

Mechanisms of neuronal synchronization in Hindmarsh-Rose network with time-varying connectivity

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Abstract

This project investigates how time-varying long-range connectivity shapes neuronal synchronization in a network of Hindmarsh–Rose (HR) model neurons coupled via electrical synapses on a small-world Watts–Strogatz graph. Using a fourth-order Runge–Kutta integrator implemented in JAX, we simulate networks of $N = 50$ or $N = 200$ HR neurons for up to 5×10^5 time steps while systematically varying the rewiring probability p_r , the power-law decay exponent α governing long-range interactions, and the coupling strength ε . Synchrony is quantified by a time-averaged Euclidean distance between neuronal state vectors. After identifying and correcting a subtle implementation error in the Watts–Strogatz rewiring algorithm, our simulations reproduce the key qualitative findings of Rakshit et al.: increasing p_r robustly promotes synchrony, larger α (stronger suppression of long-range couplings) hinders or delays global synchronization, and stronger ε facilitates near-complete synchrony with errors as low as $E \approx 0.03$. These results highlight how dynamic small-world topology and long-range interactions jointly control emergent synchronization in HR networks and provide a reusable JAX-based framework for exploring synchronization mechanisms in more realistic brain-inspired circuits.

1 Introduction

Synchrony in neuronal networks plays a dual role as a crucial mechanism of healthy brain functions underlying cognition and motor coordination, but also a potential indicator of pathological states, such as schizophrenia, Parkinson’s disease and epilepsy [Majhi et al., 2025]. Importantly, synchronous neuronal dynamics is an emergent phenomenon arising from the interacting dynamics of multiple neuronal and non-neuronal cells, including both pairwise interactions and higher-order correlations in neuronal ensembles [Majhi et al., 2025]. Past literature suggest various forms of neuronal synchrony can be driven by the connectivity mechanisms. The algebraic connectivity of a network (i.e., the second minimal eigenvalue of Laplace matrix) affects the synchronization of nonlinear networks under diffusive coupling [Plotnikov, 2021]. Through spike-timing-dependent plasticity (STDP), neurons receiving coherent input could temporally associate [Gansel, 2022]. This project focuses on the synchronization in networks of Hindmarsh–Rose systems.

The Hindmarsh–Rose (HR) system describes a biological neuron in terms of a nonlinear system of third-order differential equation with three variables [Hindmarsh and Rose, 1984]:

$$\dot{x} = -ax^3 + bx^2 + y - z + I \quad (1)$$

$$\dot{y} = c - dx^2 - y \quad (2)$$

$$\dot{z} = r(s(x + w) - z) \quad (3)$$

where x represents the dynamics of membrane potential, while y describes the dynamics of fast sodium currents and z describes the dynamics of slow potassium currents. The Hindmarsh-Rose (HR) model based on pairwise neuronal interactions has been used to understand brain network synchronization in children with ADHD, which showed that ADHD brains react more rapidly and strongly to emotional stimuli [Ansarinab et al., 2023].

My aim here is to explore the synchronizing impact of long-range connections under time-varying connectivity via electrical synapses in the HR model. The primary goal is to replicate/confirm the core finding from the Ref. [Rakshit et al., 2021] of the dependence of synchronization on coupling strength and long-range interaction delays in order to enhance my own understanding of the subject.

2 Method

2.1 Hindmarsh-Rose Model

Suppose a network of N neurons connected through temporal long-range electrical synapses. Let x_i represent the membrane potential of the i -th neuron, y_i represents to the fast Na^+ or K^+ current, and z represent the slow Ca^+ current. The network according to the Ref. [Rakshit et al., 2021] can be mathematically expressed as the following:

$$\dot{x}_i = y_i - ax_i^3 + bx_i^2 - z_i + I + \epsilon \sum_{k=1}^{k_{\max}(t)} \frac{1}{k^\alpha} \sum_{j=1}^N \mathcal{A}_{ij}^{[k]}(t)(x_j - x_i) \quad (4)$$

$$\dot{y}_i = c - dx_i^2 - y_i \quad (5)$$

$$\dot{z}_i = r(s(x_i - x_0) - z_i), \quad i = 1, 2, \dots, N. \quad (6)$$

In the equation, ϵ is the coupling strength, r is the modulator of the slow dynamics. The model parameters are chosen as $a = 1$, $b = 3$, $c = 1$, $d = 5$, $r = 0.005$, $I = 3.25$, $s = 4$ and $x_0 = -1.6$ [Rakshit et al., 2021]. In addition, $\mathcal{A}_{ij}^{[k]}$ represents the adjacency matrix at time t , characterizing temporal connectivity in the network, where k describes the range of interactions. If the i th and j th nodes are k -path connected, the element $\mathcal{A}_{ij}^{[k]}(t) = 1$ and $\mathcal{A}_{ij}^{[k]}(t) = 0$ otherwise. At each time instant t , the entire network is rewired with probability p_r . Hence, the adjacency matrix $\mathcal{A}_{ij}^{[k]}(t)$ for each k -th path network is time-dependent. The i th and j th nodes are considered k -path connected if $\mathcal{A}_{ij}^{[k]}(t) = 1$ and $1 \leq k \leq k_{\max}(t)$, where $k_{\max}(t)$ is the diameter of the network at the time instance t , or the largest shortest-path length between any pair of nodes at that time. In other words, the entries of the time-varying k -path adjacency matrix $\mathcal{A}_{ij}^{[k]}(t)$ for $1 \leq k \leq k_{\max}(t)$ are as follows:

$$\mathcal{A}_{ij}^{[k]}(t) = \begin{cases} 1, & \text{if } \text{distance}(i, j) = k \text{ at time } t \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

The effective coupling strength $\epsilon_k = \epsilon/k^\alpha$ follows a power-law decay, where α controls the decay rate. The underlying network connectivity is set to observe a small-world architecture according to the Watts-Strogatz (WS) graph model. The entire network, with the architecture of a ring lattice, is rewired with probability p_r . Small p_r implies that the network is almost static, while a large value of p_r indicates a rapidly varying network. Initially, each node in the network is adjacent with k_{sw} nearest neighbors on both sides, so the degree of each node is $2k_{sw}$. These initial links are then reconnected randomly to chosen distant nodes with probability p_{sw} . That means, for each edge on the ring, with probability p_{sw} , we delete that edge and reconnect the source node to a random node, avoiding self-loop and duplicating edges.

2.2 Synchronizaton Error

The error of synchronization will be averaged over a sufficiently large time duration (10^5 time steps based on the Ref. [Rakshit et al., 2021] as follows:

$$E = \left\langle \frac{1}{N-1} \sum_{j=2}^N \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \right\rangle_t \quad (8)$$

where N is originally chosen to be 200 HR neurons. The evolution of E is examined by varying network and coupling parameters to observe the conditions under which synchronization is promoted. In the original paper, phase diagrams were created to illustrate the dependence of synchronization on the coupling strength ϵ and the small-world parameter p_{sw} for varying levels of α in Eq. 4, which control the long-range interaction decay.

3 Experiments

The original paper [Rakshit et al., 2021] simulated 200 HR neurons for 3×10^5 time steps. Due to computing resource limit, we either reduced the network size to $N = 50$ or the number of time steps to 1×10^5 . Additionally, we simulated the network in JAX for speed optimization. Equation 4 5 6 were manually integrated using the fourth order Runge-Kutta method over 3×10^5 time iterations, with $dt = 0.01$ as the integration time step. Specifically, the dynamics of y and z were simulated as written in Eq. 5 and Eq. 6. Let $F(x, y, z) = y_i - ax_i^3 + bx_i^2 - z_i + I$. And we call

$$B_{ij}(t) = \sum_{k=1}^{k_{max}(t)} \frac{1}{k^\alpha} A_{ij}^{[k]}(t)$$

the effective coupling term. Then we can rewrite Eq. 4 as follows:

$$\dot{x}_i = F(x, y, z) + \epsilon \sum_{k=1}^{k_{max}(t)} \frac{1}{k^\alpha} \sum_j A_{ij}^{[k]}(t)(x_j - x_i) \quad (9)$$

$$\dot{x}_i = F(x, y, z) + \epsilon \sum_j B_{ij}(x_j - x_i) \quad (10)$$

$$\dot{x}_i = F(x, y, z) + \epsilon \left[\sum_j B_{ij}x_j - x_i \sum_j B_{ij} \right] \quad (11)$$

We use Eq. 11 to simulate for x . For initial conditions, we tried randomly drawing from the normal distribution using `jax.random.normal`; we also tried sampling from the states of the original HR model (defined by Eq. 1, Eq. 2 and Eq. 3) after the first 50,000 time steps, where the trajectory is in a regime exhibiting chaotic attractor dynamics.

Following the Ref. [Rakshit et al., 2021], we investigated the synchronization error subject to the variation of the parameters p_{sw} , p_r and ϵ . Interestingly, our simulation resulted very different error dynamics, probably due to differences in the length of simulation and initial conditions.

3.0.1 Experiment I: Effect of α on synchronization

Fix $p_{sw} = 0.1$, $k_1 = 3$, $\epsilon = 0.5$, varying α, p_r . Expect to see increasing p_r promotes neuronal synchronization irrespective of the chosen values of α , though α still significantly affects where synchronization emerges. Specifically, the higher the value of α , the higher the value of p_r required to achieve complete synchronization where $E \approx 0$.

Primary Results We first simulated the network with $N = 200$, experimenting with different state initializations, either drawing (x_0, y_0, z_0) from chaos (Fig. 1a) or from normal distribution (Fig. 1b). Our simulation in Fig 1 shows as p_r increases, synchronization error decreases and stabilizes under both initializations, approaching perfect synchrony, which is consistent with our anticipation. However, unlike what the Ref.[Rakshit et al., 2021] claims in their paper, the α value does not seem to strongly modulate where synchronization occurs.

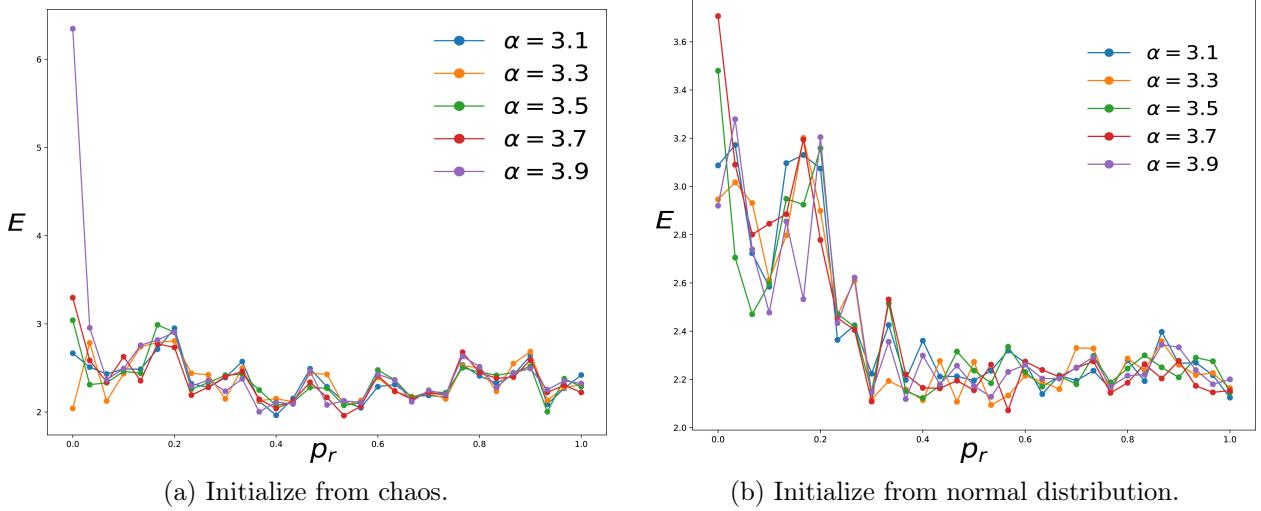


Figure 1: Synchronization error E as a function of p_r for different α .

Additional Observations When we reduced the network size to $N = 50$ but increased the stimulation time steps to 5×10^5 , E no longer exhibits a decreasing trend. Still we did not see α differentiates the synchronization dynamics under the normal distribution initialization.

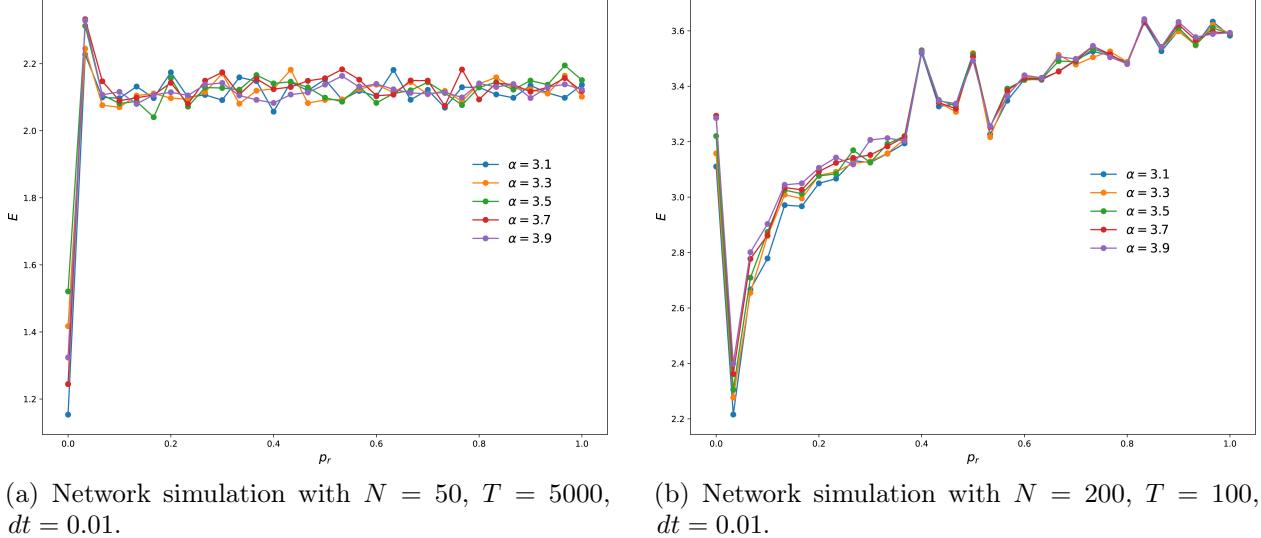


Figure 2: Synchronization error E as a function of p_r for different α .

3.0.2 Experiment II: Effect of ϵ on synchronization

Fix $\alpha = 2.5$, $k_{sw} = 3$, $p_{sw} = 0.1$, varying ϵ, p_r . Expect to see an increasing p_r favors synchrony. In contrast to the effect of α on synchronization, here we expect to observe increasing values of ϵ enhances the appearance of neuronal synchrony.

Primary Results We simulated an HR network of size $N = 200$ for 1×10^5 time steps starting from an initial state randomly sampled from trajectory in a chaotic regime according to the classic HR model. The effect of p_r is consistent with our expectation: higher p_r values lead to lower E , because as the network wiring switches more rapidly, the neurons get the opportunity of interacting with the otherwise-noninteracting neurons [Rakshit et al., 2021]. But we still do not see an obvious pattern in how the network modulation parameters affect synchronization, unlike what the Ref. [Rakshit et al., 2021] predicted. At this point, we think it is reasonable to suspect that our simulation may have some critical difference with the original HR network described. We will delve into this in the Discussion section.

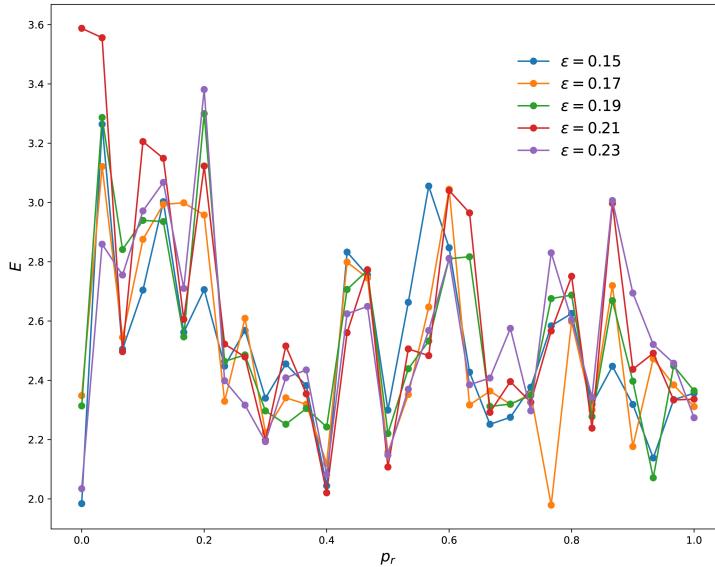


Figure 3: Synchronization error E as a function of p_r for different ϵ

4 Discussion

After scrutinizing over our code, we found one minor but critical mistake in our implementation of the canonical Watts-Strogatz (WS) graph model, which the original paper [Rakshit et al., 2021] unfortunately does not describe in sufficient detail. In the canonical WS model for an undirected graph, the adjacency (or connectivity) matrix starts from a regular ring lattice, where each node i is connected to its k nearest neighbors. For each edge (i, j) in some ordered subset, with probability p_{sw} , it gets rewired according to the following rules: (1) remove edge (i, j) ; (2) choose a new node j' uniformly at random from nodes not equal to node i and not already connected to i ; (3) add edge (i, j') . Under this algorithm, the number of edges and the degree of every node are *always preserved* (plus, there are no self-loops or multi-edges). In our implementation, when updating the connectivity of an existing edge with probability p_{sw} , we failed to put any constraint on where the new edges

could be drawn from in addition to inhibiting self-loops. Suppose there are N nodes in the network. As a result, the new node j' can be any node $0, \dots, N - 1$. If $j' = i$, we would skip adding any new edge (i.e., the original edge is removed and **not replaced**), which leads to decreasing the total edge count in the graph, thus changing the degree of the node. If j' is already a neighbor of i , then we would remove edge (i, j) and add a "new" edge that already exists, which effectively results in losing one edge and decreasing the degree of both i and j . Therefore, the degree distribution is slightly changing over time. To fix this, we instead chose the new node j' uniformly from the set $\{0, \dots, N - 1\} / (\{i\} \cup \text{neighbors}(i))$.

We re-ran experiment I to explore the effect of α on synchronization. Previously with the erroneous implementation of the WS graph model, we did not observe α modulating when network synchronization occurs, and the network never achieved perfect synchrony (i.e., $E \approx 0$) with any of the (p_r, α) combinations we tried. Initializing a network with $N = 200$ and simulated for 1×10^5 iterations yielded an oscillating $E \approx 2.3$ for all α 's as we increased p_r . We first simulated a network with $N = 50$ for 5×10^5 iterations, initialized from `jax.random.normal` (Fig. 4). We found exactly what the Ref. [Rakshit et al., 2021] concluded: higher α resulted in higher synchronization error; in addition, the network with only 50 HR neurons were able to achieve $E \approx 0.08$, a significant reduction from the previous $E \approx 2.3$.

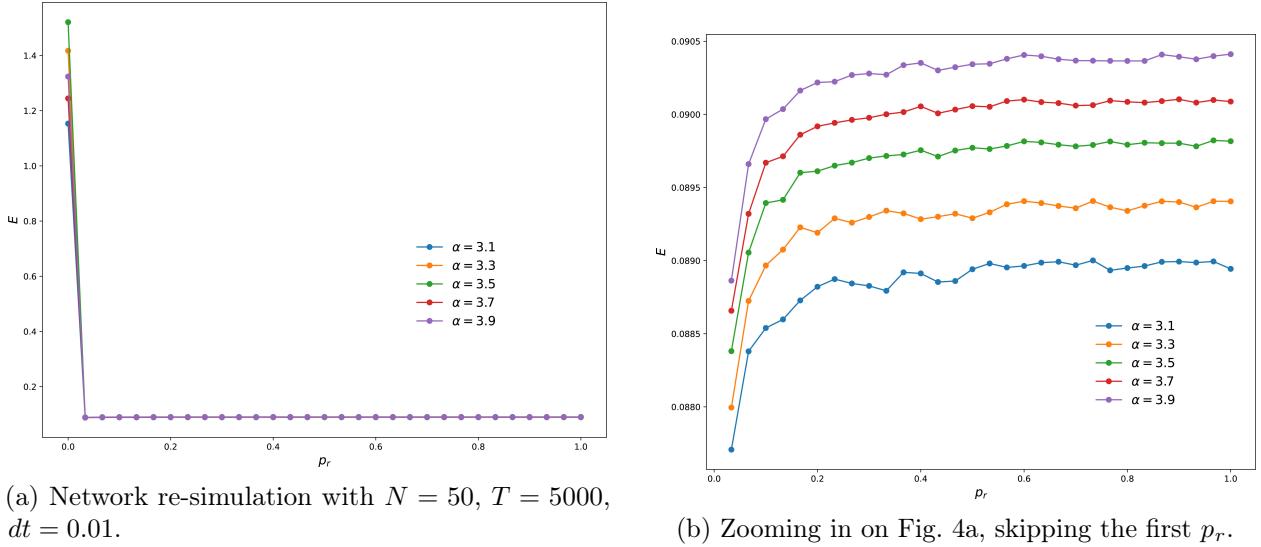


Figure 4: Synchronization error E as a function of p_r for different α .

Next, we simulated a larger network with $N = 200$ for 1×10^5 iterations, initialized from chaos (Fig. 5). With more neurons, the network has synchronization error further reduced to $E \approx 0.03$, achieving almost perfect synchrony, while we still observed that increasing α hinders synchrony, just as the Ref. [Rakshit et al., 2021] had shown.

Experiment II explores the effect of ϵ on synchronization. The Ref [Rakshit et al., 2021] observed that increasing ϵ enhances synchrony, opposite to the effect of increasing α . We simulated a network of $N = 200$ for 1×10^5 time steps, initialized from chaos and integrated with the fourth order Runge-Kutta as before. Our results beautifully agree with this conclusion (Fig. 6).

5 Appendix

All code, results and figures can be found in <https://github.com/aletheia88/hrsyncx/tree/main>.

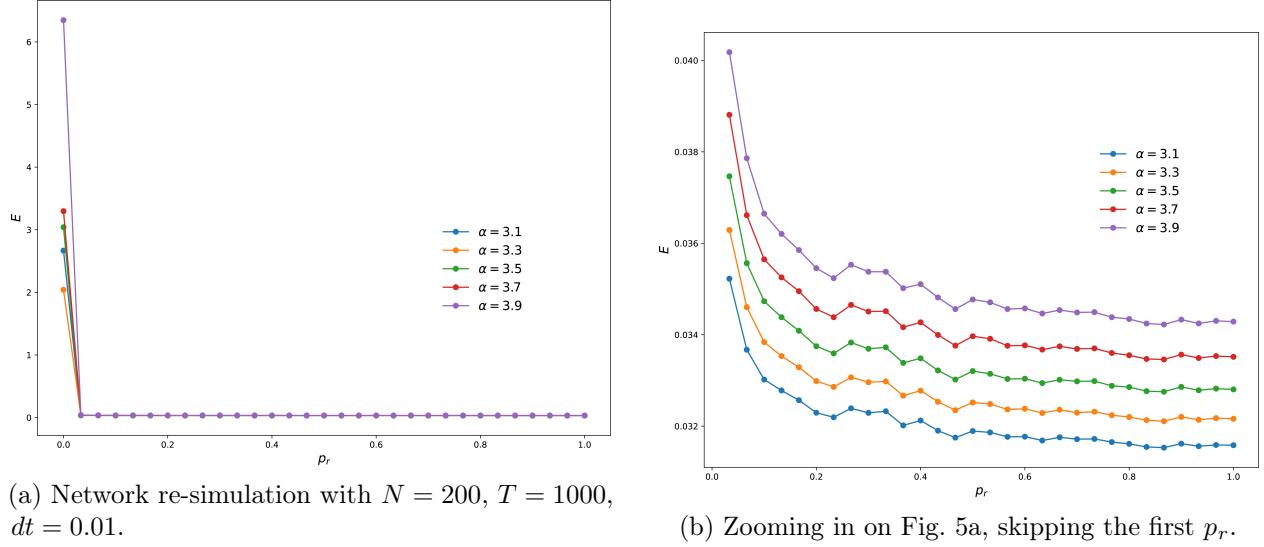


Figure 5: Synchronization error E as a function of p_r for different α .

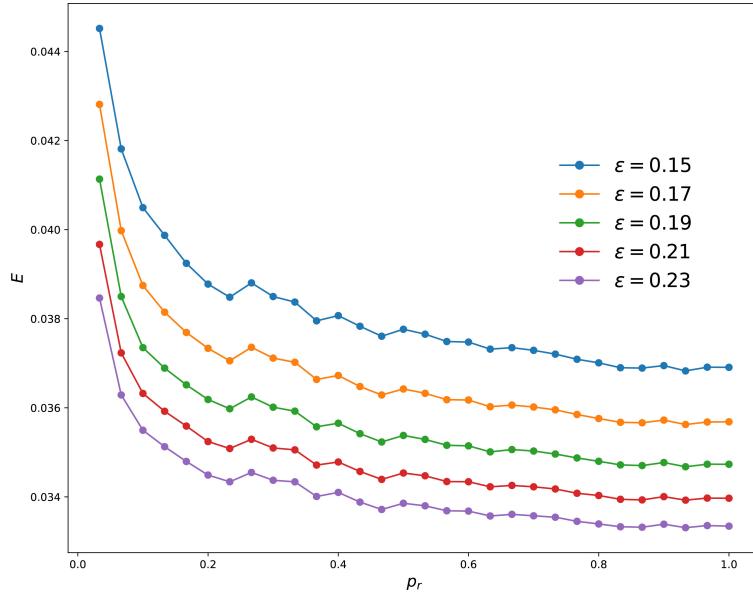


Figure 6: Synchronization error E as a function of p_r for different ϵ

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