Categories of Relations

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1 Introduction

1.1 Idea

A category can be thought of as an abstract calculus of functions with the category axioms motivated by the behaviour of functions between sets. Functions between sets are a special case of relations between sets, which themselves form a category. Relations between objects can be defined in any category, and under suitable conditions, these relations themselves form a category, which shares some of the basic structure of the categories of sets and relations. An allegory is an axiomization of this basic structure, so allegories can be thought of as an abstract calculus of relations. Every allegory has a subcategory of those morphisms that "behave like functions", which we will call maps. We will investigate how properties of an allegory are closely related to properties of its subcategory of maps. Allegories that are categories of relations of some category have "a lot" of maps, which will lead to the definition of tabular allegories. We will then prove that, in fact, any tabular allegory is the category of relations of its own subcategory of maps. By considering allegories with additional structure and seeing how this structure affects the subcategory of maps, we will arrive at the following correspondence:

Locally regular category	Tabular allegory
Regular category	Tabular allegory with a unit
Coherent category	Tabular union allegory with a unit
Heyting category	Tabular union division allegory with a unit
Topos	Tabular power allegory with a unit

1.2 Word on Sources

[1] and [2] were the sole sources of information for the ideas, proofs, and definitions in this paper. In our development of the category of internal relations, we do not assume the categories have products, so our definitions in this part are closer to those in [2]. This is mentioned in a remark on page 141 of [1], and in this paper we expand this remark and fill in the detailed needed to do so. Most of the lemmas about categories of spans proved in sub-sections 3.1 and 3.2 are not from the sources, but are direct generalization of simple results about slice categories. The definition of images of spans given here is a direct generalization of the definition given for images of single morphisms. The proof of proposition 3.1 follows [1] for one direction, and [2] for the other since terminal objects are not assumed. Lemma 3.8 is stated without proof in [2] but is proved here, and similarly a full proof of lemma 3.9 (i) and (ii) is given here though it is omitted in the sources. The proof of proposition 3.2 follows the proof given in [1]. Lemma 4.1 is not stated explicitly in the sources, but is a generalization of lemma 3.2.6 in [1]. Proposition 4.1 is not stated in this way in any of the sources, but again is a direct generalization of the proofs given there for the existence of images, pullbacks, and equalizers (in the case of [2]). Sections 5 and 6 follow the sources more closely.

2 Allegories

We start by looking at the category of sets and relations to gain some insight into how relations behave.

2.1 The Category Rel

Definition 2.1: **Rel** is the category whose objects are sets and whose morphisms are binary relations. A morphism $A \to B$ is a subset of $A \times B$, the identity morphism on A is the equality relation on A, and composition is defined by $R'R = \{ (a,c) \in A \times C \mid \exists b \in B \, , \, aRb \wedge bR'c \}$ for $R : A \to B$, $R' : B \to C$, where we write aRb for $(a,b) \in R$.

We list properties of **Rel** that will motivate our definition of an allegory:

1. The hom-sets of **Rel** are posets since $\mathbf{Rel}(A,B) = \mathcal{P}(A \times B)$. Composition of relations is order preserving.
2. Every relation has a reciprocal. Given a relation $R \subset A \times B$, there is a relation $R^\circ \subset B \times A$ given by $bR^\circ a$ iff aRb. $(-)^\circ$ is a lattice isomorphism between $\mathcal{P}(A \times B)$ and $\mathcal{P}(B \times A)$, and defines a functor $(-)^\circ : \mathbf{Rel}^{op} \to \mathbf{Rel}$.
3. **Rel** has **Set** as a subcategory with the same objects and with only those relations that correspond to functions, i.e. those relations $R \subset A \times B$ for which $\forall x \exists ! y, xRy$. This condition can be written equationally as $1_A \subset R^\circ R$ and $RR^\circ \subset 1_B$.
4. There is one more property of **Rel** that will appear in the definition of an allegory that feels somewhat less natural than those considered so far, the modular law. It describes a way in which the operations of composition, intersection, and reciprocation relate to each other.

Remark 2.1*:* Many properties of relations can be expressed as quantifier free equations using the operations of intersection, reciprocation, and composition, because composition has an existential quantifier in its definition.

2.2 Allegory Axioms

Abstracting from these observations, we give the following definitions:

Definition 2.2: A category $\mathcal{C}$ is *locally posetal* if it is enriched over **Pos**, the category of posets and monotone maps. In other words, the hom-sets of $\mathcal{C}$ are posets and composition is order preserving. We say a functor $F : \mathcal{C}^{op} \to \mathcal{C}$ is an *anti-involution* if, when viewed as a mapping $\mathcal{C} \to \mathcal{C}$ that reverses arrow, it is its own inverse.

Definition 2.3: An *allegory* $\mathcal{A}$ is a locally posetal category equip with an anti-involution $(-)^\circ : \mathcal{A}^{op} \to \mathcal{A}$ such that:

(i) $(-)^\circ$ is the identity on objects and is order preserving on hom-posets.

(ii) The hom-posets of $\mathcal{A}$ have binary meets.

(iii) The modular law holds: Whenever $A \xrightarrow{R_1} B$, $B \xrightarrow{R_2} C$, $A \xrightarrow{R_3} C$, $R_2R_1 \cap R_3 \leq (R_2 \cap R_3R_1^\circ)R_1$.

We call R° the *reciprocal* of R.

Remark 2.2*:* Notice that while the hom-posets have binary meets, $\mathcal{A}$ is not enriched over this additional structure. That is, composition need not preserve meets. This definition implies that allegories are locally small, which is not a necessary restriction. The morphisms between any two fixed objects could be a “large” poset. When we say a category is a preorder or a poset we do not mean that category is actually a set. These size consideration will not be important throughout this paper.

We note the following immediate consequences of the axioms:

Lemma 2.1: *In any allegory, the following equations hold whenever they are defined:*

(i) $(R \cap S)^\circ = R^\circ \cap S^\circ$.

(ii) $S(R_1 \cap R_2) \leq SR_1 \cap SR_2$ *and* $(R_1 \cap R_2)S \leq R_1S \cap R_2S$.

(iii) Whenever $A \xrightarrow{R_1} B$, $B \xrightarrow{R_2} C$, $A \xrightarrow{R_3} C$, $R_2R_1 \cap R_3 \leq R_2(R_1 \cap R_2^\circ R_3)$, *and this is equivalent to the modular law.*

(iv) $R \leq RR^\circ R$.

(v) $(R \cap S)^\circ (R \cap S) \geq S^\circ R \cap 1_A$ *where* $A = dom(R \cap S)$.

Proof:

(i) By monotonicity of $(-)^\circ$, $(R \cap S)^\circ \leq R^\circ$ and $(R \cap S)^\circ \leq S^\circ$. If $X \leq R^\circ$ and $X \leq S^\circ$ then $X^\circ \leq R$ and $X^\circ \leq S$ so $X \leq (R \cap S)^\circ$.

(ii) $R_1 \cap R_2 \leq R_1$ and $R_1 \cap R_2 \leq R_2$, so $S(R_1 \cap R_2) \leq SR_1$ and $S(R_1 \cap R_2) \leq SR_2$, hence $S(R_1 \cap R_2) \leq SR_1 \cap SR_2$.

(iii) If $A \xrightarrow{R_1} B$, $B \xrightarrow{R_2} C$, $A \xrightarrow{R_3} C$, then $C \xrightarrow{R_2^\circ} B$, $B \xrightarrow{R_1^\circ} A$, $C \xrightarrow{R_3^\circ} A$, so applying the modular we get $R_1^\circ R_2^\circ \cap R_3^\circ \leq (R_1^\circ \cap R_3^\circ R_2)R_2^\circ$ which by (i) and (ii) is equivalent to what is stated. Similarly for the converse.

(iv) Applying the modular law to $A \underset{1_A}{\longrightarrow} A \underset{R}{\longrightarrow} B$ we get $R = R \cap R1_A \leq (1_A \cap RR^\circ)R \leq RR^\circ R$.

(v) By two applications of the modular law (one in the from of (iii), the other in the original form):

$$(R \cap S)^\circ (R \cap S) = (R \cap S)^\circ (R \cap (R \cap S)^{\circ\circ} 1_A) \geq (R \cap S)^\circ R \cap 1_A = (S^\circ \cap 1_A R^\circ)R \cap 1_A \geq S^\circ R \cap 1_A \cap 1_A$$

□

2.3 Maps in an Allegory

Definition 2.4: A morphism $R : A \to B$ in an allegory is:

- *total* if $1_A \leq R^\circ R$.
- a *partial map* if $RR^\circ \leq 1_B$.
- a *map* if it is total and a partial map.

We denote maps in an allegory with lower case letters.

The composite of total morphisms is total, the composite of partial maps is a partial map and the identity morphisms in an allegory are maps.

Definition 2.5: For $\mathcal{A}$ an allegory, $\mathcal{M}ap(A)$ is the subcategory with the same objects as $\mathcal{A}$ and whose morphisms are the maps in $\mathcal{A}$.

In a locally posetal category, we say $f : A \to B$ is *left adjoint* to $g : B \to A$ (and g *right adjoint* to f) and write $f \dashv g$ if $1_A \leq gf$ and $fg \leq 1_B$. If $f \dashv g$ and $f \dashv g'$, then $g \leq g'fg \leq g'$ and $g' \leq gfg' \leq g$, so right (and dually left) adjoints are unique. By definition, a morphism f in an allegory is a map iff $f \dashv f^\circ$. In turns out that maps in an allegory are precisely those morphisms that have a right adjoints.

Lemma 2.2: *In any allegory, if a morphism R has a right adjoint, then the right adjoint is R°. In particular, maps in an allegory are precisely those morphisms that have a right adjoint.*

PROOF: Suppose $A \underset{R}{\overset{S}{\rightleftarrows}} B$ and $R \dashv S$ in an allegory. $(S \cap R^\circ)R = (S \cap 1_A R^\circ)R \geq SR \cap 1_A = 1_A$ by the modular law and since $1_A \leq SR$. Moreover $R(S \cap R^\circ) \leq RS \cap RR^\circ \leq RS \leq 1_B$, so $R \dashv (S \cap R^\circ)$. By uniqueness of right adjoints we conclude that $(S \cap R^\circ) = S$, i.e. $R^\circ \leq S$. We have shown that $R \dashv S$ implies that $R^\circ \leq S$ for arbitrary R, S. But $1_A \leq SR$ and $RS \leq 1_B$ imply that $1_A \leq R^\circ S^\circ$ and $S^\circ R^\circ \leq 1_B$, so $S^\circ \dashv R^\circ$, hence $S \leq R^\circ$, and so $S = R^\circ$ as desired. □

Corollary 2.1: *In an allegory, an isomorphism is a map and its inverse is its reciprocal.*

PROOF: If R has an inverse then $R \dashv R^{-1}$ so $R^\circ = R^{-1}$. □

Lemma 2.3: *Let $\mathcal{A}$ be an allegory. Then:*

(i) The maps of $\mathcal{A}$ are discreetly ordered. ($g \leq f$ iff $g = f$).

(ii) The intersection of distinct maps is not a map.

(iii) The reciprocal of a map that is not an isomorphism is not a map.

PROOF: If $g \leq f$ then $g^\circ g f^\circ \leq g^\circ f f^\circ$. But f and g are maps so this extends to $f \leq g^\circ g f^\circ \leq g^\circ f f^\circ \leq g$, so $f \leq g$. If $g \cap f$ is a map then $g \cap f \leq f$ and $g \cap f \leq g$ so $f = g = f \cap g$. (iii) is immediate from the definitions.

□

So in some sense, $\mathcal{M}ap(A)$ loses the allegory structure of $\mathcal{A}$.

Lemma 2.4: *If $R : A \to B$ is a partial map in an allegory then $(S_1 \cap S_2)R = S_1 R \cap S_2 R$.*

PROOF: By the modular law we have $S_1 R \cap S_2 R \leq (S_1 \cap S_2 R R^\circ)R$ and since $RR^\circ \leq 1_B$, $(S_1 \cap S_2 R R^\circ)R \leq (S_1 \cap S_2)R$. The other direction is lemma 2.1(ii). □

Remark 2.3*:* Notice that the order of composition mattered.

2.4 Examples

Example 2.1: A *modular lattice* is a lattice in which a weak form of distributivity of joins over meets holds:

$$\forall a \, \forall b \forall c \quad a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Or equivalently:

$$\forall a \, \forall b \forall c \quad a \leq c \Rightarrow (a \vee b) \wedge c \leq a \vee (b \wedge c)$$

Let L be a lattice (poset with binary meets and joins) with a bottom element 0. L forms a commutative monoid with join as composition and 0 as the identity, so we can view L as a one object category in which morphisms have binary meets. $\mathcal{A}$ is locally posetal since $a \leq b \Rightarrow a \vee c \leq b \vee c$, and the identity functor is an isomorphism between $\mathcal{A}$ and $\mathcal{A}^{op}$ because composition is commutative. Using lemma 2.1 (iii), the modular law becomes: for all a, b, c in L, $(a \vee b) \wedge c \leq a \vee (b \wedge (a \vee c))$. But $a \leq c$ iff $c = a \vee c$, so $\mathcal{A}$ is an allegory iff L is a modular lattice. Viewing a modular lattice as an allegory in this way, an element a is a map iff $0 \leq a^\circ a$ and $aa^\circ \leq 0$. $a^\circ a = aa^\circ = a$, so the only map is the bottom element.

Example 2.2: A *frame* is a poset that has arbitrary joins and finite meets (includes top element 1 and bottom element 0), in which binary meets distribute over arbitrary joins. Viewing a relation $R \subset A \times B$ as its characteristic function $A \times B \xrightarrow{R} \{0, 1\}$, and viewing $\{0, 1\}$ as the two element frame, the composite of $A \times B \xrightarrow{R} \{0, 1\}$, $B \times C \xrightarrow{R'} \{0, 1\}$ can be written as:

$$R'R(a, c) = \bigvee_{b \in B} (\, R(a, b) \, \wedge R'(b, c)).$$

The arbitrary disjunction serves the purpose of the existential quantifier.

Similarly, for any lattice $\mathcal{V}$ with arbitrary joins and binary meets, we can define a *$\mathcal{V}$-valued relation* on sets A and B as a function $A \times B \to \mathcal{V}$ with composition as described above. If $\mathcal{V}$ is a frame, this composition is associative and we obtain a category $\mathbf{Mat}(\mathcal{V})$ with identity morphisms mapping diagonal elements to 1 and everything else to 0. Moreover, $\mathbf{Mat}(\mathcal{V})$ has an allegory structure, with the order being defined point-wise as $\phi \leq \psi \Leftrightarrow \forall\, a, b\, \phi(a,b) \leq \psi(a,b)$ intersection given by $(\phi \cap \psi)(a,b) = \phi(a,b) \cap \psi(a,b)$, and $\phi^\circ(b,a) = \phi(a,b)$. In particular, $\mathbf{Rel} \cong Mat(\mathbf{2})$ where $\mathbf{2}$ is the two element frame. Let $\phi : A \to B$ be a $\mathcal{V}$-valued relation.

$$
\begin{aligned}
&1_A(a,a') = 0 \leq \phi^\circ\phi(a,a')\, \forall\, a \neq a' \\
&1_A(a,a) \leq \phi^\circ\phi(a,a) \Leftrightarrow 1 \leq \bigvee_{b \in B} \phi(a,b) \Leftrightarrow 1 = \bigvee_{b \in B} \phi(a,b)
\end{aligned}
$$

and

$$
\begin{aligned}
&1 = 1_B(b,b) \geq \phi\phi^\circ(b,b) \\
&\phi\phi^\circ(b,b') \leq 1_B(b,b') \Leftrightarrow \bigvee_{a \in A} (\phi(a,b) \wedge \phi(a,b')) \leq 0 \Leftrightarrow \forall a \in A \quad \phi(a,b) \wedge \phi(a,b') = 0
\end{aligned}
$$

Combining these facts we get that ϕ is a map iff

$$\forall a \in A, \quad \bigvee_{b \in B} \phi(a,b) = 1 \tag{1}$$

$$\forall a \in A\ \forall b, b' \in B \quad b \neq b' \Rightarrow \phi(a,b) \wedge \phi(a,b') = 0 \tag{2}$$

In particular, if the only joins in $\mathcal{V}$ that equal 1 are joins with 1 itself (for example, a total order), then condition (1) becomes $\forall a \in A$, $\exists b \in B\ \phi(a,b) = 1$ and condition (2) forces $\phi(a,b') = 0$ for any other $b' \in B$. So we can identify ϕ with an ordinary function in $\mathbf{Set}$. In particular, $\mathcal{M}ap(Mat(\mathcal{V})) \cong \mathcal{M}ap(Mat(\mathbf{2}))$, even though $Mat(\mathcal{V}) \ncong \mathbf{Set}$. (If $\mathcal{V}$ has more than 2 elements this already fails for finite sets just for cardinality reasons). If we take $\mathcal{V}$ to be the real interval $[0,1]$, we get the category of *fuzzy relations*. We can think of such a relation as associating a degree of which two elements are related.

If we only consider $\mathcal{V}$-valued relations on finite sets, we can relax the condition that $\mathcal{V}$ is a frame to just a distributive lattice. We can view the set of $\mathcal{V}$-valued relation between two finite sets A and B as the set of $|B| \times |A|$ matrices with entries in $\mathcal{V}$. Composition becomes matrix multiplication when meets are interpreted as multiplication and joins as addition. The identity matrix is then the usual identity matrix with 1's along the diagonal and 0 everywhere else. We write $\mathbf{Mat}_f(\mathcal{V})$ for the category of $\mathcal{V}$ valued relations between finite sets.

3 Allegories of Internal Relations

In this section we define relations between objects in an arbitrary categories and see under what conditions these relations form an allegory. In the category of sets, relations are subsets of products. We avoid requiring that a category has products by using spans and jointly monic families.

3.1 Spans and Jointly Monic Families

Definition 3.1: Let $\mathcal{C}$ be a category. An *n-span* in $\mathcal{C}$ is an ordered tuple of n morphisms with a common domain. We write $(A, f_1, \ldots, f_n)$ for an n-span in which all the f_i's have domain A.
For n-spans $(A, f_1, \ldots, f_n)$ and $(A', f'_1, \ldots, f'_n)$ such that f_i and f'_i have the same codomain for all i, we define a morphism of spans to be a morphism $t : A \to A'$ in $\mathcal{C}$ that satisfies $f_i = f'_i t$ for all i. For fixed objects $A_1, \ldots, A_n$ we obtain a category $\mathit{Span}(A_1, \ldots, A_n)$ of spans over $A_1, \ldots, A_n$.

Definition 3.2: An object A in a category $\mathcal{C}$ is *subterminal* if for any object X of $\mathcal{C}$ there is at most one morphism $X \to A$. A subterminal object in $\mathit{Span}(A_1, \ldots, A_n)$ is called a *jointly monic family*, and we write $Mon(A_1, \ldots, A_n)$ for the full subcategory of jointly monic families in $\mathit{Span}(A_1, \ldots, A_n)$. When $n = 2$ we say *jointly monic pair.*

Remark 3.1*:* $(A, f_1, \ldots, f_n)$ is a jointly monic family iff any morphisms $X \underset{g_2}{\overset{g_1}{\rightrightarrows}} A$ in $\mathcal{C}$ such that $f_i g_1 = f_i g_2$ for all i are equal. If $(A, f_1, \ldots, f_n)$ is a span and some f_i is monic, then the span is jointly monic, but $(A, f_1, \ldots, f_n)$ being jointly monic does not imply any f_i is monic. Note that an isomorphism of spans is an isomorphism in $\mathcal{C}$.

Definition 3.3: We say a span S *factors through* a span S' in $\mathit{Span}(A_1, \ldots, A_n)$ if there exists a morphism of spans $S \to S'$.

$Mon(A_1, \ldots, A_n)$ is a preorder category, because all objects are weakly initial, with $m_1 \leq m_2$ iff m_1 factors through m_2. Such a $t : m_1 \to m_2$ in $Mon(A_1, \ldots, A_n)$ is necessarily monic in $\mathcal{C}$. In fact, any morphism of spans $t : (X, f_1 \ldots f_n) \to (Y, g_1, \ldots, g_n)$ whose domain is a jointly monic family is a monomorphism in $\mathcal{C}$ because if $ta = tb$ then $f_i a = g_i ta = g_i tb = f_i b$ for all i so $a = b$. For $n = 1$, $\mathit{Span}(A)$ is the comma category $(1_C \downarrow A)$ and is called the *slice category*, written $\mathcal{C}/A$. A jointly monic family is then just a monomorphism. Note that the order of the tuples matters. For $f \neq g$, $(A, f, g) \neq (A, g, f)$ as spans. When we write a span as $A \xleftarrow{f} X \xrightarrow{g} B$, we mean the span (A, f, g).

Lemma 3.1: *A terminal object in $\mathit{Span}(A_1, \ldots, A_n)$ is a product $A_1 \times \cdots \times A_n$, and if a product exists, then a span is a jointly monic family iff the induced map into the product is a monomorphism. Moreover, $Mon(A_1, \ldots, A_n) \cong Mon(A_1 \times \cdots \times A_n)$ as categories.*

Proof: A terminal object is a product by definition. If the product exists, define
$F : Mon(A_1 \times \cdots \times A_n) \to Mon(A_1, \ldots, A_n)$ to be the identity on morphisms and on objects as $F(t) = (\mathrm{dom}(t), \pi_1 t, \ldots \pi_n t)$ where π_i are the products projections. If f and g are morphisms such that $\pi_i t f = \pi_i t g$ for all i then $tf = tg$ by definition of product, and $f = g$ since t is monic. F has an inverse given by sending a jointly monic family to the induced morphism into the product, which by a similar argument is a monomorphism. □

Just by comparing definitions, it can be seen that a pullback of a pair of morphisms is the same thing as their product in the slice category of their codomain. For binary spans, we get the following:

Lemma 3.2: *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ has pullbacks and equalizers iff it has limits of shape*

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \\ & \bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Moreover, a limit of such a diagram is the same thing as a product of the spans that are in it.

Proof: Given spans $A \xleftarrow{f} X \xrightarrow{g} B$ and $A \xleftarrow{f'} X' \xrightarrow{g'} B$, a cone over the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \\ & X' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

in $\mathcal{C}$ is the same as an object of $Span(A,B)$ with morphisms of spans to $A \xleftarrow{f} X \xrightarrow{g} B$ and $A \xleftarrow{f'} X' \xrightarrow{g'} B$, so a limit of this diagram in $\mathcal{C}$ is the same as a product of these spans in $Span(A,B)$. If $\mathcal{C}$ has pullbacks and equalizers, it has limits of this diamond shape because it has all finite connected limits. The limit can be constructed explicitly by taking a pullback of the pair (f,f') and then the equalizer of the composites of the pullbacks projections with g and g'. Conversely, by writing out definitions, a pullback of $A \xrightarrow{f} B \xleftarrow{g} C$ can be constructed as a product of $B \xleftarrow{f} A \xrightarrow{f} B$ and $B \xleftarrow{g} C \xrightarrow{g} B$, and an equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ can be constructed as a product of the spans $A \xleftarrow{1_A} A \xrightarrow{f} B$ and $A \xleftarrow{1_A} A \xrightarrow{g} B$. $\square$

Lemma 3.3: *Let $\mathcal{C}$ be a category. If $((P,p_1,\ldots,p_n), P \xrightarrow{\pi_X} X, P \xrightarrow{\pi_M} M)$ is a product of $(X,f_1,\ldots,f_n)$ and $(M,g_1,\ldots,g_n)$ in $Span(A_1,\ldots,A_n)$ and $(M,g_1,\ldots,g_n)$ is a jointly monic family, then π_X is a monomorphism in $\mathcal{C}$. By the previous remarks, this can also be stated as: if (P,π_X,π_M) is a limit cone for the diamond diagram obtained from combining these spans, then π_X is monic.*

Proof: If $\pi_X a = \pi_X b$, then

$$\begin{tikzcd} & & & X \arrow[dl, "f_1"] \arrow[drr, "f_n"] & & \\ Z \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[urrr, dashed, bend left, "\pi_X a = \pi_X b"] & P \arrow[urr, "\pi_X"] \arrow[drr, "\pi_M"'] & A_1 & \cdots\cdots & & A_n \\ & & & M \arrow[ul, "g_1"] \arrow[urr, "g_n"'] & & \end{tikzcd}$$

$g_i\pi_M a = f_i\pi_X a = f_i\pi_X b = g_i\pi_M b$ for all i, and since $(M,g_1,\ldots,g_n)$ is a monic family, $\pi_M a = \pi_M b$. so $a = b$ since π_M and π_X are the projections of a limit. $\square$

Corollary 3.1: *In any category $\mathcal{C}$, a product of jointly monic families is a jointly monic family.*

Proof: If $((P,p_1,\ldots,p_n), P \xrightarrow{\pi_M} M, P \xrightarrow{\pi_N} N)$ is a product of jointly monic families $(M,f_1,\ldots,f_n)$, $(M',g_1,\ldots,g_n)$, then $(P,p_1,\ldots,p_n) = (P,g_1\pi_N,\ldots,g_n\pi_N)$ with π_N monic by lemma 3.3. So if $g_i\pi_N a = g_i\pi_N b$ for all i then $\pi_N a = \pi_N b$ since $(M,f_1,\ldots,f_n)$ is jointly monic, and $a = b$ since π_N is monic. $\square$

Definition 3.4: We say a property P that a morphism may have in a category is *stable under pullback* if whenever f has property P, and

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

is a pullback diagram, then g has property P.

Lemma 3.3 for the case $n = 1$ says that being a monomorphism is stable under pullback in any category.

3.2 Images and Covers

Definition 3.5: We say a category $\mathcal{C}$ *has n-ary images* if for all objects $A_1, \ldots A_n$, the inclusion functor $Mon(A_1, \ldots, A_n) \xhookrightarrow{inc} Span(A_1, \ldots, A_n)$ has a left adjoint $Span(A_1, \ldots, A_n) \xrightarrow{im} Mon(A_1, \ldots, A_n)$.

If S is a span, the unit of this adjunction is a morphism of spans $\eta_S : S \to im(S)$ that is initial in $(S \downarrow inc)$. That is, if S factors through a jointly monic family M, then $im(S) \leq M$. So $im(S)$ is a smallest monic family through which S factors. We call any monic family with this property an *image* of S. In particular, images are unique up to isomorphism, and by functoriality we have that if S factors through S', then $im(S) \leq im(S')$.

Definition 3.6: A morphism is a *cover* if the only monomorphisms it factors through are isomorphisms. We denote a cover by $\mapsto$.

Images and covers are closely related by the following fact:

Lemma 3.4: *Let $(X, f_1, \ldots, f_n)$ be a span in $\mathcal{C}$ with image $(I, g_1, \ldots, g_n)$. Then any morphism of spans $X \xrightarrow{c} I$ in $\mathcal{C}$ is a cover.*

Proof: Suppose c factors through a monomorphism m:

$$\begin{tikzcd}
& & X \arrow[ddll, "f_1"'] \arrow[ddrr, "f_n"] \arrow[d, "k"] \arrow[dd, bend right, "c"'] & & \\
& & Z \arrow[d, tail, "m"] & & \\
& & I \arrow[dll, "g_1"] \arrow[drr, "g_n"'] & & \\
A_1 & & \cdots\cdots & & A_n
\end{tikzcd}$$

Since m is monic, $(Z, g_1 m, \ldots, g_n m)$ is a jointly monic family, and k becomes a morphism of spans, so $(Z, g_1 m, \ldots, g_n m) \leq (I, g_1, \ldots, g_n)$ and by the property of images $(Z, g_1 m, \ldots, g_n m) \geq (I, g_1, \ldots, g_n)$. Thus m is an isomorphism of spans, so it is an isomorphism in $\mathcal{C}$. $\square$

The following small facts about covers will be useful:

Lemma 3.5: *In any category:*

(i) A morphism is an isomorphism iff it is monic and a cover.

(ii) A split epimorphism is a cover.

(iii) If gf is a cover, then g is a cover.

(iv) If pullbacks exist, covers are closed under composition.

Proof:

(i) If an isomorphism $t : A \to B$ factors as $t = ma$ with m monic, then $1_B = m(at^{-1})$ and $m = m(at^{-1})m$. Since m is monic, $(at^{-1})m = 1_{dom(m)}$ so m is an isomorphism. Conversely if t is monic and a cover then its an isomorphism since it factors through itself.

(ii) Suppose $fs = 1$ and $f = mc$ with m monic. Then $m(cs) = fs = 1$ so m is also split epi, so an isomorphism (as shown in (i)).

(iii) If gf is a cover and $g = mc$ with m monic then $gf = mcf$ so m is an isomorphism.

(iv) Suppose we are given covers $A \mapsto B$ and $B \mapsto C$, and suppose their composite factors through a monomorphism $D \rightarrowtail C$. Forming the pullback, we get the following with the morphism $P \rightarrowtail B$ monic since monomorphisms are stable under pullback:

$$\begin{tikzcd} P \arrow[drr, tail] \arrow[dd, shift left] & & \\ & A \arrow[ul] \arrow[r, mapsto] \arrow[d, mapsto] & B \arrow[d, mapsto] \\ & D \arrow[r, tail] & C \end{tikzcd}$$

$A \mapsto B$ is a cover so $P \rightarrowtail B$ is an isomorphism, and so $B \mapsto C$ factors through $D \rightarrowtail C$, which implies, $D \rightarrowtail C$ is an isomorphism, as desired.

□

We have seen that the unit of the adjunction in an image factorization is a cover. The next lemma shows that if the category has the certain limits, the converse is also true: any factorization through a jointly monic family via a cover is an image factorization.

Lemma 3.6: *Let $\mathcal{C}$ be a category:*

(i) Suppose $\mathcal{C}$ has pullbacks. If a morphism $f : A \to B$, factors as

$$\begin{tikzcd} A \arrow[d, mapsto, "c"'] \arrow[dr, "f"] & \\ I \arrow[r, tail, "i"'] & B \end{tikzcd}$$

where i is monic and e is a cover, then i is an image of f.

(ii) Suppose $\mathcal{C}$ has pullbacks and equalizers. If a span $A \xleftarrow{f} X \xrightarrow{f'} B$, factors as

$$\begin{tikzcd} & I \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ A & X \arrow[l, "f"'] \arrow[r, "f'"] \arrow[u, mapsto, "c"] & B \end{tikzcd}$$

where (I, g, g') is a jointly monic pair and e is a cover, then (I, g, g') is an image of (X, f, f').

Proof:

(i) Suppose f factors through a monomorphism m. Form the pullback (P, p, p') of m and i to get:

$$\begin{tikzcd} P \arrow[drr, tail, "p'"] \arrow[ddr, "p"'] & & \\ & A \arrow[ul, dashed, "\exists! t"'] \arrow[r] \arrow[d, mapsto, "e"'] \arrow[dr, "f"] & C \arrow[d, tail, "m"] \\ & I \arrow[r, tail, "i"'] & B \end{tikzcd}$$

p is monic since m is, so p is an isomorphism as it is part of a factorization of the cover e. Now $m(p'p^{-1}) = i$, i.e. $i \leq m$ in $Mon(B)$ as desired.

(ii) This case is basically the same thing as (i). Suppose (X, f, f') factors through a jointly monic pair (M, k, k'). Since $\mathcal{C}$ has pullbacks and equalizers, we can form the product of the spans (M, k, k') and (I, g, g') to get:

$$\begin{tikzcd}
 & & I \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
P \arrow[urr, tail, bend left, "p"] \arrow[drr, bend right, "p'"'] & A & X \arrow[l, "f"'] \arrow[r, "f'"] \arrow[u, "c"] \arrow[d, "b"] \arrow[ll, dashed, bend left, "\exists! t"] & B \\
 & & M \arrow[ul, "k"] \arrow[ur, "k'"'] &
\end{tikzcd}$$

Where p is monic by lemma 3.3. $t : X \to P$ is morphism of spans induced by the universal property of the product. The cover c factors through the monomorphism p, so p is an isomorphism, and so $p'p^{-1}$ is a morphism of spans. So $(I, g, g') \leq (M, k, k')$ as desired.

□

Corollary 3.2: *If $\mathcal{C}$ has pullbacks and equalizers, and a span $A \xleftarrow{f} X \xrightarrow{f'} B$ factors through $A \xleftarrow{g} Y \xrightarrow{g'} B$ by a cover $c : X \to Y$, then any image of $A \xleftarrow{g} Y \xrightarrow{g'} B$ is an image of $A \xleftarrow{f} X \xrightarrow{f'} B$.*

Proof: If (I, i, i') is an image of (Y, g, g')), there is a morphism of spans that is a cover $d : Y \to I$, and so $dc : X \to I$ is a cover and a morphism of spans and (I, i, i') is jointly monic. □

3.3 Relations in a Category

We have now developed enough preliminaries to define and investigate relations in an arbitrary category.

Definition 3.7: An *n-ary relation* in a category $\mathcal{C}$ is an equivalence class of objects in $Mon(A_1, \ldots, A_n)$ for objects A_i in $\mathcal{C}$. We write $Rel(A_1, \ldots, A_n)$ for the category of relations on $A_1, \ldots, A_n$. A unary relation is called a *subobject*, and we write $Sub(A)$ for $Rel(A)$. For $n = 2$ we say an object R in $Rel(A_1, A_2)$ is a relation *from A to B* and write $R : A_1 \nrightarrow A_2$.

Remark 3.2: Since $Mon(A_1, \ldots, A_n)$ is a preorder category, $Rel(A_1, \ldots, A_n)$ is a poset category. That is, the only isomorphisms are identities.

Corollary 3.3: *Let $\mathcal{C}$ be a category:*

(i) If $\mathcal{C}$ has binary products, then $Rel(A_1, \ldots, A_n) \cong Sub(A_1 \times \cdots \times A_n)$

(ii) If $\mathcal{C}$ has pullbacks then $Sub(A)$ has binary meets.

(iii) If $\mathcal{C}$ has pullbacks and equalizers then $Rel(A_1, A_2)$ has binary meets.

Proof: (i) is immediate from lemma 3.1. (ii) follows from the remark that pullbacks are the same as products of spans, and (iii) follow from lemmas 3.2 and 3.1. □

We use $\cap$ so denote meets of relations. From this point on we will restrict our attention only to binary relations, so when we say relation we will always mean binary relation.

Definition 3.8: Let R be a relation in a category $\mathcal{C}$. We call the jointly monic pairs that are representatives of R *tabulations* of R, and say that a jointly monic pair *tabulates* R and R is *tabulated* by a jointly monic pair. Given a relation $R : A \nrightarrow B$ tabulated by a jointly monic pair $A \xleftarrow{f} X \xrightarrow{g} B$, its *reciprocal* $R^\circ : B \nrightarrow A$ is the relation tabulated by $A \xleftarrow{g} X \xrightarrow{f} B$.

Remark 3.3*:* Notice that if $R \leq S$ then $R^\circ \leq S^\circ$ since a morphism of spans between any tabulations of R and S remains a morphism of spans when we interchange the order of the morphisms.

Definition 3.9: Given a morphism $f : A \to B$ in $\mathcal{C}$, its *graph* is the relation $f_\bullet : A \looparrowright B$ tabulated by the monic pair $A \xleftarrow{1_A} A \xrightarrow{f} B$. We write $f^\bullet$ for $(f_\bullet)^\circ$. We call a relation a *map* if it is the graph of some morphism.

Lemma 3.7: *A monic pair $A \xleftarrow{f} T \xrightarrow{g} B$ tabulates a map iff f is an isomorphism.*

Proof: If f is an isomorphism then f^{-1} is an isomorphism of spans $(T, f, g) \to (A, 1_A, gf^{-1})$. If (T, f, g) tabulates a map, it is isomorphic to a graph and the isomorphism is the inverse of f. $\square$

Definition 3.10: Let $\mathcal{C}$ be a category with pullbacks, and images of monic pairs. Given relations $R : A \looparrowright B$ and $R' : B \looparrowright C$ tabulated by (X, f, g) and (Y, f', g') respectively, define the composite $R'R : A \looparrowright C$ to be the relation tabulated by an image of the span $(P, fp, g'p')$ where (P, p, p') is a pullback of g and f'.

$$\begin{tikzcd}
 & & I \arrow[ddll] \arrow[ddrr] & & \\
 & & P \arrow[u, hook] \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & Y \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
A & & B & & C
\end{tikzcd}$$

Since pullbacks, images, and tabulations (by definition) are unique up to isomorphism, this definition does not depend on any choice of these. This composition agrees with the composition defined in **Rel**, in which case I is the direct image of the pullback $P \subset X \times Y$ along the function $(f \times g') : X \times Y \to A \times C$:

$$\{\, (a, c) \mid \exists b \in B,\ aRb \wedge bR'c \,\} = \{\, (f(x), g'(y)) \mid g(x) = f'(y) \,\} = (f \times g')(\{\, (x, y) \mid g(x) = f'(y) \,\})$$

since a tabulation (X, f, g) of R is a pair of functions that satisfy aRb iff $\exists! x \in X$ such that $f(x) = a$ and $g(x) = b$.

Definition 3.11: Let A be an object in a category $\mathcal{C}$ with pullbacks. Define the *identity relation on* A to be the graph of 1_A. We will denote the identity relation by $\iota_A : A \looparrowright A$ to distinguish it from the identity morphism $1_A : A \to A$.

Remark 3.4*:* If $R : B \looparrowright A$ is tabulated by (X, f, g), then the following diagram

$$\begin{tikzcd}
 & & X \arrow[ddll, bend right, "f"'] \arrow[ddrr, bend left, "g"] & & \\
 & & X \arrow[u, hook, "1_X"'] \arrow[dl, "1_X"'] \arrow[dr, "g"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & A \arrow[dl, "1_A"'] \arrow[dr, "1_A"] & \\
B & & A & & A
\end{tikzcd}$$

shows that $\iota_A R = R$ (if $\mathcal{C}$ has equalizers), because the diamond is a pullback and the identity morphism is a cover, so it is an image factorization. Similarly for composition on the other side.

As mentioned in the case of **Rel** in remark 2.1, composition of relations allows us to express properties of morphism using quantifier free equations.

Lemma 3.8: *Let $R : A \nrightarrow B$ be a relation tabulated by a jointly monic pair (X, a, b) in a category with pullbacks, equalizers, and images of binary spans. Then:*

(i) a is a cover iff $\iota_A \le R^\circ R$.

(ii) a is monic iff $RR^\circ \le \iota_B$.

PROOF:

(i) Forming $R^\circ R$ gives the diagram:

$$\begin{tikzcd}
& & I \arrow[dddll, "i"'] \arrow[dddrr, "i'"] & & \\
& & P \arrow[u, "c"] \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
& X \arrow[dl, "a"'] \arrow[dr, "b"] & X \arrow[u, "s"] \arrow[l, "1_X"] \arrow[r, "1_X"'] & X \arrow[dl, "b"'] \arrow[dr, "a"] & \\
A & & B & & A \\
& & A \arrow[ull, "1_A"] \arrow[urr, "1_A"'] & &
\end{tikzcd}$$

Notice that a is a morphism of spans $(X, a, a) \to (A, 1_A, 1_A)$, so if a is a cover, $(A, 1_A, 1_A)$ is an image of (X, a, a). But since $b1_X = b1_X$, $\exists! s : X \to P$ such that $ps = p's = 1_X$, so $ics = aps = a$ and $i'cs = a'ps = a$, so cs is a morphism of spans $(X, a, a) \to (I, i, i')$. But (I, i, i') is jointly monic, so $(A, 1_A, 1_A) \le (I, i, i')$, which means $\iota_A \le R^\circ R$. Conversely, if $\iota_A \le R^\circ R$ then there exists a morphism of spans $(A, 1_A, 1_A) \xrightarrow{t} (I, i, i')$. So i is split epi hence a cover, which means $ap = ic$ is a composite of covers, so a cover, and so a is a cover.

(ii) Forming RR° gives the diagram:

$$\begin{tikzcd}
& & I \arrow[dddll, "i"'] \arrow[dddrr, "i'"] & & \\
& & P \arrow[u, "c"] \arrow[dl, "p"'] \arrow[dr, "p'"] & & \\
& X \arrow[dl, "b"'] \arrow[dr, "a"] & & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\
B & & A & & B \\
& & B \arrow[ull, "1_B"] \arrow[urr, "1_B"'] & &
\end{tikzcd}$$

If a is monic than any cone over the pullback diagram (a, a) must have its projections equal, which means $(X, 1_X, 1_X)$ is a terminal cone, so we may assume $(P, p, p') = (X, 1_X, 1_X)$. So RR° is tabulated by an image of (X, b, b). But (X, b, b) factors through $(B, 1_B, 1_B)$, so $im(X, b, b) \le im(B, 1_B, 1_B) = (B, 1_B, 1_B)$, hence $RR^\circ \le \iota_B$. Conversely, if $RR^\circ \le \iota_B$, then then there exists a morphism of spans $(I, i, i') \xrightarrow{t} (B, 1_B, 1_B)$, so $i = i'$, and in turn $bp = bp'$. But we also have $ap = ap'$ from the pullback square, and since (X, a, b) is jointly monic, $p = p'$, which then forces a to be monic by the universal property of the pullback.

$\square$

Lemma 3.9: *Let $\mathcal{C}$ be a category with pullbacks, equalizers, and images of binary spans:*

(i) A relation $\phi : A \looparrowright B$ is a map iff $\iota_A \leq \phi^\circ\phi$ and $\phi\phi^\circ \leq \iota_B$.

(ii) A relation $\phi : A \looparrowright B$ is tabulated by $A \xleftarrow{a} X \xrightarrow{b} B$ iff $\phi = b_\bullet a^\bullet$ and $a^\bullet a_\bullet \cap b^\bullet b_\bullet = \iota_X$.

(iii) A morphism $f : A \to B$ is a cover iff $f_\bullet f^\bullet = \iota_B$.

(iv) A morphism $f : A \to B$ is monic iff $f^\bullet f_\bullet = \iota_A$.

PROOF:

(i) Suppose (A, a, b) is a tabulation of ϕ. Then ϕ is a map iff a is an isomorphism (lemma 3.7) iff a is a cover and monic iff $\iota_A \leq \phi^\circ\phi$ and $\phi\phi^\circ \leq \iota_B$.

(ii) If (A, a, b) is jointly monic, it is its own image, so $b_\bullet a^\bullet$ is the relation tabulated by (A, a, b). Hence it tabulates ϕ iff $\phi = b_\bullet a^\bullet$. It remains to show that (A, a, b) is jointly monic iff $a^\bullet a_\bullet \cap b^\bullet b_\bullet = \iota_X$.

$a^\bullet a_\bullet$ is tabulated by the pullback of $X \xrightarrow{a} A \xleftarrow{a} X$ because a pullback is already a jointly monic pair, and similarly $b^\bullet b_\bullet$ is tabulated by the pullback of $X \xrightarrow{b} B \xleftarrow{b} X$. Let (P, p_1, p_2) and (Q, q_1, q_2) be these pullbacks respectively. Then $a^\bullet a_\bullet \cap b^\bullet b_\bullet$ is tabulated by a product of the spans (P, p_1, p_2) and (Q, q_1, q_2). If (K, k_1, k_2) is any span that admits morphisms of spans $t_1 : (K, k_1, k_2) \to (P, p_1, p_2)$ and $t_2 : (K, k_1, k_2) \to (Q, q_1, q_2)$, it follows that $ak_1 = ap_1t_1 = ap_2t_1 = ak_2$ and similarly $bk_1 = bk_2$.

If (X, a, b) is jointly monic, $k_1 = k_2$, in which case (K, k_1, k_2) factors uniquely through $(X, 1_X, 1_X)$, which itself has morphisms of spans into (P, p_1, p_2) and (Q, q_1, q_2) given by the property of the pullbacks. So $(X, 1_X, 1_X)$ tabulates $a^\bullet a_\bullet \cap b^\bullet b_\bullet$, as desired.

Conversely, suppose $a^\bullet a_\bullet \cap b^\bullet b_\bullet = \iota_X$. So $(X, 1_X, 1_X)$ is a product of the spans previously mentioned. But then if $X \xleftarrow{n} Y \xrightarrow{m} X$ are morphisms such that $an = bn$ and $am = bm$, then the universal properties of the pullbacks gives morphisms of spans into (P, p_1, p_2) and (Q, q_1, q_2), which in turn induces a morphism of spans into the product $(X, 1_X, 1_X)$ which forces $n = m$, so (X, a, b) is jointly monic.

For (iii) and (iv), since $f_\bullet$ is map, by (i) we have $\iota_A \leq f^\bullet f_\bullet$ and $f_\bullet f^\bullet \leq \iota_B$. $B \xleftarrow{f} A \xrightarrow{1_A} A$ tabulates $f^\bullet$, so by lemma 3.8, f is a cover iff $\iota_B \leq f_\bullet f^\bullet$, and f is monic iff $f^\bullet f_\bullet \leq \iota_A$. $\square$

Notice that these properties did not depend on composition being associative.

Proposition 3.1: *Let $\mathcal{C}$ be a category with pullbacks, equalizers, and images of binary spans. Then composition of relations is associative iff covers are stable under pullback.*

PROOF: $\Longleftarrow$: Let $R_1 : A \looparrowright B$, $R_2 : B \looparrowright C$, $R_3 : C \looparrowright D$ be relations tabulated by

$$\begin{tikzcd}
& X \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & Y \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & Z \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

$(R_1R_2)R_3$ is tabulated by the image of (S, is, g_3s') where (P, p, p') is a pullback of (g_1, f_2), (I, i, i') is the image of (P, f_1p, g_2p'), and (S, s, s') is the pullback of (i', f_3):

$$\begin{tikzcd}
& & P \arrow[dl, "p"'] \arrow[d, "e"] \arrow[dr, "p'"] & S \arrow[dl, "s"'] \arrow[drr, "s'"] & & & \\
& X \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & I \arrow[dll, "i"'] \arrow[drr, "i'"] & Y \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & Z \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

We could also form the following diagram just by taking pullbacks:

$$\begin{tikzcd}
 & & & R \arrow[dl, "r"'] \arrow[dr, "r'"] & & & \\
 & & P \arrow[dl, "p"'] \arrow[dr, "p'"] & & Q \arrow[dl, "q"'] \arrow[dr, "q'"] & & \\
 & X \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & Y \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & Z \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

We will show that any image of (S, is, g_3s') is an image of $(R, f_1pr, g_3q'r')$. By symmetry, the other direction of composition will have the same image, so the composites will be equal as relations. Combining these diagrams we get:

$$\begin{tikzcd}
 & K \arrow[dr, dashed, "k_1"'] \arrow[drr, dashed, "k_2"] & & R \arrow[dl, "r"'] \arrow[d, "\exists! d"] \arrow[dr, "r'"] & & & \\
 & & P \arrow[dl, "p"'] \arrow[d, tail, "e"] \arrow[dr, "p'"] & S \arrow[dl, "s"] \arrow[drr, "s'"] & Q \arrow[dl, "q"] \arrow[dr, "q'"] & & \\
 & R_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & I \arrow[dll, tail, "i"] \arrow[drr, tail, "i'"] & R_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & & R_3 \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A & & B & & C & & D
\end{tikzcd}$$

where d is induced by the universal property of the pullback (S, s, s') since $i'(er) = g_2p'r = f_3(q'r')$, so $sd = er$ and $s'd = q'r'$. Moreover, the square $sd = er$ is a pullback square because if $P \xleftarrow{k_1} K \xrightarrow{k_2} S$ satisfies $ek_1 = sk_2$, then $f_3s'k_2 = i'sk_2 = i'ek_1 = g_2p'k_1$. $(R, r, q'r')$ is a pullback of f_3 and g_2p', so we get a unique morphism $t : K \to R$ such that $q'r't = s'k_2$ and $rt = k_1$. But notice that $sk_2 = ek_1 = ert = sdt$ and $s'k_2 = q'r't = s'dt$. Since s and s' are the projections of a limit cone, it follows that $k_2 = dt$ as desired. If $t' : K \to R$ satisfies $k_1 = rt'$ and $k_2 = dt'$, then $q'r't' = s'dt' = s'k_2$ and so $t' = t$.

Covers are stable under pullback, so d is a cover, and it is a morphism of spans $(R, f_1pr, g_3q'r') \xrightarrow{d} (S, is, g_3s')$, so by corollary 3.2 these spans have isomorphic images.

$\Longrightarrow$: Suppose

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "p'"'] & C \arrow[d, "g"] \\
A \arrow[r, mapsto, "f"'] & B
\end{tikzcd}$$

is a pullback square with f a cover. By lemma 3.8, $(f_\bullet f^\bullet)g_\bullet = \iota_B g_\bullet = g_\bullet$. So by associativity, $f_\bullet(f^\bullet g_\bullet) = g_\bullet$. Computing the composite we get:

$$\begin{tikzcd}
 & & & J \arrow[dlllll, bend right, "j'"'] \arrow[dddddrrr, bend left, "j"] & & & \\
 & & & I \arrow[u, "c'"] \arrow[dl, "1_I"'] \arrow[dddrr, "i"] & & & \\
 & & I \arrow[dddll, "i'"'] \arrow[dddrr, "i"] & & & & \\
 & & P \arrow[u, "c"] \arrow[dl, "p"'] \arrow[dr, "p'"] & & & & \\
 & C \arrow[dl, "1_C"] \arrow[dr, "g"] & & A \arrow[dl, "f"] \arrow[dr, "1_A"] & & A \arrow[dl, "1_A"] \arrow[dr, "f"] & \\
C & & B & & A & & B
\end{tikzcd}$$

and $C \xleftarrow{j'} J \xrightarrow{j} B$ is a tabulation of $f_\bullet(f^\bullet g_\bullet) = g_\bullet$ which is a map. So by lemma 3.7, j' is an isomorphism, hence by lemma 3.5 it is a cover. But notice that $p = j'c'c$ which is a composite of three covers, so by lemma 3.5, p is a cover as desired. $\square$

3.4 Categories of Relations

Definition 3.12: A category $\mathcal{C}$ is *locally regular* if it has pullbacks, equalizers, images of binary spans, and covers are stable under pullback. (Equivalently, finite connected limits instead of pullbacks and equalizers).

By proposition 3.1 and remark 3.4, if a category $\mathcal{C}$ is locally regular, then the relations in $\mathcal{C}$ form a category with the same objects as $\mathcal{C}$. We denote this category by $\mathcal{R}el(\mathcal{C})$.

Proposition 3.2: *For a locally regular category $\mathcal{C}$, $\mathcal{R}el(\mathcal{C})$ is an allegory.*

Proof: For this proof, whenever $R : A \looparrowright B$ is a relation, we will also denote the domain of morphisms that tabulate it by R. In any diagram, for any letter l, morphisms named l_1 and l_2 will always share a common domain and will be part of a span (L, l_1, l_2). We will reserve the letter P for the apex of a limit cone for a pullback, and the letter p for the pullbacks projections. We will reserve π to denote projections from a product cone. In diagrams, dashed morphisms will denote morphisms that are projections of some limit cone.

- Locally posetal: $\mathcal{R}el(\mathcal{C})$ is ordered by $R \leq R'$ iff R factors through R', so any tabulation of R factors through a tabulation of R'. If $R \leq R' \in Rel(A, B)$ and $S : B \looparrowright C$, choosing tabulations and forming the pullbacks for the composites SR and SR' we get:

$$\begin{tikzcd}
 & & P \arrow[dl, dashed, "p_1"'] \arrow[ddr, dashed, "p_2"] & & \\
 & R \arrow[d, "t"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & P' \arrow[dl, dashed, "p'_1"'] \arrow[dr, dashed, "p'_2"] & & \\
 & R' \arrow[dl, "g_1"] \arrow[dr, "g_2"'] & & S \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
A & & B & & C
\end{tikzcd}$$

  and tp_1, p_2 induce a morphism (and it is a morphism of spans) $P \to P'$ since P' is the apex of a pullback. $\mathcal{C}$ has images, so they are functorial (by definition), so $SR \leq SR'$. Same for other direction.

- Anti-involution: Define $(-)^\circ : \mathcal{R}el(\mathcal{C})^{op} \to \mathcal{R}el(\mathcal{C})$ to be the identity on objects and map morphisms to their reciprocal. This is functorial and order preserving as previously mentioned.

- Binary meets: By corollary 3.3.
- Modular Law: Let $\phi : A \nrightarrow B$, $\psi : B \nrightarrow C$, $\chi : A \nrightarrow C$. We want to show that $\psi\phi \cap \chi \leq (\psi \cap \chi\phi^\circ)\phi$.

$\mathcal{C}$ has pullbacks and equalizers, so it has all finite connected limits. We start by choosing tabulations for the relations and forming the limit (L, x, y, z) of the diagram given by the tabulations:

$$\begin{tikzcd}
L \arrow[drr, dashed, "x"] \arrow[ddrrr, dashed, "y"] \arrow[dddr, dashed, "z"'] & & & \\
& & \phi \arrow[r, "f_2"] \arrow[dl, "f_1"'] & B \\
& A & & \psi \arrow[u, "g_1"'] \arrow[dl, "g_2"] \\
& \chi \arrow[u, "h_1"'] \arrow[r, "h_2"'] & C &
\end{tikzcd}$$

The intersections of spans is their product in the span category. So following all the definitions, we form $\psi\phi \cap \chi$ as:

$$\begin{tikzcd}
& & & \psi\phi \arrow[dddll, "i_1"'] \arrow[dddrr, "i_2"] & & \\
& & & P \arrow[u, "c"] \arrow[dl, dashed, "p_1"'] \arrow[dr, dashed, "p_2"] & & \\
& & \phi \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & L \arrow[u, dashed, "\exists! s"] \arrow[l, dashed, "x"'] \arrow[r, dashed, "y"] & \psi \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\
\psi\phi \cap \chi \arrow[r, "k_1"'] \arrow[uuurrr, dashed, "\pi_{\psi\phi}"] \arrow[drrr, dashed, "\pi_\chi"'] \arrow[rrrrr, bend right=50, "k_2"'] & A & & B & & C \\
& & & \chi \arrow[ull, "h_1"'] \arrow[urr, "h_2"'] & &
\end{tikzcd}$$

where (P, p_1, p_2) is a pullback, $(\psi\phi, i_1, i_2)$ is the image of (P, f_1p_1, g_2p_2), and $(\psi\phi \cap \chi, k_1, k_2)$ is the product of the spans $(\psi\phi, i_1, i_2)$ and (χ, h_1, h_2) with projections π_χ and $\pi_{\psi\phi}$. Since $f_2x = g_1y$, the pullback property of (P, p_1, p_2) induces a morphism $s : L \to P$ such that $x = p_1s$, $y = p_2s$. The composite cs is then a morphism of spans $cs : (L, f_1x, g_2y) \to (\psi\phi, i_1, i_2)$ because $i_1cs = f_1p_1s = f_1x$ and similarly $i_2cs = g_2p_2s = g_2y$. $z : L \to \chi$ is also a morphism of spans $z : (L, f_1x, g_2y) \to (\chi, h_1, h_2)$ by definition of (L, x, y, z). So there is an induced morphism of spans $(cs, z) : (L, f_1x, g_2y) \to (\psi\phi \cap \chi, k_1, k_2)$ by definition of product such that $cs = \pi_{\psi\phi}(cs, z)$ and $z = \pi_\chi(cs, z)$. We now show that the square $cs = \pi_{\psi\phi}(cs, z)$ is in fact a pullback. To see this, we put all the relevant information into one diagram:

$$
\begin{tikzcd}
W \arrow[dr, "\exists! t"] \arrow[drrrr, "w_2"] \arrow[ddddr, "w_1"'] & & & & & \\
& L \arrow[rrrr, "s"] \arrow[dddd, "{(cs,z)}"] \arrow[drr, dashed, "x"] \arrow[ddrrr, dashed, "y"] \arrow[dddr, dashed, "z"] & & & & P \arrow[dll, dashed, "p_1"'] \arrow[ddl, dashed, "p_2"] \arrow[dddd, "c"] \\
& & & \phi \arrow[r, "f_2"] \arrow[dl, "f_1"'] & B & \\
& & A & & \psi \arrow[u, "g_1"'] \arrow[dl, "g_2"] & \\
& & \chi \arrow[u, "h_1"'] \arrow[r, "h_2"] & C & & \\
& \psi\phi \cap \chi \arrow[uuur, "k_1"] \arrow[uurr, "k_2"'] \arrow[ur, dashed, "\pi_\chi"] \arrow[rrrr, dashed, "\pi_{\psi\phi}"] & & & & \psi\phi \arrow[uulll, bend right, "i_1"'] \arrow[ull, "i_2"]
\end{tikzcd}
$$

Now if $\pi_{\psi\phi} w_1 = c w_2$ then:

$$
\begin{aligned}
h_1(\pi_\chi w_1) &= k_1 w_1 = i_1 \pi_{\psi\phi} w_1 = i_1 c w_2 = f_1(p_1 w_2)\\
h_2(\pi_\chi w_1) &= k_2 w_1 = i_2 \pi_{\psi\phi} w_1 = i_2 c w_2 = g_2(p_2 w_2)\\
f_2(p_1 w_2) &= g_1(p_2 w_2)
\end{aligned}
$$

So $(W, \pi_\chi w_1, p_1 w_2, p_2 w_2)$ is a cone over the diagram of which (L, x, y, z) is a limit of, so $\exists! t : W \to L$ such that $\pi_\chi w_1 = zt$, $p_1 w_2 = xt$, $p_2 w_2 = yt$. But then $p_1 st = xt = p_1 w_2$ and $p_2 st = yt = p_2 w_2$ so $st = w_2$. On the other side $\pi_\chi (cs, z)t = zt = \pi_\chi w_1$ and $i_2 \pi_{\psi\phi}(cs, z)t = k_2(cs, z)t = h_2 \pi_\chi (cs, z)t = h_2 \pi_\chi w_1 = k_2 w_1 = i_2 \pi_{\psi\phi} w_1$. For the exact same reason $i_1 \pi_{\psi\phi}(cs, z)t = i_1 \pi_{\psi\phi} w_1$, so $\pi_{\psi\phi}(cs, z)t = \pi_{\psi\phi} w_1$ (since $(\psi\phi, i_1, i_2)$ is jointly monic), and so $(cs, z)t = w_1$. Uniqueness holds as any such t will induce a limit cone over the original diagram of tabulations. Since covers are stable under pullbacks, (cs, z) is a cover. So $(cs, z) : (L, f_1 x, g_2 y) \to (\psi\phi \cap \chi, k_1, k_2)$ is an image factorization of $(L, f_1 x, g_2 y)$ since $(\psi\phi \cap \chi, k_1, k_2)$ is jointly monic.

Let $(\psi \cap \chi\phi^\circ, m_1, m_2)$ be a tabulation of the product of the spans (ψ, g_1, g_2) and a tabulation of $\chi\phi^\circ$, and let π_ψ be the projection of the product onto (ψ, g_1, g_2). As before, $(L, g_1 y, g_2 y)$ admits morphisms of spans to the spans of which $(\psi \cap \chi\phi^\circ, m_1, m_2)$ is a product, with $y : L \to \psi$ being the morphism into (ψ, g_1, g_2). Let $d : (L, g_1 y, g_2 y) \to (\psi \cap \chi\phi^\circ)$ be the induced morphism of spans, so that $y = \pi_\psi d$. In other words, the following diagram commutes:

$$
\begin{tikzcd}
& \psi \cap \chi\phi^\circ \arrow[d, dashed, "\pi_\psi"] \arrow[ddl, "m_2"'] \arrow[ddr, "m_1"] & L \arrow[l, "d"'] \arrow[dl, "y"] \\
& \psi \arrow[dl, "g_2"] \arrow[dr, "g_1"'] & \\
C & & B
\end{tikzcd}
$$

Taking this into account when forming the composite $(\psi \cap \chi\phi^\circ)\phi$ we get:

$$
\begin{tikzcd}
 & & (\psi \cap \chi\phi^\circ)\phi & \geq & \psi\phi \cap \chi & \\
 & & P' \arrow[u] \arrow[dl, dashed, "p_1'"'] \arrow[drr, dashed, "p_2'"] & & L \arrow[u] \arrow[ll, "t'"'] \arrow[dlll, "x"] \arrow[d, dashed, "d"] & \\
 & \phi \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & & \psi \cap \chi\phi^\circ \arrow[dll, "m_1"'] \arrow[dr, "m_2"] & \\
A & & B & & & C
\end{tikzcd}
$$

where t' is induced by the fact that $xf_2 = g_1 y = g_1 \pi_\psi d = m_1 d$. t' is then a morphism of spans $t' : (L, f_1 x, g_2 y) \to (P', f_1 p_1', m_2 p_2')$ since $f_1 p_1' t' = f_1 x$ and $m_2 p_2' t' = m_2 d = g_2 \pi_\psi d = g_2 y$ and (P', p_1', p_2') is a pullback of (f_2, m_1). By functoriality of images, $im(L, f_1 x, g_2 y) \leq im(P', f_1 p_1', m_2 p_2')$. An image of $(P', f_1 p_1', m_2 p_2')$ is a tabulation of $(\psi \cap \chi\phi^\circ)\phi$ by definition, and we have seen that an image of $(L, f_1 x, g_2 y)$ is a tabulation of $\psi\phi \cap \chi$.

□

4 Tabular Allegories

Allegories of the form $\mathcal{R}el(\mathcal{C})$ have a lot of maps. For any morphism $R : A \nrightarrow B$, a tabulation $A \xleftarrow{f} X \xrightarrow{g} B$ for R gives a jointly monic pair of maps $f_\bullet$ and $g_\bullet$ such that $g_\bullet f^\bullet = R$.

Definition 4.1: Let $\mathcal{A}$ be an allegory and $R : A \to B$ a morphism in $\mathcal{A}$. A *tabulation* of R is a pair of maps $A \xleftarrow{f} X \xrightarrow{g} B$ such that $gf^\circ = R$ and $f^\circ f \cap g^\circ g = 1_X$. $\mathcal{A}$ is *tabular* if every morphism has a tabulation.

By lemma 3.9, the definition of tabulation in a locally regular category coincides with the definition in its allegory of relations. In particular, the allegory of relations of a locally regular category is tabular. We will show that any tabular allegory is the allegory of relations of its category of maps, which is locally regular. Not all allegories are tabular. In a modular lattice, only the bottom element has a tabulation. The maps in the allegory of fuzzy relations are non-fuzzy (crisp) functions, and so the only relations that have tabulations are the crisp relations.

Lemma 4.1: *Let $\mathcal{A}$ be a tabular allegory and $A \xleftarrow{f} X \xrightarrow{g} B$ a pair of maps. Then the following are equivalent:*

- *(i) $A \xleftarrow{f} X \xrightarrow{g} B$ is a jointly monic pair in $\mathcal{M}ap(\mathcal{A})$.*
- *(ii) $f^\circ f \cap g^\circ g = 1_X$.*
- *(iii) $A \xleftarrow{f} X \xrightarrow{g} B$ is a tabulation of gf°.*

Proof: $(i) \Rightarrow (ii)$: Suppose $A \xleftarrow{f} X \xrightarrow{g} B$ is a jointly monic pair in $\mathcal{M}ap(\mathcal{A})$ and let $X \xleftarrow{h} T \xrightarrow{k} X$ be a tabulation of $f^\circ f \cap g^\circ g$. h is a map so $fk \leq fkh^\circ h$, and $kh^\circ = f^\circ f \cap g^\circ g$, so $fk \leq fkh^\circ h \leq f(f^\circ f \cap g^\circ g)h \leq f(f^\circ f)h = (ff^\circ)fh \leq fh$, hence $fh = fk$ since maps are discreetly ordered. Similarly $gk \leq gkh^\circ h \leq g(f^\circ f \cap g^\circ g)h \leq g(g^\circ g)h = (gg^\circ)gh \leq gh$ so $gh = gk$, and since (X, f, g) is jointly monic, $k = h$. So $f^\circ f \cap g^\circ g = kh^\circ = hh^\circ \leq 1_X$. $1_X \leq f^\circ f \cap g^\circ g$ since f and g are both maps, hence $f^\circ f \cap g^\circ g = 1_X$.
$(ii) \Rightarrow (iii)$ is immediate from the definition.
$(iii) \Rightarrow (i)$: If $A \xleftarrow{f} X \xrightarrow{g} B$ is a tabulation of gf°, then by definition $f^\circ f \cap g^\circ g = 1_X$. If h, h' are maps such that $fh = fh'$ and $gh = gh'$, then $h = (f^\circ f \cap g^\circ g)h = f^\circ fh \cap g^\circ gh = f^\circ fh' \cap g^\circ gh' = (f^\circ f \cap g^\circ g)h' = h'$, where we used lemma 2.4 twice.

□

Remark 4.1*:* Note that the proof of $(iii) \Rightarrow (i)$ is actually just a proof of $(ii) \Rightarrow (i)$ in an arbitrary allegory.

The following small results will be useful:

Lemma 4.2: *Let $R : A \to A$ be an endomorphism in an allegory tabulated by $A \xleftarrow{t} T \xrightarrow{t'} A$. If $R \leq 1_A$, then $t = t'$ and t is split monic.*

PROOF: Since $t't^\circ = R \leq 1_A$ and t is a map we get $t' \leq t't^\circ t \leq t$, so $t = t'$ since maps are discreetly ordered. So $1_A = t'^\circ t' \cap t^\circ t = t^\circ t \cap t^\circ t = t^\circ t$. $\square$

Lemma 4.3: *In any allegory $\mathcal{A}$:*

(i) If f, g, h, k are maps, then $hf = kg$ iff $gf^\circ \leq k^\circ h$.

(ii) If f factors through g in $\mathcal{M}ap(\mathcal{A})$ then $ff^\circ \leq gg^\circ$.

PROOF:

(i) If $hf = kg$, then $gf^\circ \leq k^\circ kgf^\circ = k^\circ hff^\circ \leq k^\circ h$. Conversely, if $gf^\circ \leq k^\circ h$ then $kg \leq kgf^\circ f \leq kk^\circ hf \leq hf$, so $kg = hf$ by lemma 2.3.

(ii) If $f = gh$ and h is a map, then by (i) $hf^\circ \leq g^\circ$ since the identity is a map. But $ff^\circ = ghf^\circ$, so $ff^\circ \leq gg^\circ$.

$\square$

4.1 Properties of Tabulations

In this section we will see how tabulations can be used to construct finite connected limits and images in the category of maps of an allegory. These constructions are all consequence of the following lemma:

Lemma 4.4 (Universal Property of Tabulations)**:** *Let $\mathcal{A}$ be an allegory. If a morphism $R : A \to B$ is tabulated by a pair of maps $A \xleftarrow{f} T \xrightarrow{g} B$, then for any pair of maps $A \xleftarrow{x} X \xrightarrow{y} B$, $yx^\circ \leq R$ iff there exists a map $h : X \to T$ such that $x = fh$ and $y = gh$. Moreover, such an h is unique when it exists.*

PROOF: By remark 4.1, $A \xleftarrow{f} T \xrightarrow{g} B$ is jointly monic in $\mathcal{M}ap(\mathcal{A})$, so any map $h : X \to T$ that is a morphism of spans is unique. It remains to show that such an h exists iff $yx^\circ \leq R$. If $h : X \to T$ is a map and a morphism of spans, then $yx^\circ = (gh)(fh)^\circ = ghh^\circ f^\circ \leq gf^\circ = R$. Conversely, consider the morphism $h = f^\circ x \cap g^\circ y$. Without any assumptions, $fh = f(f^\circ x \cap g^\circ y) \leq ff^\circ x \leq x$, and similarly $gh \leq y$. Since maps are linearly ordered, it remains to show h is a map. $A \xleftarrow{f} T \xrightarrow{g} B$ is jointly monic implies that h is a partial map because:

$$\begin{aligned}
hh^\circ &= (f^\circ x \cap g^\circ y)(f^\circ x \cap g^\circ y)^\circ \\
&= (f^\circ x \cap g^\circ y)(x^\circ f \cap y^\circ g) \\
&\leq f^\circ xx^\circ f \cap f^\circ xy^\circ g \cap g^\circ yx^\circ f \cap g^\circ yy^\circ g \\
&\leq f^\circ xx^\circ f \cap g^\circ yy^\circ g \\
&\leq f^\circ f \cap g^\circ g = 1_T
\end{aligned}$$

By lemma 2.1 (iv) we have $h^\circ h \geq 1_X \cap y^\circ gf^\circ x$. If $yx^\circ \leq R = gf^\circ$, then $1_X \leq y^\circ yx^\circ x \leq y^\circ gf^\circ x$ because x and y are maps, so $h^\circ h \geq 1_X \cap y^\circ gf^\circ x = 1_X$ as desired. $\square$

Corollary 4.1: *Any two tabulations of a morphism in an allegory $\mathcal{A}$ are isomorphic as spans.*

PROOF: Given two tabulations, the universal property gives morphisms of spans between the two in each direction and their composites are forced to be the identity. $\square$

Remark 4.2*:* Let A, B be objects of an allegory $\mathcal{A}$. Define a functor

$$M : Span_{\mathcal{M}ap(\mathcal{A})}(A, B) \to \mathcal{A}(A, B), \qquad M(A \xleftarrow{x} X \xrightarrow{y} B) = yx^\circ.$$

$(X, a, b) \xrightarrow{t} (Y, f, g)$ implies that $ba^\circ = gtt^\circ f^\circ \leq gv^\circ$ (t is a map) and functoriality follows by uniqueness. The universal property of tabulations says that for all morphisms $R : A \to B$, a tabulation of R is a terminal object in $(M \downarrow R)$. So if $\mathcal{A}$ is tabular, M has a right adjoint T that sends a morphism to a tabulation of it. By definition of tabulation, MT is the identity functor, so $Fix(MT) = \mathcal{A}(A, B)$. By lemma 4.1, $Fix(TM) = Mon_{\mathcal{M}ap(\mathcal{A})}(A, B)$. So the adjunction $M \dashv T$ restricts to an equivalence $Mon_{\mathcal{M}ap(\mathcal{A})}(A, B) \simeq \mathcal{A}(A, B)$, and in turn an isomorphism $Rel_{\mathcal{M}ap(\mathcal{A})}(A, B) \cong \mathcal{A}(A, B)$.

Lemma 4.5: *Let $\mathcal{A}$ be a tabular allegory, and $f : A \to B$ a map.*

(i) f is monic in $\mathcal{M}ap(\mathcal{A})$ iff $f^\circ f = 1_A$.

(ii) f is a cover in $\mathcal{M}ap(\mathcal{A})$ iff $ff^\circ = 1_B$.

Proof:

(i) Immediate from lemma 4.1 since f is monic iff (A, f, f) is jointly monic.

(ii) If f is a cover, let (T, t, t) be a tabulation of ff° with t monic. Such a tabulation exists by lemma 4.2 as $ff^\circ \leq 1_B$. Since $ff^\circ = tt^\circ$, the universal property of tabulations gives a morphism of spans $(A, f, f) \to (T, t, t)$, which in turn induces a factorization of f through the monomorphism t. But f is a cover, so t is an isomorphism and $t^\circ = t^{-1}$ and so $ff^\circ = tt^\circ = 1_B$. Conversely, if $ff^\circ = 1_B$ and f factors through a monomorphism $m : X \rightarrowtail B$ in $\mathcal{M}ap(\mathcal{A})$, then by lemma 4.3 (ii) and since m is a map, $1_B = ff^\circ \leq mm^\circ \leq 1_B$. So $mm^\circ = 1_B$ and by (i), $m^\circ m = 1_A$, so m is an isomorphism.

$\square$

Proposition 4.1: *Let $\mathcal{A}$ be a tabular allegory.*

(i) Let $A \xleftarrow{f} T \xrightarrow{g} B$ be maps. Then a span $A \xleftarrow{i} I \xrightarrow{j} B$ in $\mathcal{M}ap(\mathcal{A})$ is a tabulation of gf° iff it is an image of $A \xleftarrow{f} T \xrightarrow{g} B$ in $\mathcal{M}ap(\mathcal{A})$.

(ii) Let

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \\ & Y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

be maps. Then a span $X \xleftarrow{x} T \xrightarrow{y} Y$ is a limit of this diagram in $\mathcal{M}ap(\mathcal{A})$ iff it is a tabulation of $h^\circ f \cap k^\circ g$.

Proof:

(i) $\Rightarrow$: Let $A \xleftarrow{f} T \xrightarrow{g} B$ be a span in $\mathcal{M}ap(\mathcal{A})$ and let $A \xleftarrow{i} I \xrightarrow{j} B$ be a tabulation of gf°. By lemma 4.1, $A \xleftarrow{i} I \xrightarrow{j} B$ is a jointly monic pair. By the universal property of tabulations, $gf^\circ = ji^\circ$ implies that there exists a morphism of spans $h : T \to I$. It remains to show that $A \xleftarrow{i} I \xrightarrow{j} B$ is the minimal jointly monic pair through which $A \xleftarrow{f} T \xrightarrow{g} B$ factors. Suppose there is another jointly monic pair $A \xleftarrow{a} M \xrightarrow{b} B$ and a morphism of spans $k : T \to M$, i.e. $f = ka$ and $g = bk$. By lemma 4.1, $A \xleftarrow{a} M \xrightarrow{b} B$ tabulates ba°, so by the universal property of tabulations we get that $gf^\circ \leq ba^\circ$. But $gf^\circ = ji^\circ$ by definition of tabulation, so again by the universal property of tabulations, there exists a morphism of spans $s : I \to M$, as desired
$\Leftarrow$: Suppose (I, i, j) is an image of (T, f, g) in $\mathcal{M}ap(\mathcal{A})$. By lemma 4.1, since (I, i, j) is jointly monic it remains to verify that $ji^\circ = gf^\circ$. Let $c : X \mapsto I$ be the cover in the image factorization. Then $gf^\circ = jc(ic)^\circ = jcc^\circ i^\circ = ji^\circ$ since cc° is the identity because c is a cover.

(ii) $\Rightarrow$: Suppose (T, x, y) is a tabulation of $h^\circ f \cap k^\circ g$. $yx^\circ = h^\circ f \cap k^\circ g$ implies that $yx^\circ \leq h^\circ f$ and $yx^\circ \leq k^\circ g$, so $fx = hy$ and $gx = ky$. If $X \xleftarrow{z_1} Z \xrightarrow{z_2} Y$ is any other pair of maps such that $fz_1 = hz_2$ and $gz_1 = kz_2$, then $z_2 z_1^\circ \leq h^\circ f$ and $z_2 z_1^\circ \leq k^\circ g$, so $z_2 z_1^\circ \leq h^\circ f \cap k^\circ g$, hence by the universal property of tabulations, there exists a unique morphism $t : Z \to T$ such that $z_1 = xt$ and $z_2 = yt$, which is precisely the definition of the limit.
$\Leftarrow$: If (T, x, y) is the limit of this diagram in $\mathcal{M}ap(\mathcal{A})$, $fx = hy$ and $gx = ky$ implies that $yx^\circ \leq h^\circ f$ and $yx^\circ \leq k^\circ g$. If $R : X \to Y$ is any morphism tabulated by (T', r, s), such that $R \leq h^\circ f$ and $R \leq k^\circ g$, then $rs^\circ \leq h^\circ f$ and $rs^\circ \leq k^\circ g$ imply that $fs = hr$ and $gs = kr$, so by the universal property of the limit, there exists a unique morphism of spans $t : (T', r, s) \to (T, x, y)$. So $R = rs^\circ = xtt^\circ y^\circ \leq xy^\circ = h^\circ f \cap k^\circ g$, so $xy^\circ = h^\circ f \cap k^\circ g$.

□

Remark 4.3*:* In particular, by lemma 3.2, a pair of maps $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ is a pullback of $A \xrightarrow{f} X \xleftarrow{g} B$ iff (P, p_1, p_2) is a tabulation of $g^\circ f \cap g^\circ f = g^\circ f$, and $e : E \to X$ is an equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ iff (E, e, e) is a tabulation of $g^\circ f \cap 1_X$.

Proposition 4.2: *If $\mathcal{A}$ is a tabular allegory then $\mathcal{M}ap(\mathcal{A})$ is locally regular.*

Proof: We have shown the existence of images of jointly monic pairs, and by lemma 3.2 and the previous result, $\mathcal{M}ap(\mathcal{A})$ has pullbacks and equalizers. It remains to show that covers are stable under pullback. Suppose h is a cover. By lemma 4.5 $hh^\circ = 1_C$. Suppose

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "h"] \\ B \arrow[r, "k"'] & C \end{tikzcd}$$

is a pullback square. The square commutes so $h^\circ k = gf^\circ$, which implies that $1_B \leq k^\circ k = k^\circ 1_C k = k^\circ hh^\circ k = gf^\circ fg^\circ$. Using the modular law we get $1_B = 1_B \cap gf^\circ fg^\circ \leq (g \cap gf^\circ f)g^\circ \leq gg^\circ$. So $gg^\circ = 1_B$ as desired. □

4.2 Main Theorem of Tabular Allegories

Definition 4.2: A functor $F : \mathcal{A} \to \mathcal{B}$ between allegories is a *morphism of allegories* if $F(R \cap S) = FS \cap FR$ and $F(R^\circ) = (FR)^\circ$. A *locally regular functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves finite connected limits and covers.

Remark 4.4*:* A morphism of allegories also preserves the ordering of morphisms since $R \leq S \Leftrightarrow R = R \cap S$. If $\mathcal{C}$ and $\mathcal{D}$ are locally regular categories, then a locally regular functor also preserves images of spans. In any category, an object X is sub-terminal iff $(X, 1_X, 1_X)$ is a product cone. The product of spans is the same as the diamond shaped limit we have seen which is preserved by a locally regular functor. Such a functor also preserves covers, and we assume the categories have pullbacks, so it preserves images.

Theorem 4.1: *Let $\mathcal{C}$ be a category and let $\mathcal{A}$ be an allegory.*

(i) If $\mathcal{C}$ is locally regular, then $\mathcal{R}el(\mathcal{C})$ is tabular and $\mathcal{C} \cong \mathcal{M}ap(\mathcal{R}el(\mathcal{C}))$.

(ii) If $\mathcal{A}$ is tabular, then $\mathcal{M}ap(\mathcal{A})$ is locally regular and $\mathcal{A} \cong \mathcal{R}el(\mathcal{M}ap(\mathcal{A}))$.

Proof:

(i) $(-)_\bullet : \mathcal{C} \to \mathcal{M}ap(\mathcal{R}el(\mathcal{C}))$ defines a functor that sends every morphism to its graph and is the identity on objects. Its inverse sends a map ϕ to the unique morphism f such that $\cdot \xleftarrow{1} \cdot \xrightarrow{f} \cdot$ tabulates ϕ.

(ii) Define a functor $M : \mathcal{R}el(\mathcal{M}ap(\mathcal{A})) \to A$ as the identity on objects and for $R : A \looparrowright B$, $M(R) = ba^\circ$ where (X, a, b) is a representative of R. This does not depend on the choice of representative because (X, a, b) and (Y, a', b') represent the same relation iff there is an isomorphism $t : X \to Y$ such that $a = a't$ and $b = b't$ so $ba^\circ = a'tt^\circ b'^\circ = a'b'^\circ$. If $R : A \looparrowright B$ and $S : B \looparrowright C$ are represented by (X, r, r'), (Y, s, s') respectively, then SR is represented by (I, i, i'), the image of $(P, rp, s'p')$ where (P, p, p') is a pullback of (r', s). By lemma 4.1, $M(SR) = i'i^\circ = s'p'p^\circ r^\circ$, and by remark 4.3, $p^\circ p' = s^\circ r'$, so $i'i^\circ = s's^\circ r'r^\circ = M(S)M(R)$, so M is functorial. Moreover, M is full because $\mathcal{A}$ is tabular, and faithful since if (X, a, b), (Y, a', b') are monic spans such that $ba^\circ = b'a'^\circ$, then the universal property of tabulations implies they are isomorphic since they both tabulated the morphism $ba^\circ = b'a'^\circ$. (We can also see M is fully faithful from remark 4.2). So M is an isomorphism. M is order preserving since if $t : (X, a, b) \to (Y, a', b')$ is a map then $ba^\circ = a'tt^\circ b'^\circ \le a'b'^\circ$, and preserves $(-)^\circ$, so it is a morphism of allegories.

□

Remark 4.5*:* Any morphism of allegories between tabular allegories restricts to a locally regular functor between the categories of maps. Maps, tabulations, and being a cover (in the subcategory of maps) are all properties that we have seen are equational in the language of allegories, so they are preserved by morphisms of allegories. By lemma 4.1, equalizers and pullbacks are special cases of tabulations, so they too are preserved. Conversely, any locally regular functor between locally regular categories induces a morphism of allegories between the allegories of relations.

In the following chapters, we expand the result of the main theorem by adding more **Rel**-like structure to the allegories we consider, and examining how the additional structure affects the category of maps.

5 Allegories with Additional Structure

5.1 Regular Categories and Units

Definition 5.1: A *regular category* is a category with finite limits and images (of single morphisms) in which covers are stable under pullback.

Lemma 5.1: *A locally regular category has images of single morphisms.*

Proof: By the main theorem, $\mathcal{R}el(\mathcal{C})$ is tabular. Given a morphism $f : A \to B$, by lemma 4.2, the span $(A, f_\bullet, f_\bullet)$ in $\mathcal{M}ap(\mathcal{A})$ is tabulated by $(I, m_\bullet, m_\bullet)$ with $m_\bullet$ monic. By theorem 4.1, this tabulation is an image, so $m_\bullet$ is an image of $f_\bullet$. Apply the isomorphism of the main theorem. □

If P is a property that a category may have, then a category is *locally* P if all of its slice categories have property P. It can be shown that any slice category of a locally regular category is locally regular, and since slice categories always have terminal objects (the identity), all slice categories of a locally regular category are regular. However, this is not an equivalence. If a category has images of single morphisms, pullbacks, and covers are stable under pullback, then all of its slice categories are regular, but there is no guarantee it has equalizers. In other sources, locally regular may be defined differently. We now see how a terminal object affects the allegory of relations.

Definition 5.2: Let $\mathcal{A}$ be an allegory. A *unit* is an object U of $\mathcal{A}$ such that 1_U is the largest endomorphism of U, and for any object A, there exists a total morphism $R : A \to U$.

Lemma 5.2: *If U is a unit in an allegory $\mathcal{A}$, then it is a terminal object in $\mathcal{M}ap(\mathcal{A})$. If $\mathcal{A}$ is tabular, the converse holds.*

Proof: Suppose U is a unit. For any object A, there is a morphism $R : A \to U$ that is total, but it is also a partial map since RR° is an endomorphism on U, so $RR^\circ \le 1_U$. If $A \underset{g}{\overset{f}{\rightrightarrows}} U$, then because g is a map and fg° is an endomorphism on U we get $f \leq fg^\circ g \leq g$, so $f = g$ since maps are discreetly ordered. Conversely, if $\mathcal{A}$ is tabular and U terminal in $\mathcal{M}ap(\mathcal{A})$, for any A there is a map $f : A \to U$. It remains to show 1_U is maximal. But any tabulation $U \xleftarrow{f} X \xrightarrow{g} U$ of an endomorphism on U must have $f = g$ since U is terminal, so $gf^\circ = ff^\circ \leq 1_U$. □

In particular, we get the same kind of correspondence between regular categories and tabular allegories with a unit as in theorem 4.1.

5.2 Inverse Images

Let $\mathcal{C}$ be a category with pullbacks. For each morphism $f : A \to B$ there is a functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ called the *pullback (along f) functor* defined on objects by the following square being a pullback:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f^*(g)"] & X \arrow[d, "g"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

and on morphisms by:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f^*(g)"] \arrow[dd, bend right=60, "f^*(t)"'] & X \arrow[d, "g"] \arrow[dd, bend left=60, "t"] \\
A \arrow[r, "f"] & B \\
Y' \arrow[r] \arrow[u, "f^*(g')"] & X' \arrow[u, "g'"]
\end{tikzcd}$$

where $f^*(t)$ is induced by the universal property of the lower pullback square.
Since monos are stable under pullback, this restricts to a functor $f^* : Sub(B) \to Sub(A)$ which we will call the *inverse image* functor, since this is precisely what it cooresponds to in **Set**. In **Set**, this functor has a left adjoint, namely the direct image.

Lemma 5.3: *Let $\mathcal{C}$ be a category with pullbacks. Then $\mathcal{C}$ has (unary) images iff for all $f : A \to B$, f^* has a left adjoint $\exists_f$.*

Proof: Suppose $\mathcal{C}$ has images. For $f : A \to B$, define $\exists_f : Sub(A) \to Sub(B)$ by precomposing (a representative) with f and the taking the (isomorphism class of an) image of the resulting morphism. (This is analogous to the case in **Set** where we "apply" a function to a subset). This is functorial because it is a composite of functors:

$$Sub(A) \to \mathcal{C}/A \xrightarrow{f(-)} \mathcal{C}/B \xrightarrow{im} Sub(B)$$

The unit and counit are then the conditions $h \leq f^*\exists_f(h)$ and $\exists_f f^*(k) \leq k$ which follow from the diagrams:

$$\begin{tikzcd}
X \arrow[d, tail, "h"] \arrow[dr, dashed, "\exists! t"] \arrow[ddr, bend left=50, "c"] & \\
A \arrow[d, "f"] & P \arrow[l, tail, "f^*\exists_f(h)"] \arrow[d] \\
B & I \arrow[l, tail, "\exists_f(h)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[ddl, tail, bend right=50, "\exists_f f^*(k)"'] \arrow[ddr, dashed, bend left=40, "\exists! t"] & \\
A \arrow[d, "f"] & & P \arrow[ll, tail, "f^*(k)"] \arrow[ul, tail, "c"'] \arrow[d] \\
B & & X \arrow[ll, tail, "k"]
\end{tikzcd}$$

where in the left diagram t comes from the universal property of the pullback, and in the right diagram from the universal property of the image.

Conversely, $\exists_f(1_A)$ can be seen to satisfy the conditions of an image of f. □

Remark 5.1*:* If we compose a relation R tabulated by $B \xleftarrow{b} R \xrightarrow{c} C$ with the graph of a morphism $f : A \to B$, the result is tabulated by the pullback:

$$\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
A \arrow[dr, "f"] & & R \arrow[dl, "b"'] \arrow[dr, "c"] & \\
 & B & & C
\end{tikzcd}$$

since (p_1, cp_2) is necessarily jointly monic. In a category with finite products, we can express this in a single pullback square:

$$\begin{tikzcd}
Q \arrow[r, "q_2"] \arrow[d, tail, "q_1"] & R \arrow[d, tail, "(b,c)"] \\
A \times C \arrow[r, "f \times 1_C"] & B \times C
\end{tikzcd}$$

where (p_1, cp_2) and q_1 both represent the same subobject of $A \times C$. Viewing R as the monomorphism (b, c), we can write $Rf_\bullet = (f \times 1_C)^*(R)$.
If we compose R with the reciprocal of a graph of a morphism $g : B \to D$ we get:

$$\begin{tikzcd}
 & & R \arrow[dl, "b"'] \arrow[dr, "1_R"] & & \\
 & B \arrow[dl, "g"'] \arrow[dr, "1_B"] & & R \arrow[dl, "b"'] \arrow[dr, "c"] & \\
D & & B & & C
\end{tikzcd}$$

and the image of the span $D \xleftarrow{gb} R \xrightarrow{c} C$ coorresponds to the image of the monomorphism $(gb, c) : R \rightarrowtail D \times C$. But $(gb, c) = (g \times 1_C)(b, c)$ by basic properties of products. So $Rg^\bullet = im((gb, c)) = im((g \times 1_C)(b, c)) = \exists_{(g \times 1_C)}(R)$.
So for any relation $R : A \nrightarrow B$ tabulated by $A \xleftarrow{f} T \xrightarrow{g} B$, and $S : B \nrightarrow C$:

$$SR = Sg_\bullet f^\bullet = (g \times 1_C)^*(S)f^\bullet = \exists_{(f \times 1_C)}(g \times 1_C)^*(S).$$

Right composition by R is then given by the composite of functors:

$$\begin{tikzcd}
Sub(B \times C) \arrow[r, "(-)g_\bullet = (g \times 1_C)^*"] \arrow[rr, bend left, "(-)R"] & Sub(T, C) \arrow[r, "(-)f^\bullet = \exists_{f \times 1_C}"] & Sub(A, C)
\end{tikzcd}$$

5.3 Union Allegories and Coherent Categories

Definition 5.3: A *union allegory* is an allegory whose hom-posets have finite unions that are preserved by composition.

Remark 5.2*:* If the allegory is not tabular, unions and intersections may not distribute. Any modular lattice that is not a distributive lattice is an example of this.

Definition 5.4: A *coherent category* is a regular category such that $Sub(A)$ has finite unions (coproducts) that are preserved by f^*.

Proposition 5.1: *Let $\mathcal{A}$ be a tabular allegory with a unit. Then $\mathcal{M}ap(\mathcal{A})$ is a coherent category iff $\mathcal{A}$ is a union allegory.*

Proof: By theorem 4.1 it suffices to show that for a regular category $\mathcal{C}$, $\mathcal{R}el(\mathcal{C})$ is a union allegory iff $\mathcal{C}$ is coherent. If $\mathcal{C}$ is coherent, since $Rel(A,B) \cong Sub(A \times B)$, it has unions. Given $A \xrightarrow{Q} B \underset{S}{\overset{R}{\rightrightarrows}} C$, by remark 5.1, $(R \cup S)Q = \exists_{(f \times 1_C)}(g \times 1_C)^*(R \cup S)$ for (f,g) a tabulation of Q. But $\exists_{(f \times 1_C)}$ preserves unions because it is a left adjoint, and $(g \times 1_C)^*$ preserves unions by assumption. The anti-involution implies the same for composition on the other side. Conversely, if $\mathcal{R}el(\mathcal{C})$ is a union allegory, then for any object A, $Sub(A) \cong Rel(A,1)$ gives unions, and by remark 5.1, the pullback functor f^* cooresponds to composition with $f_\bullet$, which preserves unions by assumption.

$\square$

6 Power Allegories

6.1 Power Objects

In the category of sets, every relation R can be viewed as a function, $f_R : A \to \mathcal{P}(B)$ that maps an element of A to all elements in B it is related to via R (i.e. the "curried" characteristic function). So $aRb \Leftrightarrow b \in f_R(a)$. Viewing $\in$ as a relation $\in \subset \mathcal{P}(B) \times B$, we can write $R = \in \circ f_R$ in **Rel**. Abstracting from this, the next structure we consider is an allegory $\mathcal{A}$ for which the inclusion $\mathcal{M}ap(\mathcal{A}) \hookrightarrow \mathcal{A}$ has a right adjoint.

Let us examine the structure that this adjunction imposes on $\mathcal{M}ap(\mathcal{A})$ when $\mathcal{A}$ is tabular. In this case, the inclusion $\mathcal{M}ap(\mathcal{A}) \hookrightarrow \mathcal{A}$ cooresponds to the functor that sends morphisms to their graphs in a locally regular category. Let $\mathcal{C}$ be locally regular and $\Gamma : \mathcal{C} \to \mathcal{R}el(\mathcal{C})$ be the functor that sends morphisms to their graphs. Γ has a right adjoint iff $(\Gamma \downarrow A)$ has a terminal object for all objects A of $\mathcal{C}$. This means that for any object A, there is an object PA and a relation $\in_A$: $PA \looparrowright A$ such that for any relation $\phi : B \looparrowright A$, there exists a unique morphism $f : B \to PA$ such that $\phi = \in_A f_\bullet$:

$$\begin{tikzcd} PA \arrow[d, "\in_A"'] & B \arrow[l, dashed, "\exists! f_\bullet"'] \arrow[dl, "\phi"] \\ A & \end{tikzcd}$$

This composition of relations is particularly simple as it does not require taking images, since the pullback will necessarily be a jointly monic pair, which allows this property to be stated as follows:

Definition 6.1: Let $\mathcal{C}$ be a category with pullbacks. A *power object* of an object A is an object PA together with a jointly monic pair $PA \leftarrow \in_A \rightarrow A$ such that for any other jointly monic pair $B \leftarrow R \rightarrow A$, there exists a unique morphism $f : R \to PA$ for which there is a pullback square:

$$\begin{tikzcd} B \arrow[d, "f"'] & R \arrow[l] \arrow[d] \arrow[dr] & \\ PA & \in_A \arrow[l] \arrow[r] & A \end{tikzcd}$$

We say a category *has power objects* if every object has one.

Lemma 6.1: *If $\mathcal{A}$ is a tabular allegory, the inclusion $\mathcal{M}ap(\mathcal{A}) \hookrightarrow \mathcal{A}$ has a right adjoint iff $\mathcal{M}ap(\mathcal{A})$ has power objects.*

Proof: Immediate from definitions and theorem 4.1. $\square$

6.2 Division Allegories and Heyting Categories

Our goal now is to give an axiomization of allegories such that the inclusion $\mathcal{M}ap(\mathcal{A}) \hookrightarrow \mathcal{A}$ has a right adjoint in some equational way. The obvious approach would be to add to the language a symbol $\in_{(-)}$ relating objects to morphisms, and a symbol $\lceil(-)\rceil$ which would relate morphisms to maps, such that for all morphisms R, $\lceil R \rceil$ is the unique map for which $R = \in_{(cod(R))} \lceil R \rceil$. First we introduce an operation that will simplify this axiomization in a way that $\lceil(-)\rceil$ need not be taken as primitive.

Definition 6.2: A *division allegory* is an allegory $\mathcal{A}$ such that for every morphism $\phi : A \to B$ and every object C, the order preserving map $(-)\phi : \mathcal{A}(B,C) \to \mathcal{A}(A,C)$ has a right adjoint, $(-)/\phi : \mathcal{A}(A,C) \to \mathcal{A}(B,C)$. We call this operation *right division* by ϕ.

$$\begin{tikzcd} \mathcal{A}(B,C) \arrow[rr, bend left, "(-)\phi"] \arrow[rr, phantom, "\perp"] & & \mathcal{A}(A,C) \arrow[ll, bend left, "(-)/\phi"] \end{tikzcd}$$

The hom-set definition of the adjunction says that right division is an operation on a pair a morphisms $C \xleftarrow{R} A \xrightarrow{\phi} B$ such that $R/\phi : B \to C$ is the optimal morphism that fits into the following semi-commuting diagram as indicated by the inequality:

$$\begin{tikzcd} & A \arrow[dl, "\phi"'] \arrow[dd, "R"] \\ B \arrow[dr, dashed] \arrow[r, phantom, "\leq"] & {} \\ & C \end{tikzcd}$$

$$\forall\, T : B \to C \quad T\phi \leq R \Leftrightarrow T \leq R/\phi$$

Since right adjoints preserve limits, we have $(R \cap R')/\phi = R/\phi \cap R'/\phi$. The unit of the adjunction is the condition $R \leq (R\phi)/\phi$ for all $R : B \to C$, and the counit is $(R/\phi)\phi \leq R)$ for all $R : A \to C$.
The anti-involution implies that $\phi(-) : \mathcal{A}(X,A) \to \mathcal{A}(X,B)$ has a right adjoint $\phi \setminus (-)$ that we call *left division* by ϕ.

$$\begin{tikzcd} \mathcal{A}(X,A) \arrow[rr, bend left, "\phi(-)"] \arrow[rr, phantom, "\perp"] & & \mathcal{A}(X,B) \arrow[ll, bend left, "\phi\setminus(-)"] \end{tikzcd}$$

It is given by $\phi \setminus R = (R^\circ/\phi^\circ)^\circ$, and the unit $R \leq \phi \setminus (\phi R)$ and counit $\phi(\phi \setminus R \leq R$ follow from the fact that $(-)^\circ$ preserves order and is an anti-involution. Left division is an operation on a pair of morphisms with common codomain that gives the universal solution to the following semi-commuting diagram:

$$\begin{tikzcd} & B \\ A \arrow[ur, "\phi"] \arrow[r, phantom, "\leq"] & {} \\ & X \arrow[ul, dashed] \arrow[uu, "R"'] \end{tikzcd}$$

$$\forall\, T : X \to A \quad \phi T \leq R \Leftrightarrow T \leq \phi \setminus R$$

In **Set**, this relation is given by $x(\phi \setminus R)a \Leftrightarrow \forall b \in B\ a\phi b \Rightarrow xRb$.

Definition 6.3: A *Heyting category* is a coherent category such that for every morphism $f : A \to B$, the inverse image functor $f^* : Sub(B) \to Sub(A)$ has a right adjoint $\forall_f$.

Lemma 6.2: *Let $\mathcal{A}$ be a tabular union allegory with a unit. Then $\mathcal{A}$ is a division allegory iff $\mathcal{M}ap(\mathcal{A})$ is a Heyting category.*

Proof: By our previous results, it suffices to show that if $\mathcal{C}$ is a coherent category, then it is a Heyting category iff $\mathcal{R}el(\mathcal{C})$ is a division allegory. By remark 5.1, if $\mathcal{C}$ is a Heyting category and $\phi : A \nrightarrow B$ is tabulated by (f, g), then

$$\begin{tikzcd}
Sub(B \times C) \arrow[rr, bend left=20, "(-)g_\bullet = (g \times 1_C)^*"] \arrow[rrrr, bend left=40, "(-)\phi"] \arrow[rr, phantom, "\perp"] & & Sub(T, C) \arrow[ll, bend left=20, "\forall_{g \times 1_C}"] \arrow[rr, bend left=20, "(-)f^\bullet = \exists_{f \times 1_C}"] \arrow[rr, phantom, "\perp"] & & Sub(A, C) \arrow[ll, bend left=20, "(f \times 1_C)^*"]
\end{tikzcd}$$

and the composites form an adjunction. Conversely, we have $Sub(A) \cong Rel(A, 1)$, and precomposing a relation with a graph $f_\bullet$ corresponds to applying the inverse image functor f^* which is assumed to have a right adjoint. $\square$

The following fact will be useful.

Lemma 6.3: *In a division allegory, $(\phi \setminus \psi)(\psi \setminus \chi) \leq \phi \setminus \chi$ whenever defined.*

Proof: $\phi(\phi \setminus \psi)(\psi \setminus \chi) \leq \psi(\psi \setminus \chi) \leq \chi$ by the counit of the adjunction, so $(\phi \setminus \psi)(\psi \setminus \chi) \leq \phi \setminus \chi$ $\square$

6.3 Power Allegories

Given morphisms $\phi : B \to A$, $\psi : C \to A$ with common codomain, we can form the right divisions in both ways:

$$\begin{tikzcd}
B \arrow[r, "\phi"] \arrow[d, shift right, "\psi \setminus \phi"'] & A \\
C \arrow[u, shift right, "\phi \setminus \psi"'] \arrow[ur, "\psi"']
\end{tikzcd}$$

The following derived operation will be what we use to define the axiomization we are seeking.

Definition 6.4: Given morphisms $\phi : B \to A$, $\psi : C \to A$ in a division allegory, their *symmetric division* is the relation $(\phi|\psi) : C \to B$ given by $(\phi|\psi) = (\phi \setminus \psi) \cap (\psi \setminus \phi)^\circ$.

This relation is "symmetric" in the sense that $(\phi|\psi)^\circ = (\psi|\phi)$. In **Set**, $c(\phi|\psi)b$ iff c and b are related to the same elements of A. In particular, $(\phi|\phi) = 1_B$ would mean that different elements of B are related to different elements of A under ϕ.

Definition 6.5: A *power allegory* is a division allegory such that for every object A, there is a morphism $\in_A : PA \to A$ such that $(\in_A \mid \in_A) = 1_{PA}$ and $1_B \leq (R \setminus \in_A)(\in_A \setminus R)$ for all $R : B \to A$.

In **Set**, $(\in_A \mid \in_A) = 1_{PA}$ is saying that subsets are equal iff they have the same elements, also known as the axiom of extensionality. Unraveling the definitions in the condition $1_B \leq (R \setminus \in_A)(\in_A \setminus R)$, it becomes $\forall a \in A\, \exists X \in \mathcal{P}(A)\; a \in X \Leftrightarrow bRa$, that is, there exists a subset of A of all those elements related to b by R (note the we flipped the direction of $\in$ to its usual one here). In other words, this is the axiom of comprehension.

The next lemma shows that this axiomization gives us exactly what we wanted:

Lemma 6.4: *Let $R : B \to A$ be a morphism in a power allegory, and let $\lceil R \rceil = (\in_A | R) : B \to PA$. then:*

(i) *$\lceil R \rceil$ is a map.*

(ii) *$\in_A \lceil R \rceil = R$*

(iii) *$\lceil R \rceil$ is the unique map satisfying (ii).*

Proof:

(i) Using lemma 6.3 and unraveling definitions we get:

$$\begin{aligned} \lceil R \rceil \lceil R \rceil^\circ &= (\in_A | R)(R | \in_A) = ((\in_A \backslash R) \cap (R \backslash \in_A)^\circ)((R \backslash \in_A) \cap (\in_A \backslash R)^\circ) \\ &\leq (\in_A \backslash R)(R \backslash \in_A) \cap (R \backslash \in_A)^\circ (\in_A \backslash R)^\circ \\ &= (\in_A \backslash R)(R \backslash \in_A) \cap (\in_A^\circ / R^\circ)(R^\circ / \in_A^\circ) \\ &\leq (\in_A \backslash \in_A) \cap (\in_A^\circ / \in_A^\circ) \\ &= (\in_A \backslash \in_A) \cap (\in_A \backslash \in_A)^\circ \\ &= (\in_A | \in_A) = 1_{PA} \end{aligned}$$

Setting $X = (\in_A \backslash R)$, $Y = (R \backslash \in_A)^\circ$, we have $\lceil R \rceil = X \cap Y$, and so by lemma 2.1 (iv) we have

$$\lceil R \rceil^\circ \lceil R \rceil = (X \cap Y)^\circ (X \cap Y) \geq Y^\circ X \cap 1_B = (R \backslash \in_A)(\in_A \backslash R) \cap 1_B = 1_B$$

since by assumption $1_B \leq (R \backslash \in_A)(\in_A \backslash R)$.

(ii) By the counit in the definition of division we get:

$$\in_A \lceil R \rceil = \in_A ((\in_A \backslash R) \cap (R \backslash \in_A)^\circ) \leq \in_A (\in_A \backslash R) \leq R$$

and since $\lceil R \rceil$ is a map and $\lceil R \rceil^\circ = (\in_A \backslash R)^\circ \cap (R \backslash \in_A) \leq (R \backslash \in_A)$:

$$R \leq R \lceil R \rceil^\circ \lceil R \rceil \leq R(R \backslash \in_A) \lceil R \rceil \leq \in_A \lceil R \rceil.$$

(iii) If $f : B \to PA$ is a map such that $\in_A f = R$, then $\in_A f \leq R$ and so by universal property of left division, $f \leq (\in_A \backslash R)$. In addition, $f R^\circ = f f^\circ \in_A^\circ \leq \in_A^\circ$, so by the universal property of right division, $f \leq (\in_A^\circ / R^\circ) = (\in_A \backslash R)^\circ$. $f \leq \lceil R \rceil$ so they are equal since maps are discreetly ordered.

□

Corollary 6.1: *In a power allegory $\mathcal{A}$, the inclusion $\mathcal{M}ap(\mathcal{A}) \overset{i}{\hookrightarrow} \mathcal{A}$ has a right adjoint.*

Proof: By the previous lemma, for all objects A, $\in_A$ is a terminal object in $(i \downarrow A)$. In particular $\in$ can be taken to be the counit of the adjunction, and the object map can be taken to be $B \mapsto PB$. □

By what we have shown so far, the category of maps of a tabular power allegory with a unit is a regular category with power objects. It turns out that this is redundant:

Definition 6.6: A *topos* is a category with finite limits and power objects.

The following result is our final classification for the structure we considered for allegories. One direction follows immediately by our previous results and definition, and the other direction follows from the fact that a topos is a Heyting category, which we have not proven here, so we just state the result without a full proof:

Fact: *If $\mathcal{A}$ is a tabular allegory with a unit, then $\mathcal{A}$ is a power allegory iff $\mathcal{M}ap(\mathcal{A})$ is a topos.*

Remark 6.1*:* Notice that the category of maps of the allegory of fuzzy relations is a topos (**Set**), but it is not tabular. So being tabular is not a consequence of division or being a power allegory.

7 Conclusion

Constructing categories of relations, we arrived at the notions of locally regular categories and tabular allegories. We then added structure to the allegories and examined how it affected the category of maps when the allegories were tabular. Additional information can be found in in [1] A.3 and [2]. The important result that every topos is a Heyting category which we did not prove can be found in [1]A.2. Another omitted result concerns the splitting of symmetric idempotents. Similarly to how the Karoubian completion is the optimal way to split all idempotent in a category, if an allegory is pre-tabular (every morphism is contained in one that has a tabulation), the process of splitting symmetric ($R = R^{\circ}$) idempotents gives the optimal way to make a pre-tabular category tabular. This is discussed in [1]A.3.3 and in [2] throughout the section on allegories. For example, $Mat(\mathcal{V})$ is pre-tabular, since the constant functions are maximal in each hom-set and have tabulations.

References

[1] P.T. Johnstone, *Sketches of an Elephant: A Topos Theory Compendium* (Clarendon Press, 2002)

[2] P. J. Freyd and A. Scedrov, *Categories, Allegories* (North-Holland, 1990)