1 Introduction

1.1 What is analysis?

- **Analysis**: the rigorous study of objects such as real numbers, complex numbers, etc.

1.2 Why do analysis?

- To understand how these objects work so that you can apply them intelligently

2 The Natural Numbers

2.1 The Peano Axioms

- **Axiom 2.1** 0 is a natural number
- **Axiom 2.2** If n is a natural number, then $n++$ is also a natural number
- **Axiom 2.3** 0 is not the successor of any natural number
  - This prevents "wrap-around"
- **Axiom 2.4** Different natural numbers must have different successors: if n, m are natural numbers and $n \neq m$, then $n++ \neq m++$. Equivalently, if $n++ = m++$ then $n = m$
  - This prevents incrementation hitting a "ceiling"
- **Axiom 2.5 (Principle of mathematical induction)** Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true $\forall$ natural n.
  - This prevents "rogue" elements (like rational numbers) from being in the naturals
- This definition is *axiomatic* and not *constructive*, they lay out what you can do with the naturals and properties they have, rather than what they are

2.2 Addition

- **Def 2.2.1 (Addition)** Let m be a natural number. To add zero to m, we define $0 + m := m$. Now by induction, suppose we know how to add n to m. Adding $n++$ to m can the be defined as

$$(n++) + m := (n + m)++$$

- **Lemma 2.2.2** For any natural number n, $n + 0 = n$
  - This cannot be proven from $0 + m = m$, as we haven't proven commutativity
- **Lemma 2.2.3** For any natural numbers n and m, $n + (m++) = (n + m)++$.
- **Proposition 2.2.4 (Addition is commutative)** For any natureal numbers n, m, $n + m = m + n$
- **Proposition 2.2.5 (Addition is associative)** For any natural numbers a, b, c, we have $(a+b)+c = a + (b + c)$

- **Proposition 2.2.6 (Cancellation law)** Let a, b, c be natural numbers such that $a + b = a + c$. Then we have that $b = c$
- **Definition 2.2.7 (Positive natural numbers)** A natural number n is positive iff it is not equal to 0
- **Proposition 2.2.8** If a is positive and b is a natural number, then $a + b$ is positive
- **Corollary 2.2.9** If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$
- **Lemma 2.2.10** Let a be a positive number. Then there exists only one natural number b such that $b{+}{+} = a$
- **Definition 2.2.11 (Ordering of the natural numbers)** Let n, m be natural numbers. We say that n is greater than or equal to m if $n = m + a$ for some natural number a
- **Proposition 2.2.12** Order is
  1. reflexive
  2. transitive
  3. anti-symmetric ($a \geqslant b, b \geqslant a \implies a = b$)
  4. preserved by addition
  5. $a < b$ iff $a{+}{+} \leqslant b$
  6. $a < b$ iff $b = a + d$ for some positive number d
- **Proposition 2.2.13** Let a, b be natural numbers. Either $a < b, a = b$ or $a > b$

2.3 Multiplication

- **Definition 2.3.1**
- **Lemma 2.3.2**
- **Lemma 2.3.3**
- **Proposition 2.3.4**
- **Proposition 2.3.5**
- **Proposition 2.3.6**
- **Corollary 2.3.7**
- **Proposition 2.3.9**
- **Definition 2.3.11**

3 Set Theory

3.1 Fundamentals

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