Homework 1

Problem 2.2.2. Let a be a positive number. Prove there exists exactly one natural number b such that $b++=a$.

Proof. Proof by contradiction. Assume natural numbers $b \neq c$ such that $b++=a$ and $c++=a$. From Axiom 2.4, we know that if $b \neq c$, then $n++ \neq m++$. This implies that $a \neq a$, which is not true. □

Problem 3.1.6. Let A, B, C be sets, and let X be a set containing A, B, C as subsets. Prove De Morgan's law for sets.

1. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$
2. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Parts a-g of proposition 3.1.27 can be assumed.

Proof. .

1. Let $X = \{A \cup B \cup C \cup Y\}, Y$, where Y is everything else in X that is not $A \cup B \cup C$. Then

$$\begin{aligned} X \setminus (A \cup B) &= C \setminus (A \cup B) \cup Y \\ X \setminus A &= B \setminus A \cup C \setminus A \cup Y \\ X \setminus A &= B \setminus A \cup C \setminus A \cup Y \end{aligned}$$

□

Problem 3.3.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly show that if f and g are both surjective, then so is $g \circ f$.

Proof. □

Problem 3.4.2. Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general can one say about $f^{-1}(f(S))$ and S. What about $f(f^{-1}(U))$ and U. Prove your assertions.

Proof. □

Problem 3.5.10. If $f : X \to Y$ is a function, define the *graph* of f to be the subset of $X \times Y$ defined by $\{x, f(x)\} : x \in X$. Show that two functions $f : X \to Y$. $\bar{f} : X \to Y$ are equal if and only if they share the same graph. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.

Proof. □

Problem 3.6.10. Let $A_1, \cdots, A_n$ be finite sets such that $\#(\bigcup_{i\in\{1,\cdots,n\}} A_i) > n$. Show that there exists $i \in \{i, \cdots, n\}$ such that $\#(A_i) \geq 2$. (This is known as the pidgeonhole principle.)

Proof. □