Chapter 11: Rings

11.1: Definition of a Ring

- **Ring**: A *ring* R is a set with two laws of composition $+$ and $\times$, called addition and multiplication, that satisfy these axioms:

  (a) With the law of composition $+$, R is an abelian group that we denote by R^+; its identity is denoted by 0.

  (b) Multiplication is commutative and associative, and has an identity denoted by 1.

  (c) *Distributive law*: For all a, b and c in R, $(a + b)c = ac + bc$.

  - **Subring**: Subset which is closed under addition, subtraction, multiplication and which contains 1.
  - **Non-commutative Ring**: Satisfies all of the above axioms, except for the commutative law for multiplication.

- **Gauss integers**: The complex numbers of the form $a + bi$ where a and b are integers form a subring of $\mathbb{C}$ that we denote by $\mathbb{Z}[i] = \{a + bi \mid b, b \in \mathbb{Z}\}$. Its elements are points of a square lattice in the complex plane.

  - $\mathbb{Z}[\alpha]$ **subring**: Contains every complex number $\beta = a_n\alpha^n + \cdots + a_1\alpha + a_0$ where a_i are in $\mathbb{Z}$ and α is a complex number.
    * Analogous to the ring of Gauss integers.
    * Subring generated by α
    * Usually not represented as a lattice in the complex plane

- A complex number α is **algebraic** if it is a root of a (nonzero) polynomial with integer coefficients (i.e. if some expression of the form $a_n\alpha^n + \cdots + a_1\alpha + a_0$ evaluates to 0)

  - When α is algebraic there will be many polynomial expressions that represent the same complex number.

- If there is no polynomial with integer coefficients having α as a root, α is **transcendental**

  - When α is transcendental, two distinct polynomial expressions represent distinct complex numbers, and the elements of the ring $\mathbb{Z}[\alpha]$ correspond bijectively to polynomials $p(x)$ with integer coefficients.

- A polynomial in x with coefficients in a ring R is an expression of the form

$$a_nx^n + \cdots + a_1x + a_0$$

with a_i in R.

- **Zero Ring**: A ring containing only the element 0.
  - A ring R in which the elements 1 and 0 are equal is the zero ring.
- **Unit**: A *unit* of a ring is an element that has a multiplicative inverse (if it exists, it is unique)
  - Units in the ring of integers are 1 and -1
  - Units in the ring of Gauss integers are ±1 and ±i
  - Units in the ring $\mathbb{R}[x]$ of real polynomials are the nonzero constant polynomials
  - The identity element 1 of a ring is always a unit

11.2: Polynomial Rings

- **Formal Polynomial**: A polynomial with coefficients in a ring R is a (finite) linear combination of powers of the variable: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where the coefficients a_i are elements of R.
  - The set of polynomials with coefficients in a ring R will be denoted $R[x]$
  - Thus $\mathbb{Z}[x]$ is the set of *integer polynomials*
- The *monomials* x^i are considered independent, so if ∃ another polynomial with coefficients in R, then $f(x) = g(x)$ only if $a_i = b_i$ for all $i = 0, 1, 2, ...$
- **Degree**: The *degree* of a nonzero polynomial (denoted deg f) is the largest integer n such that the coefficient a_n of x_n is not zero
  - A polynomial of degree zero is called a *constant* polynomial
  - The zero polynomial is also a constant polynomial, but its degree will not be defined
- **Leading Coefficient**: The nonzero coefficient of highest degree of a polynomial
  - **Monic Polynomial**: Polynomial with a leading coefficient of 1
- A polynomial is determined by its vector of coefficients a_i: $a = (a_0, a_1, ...)$ where a_i are elements of R, all but a finite number zero.
- When R is a field, these infinite vectors form the vector space Z with the infinite basis e_i. The vector e_i corresponds to the monomial x_i, and the monomials form a basis of the space of all polynomials.

- **Addition of polynomials**: $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + ...$ where $(a_i + b_i)$ is addition in R
- **Multiplication of polynomials**: $f(x)g(x) = (a_0 + a_1 x + ...)(b_0 + b_1 x + ...)$ where $a_i b_j$ are to be evaluated in the ring R.
- There is a unique commutative ring structure on the set of polynomials $R[x]$ having these properties:
  - Additions of polynomials as defined above
  - Multiplication of polynomials as defined above
  - The ring R becomes a subring of $R[x]$ when the elements of R are identifies with the constant polynomials
- **Division with Remainder**: Let R be a ring, f is a monic polynomial, and g is any polynomial, both with coefficients in R. There are uniquely determined polynomials q and r in $R[x]$ s.t. $g(x) = f(x)q(x) + r(x)$ where r has degree $\geqslant 0$ and $\leqslant f$
  - Division with remainder can be done whenever the leading coefficient of f is a unit
  - If $g(x)$ is a polynomial in $R[x]$ and α is an element of R, the remainder of division of $g(x)$ by $x - \alpha$ is $g(\alpha)$. Thus $x - \alpha$ divides g in $R[x]$ iff $g(\alpha) = 0$
- **Monomial**: a formal product of some variables $x_1, ..., x_n$ of the form

$$x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$$

  where i_v are non-negative integers.
  - **Degree**: the sum $i_1 + \cdots + i_n$, sometimes called *total degree*
  - **Multi-index**: an n-tuple that can be represented with vector notation e.g. $i = (i_1, \cdots i_n)$.
  - A monomial can be written as x^i $(= x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})$ using multi-index form
  - The monomial x^0 is denoted by 1
- With multi-index notation, a polynomial $f(x) = f(x_1, ..., x_n)$ can be written in exactly one way in the form

$$f(x) = \sum_i a_i x^i$$

  where i runs through all multi-indices $(i_1, ..., i_n)$, the coefficients a_i are in R and only finitely many of these coefficients are not 0.
- **Homogeneous Polynomial**: A polynomial in which all monomials with nonzero coefficients have degree d

11.3: Homomorphisms and Ideals

- **Ring Homomorphism**: A *ring homomorphism* $\phi : R \to R'$ is a map from one ring to another which is compatible with the laws of composition and which carries the unit element 1 of R to the unit element 1 of R' - a map such that for all a and b in R,

$$\phi(a+b) = \phi(a) + \phi(b), \quad \phi(ab) = \phi(a)\phi(b), \quad and \quad \phi(1) = 1$$

  - The map $\phi : \mathbb{Z} \to \mathbb{F}_p$ that send an integer to its congruence class modulo p is a ring homomorphism.

- **Isomorphism**: An *isomorphism* of rings is a bijective homomorphism, denoted $R \approx R'$

- Evaluation of real polynomials at a real number a defines a homomorphism

$$\mathbb{R}[x] \to \mathbb{R}, \quad \text{that sends} \quad p(x) \rightsquigarrow p(a)$$

- **Substitution Principle**: Let $\phi : R \to R'$ be a ring homomorphism, and let $R[x]$ be the ring of polynomials with coefficients in R.
  (a) Let α be an element of R'. There is a unique homomorphism $\Phi : R[x] \to R'$ that agrees with the map ϕ on constant polynomials, and that send $x \rightsquigarrow a$
  (b) Given elements $\alpha_1, ..., \alpha_n$ of R', there is a unique homomorphism $\Phi : R[x_1, ..., x_n] \to R'$, from the polynomial ring in n variables to R', that agrees with ϕ on constant polynomials and that send $x_v \rightsquigarrow \alpha_v$, for $v = 1, ..., n$.

- Let R be any ring, and let P be the polynomial ring $R[x]$. One can use the substitution principle to construct an isomorphism

$$R[x, y] \to P[y] = (R[x])[y]$$

  This statement is a formalization of the procedure of collecting terms of like degree in y in a polynomial $f(x, y)$. For example:

$$x^4y + x^3 - 3x^2y + y^2 + 2 = y^2 + (x^4 - 3x^2)y + (x^3 + 2)$$

- Let $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_n)$ denote sets of variables. There is a unique isomorphism $R[x, y] \to R[x][y]$, which is the identity on R and sends the variables to themselves.

- Let $f(x, y)$ and $g(x, y)$ be polynomials in two variables, elements of $R[x, y]$. Suppose that f is a monic polynomial of degree m (grouped by y). There are uniquely determined polynomials $q(x, y)$ and $r(x, y)$ such that $g = fq + r$ and $0 \leqslant r(x, y) < m$

- There is exactly one homomorphism $\phi : \mathbb{Z} \to R$, defined for $n \geqslant 0$ where $\phi(n) = 1 + \cdots + 1$ (for n terms) and $\phi(-n) = -\phi(n)$

- **Kernel**: The *kernel* of ϕ is the set of elements R that map to zero:

$$\ker\phi = \{s \in R \mid \phi(s) = 0\}$$

  - If s is in $ker\phi$, then for every element r of R, rs is in $ker\phi$

- **Ideal**: An *ideal* I of a ring R is a nonempty subset of R with these properties:

  (a) I is closed under addition, and
  (b) If s is in I and r is in R, then rs is in I

  - **Principal Ideal**: The ideal formed by multiples of a particular element a, also defined as:

  $$(a) - aR = Ra = \{ra \mid r \in R\}$$

  - **Unit Ideal**: The ring R is the principal ideal (1), and is called the *unit ideal*
  - **Zero Ideal**: The principal ideal (0)
  - **Proper Ideal**: An ideal that is neither the unit or zero ideal

- The kernel of a ring homomorphism is an ideal

- An ideal is not a subring unless the ideal I is equal to the whole ring R

- The ideal *generated by a set of elements* $\{a_1, ..., a_n\}$ of a ring R is the smallest ideal that contains those elements. This ideal is often denoted as $(a_1, ..., a_n)$:

$$(a_1, ..., a_n) = \{r_1a_1 + \cdots + r_na_n \mid r_i \in R\}$$

- The only ideals of a field are the zero ideal and the unit ideal

- A ring that has exactly two ideals is a field

- Every homomorphism $\phi : F \to R$ from a field F to a nonzero ring R is injective

- The ideals in the ring of integers are the subgroups of $\mathbb{Z}^+$, and they are principal ideals

- Every ideal in the ring $F[x]$ of polynomials in one variable x over a field F is a principal ideal. A nonzero ideal I in $F[x]$ is generated by the unique monic polynomial of lower degree that it contains.

- Let f be a monic integer polynomial, and let g be another integer polynomial. If $f \mid g$ in $\mathbb{Q}[x]$, $f \mid g$ in $\mathbb{Z}[x]$

- **Greatest Common Divisor**: Let R denote the polynomial ring $F[x]$ in one variable over a field F, and let f and g be elements of R, not both zero. Their *greatest common divisor* $d(x)$ is the unique monic polynomials that generates the ideal (f, g). It has these properties:
  (a) $Rd = Rf + Rg$
  (b) d divides f and g
  (c) If a polynomial $e = e(x)$ divides both f and g, it also divides d
  (d) There are polynomials p and q such that $d = pf + qg$
- **Characteristic**: The non-negative integer n that generates the kernel of the homomorphism $\phi : \mathbb{Z} \to R$
  1. If $n = 0$, this means that no positive multiple of 1 in R is equal to zero. Otherwise n is the smallest positive integer s.t. "n times 1" is zero in R

11.4: Quotient Rings

- Let I be an ideal of a ring R. There is a unique ring structure on the set $\bar{R}$ (R/I) of additive cosets of I such that the map $\pi : R \to \bar{R}$ that send $a \rightsquigarrow \bar{a} = [a + I]$ (the coset generated with the subgroup I) is a ring homomorphism. The kernel of π is I.
  - **Canonical Map**: π
  - **Quotient Ring**: $\bar{R}$
  - **Residue**: The image $\bar{a}$ of a
- **Mapping Property of Quotient Rings**: Let $f : R \to R'$ be a ring homomorphism with kernel K and let I be another ideal. Let $\pi : R \to \bar{R}$ be the canonical map from R to $\bar{R} = R/I$
  (a) If $I \subset K$, there is a unique homomorphism $\bar{f} : \bar{R} \to R'$ such that $\bar{f}\pi = f : R \to R/I \to R'$.
  (b) **First Isomorphism Theorem**: If f is onto and $I = K$, is an isomorphism.
- **Correspondence Theorem**: Let $\phi : R \to \mathcal{R}$ be an onto ring homomorphism with kernel K. There is a bijective correspondence between the set of all ideals of $\mathcal{R}$ and the set of ideals of R that contain K. It is defined as follows:
  - If I is an ideal of R and of $K \subset I$, the corresponding ideal of $\mathcal{R}$ is $\phi(I)$
  - If $\mathcal{I}$ is am ideal of $\mathcal{R}$, the corresponding ideal of R is $\phi^{-1}(\mathcal{I})$
- Of the ideal I of R corresponds to the ideal $\mathcal{I}$ of $\mathcal{R}$, the quotient rings R/I and $\mathcal{R}/\mathcal{I}$ are naturally isomorphic.

- The image of a subgroup is a subgroup
- We reinterpret the quotient ring construction when the ideal is principal ($I = (a)$). In this situation, $\bar{R} = R/I$ as the "killing" of a by imposing the relation $a = 0$ on R
  - Imposing the relation $a = 0$ on R forces us to set $b = b + ra$ for all b, r in R
  - Two elements b and b' of R have the same image in $\bar{R}$ iff b' had the form $b + r_1 a_1 + \cdots + r_n a_n$ for some $r_i \in R$

11.5: Adjoining Elements

- **Ring Extension**: A ring that contains another ring as a subring
- **Adjoining an Element to a Ring**: We want to adjoin an element α to a ring R and we want α to satisfy the polynomial relation $f(x) = 0$, where

  $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_i x + a_0 \quad \text{with } a_i \in R$$

  The solution is $R' = R[x]/(f)$ where (f) is the principal ideal of $R[x]$ generated by f.
- We let α denote the residue $\bar{x}$ of x in R'. Then because the map $\pi : R[x] \to R[x]/(f)$ is a homomorphism,

  $$\pi(f(x)) = \overline{f(x)} = \bar{a}_n \alpha^n + \cdots + \bar{a}_0 = 0$$

  Where $\bar{a}_i$ is the image in R' of the constant polynomial a_i. So α satisfies the relation $f(\alpha) = 0$
- Let R be a ring, and let $f(x)$ be a monic polynomial of positive degree n with coefficients in R. Let $R[\alpha]$ denote the ring $R[x]/(f)$ obtained by adjoining an element satisfying the relation $f(\alpha) = 0$:
  (a) The set $(1, \alpha, ..., \alpha^{n-1})$ is a *basis* of $R[\alpha]$ over R: every element of $R[\alpha]$ can be written uniquely as a linear combination of this basis, with coefficients in R.
  (b) Addition of two linear combination is vector addition
  (c) Multiplication of linear combinations is as follows: Let β_1 and β_2 be elements of $R[\alpha]$, and let $g_1(x)$ and $g_2(x)$ be polynomials s.t. $\beta_1 = g_1(\alpha)$ and $\beta_2 = g_2(\alpha)$. One divides the product polynomial $g_1 g_2$ by f, say $g_1 g_2 = fq + r$, where the remainder $0 \leqslant r(x) < n$. Then $\beta_1 \beta_2 = r(\alpha)$.
- Let f be a *monic* polynomial of degree n in a polynomial ring $R[x]$. Every nonzero element of (f) has degree of at least n.

- The kernel of ψ (homomorphism that takes $R \to R'$ by restricting the canonical map π to just the constant polynomials in $R[x]$) is the set of constant polynomials in the ideal:

$$\ker\,\psi = R \cap (f)$$

ker ψ will likely be zero because f will have positive degree, and we would need to make a polynomial multiple of f have degree zero.

11.6: Product Rings

- **Product Ring**: Let R and R' be rings.
    (a) The product set $R \times R'$ is a ring called the *product ring*, with component-wise addition and multiplication.
    (b) The additive and multiplicative identities are $(0, 0)$ and $(1, 1)$.
    (c) The projections $\pi : R \times R' \to R$ and $\pi' : R \times R' \to R'$ defined by $\pi(x, x') = x$ and $\pi'(x, x') = x'$ are ring homomorphisms. The kernels are the ideals $\{0\} \times R'$ and $R \times \{0\}$ of $R \times R'$.
    (d) The kernel of π' is a ring with multiplicative identity $e = (1, 0)$. It is not a subring of of $R \times R'$ unless R' is the zero ring. The same holds for the kernel of π, but the identity is $(0, 1)$.
- To see if a ring is isomorphic to a product ring, you must find the elements that would be $(0, 1)$ and $(1, 0)$. These elements are idempotent.
- **Idempotent**: An element e is *idempotent* if $e^2 = e$
- Let e be an idempotent element of the ring S.
    (a) The element $e' = 1 - e$ is also idempotent, $e + e' = 1$ and $ee' = 0$
    (b) The principal ideal eS is a ring with identity element e and multiplication by e defines a ring homomorphism $S \to eS$
    (c) The ideal eS is not a subring of S unless e is the unit element 1 of S and $e' = 0$
    (d) The ring S is isomorphic to the product ring $eS \times e'S$

11.7: Fractions

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Chapter 11 Exercises

Problem 11.1.1: Prove that $7+\sqrt[3]{2}$ and $\sqrt{3}+\sqrt{-5}$ are algebraic numbers

Proof. We need to show that they are roots of a nonzero polynomial with integer coefficients. We can show that $(7+\sqrt[3]{2})^3 - 21(7+\sqrt[3]{2})^2 + 147(7+\sqrt[3]{2}) - 345 = 0$. This means it can be represented as the root of a polynomial, namely, $x^3 - 21x^2 + 147x - 345$. For $\sqrt{3}+\sqrt{-5}$, let $x = \sqrt{3}+\sqrt{-5}$.

$$\begin{aligned}
x^2 &= (\sqrt{3}+\sqrt{-5})(\sqrt{3}+\sqrt{-5}) \\
x^2 &= 3 + 2\sqrt{-15} - 5 \\
x^2 &= 2\sqrt{-15} - 2 \\
x^2 + 2 &= 2\sqrt{-15} \\
(x^2+2)^2 &= -60 \\
(x^2+2)^2 + 60 &= 0
\end{aligned}$$

This means that $\sqrt{3}+\sqrt{-5}$ can be represented as the root of a polynomial. □

Problem 11.1.3: Let $\mathbb{Q}[\alpha, \beta]$ denote the smallest subring of $\mathbb{C}$ containing the rational numbers $\mathbb{Q}$ and the elements $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}$. Let $\gamma = \alpha + \beta$. Is $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$? Is $\mathbb{Z}[\alpha, \beta] = \mathbb{Z}[\gamma]$?

Proof. $\mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$. To show this, we need to show that $\mathbb{Q}[\alpha, \beta] \subseteq \mathbb{Q}[\gamma]$ and $\mathbb{Q}[\gamma] \subseteq \mathbb{Q}[\alpha, \beta]$. By definition of a subring, we know that $(\alpha + \beta) \in \mathbb{Q}[\alpha, \beta]$, so we know that $\mathbb{Q}[\gamma] \subseteq \mathbb{Q}[\alpha, \beta]$. Now we need to show that α and β are in $\mathbb{Q}[\gamma]$. Since $\gamma = \alpha + \beta$, we know that $\gamma^3 = 11\alpha + 9\beta$ is also in $\mathbb{Q}[\gamma]$.

$$\begin{aligned}
\gamma^3 - 9\gamma &= 2\alpha \\
\frac{1}{2}\left[\gamma^3 - 9\gamma\right] &= \alpha
\end{aligned}$$

Since $\frac{1}{2}$ is in $\mathbb{Q}$, we know that α is in $\mathbb{Q}[\gamma]$. A similar argument can be made to show that β is in $\mathbb{Q}[\gamma]$. Since we have shown that α and β are in $\mathbb{Q}[\gamma]$, we know that $\mathbb{Q}[\gamma] \subseteq \mathbb{Q}[\alpha, \beta]$, $\therefore \mathbb{Q}[\alpha, \beta] = \mathbb{Q}[\gamma]$.

$\mathbb{Z}[\alpha, \beta] \neq \mathbb{Z}[\gamma]$, but I don't know how to prove it. My intuition is that the difference between the two coefficients in a $x\alpha + y\beta$ term will never be 1, and we aren't able to use fractions, so we'll never be able to get α or β on its own. □

Problem 11.1.6: Decide whether or not S is a subring of R, when

(a) S is the set of all rational numbers a/b, where b is not divisible by 2, and $R = \mathbb{Q}$

Proof. S is closed under multiplication because if we multiply $\frac{a}{b}\frac{c}{d}$, we get $\frac{ac}{bd}$, and we know there is no 3 to factor out of the denominator by definition. S is closed under addition because $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$, where again, a 3 cannot be factored out of the denominator. A similar argument can be made for subtraction (since the denominator is the same). S obviously contains 1 ($\frac{1}{1}$), so S is a subring of $\mathbb{Q}$. □

(b) S is the set of functions which are linear combinations with integer coefficients of the functions 1, cos nt, sin nt, n $\in \mathbb{Z}$ and R is the set of all real valued functions of t.

Proof. S is not a subring of R because it is not closed under multiplication. $sin(x)cos(x) = \frac{1}{2}sin(2x)$. Since you can't write this as a linear combination of the other functions, you know that it is not in R and S is not closed under multiplication. □

Problem 11.1.7

(a) *Proof.* □

(b) *Proof.* □

Problem 11.1.8

Proof. □

Problem 11.2.2

Proof. □

Problem 11.3.1

Proof. □

Problem 11.3.2

Proof. □

Problem 11.3.3

Proof. □

Problem 11.3.5

Proof. □

Problem 11.3.6

Proof. □

Problem 11.3.7

Proof. □

Problem 11.3.8

Proof. □

Problem 11.3.9

Proof. □

Problem 11.4.1

Proof. □

Problem 11.4.2

Proof. □

Problem 11.5.1

Proof. □

Problem 11.5.2

Proof. □

Problem 11.5.3

Proof. □

Problem 11.5.6

Proof. □

Problem 11.5.7

Proof. □

Problem 11.6.2

Proof. □

Problem 11.6.2

Proof. □

Problem 11.6.8

Proof. □

Problem 11.7.1

Proof. □

Problem 11.7.2

Proof. ∎

Problem 11.7.5

Proof. ∎

Problem 11.8.1

Proof. ∎

Problem 11.8.2

Proof. ∎

Problem 11.8.4

Proof. ∎

Problem 11.9.1

Proof. ∎

Problem 11.9.2

Proof. ∎

Problem 11.9.3

Proof. ∎

Problem 11.9.4

Proof. ∎

Problem 11.9.5

Proof. ∎

Problem 11.9.6

Proof. ∎

Problem 11.9.9

Proof. ∎

Problem 11.9.10

Proof. ∎

Problem 11.9.11

Proof. □

Problem 11.9.12

Proof. □

Problem 11.9.12

Proof. □

Problem 11.M.1

Proof. □

Problem 11.M.2

Proof. □

Problem 11.M.3

Proof. □

Problem 11.M.5

Proof. □

Problem 11.M.6

Proof. □