Chapter 11: Rings

11.1: Definition of a Ring

- **Ring**: A *ring* R is a set with two laws of composition $+$ and $\times$, called addition and multiplication, that satisfy these axioms:
  (a) With the law of composition $+$, R is an abelian group that we denote by R^+; its identity is denoted by 0.
  (b) Multiplication is commutative and associative, and has an identity denoted by 1.
  (c) *Distributive law*: For all a, b and c in R, $(a+b)c = ac + bc$.
  - **Subring**: Subset which is closed under addition, subtraction, multiplication and which contains 1.
  - **Non-commutative Ring**: Satisfies all of the above axioms, except for the commutative law for multiplication.
- **Gauss integers**: The complex numbers of the form $a + bi$ where a and b are integers form a subring of $\mathbb{C}$ that we denote by $\mathbb{Z}[\mathrm{i}] = \{a+bi \mid b, b \in \mathbb{Z}\}$. Its elements are points of a square lattice in the complex plane.
  - $\mathbb{Z}[\alpha]$ **subring**: Contains every complex number $\beta = a_n\alpha^n + ... + a_1\alpha + a_0$ where a_i are in $\mathbb{Z}$ and α is a complex number.
    * Analogous to the ring of Gauss integers.
    * Subring generated by α
    * Usually not represented as a lattice in the complex plane
- A complex number α is **algebraic** if it is a root of a (nonzero) polynomial with integer coefficients (i.e. if some expression of the form $a_n\alpha^n + ... + a_1\alpha + a_0$ evaluates to 0)
  - When α is algebraic there will be many polynomial expressions that represent the same complex number.
- If there is no polynomial with integer coefficients having α as a root, α is **transcendental**
  - When α is transcendental, two distinct polynomial expressions represent distinct complex numbers, and the elements of the ring $\mathbb{Z}[\alpha]$ correspond bijectively to polynomials $p(x)$ with integer coefficients.
- A polynomial in x with coefficients in a ring R is an expression of the form

$$a_nx^n + ... + a_1x + a_0$$

with a_i in R.

- **Zero Ring**: A ring containing only the element 0.
  - A ring R in which the elements 1 and 0 are equal is the zero ring.
- **Unit**: A *unit* of a ring is an element that has a multiplicative inverse (if it exists, it is unique)
  - Units in the ring of integers are 1 and -1
  - Units in the ring of Gauss integers are ±1 and ±i
  - Units in the ring $\mathbb{R}[x]$ of real polynomials are the nonzero constant polynomials
  - The identity element 1 of a ring is always a unit

11.2: Polynomial Rings

- **Formal Polynomial**: A polynomial with coefficients in a ring R is a (finite) linear combination of powers of the variable: $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where the coefficients a_i are elements of R.
  - The set of polynomials with coefficients in a ring R will be denoted $R[x]$
  - Thus $\mathbb{Z}[x]$ is the set of *integer polynomials*
- The *monomials* x^i are considered independent, so if ∃ another polynomial with coefficients in R, then $f(x) = g(x)$ only if $a_i = b_i$ for all $i = 0, 1, 2, ...$
- **Degree**: The *degree* of a nonzero polynomial (denoted deg f) is the largest integer n such that the coefficient a_n of x_n is not zero
  - A polynomial of degree zero is called a *constant* polynomial
  - The zero polynomial is also a constant polynomial, but its degree will not be defined
- **Leading Coefficient**: The nonzero coefficient of highest degree of a polynomial
  - **Monic Polynomial**: Polynomial with a leading coefficient of 1
- A polynomial is determined by its vector of coefficients a_i: $a = (a_0, a_1, ...)$ where a_i are elements of R, all but a finite number zero.
- When R is a field, these infinite vectors form the vector space Z with the infinite basis e_i. The vector e_i corresponds to the monomial x_i, and the monomials form a basis of the space of all polynomials.

- **Addition of polynomials**: $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + ...$ where $(a_i + b_i)$ is addition in R
- **Multiplication of polynomials**: $f(x)g(x) = (a_0 + a_1x + ...)(b_0 + b_1x + ...)$ where a_ib_j are to be evaluated in the ring R.
- There is a unique commutative ring structure on the set of polynomials $R[x]$ having these properties:
  - Additions of polynomials as defined above
  - Multiplication of polynomials as defined above
  - The ring R becomes a subring of $R[x]$ when the elements of R are identifies with the constant polynomials
- **Division with Remainder**: Let R be a ring, f is a monic polynomial, and g is any polynomial, both with coefficients in R. There are uniquely determined polynomials q and r in $R[x]$ s.t. $g(x) = f(x)q(x) + r(x)$ where r has degree $\geqslant 0$ and $\leqslant f$
  - Division with remainder can be done whenever the leading coefficient of f is a unit
  - If $g(x)$ is a polynomial in $R[x]$ and α is an element of R, the remainder of division of $g(x)$ by $x - \alpha$ is $g(\alpha)$. Thus $x - \alpha$ divides g in $R[x]$ iff $g(\alpha) = 0$
- **Monomial**: a formal product of some variables $x_1, ..., x_n$ of the form

  $$x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$$

  where i_v are non-negative integers.
  - **Degree**: the sum $i_1 + ... + i_n$, sometimes called *total degree*
  - **Multi-index**: an n-tuple that can be represented with vector notation e.g. $i = (i_1, ...i_n)$.
  - A monomial can be written as x^i $(= x_1^{i_1} x_2^{i_2} ... x_n^{i_n})$ using multi-index form
  - The monomial x^0 is denoted by 1
- With multi-index notation, a polynomial $f(x) = f(x_1, ..., x_n)$ can be written in exactly one way in the form

  $$f(x) = \sum_i a_i x^i$$

  where i runs through all multi-indices $(i_1, ..., i_n)$, the coefficients a_i are in R and only finitely many of these coefficients are not 0.
- **Homogeneous Polynomial**: A polynomial in which all monomials with nonzero coefficients have degree d

11.3: Homomorphisms and Ideals

- **Ring Homomorphism**: A *ring homomorphism* $\phi : R \to R'$ is a map from one ring to another which is compatible with the laws of composition and which carries the unit element 1 of R to the unit element 1 of R' - a map such that for all a and b in R,

$$\phi(a+b) = \phi(a) + \phi(b), \quad \phi(ab) = \phi(a)\phi(b), \quad \textit{and} \quad \phi(1) = 1$$

  - The map $\phi : \mathbb{Z} \to \mathbb{F}_p$ that send an integer to its congruence class modulo p is a ring homomorphism.

- **Isomorphism**: An *isomorphism* of rings is a bijective homomorphism, denoted $R \approx R'$

- Evaluation of real polynomials at a real number a defines a homomorphism

$$\mathbb{R}[x] \to \mathbb{R}, \quad \text{that sends} \quad p(x) \rightsquigarrow p(a)$$

- **Substitution Principle**: Let $\phi : R \to R'$ be a ring homomorphism, and let $R[x]$ be the ring of polynomials with coefficients in R.
  (a) Let α be an element of R'. There is a unique homomorphism $\Phi : R[x] \to R'$ that agrees with the map ϕ on constant polynomials, and that send $x \rightsquigarrow a$
  (b) Given elements $\alpha_1, ..., \alpha_n$ of R', there is a unique homomorphism $\Phi : R[x_1, ..., x_n] \to R'$, from the polynomial ring in n variables to R', that agrees with ϕ on constant polynomials and that send $x_v \rightsquigarrow \alpha_v$, for $v = 1, ..., n$.

- Let R be any ring, and let P be the polynomial ring $R[x]$. One can use the substitution principle to construct an isomorphism

$$R[x, y] \to P[y] = (R[x])[y]$$

  This statement is a formalization of the procedure of collecting terms of like degree in y in a polynomial $f(x, y)$. For example:

$$x^4y + x^3 - 3x^2y + y^2 + 2 = y^2 + (x^4 - 3x^2)y + (x^3 + 2)$$

- Let $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_n)$ denote sets of variables. There is a unique isomorphism $R[x, y] \to R[x][y]$, which is the identity on R and sends the variables to themselves.

- Let $f(x, y)$ and $g(x, y)$ be polynomials in two variables, elements of $R[x, y]$. Suppose that f is a monic polynomial of degree m (grouped by y). There are uniquely determined polynomials $q(x, y)$ and $r(x, y)$ such that $g = fq + r$ and $0 \leqslant r(x, y) < m$

- There is exactly one homomorphism $\phi : \mathbb{Z} \to R$, defined for $n \geqslant 0$ where $\phi(n) = 1 + ... + 1$ (for n terms) and $\phi(-n) = -\phi(n)$
- **Kernel**: The *kernel* of ϕ is the set of elements R that map to zero:

$$\ker\phi = \{s \in R \mid \phi(s) = 0\}$$

  - If s is in $ker\phi$, then for every element r of R, rs is in $ker\phi$

- **Ideal**: An *ideal* I of a ring R is a nonempty subset of R with these properties:
  (a) I is closed under addition, and
  (b) If s is in I and r is in R, then rs is in I
  - **Principal Ideal**: The ideal formed by multiples of a particular element a, also defined as:

$$(a) - aR = Ra = \{ra \mid r \in R\}$$

  - **Unit Ideal**: The ring R is the principal ideal (1), and is called the *unit ideal*
  - **Zero Ideal**: The principal ideal (0)
  - **Proper Ideal**: An ideal that is neither the unit or zero ideal

- The kernel of a ring homomorphism is an ideal
- An ideal is not a subring unless the ideal I is equal to the whole ring R
- The ideal *generated by a set of elements* $\{a_1, ..., a_n\}$ of a ring R is the smallest ideal that contains those elements. This ideal is often denoted as $(a_1, ..., a_n)$:

$$(a_1, ..., a_n) = \{r_1 a_1 + ... + r_n a_n \mid r_i \in R\}$$

- The only ideals of a field are the zero ideal and the unit ideal
- A ring that has exactly two ideals is a field
- Every homomorphism $\phi : F \to R$ from a field F to a nonzero ring R is injective
- The ideals in the ring of integers are the subgroups of $\mathbb{Z}^+$, and they are principal ideals
- Every ideal in the ring $F[x]$ of polynomials in one variable x over a field F is a principal ideal. A nonzero ideal I in $F[x]$ is generated by the unique monic polynomial of lower degree that it contains.
- Let f be a monic integer polynomial, and let g be another integer polynomial. If $f \mid g$ in $\mathbb{Q}[x]$, $f \mid g$ in $\mathbb{Z}[x]$

- **Greatest Common Divisor**: Let R denote the polynomial ring $F[x]$ in one variable over a field F, and let f and g be elements of R, not both zero. Their *greatest common divisor* $d(x)$ is the unique monic polynomials that generates the ideal (f, g). It has these properties:
  (a) $Rd = Rf + Rg$
  (b) d divides f and g
  (c) If a polynomial $e = e(x)$ divides both f and g, it also divides d
  (d) There are polynomials p and q such that $d = pf + qg$

- **Characteristic**: The non-negative integer n that generates the kernel of the homomorphism $\phi : \mathbb{Z} \to R$
  1. If $n = 0$, this means that no positive multiple of 1 in R is equal to zero. Otherwise n is the smallest positive integer s.t. "n times 1" is zero in R

11.4: Quotient Rings