# Fourier Transform

01URO - Mathematical methods for Computer Science
2023/2024 course

Version: April 2024

**Disclaimer**: This is a translation of already existing notes written in Italian. **These notes cannot substitute classroom lectures, but only integrate them, it is heavily advised to go to the lectures**.

# Contents

# 1 Fourier transform for functions

## 1.1 Theory

Let $g : \mathbb{R} \to \mathbb{C}$ be a summable function, so a function with a convergent integral:

$$\int_{-\infty}^{+\infty} |g(t)|dt < +\infty \tag{1}$$

If so, we can define the **Fourier transform** of g as:

$$\mathscr{F}[g(t)](\nu) = \int_{-\infty}^{+\infty} g(t) \cdot e^{-2\pi i\nu t}\, dt = \hat{g}(\nu) \qquad \textit{with} \quad \nu \in \mathbb{R} \tag{2}$$

$$e^{-2\pi i\nu t} \text{ is know as Fourier nucleus}$$

The function $\hat{g}(\nu)$ is no longer t dependant, but rather $\nu$ dependant, in fact, the function varies as the frequency $\nu$ varies, sometimes, we may correctly write only $\mathscr{F}[g]$ or $\mathscr{F}(g)$.
The Fourier transform has the following proprieties:

- $\mathscr{F}[\lambda \cdot g(t) + \mu \cdot h(t)](\nu) = \lambda \cdot \mathscr{F}[g(t)](\nu) + \mu \cdot \mathscr{F}[h(t)](\nu)$
  with: $\lambda, \mu \in \mathbb{C} \qquad h : \mathbb{R} \to \mathbb{C}$ summable
- $\mathscr{F}[e^{2\pi i\nu_0 t} \cdot g(t)](\nu) = \mathscr{F}[g(t)](\nu - \nu_0) \qquad \nu_0 \in \mathbb{R}$
- $\mathscr{F}[g(t - t_0)](\nu) = e^{-2\pi i t_0 \nu} \cdot \mathscr{F}[g(t)](\nu) \qquad t_0 \in \mathbb{R}$
- $\mathscr{F}[g(at)](\nu) = \frac{1}{|a|}\mathscr{F}[g(t)]\left(\frac{\nu}{a}\right) \qquad a \in \mathbb{R}/\{0\}$
- $\mathscr{F}[g(-t)](\nu) = \mathscr{F}[g(t)](-\nu)$
- $\mathscr{F}\left[\frac{d^n}{dt^n} g(t)\right](\nu) = \mathscr{F}[g^{(n)}(t)](\nu) = (2\pi i\nu)^n \mathscr{F}[g(t)](\nu) \qquad n \in \mathbb{N} \qquad g^{(n)}(t)$ summable
- $\mathscr{F}[t^n \cdot g(t)](\nu) = \left(-\frac{1}{2\pi i}\right)^n \mathscr{F}^n[g(t)](\nu) \qquad t^n \cdot g(t)$ summable

A common mistake is correlated to using the time shifting property together with the time scaling property, which, can be used together, but, with caution. To use them together correctly, we can use the following property:

$$\mathscr{F}[g(a \cdot t - t_0)](\nu) = \frac{1}{|a|} \cdot e^{-2\pi i \frac{t_0}{a}\nu} \mathscr{F}[g(t)]\left(\frac{\nu}{a}\right) \qquad a \in \mathbb{R}/\{0\},\ t_0 \in \mathbb{R} \tag{3}$$

# 2 Exercises

## 2.1 Exercise 1

Evaluate the Fourier transform of:

$$g(t) = e^{-2t^2+4t}$$

*Solution*
The function is summable, since:

$$|g(t)| \sim e^{-2t^2} \qquad |t| \to +\infty$$

The Fourier transform that can be used in this case is:

$$h(t) = e^{-a\cdot t^2} \to \mathscr{F}[h(t)](\nu) = \sqrt{\frac{\pi}{a}} \cdot e^{-\pi^2 \frac{\nu^2}{a}} \tag{4}$$

We can do the following operations to create the perfect square needed to apply the formula:

$$g(t) = e^{-2t^2+4t} = e^{-2\cdot(t^2-2t)} = e^{-2\cdot(t^2-2t-1+1)} = e^{-2\cdot[(t-1)^2-1]} = e^2 \cdot e^{-2\cdot(t-1)^2}$$

Now g(t) is much more similar to h(t), defined here, we can now evaluate both parameters: $a = 2$ and $t_0 = 1$, so

$$g(t) = e^2 \cdot e^{-2\cdot(t-1)^2} = e^2 \cdot h(t-1)$$

Using the time shifting property stated before 1.1, we can finally Fourier transform g(t):

$$\mathscr{F}[g(t)](\nu) = e^2 \cdot \mathscr{F}[h(t-t_0)] = e^2 \cdot e^{-2\pi i t_0 \nu} \cdot \mathscr{F}[h(t)](\nu) = e^2 \cdot e^{-2\pi i\nu} \cdot \sqrt{\frac{\pi}{2}} \cdot e^{-\pi^2 \frac{\nu^2}{2}} =$$

$$= \sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\pi^2\cdot\nu^2}{2} - 2\pi i\nu + 2} = \sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\pi^2}{2}\cdot\left(\nu^2 + \frac{4i}{\pi}\cdot\nu - \frac{4}{\pi^2}\right)} = \sqrt{\frac{\pi}{2}} \cdot e^{-\frac{\pi^2}{2}\cdot\left(\nu + \frac{2i}{\pi}\right)^2}$$

## 2.2 Exercise 2

Evaluate the Fourier transform of:

$$g(t) = |t| \cdot e^{-|t|}$$

*Solution*
The function is summable, since:

$$\int_{-\infty}^{+\infty} |t|{\cdot}e^{-|t|}\ dt \to \text{even function on symmetrical domain} \to 2{\cdot}\int_{0}^{+\infty} t{\cdot}e^{-t}\ dt =$$

$$= 2\cdot\left(\left[-e^{-t}\cdot t\right]_0^{+\infty} + \int_0^{+\infty} e^{-t}\ dt\right) = 2 < +\infty$$

To correctly transform the function, we must divide the function definition into two disjoint parts due to the absolute values:

$$g(t) = \begin{cases} t\cdot e^{-t} & \text{if } t > 0 \\ -t\cdot e^{t} & \text{if } t < 0 \end{cases}$$

g(t) can also be defined as a sum of two terms, each multiplied by a different Heaviside function to have the two disjoint functions previously defined:

$$g(t) = t\cdot e^{-t}\cdot H(t) - t\cdot e^{t}\cdot H(-t)$$

The first part of g(t) is non zero only if t $\geq$ 0, whilst, the second part is non zero only if $t < 0$.
To correctly Fourier transform g(t) we must use the following formula:

$$h(t) = e^{-a\cdot t}\cdot H(t) \to \mathscr{F}[h(t)](\nu) = \frac{1}{a + 2\pi i\nu}$$

with $a = 1$. Moreover, g(t) can be written as:

$$g(t) = t\cdot h(t) - t\cdot h(-t)$$

Now we can finally evaluate the F-transform of g(t):

$$\mathscr{F}[g(t)](\nu) = -\frac{1}{2\pi i}\frac{d}{d\nu}\mathscr{F}[h(t)](\nu) - \left(-\frac{1}{2\pi i}\right)\frac{d}{d\nu}\mathscr{F}[h(-t)](\nu) =$$

$$= -\frac{1}{2\pi i}\frac{d}{d\nu}\hat{h}(\nu) + \frac{1}{2\pi i}\frac{d}{d\nu}\hat{h}(-\nu) = -\frac{1}{2\pi i}\frac{d}{d\nu}\frac{1}{1+2\pi i\nu} + \frac{1}{2\pi i}\frac{d}{d\nu}\frac{1}{1-2\pi i\nu} =$$

$$= \frac{1}{(1+2\pi i\nu)^2} + \frac{1}{(1-2\pi i\nu)^2}$$

### 2.3 Exercise 3

Evaluate the Fourier transform of:

$$g(t) = \frac{t}{(9+4t^2)^2}$$

*Solution*
The function is summable, since:

$$|g(t)| \sim \frac{1}{16 \cdot |t^3|} \qquad |t| \to +\infty$$

To solve this exercise we may use the following formula:

$$h(t) = \frac{1}{a^2+t^2} \to \mathscr{F}[h(t)](\nu) = \frac{\pi}{a} \cdot e^{-2\pi a \cdot |\nu|}$$

The only problem with doing so, is that the denominator of g(t) is squared, whilst, the denominator of h(t) is raised to the power of one.
This is a non-issue, since, g(t) can be seen as the first derivative of a function similar to h(t).
We can find that:

$$h'(t) = \frac{d}{dt}h(t) = \frac{d}{dt}\frac{1}{a^2+t^2} = -\frac{2t}{(a^2+t^2)^2}$$

Since g(t) has $4t^2$, we want to evaluate h'(t) using 2t instead of t:

$$h'(2t) = -\frac{4t}{(a^2+4t^2)^2}$$

Now, choosing $a = 3$, h'(t) is proportional to g(t):

$$h'(2t) = -4\frac{2t}{(9+4t^2)^2} = -8 \cdot g(t) \to g(t) = -\frac{1}{8} \cdot h'(2t)$$

Now, we can finally evaluate the F-transform:

$$\mathscr{F}[g(t)](\nu) = -\frac{1}{8} \cdot \mathscr{F}[h'(2t)](\nu) = -\frac{1}{8} \cdot \frac{1}{2}\mathscr{F}\left[\frac{d}{dt}h(t)\right]\left(\frac{\nu}{2}\right) =$$

$$= -\frac{1}{16} \cdot 2\pi i\nu \cdot \mathscr{F}[h(t)]\left(\frac{\nu}{2}\right) = -\frac{\pi i\nu}{16} \cdot \frac{\pi}{3} \cdot e^{-3\pi|\frac{\nu}{2}|} = -\frac{\pi^2 i}{48} \cdot \nu \cdot e^{-3\pi|\frac{\nu}{2}|}$$

# 3 Fourier transform for distributions

## 3.1 Theory

The aim is to extend the Fourier transform to distributions, to do so we must ease the hypothesis about the summability of the function that we want to transform, being able to work with periodic signals as an example.
The objective is to use $T$ distributions and $\varphi$ functions for which the following equality is true:

$$\langle \mathscr{F}(T), \varphi \rangle = \langle T, \mathscr{F}(\varphi) \rangle \tag{5}$$

Unfortunately, using the usual test function in the set $\mathscr{D}$ and the distributions of the set $\mathscr{D}'$ is not a valid solution: **the Fourier transform of a test function is not a test function**. Moreover,

$$\varphi \in \mathscr{D} \implies \mathscr{F}[\varphi](\nu) \in \mathbf{C}^{\infty}(\mathbb{R})$$

So the previously defined equality 5 is not always true. It is only true if the given distributions has a compact support, which is too strict as a requirement to be viable.
So, instead of test functions, we will use rapidly decreasing functions, which are functions defined as it follows:

$$\mathscr{S} := \left\{ \varphi \in \mathbf{C}^{\infty}(\mathbb{R}) : \lim_{t \to \pm\infty} t^p \cdot \varphi^q(t) = 0 \qquad \forall p, q \in \mathbb{N} \right\} \tag{6}$$

So functions that are in **Schwartz's space**.
Although the definition might be seem as difficult at first, we can say that the characteristic of the functions in the set $\mathscr{S}$ (and their derivatives) is the fact that they decrease more rapidly than any polynomial term.
The most important thing about the functions included in this space is the fact that they are stable with respect to their the Fourier transform:

$$\varphi \in \mathscr{S} \implies \mathscr{F}[\varphi] \in \mathscr{S}$$

The distributions that act upon the functions in $\mathscr{S}$ (so the functional that are continuous and linear $\forall \varphi \in \mathscr{S}$) are called **tempered distributions** and form the space $\mathscr{S}'$.
The following three conditions are sufficient (but not necessary) so that a distributions T is tempered:

- if $f : \mathbb{R} \to \mathbb{C}$ is summable, then $T_f$ is a tempered distribution
- if $f : \mathbb{R} \to \mathbb{C}$ is locally summable and is slowly increasing (so $|f(t)| \leq A \cdot (1 + |x|^p)$ then $T_f$ is a tempered distribution. An example of functions $f$ that meet this requirement are polynomial functions and limited functions.
- if $T \in \mathscr{D}'$ has a compact support, then T is a tempered distribution.

Using $\varphi \in \mathscr{S}$ and $T \in \mathscr{S}'$, the previously defined expression 5 is well defined. When we want to transform a non summable function, we may use 5 using the regular distribution induced by the given function.
To further simplify the used notation, regular distributions $T_f$ will be now noted as $\mathscr{F}[T_f] = \mathscr{F}[f]$.
Even if the Fourier transform is applied to distributions, the same proprieties stated previously can be applied whenever needed, the main difference is that derivatives are now distributional derivatives.

# 4 Exercises

## 4.1 Exercise 1

Evaluate the Fourier transform of:

$$g(t) = cos(2t + 1)$$

*Solution*
The function g(t) is not summable, however, the regular distribution induced by the function is tempered since we are dealing with a strictly limited function (the second condition is met 3.1).
The Fourier transform $T_g$ can be computed as previously stated, we can write that $\mathscr{F}[T_g] = \mathscr{F}[g(t)] = \mathscr{F}[cos(2t + 1)]$.
Using both the proprieties of time scaling and time shifting 3 with parameters a = 2 and $t_0 = -1$ we can find:

$$\mathscr{F}[cos(2t + 1)](\nu) = \frac{1}{2}e^{\pi i\nu} \cdot \mathscr{F}[cos(t)]\left(\frac{\nu}{a}\right)$$

Now, we just need to evaluate the Fourier transform of cos(t) which can be further transformed using Euler's formula:

$$cos(t) = \frac{1}{2}(e^{it} + e^{-it})$$

Recalling that:

$$\mathscr{F}[e^{2\pi i x_0 t}] = \delta_{x_0} \tag{7}$$

we can now write:

$$cos(t) = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}\left(e^{2\pi\left(\frac{1}{2\pi}\right)it} + e^{-2\pi\left(-\frac{1}{2\pi}\right)it}\right)$$

Which is the previously stated Fourier transform 7 with $x_0 = \frac{1}{2\pi}$ for the first term and $x_0 = -\frac{1}{2\pi}$ for the second term. So

$$\mathscr{F}[cos(t)] = \frac{1}{2}\left(\delta_{\frac{1}{2\pi}} + \delta_{-\frac{1}{2\pi}}\right)$$

$$\mathscr{F}[cos(2t + 1)] = \frac{1}{4} \cdot e^{\pi i\nu} \cdot 2\left(\delta_{\frac{1}{\pi}} + \delta_{-\frac{1}{\pi}}\right) = \frac{1}{2} \cdot \left(e^{i} \cdot \delta_{\frac{1}{\pi}} + e^{-i} \cdot \delta_{-\frac{1}{\pi}}\right)$$

Using a well-known proprieties of Dirac's delta:

$$\delta_{x_0}(\alpha x) = \frac{1}{|\alpha|} \cdot \delta_{\frac{x_0}{\alpha}}$$

## 4.2 Exercise 2

Evaluate the Fourier transform of:

$$g(t) = \frac{k + t^2}{1 + t^2}$$

*Solution*
The function g(t) is not summable, however, the regular distribution induced by the function is tempered since we are dealing with a strictly limited function (the second condition is met 3.1).
There is a well-known Fourier transform of a similar function that might prove useful:

$$h(t) = \frac{1}{a^2 + t^2} \to \mathscr{F}[h(t)](\nu) = \frac{\pi}{a} \cdot e^{-2\pi a|\nu|}$$

If we consider that $k = (k - 1) + 1$, then we can modify g(t) as it follows:

$$g(t) = \frac{(k-1) + 1 + t^2}{1 + t^2} = \frac{k-1}{1+t^2} + \frac{1+t^2}{1+t^2} = \frac{k-1}{1+t^2} + 1$$

Now we can easily transform g(t): the first term can be transformed as previously written 4.2 (you must consider k-1 as a constant that multiplies the fraction), whilst, the second term can be transformed as done here with $x_0 = 0$.

$$\tilde{g}(\nu) = (k-1) \cdot \pi e^{-2\pi|\nu|} + \delta_0$$

## 4.3 Exercise 3

Evaluate the Fourier transform of the following distribution:

$$T = t^4 \cdot \delta_2 + \delta_0''$$

*Solution*
Recalling one of the proprieties of Dirac's delta, which defines what happens when Dirac's delta multiplies a function that is $\mathbb{C}^\infty(\mathbb{R})$, we can write:

$$t^4 \cdot \delta_2 = 2^4 \cdot \delta_2 = 16 \cdot \delta_2$$

After doing this, the distribution obtained is a sum of distributions that have a compact support, so it is a sum of tempered distributions. We can conclude that $T \in \mathscr{S}'$.
Now, using both the derivative and the linearity property as well as the following well-known Fourier transform, we can conclude the exercise:

$$\mathscr{F}[\delta_{x_0}] = e^{-2\pi i x_0 \nu}$$

$$\tilde{T} = 16 \cdot \mathscr{F}[\delta_2] + \mathscr{F}[\delta_0''] = 16 \cdot e^{-4\pi i\nu} + (2\pi i\nu)^2 \cdot \mathscr{F}[\delta_0] = 16 \cdot e^{-4\pi i\nu} - 4\pi^2\nu^2$$

### 4.4 Exercise 4

Evaluate the Fourier transform of:

$$g(t) = t \cdot H(t)$$

*Solution*
The given function is not summable, however, $T_g$ is a tempered distribution (using the second proprieties 3.1), in particular:

$$|g(t)| \leq A \cdot (1 + |t|^p), \qquad p \geq 2$$

Using the product property:

$$\tilde{g}(\nu) = -\frac{1}{2\pi i} \cdot \frac{d}{d\nu} \mathscr{F}[H(t)](\nu) = -\frac{1}{2\pi i} \cdot \frac{d}{d\nu} \left[ \frac{1}{2\pi i} v.p. \frac{1}{\nu} + \frac{\delta_0}{2} \right] =$$

$$= \frac{1}{4\pi^2} \cdot \frac{d}{d\nu} v.p. \frac{1}{\nu} + \frac{i}{4\pi} \cdot \delta_0'$$