Fundamentals of AI and KR - Module 3

2. Bayesian network representation

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Notice

Credits

The present slides are largely an adaptation of existing material, including:

- slides from Russel & Norvig
- slides by Daphne Koller on Probabilistic Graphical Models
- slides by Fabrizio Riguzzi on Data Mining and Analytics

I am especially grateful to these authors.

Downloading and ~~sharing~~

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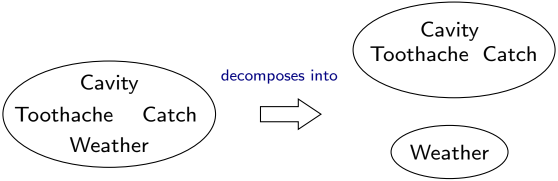
Table of Contents

Independence

Independence

A and B are independent, denoted $\mathbf{P} \models (A \perp B)$, iff
$\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$



$\mathbf{P}(Toothache, Catch, Cavity, Weather)$
$= \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$
32 entries reduced to 12; for n independent biased coins, $2^n \rightarrow n$

Absolute (marginal) independence powerful but rare.
Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

A C B

$\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

(1) $P(catch|toothache, cavity) = P(catch|cavity)$

The same independence holds if I haven't got a cavity:

(2) $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

Catch is conditionally independent of *Toothache* given *Cavity*:

$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$

Equivalent statements:

- $\mathbf{P}(Toothache|Catch, Cavity) = \mathbf{P}(Toothache|Cavity)$
- $\mathbf{P}(Toothache, Catch|Cavity) = \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)$

Notation: $\mathbf{P} \models (Catch \perp Toothache|Cavity)$

Conditional independence

Write out full joint distribution using chain rule:

$\mathbf{P}(Toothache, Catch, Cavity)$
$= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$
$= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)$

i.e., $2 + 2 + 1 = 5$ independent numbers

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

A, B 2 RANDOM VARIABLES

how b influnces a?

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

a influences b

$$\Rightarrow \text{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

70% of cavity cause toothache, (a = cavity) (b = toothache). Now we want to know he opposite, the probabily of cavity given toothache so P(a|b)

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

Example of diagnosis using Bayes' Rule

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

Say 1 individual in 50,000 suffers from meningitis, 1% from a stiff neck, and 70% of the times meningitis causes a stiff neck. *What is the probability that an individual with a stiff neck has meningitis?*

P(Meningitis | Stiff neck) = P(Stiff Neck | Menegitis) P (Menegitis) / P(Stiff Neck) = 0,7 x 1/50 000 / 0,01

Example of diagnosis using Bayes' Rule

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

Say 1 individual in 50,000 suffers from meningitis, 1% from a stiff neck, and 70% of the times meningitis causes a stiff neck. *What is the probability that an individual with a stiff neck has meningitis?*

Let M be meningitis and S be stiff neck.
$P(m) = 1/50,000$, $P(s) = 0.01$, $P(s|m) = 0.7$.

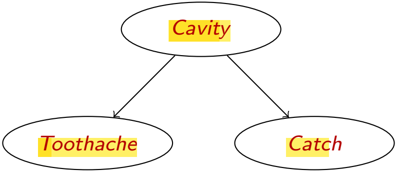
$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times 1/50,000}{0.01} = 0.0014$$

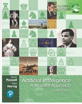
Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

toothache and cath given cavity

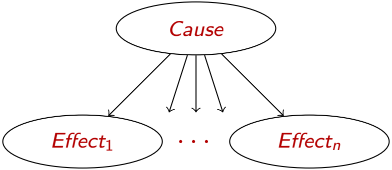
$$
\begin{aligned}
&\mathbf{P}(Cavity \mid toothache \wedge catch) \\
&\quad = \alpha\, \mathbf{P}(toothache \wedge catch \mid Cavity)\mathbf{P}(Cavity) \\
&\quad = \alpha\, \mathbf{P}(toothache \mid Cavity)\mathbf{P}(catch \mid Cavity)\mathbf{P}(Cavity)
\end{aligned}
$$





Bayes' Rule and conditional independence

In these cases we knnow the effect but we don't kow the cause. So to find the most probability cause we can estimate the joint distribution of the cause and the effect, and we take the higher numner, our best guest.



Example: Document classification
We want to classify articles
We define the class of interest
c1= Sport
c2 = Politics
So we wat to find the probability of some words inside documents

P(race | c1) = p
P(race| c2) = q

This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i | Cause)$$

Total number of parameters is **linear** in n

Every effecet is conditional indipendet given the cause from the others effects

Summary so far

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools

Bayesian network representation

Bayesian networks

A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions.

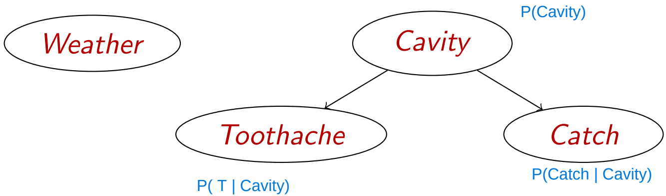
Syntax:

- a set of nodes, one per variable
- a directed, acyclic graph (link $\approx$ "directly influences")
- a conditional distribution for each node given its parents: $\mathbf{P}(X_i | Parents(X_i))$

In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

Topology of network encodes conditional independence assertions:



- *Weather* is independent of the other variables
- *Toothache* and *Catch* are conditionally independent given *Cavity*

Example

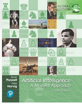
I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

All binary variables

Variables: *Burglar*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

Network topology reflects "causal" knowledge:

- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call
- The alarm can cause John to call

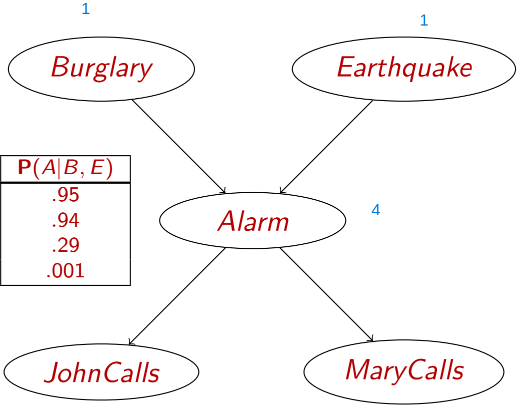


Example

The numbers represent the number of parmater to express the condinitional probability of the node in question

$\mathbf{P}(B)$
0.001

$\mathbf{P}(E)$
.002



B	E	$\mathbf{P}(A \mid B, E)$
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	$\mathbf{P}(J \mid A)$
T	.90
F	.05

A	$\mathbf{P}(M \mid A)$
T	.70
F	.01

This is equal to the probability of Alarm given Bulgary or Earthquake
So it is 2, (there is only one arrow and alarm is binary so 2)

This is the same so 2

Reasoning patterns

The student network

A student's grade depends on intelligence and on the difficulty of the course. SAT scores are correlated with intelligence. A professor writes recommendation letters by only looking at grades.

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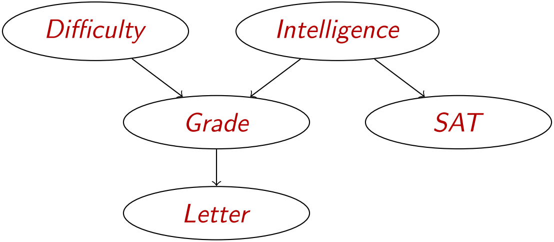
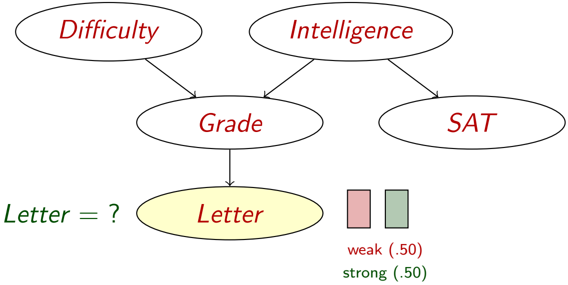
A student's grade depends on intelligence and on the difficulty of the course. SAT scores are correlated with intelligence. A professor writes recommendation letters by only looking at grades.



Reasoning patterns

- Causal: will George get a strong reference letter? (prediction)

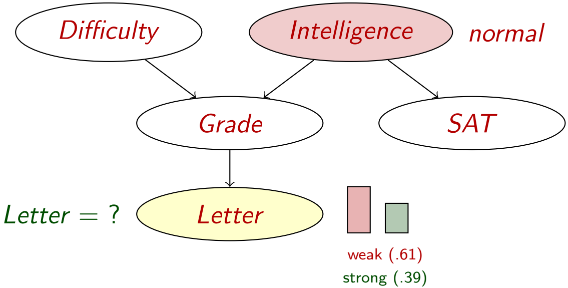
this is top-down research



We want to look at the probability of letter given some evidence

Reasoning patterns

- Causal: will George get a strong reference letter? (prediction)



Our prediction is influenced by our hipothesys/ evidence

Reasoning patterns

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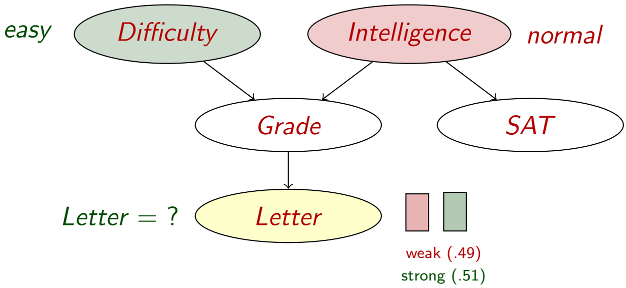
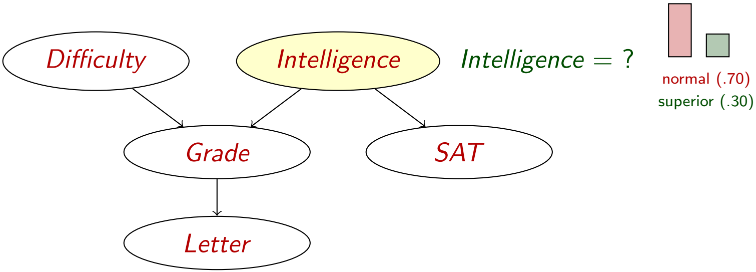


Reasoning patterns

- Causal: will George get a strong reference letter? (prediction)
- Evidential: is George a good potential recruit? (explanation)

this is used for medical diagnosis



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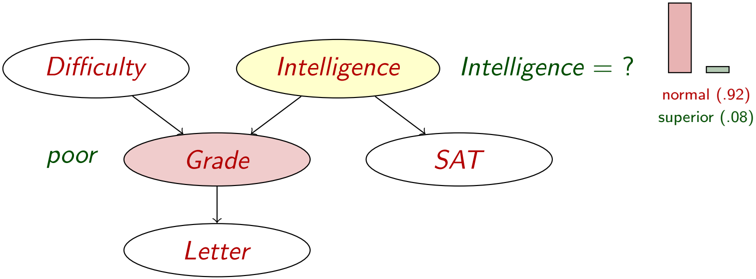

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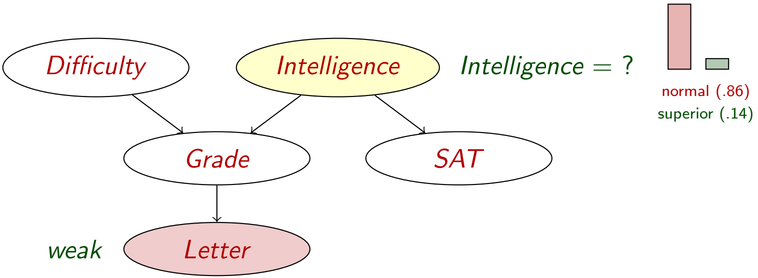

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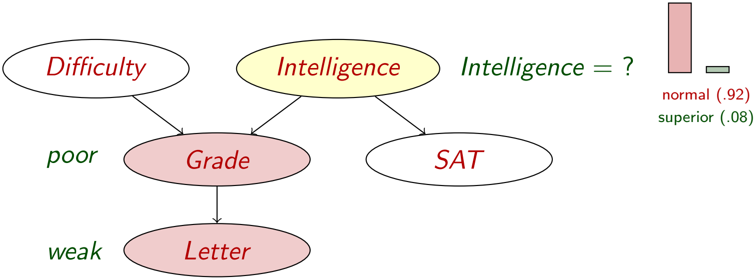

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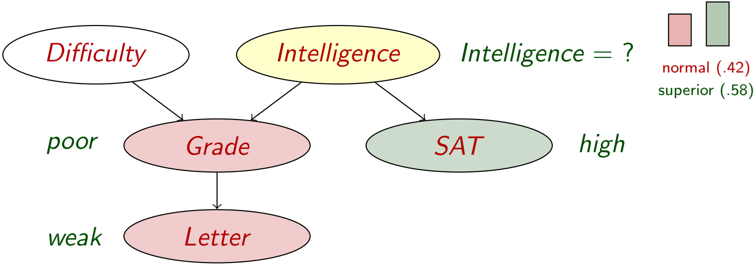

Reasoning patterns

- Causal: will George get a strong reference letter? (prediction)
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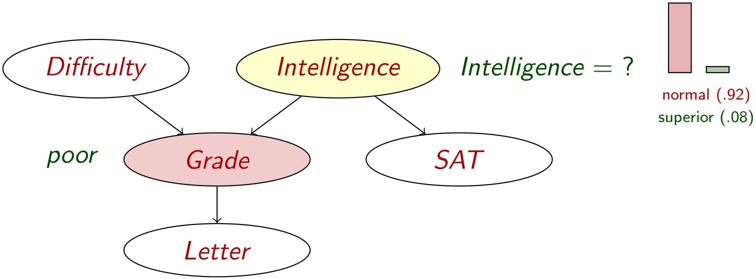
- Intercausal: why did George score low/high? (explaining away)



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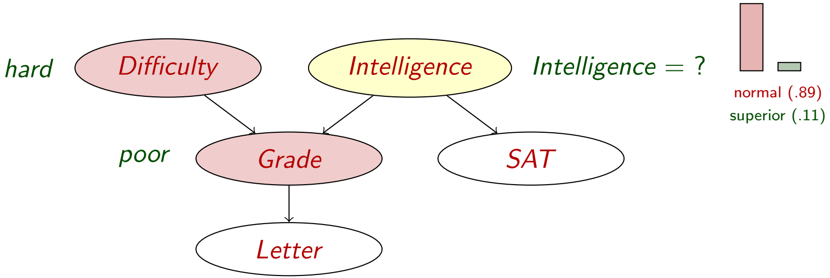

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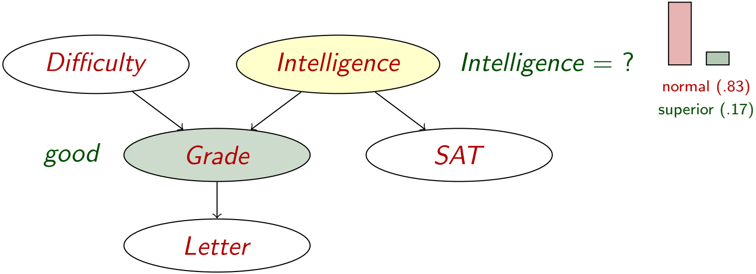

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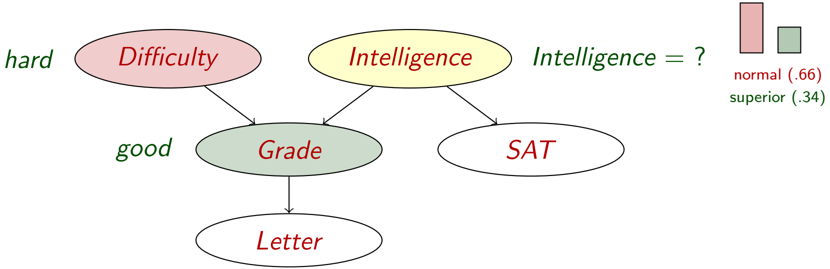
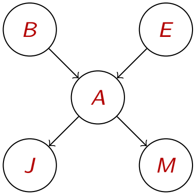


Compactness

A CPT for Boolean X_i with k Boolean parents
has 2^k rows for the combinations of parent values
Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1 - p$)



If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers

I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution

For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)

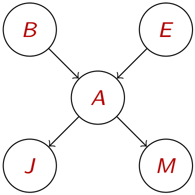
For student net, $1 + 1 + 8 + 2 + 3 = 15$ numbers (vs. $2^4 \times 3 - 1 = 47$)

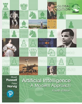
Global semantics

Global semantics defines the full joint distribution as the product of the local conditional distributions:

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \land m \land a \land \neg b \land \neg e)$





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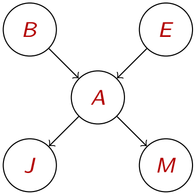
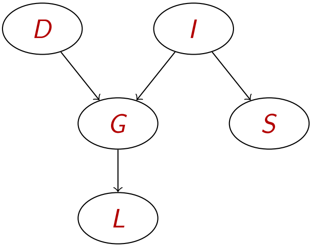


$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$

e.g., $P(j \land m \land a \land \neg b \land \neg e)$

$$\begin{aligned} &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Basic independences in the Student network



What independences?

- $\mathbf{P} \models (L \perp \ldots ?$
- $\mathbf{P} \models (S \perp \ldots ?$
- $\mathbf{P} \models (G \perp \ldots ?$
- $\mathbf{P} \models (I \perp \ldots ?$
- $\mathbf{P} \models (D \perp \ldots ?$

G is conditionally indipendend of S given I

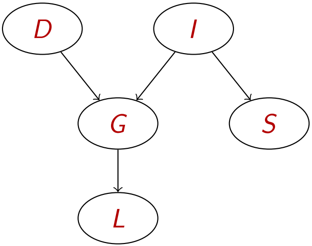
L is indipendence of D,I,S given G (conditionally indipendent)

S is indipendent of D

L and S are not absolute independet beetween each other

D is not indipendent by S given the grade, even if we don't know the grade and we know the letter. We will have the same result because with the letter we can know some information about the grade

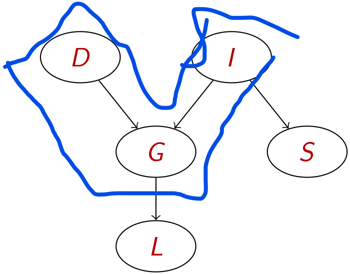
Basic independences in the Student network



What independences?

- $\mathbf{P} \models (L \perp I, D, S | G)$
- $\mathbf{P} \models (S \perp G, D, L | I)$
- $\mathbf{P} \models (G \perp S | I)$
- $\mathbf{P} \models (I \perp D)$
- $\mathbf{P} \models (D \perp I, S)$
- ...

Flow of probabilistic influence



I in this case is the common cause of G and S

In the V structure (common effect) is we know something is ok, if we don't know anything it broks the flow

Or else, when could X influence Y?

- $X \rightarrow Y$ (direct cause)
- $X \leftarrow Y$ (direct effect)
- $X \rightarrow Z \rightarrow Y$ (causal trail)
- $X \leftarrow Z \leftarrow Y$ (evidential trail)
- $X \leftarrow Z \rightarrow Y$ (common cause)
- $X \rightarrow Z \leftarrow Y$ (common effect)

Definition (active two-edge trail)

If influence can flow from X to Y via Z, the trail $X \rightleftharpoons Z \rightleftharpoons Y$ is active

Flow of probabilistic influence: active trails

Consider a longer trail $X_1 \rightleftharpoons \cdots \rightleftharpoons X_n$.

For influence to flow from X_1 to X_n, it needs to flow through every single node on the trail

This is true if and only if every two-edge trail $X_{i-1} \rightleftharpoons X_i \rightleftharpoons X_{i+1}$ along the trail allows influence to flow

Definition (active trail)

Let $\mathbf{Z}$ be a subset of observed variables.
The trail $X_{i-1} \rightleftharpoons X_i \rightleftharpoons X_{i+1}$ is active given $\mathbf{Z}$ if

- $\forall X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$, X_i or one of its descendants are in $\mathbf{Z}$
- no other node along the trail is in $\mathbf{Z}$

Flow of probabilistic influence: direct separation

Definition (d-separation)

Two sets of nodes **X**, **Y** are d-separated given **Z** if there is no active trail between any $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$ given **Z**

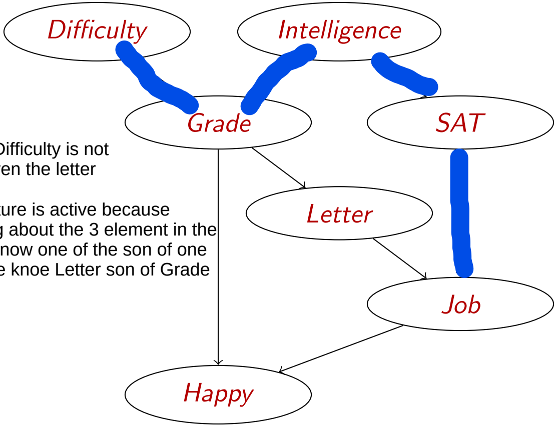
To determine if X and Y are **independent** given **Z**:

1. traverse the graph bottom-up marking all nodes in **Z** or having descendants in given **Z**
2. traverse the graph from X to Y, stopping if we get to a blocked node
3. if we can't reach Y, then X and Y are independent

A node is blocked if either the middle of an unmarked v-structure, or in **Z** (not both)

Example

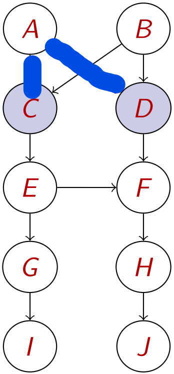
Question: is difficulty indipendent by Job given the letter?



This trail is active, so Difficulty is not Indipendent by Job given the letter

The trial in the V structure is active because we dont know anything about the 3 element in the V structures BUT we know one of the son of one of these 3 element. We knoe Letter son of Grade

Example



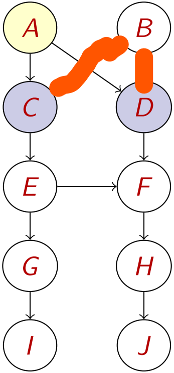
What independences?

- $\mathbf{P} \models (C \perp D)$?

(C and D absolute indipendence)

This trail is active so they are not indipendent

Example

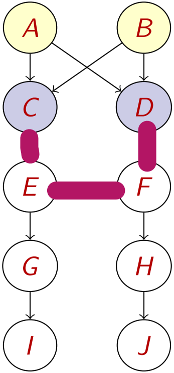


What independences?

- $\mathbf{P} \models (C \perp D)$?
- $\mathbf{P} \models (C \perp D | A)$?

Now we know A, so the trail before is blocked, but the other trail is active

Example

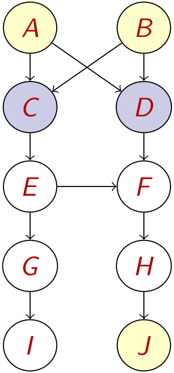


What independences?

- $\mathbf{P} \models (C \perp D)$?
- $\mathbf{P} \models (C \perp D | A)$?
- $\mathbf{P} \models (C \perp D | A, B)$?

Now we know A and Bm so the trails before are blocked and also the other train is not active. So yes in this case they are indipendent

Example



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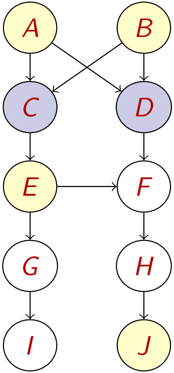
- $\mathbf{P} \models (C \perp D)$?
- $\mathbf{P} \models (C \perp D | A)$?
- $\mathbf{P} \models (C \perp D | A, B)$?
- $\mathbf{P} \models (C \perp D | A, B, J)$?

Example



What independences?

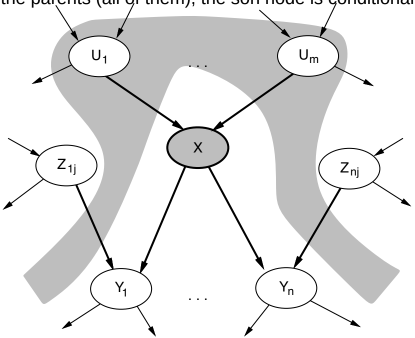
- $\mathbf{P} \models (C \perp D)$?
- $\mathbf{P} \models (C \perp D | A)$?
- $\mathbf{P} \models (C \perp D | A, B)$?
- $\mathbf{P} \models (C \perp D | A, B, J)$?
- $\mathbf{P} \models (C \perp D | A, B, E, J)$?

Yes they are indipendent, because all the trails are blocked by evidences

Local semantics

Local semantics: each node is conditionally independent of its nondescendants given its parents

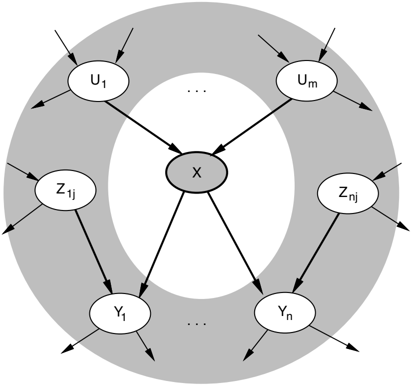
This says thatif we only know the parents (all of them), the son node is conditionally indipendent by al the other nodes nondescendats



Theorem: Local semantics $\Leftrightarrow$ global semantics

Markov blanket

Each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents



Questions?