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# A modified Lavrentiev iterative regularization method for analytic continuation

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ABSTRACT

We consider the problem of numerical analytic continuation of an analytic function $f(z) = f(x+iy)$ on a strip domain $\Omega_+ = \{z = x+iy \in \mathbb{C} | x \in \mathbb{R}, 0 < y < y_0\}$, where the data is given approximately only on the line $y = 0$. This is a severely ill-posed problem. Motivated by the advantage of iterative methods for solving ill-posed problems, we propose a new modified iterative method to solve this problem under both a priori and a posteriori parameter choice rules. Moreover, some sharp error estimates between the exact solution and its approximation are proved. Some interesting numerical examples are conducted for showing that the newly-developed method works well.



## 1. Introduction

The analytic continuation is an old yet persistent problem, which arises from many practical applications [1–4], moreover, analytic continuation is a very useful tool for solving ill-posed problems, especially in medical imaging [5,6] and integral transformation [7].

In this study, we consider the following problem:

*Let the function* $f(z) := f(x+iy)$ *be an analytic function in* $\overline{\Omega}_+$. *And* $\Omega_+ = \{z = x+iy \in \mathbb{C} | x \in \mathbb{R}, 0 < y < y_0, y_0$ *is a positive constant*$\}$, *where* $i$ *is the imaginary unit. The data is only given on the real axis, i.e.,* $f(z)|_{y=0} = f(x)$ *is known approximately, and the noisy data is denoted by* $f^\delta(x)$. *We want to extend* $f(z)$ *analytically from this data to the whole domain* $\Omega_+$.

Recently the problem has been well studied. It is well-known that this problem is seriously ill-posed problem [8]. Some regularization theory [9–12] should be applied for solving this problem. In [13], Hào presented a mollification method for solving this problem, in [14], Deng considered a new mollification method, in [15], Fu gave a modified Tikhonov method based on Fourier transform, in [16], Fu presented a spectral cut-off method, in [17], Feng considered a wavelet method for solving this problem, in [18], Cheng investigated the Landweber-type iteration method. Until now the theoretical results of most of these works are limited to the following cases: (I) the a priori regularization parameter choice rules; (II) non-iterative method. Motivated by the advantage of iterative methods for this problem, in this paper, we investigate a new iteration method which is derived from the iterated Tikhonov method [8]. We will show that the new method is of optimal order under the usual a priori and a posteriori parameter choice rules.

Consider that the iterated Tikhonov method was successful in dealing with many ill-posed problems. However the iteration Tikhonov method requires the numerical computation of the adjoint operator $A^*$ of the forward operator $A$. In order to avoid computing the adjoint operator $A^*$, we appeal to the Lavrentiev method. Based on the idea of [15,16] and the fact that the forward operator is self-adjoint, we devise a new modified Lavrentiev iteration method to deal with the

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problem. Our contributions are twofold. On the one hand, we make a systematic study on the new Lavrentiev iteration method including the a priori and a posteriori parameter choice rules. On the other hand, the new method improves over the classical methods, please refer to Section 4.

Let $\hat{g}$ denote the Fourier transform of function $g(x)$ defined by

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x)dx \tag{1.1}$$

and $\|\cdot\|_p$ denotes the norm in the Sobolev space $H^p(\mathbb{R})$ defined by

$$\|g\|_p := \left( \int_{-\infty}^{\infty} (1+\xi^2)^p |\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{1.2}$$

When $p = 0$, $\|\cdot\|_0 := \|\cdot\|$ denotes the $L^2(\mathbb{R})$ norm, or equivalently, by the Parseval formula

$$\|g\|_{L^2(\mathbb{R})} = \left( \int_{-\infty}^{\infty} |\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{1.3}$$

The subscript $L^2(\mathbb{R})$ will be left out below when there is no confusion.

Throughout this paper, we assume that the solution which is to seek

$$f(\cdot + iy) \in L^2(\mathbb{R}) \quad \text{for} \quad 0 < y < y_0.$$

The problem can be solved as follows:

$$\widehat{f(\cdot + iy)}(\xi, y) = e^{-y\xi} \hat{f}(\xi), \tag{1.4}$$

or

$$e^{y\xi} \widehat{f(\cdot + iy)}(\xi, y) = \hat{f}(\xi). \tag{1.5}$$

By using the inverse Fourier transform technique with respect to the variable $x$, we have

$$f(z) = f(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(x+iy)\xi} \hat{f}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-y\xi} \hat{f}(\xi) d\xi,\ 0 < y < y_0. \tag{1.6}$$

Especially at the boundary $y = y_0$, we have the formula

$$\widehat{f(\cdot + iy_0)}(\xi, y_0) = e^{-y_0\xi} \hat{f}(\xi), \tag{1.7}$$

and therefore

$$\hat{f}(\xi) = e^{y_0\xi} \widehat{f(\cdot + iy_0)}(\xi, y_0). \tag{1.8}$$

Since the factor $e^{-y\xi} \to +\infty$ as $\xi \to -\infty$, and considering the boundedness of $f(x+iy)$, the formula (1.6) implies a rapid decay of the data $f(x)$. But such a decay is not likely to occur in the measured noisy data $f^\delta(x)$. So, a small perturbation of $f(x)$ can blow up and completely destroy the solution $f(x+iy)$. It is easy to see the ill-posedness of the problem, which is caused by the negative high frequencies. It is impossible to stably solve the problem using classical methods and requires regularization theory. In the present paper, we will study problem (1.5) by using the modified Lavrentiev iteration method.

The organization of this paper is as follows. In Section 2, we will obtain the Hölder-type error estimate for the a priori choice rule. The a posterior choice rule is given in Section 3 and we provide some numerical examples in Section 4 to show the effectiveness of the proposed method.

## 2. The error estimate with a priori parameter choice

We assume the exact data $f(x)$ and the measured data $f^\delta(x)$ both belong to $L^2(\mathbb{R})$, and satisfy

$$\|f - f^\delta\| \le \delta, \tag{2.1}$$

where $\delta > 0$ denotes the noisy level.

For simplicity, we assume only the following a priori information for the exact solution holds:

$$\|f(\cdot + iy_0)\| = \|\widehat{f(\cdot + iy_0)}\| \le E, \tag{2.2}$$

where $E$ is a fixed positive constant.

In order to solve this problem, from (1.4) we need to consider the problem as an operator equation which is given by

$$A_y \widehat{f(\cdot + iy)}(\xi, y) = \hat{f}(\xi), \tag{2.3}$$

where $A_y = e^{y\xi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a multiplication operator.

Based on the operator equation (2.3), we introduce the following modified Lavrentiev iteration:

$$\widehat{f_k^\delta(\cdot+iy)}(\xi)=\widehat{f_{k-1}^\delta(\cdot+iy)}(\xi)+(\alpha I+A_y)^{-1}(\hat{f}^\delta(\xi)-A_y\widehat{f_{k-1}^\delta(\cdot+iy)}(\xi)),\ k=1,2,\ldots, \tag{2.4}$$

with initial guess $\widehat{f_0^\delta(\cdot+iy)}(\xi)$ and a positive constant $\alpha$.

From formula (2.4), by using an argument of induction, we can obtain the expression of the iteration scheme in the frequency domain:

$$\begin{aligned}\widehat{f_k^\delta(\cdot+iy)}(\xi)&=\frac{\alpha}{\alpha+e^{y\xi}}\widehat{f_{k-1}^\delta(\cdot+iy)}(\xi)+\frac{1}{\alpha+e^{y\xi}}\hat{f}^\delta(\xi)\\&=\left(\frac{\alpha}{\alpha+e^{y\xi}}\right)^k\widehat{f_0^\delta(\cdot+iy)}(\xi)+\frac{\hat{f}^\delta(\xi)}{\alpha+e^{y\xi}}\sum_{i=1}^{k-1}\left(\frac{\alpha}{\alpha+e^{y\xi}}\right)^i\\&=\left(\frac{\alpha}{\alpha+e^{y\xi}}\right)^k\widehat{f_0^\delta(\cdot+iy)}(\xi)+\left[1-\left(\frac{\alpha}{\alpha+e^{y\xi}}\right)^k\right]e^{-y\xi}\hat{f}^\delta(\xi).\end{aligned}\tag{2.5}$$

We let $\widehat{f_0^\delta(\cdot+iy)}(\xi)=0$ for the convenience of computation.

Then

$$\widehat{f_k^\delta(\cdot+iy)}(\xi)=\left[1-\left(\frac{\alpha}{\alpha+e^{y\xi}}\right)^k\right]e^{-y\xi}\hat{f}^\delta(\xi). \tag{2.6}$$

Comparing expression (2.6) with (1.4), we see that the filter $1-(\frac{\alpha}{\alpha+e^{y\xi}})^k$ attenuates as the negative frequency, we can use another better decay filter $1-(\frac{\alpha}{\alpha+e^{y_0\xi}})^k$ to replace the original one. Then from the forthcoming sections, we will obtain a better convergence result. Thus we get a novel iteration solution:

$$\widehat{f_k^\delta(\cdot+iy)}(\xi)=\left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right]e^{-y\xi}\hat{f}^\delta(\xi). \tag{2.7}$$

Or equivalently, the iterative regularization solution of the problem is given by

$$f_k^\delta(x+iy)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}\widehat{f_k^\delta(\cdot+iy)}(\xi)d\xi. \tag{2.8}$$

**Theorem 2.1.** *Let $f(x+iy)$ be the solution with exact data, and $f_k^\delta(x+iy)$ be its regularization approximation given by (2.8). Let assumptions (2.1) and (2.2) be satisfied and take $k=[(\frac{\alpha E}{\delta})^{\frac{y_0}{2y_0-y}}]$, where $[t]$ denotes the largest integer not exceeding $t$, then there holds the estimate*

$$\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\|\le C\delta^{1-\frac{y_0}{2y_0-y}}E^{\frac{y_0}{2y_0-y}}[1+o(1)], \tag{2.9}$$

*where $C=2\alpha^{\frac{(y_0-y)^2}{y_0(2y_0-y)}}$ is a constant independent of $\delta$ and $E$.*

**Proof.** By using the Parseval formula in Fourier analysis and the triangle inequality, we have

$$\begin{aligned}\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\|&=\|\widehat{f(\cdot+iy)}(\cdot,y)-\widehat{f_k^\delta(\cdot+iy)}(\cdot,y)\|\\&\le\|\widehat{f(\cdot+iy)}(\cdot,y)-\widehat{f_k(\cdot+iy)}(\cdot,y)\|\\&\quad+\|\widehat{f_k(\cdot+iy)}(\cdot,y)-\widehat{f_k^\delta(\cdot+iy)}(\cdot,y)\|\\&:=I_1+I_2.\end{aligned}\tag{2.10}$$

For the first term $I_1$ on the right-hand side of (2.10), due to (1.8), we have

$$\begin{aligned} I_1^2 &= \|f(\cdot+iy)-f_k^\delta(\cdot+iy)\|^2 \\ &= \int_{-\infty}^{\infty}\left|e^{-y\xi}\left[1-1+\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right]\hat{f}(\xi)\right|^2 d\xi \\ &= \int_{-\infty}^{\infty}\left|\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k e^{-y\xi}\hat{f}(\xi)\right|^2 d\xi \\ &= \int_{-\infty}^{\infty}\left|\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k e^{(y_0-y)\xi}\widehat{f(\cdot+iy_0)}\right|^2 d\xi \\ &\le E^2\max_{\xi\in\mathbb{R}}\left[\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k e^{(y_0-y)\xi}\right]^2. \end{aligned} \tag{2.11}$$

Denoting $g(\xi)=(\frac{\alpha}{\alpha+e^{y_0\xi}})^k e^{(y_0-y)\xi}$, then

$$\begin{aligned} g'(\xi) &= k\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^{k-1}\left[-\frac{\alpha y_0 e^{y_0\xi}}{(\alpha+e^{y_0\xi})^2}\right]e^{(y_0-y)\xi}+\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k(y_0-y)e^{(y_0-y)\xi} \\ &= \left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k e^{(y_0-y)\xi}\left[-\frac{ky_0e^{y_0\xi}}{\alpha+e^{y_0\xi}}+y_0-y\right]. \end{aligned}$$

Setting $g'(\xi)=0$, we have $\xi=\frac{1}{y_0}\ln\frac{\alpha(y_0-y)}{(k-1)y_0+y}$, then $g(\xi)$ has a unique maximal value at $\xi=\frac{1}{y_0}\ln\frac{\alpha(y_0-y)}{(k-1)y_0+y}$. Therefore,

$$\begin{aligned} \max_{\xi\in\mathbb{R}} g(\xi) &= \left(\frac{(k-1)y_0+y}{ky_0}\right)^k\left[\frac{\alpha(y_0-y)}{(k-1)y_0+y}\right]^{1-\frac{y}{y_0}} \\ &= \left(\frac{(k-1)y_0+y}{ky_0}\right)^{k-1}\frac{(k-1)y_0+y}{ky_0}\left[\frac{\alpha(y_0-y)}{(k-1)y_0+y}\right]^{1-\frac{y}{y_0}} \\ &= \left[1-\frac{y_0-y}{ky_0}\right]^{k-1}\frac{(\alpha(y_0-y))^{1-\frac{y}{y_0}}}{ky_0[(k-1)y_0+y]^{-\frac{y}{y_0}}} \\ &\le [\alpha(y_0-y)]^{1-\frac{y}{y_0}}\left(\frac{1}{ky_0}\right)^{1-\frac{y}{y_0}} \\ &\le \alpha^{1-\frac{y}{y_0}}\left(\frac{1}{k}\right)^{1-\frac{y}{y_0}}. \end{aligned} \tag{2.12}$$

Then

$$I_1^2 \le \alpha^{2(1-\frac{y}{y_0})}\left(\frac{1}{k}\right)^{2(1-\frac{y}{y_0})}E^2. \tag{2.13}$$

For the second term $I_2$ on the right-hand side of (2.10) and in terms of (2.7), we have

$$\begin{aligned} I_2 &= \|f_k\widehat{(\cdot+iy)}(\cdot,y)-f_k^\delta\widehat{(\cdot+iy)}(\cdot,y)\| \\ &= \left[\int_{-\infty}^{\infty}\left|\left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right]e^{-y\xi}(\hat{f}(\xi)-\hat{f^\delta}(\xi))\right|^2 d\xi\right]^{\frac{1}{2}} \\ &\le \delta\max_{\xi\in\mathbb{R}}\left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right]e^{-y\xi}. \end{aligned} \tag{2.14}$$

It is easy to verify the inequality: $(1+x)^k\ge 1+kx,\ k>0, x>0$. Therefore,

$$\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k=\left(1-\frac{e^{y_0\xi}}{\alpha+e^{y_0\xi}}\right)^k\ge 1-\frac{ke^{y_0\xi}}{\alpha+e^{y_0\xi}}. \tag{2.15}$$

Then

$$I_2 \le \delta \max_{\xi\in\mathbb{R}} \frac{ke^{(y_0-y)\xi}}{\alpha+e^{y_0\xi}}. \tag{2.16}$$

Now we denote $h(\xi)=\frac{e^{(y_0-y)\xi}}{\alpha+e^{y_0\xi}}$, then $h'(\xi)=\frac{e^{(y_0-y)\xi}[(y_0-y)\alpha-ye^{y_0\xi}]}{(\alpha+e^{y_0\xi})^2}$.

Setting $h'(\xi)=0$, we have $\xi=\frac{1}{y_0}\ln\frac{\alpha(y_0-y)}{y}$. Then $h(\xi)$ has a unique maximal value at $\xi=\frac{1}{y_0}\ln\frac{\alpha(y_0-y)}{y}$. Therefore

$$\begin{aligned}\max_{\xi\in\mathbb{R}} h(\xi) &= \frac{\left[\frac{\alpha(y_0-y)}{y}\right]^{\frac{y_0-y}{y_0}}}{\alpha+\frac{\alpha(y_0-y)}{y}} = \frac{y_0-y}{y_0}\left[\frac{\alpha(y_0-y)}{y}\right]^{-\frac{y}{y_0}} \\ &\le \frac{y_0-y}{y_0}\left[\frac{\alpha(y_0-y)}{y_0}\right]^{-\frac{y}{y_0}} = \alpha^{-\frac{y}{y_0}}\left(1-\frac{y}{y_0}\right)^{1-\frac{y}{y_0}} \le \alpha^{-\frac{y}{y_0}},\end{aligned} \tag{2.17}$$

i.e.,

$$I_2 \le k\delta\alpha^{-\frac{y}{y_0}}. \tag{2.18}$$

Combining inequality (2.13) and (2.18) with (2.10), the error estimate between the approximation solution and exact solution is given by

$$\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\| \le \alpha^{1-\frac{y}{y_0}}\left(\frac{1}{k}\right)^{1-\frac{y}{y_0}}E + k\delta\alpha^{-\frac{y}{y_0}}. \tag{2.19}$$

Minimizing the right side of (2.19), simply we let $\alpha^{1-\frac{y}{y_0}}(\frac{1}{k})^{1-\frac{y}{y_0}}E = k\delta\alpha^{-\frac{y}{y_0}}$. By a simple computation, we obtain $k=(\frac{\alpha E}{\delta})^{\frac{y_0}{2y_0-y}}$ and choose the regularization parameter

$$k=\left[\left(\frac{\alpha E}{\delta}\right)^{\frac{y_0}{2y_0-y}}\right], \tag{2.20}$$

then $(\frac{\alpha E}{\delta})^{\frac{y_0}{2y_0-y}} \le k < (\frac{\alpha E}{\delta})^{\frac{y_0}{2y_0-y}}+1$, therefore,

$$\begin{aligned}\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\| &\le \alpha^{1-\frac{y}{y_0}}\left[\left(\frac{\delta}{\alpha E}\right)^{\frac{y_0}{2y_0-y}}\right]^{1-\frac{y}{y_0}}E + \left[\left(\frac{\alpha E}{\delta}\right)^{\frac{y_0}{2y_0-y}}+1\right]\delta\alpha^{-\frac{y}{y_0}} \\ &= 2\alpha^{\frac{(y_0-y)^2}{y_0(2y_0-y)}}\delta^{1-\frac{y_0}{2y_0-y}}E^{\frac{y_0}{2y_0-y}}+\delta\alpha^{-\frac{y}{y_0}} \\ &= C\delta^{1-\frac{y_0}{2y_0-y}}E^{\frac{y_0}{2y_0-y}}[1+o(1)],\end{aligned} \tag{2.21}$$

where $C=2\alpha^{\frac{(y_0-y)^2}{y_0(2y_0-y)}}$ is a positive constant independent of $\delta$ and $E$. From (2.21), it is obvious that

$$\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\| \to 0, \quad \text{for} \quad \delta\to 0.$$

This completes the proof. □

**Remark 2.2.** In general, the a priori bound $E$ in (2.2) is unknown exactly in practice. In this case, replacing $k$ in (2.20) by

$$k^* := \left[\left(\frac{\alpha}{\delta}\right)^{\frac{y_0}{2y_0-y}}\right], \tag{2.22}$$

then there holds the estimate

$$\|f(\cdot+iy)-f_k^\delta(\cdot+iy)\| \le C^*\delta^{1-\frac{y_0}{2y_0-y}}[E+1+o(1)], \tag{2.23}$$

where $C^*=\alpha^{\frac{(y_0-y)^2}{y_0(2y_0-y)}}$, $E$ is only a bounded positive constant and it is not necessary to be known exactly. This choice is helpful in concrete computation.

## 3. The a posteriori parameter choice rule and error estimates

In this subsection, we will consider the a posteriori stopping rule for iteration scheme (2.7). Since the a priori stopping rule contains the a priori bound (2.2) in the iteration process, the a posteriori stopping rule is necessary. We introduce the widely-used discrepancy principle due to Morozov's work. By the Parseval equality, we have the discrepancy principle in the frequency domain (by Fourier transform):

$$\|f^\delta(\cdot) - f_k^\delta(\cdot + iy)|_{y=0}\| = \|\hat{f}^\delta - \widehat{f_k^\delta(\cdot + iy)}(\xi)|_{y=0}\| = \tau\delta, \tag{3.1}$$

where $\tau > 1$ is a constant and $k$ denotes the regularization parameter. In the numerical experiments, we can take the iteration step $k$ stopped at

$$\tilde{k} = \arg\min_{[k],[k]+1} \|\hat{f}^\delta - \widehat{f_k^\delta(\cdot + iy)}(\xi)|_{y=0}\|, \tag{3.2}$$

and $\widehat{f_0^\delta(\cdot + iy)}(\xi) = 0$, thus (3.1) can be simplified to

$$\|\hat{f}^\delta - \widehat{f_k^\delta(\cdot + iy)}(\xi)|_{y=0}\| = \left\|\hat{f}^\delta(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right]\hat{f}^\delta(\xi)\right\| = \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k \hat{f}^\delta(\xi)\right\| = \tau\delta. \tag{3.3}$$

To establish existence and uniqueness of the solution of Eq. (3.1), we need the following lemma:

**Lemma 3.1.** *Set* $\rho(k) = \|\hat{f}^\delta - \widehat{f_k^\delta(\cdot + iy)}(\xi)|_{y=0}\| = \|(\frac{\alpha}{\alpha+e^{y_0\xi}})^k \hat{f}^\delta(\xi)\|$. *If* $\delta > 0,\ 0 < \tau\delta < \|\hat{f}^\delta\|$, *then there hold*

(a) $\rho$ *is a continuous function,*
(b) $\lim_{k\to 0^+}\rho(k) = \|\hat{f}^\delta\|$,
(c) $\lim_{k\to+\infty}\rho(k) = 0$,
(d) $\rho$ *is a strictly decreasing function.*

The proof is very obvious and we omit it here.

**Remark 3.2.** To ensure the existence and uniqueness of the solution of Eq. (3.1), by Lemma 3.1, we can choose appropriate $\tau$ such that $\delta k \le \frac{\alpha E}{\tau - 1}$.

From (3.3), and noting that $k$ should be greater than one in practical computation, due to Eqs. (1.8), (2.1) and (3.1), and the triangle inequality, we know

$$\begin{aligned}
\tau\delta &= \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k \hat{f}^\delta(\xi)\right\| = \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k (\hat{f}^\delta(\xi) - \hat{f}(\xi) + \hat{f}(\xi))\right\| \\
&\le \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k (\hat{f}^\delta(\xi) - \hat{f}(\xi))\right\| + \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k \hat{f}(\xi)\right\| \\
&\le \delta + \left\|\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k e^{y_0\xi}\widehat{f(\cdot + iy_0)}(\xi)\right\| \\
&\le \delta + E\max_{\xi\in\mathbb{R}}\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k e^{y_0\xi}.
\end{aligned} \tag{3.4}$$

If we denote $m(\xi) = (\frac{\alpha}{\alpha+e^{y_0\xi}})^k e^{y_0\xi}$, then

$$\begin{aligned}
m'(\xi) &= k\left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^{k-1}\left[-\frac{\alpha y_0 e^{y_0\xi}}{(\alpha + e^{y_0\xi})^2}\right]e^{y_0\xi} + \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k y_0 e^{y_0\xi} \\
&= \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\left(-\frac{ky_0 e^{2y_0\xi}}{\alpha + e^{y_0\xi}}\right) + \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k y_0 e^{y_0\xi} \\
&= \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k y_0 e^{y_0\xi}\left[-\frac{ke^{2y_0\xi}}{\alpha + e^{y_0\xi}} + 1\right].
\end{aligned} \tag{3.5}$$

Setting $m'(\xi) = 0$, we have $\xi = \frac{1}{y_0}\ln\frac{\alpha}{k-1}$, $m(\xi)$ has a unique maximal value at $\xi = \frac{1}{y_0}\ln\frac{\alpha}{k-1}$. Therefore

$$\begin{aligned}
\max_{\xi\in\mathbb{R}}(\xi) &= \left(\frac{\alpha}{\alpha + \frac{\alpha}{k-1}}\right)^k \frac{\alpha}{k-1} = \left(\frac{k-1}{k}\right)^k \frac{\alpha}{k-1} \\
&= \left(\frac{k-1}{k}\right)^{k-1}\frac{\alpha}{k} \le \frac{\alpha}{k}.
\end{aligned} \tag{3.6}$$

Then we have

$$\tau\delta \le \delta + \frac{\alpha E}{k},$$

i.e.,

$$\delta k \leq \frac{\alpha E}{\tau - 1}. \tag{3.7}$$

Denote

$$\omega_k^\delta(\cdot, y) := f\widehat{(\cdot + iy)}(\xi) - f_k^\delta\widehat{(\cdot + iy)}(\xi). \tag{3.8}$$

**Lemma 3.2.** *The following inequality holds*

$$\|\omega_k^\delta(\cdot, y)\| \leq \|\omega_k^\delta(\cdot, 0)\|^{1-\frac{y}{y_0}} \|\omega_k^\delta(\cdot, y_0)\|^{\frac{y}{y_0}}. \tag{3.9}$$

**Proof.** By (3.8), it is easy to see that

$$\omega_k^\delta(\cdot, y) = e^{-y\xi}\hat{f}(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right] e^{-y\xi}\hat{f}^\delta(\xi).$$

Then

$$\omega_k^\delta(\cdot, 0) = \hat{f}(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right] \hat{f}^\delta(\xi),$$

$$\omega_k^\delta(\cdot, y_0) = e^{-y_0\xi}\hat{f}(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right] e^{-y_0\xi}\hat{f}^\delta(\xi).$$

By using the Hölder inequality, we have the inequality

$$\begin{aligned}
\|\omega_k^\delta(\cdot, y)\|^2 &= \int_{-\infty}^{\infty} \left| e^{-y\xi}\hat{f}(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right] e^{-y\xi}\hat{f}^\delta(\xi)\right|^2 d\xi \\
&= \int_{-\infty}^{\infty} e^{-2y\xi} |\Theta_f(k, \xi, \delta)|^2 d\xi \\
&= \int_{-\infty}^{\infty} e^{-2y\xi} |\Theta_f(k, \xi, \delta)|^{\frac{2y}{y_0}} |\Theta_f(k, \xi, \delta)|^{2(1-\frac{y}{y_0})} d\xi \\
&\leq \left[\int_{-\infty}^{\infty} (e^{-2y\xi} |\Theta_f(k, \xi, \delta)|^{\frac{2y}{y_0}})^{\frac{y_0}{y}} d\xi\right]^{\frac{y}{y_0}} \left[\int_{-\infty}^{\infty} (|\Theta_f(k, \xi, \delta)|^{2(1-\frac{y}{y_0})})^{\frac{y_0}{y_0 - y}} d\xi\right]^{\frac{y_0 - y}{y_0}} \\
&= \left[\int_{-\infty}^{\infty} |e^{-y_0\xi}\Theta_f(k, \xi, \delta)|^2 d\xi\right]^{\frac{y}{y_0}} \left[\int_{-\infty}^{\infty} |\Theta_f(k, \xi, \delta)|^2 d\xi\right]^{1-\frac{y}{y_0}} \\
&= \|\omega_k^\delta(\cdot, y_0)\|^{\frac{2y}{y_0}} \|\omega_k^\delta(\cdot, 0)\|^{2(1-\frac{y}{y_0})}
\end{aligned} \tag{3.10}$$

where $\Theta_f(k, \xi, \delta) = \hat{f}(\xi) - [1 - (\frac{\alpha}{\alpha + e^{y_0\xi}})^k]\hat{f}^\delta(\xi)$. Thus we obtain the result. □

**Lemma 3.3.** *The following inequality holds:*

$$\|\omega_k^\delta(\cdot, 0)\| \leq (\tau + 1)\delta, \tag{3.11}$$

$$\|\omega_k^\delta(\cdot, y_0)\| \leq \frac{\tau E}{\tau - 1}. \tag{3.12}$$

**Proof.** Due to the triangle inequality and (1.7), (2.1), (2.2), (3.3), there hold

$$\begin{aligned}
\|\omega_k^\delta(\cdot, 0)\| &= \left\| \hat{f}(\xi) - \left[1 - \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k\right] \hat{f}^\delta(\xi)\right\| \\
&\leq \|\hat{f}(\xi) - \hat{f}^\delta(\xi)\| + \left\| \left(\frac{\alpha}{\alpha + e^{y_0\xi}}\right)^k \hat{f}^\delta(\xi)\right\| \\
&\leq \delta + \tau\delta = (\tau + 1)\delta
\end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
\|\omega_k^\delta(\cdot,y_0)\| &= \left\| e^{-y_0\xi}\hat{f}(\xi) - \left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi}\hat{f^\delta}(\xi)\right\| \\
&= \left\| \left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi}(\hat{f^\delta}(\xi)-\hat{f}(\xi)) + \left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi}\hat{f}(\xi) - e^{-y_0\xi}\hat{f}(\xi)\right\| \\
&\le \left\| \left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi}(\hat{f^\delta}(\xi)-\hat{f}(\xi))\right\| + \left\|\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k e^{-y_0\xi}\hat{f}(\xi)\right\| \\
&\le \delta \max_{\xi\in\mathbb{R}} \left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi} + E.
\end{aligned} \tag{3.14}
$$

Using the inequality (2.15), we have

$$
\left[1-\left(\frac{\alpha}{\alpha+e^{y_0\xi}}\right)^k\right] e^{-y_0\xi} \le \frac{k}{\alpha+e^{y_0\xi}} \le \frac{k}{\alpha}. \tag{3.15}
$$

According to the inequalities (3.7) and (3.14), we arrive

$$
\|\omega_k^\delta(\cdot,y_0)\| \le \frac{k\delta}{\alpha} + E \le \frac{\alpha E}{\alpha(\tau-1)} + E = \frac{\tau E}{\tau-1}. \quad \square \tag{3.16}
$$

**Theorem 3.4.** *Assume the conditions (2.1), (2.2) hold. Let $f(x+iy)$ be the exact solution, and $f_k^\delta(x+iy)$ be its regularization approximation defined by (2.7). If the regularization parameter $k$ is chosen as the solution of (3.1), i.e., the iteration (2.6) is stopped by the discrepancy principle (3.1), then there holds the following estimate:*

$$
\|f(\cdot+iy) - f_k^\delta(\cdot+iy)\| \le C E^{\frac{y}{y_0}} \delta^{1-\frac{y}{y_0}}, \tag{3.17}
$$

*where $C = (\frac{\tau}{\tau-1})^{\frac{y}{y_0}} (\tau+1)^{1-\frac{y}{y_0}}$.*

**Proof.** Substituting (3.11) and (3.12) into (3.9), we obtain

$$
\|\omega_k^\delta(\cdot,y)\| \le \left(\frac{\tau E}{\tau-1}\right)^{\frac{y}{y_0}} [(\tau+1)\delta]^{1-\frac{y}{y_0}} = C E^{\frac{y}{y_0}} \delta^{1-\frac{y}{y_0}}, \tag{3.18}
$$

where $C = (\frac{\tau}{\tau-1})^{\frac{y}{y_0}} (\tau+1)^{1-\frac{y}{y_0}}$.

The proof of Theorem 3.4 is completed. $\square$

## 4. Numerical experiment

The modified Lavrentiev iterative method can be easily implemented numerically by the fast Fourier transform. In this section some numerical examples are devised to verify the validity of the regularization method discussed in Sections 2 and 3. In these numerical experiments we always take $y_0 = 1$ and fix the domain

$$\{z = x+iy \in \mathbb{C} \mid |x| \le 10, 0 < y < 1\}.$$

Suppose the vectors $F$ and $F(\cdot+iy)$ represent samples from the functions $f(x)$ and $f(x+iy)$, respectively. Then we add a perturbation to the input data $F$ and get the perturbation data

$$
F^\delta = F + \varepsilon\, \underline{randn}(size(F)), \tag{4.1}
$$

where the function "$\underline{randn}(\cdot)$" generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$. We subdivide the interval $[-10, 10]$ into $M$ equal parts. Then the noisy level $\delta$ can be calculated by (Root-Mean-Square Error (RMSE))

$$
\delta = \|F^\delta - F\|_{l^2} := \sqrt{\frac{1}{M+1}\sum_{n=1}^{M+1} |F^\delta(n) - F(n)|^2}. \tag{4.2}
$$

In the present paper, we fix $M = 200$ and $F_k^\delta(x+iy)$ to represent the discrete regularization solution of $F(x+iy)$.

Meanwhile, we would like to compare the a priori parameter choice rule (2.22) with the a posteriori parameter choice rule (3.1).

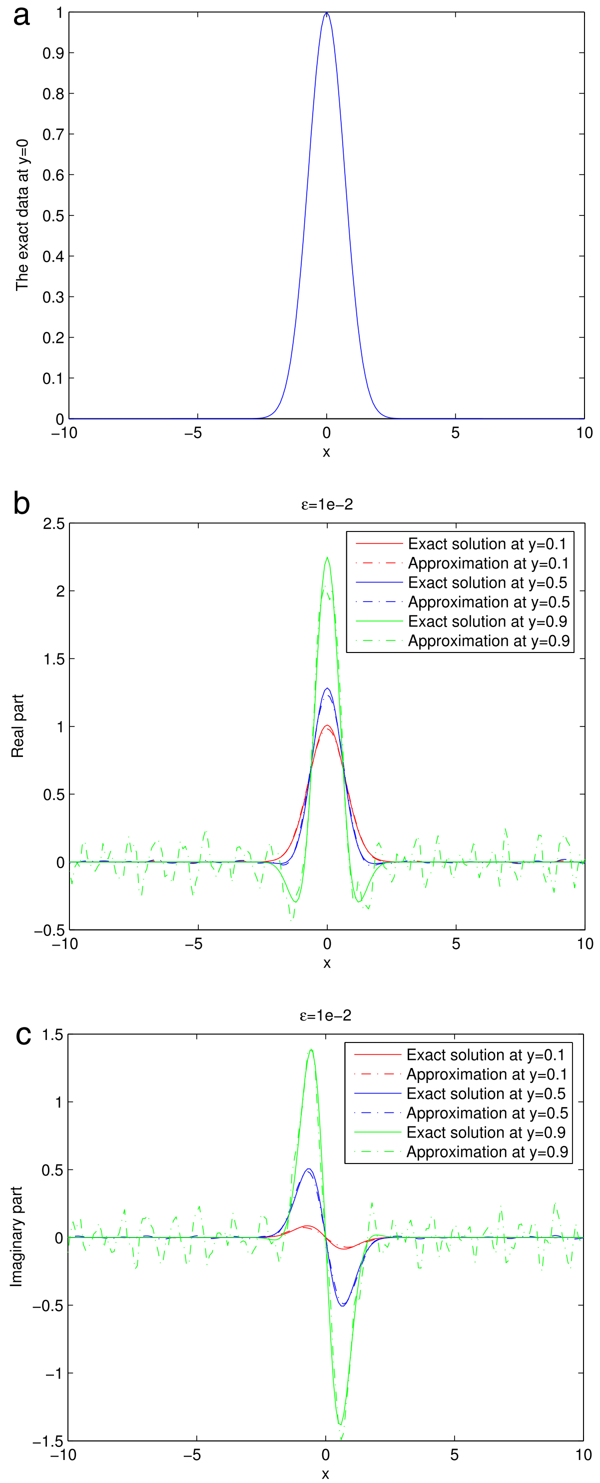


**Fig. 1.** Example 4.1. (1a) Input data, (1b) the reconstructed result for real part, (1c) the reconstructed result for imaginary part.

Then, the numerical examples were constructed in the following way: First we chose the exact solution $F(x+iy)$ and obtained the exact data function $F(x)$. Secondly, we added a normally distributed perturbation to each data function giving vectors $F^{\delta}(x)$. Finally we obtained the regularization solutions using (2.6) via inverse Fourier transform.

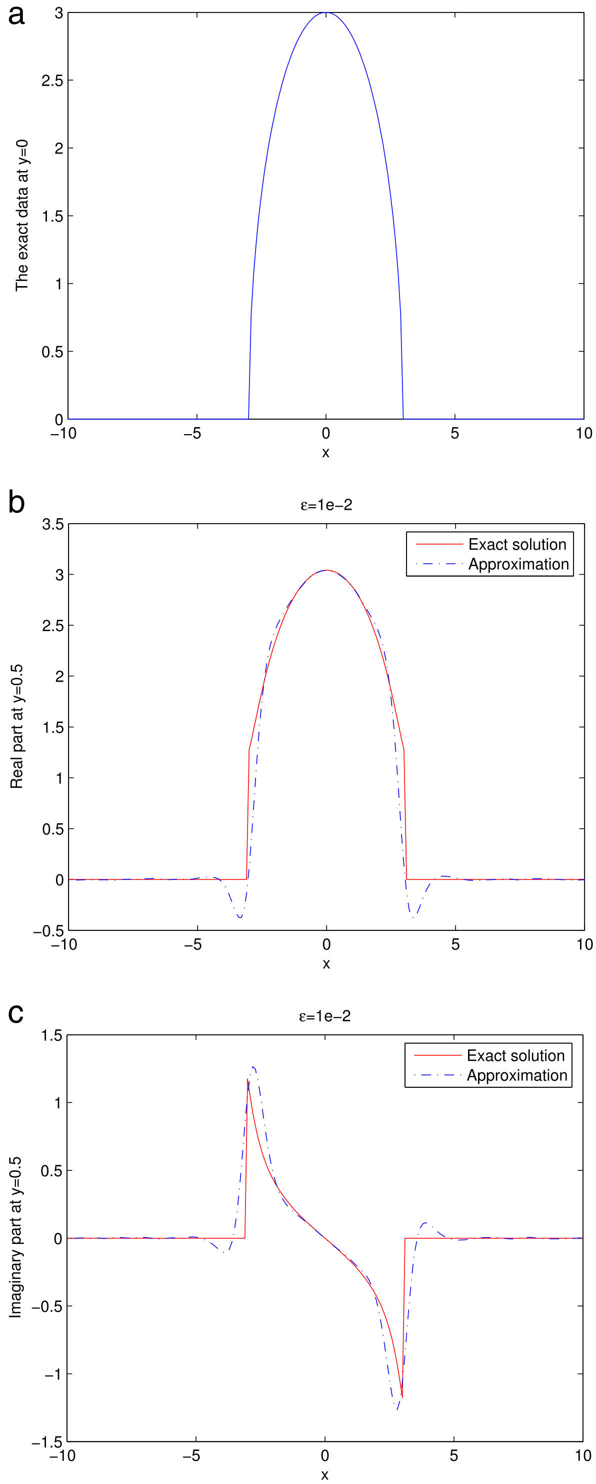


**Fig. 2.** Example 4.2 tested by our method under the a priori rule. (2a) Input data, (2b) the reconstructed result for real part, RMSE = 0.03, (2c) the reconstructed result for imaginary part, RMSE = 0.03.

In all numerical experiments, the a priori parameter $k^* := [(\frac{\alpha}{\delta})^{\frac{y_0}{2y_0-y}}]$ and the a posteriori parameter $\tilde{k}$ according to (3.1) are used for calculation.

**Numerical examples.** We conduct two numerical examples to show the effectiveness of the proposed method.

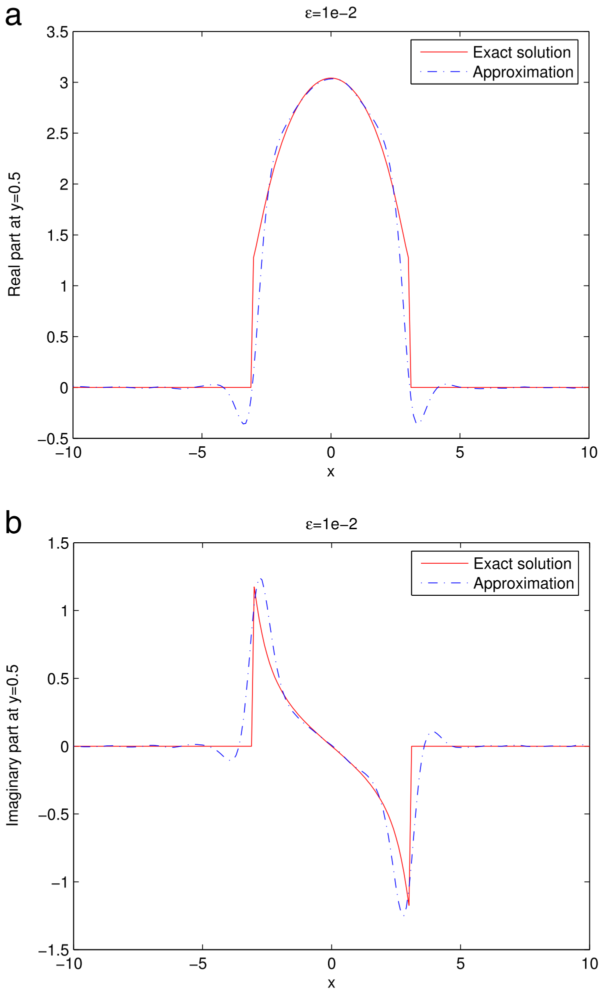


**Fig. 3.** Example 4.2 tested by our method under the post-priori rule. (3a) the reconstructed result for real part, RMSE $= 0.02$, (3b) the reconstructed result for imaginary part, RMSE $= 0.02$.

**Example 4.1.** The function

$$f(z) = e^{-z^2} = e^{-(x+iy)^2} = e^{y^2-x^2}(\cos 2xy - i \sin 2xy)$$

is analytic in the domain

$$\Omega = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, 0 < y \leq 1\}$$

with

$$f(z)|_{y=0} = e^{-x^2} \in L^2(\mathbb{R}),$$

and

$$\mathrm{Re}\, f(z) = e^{y^2-x^2} \cos 2xy,$$

$$\mathrm{Im}\, f(z) = -e^{y^2-x^2} \sin 2xy.$$

Fig. 1 gives the reconstruction results for the real parts and imaginary parts of the exact solution $f(z)$ and the regularized solution for $y = 0.1, 0.5, 0.9$ with noisy level $\varepsilon = 10^{-2}$. In this example, we used the a priori parameter choice rule (2.22), the regularized parameters are $k^* = 4, 5, 9$ with the parameter $\alpha = 0.1$. From Fig. 1, we can see that the smaller the $y$, the better the reconstructed results. This is due to the formula (1.6).

In the following example, we consider the numerical effect for the a priori rule and a posteriori rule.

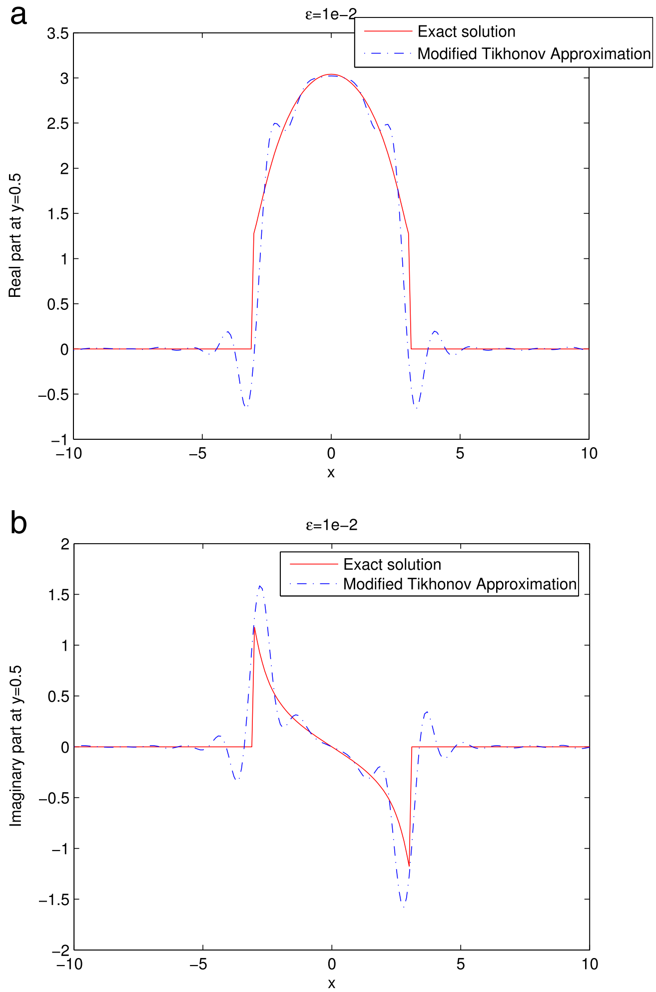


**Fig. 4.** Example 4.2 tested by Modified Tikhonov method. (4a) the reconstructed result for real part, RMSE $= 0.05$, (4b) the reconstructed result for imaginary part, RMSE $= 0.04$.

**Example 4.2.** If the function $f$ is given by

$$f(z) = \begin{cases} \sqrt{9 - z^2} = \sqrt{9 - (x + iy)^2}, & \text{if } |x| < 3, \\ 0, & \text{if } |x| \geq 3. \end{cases}$$

It is a piecewise analytic function and $\sqrt{9 - (x + iy)^2}$ has a single-valued determination in the complex plane minus the set $\{x : |x| \geq 3\}$.

Fig. 2 gives the comparison of the exact solution $f(z)$ and its approximation with $y = 0.5,\ k^* = 18,\ \alpha = 0.8$ and $\varepsilon = 0.01$ under the a priori parameter choice rule.

Fig. 3 gives the comparison of the exact solution $f(z)$ and its approximation with $y = 0.5,\ \tau = 80,\ \alpha = 0.8$ and $\varepsilon = 0.01,\ \delta = 0.01$ under the posteriori parameter choice rule.

In addition, from Examples 4.1 and 4.2, we can see that the smaller the $y$, the better the approximate effect of $f(x + iy)$.

**Comparing with other methods.** For the sake of comparison, we give some existing classical methods. We test Example 4.2 by two important methods, respectively.

**1. Modified Tikhonov method.**

Tikhonov method is an important method for ill-posed problems. A modified version of Tikhonov method [15] produces a regularization solution $f(x + iy)$ in the Fourier space defined as follows:

$$\widehat{f_T^{\delta}(\xi + iy)} = \frac{e^{-y\xi}}{1 + \alpha e^{-2y_0\xi}} \hat{f}_\delta(\xi), \tag{4.3}$$

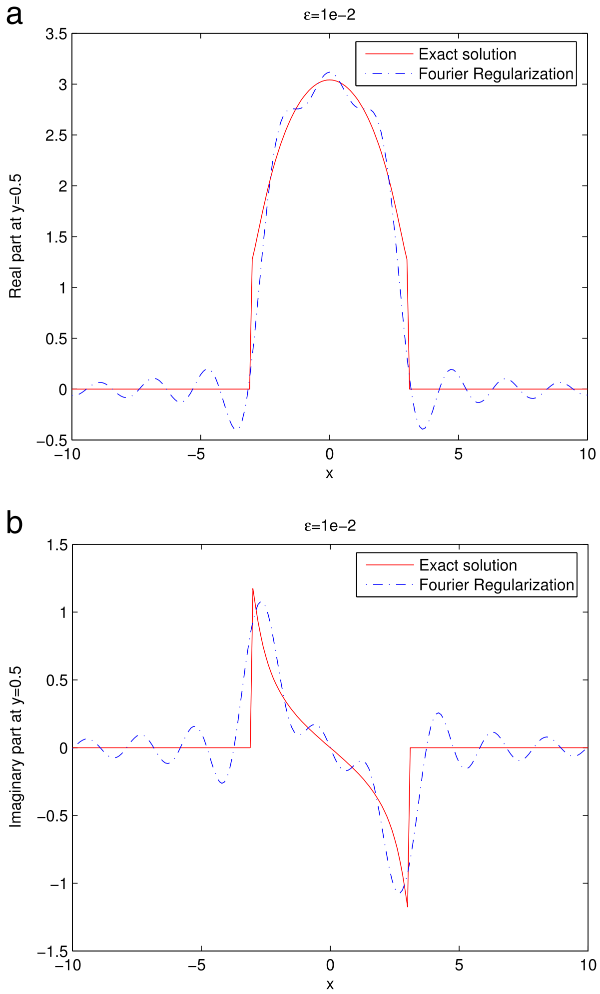


**Fig. 5.** Example 4.2 tested by Fourier method. (5a) the reconstructed result for real part, RMSE = 0.05, (5b) the reconstructed result for imaginary part, RMSE = 0.05.

where $\alpha$ is the regularization parameter. According to Remark 3.2 in [15], in numerical computation we take $\alpha = \delta^2$. Fig. 4 shows the results by the modified Tikhonov method with $y = 0.5,\ \alpha = 0.0001$ and $\varepsilon = 0.01,\ \delta = 0.01$.

**2. Fourier method.**

Following the idea of truncated singular value decomposition (TSVD) in the discrete setting for deblurring problems, for the present problem, the Fourier cut-off regularization solution can be given [16].

Fourier method:

$$\widehat{f_F^\delta(\xi + iy)} = e^{-y\xi}\hat{f}_\delta(\xi)\chi^+_{\max}, \tag{4.4}$$

where $\chi^+_{\max}$ is given by

$$\chi^+_{\max}(\xi) = \begin{cases} 1, & \xi \geq -\xi_{\max}, \\ 0, & \xi < -\xi_{\max}, \end{cases} \tag{4.5}$$

$\xi_{\max}$ is the regularization parameter. According to the Remark 2.1 in [16], in numerical computation we take $\xi_{\max} = \log(1/\delta)$. Fig. 5 shows the results by the Fourier method with $y = 0.5,\ \xi_{\max} = 3$ and $\varepsilon = 0.01,\ \delta = 0.01$.

In the same noisy level $\delta = 0.01$ and the same reconstruction location $y = 0.5$, we have tested three methods. From Fig. 3 to Fig. 5, we can see that the proposed method is better than the modified Tikhonov method and Fourier method.

## 5. Conclusion

In this paper, a modified Lavrentiev iteration regularization method is proposed to give a stable numerical analytic continuation of analytic function on a strip domain. This method can be numerically implemented by a fast Fourier transform.

For the a priori and the a-postpriori parameter choice rules, the convergence error estimates for the approximation are obtained. The numerical experiments show that the proposed method works well and is better than the classical methods including modified Tikhonov method and Fourier method in some situation.

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