d}x \\ &\leqslant \int_a^b |f(x)| \sup_{x\in[a,b]} |\varphi(x)|\,\mathrm{d}x = \int_a^b |f(x)|\|\varphi\|_\infty\,\mathrm{d}x \\ &= \|\varphi\|_\infty \int_a^b |f(x)|\,\mathrm{d}x < \infty, \end{aligned}$$

because f is locally summable. Hence we have proved that the integral is finite. Now we check that T is a distribution.

a) T_f *is linear*:
For every $\varphi, \psi \in \mathscr{D}(\mathbb{R})$, $\lambda, \mu \in \mathbb{C}$ we have, by the linearity of the integral,

$$\begin{aligned} \langle T_f, \lambda\varphi + \mu\psi\rangle &= \int_{\mathbb{R}} f(x)(\lambda\varphi(x) + \mu\psi(x))\,\mathrm{d}x \\ &= \lambda \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x + \mu \int_{\mathbb{R}} f(x)\psi(x)\,\mathrm{d}x = \lambda\langle T_f, \varphi\rangle + \mu\langle T_f, \psi\rangle \end{aligned}$$

b) T_f *is continuous*:
We use Lemma 3.1: if $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$, then there are $a, b \in \mathbb{R}$, $a < b$, such that $\varphi_n(x) = 0$ for every $x \notin [a,b]$. Moreover $\varphi_n \to \varphi$ uniformly on $\mathbb{R}$, hence

$$\begin{aligned} |\langle T_f, \varphi_n\rangle| &= \left|\int_{\mathbb{R}} f(x)\varphi_n(x)\,\mathrm{d}x\right| = \left|\int_a^b f(x)\varphi_n(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)||\varphi_n(x)|\,\mathrm{d}x \\ &\leqslant \int_a^b |f(x)|\|\varphi_n\|_\infty\,\mathrm{d}x = \|\varphi_n\|_\infty \int_a^b |f(x)|\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty \end{aligned}$$

since $\int_a^b |f(x)|\,\mathrm{d}x < +\infty$ and $\|\varphi_n\|_\infty \to 0$ by the uniform convergence. Hence we have shown that $\lim_{n\to\infty}\langle T_f, \varphi_n\rangle = 0$ and by Lemma 3.1 we infer that T_f is continuous.

Definition 3.3. We say that $T \in \mathscr{D}'(\mathbb{R})$ is called a *regular distribution* if there exists a locally summable function $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $T = T_f$. In this case, if no confusion may arise, the distribution T_f is sometimes denoted simply by f.

Proposition 3.2. *If $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ are continuous and $T_f = T_g$, then $f = g$.*

Proof. Let us assume by contradiction that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq g(x_0)$. Then by the continuity of f and g there exists $\delta > 0$ such that $f(x) \neq g(x)$ for every $x \in [x_0 - \delta, x_0 + \delta]$. Let $\varphi \in \mathscr{D}(\mathbb{R})$ be a test function such that $\varphi(x) > 0$ whenever $x \in [x_0 - \delta, x_0 + \delta]$ (it is enough to shrink and translate the graph of a bell shaped function). We have

$$\int_{\mathbb{R}} (f(x) - g(x))\varphi \,\mathrm{d}x = \int_{x_0-\delta}^{x_0+\delta} (f(x) - g(x))\varphi(x) \,\mathrm{d}x \neq 0$$

that is $T_f \neq T_g$. □

The previous proposition allows us to identify the space of continuous functions $C(\mathbb{R})$ with a subset of distributions, therefore we can write

$$\boxed{C(\mathbb{R}) \subseteq \mathscr{D}'(\mathbb{R}).}$$

More generally, if f and g are not continuous and $T_f = T_g$, it may happen that $f \neq g$. A trivial example being $f(x) := 0$ and $g(x) := \mathbb{1}_{\{0\}}(x)$, $x \in \mathbb{R}$. Nevertheless it can be proved that if $T_f = T_g$, then the set of points x where $f(x) \neq g(x)$ is "negligible" with respect to integration. This notion could be made precise and rigorous, but what is important to us is that we can consider two locally summable functions essentially as the same function if $T_f = T_g$. From this point of view the set of locally summable functions may be regarded as (identified with) a subset of $\mathscr{D}'(\mathbb{R})$.

Remark 3.1. Assume that f and g are two locally summable functions that agree everywhere except for finitely many points. Then $T_f = T_g$, indeed the integral of a function does not change if we modify the values of this function at finitely many points, thus

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x) \,\mathrm{d}x = \int_{\mathbb{R}} g(x)\varphi(x) \,\mathrm{d}x = \langle T_g, \varphi \rangle.$$

Thus if $h : \mathbb{R} \smallsetminus \{x_1, \dots, x_m\} \longrightarrow \mathbb{C}$ is locally summable, it generates a regular distribution T_h such that $\langle T_h, \varphi \rangle := \int_{\mathbb{R}} h(x)\varphi(x) \,\mathrm{d}x$, since $x_1, \dots, x_m$ does not affect the value of this integral. ♢

In the following important example we define a distribution that is not regular: the *Dirac delta*.

Example 3.1. If $x_0 \in \mathbb{R}$ the *Dirac delta with center* x_0 is the distribution $\delta_{x_0} : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle \delta_{x_0}, \varphi \rangle := \varphi(x_0), \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{3.2}$$

If $x_0 = 0$ the notation $\delta := \delta_0$ is also used. Let us verify that δ_{x_0} is a distribution:

a) δ_{x_0} *is linear*:
   If $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ and $\lambda, \mu \in \mathbb{C}$ then

$$\begin{aligned} \langle \delta_{x_0}, \lambda\varphi + \mu\psi \rangle &= \lambda\varphi(x_0) + \mu\psi(x_0) \\ &= \lambda\langle \delta_{x_0}, \varphi \rangle + \mu\langle \delta_{x_0}, \psi \rangle. \end{aligned}$$

b) δ_{x_0} *is continuous*:
   Assume that $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$. Then in particular $\varphi_n(x_0) \to 0$ as $n \to \infty$, hence

$$\lim_{n\to\infty} \langle \delta_{x_0}, \varphi_n \rangle = \lim_{n\to\infty} \varphi_n(x_0) = 0$$

   so δ_{x_0} is continuous.

It is possible to prove that δ_{x_0} is not a regular distribution. ♡

Example 3.2.

(a) Let $T : \mathscr{D} \longrightarrow \mathbb{C}$ be defined by $\langle T, \varphi\rangle := \int_{-\infty}^{+\infty} \arctan x\, \varphi(x)\,\mathrm{d}x$. Then T is a distribution because it is the regular distribution $T = T_f$ with $f(x) := \arctan x$, $x \in \mathbb{R}$, and f is locally summable (it is continuous).

(b) Let $T : \mathscr{D} \longrightarrow \mathbb{C}$ be defined by $\langle T, \varphi\rangle := \int_0^1 x^2\varphi(x)\,\mathrm{d}x$. Then T is a distribution because

$$\langle T, \varphi\rangle = \int_0^1 x^2\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \mathbb{1}_{[0,1]}(x)x^2\varphi(x)\,\mathrm{d}x.$$

Thus $T = T_f$ with $f(x) := \mathbb{1}_{[0,1]}(x)x^2$, $x \in \mathbb{R}$, which is a locally summable function.

(c) Let $T : \mathscr{D} \longrightarrow \mathbb{C}$ be defined by $\langle T, \varphi\rangle := \int_{-2}^{+2} \varphi'(x)\,\mathrm{d}x$. Then thanks to the fundamental theorem of calculus we get

$$\langle T, \varphi\rangle = \int_{-2}^{+2} \varphi'(x)\,\mathrm{d}x = \varphi(2) - \varphi(-2) = \langle \delta_2, \varphi\rangle - \langle \delta_{-2}, \varphi\rangle$$

hence $T = \delta_2 - \delta_{-2} \in \mathscr{D}'(\mathbb{R})$.

(d) Let $T : \mathscr{D} \longrightarrow \mathbb{C}$ be defined by $\langle T, \varphi\rangle := \int_0^2 x\varphi'(x)\,\mathrm{d}x$. Then integrating by parts we obtain

$$\langle T, \varphi\rangle = \int_0^2 x\varphi'(x)\,\mathrm{d}x = 2\varphi(2) - \int_0^2 \varphi(x)\,\mathrm{d}x = \langle 2\delta_2, \varphi\rangle - \int_{-\infty}^{+\infty} \mathbb{1}_{[0,2]}(x)\varphi(x)\,\mathrm{d}x$$

hence $T = 2\delta_2 - T_{\mathbb{1}_{[0,2]}} \in \mathscr{D}'(\mathbb{R})$.

♡

Example 3.3.

(a) Let us define the functional $T : \mathscr{D} \longrightarrow \mathbb{C}$ by

$$\langle T, \varphi\rangle := \|\varphi\|_2 = \|\varphi\|_{2,\mathbb{R}} := \left(\int_{-\infty}^{\infty} |\varphi(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Then T is not a distribution because T is not linear, indeed let us take $\varphi \in \mathscr{D}$ such that $\varphi \neq 0$. Then since φ is continuous we have

$$\langle T, \varphi\rangle = \left(\int_{-\infty}^{\infty} |\varphi(x)|^2\,\mathrm{d}x\right)^{1/2} > 0.$$

On the other hand

$$\langle T, -\varphi\rangle = \left(\int_{-\infty}^{\infty} |-\varphi(x)|^2\,\mathrm{d}x\right)^{1/2} = \|\varphi\|_2 = \langle T, \varphi\rangle,$$

but this is not possible, since if $\varphi \neq 0$ and if T were linear we should find $\langle T, -\varphi\rangle = -\langle T, \varphi\rangle$.

(b) Let us define the functional $T : \mathscr{D} \longrightarrow \mathbb{C}$ by

$$\langle T, \varphi\rangle := 2013, \qquad \forall \varphi \in \mathscr{D}.$$

Then T is not a distribution bacause T is not linear, indeed if T were linear we should have $\langle T, 0\rangle = 0 \neq 2013$.

♡

4 Operations on distributions

4.1 Differentiation

Let us consider a C^1 function $f : \mathbb{R} \longrightarrow \mathbb{C}$. It follows that $f' : \mathbb{R} \longrightarrow \mathbb{C}$ is locally summable, and we want to see what is $T_{f'}$. For a test function $\varphi \in \mathscr{D}(\mathbb{R})$, if $\mathrm{supp}(\varphi) \subseteq [a,b]$, we have

$$\begin{aligned}\langle T_{f'}, \varphi\rangle &= \int_{-\infty}^{+\infty} f'(x)\varphi(x)\,\mathrm{d}x = \int_a^b f'(x)\varphi(x)\,\mathrm{d}x \\ &\overset{\text{by parts}}{=} [f(x)\varphi(x)]_{x=a}^{x=b} - \int_a^b f(x)\varphi'(x)\,\mathrm{d}x \\ &\overset{\varphi(a)=\varphi(b)=0}{=} -\int_a^b f(x)\varphi'(x)\,\mathrm{d}x \\ &\overset{\mathrm{supp}(\varphi')\subseteq[a,b]}{=} -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,\mathrm{d}x = -\langle T_f, \varphi'\rangle,\end{aligned}$$

thus we have proved that

$$\boxed{f \in C^1(\mathbb{R}) \quad \Longrightarrow \quad \langle T_{f'}, \varphi\rangle = -\langle T_f, \varphi'\rangle \quad \forall \varphi \in \mathscr{D}(\mathbb{R}),} \tag{4.1}$$

In fact the previous formula also holds if f is continuous and f' is only locally summable. Motivated by this formula we can define the derivative of a distribution.

Definition 4.1 (Derivative)**.** If $T \in \mathscr{D}'(\mathbb{R})$ is given, we say that the *(distributional) derivative of* T is the distribution $T' : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle T', \varphi\rangle := -\langle T, \varphi'\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{4.2}$$

Observe that we have to prove that T' is actually a distribution. We check this fact in the following

Proposition 4.1. *If $T \in \mathscr{D}'(\mathbb{R})$, then the functional $T' : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined in* (4.2) *is a distribution.*

Proof. If $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ and $\lambda, \mu \in \mathbb{C}$ then

$$\langle T', \lambda\varphi + \mu\psi\rangle = -\langle T, \lambda\varphi' + \mu\psi'\rangle = -\lambda\langle T, \varphi'\rangle - \mu\langle T, \psi'\rangle = \lambda\langle T', \varphi\rangle + \mu\langle T', \psi\rangle$$

therefore T' is linear. Hence, in order to check the continuity we can consider $\varphi_n \to 0$ in $\mathscr{D}(\mathbb{R})$. In particular $\varphi_n' \to 0$ in $\mathscr{D}(\mathbb{R})$, so

$$\lim_{n\to\infty} \langle T', \varphi_n\rangle = -\lim_{n\to\infty} \langle T, \varphi_n'\rangle = 0,$$

and the continuity follows from Lemma 3.1. □

Observe that from (4.1) and from the definition of distributional derivative we infer that

$$f \in C^1(\mathbb{R}) \quad \Longrightarrow \quad (T_f)' = T_{f'}.$$

It is also easy to see that if $T, S \in \mathscr{D}'(\mathbb{R})$ and $\lambda, \mu \in \mathbb{C}$, then

$$(\lambda T + \mu S)' = \lambda T' + \mu S'.$$

Example 4.1.

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geqslant 0 \end{cases}$$

Then if $\varphi \in \mathscr{D}$, integrating by parts we have

$$\begin{aligned} \langle T_f', \varphi \rangle &= -\langle T_f, \varphi' \rangle = -\int_{-\infty}^{+\infty} f(x)\varphi'(x)\,\mathrm{d}x = -\int_0^{+\infty} x\varphi'(x)\,\mathrm{d}x \\ &= -\left[x\varphi(x)\right]_{x=0}^{x\to+\infty} + \int_0^{+\infty} \varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \mathbb{1}_{[0,+\infty[}(x)\varphi(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} H(x)\varphi(x)\,\mathrm{d}x = \langle T_H, \varphi \rangle \end{aligned} \tag{4.3}$$

where H is the Heaviside function. Thus $T_f' = T_H$. Observe that the function f is not differentiable at $x = 0$.

(b) Let us compute the distributional derivative of the Heaviside function H, i.e. the derivative of T_H. If $\varphi \in \mathscr{D}$ we get

$$\langle (T_H)', \varphi \rangle = -\langle T_H, \varphi' \rangle = -\int_0^{\infty} \varphi'(x)\,\mathrm{d}x = -(0 - \varphi(0)) = \varphi(0) = \langle \delta_0, \varphi \rangle. \tag{4.4}$$

Therefore

$$\boxed{T_H' = \delta_0}$$

We could consider the function f of the previous example (a) as the law of motion of a particle at rest receiving an impulse from a force at the time $x = 0$ and then moving with velocity 1. Its first distributional derivative is H, the velocity: this is not surprising and distributional derivative does not seem to be a useful tool in this case. More interesting is the second derivative. We get $(T_f)'' = (T_H)' = \delta_0$, the impulsive force.

♡

Example 4.2. Let us compute the derivative of δ_{x_0}, $x_0 \in \mathbb{R}$. For $\varphi \in \mathscr{D}$ we have

$$\langle \delta_{x_0}', \varphi \rangle = -\langle \delta_{x_0}, \varphi' \rangle = -\varphi'(x_0). \tag{4.5}$$

More generally, if $p \in \mathbb{N}$, we have

$$\langle \delta_{x_0}^{(p)}, \varphi \rangle = (-1)^p \langle \delta_{x_0}, \varphi^{(p)} \rangle = (-1)^{(p)} \varphi^{(p)}(x_0).$$

♡

Remark 4.1. Assume that $a, b \in \mathbb{R}$ and $a < b$. Let $\varphi \in C^1([a,b])$ and let $f : [a,b] \longrightarrow \mathbb{C}$ such that

$$\exists f(a+) := \lim_{x\to a+} f(x) \in \mathbb{R}, \qquad \exists f(b-) := \lim_{x\to b-} f(x) \in \mathbb{R}$$

(these limits must be finite) and there exists $f'(x)$ for every $x \in (a,b)$ such that f' summable on $[a,b]$. Then

$$\int_a^b f(x)\varphi'(x)\,\mathrm{d}x = f(b-)\varphi(b) - f(a+)\varphi(a) - \int_a^b f'(x)\varphi(x)\,\mathrm{d}x, \tag{4.6}$$

indeed in order to apply the integration by parts formula in the correct way we have to consider $f(b-)$ and $f(a+)$ instead of $f(b)$ and $f(a)$. ♢

Now we can prove the following useful theorem concerning the distributional derivative of piecewise regular functions.

Theorem 4.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a locally summable function which is differentiable in $\mathbb{R} \setminus \{x_1, x_2, \dots, x_m\}$. Assume that f has a jump point at every x_k (i.e. $f(x_k-)$ and $f(x_k+)$ exist and are finite for every $k = 1, \dots, m$). Moreover assume that $f' : \mathbb{R} \setminus \{x_1, x_2, \dots, x_m\} \longrightarrow \mathbb{C}$ is locally summable in $\mathbb{R}$ (we can arbitrarily define f' at the points x_k). Then*

$$(T_f)' = T_{f'} + \sum_{k=1}^{m} [f(x_k+) - f(x_k-)]\delta_{x_k} \tag{4.7}$$

Proof. Let us prove the theorem in the case $m = 1$, i.e. f has a single jump point x_1. Then, integrating by parts and recalling also Remark 4.1,

$$\begin{aligned}
\langle (T_f)', \varphi \rangle &= -\langle T_f, \varphi' \rangle = -\int_{\mathbb{R}} f(x)\varphi'(x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{x_1} f(x)\varphi'(x)\,\mathrm{d}x - \int_{x_1}^{\infty} f(x)\varphi'(x)\,\mathrm{d}x \\
&= -[f(x)\varphi(x)]_{x=-\infty}^{x=x_1-} + \int_{-\infty}^{x_1} f'(x)\varphi(x)\,\mathrm{d}x \\
&\quad - [f(x)\varphi(x)]_{x=x_1+}^{x=+\infty} + \int_{x_1}^{\infty} f'(x)\varphi(x)\,\mathrm{d}x \\
&= -f(x_1-)\varphi(x_1) + \int_{-\infty}^{x_1} f'(x)\varphi(x)\,\mathrm{d}x + f(x_1+)\varphi(x_1) + \int_{x_1}^{\infty} f'(x)\varphi(x)\,\mathrm{d}x \\
&= [f(x_1+) - f(x_1-)]\varphi(x_1) + \int_{\mathbb{R}} f'(x)\varphi(x)\,\mathrm{d}x \\
&= [f(x_1+) - f(x_1-)]\langle \delta_{x_1}, \varphi \rangle + \langle T_{f'}, \varphi \rangle = \langle [f(x_1+) - f(x_1-)]\delta_{x_1} + T_{f'}, \varphi \rangle.
\end{aligned}$$

□

Observe that from the previous theorem it follows that if $x_1, \dots, x_m \in \mathbb{R}$, then

$$\begin{cases} f \in C(\mathbb{R}), \\ \exists f' \text{ in } \mathbb{R} \setminus \{x_1, \dots, x_m\} \\ f' \text{ locally summable on } \mathbb{R} \end{cases} \qquad \Longrightarrow \qquad (T_f)' = T_{f'} \tag{4.8}$$

Example 4.3.

(a) Compute the derivative of T_f, where $f(x) = \mathbb{1}_{[-3,3]}(x)e^{-|2x|}$.
The function f is locally summable and has two jumps at the points $x_1 = -3$, $x_2 = 3$. The jump at x_1 is $f(-3+) - f(-3-) = e^{-6}$, the jump at x_2 is $f(3+) - f(3-) = -e^{-6}$. Concerning the pointwise derivative we have

$$f'(x) = \begin{cases} 0 & \text{if } |x| > 3 \\ -e^{-|2x|}\,\mathrm{sign}(2x)2 = -2\,\mathrm{sign}(x)e^{-|2x|} & \text{if } |x| < 3,\ x \neq 0 \end{cases}$$

(at the point $x = 0$ there does not exist the derivative, you could consider $x = 0$ as a jump point with jump equal to zero). Since $f' : \mathbb{R} \setminus \{-3, 0, 3\} \longrightarrow \mathbb{R}$ is locally summable we have that

$$(T_f)' = T_{f'} + e^{-6}\delta_{-3} - e^{-6}\delta_3$$

(b) Compute the derivative of T_f, where $f(x) = |x| - 1 + p_2(x)$.
f is locally summable and $f'(x) = \operatorname{sign}(x)$ for every $x \in \mathbb{R} \smallsetminus \{-1, 0, 1\}$. Since f' is locally summable, we have that

$$T_f' = T_{\text{sign}} + \delta_{-1} - \delta_1.$$

(c) Compute the derivative of T_f, where $f(x) = e^x H(-x) + (e^x + 1)p_2(x-1)$.
If $f_1(x) = e^x H(-x)$ then $f_1'(x) = e^x H(-x)$ for $x \neq 0$. At $x = 0$ the jump of f_1 is -1, thus $T_{f_1}' = T_{f_1'} - \delta_0$. If $f_2(x) = (e^x + 1)p_2(x-1)$ then $f_2'(x) = e^x p_2(x-1)$ for $x \neq 0$ and $x \neq 1$. There are two jumps: at $x = 0$ the jump of f_2 is 2; at $x = 1$ the jump of f_2 is $-(e^2 + 1)$, thus $T_{f_2}' = T_{f_2'} + 2\delta_0 - (e^2 + 1)\delta_2$. Therefore

$$T_f' = T_{e^x H(-x) + e^x p_2(x-1)} + \delta_0 - (e^2 + 1)\delta_2.$$

♡

The following important example shows the classical pointwise derivative can be different from the distributional derivative:

Example 4.4. Let us consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) := \log|x|$ which is locally summable.[1] Let us find its distributional derivative, i.e. let us compute T_f'. Observe that $\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = 1/x$ for every $x \neq 0$, but $1/x$ is not locally summable on $\mathbb{R}$. Thus the symbol $T_{1/x}$ makes no sense. If $\varphi \in \mathscr{D}$ there exists some $R > 0$ such that $\operatorname{supp}(\varphi) \subseteq [-R, R]$, and we find

$$\begin{aligned}
\langle T_{\log|x|}', \varphi\rangle &= -\langle T_{\log|x|}, \varphi'\rangle = -\int_{-\infty}^{+\infty} \log|x|\ \varphi'(x)\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0+} \left(\int_{-R}^{-\varepsilon} \log|x|\ \varphi'(x)\,\mathrm{d}x + \int_{\varepsilon}^{R} \log|x|\ \varphi'(x)\,\mathrm{d}x \right) \\
&\overset{\text{by parts}}{=} -\lim_{\varepsilon\to 0+} \left([\log|x|\ \varphi(x)]_{x=-R}^{x=-\varepsilon} - \int_{-R}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + [\log|x|\ \varphi(x)]_{x=\varepsilon}^{x=R} - \int_{\varepsilon}^{R} \frac{\varphi(x)}{x}\,\mathrm{d}x \right) \\
&= -\lim_{\varepsilon\to 0+} \left(\log\varepsilon\ \varphi(-\varepsilon) - \int_{-R}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x - \log\varepsilon\ \varphi(\varepsilon) - \int_{\varepsilon}^{R} \frac{\varphi(x)}{x}\,\mathrm{d}x \right) \\
&= -\lim_{\varepsilon\to 0+} \left(\log\varepsilon\ (\varphi(\varepsilon) - \varphi(-\varepsilon)) - \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + - \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \right).
\end{aligned}$$

Now observe that

$$\lim_{\varepsilon\to 0+} \log\varepsilon\ (\varphi(\varepsilon) - \varphi(-\varepsilon)) = \lim_{\varepsilon\to 0+} \varepsilon\log\varepsilon\ \frac{\varphi(\varepsilon) - \varphi(-\varepsilon)}{\varepsilon} = 0 \cdot 2\varphi'(0) = 0.$$ [2]

Hence we get

$$\langle T_{\log|x|}', \varphi\rangle = \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x. \tag{4.9}$$

It follows that the right hand side of the previous formula defines a distribution that we call *principal value of* $1/x$ and we denote by $\mathrm{p.v.}\frac{1}{x} \in \mathscr{D}'(\mathbb{R})$:

$$\boxed{\langle \mathrm{p.v.}\frac{1}{x}, \varphi\rangle := \lim_{\varepsilon\to 0+} \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x, \qquad \varphi \in \mathscr{D}.} \tag{4.10}$$

Summarizing we have

$$\boxed{(T_{\log|x|})' = \mathrm{p.v.}\frac{1}{x}.} \tag{4.11}$$

♡

[1] Since $\lim_{x\to 0} |x|^\alpha \log|x| = 0$ for every $\alpha > 0$, we have that $|x|^\alpha |\log|x|| \leqslant 1$ in a neighborhood of $x = 0$, hence $|\log|x|| \leqslant 1/|x|^\alpha$ in such neighborhood. Taking $\alpha = 1/2$ we obtain the absolute convergence of the improper integrals $\int_0^1 \log|x|\,\mathrm{d}x$, $\int_{-1}^0 \log|x|\,\mathrm{d}x$.

[2] $\lim_{\varepsilon\to 0+} \frac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0+} \frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon} + \frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon} = 2\varphi'(0)$, but you can also use the L'Hopital rule.

4.2 Translation

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be locally summable and $x_0 \in \mathbb{R}$. Let us consider the function $g : \mathbb{R} \longrightarrow \mathbb{C}$ defined by $g(x) := f(x - x_0)$, $x \in \mathbb{R}$. The graph of g is obtained by translating the graph of f. Thanks to a change of variable, for every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$\langle T_g, \varphi\rangle = \int_{\mathbb{R}} f(x - x_0)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\varphi(x + x_0)\,\mathrm{d}x = \langle T_f(x), \varphi(x + x_0)\rangle \quad (4.12)$$

where we used the (incorrect, but not ambiguous) notation $\langle T_f(x), \varphi(x+x_0)\rangle$, meaning that we are evaluating the functional T_f at the test function $\psi(x) := \varphi(x - x_0)$.

This example suggests the following

Definition 4.2 (Translation)**.** If $T \in \mathscr{D}'(\mathbb{R})$ and $a \in \mathbb{R}$, the *translation of* T *by* a is the functional $T(x - a) : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle T(x - a), \varphi\rangle := \langle T(x), \varphi(x + a)\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \quad (4.13)$$

It is easy to check that $T(x - a)$ is a distribution. Let us stress the fact that $T(x-a)$ is just a symbol, a notation for the distribution defined by $\langle T(x), \varphi(x+a)\rangle$: $x-a$ is not the independent variable of T, since T *is not* a function of a real variable.

Example 4.5. If $x_0 \in \mathbb{R}$ and $a \in \mathbb{R}$, then

$$\langle \delta_{x_0}(x - a), \varphi\rangle = \langle \delta_{x_0}, \varphi(x + a)\rangle = \varphi(x_0 + a) = \langle \delta_{x_0+a}, \varphi\rangle.$$

In particular

$$\boxed{\delta_0(x - a) = \delta_a}$$

♡

4.3 Rescaling

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be locally summable and let $a \in \mathbb{R}\setminus\{0\}$ be given. Let $g : \mathbb{R} \longrightarrow \mathbb{C}$ be the defined by $g(x) := f(ax)$, $x \in \mathbb{R}$. By the change of variable $y = ax$, for every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$\langle T_g, \varphi\rangle = \int_{-\infty}^{+\infty} f(ax)\varphi(x)\,\mathrm{d}x = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} f(y)\varphi\left(\frac{y}{a}\right)\frac{1}{a}\,\mathrm{d}y & \text{if } a > 0 \\[2ex] \displaystyle -\int_{-\infty}^{+\infty} f(y)\varphi\left(\frac{y}{a}\right)\frac{1}{a}\,\mathrm{d}y & \text{if } a < 0 \end{cases}$$

$$= \int_{-\infty}^{+\infty} f(y)\varphi\left(\frac{y}{a}\right)\frac{1}{|a|}\,\mathrm{d}y = \left\langle T_{f(x)}, \frac{1}{|a|}\varphi\left(\frac{x}{a}\right)\right\rangle.$$

Hence

$$\langle T_{f(xa)}, \varphi(x)\rangle = \left\langle T_{f(x)}, \frac{1}{|a|}\varphi\left(\frac{x}{a}\right)\right\rangle$$

Therefore we are lead to the following

Definition 4.3 (Rescaling)**.** If $T \in \mathscr{D}'(\mathbb{R})$ and $a \in \mathbb{R} \smallsetminus \{0\}$, we call *rescaling of T by the factor a* the distribution $T(ax) : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle T(ax), \varphi \rangle := \left\langle T(x), \frac{1}{|a|} \varphi \left(\frac{x}{a} \right) \right\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{4.14}$$

Also in this case it is easy to check that $T(ax)$ is actually a distribution. A relevant particular case of the previous definition is given by $a = -1$:

$$\langle T(-x), \varphi \rangle := \langle T(x), \varphi(-x) \rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{4.15}$$

Example 4.6. If $x_0 \in \mathbb{R}$ and $a \neq 0$ we have

$$\langle \delta_{x_0}(ax), \varphi \rangle = \left\langle \delta_{x_0}, \frac{1}{|a|} \varphi \left(\frac{x}{a} \right) \right\rangle = \frac{1}{|a|} \varphi \left(\frac{x_0}{a} \right) = \left\langle \frac{1}{|a|} \delta_{\frac{x_0}{a}}, \varphi \right\rangle$$

hence

$$\delta_{x_0}(ax) = \frac{1}{|a|} \delta_{x_0/a}. \tag{4.16}$$

If $a = -1$

$$\boxed{\delta_{x_0}(-x) = \delta_{-x_0}.} \tag{4.17}$$

♡

4.4 Multiplication by a C^∞ function

Let us consider $f : \mathbb{R} \longrightarrow \mathbb{C}$ locally summable and $h \in C^\infty(\mathbb{R})$. Let $hf : \mathbb{R} \longrightarrow \mathbb{C}$ be defined by $hf(x) := h(x)f(x)$, $x \in \mathbb{R}$, locally summable. For every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$\langle T_{hf}, \varphi \rangle = \int_{\mathbb{R}} h(x) f(x) \varphi(x) \, \mathrm{d}x = \int_{\mathbb{R}} f(x) h(x) \varphi(x) \, \mathrm{d}x = \langle T_f, h\varphi \rangle \tag{4.18}$$

which makes sense, since $h\varphi \in \mathscr{D}(\mathbb{R})$ ($\operatorname{supp}(h\varphi) \subseteq \operatorname{supp}(\varphi)$ and $h\varphi \in C^\infty$ by the Leibniz formula for the derivative of a product). The formula for T_{hf} suggests the following

Definition 4.4 (Multiplication)**.** If $T \in \mathscr{D}'(\mathbb{R})$ and $h \in C^\infty(\mathbb{R})$, we call *multiplication of T by h* the functional $hT : \mathscr{D}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\langle hT, \varphi \rangle := \langle T, h\varphi \rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{4.19}$$

Example 4.7. If $h \in C^\infty(\mathbb{R})$ we get

$$\langle h\delta_{x_0}, \varphi \rangle = \langle \delta_{x_0}, h\varphi \rangle = h(x_0)\varphi(x_0) = \langle h(x_0)\delta_{x_0}, \varphi \rangle$$

i.e.

$$\boxed{h\delta_{x_0} = h(x_0)\delta_{x_0}.} \tag{4.20}$$

For instance $\cos(x)\delta_{\pi/4} = \frac{1}{\sqrt{2}}\delta_{\pi/4}$. ♡

Now some properties showing the connection between the derivative and the previous operations. The proofs are left as an exercise.

Proposition 4.2. *If $T \in \mathscr{D}'$, $a \in \mathbb{R}$, and $h \in C^\infty(\mathbb{R})$, then*

$$\begin{aligned} &(T(x-a))' = T'(x-a), \\ &(T(ax))' = aT'(ax) \qquad (a \neq 0), \\ &(hT)' = h'T + hT'. \end{aligned}$$

5 Convergence of distributions

Definition 5.1 (Convergence of distributions)**.** Let $T_n \in \mathscr{D}'(\mathbb{R})$ be a sequence of distributions. We say that T_n *converges to $T \in \mathscr{D}'(\mathbb{R})$ in the sense of distributions* (and we write $T_n \to T$ *in* $\mathscr{D}'$) if

$$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \quad \text{as } n \to \infty, \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}). \tag{5.1}$$

It is easy to prove the following

Proposition 5.1. *If $T_n \to T$ and $S_n \to S$ in $\mathscr{D}'(\mathbb{R})$, then $\lambda T_n + \mu S_n \to \lambda T + \mu S$ for every $\lambda, \mu \in \mathbb{C}$.*

Example 5.1.

(a) Assume that $T_n = \delta_n$ for every $n \in \mathbb{N}$. If $\varphi \in \mathscr{D}$ is arbitrarily fixed, then $\operatorname{supp}(\varphi) \subseteq [a, b]$, for some $a, b \in \mathbb{R}$ and we have

$$\langle \delta_n, \varphi \rangle = \varphi(n) \qquad \forall n.$$

Now $\varphi(n) = 0$ for every $n > b$, thus $\lim_{n\to\infty} \varphi(n) = 0$ and

$$\langle \delta_n, \varphi \rangle \to 0 = \langle 0, \varphi \rangle \qquad \text{as } n \to \infty.$$

Therefore $\delta_n \to 0$ in $\mathscr{D}'(\mathbb{R})$.

(b) Define $T_n = \delta_n - \delta_{1/n}$. If $\varphi \in \mathscr{D}$ then

$$\langle T_n, \varphi \rangle = \langle \delta_n, \varphi \rangle - \langle \delta_{1/n}, \varphi \rangle = \varphi(n) - \varphi(1/n) \to 0 - \varphi(0)$$

as φ is continuous at $x = 0$. Thus

$$\langle T_n, \varphi \rangle \to -\varphi(0) = -\langle \delta_0, \varphi \rangle$$

i.e. $\delta_n - \delta_{1/n} \to -\delta_0$ in $\mathscr{D}'(\mathbb{R})$.

(c) Set $T_n = \delta_{(-2)^{3n}\log(2n)}$ for $n > 0$. If $\varphi \in \mathscr{D}$ then $\operatorname{supp}(\varphi) \subseteq [a, b]$, for some $a, b \in \mathbb{R}$ and we have

$$\langle T_n, \varphi \rangle = \varphi\left((-2)^{3n}\log(2n)\right).$$

The limit of the sequence $(-2)^{3n}\log(2n)$ does not exist, but $|(-2)^{3n}\log(2n)| \to +\infty$ as $n \to \infty$, therefore $(-2)^{3n}\log(2n) \notin \operatorname{supp}(\varphi)$ for all n greater than a suitable n_0. It follows that $\varphi\left((-2)^{3n}\log(2n)\right) = 0$ for every $n > n_0$ and

$$\langle T_n, \varphi \rangle \to 0 \qquad \text{as } n \to \infty$$

i.e. $T_n \to 0$ in $\mathscr{D}'(\mathbb{R})$.

(d) If $T_n = \delta_{(-1)^n}$ then for every $\varphi \in \mathscr{D}$ we have

$$\langle \delta_{(-1)^n}, \varphi \rangle = \varphi((-1)^n).$$

Now let us consider a test function φ_0 such that $\varphi_0(-1) = 1$ and $\varphi_0(1) = 0$. It follows that the limit of $\varphi_0((-1)^n)$ does not exists, thus $\delta_{(-1)^n}$ does not converge in $\mathscr{D}'(\mathbb{R})$. ♡

Proposition 5.2. *Assume that $f_n, f : \mathbb{R} \longrightarrow \mathbb{C}$ are given for every n and that*

$$f_n \to f \qquad \text{uniformly on } [a,b] \quad \forall a,b \in \mathbb{R},\ a < b$$

($[a,b]$ bounded). Then $T_{f_n} \to T_f$ in $\mathscr{D}'$.

Proof. We have to proof that $\langle T_{f_n}, \varphi \rangle \to \langle T_f, \varphi \rangle$ for every $\varphi \in \mathscr{D}$, i.e. that $\langle T_{f_n} - T_f, \varphi \rangle \to 0$ as $n \to \infty$. Therefore we are going to estimate $|\langle T_{f_n} - T_f, \varphi \rangle|$ with a sequence converging to zero. If $\mathrm{supp}(\varphi) \subseteq [c,d]$, then

$$\begin{aligned} |\langle T_{f_n} - T_f, \varphi \rangle| &= \left| \int_{-\infty}^{+\infty} (f_n(x) - f(x))\varphi(x)\,\mathrm{d}x \right| \leqslant \int_{-\infty}^{+\infty} |f_n(x) - f(x)||\varphi(x)|\,\mathrm{d}x \\ &\leqslant \int_{-\infty}^{+\infty} \|f_n - f\|_{\infty,[c,d]} \|\varphi\|_\infty \,\mathrm{d}x = \|f_n - f\|_{\infty,[c,d]} \|\varphi\|_\infty (d-c) \to 0 \end{aligned}$$

as $n \to \infty$ since $f_n \to f$ uniformly on $[c,d]$. □

Example 5.2. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f_n(x) = \dfrac{\cos(nx)}{n}$, $x \in \mathbb{R}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$, indeed

$$\|f_n\|_\infty = \sup_{x\in\mathbb{R}} \left| \frac{\cos(nx)}{n} \right| \leqslant \sup_{x\in\mathbb{R}} \frac{1}{n} = \frac{1}{n} \to 0$$

as $n \to \infty$. Therefore thanks to the previous theorem $T_{f_n} \to 0$ in $\mathscr{D}'(\mathbb{R})$. Of course it is easy to find directly the distributional limit: if $\varphi \in \mathscr{D}$ and $\mathrm{supp}(\varphi) \subseteq [a,b]$, then

$$|\langle T_{f_n}, \varphi \rangle| = \left| \int_a^b \frac{\cos(nx)}{n} \varphi(x)\,\mathrm{d}x \right| \leqslant \int_a^b \left| \frac{\cos(nx)}{n} \right| |\varphi(x)|\,\mathrm{d}x \leqslant \frac{\|\varphi\|_\infty}{n}(b-a) \to 0 \tag{5.2}$$

as $n \to \infty$ (these are the same computations as in the previous proof). ♡

Example 5.3. If $f_n(x) = \mathbb{1}_{[n,n+1]}(x)$ then $f_n(x) \to 0$ for every $x \in \mathbb{R}$ fixed, indeed

$$\mathbb{1}_{[n,n+1]}(x) = \begin{cases} 1 & \text{if } n \leqslant x \leqslant n+1 \\ 0 & \text{if } x \notin [n,n+1] \end{cases}$$

hence $f_n(x) = 0$ for every $n > x$. This convergence is not uniform on $\mathbb{R}$, but it is uniform on every bounded interval $[a,b]$:

$$\|f_n\|_\infty = \sup_{x\in[a,b]} |\mathbb{1}_{[n,n+1]}(x)| = 0 \qquad \forall n > b.$$

Hence $T_{f_n} \to 0$ in $\mathscr{D}'(\mathbb{R})$. Also in this case it is maybe easier to check the distributional convergence directly by means of the definition. ♡

Example 5.4. Let $f_n(x) = np_{1/n}(x)$ for every $x \in \mathbb{R}$. It is easy to see that $\lim_{n\to\infty} f_n(x) = 0$ if $x \neq 0$ and $f_n(0) = n \to +\infty$, so the convergence is not uniform. Let us verify if T_{f_n} converges in $\mathscr{D}'(\mathbb{R})$. If $\varphi \in \mathscr{D}$ then

$$\langle T_{f_n}, \varphi \rangle = \int_{-1/2n}^{1/2n} n\varphi(x)\,\mathrm{d}x = \frac{1}{1/n} \int_{-1/2n}^{1/2n} \varphi(x)\,\mathrm{d}x = \varphi(x_n) \tag{5.3}$$

for a suitable $x_n \in]-1/2n, 1/2n[$, by virtue of the integral mean value theorem. Since $-1/2n < x_n < 1/2n$, we have that $\lim_{n\to\infty} x_n = 0$, thus by the continuity of φ at $x = 0$, we find $\lim_{n\to\infty} \varphi(x_n) = \varphi(0)$. Then

$$\lim_{n\to\infty} \langle T_{f_n}, \varphi\rangle = \lim_{n\to\infty} \varphi(x_n) = \varphi(0) = \langle \delta_0, \varphi\rangle$$

and $T_{np_{1/n}} \to \delta_0$ in $\mathscr{D}'(\mathbb{R})$. ♡

6 Distributions with compact support

Definition 6.1. Assume that $T \in \mathscr{D}'(\mathbb{R})$.

(i) We say that T *vanishes in* $]a, b[\subseteq \mathbb{R}$, if $\langle T, \varphi\rangle = 0$ for every $\varphi \in \mathscr{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq]a, b[$.

(ii) We call *support of* T the set $\mathrm{supp}(T) := \mathbb{R}\setminus N_T$, where N_T is the union of all the intervals where T vanishes.

From the previous definition it follows that T has compact support if and only if

$$\exists a, b \in \mathbb{R},\ a < b,\ :\ T \text{ vanishes on } \mathbb{R}\setminus[a, b]\,.$$

Example 6.1. Let us find the support of δ_{x_0}, where $x_0 \in \mathbb{R}$.
It is clear that δ_{x_0} does not vanish on any open interval containing x_0. Now take $a, b \in \mathbb{R}$, $a < b$, and $\varphi \in \mathscr{D}(\mathbb{R})$. Then

$$]a, b[\subseteq \mathbb{R}\setminus\{x_0\},\ \mathrm{supp}(\varphi) \subseteq]a, b[\quad \Longrightarrow \quad \langle \delta_{x_0}, \varphi\rangle = \varphi(x_0) = 0,$$

thus δ_{x_0} vanishes in $]a, b[$. Hence $N_{\delta_{x_0}} = \mathbb{R}\setminus\{x_0\}$ and we have

$$\mathrm{supp}(\delta_{x_0}) = \{x_0\},$$

in particular δ_{x_0} has compact support. ♡

Proposition 6.1. *If* $T \in \mathscr{D}'(\mathbb{R})$ *then* $\mathrm{supp}(T') \subseteq \mathrm{supp}(T)$.

Proof. If T vanishes in $]a, b[$, $\varphi \in \mathscr{D}(\mathbb{R})$ and $\mathrm{supp}(\varphi) \subseteq]a, b[$ then $\mathrm{supp}(\varphi') \subseteq]a, b[$, thus $\langle T', \varphi\rangle = -\langle T, \varphi'\rangle = 0$. It follows that $N_T \subseteq N_{T'}$, therefore $\mathrm{supp}(T') = \mathbb{R}\setminus N_{T'} \subseteq \mathbb{R}\setminus N_T = \mathrm{supp}(T)$. □

Example 6.2. From the previous proposition we infer that $\mathrm{supp}(\delta^{(p)}_{x_0}) \subseteq \{x_0\}$ for every $p \in \mathbb{N}$. On the other hand, if we take a test function such that $\varphi^{(p)}(x_0) \neq 0$ we obtain that $\langle \delta^{(p)}_{x_0}, \varphi\rangle = (-1)^p \varphi^{(p)}(x_0) \neq 0$, thus $x_0 \notin N_{\delta^{(p)}_{x_0}}$. Therefore

$$\mathrm{supp}(\delta^{(p)}_{x_0}) = \{x_0\} \qquad \forall p \in \mathbb{N}.$$

♡

Definition 6.2. We say that $T \in \mathscr{D}'(\mathbb{R})$ *is with compact support* if $\mathrm{supp}(T)$ is compact.

When $T \in \mathscr{D}'$ and $\operatorname{supp}(T)$ is compact, it is possible to give a meaning to $\langle T, \varphi \rangle$, where $\varphi \in C^\infty$, but its support is not compact.

Definition 6.3. Let $T \in \mathscr{D}'(\mathbb{R})$ have compact support, and let $a, b \in \mathbb{R}$ be such that $\operatorname{supp}(T) \subseteq \,]a, b[$. Then we set

$$\langle T, \psi \rangle := \langle T, \varphi_0 \psi \rangle \qquad \forall \psi \in C^\infty(\mathbb{R}), \tag{6.1}$$

where $\varphi_0 \in \mathscr{D}(\mathbb{R})$ is a test function such that $\varphi_0(x) = 1$ for every $x \in \,]a, b[$. In this way we extend the definition of T to $C^\infty(\mathbb{R})$, in other words we can write $T : C^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$.

The previous definitions makes sense, since it is possible to prove that it does not depend on the choice of φ_0.

7 Convolution

7.1 Convolution of functions

Definition 7.1. If $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ are locally summable, we call *convolution of f and g* the function

$$(f * g)(x) := \int_{-\infty}^{+\infty} f(x-y) g(y) \, \mathrm{d}y \tag{7.1}$$

defined for the real numbers x such that the integral is convergent.

Let us observe that by a change of variable it is easily seen that

$$\boxed{f * g = g * f} \tag{7.2}$$

indeed, if we set $t = x - y$, we have $\mathrm{d}t = -\,\mathrm{d}y$ and

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{+\infty} f(x-y) g(y) \, \mathrm{d}y = -\int_{+\infty}^{-\infty} f(t) g(x-t) \, \mathrm{d}y \\ &= \int_{-\infty}^{+\infty} f(t) g(x-t) \, \mathrm{d}y = (g * f)(x).\end{aligned}$$

Now we give some sufficient conditions ensuring that $(f * g)(x)$ exists.

Proposition 7.1. *Let $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ be given.*

(a) f summable, g locally summable, g bounded $\Longrightarrow$ $(f * g)(x)$ exists.

(b) f locally summable, $g \in C(\mathbb{R})$, $\operatorname{supp}(g)$ compact $\Longrightarrow$ $(f * g)(x)$ exists.

(c) $|f|^2, |g|^2$ summable $\Longrightarrow$ $(f * g)(x)$ exists.

Proof.

(a) For every $y \in \mathbb{R}$ we have $|f(x-y)g(y)| \leqslant |f(x-y)|\|g\|_\infty$ and

$$\int_{-\infty}^{+\infty} |f(x-y)|\|g\|_\infty \,\mathrm{d}y = \|g\|_\infty \int_{-\infty}^{+\infty} |f(t)| \,\mathrm{d}t < \infty.$$

(b) Since g is continuous and its support is contained in some interval $[a,b]$, $a,b \in \mathbb{R}$, by the Weierstrass theorem $\|g\|_\infty$ is finite, hence $|f(x-y)g(y)| \leqslant |f(x-y)|\|g\|_\infty$. Moreover the support of the function $F(y) = f(x-y)g(y)$ is contained in $[a,b]$, therefore

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(x-y)g(y)| \,\mathrm{d}y &= \int_a^b |f(x-y)g(y)| \,\mathrm{d}y \\ &\leqslant \int_a^b |f(x-y)|\|g\|_\infty \,\mathrm{d}y = \|g\|_\infty \int_{x-b}^{x-a} |f(t)| \,\mathrm{d}t < \infty\end{aligned}$$

as f is locally summable.

(c) Observe that f and g are locally summable, indeed $|f(x)| \leqslant \frac{|f(x)|^2}{2} + \frac{1}{2}$ [3] and if $-\infty < a < b < +\infty$ we have

$$\begin{aligned}\int_a^b |f(x)| \,\mathrm{d}x &\leqslant \int_a^b \left(\frac{|f(x)|^2}{2} + \frac{1}{2}\right) \mathrm{d}x = \int_a^b \frac{|f(x)|^2}{2} \,\mathrm{d}x + \frac{b-a}{2} \\ &\leqslant \frac{1}{2}\int_{-\infty}^{+\infty} |f(x)|^2 \,\mathrm{d}x + \frac{b-a}{2} < \infty.\end{aligned}$$

It follows that $F(y) = f(x-y)g(y)$ is locally summable. We have

$$\int_{-\infty}^{+\infty} |f(x-y)g(y)| \,\mathrm{d}x \leqslant \int_{-\infty}^{+\infty} \frac{|f(x-y)|^2}{2} \,\mathrm{d}y + \int_{-\infty}^{+\infty} \frac{|g(y)|^2}{2} \,\mathrm{d}y < \infty.$$

□

Proposition 7.2. *If $p \in \mathbb{N}$, $f : \mathbb{R} \longrightarrow \mathbb{C}$ is locally summable, $g \in C^p(\mathbb{R})$, and* $\mathrm{supp}(g)$ *is compact, then $f * g \in C^p(\mathbb{R})$ and*

$$(f*g)^{(p)}(x) = (f * g^{(p)})(x) \qquad \forall x \in \mathbb{R}, \qquad p \geqslant 1.$$

Proof. Thanks to Proposition 7.1-(b), the convolution exists. It is possible to prove that one can pass to the limit under the integral sign, thus for every $x_n \to x$ one has, using the continuity of g,

$$\begin{aligned}\lim_{n\to\infty}(f*g)(x_n) &= \lim_{n\to\infty}(g*f)(x_n) = \lim_{n\to\infty}\int_{\mathbb{R}} g(x_n-y)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \lim_{n\to\infty} g(x_n-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}} g(x-y)f(y)\,\mathrm{d}y = (g*f)(x) = (f*g)(x).\end{aligned}$$

It follows that $(f*g)$ is continuous at x. It is also possible to differentiate under the integral sign, hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(f*g)(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(g*f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} g(x-y)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x} g(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}} g'(x-y)f(y)\,\mathrm{d}y = (g'*f)(x) = (f*g')(x).\end{aligned}$$

The statement for $p \in \mathbb{N}$ follows arguing by induction. □

[3] for every $\alpha, \beta \in \mathbb{R}$ one has $\alpha\beta \leqslant (\alpha^2+\beta^2)/2$, since $0 \leqslant (\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

Lemma 7.1. *Let $\rho_n : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined in Example 2.2. If $f \in C(\mathbb{R})$ then for every $a, b \in \mathbb{R}$*

$$\rho_n * f \to f \qquad \text{uniformly on } [a,b].$$

Proof. For every $x \in \mathbb{R}$ we have

$$\begin{aligned}(\rho_n * f)(x) - f(x) = (f * \rho_n)(x) - f(x) &= \int_{\mathbb{R}} \rho_n(y) f(x-y)\,\mathrm{d}y - \int_{\mathbb{R}} \rho_n(y) f(x)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \rho_n(y)(f(x-y) - f(x))\,\mathrm{d}y = \int_{-1/n}^{1/n} \rho_n(y)(f(x-y) - f(x))\,\mathrm{d}y.\end{aligned}$$

Since f is uniformly continuous on $[a,b]$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x-y) - f(x)| < \varepsilon$ whenever $|y| < \delta$. Thus for $n > 1/\delta$ we have

$$\begin{aligned}\|\rho_n * f - f\|_{\infty,[a,b]} &= \sup_{x\in[a,b]} |(\rho_n * f)(x) - f(x)| \\ &\leqslant \int_{-1/n}^{1/n} \rho_n(y)|f(x-y) - f(x)|\,\mathrm{d}y < \varepsilon \int_{-1/n}^{1/n} \rho_n(y) = \varepsilon.\end{aligned}$$

□

Corollary 7.1. *If $\rho_n : \mathbb{R} \longrightarrow \mathbb{R}$ is the function defined in Example 2.2, then $T_{\rho_n} \to \delta_0$ in $\mathscr{D}'(\mathbb{R})$.*

Proof. If $\varphi \in \mathscr{D}$ by the previous lemma we have that $\rho_n * \varphi \to \varphi$ uniformly on bounded intervals, in particular $(\rho_n * \varphi)(0) \to \varphi(0)$, hence

$$\begin{aligned}\langle T_{\rho_n}, \varphi\rangle &= \int_{-\infty}^{+\infty} \rho_n(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} \rho_n(0-x)\varphi(x)\,\mathrm{d}x \\ &= (\rho_n * \varphi)(0) \to \varphi(0) = \langle \delta_0, \varphi\rangle.\end{aligned}$$

□

7.2 Convolution of distributions

Let us consider two functions f, g such that the convolution $f * g$ is locally summable and let us compute T_{f*g}. For every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$\begin{aligned}\langle T_{f*g}, \varphi\rangle &= \int_{-\infty}^{+\infty} (f*g)(x)\varphi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y)g(x-y)\,\mathrm{d}y\ \varphi(x)\,\mathrm{d}x \\ &\overset{\text{Fubini}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y)g(x-y)\ \varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} f(y) \int_{-\infty}^{+\infty} g(x-y)\ \varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\overset{t=x-y}{=} \int_{-\infty}^{+\infty} f(y) \int_{-\infty}^{+\infty} g(t)\ \varphi(y+t)\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} f(y) \int_{-\infty}^{+\infty} g(x)\ \varphi(y+x)\,\mathrm{d}x\,\mathrm{d}y = \Big\langle T_f(y), \langle T_g(x), \varphi(x+y)\rangle \Big\rangle\end{aligned}$$

(let us recall that Fubini theorem allows to exchange the order of the integrals). This computation may lead to define the convolution of two distributions $T, S \in \mathscr{D}'$ in the following way:

$$\langle T * S, \varphi\rangle := \Big\langle T(y), \big\langle S(x), \varphi(x+y)\big\rangle \Big\rangle, \tag{7.3}$$

but we have to check if the right hand side makes sense. The following proposition can be proved

Proposition 7.3. *If $x \in \mathbb{R}$ is fixed let $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ be the function defined by*

$$\psi(y) := \big\langle S(x), \varphi(x+y)\big\rangle, \qquad y \in \mathbb{R}$$

Then

$$\psi \in C^\infty(\mathbb{R}), \tag{7.4}$$

$$\operatorname{supp}(S) \text{ compact} \quad \Longrightarrow \quad \operatorname{supp}(\psi) \text{ compact}. \tag{7.5}$$

Proof. . □

Thus we have two possibilites:

(1.) If $\operatorname{supp}(S)$ is compact, then, thanks to (7.4)-(7.5), $\psi \in \mathscr{D}(\mathbb{R})$ and (7.3) makes sense.

(2.) If instead $\operatorname{supp}(T)$ is compact, from (7.5), we infer that (7.3) makes sense according to Definition 6.3.

Therefore we can finally give the following

Definition 7.2. Assume that $T, S \in \mathscr{D}'(\mathbb{R})$ and that $\operatorname{supp}(T)$ is compact or $\operatorname{supp}(S)$ is compact. The *convolution of T and S* is the distribution $T * S \in \mathscr{D}'(\mathbb{R})$ defined by

$$\langle T * S, \varphi\rangle := \Big\langle T(y), \big\langle S(x), \varphi(x+y)\big\rangle \Big\rangle, \qquad \varphi \in \mathscr{D}(\mathbb{R}). \tag{7.6}$$

It is possible to prove that $T * S$ is linear and continuous, i.e. it is in fact a distribution.

Proposition 7.4. *If $S, T, U \in \mathscr{D}'(\mathbb{R})$ then*

(i) $S * T = T * S$

(ii) $S * (T * U) = (S * T) * U$

(iii) $S * (\lambda T + \mu U) = \lambda(S * T) + \mu(S * U)$

(iv) $(S * T)(x - a) = S(x - a) * T = S * T(x - a)$

(v) $(S * T)' = S' * T = S * T'$

when the convolutions make sense.

Proof. $\square$

Example 7.1. If $T \in \mathscr{D}'$ and $x_0 \in \mathbb{R}$ then

$$\langle T * \delta_{x_0}, \varphi \rangle = \Big\langle T(y), \langle \delta_{x_0}(x), \varphi(x+y) \rangle \Big\rangle = \langle T(y), \varphi(x_0 + y) \rangle = \langle T(y - x_0), \varphi(y) \rangle$$

where in the last equality we used the definition of $T(x - x_0)$, hence

$$\boxed{T * \delta_{x_0} = T(x - x_0)}$$

in particular for $x_0 = 0$

$$\boxed{T * \delta_0 = T}$$

$\heartsuit$

Example 7.2. If $T \in \mathscr{D}'$ and $x_0 \in \mathbb{R}$ then

$$(T * \delta_{x_0})^{(p)} = T^{(p)} * \delta_{x_0} = T^{(p)}(x - x_0).$$

$\heartsuit$

Proposition 7.5. *If $T_n, T, S \in \mathscr{D}'(\mathbb{R})$, $\operatorname{supp}(S)$ is compact and $T_n \to T$ in $\mathscr{D}'(\mathbb{R})$, then $T_n * S \to T * S$ in $\mathscr{D}'(\mathbb{R})$.*

Proof. For every $\varphi \in \mathscr{D}$ we have

$$\langle T_n * S, \varphi \rangle = \Big\langle T_n(y), \langle S(x), \varphi(x+y) \rangle \Big\rangle \to \Big\langle T(y), \langle S(x), \varphi(x+y) \rangle \Big\rangle = \langle T * S, \varphi \rangle$$

as $n \to \infty$. $\square$

Proposition 7.6. *Let $\rho_n : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined in Example 2.2. If $T \in \mathscr{D}'(\mathbb{R})$, then $T * \rho_n$ is a regular distribution associated to a C^∞ function and $T * \rho_n \to T$ in $\mathscr{D}'(\mathbb{R})$.*

Proof. $\square$

We conclude this chapter with two useful theorems (we omit their proofs).

Theorem 7.1. *If $T \in \mathscr{D}'(\mathbb{R})$ and $xT(x) = 0$ then there exists a constant c such that $T = c\delta_0$.*

Theorem 7.2. *If $T \in \mathscr{D}'(\mathbb{R})$ and $T' = 0$ then T is a constant, more precisely there exists a constant c such that $T = T_c$.*

Exercise set (from the old Complex Analysis lecture notes)

1. Which of the following functionals are distributions?

$$\langle T_1, \varphi\rangle = \int_0^1 \ln(x+1)\varphi(x)\,dx\,, \qquad \langle T_2, \varphi\rangle = \int_0^1 |\varphi(x)|^2\,dx\,,$$
$$\langle T_3, \varphi\rangle = \int_0^1 \varphi'(x)\,dx\,, \qquad \langle T_4, \varphi\rangle = |\varphi(5)|$$
$$\langle T_5, \varphi\rangle = \varphi(1)-\varphi(2)+\varphi(3)-\varphi(4)\,, \qquad \langle T_6, \varphi\rangle = \int_{-4}^4 \sin x\varphi(x)\,dx + 6\varphi(4)$$

2. Which of the following functionals are distributions?

$$\langle T_1, \varphi\rangle = \int_{-1}^1 x^2\varphi(x)\,dx + \int_{-2}^3 e^x\varphi(x)\,dx\,, \qquad \langle T_2, \varphi\rangle = \int_0^1 \varphi(x)^3\,dx$$
$$\langle T_3, \varphi\rangle = \int_0^1 x\varphi'(x)\,dx\,, \qquad \langle T_4, \varphi\rangle = \varphi(5)\varphi(3)$$
$$\langle T_5, \varphi\rangle = \int_{-\infty}^{\infty} (\sinh x - 4x)\varphi(x)\,dx + e^{12}\varphi(e)\,, \qquad \langle T_6, \varphi\rangle = 1$$

3. Compute the distributional derivative of the regular distributions associated to the following locally summable functions:

$$(5x+3)H(x)\,, \qquad \operatorname{sgn}(x)+2x\,, \quad |x^2-1|$$
$$(x^2-1)H(-x)\,, \qquad \sin xH(x)\,, \quad \arctan\tfrac{1}{x-1}$$

4. Compute the distributional derivative of the following distributions:

$$T_{H(2x)} + 5\delta_3(2x)\,, \quad e^{x^2}\delta_{-1} + T_{3\,\mathrm{sign}(-x)}\,, \quad x^2 T_{\mathbb{1}_{[-1,1]}(x)}$$

5. Let $\varphi \in \mathscr{D}$ be a test function such that $\varphi'(0) = -2$. Compute

$$\langle \sin x\delta_0'', \varphi\rangle\,.$$

6. Find all the distributions $T \in \mathscr{D}'$ such that $T' = \delta_0 + \delta_2 - 2\delta_1'$.
7. Show that

$$n^n\delta_n \to 0\,, \quad \delta_n^{(n)} \to 0\,, \quad e^{-1/n}\delta_{1/n} \to \delta_0$$

in $\mathscr{D}'$ as $n \to \infty$.

8. Show that

$$T_n = n(\delta_{1/n} + \delta_0)$$

is not convergent in $\mathscr{D}'$.

9. Find the limit, if it exists, of the following sequences of distributions:

$$n(\delta_{1/n} - \delta_{-1/n}), \quad \sqrt{n}(\delta_{1/n} - \delta_{-1/n}), \quad n^2(\delta_{1/n} - \delta_{-1/n}).$$

10. If

$$f_n(x) = \mathbb{1}_{[2(-1)^n, 2(-1)^n+1]}(x),$$

show that T_{f_n} does not converge in the sense of distributions.

11. If

$$f_n(x) = n^2 p_{1/n}(x),$$

show that T_{f_n} does not converge in $\mathscr{D}'$.

Answers

1. T_1 is a distribution since $T_1 = T_g$ with $g(x) = \mathbb{1}_{[0,1]}(x)\ln(x+1)$. T_2 is not a distribution because it is not linear (for instance $\langle T_2, -\varphi\rangle = \langle T_2, \varphi\rangle$ for every $\varphi \in \mathscr{D}$). T_3 is a distribution since by the fundamental theorem of calculus

$$\langle T_3, \varphi\rangle = \varphi(1) - \varphi(0) = \langle \delta_1 - \delta_0, \varphi\rangle.$$

Hence $T_3 = \delta_1 - \delta_0$. T_4 is not a distribution because it is not linear. T_5 is a distribution since $T_5 = \delta_1 - \delta_2 + \delta - 3 - \delta_4$. T_6 is a distribution since $T_6 = T_g + 6\delta_4$ with $g(x) = \mathbb{1}_{[-4,4]}(x)\sin x$.

2. T_1, T_3 and T_5 are distributions. The other functionals are not distributions.

3. $$5T_H + 3\delta_0, \qquad 2\delta_0 + T_2, \quad T_{2x\ \mathrm{sign}(x^2-1)}$$

$$T_{2xH(-x)} + \delta_0, \qquad T_{\cos x H(x)}, \quad T_{\frac{-1}{(x-1)^2+1}} + \pi\delta_1$$

4. $$\delta_0 + 5/2\delta'_{3/2}, \quad e\delta'_{-1} - 6\delta_0, \quad 2xT_{\mathbb{1}_{[-1,1]}(x)} + \delta_{-1} - \delta_1$$

5. -4.

6. $T = T_{H(x)} + T_{H(x-2)} - 2\delta_1 + T_c$ where c is a constant.

7. The solution is similar to the exercises studied during the lectures.

8. If φ is a test function then

$$\langle n(\delta_{1/n} + \delta_0), \varphi\rangle = n\varphi(1/n) + n\varphi(0).$$

If we choose a particular φ such that $\varphi(0) = 1$ we obtain that $n\varphi(1/n) + n\varphi(0) \to +\infty$ as $n \to \infty$ and we are done.

9. $-2\delta'_0$, 0, the third sequence does not converge.

10. The sequence f_n is $\mathbb{1}_{[-2,-1]},\ \mathbb{1}_{[2,3]},\ \mathbb{1}_{[-2,-1]},\ \mathbb{1}_{[2,3]}, \ldots$, Thus if $\varphi \in \mathscr{D}$ then the sequence $\langle T_{f_n}, \varphi\rangle$ is

$$\int_{-2}^{-1} \varphi(x)\,\mathrm{d}x,\ \int_{2}^{3} \varphi(x)\,\mathrm{d}x,\ \int_{-2}^{-1} \varphi(x)\,\mathrm{d}x,\ \int_{2}^{3} \varphi(x)\,\mathrm{d}x, \ldots$$

If we choose a test function $\varphi_0 \neq 0$ such that $\operatorname{supp}\varphi_0 \subseteq [-2,-1]$ and, for instance, $\int_{\mathbb{R}} \varphi_0(x)\,\mathrm{d}x = 1$, then

$$\langle T_{f_n}, \varphi_0\rangle = 1, 0, 1, 0, \ldots$$

thus T_{f_n} does not converge in the distributional sense.

11. See classroom lectures (use the integral mean value theorem).

Changes from version May 24, 2016 to version June 1, 2016:

1. Page 11, Example 3.2-d): $\varphi(2) \rightsquigarrow 2\varphi(2)$; $-\langle \delta_2, \varphi \rangle \rightsquigarrow \langle 2\delta_2, \varphi \rangle$; $\delta_2 \rightsquigarrow 2\delta_2$.
2. Example 4.4, line 1 of the first formula: $\int_{-\infty}^{\infty} \rightsquigarrow -\int_{-\infty}^{\infty}$