Proxy Variables and Feedback Effects in Decision Making*

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Abstract

When using data, often an analyst only has access to proxies or measurements of the true variables of interest. I propose a framework that models economic decision makers as 'flawed statisticians' who assume potentially noisy proxy variables are perfectly measured. Due to feedback from the decision maker's choices to the distribution over variables, a notion of equilibrium is required to close the model. I illustrate the concept with applications to policing/crime and market entry. In these examples, we see that very small imperfections in the proxy variable can lead to large distortions in beliefs. I characterize all strategies that can arise as equilibria when measurement is arbitrarily close to perfect.

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1 Introduction

The analysis of quantitative data in order to inform decisions is increasingly important to organizations and firms. For the case of private companies, McAfee et al. (2012) argue that improvements in the ability of managers to measure, store and collect information about their business can result in large performance gains. However, data used in economic decision making is often an imperfect measurement or proxy of the underlying variables.

Examples of proxy variables that play an important role in driving allocation of economic resources abound. GDP per capita guides entrepreneurs and traders in assessing the relative economic vitality of countries in which they are considering investment, and is used by governments in determining important policy decisions. Yet as a proxy for living standards it has come in for criticism[1]. The use of citation metrics is another case. Governments and academic institutions find these metrics valuable for assessing academic impact, but there is debate around the extent to which they are truly good measures[2].

In this paper I propose a framework for modelling decision makers who naively use possibly mismeasured proxies. The decision makers (henceforth DMs) in this framework are assumed to form expectations about the impact of their actions from the proxy variables they have available, treating the proxy variables as if they were *exactly identical* to the true variables. This follows a long tradition in economics and psychology of modelling economic agents as 'flawed statisticians', for example early work in behavioural economics on the 'law of small numbers' by Tversky and Kahneman (1971) to more recent work testing the 'What You See Is All There Is' heuristic in experiments (Enke, 2020).

The structure of the agent's problem is as follows. First they draw the realizations of a circumstance variable s and a signal variable z. They then choose an action variable x, and both the chosen action and signal then affect the realization

[1] See Coyle (2015) for an outline of various arguments in this debate.

[2] See Borchardt and Hartings (2018) for discussion and reference to work (Borchardt et al., 2018) providing evidence that for academic chemists there are significant differences between the level of citations a paper receives and perceived academic impact among scientists.

of an outcome variable y. Agents are assumed to have a vNM utility function over the circumstance, signal and action variables. Thus, the circumstance variable affects utility but not the outcome while the signal affects the outcome but does not affect utility directly. Finally, a vector of proxy variables $(z^\bullet, x^\bullet, y^\bullet)$ is drawn from a distribution π that depends on (z, x, y), where each of the true variables has a corresponding proxy. The DMs only have access to data that gives them knowledge of the joint distribution of the proxies. Due to the possible imperfections in the variables available, choices made by these decision makers can affect the data they use to form beliefs. I define a notion of Proxy Equilibrium which ensures consistency between these choices and the data upon which they are based.

I present two examples illustrating the framework. The first demonstrates the possibility of stark discontinuity between the extent of imperfection in proxy variables and the extent of the bias in the beliefs of the DMs. A municipality chooses the number of police in order to affect crime. In order to do so, the municipality needs to learn the relationship between police numbers and crime. If they knew this relationship perfectly, municipalities would vary police numbers in a way that depends on crime's responsiveness to policing. However, if proxy variables are used to learn this relationship, measurement error in recorded police numbers results in downward attenuation bias. If there is little or no variation in police numbers then the extent of this attenuation bias is greater. This is because more of the variation in the policing measure is then coming from measurement error. As such, there exists a Proxy Equilibrium in which municipalities do not systematically vary police numbers at all, regardless of how small the measurement error is.

The second example examines how the use of imperfect proxies results in excessive entry into markets by firms. Here there is a clear ordering over proxy 'noise' in terms of the welfare of firms, in which more noise results in a greater extent of excessive entry. Without equilibrium feedback effects, the effect of noise on entry is ambiguous. This is in contrast to other work on the behavioural bias caused by selection effects, such as Jehiel (2018) and Esponda and Pouzo (2017), in which there is no such ordering both in and out of equilibrium. The ordering results from features of how beliefs are formed under Proxy Equilibrium.

Finally, I give a characterization of all strategies that could arise as Proxy Equilibria when the proxy variables are imperfectly measured but arbitrarily close to being perfect measurements. The characterization imposes different conditions on actions that are in the support of the strategy and actions that are not. Actions in the support must give higher true expected utility than the other actions in the support at least at one circumstance realization. Actions that are not in the support must be worse that some action in the support for some full-support belief. A second result demonstrates how the multiplicity of possible equilibria is a result of strategies lacking full support over actions. When a strategy satisfies a full-support condition then proximity of proxies to perfect measurement results in beliefs of the DMs that are close to what they would be under perfect measurement.

The concept draws a distinction between the fact that the DM knows the realization the circumstance, signal and the action they have chosen but does not know how these variables covary with the outcome they wish to forecast. The story I have in mind for this is that the joint distribution over proxies is generated as a long run steady state of some learning process. The learning process is not that of a long lived agent who repeats the same decision problem enough times to generate an asymptotic sample, but instead a short-lived agent who does not generate enough experiences of the effect of their own action and signals and has to rely on a large public dataset of potentially mismeasured proxies. For concreteness, in the policing example we can imagine a sequence of short lived municipal leaders. The data generated from each municipal leader's tenure is too sparse to apply the law of large numbers, so the leaders have to draw inference from the experiences of other municipalities in other time periods by using a national dataset designed for social scientists researching crime.

The contribution of this paper is twofold. First, it contributes to the literature on solution concepts with bounded rational expectations by considering issues of measurement and proxies in an equilibrium framework. The concept generally does not fall neatly in others in the literature, and I explore these connections in Section 6. The concept demonstrates how issues of belief distortion similar to that explored in the growing literature on misspecified models can occur purely as a result of the neglect of measurement noise. Secondly, it develops applications of the concept to a variety of settings in which organizations use data to form beliefs.

2 Modelling Set Up

The space of variables V can be divided into four dimensions. There is a space of circumstances $S \subseteq \mathbb{R}$ with realization s, a space of signals $Z \subseteq \mathbb{R}$ with realization z, a space of actions $X \subseteq \mathbb{R}$ with realization x and a space of outcomes $Y \subseteq \mathbb{R}$ with realization y. The variable space can thus be described as $V = S \times Z \times X \times Y$.

The idea will be that our decision maker learns the realization of the circumstance variable s and the signal variable z, before choosing an action x resulting in a distribution over the outcome variable y. The payoff of the decision maker (henceforth DM) is defined over the circumstance, action and outcome variables by utility function $u : S \times X \times Y \to \mathbb{R}$. In examples, when a variable only takes on a single value we suppress that variable in notation.

Let $P \in \Delta(V)$ be the objective distribution over the variables. We assume throughout that this distribution admits a density p with respect to some σ-finite measure μ[3]. The causal structure between the variables is represented by the graph in Figure 2. The signal and circumstance variables affect the action variable, and in turn the signal and action variables affect the outcome variable. Using this structure, the joint density over all the variables can be factorized as follows.

$$p(y, x, z, s) = p(y|x, z)p(x|z, s)p(z, s) \tag{1}$$

The DM wants to form the conditional distribution $p(y|x, z)$. Doing this, they can calculate their objective expected utility:

[3]In all examples in this paper this will either be the counting measure for the finite case or Lebesgue measure for the continuum case.

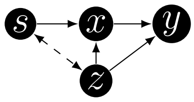

Figure 1: Causal Structure

$$U(x, z, s) \equiv \int_Y u(y, x, s) p(y|x, z) d\mu(y) \tag{2}$$

The objective conditional distribution $p(y|x, z)$ is invariant to the conditional distribution over actions $p(x|z, s)$. Thus there is no need to account for equilibrium effects and the problem of maximizing (2) boils down to a standard single agent decision problem.

2.1 Proxy variables

In order to form beliefs about the distribution of outcomes conditional on a given action being taken when a given signal is realized, the DM needs to have information on the joint distribution of (y, x, z). We assume that the DM can only access the joint distribution over proxies for these variables. Each of the three variables has a respective proxy that can take any of the values the variable it is a proxy for can take. We denote a realization of the proxy for the signal, action and outcome by $(z^\bullet, x^\bullet, y^\bullet) \in Z \times X \times Y$ respectively. We define a *proxy mapping* $\pi : Y \times X \times Z \to \Delta(Y \times X \times Z)$ that induces a distribution over the proxies for any realization of the true variables. Denote any Borel subset of the variable space $Y \times X \times Z$ by $W \in \mathcal{Y} \times \mathcal{X} \times \mathcal{Z}$. The induced distribution over proxy variables is then:

$$P_\pi(W) = \int_{Y \times X \times Z} \pi(W|y, x, z) p(y, x, z) d\mu \tag{3}$$

We assume that the proxy mapping is such that the induced distribution over proxies P_π admits a density p_π. In order to form beliefs about how their action affects the distribution over outcomes, the DM needs to form conditional beliefs:

$$p_\pi(y^\bullet|z^\bullet, x^\bullet) = \frac{p_\pi(y^\bullet, x^\bullet, z^\bullet)}{p_\pi(x^\bullet, z^\bullet)} \tag{4}$$

To ensure this is well defined we assume that the proxy mapping and the distribution over true variables is such that $p_\pi(x^\bullet, z^\bullet) > 0$ for any realization $(x^\bullet, z^\bullet)$. Given this distorted belief distribution, the agent in circumstance s with signal z chooses an action x to maximize the perceived utility given below.

$$V(x, z, s; p_\pi) = \int_{Y^\bullet} u(y = y^\bullet, x, s) p_\pi(y^\bullet|z^\bullet = z, x^\bullet = x) d\mu(y^\bullet) \tag{5}$$

2.1.1 The proxy mapping

Consider that i is any of the three variables, and that we can denote a realization of the true variable by v_i and a realization of the proxy by $v_i^\bullet$. It is possible that a proxy is a perfect measure for the underlying true variable, $v_i^\bullet = v_i$ almost everywhere for some variable(s) i. Indeed, for all the applications in this paper some of the variables are perfectly observed. In the case where $v_i^\bullet \neq v_i$ with nonzero probability, we say that $i^\bullet$ is a *mismeasurement* of i. In examples, for simplicity we generally avoid drawing a distinction between the true and proxy variable when the proxy is a perfect measurement.

A proxy mapping that induces an identical joint distribution over the proxy variables and the true variables —for any initial distribution of the true variables— is called the *perfect measurement* mapping. Throughout the paper, we say that the beliefs induced by the perfect measurement mapping comprise the *rational expectations benchmark* or induce *correct beliefs*. One could argue that what comprises rational expectations is ambiguous, given the DM only has knowledge of the proxies. However, what it means to correctly form beliefs under this ambiguity is not the focus of this paper.

In our examples, we illustrate the causal dependencies between the variables by Directed Acyclic Graphs (DAGs). In these graphs, a link $\rightarrow$ between two variables indicates that the variable being pointed to is independent of all other variables conditional on the variables that point into it. DAGs are used to model causal misperceptions in the related concept of Spiegler (2016), and we discuss the relationship between the two concepts further in Section 6.1.

2.2 Equilibrium

We can see how the decision maker's strategy can affect expectations by considering the case where all variables are finite. For any proxy mapping π we can write the perceived conditional distribution as follows.

$$p_\pi(y^\bullet|x^\bullet, z^\bullet) = \frac{\sum_{y,x,s,z} \pi(y^\bullet, x^\bullet, z^\bullet|y, x, z)p(y|x, z)p(x|s, z)p(s, z)}{\sum_{y^\bullet,y,x,s,z} \pi(y^\bullet, x^\bullet, z^\bullet|y, x, z)p(y|x, z)p(x|s, z)p(s, z)} \tag{6}$$

We illustrate the dependence of this distribution on the strategy $p(x|s, z)$ using the following binary version of the policing example.

Example 1. *The municipal leader learns the realization of a circumstance variable s determining whether the cost of crime is high $s = \bar{s} > 0$ or low $s = 0$, before choosing to whether to hire more police officers $x = 1$ or not $x = 0$. The hiring of police officers in turn affects whether crime is high $y = 1$ or low $y = 0$.*

Let the relationship between the policing variables and the crime variable be given by $p(y = 1|x, s) = \beta x+(1-\beta)(1-x)$ where $\beta \in (\frac{1}{2}, 1)$. The prior distribution over the cost of crime variable is $p(\bar{s}) = \frac{1}{2}$. We assume the crime variable y is perfectly measured, but the policing variable x is potentially not. We can write the simplest form of measurement error in the policing variable using the proxy mapping $\pi_x(x^\bullet = x|x) = \lambda$ where $\lambda \in (\frac{1}{2}, 1]$. As $\lambda \to 1$ we have close to perfect measurement.

The payoff function of the DM is given by $u(y, x, s) = s(x + y) - x$. Denote the ex-ante strategy as $\sigma(x = 1) = \frac{1}{2}p(x = 1|\bar{s}) + \frac{1}{2}p(x = 1|0)$ and $\sigma(x = 0) = \frac{1}{2}p(x = 0|\bar{s}) + \frac{1}{2}p(x = 0|0)$. The perceived conditional distribution is then:

$$p_\pi(y^\bullet = 1|x^\bullet = 1) = \frac{\sigma(x = 1)\lambda\beta + \sigma(x = 0)(1 - \lambda)(1 - \beta)}{\sigma(x = 1)\lambda + \sigma(x = 0)(1 - \lambda)} \tag{7}$$

This expression is clearly not invariant to the strategy. For example if $\sigma(x = 1) = \sigma(x = 0)$ it is equal to $\lambda\beta + (1 - \lambda)(1 - \beta)$ while if $\sigma(x = 1) = 1$ and

$\sigma(x = 0) = 0$ *it is equal to* β.

Thus, in general to characterize the DM's choices we need to define an equilibrium concept in order to establish consistency between strategies and beliefs. To ensure that conditional distributions are well defined we first define an equilibrium with a small trembling probability and then define an equilibrium as the limit when this probability goes to zero.

We make the following technical definitions to facilitate the description of Proxy Equilibrium. We define an *interval* as being a subset $X^{int} = \{x \in X : a \leq x \leq b\} \subset \mathbb{R}$. We say a sequence of strategies $\{\sigma\}_{j=1}^{\infty}$ *converges* to strategy $\bar{\sigma}$ if for every $s \in S$ the sequence of probability measures $\{\sigma(.|s)\}_{j=1}^{\infty}$ converges in distribution to the probability measure $\bar{\sigma}(.|s)$.

Definition 1. *A strategy mapping that admits a full-support density* $\sigma^*_{\epsilon}(x|s,z)$ *for every* $s \in S$, $z \in Z$ *is an* ϵ-***Proxy Equilibrium*** *if the following two conditions hold:*

1. *Belief density* p_π *is induced by the proxy mapping* π *and the true distribution over variables* P *according to* (3). *The true distribution admits a density* p *such that* $p(x|s,z) = \sigma^*_{\epsilon}(x|s,z)$.

2. *For every* $s \in S$, $z \in Z$ *and strategy* σ, *define the following set:*

$$X(s, z; \sigma) \equiv$$
$$\{x \in X : x \notin \arg\max \int_{Y^\bullet} u(y = y^\bullet, x, s) p_\pi(y^\bullet | x^\bullet = x, z^\bullet = z; \sigma) d\mu(y^\bullet)\}$$

Then for every interval $I \subseteq X(s, z; \sigma^*_{\epsilon})$, *the strategy mapping is such that* $\sigma^*_{\epsilon}(I|s,z) < \epsilon$

Definition 2. *A strategy* σ^* *is an* ***Proxy Equilibrium*** *if there exists a sequence* $\{\sigma^*_l\}_{l=1}^{\infty}$ *converging to* σ^* *as well as a sequence* $\epsilon^l \to 0$, *such that for every* l, σ^*_l *is an* ϵ^l*-Proxy Equilibrium.*

The first part of the definition describes how the beliefs of the DM are induced by proposed equilibrium strategy. The second part ensures that action-signal-circumstance combinations that are not in the best response correspondence must have vanishing probability under the equilibrium strategy.

When the variable space is finite, we can show the existence of at least one Proxy Equilibrium using conventional methods.

Proposition 1. *Assume the set V is finite. Then a Proxy Equilibrium exists.*

Proof. In Appendix □

3 An illustrative example: Police and Thieves

In this example we consider the leader of a municipal authority, who has to make a decision on the number of police officers to hire. The municipal leader wants to hire police officers in order to reduce crime. We assume that there is noise in the measured variable for police numbers. Concern about measurement error in police staffing figures is not unprecedented. It is argued by Chalfin and McCrary (2018) that based on discrepancies between official data and administrative and census information there is significant measurement error in police staffing numbers in the literature estimating the effect of police numbers on crime. For expositional purposes, we assume that the crime variable is measured perfectly. This turns out not to make a difference to the equilibria that we characterize.

The structure of the problem facing the municipal leader is as follows, first they learn the realization of a variable affecting the cost of crime in their municipality s. This is assumed to be distributed normally in the population of municipalities used in the dataset under consideration, $s \sim \mathcal{N}(0, \sigma_s^2)$. The municipal leader then chooses the change in the number of police officers. This affects the change in crime observed under their leadership via the relationship $y = \alpha + \beta x + u$, where $u \sim \mathcal{N}(0, \sigma_u^2)$ and $\beta < 0$. In the available data, it is assumed that changes in police numbers are measured as $x^\bullet = x + \epsilon$, where ϵ is normally distributed measurement error $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. The relationship between the variables can be characterized by the following DAG.

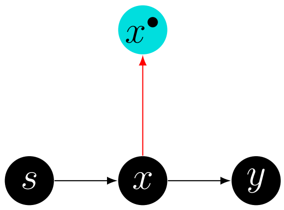

Figure 2: Crime and Policing

The utility function of the municipal leader trades-off crime and policing costs. Higher s is assumed to reflect higher costs of crime relative to altering police numbers.

$$u(y, x, s) = -s \cdot y - \frac{1}{2}x^2 \tag{8}$$

Denote the rational expectations benchmark for how policing affects crime levels in expectation by $\mathbb{E}[y|x] = f(x) = \alpha + \beta x$. We can see that by plugging this in to the utility function and calculating the best response that the optimal strategy under rational expectations for the municipal leader is to set police numbers such that $x^*(s) = -\beta s$. Thus the rational expectations benchmark is for police numbers to be increased by more when the costs of crime is larger (higher s) and the effect of police numbers on crime is greater (higher $|\beta|$). Define a *linear equilibrium* as an equilibrium in which the strategy of the policy maker can be expressed as a linear function of the cost variable, $x(s) = \theta_0 + \theta_1 s$ for some $(\theta_0, \theta_1) \in \mathbb{R}^2$. We can characterize all the linear equilibria of the model as follows[4].

Proposition 2. *There is always a linear equilibrium in which the municipal leader never changes police numbers, with best response* $x^{nv}(s) = 0$.

In addition, if $|\beta| \geq 2\frac{\sigma_\epsilon}{\sigma_s}$, *then there exist two additional linear equilibria, with best response* $x^-(s) = (-\frac{1}{2}\beta - \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$ *and* $x^+(s) = (-\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$.

There are no other linear equilibria.

Proof. In Appendix $\square$

[4]Due to the difficulty in characterizing non-linear equilibria in this setting, we do not attempt to do so.

Due to the measurement error in the police numbers proxy, there is generally downward attenuation bias in the municipal leaders estimate of the expected change in the level of crime for any given change in police numbers. However, when there is more variation in police staffing numbers the measurement error is a smaller fraction of the total variance of the proxy. This means the downward attenuation effect is lessened compared to when there is little or no variation in true police staffing numbers. This effect generates potential multiplicity of equilibria.

We illustrate the solution method and equilibria in Figure 3 below. We have the following expression for the marginal effect of increasing police numbers, given the expectations induced by the strategy $x(s) = \theta_1 \cdot s$:

$$-\frac{\partial \mathbb{E}[y^\bullet | x^\bullet]}{\partial x^\bullet} = \frac{-\beta\theta_1^2\sigma_2^2}{\theta_1^2\sigma_2^2 + \sigma_\epsilon^2} = g(\theta_1; (\beta, \sigma_s, \sigma_\epsilon)) \tag{9}$$

An equilibrium best response has to be such that $\theta_1 = g(\theta_1; (\beta, \sigma_s, \sigma_\epsilon))$. The figure below shows how this equation characterises the equilibria for two different parameter sets. We have a case in which the only equilibrium is the no variation equilibrium and a case in which all three equilibria exist.

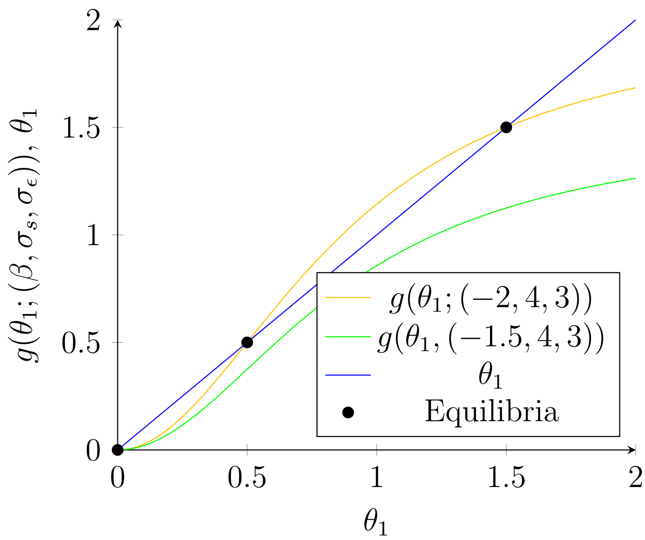

Figure 3: Best response quadratic

Adding normally distributed measurement error to the crime variable, so that $y^\bullet = y + v$ with $v \sim \mathcal{N}(0, \sigma_v^2)$, does not change either the set of Proxy Equilibria nor does it change the rational expectations benchmark. That the rational expectations benchmark is unchanged is easy to see due to linearity of expectations. The proxy-equilibrium case is due to both the linearity of the conditional expectation and the fact that the additional variance in $y^\bullet$ does not affect the marginal perceived incentive over the policing variable.

Another interesting feature is that one of the equilibria has the municipalities not varying police numbers at all, regardless of how great the effect of policing on crime is. This extreme, no variation equilibrium exists for any small amount of measurement error in the proxy $\sigma_\epsilon^2 > 0$ no matter how close to zero. For σ_ϵ^2 close to zero, the case when the proxy is close to a perfect measurement, the equilibrium with the smaller positive amount of variation in policing converges to the zero variation equilibrium while the equilibrium with the larger amount of variation converges to the rational expectations benchmark.

4 Endogenous Hubris and Market Entry

Businesses entering into new markets have high rates of failure. Using data from the US Census Bureau Haltiwanger (2015) calculates that half of new firms exit the market within 5 years. In UK data, 38 percent of enterprises newly born in 2016 survived 5 years[5]. A literature in business and economics attributes these seemingly excessive levels of market entry to overoptimism on the part of the potential market entrants, see Hayward et al. (2006), Cooper et al. (1988), Malmendier and Tate (2005).

We build an application of our solution concept that generates firms that have an upwardly biased assessment of the payoffs from entering new markets as a feature of equilibrium. Firms draw on noisily recorded data drawn from past entrants. There is a variable $z \in [0, 1] \equiv Z$, representing the location of markets in some space, which could be geographical or based on demographic information. This variable is distributed such that it admits a continuous full-support density function $p(z)$. After learning the realization of this variable, the potential market

[5] This statistic is from Office for National Statistics (2022).

entrant has to make a binary decision on whether to enter $x = 1$ or not $x = 0$. The payoff of the entrant is measured via an outcome variable $y \in \mathbb{R} \equiv Y$ representing the profitability of the enterprise, so that $u(y, x, s) = y$. The outcome variable is determined by both the entry decision and the market location variable by the following relationship.

$$\mathbb{E}[y|x, z] = \int_Y yp(y|x, z)d\mu(y) = \begin{cases} m(z) \text{ if } x = 1 \\ 0 \text{ if } x = 0 \end{cases} \tag{10}$$

We assume that the function $m : Z \to \mathbb{R}$ is strictly increasing, bounded and right-continuous, with a single point of crossing $\alpha \in [0, 1]$ such that $m(z) < 0$ for all $z \in [0, \alpha)$ and $m(z) \geq 0$ for $z \in [\alpha, 1]$. Thus for high enough realizations of the market location variable, the expected profitability of entry is always greater than the payoff of zero from not entering. We assume the potential entrant does not have data on how the market location z varies with the outcome and action variable, but instead has access to a noisy recorded proxy variable $z^{\bullet}$. The idea is that each market has a very granular definition, and in data it can only be recorded in an imprecise fashion. This could be due to data protection reasons or due to the categories available to the data recorder not being as fine-grained as the data itself.

Thus we assume the proxy variable is generated by a mapping that has the following 'window' form. There is some parameter $h \in (0, \frac{1}{2})$ such that for every $z \in [h, 1 - h]$ we have that $z^{\bullet}$ is uniformly distributed on $[z - h, z + h]$. For all $z \in [0, h)$ we have that $z^{\bullet}$ is distributed uniformly on $[0, 2h)$ and for all $z \in (1-h, 1]$ we have that $z^{\bullet}$ is distributed uniformly on $z \in (1 - 2h, 1]$. This window form of proxy noise is similar to the notion of similarity used in Steiner and Stewart's (2008) model of learning in games. The conditional independence relationships between the true variables and the proxy are illustrated in the DAG below.

Under rational expectations the best response of the potential entrant is clear; when $z \in [0, \alpha)$ $x = 0$ is optimal while for $z \in [\alpha, 1]$ the payoff from entering is above zero and therefore optimal. We are going to see that in equilibrium, there is

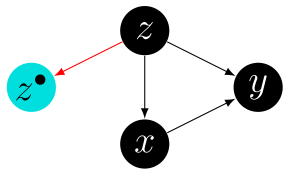

Figure 4: Market Entry

over-entry. Since in the equilibrium data there is negligible observations of entrants below a certain market location, the proxy observations for these markets are disproportionately higher location markets that have been misclassified as lower ones. This leads to an overestimate of the payoff from entering at these lower levels. However, enough over-entry reduces the extent of this proxy bias and in equilibrium the DM is indifferent between entering or not at some cut-off strength level below the cut-off they would enter at under rational expectations.

Given an induced perceived distribution over the outcome variable $p_\pi(y|x, z^\bullet)$, we give an expression for the perceived expected utility below.

$$
\begin{aligned}
U(z, x = 1; q_\pi) &= \int_Y y p_\pi(y|x = 1, z^\bullet = z) d\mu(y) \\
&= \int_Y y [\int_Z p(y|x = 1, \tilde{z}) p_\pi(\tilde{z}|z^\bullet = z, x = 1) d\mu(\tilde{z})] d\mu(y) \\
&= \int_Z [\int_Y y p(y|x = 1, \tilde{z}) d\mu(y)] p_\pi(\tilde{z}|z^\bullet = z, x = 1) d\mu(\tilde{z}) \\
&= \int_Z m(\tilde{z}) p_\pi(\tilde{z}|z^\bullet = z, x = 1) d\mu(\tilde{z}) \qquad (11)
\end{aligned}
$$

The perceived utility of $x = 0$ at z is always zero; $U(z, x = 0; q_\pi) = 0$. We can see that the perceived utility depends on the distribution $p_\pi(\tilde{z}|z^\bullet = z, x = 1)$ induced by the strategy σ. Given the distribution over proxies, this can be calculated as follows.

$$
p_\pi(z|z^\bullet, x=1) = \frac{p_\pi(z, z^\bullet, x=1)}{\int_Z p_\pi(\hat{z}, z^\bullet, x=1) d\mu(\hat{z})}
$$

$$
= \begin{cases} \frac{1_{[z\in[z^\bullet - h, z^\bullet + h]]} p(z)\sigma(x=1|z)}{\int_{z^\bullet - h}^{z^\bullet + h} p(\hat{z})\sigma(x=1|\hat{z}) d\mu(\hat{z})} & \text{if } z^\bullet \in [h, 1-h] \\ \frac{1_{[z\in[0,h]]} p(z)\sigma(x=1|z)}{\int_0^h p(\hat{z})\sigma(x=1|\hat{z}) d\mu(\hat{z})} & \text{if } z^\bullet \in [0, h) \\ \frac{1_{[z\in[1-h,1]]} p(z)\sigma(x=1|z)}{\int_{1-h}^1 p(\hat{z})\sigma(x=1|\hat{z}) d\mu(\hat{z})} & \text{if } z^\bullet \in (1-h, 1] \end{cases} \tag{12}
$$

We make the following endpoint assumptions, which ensure that the firm will not enter at low values of z and there is always an equilibrium in which the firm will enter at high values of z.

Assumption 1. *We say that $m(.)$, $p(.)$ and h satisfy the* ***endpoint assumptions*** *if $\int_h^{2h} m(z)p(z)d\mu(z) < 0$ and $\int_{1-h}^1 m(z)p(z)d\mu(z) > 0$.*

We can then show existence and characterize the Proxy Equilibria.

Proposition 3. *Assume the primitives $m(.)$, $p(.)$ and h satisfy the* ***endpoint assumptions****. Then there is a cut-off $\bar{z} \in [h, 1-h]$ such that there is a Proxy Equilibrium with strategy $\sigma(x=1|z) = 0$ for $z \in [0, \bar{z}]$ and $\sigma(x=1|z) = 1$ for $z \in (\bar{z}, 1]$.*

In addition, there is always a Proxy Equilibrium where $\sigma(x=1|z) = 0$ for all $z \in [0, 1]$.

There are no other Proxy Equilibria.

Proof. In Appendix □

We then have the following result, which gives us that in cases where there is entry in equilibrium there is excessive entry. In addition, we see that noisier proxies —in the form of higher h— always lead to greater levels of excessive entry.

Proposition 4. *The cut-off in the Proxy Equilibrium with positive probability of entry, $\bar{z}$, is always strictly less than α.*

Moreover, consider two noise parameters $1 > h_2 > h_1 > 0$, such that given $m(.)$, $p(.)$ the endpoint assumptions are satisfied for both. We have that the cut-off $\bar{z}(h_2)$ under the positive entry Proxy Equilibrium with noise parameter h_2 is strictly less that the cut-off $\bar{z}(h_1)$ under h_1.

Proof. In Appendix □

The intuition for this result is as follows. In general —for a fixed belief distribution— it is ambiguous whether larger h increases or decreases the payoff from entry at any given z. However, in equilibrium what matters is the beliefs at the pivotal cut-off $\bar{z}$. At the cut-off the DM must be indifferent between entering and not. An increase in h will always lead to greater weight on the part of the function $m(z)$ that is above the cut-off, in particular greater weight on the positive part of $m(z)$. This pushes up the expected payoff from entering strictly above zero at this cut-off, and the new cut-off at the larger h must be below in order to restore indifference.

In contrast if there are no equilibrium effects and the belief distribution is fixed, we can construct cases where entry is excessive and increasing h results in a best response with less entry.

Example 2. *Let the function $m(.)$ take the following form.*

$$m(z) = \begin{cases} z - \frac{5}{2} & \textit{if } z \in [0, \frac{1}{4}) \\ z - \frac{1}{2} & \textit{if } z \in [\frac{1}{4}, \frac{1}{2}) \\ z + \frac{1}{2} & \textit{if } z \in [\frac{1}{2}, 1] \end{cases} \tag{13}$$

Assume the distribution over signals $p(z)$ is uniform and the conditional distribution $p(.|z^\bullet, x = 1)$ is set exogenously to that which would be induced by the strategy $\sigma(x = 1|z) = 1$ for all z. This means that $p(.|z^\bullet, x = 1)$ is uniform in $z \in [z^\bullet - h, z^\bullet + h]$ for each $z^\bullet \in [h, 1 - h]$ and uniform on $z \in [0, h)$ and $z \in (1 - h, 1]$ for $z^\bullet \in [0, h)$ and $z^\bullet \in (1 - h, 1]$ respectively.

If $\frac{1}{4} < h < \frac{1}{2}$, then the cut-off signal at which the DM switches from $x = 0$ to $x = 1$ is $\bar{z}(h) = \frac{1+2h}{3+2h} \in [0, \frac{1}{2})$. The parameters are such that $0 < \bar{z}(h) - h < \frac{1}{4}$ and

$\frac{1}{2} < \bar{z}(h) + h < 1$. Since $\frac{d\bar{z}(h)}{dh} = \frac{4}{(2h+3)^2} > 0$ we can then see that as h increases by a small amount, the cut-off increases. This reduces the extent of entry at signals in $[0, \frac{1}{2})$ where it gives a negative true expected payoff.

4.0.1 Differences with Jehiel (2018)

There are similarities between this application and the model of selection bias induced overconfidence of Jehiel (2018). In his model as in ours, there is an upwardly bias distortion in beliefs in equilibrium which induces an entry action to occur at a threshold earlier than under rational expectations. However, there are subtle differences between the two models, and the result that greater 'noise' always induces greater over-entry that holds in our example under Proxy Equilibrium does not necessarily hold in Jehiel (2018).

In that paper, the reason why the effect of noise on entry is ambiguous is that a more accurate signal mechanically means that any fixed strategy is more selective, which increases the extent of upward bias at each signal. There is a countervailing effect in that for a fixed level of selection, a more accurate signal increases the expected payoff at higher signals and reduces the expected payoff at lower signals. In our application, only the latter effect is operating and thus we get the result that more noise increases over-entry, while in Jehiel (2018) which effect is larger depends on the parameterization.

5 Almost Perfect Proxies

In this section we present two results. Our first result characterizes the set of all strategies that can arise as Proxy Equilibria even as the variables are arbitrarily close to being perfectly measured. We call these strategies Potentially Implementable. If a strategy is in the set, then we can choose a particular proxy mapping that implements that strategy as an equilibrium. Moreover, this proxy mapping has very close to perfect measurement and has perfect measurement of all the outcome variables. If a strategy does not meet the conditions to be Potentially Implementable, then it cannot be implemented as a Proxy Equilibrium for any proxy mapping that is above a certain level of proximity to perfect measurement. The result is proven for the case where all variables spaces are finite, although it seems likely that the basic idea of the proof extends to non-finite spaces.

The second result gives conditions under which we have convergence to rational expectations. It shows that if the joint density over variables satisfies a full support assumption, then beliefs become arbitrarily close to rational expectations as the proxy variables become close to perfect measurements. This result concerns potentially out of equilibrium beliefs, and can be used as a diagnostic when considering equilibria in which the full-support assumption does not hold. For example, in our policing application the full-support assumption does not always hold and therefore we can have large belief distortions even as the proxy noise is close to zero.

We use the following concept of statistical distance to define a notion of proximity of the proxy variables to perfect measurement. The total variation distance between probability measures Q_1 and Q_2 on measure space $(\Omega, \mathcal{A})$ is:

$$TV(Q_1, Q_2) = \sup_{A \in \mathcal{A}} |Q_1(A) - Q_2(A)| \tag{14}$$

Denote $w = (y, x, z)$ and define the *perfect measurement proxy mapping* as $\pi_\delta : Y \times X \times Z \to \Delta(Y \times X \times Z)$ such that $P_\pi(W) = \int_{Y \times X \times Z} \pi_\delta(W^\bullet = W|w)p(w)d\mu(w) = P(W)$ for every Borel set $W \in \mathcal{Y} \times \mathcal{X} \times \mathcal{Z}$ and any P. We then have the following definition.

Definition 3. *Given $\eta > 0$, we say the proxy distortion mapping π is strongly η-close to perfect if:*

$$\sup_{w \in Y \times X \times Z} TV(\pi(.|w), \pi_\delta(.|w)) < \eta \tag{15}$$

We define the conditions required for a strategy to be Potentially Implementable below. The definition requires that the strategy meets different conditions for actions that are in the support of the strategy and actions that are not. The first condition requires that any action that is in the support is a best response against all the other strategies in the support for some realization of the circumstance variable. The second condition requires that for every action not in the support of the strategy, we can find a full support belief that results in lower perceived utility than the rational expectations benchmark utility from any action in the support, for every realization of the circumstance variable.

Define the distribution over actions at signal z induced by the strategy σ as:

$$\sigma(x|z) = \int_S \sigma(x|s,z)p(s|z)d\mu(s) \tag{16}$$

Definition 4. *Let $Y \times X \times Z \times S$ be finite. A strategy $\sigma^* : S \times Z \to \Delta(X)$ is* ***Potentially Implementable*** *if at every $z \in Z$, the following two conditions hold.*

1. *For any action $x \in supp\{\sigma^*(.|z)\}$ there exists an $s \in S$ such that, for every $x' \in supp\{\sigma^*(.|z)\}$:*

$$\sum_{y\in Y} u(y,x,s)p(y|x,z) \geq \sum_{y\in Y} u(y,x',s)p(y|x',z) \tag{17}$$

2. *For every action $x^{ns} \notin supp\{\sigma^*(.|z)\}$, there exists a full-support conditional distribution $q : X \times Z \to \Delta(Y)$ such that for any $s \in S$ and $x^s \in supp\{\sigma^*(.|s,z)\}$ we have that:*

$$\sum_{y\in Y} u(y,x^s,s)p(y|x^s,z) \geq \sum_{y\in Y} u(y,x^{ns},s)q(y|x^{ns},z) \tag{18}$$

We then have the following result.

Proposition 5. *Let $Y \times X \times Z \times S$ be finite and $supp\{p(.|x,z)\} = Y$ for every $(x,z) \in X \times Z$.*

Then $\sigma^* : S \times Z \to \Delta(X)$ is a Proxy Equilibrium under some proxy mapping that is strongly η-close to perfect for all small enough $\eta > 0$ if and only if it is a ***Potentially Implementable*** *strategy.*

Moreover, it is sufficient to construct this proxy mapping such that the outcome variable is perfectly measured.

Proof. In Appendix □

The sufficiency part of the result is proven by constructing a proxy mapping that has a small probability of randomly allocating a particular realization of the true outcome vector to the proxy of an action-signal combination that has zero probability under the proposed equilibrium strategy. The particular outcome vector is chosen so as to deter the DM from choosing that particular zero-probability action-signal combination. The necessity part follows from a stronger version of our second result, that we have convergence of beliefs to rational expectations if the proxy mapping converges to perfect measurement.

We can apply this result to our binary policing example from Section 2.2 to analyze what strategies can arise as Proxy Equilibria for arbitrarily small measurement noise.

Example 1 (Continued)**.** *The action $x = 0$ must be in the support of any Proxy Equilibrium. This is because at $s = 0$ it is a best response regardless of the beliefs of the DM about how x covaries with y. Any strategy in which $x = 0$ is not in the support must violate the second condition of Definition 4.*

There are three cases at which different strategies are potentially implementable.

1. *If $\bar{s} \in [0, \frac{1}{2\beta})$ then we must have $x = 0$ chosen at $\bar{s}$ with probability one in any potentially implementable strategy. This is because at $s = \bar{s}$ the following inequality must hold for $x = 1$ to be a best response.*

$$\bar{s}(1 + p_\pi(y = 1|x^\bullet = 1)) - 1 \geq \bar{s}p_\pi(y = 1|x^\bullet = 0) \tag{19}$$

   A strategy in which $x = 1$ is chosen with positive probability at $\bar{s}$ is one in which both actions are in the support. Thus we must check if the first condition of Definition 4 is satisfied as beliefs about both actions converge to the rational expectation benchmark as the proxy mapping tends to the perfect measurement mapping. Since it is not if $\bar{s} \in [0, \frac{1}{2\beta})$, we have our claim.

2. *If $\bar{s} \in [\frac{1}{\beta}, \infty)$ then it must be the case that $x = 1$ is chosen with probability one at $\bar{s} = 1$. For these parameters, the first condition of Definition 4 is satisfied. A strategy in which $x = 0$ at $s = \bar{s}$ with some probability is not potentially implementable. This is because satisfying the second condition would require:*

$$\bar{s}(1-\beta) \geq \bar{s}(1 + p_\pi(y = 1 | x^\bullet = 0)) - 1 \tag{20}$$

This is violated for any beliefs in which $p_\pi(y = 1|x^\bullet = 0)) > 0$ if $\bar{s} \geq \frac{1}{\beta}$.

3. *If $\bar{s} \in [\frac{1}{2\beta}, \frac{1}{\beta})$, then it is possible to implement strategies in which $x = 1$ at $s = \bar{s}$ or $x = 0$ for all s. The first type of strategy can be implemented by the perfect measurement mapping, which is trivially arbitrarily close to perfect measurement. The second type of strategy can be implemented by the following proxy mapping:*

$$\pi(y^\bullet = y, x^\bullet = x | y, x) = 1 - \eta + \eta y$$
$$\pi(y^\bullet = y, x^\bullet \neq x | y, x) = \eta(1 - y)$$

For small $\eta > 0$, this results in beliefs such that $p_\pi(y = 1|x^\bullet = 1) \to 0$ and $p_\pi(y = 1|x^\bullet = 0) \to 1 - \beta$ as the probability that $x = 1$ at $s = \bar{s}$ tends to zero. This can then sustain the proposed strategy as a proxy equilibrium as it satisfies inequality (20) for $\bar{s} \in [0, \frac{1}{\beta})$.

For our second result, we can weaken our notion of distance to perfect measurement so that it is specific to a particular distribution P. This weaker definition is sufficient.

Definition 5. *Given $\eta > 0$, we say the proxy mapping π is η-close to perfect given the distribution over true variables P if:*

$$TV(P, P_\pi) < \eta \tag{21}$$

Under the following assumptions, we can always ensure the perceived belief of the DM induced by the proxy mapping is close to rational expectations with a proxy mapping that is close enough to perfect.

Assumption 2. *The distribution F over variables in $Y \times X \times Z$ is said to satisfy the* ***full support assumption*** *if it admits a density $f(\tilde{y}, \tilde{x}, \tilde{z})$ such that $f(\tilde{x}, \tilde{z}) > 0$ for every realization $(\tilde{x}, \tilde{z}) \in X \times Z$.*

This assumption rules out zero probability events in the denominator of conditional probabilities, which then ensures the convergence of the joint distribution is passed through into the factorized conditional density. We can then show that a continuity property holds for the perceived conditional distribution of y given (x, z).

Proposition 6. *Assume the* ***full support assumption*** *holds for the true distribution P.*

Then for μ almost every $(y, x, z) \in Y \times X \times Z$, for any $\epsilon > 0$, there exists an $\eta > 0$ such that if the proxy mapping π is ***η-close to perfect*** *given true distribution P and induces a distribution over the proxy variables that satisfies the* ***full support assumption****, then $|p_\pi(y^\bullet = y | x^\bullet = x, z^\bullet = z) - p(y|x, z)| < \epsilon$.*

Proof. In Appendix □

It is clear that the policing example does not satisfy the full support assumption when the no variation equilibrium strategy is played. However, in the cases where the full support assumption is satisfied then Proposition 6 holds and we have beliefs that are close to rational expectations for proxies that are close enough to perfect measurements. We can use the Hellinger distance, convergence in which implies convergence in the Total Variation distance, to derive an expression for the distance between any joint gaussian for proxies and true variables. For example, we can obtain an expression for the square of the Hellinger distance between the distribution for the policing variable $\mathbf{x}$ and its proxy, which are distributed $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $x^\bullet \sim \mathcal{N}(\mu_x, \sigma_x^2 + \sigma_\epsilon^2)$ respectively.

$$H^2(P^{\mathbf{x}}_{\pi}, P^{\mathbf{x}}) = 1 - \sqrt{\frac{2\sigma_x\sqrt{\sigma_x^2 + \sigma_\epsilon^2}}{2\sigma_x^2 + \sigma_\epsilon^2}} \tag{22}$$

We can then see that as $\sigma_\epsilon \to 0$ we can make the two distributions arbitrarily close and thus satisfy our requirements for Proposition 6.

6 Related Literature

The strand of literature that this paper is most clearly related to is that on equilibrium solution concepts with bounded rational expectations. In particular, the work on using Bayesian Networks as a formalism to model causal misperceptions originating from Spiegler (2016) and developed to explore interactive beliefs in games (Spiegler, 2021); political narratives (Eliaz and Spiegler, 2020), (Eliaz et al., 2022); persuasion (Eliaz et al., 2021); contract theory (Schumacher and Thysen, 2022) and deception (Spiegler, 2020). This is due to the latter concept requiring that actions and signals are perfectly observed. Other solution concepts in this tradition include the Cursed Equilibrium of Eyster and Rabin (2005), the Behavioural Equilibrium of Esponda (2008) and the Analogy Based Expectation Equilibrium of Jehiel (2005), Jehiel and Koessler (2008).

The Berk-Nash equilibrium of Esponda and Pouzo (2016) supplies a framework that nests many of these concepts and provides a foundation in the literature on dynamic misspecified learning. We discuss how Proxy Equilibrium and Berk-Nash Equilibrium relate later in this section. Papers in the broader misspecified learning literature have explored overconfidence about one's ability; Heidhues et al. (2018), social learning; Bohren and Hauser (2021) and connections to Berk-Nash Equilibrium; Fudenberg et al. (2021). In particular, the work of Frick et al. (2020) on fragile social learning has a similar flavour to our paper. They show that arbitrarily small misperceptions about the distribution of other player's types can generate large breakdowns of information aggregation, similar to our results on arbitrarily small imperfections in proxies leading to large distortions in beliefs.

We can see this solution concept literature as modelling players whose actions contribute to an long-run steady state distribution of the outcomes of past decisions in the same or similar situations. In contrast, there is a literature modelling players in games as extrapolating from small samples of the equilibrium behaviour of other players, the seminal work being Osborne and Rubinstein (1998) and Osborne and Rubinstein (2003). Several recent papers developing similar ideas include Salant and Cherry (2020), Patil and Salant (2020) and Gonçalves (2022).

This paper also connects to a body of work on naive inference from selected observations as a form of decision making bias. This models of sampling investors in Jehiel (2018) and elections with retrospective voters in Esponda and Pouzo (2017). Spiegler (2017) explores a procedure in which an analyst extrapolates from a dataset with partially missing information. Fudenberg et al. (2022) presents an equilibrium concept in which agents have selective recollection of their past experience. In all of these works, agents are considering a partially missing distribution. Under Proxy Equilibrium, data does not have to be fully missing but can be distorted by measurement error instead.

Finally, there is a link between this paper and the literature on overconfidence in the sense of over-precision as discussed in Moore and Healy (2008). As in the case of over-precision, in Proxy Equilibrium agents underestimate the extent of the divergence of observable variables from true variables. The size of the overconfidence literature makes it impossible to cover fully here, but examples modelling over-precision specifically include applications to political ideology (Ortoleva and Snowberg, 2015), speculative bubbles in finance (Scheinkman and Xiong, 2003) and volatility in securities markets (Daniel et al., 1998).

6.1 Relationship to Bayesian Network Equilibrium

We can relate the proxy solution concept to both the DAG causality literature (see Pearl (2009)) and also the Bayesian Network Equilibrium (henceforth BNE) of Spiegler (2016). In Spiegler (2016) the relationship between variables is modelled using *Directed Acyclic Graphs* (henceforth DAGs). Suppose there are $m-2$ outcome variables, so there are m variables in total. Denote the vector of all variables as $v = (v_1, v_2, v_3, ..., v_m) \in V_1 \times V_2 \times V_3 \times ... \times V_m \equiv V$, and the set of variables as M. Any subset of the variables $N \subseteq M$ is denoted by $v_N = (v_j)_{j \in N} \in \prod_{j \in N} V_j$. A DAG $\mathcal{R} = (M, R)$ consists of a set of variables M and a set of directed links $R \subset M \times M$. The set R is an acyclic, irreflexive and asymmetric binary relation which we can use to define $R(i) = \{j \in M | jRi\}$ as the set of 'parents' of the variable i according to the DAG. Figure 6.1 illustrates different DAG structures for three variables, with jRi meaning that the arrow $\rightarrow$ is pointing from j to i.

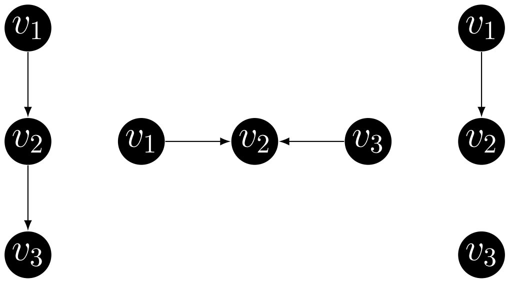

Figure 5: Illustration of three DAGs

The concept assumes there is an objective DAG R^* that that the true joint density between the variables can be factorized.

$$p(v) = p_R^*(v) = \prod_{i=1}^{m} p(v_i | v_{R^*(i)}) \tag{23}$$

Under BNE, DMs are endowed with a possibly misspecified subjective DAG R. This subjective DAG can remove or add links between the variables in M. The DM is assumed to factorize the joint density given by (23) according to their subjective DAG in way that can potentially distort beliefs and lead to equilibrium effects.

$$p_R(v) = \prod_{i=1}^{m} p(v_i | v_{R(i)}) \tag{24}$$

Let $v_1 = s$ be a signal, $v_2 = x$ be an action take after learning the realization of s and all the $m - 2$ remaining variables be outcome variables that are downstream from the action and signal according to the true DAG R^*. The DM is assumed to form beliefs $p_R(v_1, ..., v_{m-2} | x, s)$ by conditioning the distorted beliefs given by (24) on the action and signal variable. As in Proxy Equilibrium, the DM then uses these distorted beliefs to form perceived expected utility from taking an action after a given signal realization, with expectations taken with respect to a vNM utility function defined over all the variables. As in Proxy Equilibrium, there can be equilibrium effects and a trembling hand criterion in the definition of equilibrium is used to ensure that there is no conditioning on zero-probability events.

The substantive differences between the two frameworks are first that Proxy Equilibrium allows for conditioning on 'false' action and signal variables, with expectations also formed with respect to false outcome variables. This is not something that can be obtained in BNE. Secondly, BNE allows for more outcome variables and more elaborate causal structures than does Proxy Equilibrium. The idea behind Proxy Equilibrium is that the simple causal structure between the variables is understood by the DM, but the DM either neglects or does not realize there are measurement problems with the variables. BNE considers cases where the DM misunderstands the causal structure, and fits the incorrect causal structure to variables that are otherwise perfectly measured. The two concepts could be combined in a variety of ways, for example having the DM fit the incorrect DAG to the false variables.

6.2 Relationship to Berk-Nash Equilibrium

The Berk-Nash Equilibrium of Esponda and Pouzo (2016) gives a general solution concept for games in which players have to form expectations of a mapping between actions, a signal variable and outcome variables that may depend on the actions and signals of multiple players. Each player has a set of subjective models over this mapping. Under the solution concept the expectations of the players have to be such that any subjective model that is in the support of expectation of the player minimizes the Kullback-Leibler divergence between the true distribution over outcomes and that projected by the model, weighted by that player's signal and action probabilities. This is then founded as the limit of a Bayesian learning process in which the players have a prior with their set of subjective models as the support. The true model that generates the mapping between the action, signal and outcome variables may not be in the set of subjective models and thus players may have misspecified expectations.

There are similarities between Proxy Equilibrium and Berk-Nash equilibrium in that under the latter the DM dogmatically believes the proxy variables are identical to the true variables, while under Berk-Nash the players in the game dogmatically believe the objective true model is in their set of subjective models. However, it is in general not possible to nest the Proxy Equilibrium concept as an exact special case of Berk-Nash equilibrium for the following reason; the formulation of Berk-Nash equilibrium in Esponda and Pouzo (2016) assumes that players perfectly observe the joint distribution of their signals and their own equilibrium action. This means that, if the action and signal variables are imperfectly measured, any set of subjective models cannot contain models that put probability one on the proxies for actions and signal being equal to the true action and signal. This is because the Kullback-Leibler divergence is not well defined for that model, as it would place zero probability on the event that the proxies and true variable realizations differ even though they differ with positive probability. Another case in which the concepts are distinct is when imperfectly measured proxy variables enter into the utility function, as there is no allowance for the players to have models that put probability one on a payoff function that depends on the wrong dimensions in the outcome vector.

A Proofs

Proof of Proposition 1

Proof. Denote the set of all strategies conditional on circumstance s and signal z as $\Sigma(s,z)$. Unlike the main body of the paper, we make the dependence of the perceived conditional distribution on the proposed equilibrium strategy $\tilde{\sigma}$ explicit, so use notation $p_\pi(y^\bullet|z^\bullet = z, x^\bullet = x; \tilde{\sigma})$. Define the best response correspondence, given strategy $\hat{\sigma}_\xi$ and $\xi > 0$ as

$$
\begin{aligned}
&BR_\xi(\hat{\sigma}_\xi, s, z) \\
&= \{ \underset{\sigma(.|s,z)\in\Sigma(s,z)}{\arg\max} \int_X \sigma(x|s,z)[\int_{Y^\bullet} u(y = y^\bullet, x, s) p_\pi(y^\bullet|z^\bullet = z, x^\bullet = x; \hat{\sigma}_\xi) d\mu(y^\bullet)] d\mu(x) \\
&\text{ s.t } \sigma(x'|s,z) \geq \xi \ \ \forall x' \in X\}
\end{aligned}
$$

Stack the best response correspondences for each circumstance-signal combination (s,z) into a vector $BR_\xi(\hat{\sigma}) = \prod_{s,z\in S\times Z} BR_\xi(\hat{\sigma}, s, z)$. Since $p_\pi(y^\bullet|z^\bullet = z, x^\bullet = x; \tilde{\sigma})$ is continuous in $\tilde{\sigma}$ and the best response correspondence is the set of maximizers over a compact set defined by a finite set of inequalities, $BR_\xi(\hat{\sigma})$ is nonempty for any $\hat{\sigma}$. Moreover due to linearity in $\sigma(x|s,z)$, $BR_\xi(.)$ convex valued and continuity of $p_\pi(y^\bullet|z^\bullet = z, x^\bullet = x; \tilde{\sigma})$ implies continuity of the maximand, meaning $BR_\xi(.)$ has closed graph. We therefore have met all the requirements of Kakutani's fixed point theorem and a fixed point exists for any $\xi > 0$, $\sigma^*_\xi \in BR_\xi(\sigma^*_\xi)$.

For any $\epsilon > 0$, we can choose $\xi > 0$ in such a way that ensures that our ξ-fixed point is an ϵ-Proxy Equilibrium. In the finite case, the largest interval subset of $X(s,z;\sigma^*_\epsilon)$ is itself. We have that $\sigma^*_\xi(x|s,z) = \xi$ for all $x \in X(s,z;\sigma^*_\epsilon)$ and (s,z). Therefore, we can choose $\xi > 0$ to ensure that $\sum_{x\in X(s,z;\sigma^*_\epsilon)} \sigma^*_\xi(x|s,z) = |X(s,z;\sigma^*_\epsilon)|\xi < \epsilon$ for all (s,z). This ensures our fixed point, which we denote σ^*_ϵ, meets the definition of ϵ-Proxy Equilibrium.

Since finiteness ensures the space of strategies Σ is compact, we can find a convergent sequence of ϵ-equilibria as $\epsilon \to 0$, $\sigma^*_\epsilon \to \sigma^*$. $\square$

Proof of Proposition 2

Proof. We first propose a generic linear solution $x(s) = \theta_0 + \theta_1 s$, which is then used to calculate the perceived expectation $\mathbb{E}[y|x^\bullet = x]$ using the properties of the normal distribution. Under the proposed best response function, the joint normal distribution of $(y, x^\bullet)$ is:

$$\begin{pmatrix} y \\ x^\bullet \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \alpha + \beta(\theta_0 + \theta_1\mu_s) \\ \theta_0 + \theta_1\mu_s \end{pmatrix}, \begin{pmatrix} \beta^2\theta_1^2\sigma_s^2 + \sigma_u^2 & \beta\theta_1^2\sigma_s^2 \\ \beta\theta_1^2\sigma_s^2 & \theta_1^2\sigma_s^2 + \sigma_\epsilon^2 \end{pmatrix}\right)$$

Using this we can calculate the conditional expectation of y given $x^\bullet$.

$$\mathbb{E}[y|x^\bullet] = \alpha + \beta(\theta_0 + \theta_1\mu_s) + \frac{\beta\theta_1^2\sigma_s^2}{\theta_1^2\sigma_s^2 + \sigma_\epsilon^2}(x^\bullet - \theta_0 - \theta_1\mu_s)$$

Using the utility function we then get perceived expected utility $U(x, s; q) = -s\mathbb{E}[y|x^\bullet = x] - \frac{1}{2}x^2$. Solving for a maximum then gives us $x(s) = \frac{-\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2} \cdot s$. In order to have a linear equilibria, we must therefore have $\theta_0 = 0$ and $\theta_1 = \frac{-\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2}$. We can solve the latter cubic equation to get the equilibria in the statement of the proposition.

To show that these equilibria are proxy equilibria, we must show that there is a sequence of proxy equilibria that converges in distribution to them. Let $x^\epsilon(s) = \theta_0^\epsilon + \theta_1^\epsilon s + \sigma_u^2$, where $u \sim \mathcal{N}(0, \sigma_u^2)$. Then the induced distribution over $(y, x^\bullet)$ is almost identical to above except $Var(x^\bullet) = \theta_1^2\sigma_s^2 + \sigma_\epsilon^2 + \sigma_u^2$. As such the cubic equation that characterizes the proposed ϵ-Proxy Equilibria is now:

$$\theta_1^\epsilon = \frac{-\beta(\sigma_s^2(\theta_1^\epsilon)^2 + \sigma_u^2)}{(\theta_1^\epsilon)^2\sigma_s^2 + \sigma_\epsilon^2 + \sigma_u^2}$$

For each of the three proposed equilibria we can find θ_1^η and $\sigma_u^{2,\eta} > 0$ parameterized by $\eta > 0$, where the former converges to the equilibrium value and the latter converges to zero. For the no variation equilibrium, we can set:

$$\theta_1^{nv,\eta} = \frac{(-\beta\sigma_s^2 + \frac{\sigma_\epsilon^2}{\beta} - \eta) + \sqrt{(\beta\sigma_s^2 - \frac{\sigma_\epsilon^2}{\beta} + \eta)^2 - 4\sigma_s^2\eta\beta}}{2\sigma_s^2}$$
$$\sigma_u^{2,nv,\eta} = (\frac{-\sigma_\epsilon^2}{\beta} + \eta)\theta_1^{nv,\eta}$$

These parameters solve the equilibrium quadratic and $\theta_1^{nv,\eta} \to 0$, $\sigma_u^{2,\eta} \to 0$ as $\eta \to 0$. Since $\theta_1^{nv,\eta} > 0$ and $\frac{-\sigma_\epsilon^2}{\beta} + \eta > 0$ for small enough $\eta > 0$, we have that $\sigma_u^{2,\eta} > 0$ for small enough η.

Similarly, assuming $|\beta| \geq 2\frac{\sigma_\epsilon}{\sigma_s}$ holds, for the θ_1^- and θ_1^+ equilibria, we can define:

$$\theta_1^{-,\eta} = -\frac{1}{2}\beta - \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2 + \eta}}$$
$$\sigma_u^{2,-,\eta} = \theta_1^{2,-,\eta}\eta$$
$$\theta_1^{+,\eta} = -\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2 + \eta}}$$
$$\sigma_u^{2,+,\eta} = \theta_1^{2,+,\eta}\eta$$

As $\eta \to 0$, we have $(\theta_1^{-,\eta}, \theta_1^{+,\eta}) \to (\theta_1^-, \theta_1^+)$, $(\sigma_u^{2,-,\eta}, \sigma_u^{2,+,\eta}) \to (0,0)$. These parameters all solve the equilibrium quadratic, we can thus use them to define a sequence of ϵ-Proxy Equilibria.

For any $s \in S$, define $\sigma^\eta(.|s)$ as the strategy mapping implied by any of the perturbed $(\theta_1^\eta, \sigma_u^{2,\eta})$ parameters above, let $\sigma^{\eta,\delta}(.|s)$ be the Dirac measure at $x(s) = \theta_1^\eta \cdot s$ and $\sigma^*(.|s)$ be the Dirac measure at $x(s) = \theta_1^* \cdot s$ where θ_1^* is the parameter of any of the equilibria stated in the proposition. Since we have defined convergent sequence of parameters above, and the normal distribution converges to the degenerate Dirac measure as $\sigma_u^{2,\eta} \to 0$, we have that $\sigma^\eta(.|s)$ converges in distribution to $\sigma^*(.|s)$ as $(\theta_1^\eta, \sigma_u^{2,\eta}) \to (\theta_1^*, 0)$. Likewise, as $\theta_1^\eta \to \theta_1^*$ we have that $\sigma^{\eta,\delta}(.|s)$ converges in distribution to $\sigma^*(.|s)$.

Using the triangle inequality, we have that for any interval $I \subseteq X$:

$$|\sigma^{\eta,\delta}(I|s) - \sigma^{\eta}(I|s)| \leq |\sigma^{\eta,\delta}(I|s) - \sigma^{*}(I|s)| + |\sigma^{*}(I|s) - \sigma^{\eta}(I|s)|$$

For any $\epsilon > 0$, due to the fact any interval I in this application is a continuity set, the portmanteau lemma[6] and convergence in distribution of both the terms on the right hand side of the above equation means we can find an $\eta > 0$ such that $|\sigma^{\eta,\delta}(I|s) - \sigma^{\eta}(I|s)| < \epsilon$. Let $X(s; \sigma^{*}_{\eta})$ be the subset of actions that are not a best response against the beliefs induced by the linear-normal strategy given by parameters $(\theta^{\eta}_1, \sigma^{2,\eta}_u)$. Since for any interval $\tilde{I} \subseteq X(s; \sigma^{*}_{\eta})$, we have that $\sigma^{\eta,\delta}(\tilde{I}|s) = 0$, this means that $\sigma^{\eta}(\tilde{I}|s) < \epsilon$. Thus we have an ϵ-Proxy Equilibrium. This means that for any of the proposed Proxy Equilibria in the statement of the proposition we can find a sequence of ϵ-Proxy Equilibria that converge in distribution to that Proxy Equilibrium.

□

Proof of Proposition 3

We first show the following fact which is used several times in our proof.

Lemma A.1. *Let $I_{[a_1,b_1]} = [a_1, b_1]$, $I_{[a_2,b_2]} = [a_2, b_2]$ be intervals in $[0, 1]$, with $a_1 > a_2$ and $b_1 > b_2$. Then for any $z_1 > z_2$ we have that:*

$$1[z_1 \in [a_1, b_1]] \cdot 1[z_2 \in [a_2, b_2]] \geq 1[z_2 \in [a_1, b_1]] \cdot 1[z_1 \in [a_2, b_2]] \tag{25}$$

Moreover, this inequality holds strictly if $z_1 \in I_{[a_1,b_1]} \setminus I_{[a_2,b_2]}$ or $z_2 \in I_{[a_2,b_2]} \setminus I_{[a_1,b_1]}$.

These facts also hold for half-open intervals $I_{[a_1,b_1)} = [a_1, b_1)$ and $I_{[a_2,b_2)} = [a_2, b_2)$.

Proof. For the right hand side of the inequality to be equal to one requires $z_2 < z_1 \leq b_2$, $z_2 \geq a_1 > a_2$, $z_1 > z_2 \geq a_1$ and $z_1 \leq b_2 < b_1$ so $z_2 \in [a_2, b_2]$ and $z_1 \in [a_1, b_1]$ and the left hand side is also equal to one.

[6] See Billingsley (2012) Theorem 25.8 on page 358 for a proof.

The second part of the result holds by definition and the third part is clear by applying the arguments above again. □

We can then show an increasing best response property.

Lemma A.2. *Given a perceived distribution over outcomes induced by a full-support strategy σ, we have that $U(z, x = 1; p_\pi)$ is strictly increasing in $z \in [h, 1-h]$.*

Proof. For a given induced distribution p_π, we can use integration by parts to write the perceived utility of the DM as follows.

$$\begin{aligned} U(z, x = 1; p_\pi) &= \int_0^1 m(\tilde{z}) p_\pi(\tilde{z}|z^\bullet = z, x = 1) d\mu(\tilde{z}) \\ &= m(1) - \int_0^1 P_\pi(\tilde{z}|z^\bullet = z, x = 1) dM(\tilde{z}) \end{aligned}$$

Where $P_\pi(\tilde{z}|z^\bullet, x = 1)$ is the cdf of the induced distribution and M is the Lebesgue-Stieltjes measure satisfying $M((z_l, z_h]) = m(z_h) - m(z_l)$ for any $0 \leq z_l < z_h \leq 1$. Since $m(.)$ is strictly increasing and right continuous, this measure exists. Therefore, to show the result it is enough to show $P_\pi(z|z_1^\bullet, x = 1) \leq P_\pi(z|z_2^\bullet, x = 1)$ for any $1 - h \geq z_1^\bullet > z_2^\bullet \geq h$ and all z, with strict inequality for all z in some interval $[a, b] \subseteq [0, 1]$. Our assumptions about the conditional distribution of the proxies gives us the following sequence of claims.

By Lemma A.1, we have that for any $z_1 > z_2$ and $1 - h \geq z_1^\bullet > z_2^\bullet \geq h$.

$$\begin{aligned} &1[z_1 \in [z_1^\bullet - h, z_1^\bullet + h]] \cdot 1[z_2 \in [z_2^\bullet - h, z_2^\bullet + h]] \\ &\qquad \geq 1[z_2 \in [z_1^\bullet - h, z_1^\bullet + h]] \cdot 1[z_1 \in [z_2^\bullet - h, z_2^\bullet + h]] \end{aligned}$$

With strict inequality if $z_1 \in [z_1^\bullet - h, z_1^\bullet + h] \setminus [z_2^\bullet - h, z_2^\bullet + h]$ or $z_2 \in [z_2^\bullet - h, z_2^\bullet + h] \setminus [z_1^\bullet - h, z_1^\bullet + h]$, given any $1 - h \geq z_1^\bullet > z_2^\bullet > h$.

Multiplying both sides by $\frac{p(z_1)\sigma(x=1|z_1)}{\int_Z p(z)\sigma(x=1|z)p(z_1^\bullet|z)d\mu(z)} \cdot \frac{p(z_2)\sigma(x=1|z_2)}{\int_Z p(z)\sigma(x=1|z)p(z_2^\bullet|s)d\mu(z)}$, we can then write:

$$p_\pi(z_1|z_1^\bullet)p_\pi(z_2|z_2^\bullet) \geq p_\pi(z_1|z_2^\bullet)p_\pi(z_2|z_1^\bullet)$$

Integrating over both sides then gives us that $P_\pi(z|z_1^\bullet, x=1) \leq P_\pi(z|z_2^\bullet, x=1)$ for any $z_1^\bullet > z_2^\bullet$ and z. By the strict inequality case above, we have that $P_\pi(z|z_1^\bullet, x=1) < P_\pi(z|z_2^\bullet, x=1)$ for any $z \in [z_1^\bullet - h, z_1^\bullet + h] \cup [z_2^\bullet - h, z_2^\bullet + h]$. This completes the proof. □

An outline of the proof is as follows. We first show why any Proxy Equilibrium with partial entry must have a cut-off structure, and give a condition that the cut-off must satisfy in terms of perceived expected utility. We then show how we can construct a sequence of ϵ-Proxy Equilibria that converge to this cut-off structure. Finally, we combine this with the endpoint assumptions to show that any equilibria with entry at some signal has this cut-off structure.

We consider partial entry Proxy Equilibrium in which $\sigma(x=1|z) > 0$ at a strict subset of $z \in Z^> \subset [0,1]$. For such an equilibrium to exist, we must have a sequence of ϵ^l-Proxy Equilibria, $\{\sigma^*_{\epsilon^l}\}_{l=1}^\infty$ that converge in distribution to it. Given we are considering partial entry Proxy Equilibria, for large enough l we must have that the induced belief p^l_π in the ϵ^l-Proxy Equilibrium in the sequence is such that $U(z, x=1; p^l_\pi) \leq 0$ for all $z \in [0,h]$, and that $x=1$ is a best response to p^l_π for some z. By the fact perceived utility is increasing in $z \in [h, 1-h]$ from Lemma A.2 there must be a cut-off $\bar{z}^{\epsilon^l} \in [h,1]$ such that a best response is $x=0$ for $z \in [0, \bar{z}^{\epsilon^l}]$ and $x=1$ for $z \in (\bar{z}^{\epsilon^l}, 1]$.

By the definition ϵ^l-Proxy Equilibrium, we must have that $\sigma^*_{\epsilon^l}(x=1|z) < \epsilon^l$ for all $z \in [0, \bar{z}^{\epsilon^l}]$ and $\sigma^*_{\epsilon^l}(x=1|z) \geq 1-\epsilon^l$ for all $z \in (\bar{z}^{\epsilon^l}, 1]$. The perceived utility of the DM at this cut-off $\bar{z}^{\epsilon^l}$ is then:

$$\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}} m(\tilde{z}) \frac{\sigma_\epsilon(x=1|\tilde{z})p(\tilde{z})}{\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}+h} \sigma_\epsilon(x=1|\hat{z})p(\hat{z})d\mu(\hat{z})} d\mu(\tilde{z}) + \int_{\bar{z}^{\epsilon^l}}^{\bar{z}^{\epsilon^l}+h} m(\tilde{z}) \frac{\sigma_\epsilon(x=1|\tilde{z})p(\tilde{z})}{\int_{\bar{z}^{\epsilon^l}-h}^{\bar{z}^{\epsilon^l}+h} \sigma_\epsilon(x=1|\hat{z})p(\hat{z})d\mu(\hat{z})} d\mu(\tilde{z}) = 0$$

Thus as $l \to \infty$, if our sequence of ϵ-Proxy Equilibria converges it will converge to a Proxy Equilibrium with cut-off $\bar{z}^*$ such that $\sigma^*(x = 1|z) = 0$ for $z \in [0, \bar{z}^*)$ and $\sigma^*(x = 1|z) = 1$ for $z \in [\bar{z}^*, 1]$. The perceived utility at the cut-off $\bar{z}^*$ will then be:

$$\int_{\bar{z}}^{\bar{z}+h} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) = 0 \tag{26}$$

We then construct strategies that can form a sequence of ϵ-Proxy Equilibria that converge to a partial entry Proxy Equilibria. These strategies have a cut-off form where $\sigma_\xi(x|z) = \xi \in (0, \frac{1}{2})$ for $z \in [0, \bar{z})$ and $\sigma_\xi(x|z) = 1 - \xi \in (\frac{1}{2}, 1)$ for $z \in [\bar{z}, 1]$, with $\bar{z} \in [0, 1]$ as the cut-off. We can then define the following conditional density over $z \in [h, 1 - h]$ given $z^\bullet = \bar{z}$.

$$g_\xi(z|z^\bullet = \bar{z}) = \frac{(1 - \xi) 1[\tilde{z} \in [\bar{z}, \bar{z} + h]] + \xi 1[\tilde{z} \in [\bar{z} - h, \bar{z})]}{(1 - \xi) \int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z}) + \xi \int_{\bar{z}-h}^{\bar{z}} p(\hat{z}) d\mu(\hat{z})} p(\tilde{z}) \tag{27}$$

For any cut-off $\bar{z} \in [h, 1 - h]$ and k, we can choose:

$$\xi(\bar{z}, k) = \frac{k \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z})}{k \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z}) + (1 - k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})} \tag{28}$$

Which is arbitrarily small for small enough $1 > k > 0$. This ensures that:

$$\int_{\bar{z}-h}^{\bar{z}} g_\xi(\tilde{z}|z^\bullet = \bar{z}) d\mu(\tilde{z}) = \frac{\xi(\bar{z}, k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})}{(1 - \xi(\bar{z}, k)) \int_{\bar{z}}^{\bar{z}+h} p(\tilde{z}) d\mu(\tilde{z}) + \xi(\bar{z}, k) \int_{\bar{z}-h}^{\bar{z}} p(\tilde{z}) d\mu(\tilde{z})} = k$$

We can then write the perceived utility at $\bar{z} \in [h, 1 - h]$ against the beliefs induced by strategy $\sigma_{\xi(\bar{z},k)}$ with cut-off $\bar{z} \in [h, 1 - h]$ in the following way:

$$\int_0^1 m(\tilde{z}) g_{\xi(\bar{z},k)}(\tilde{z}|z^\bullet = \bar{z}) d\mu(\tilde{z}) = (1 - k) \overline{U}(\bar{z}, x = 1; \bar{z}) + k \underline{U}(\bar{z}, x = 1; \bar{z}) \tag{29}$$

Which is a linear combination of the terms:

$$\overline{U}(\bar{z}, x = 1; \bar{z}) = \int_{\bar{z}}^{\bar{z}+h} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) \tag{30}$$

$$\underline{U}(\bar{z}, x = 1; \bar{z}) = \int_{\bar{z}-h}^{\bar{z}} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}-h}^{\bar{z}} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) \tag{31}$$

We show that (29) is strictly increasing in $\bar{z} \in [h, 1-h]$ by showing (30) and (31) are strictly increasing in $\bar{z}$.

Lemma A.3. *The expressions $\overline{U}(\bar{z}, x = 1; \bar{z})$ and $\underline{U}(\bar{z}, x = 1; \bar{z})$ are strictly increasing for all $\bar{z} \in [h, 1-h]$.*

Proof. We define densities $\overline{g}(z; \bar{z}) = \frac{1_{[z \in [\bar{z}, \bar{z}+h]]} p(z)}{\int_{\bar{z}}^{\bar{z}+h} p(\hat{z}) d\mu(\hat{z})}$ and $\underline{g}(z; \bar{z}) = \frac{1_{[z \in [\bar{z}-h, \bar{z})]} p(z)}{\int_{\bar{z}-h}^{\bar{z}} p(\hat{z}) d\mu(\hat{z})}$. We show that for $z_1 > z_2$ and $1 - h \geq \bar{z}_1 > \bar{z}_2 \geq h$ we have the following inequalities:

$$\overline{g}(z_1; \bar{z}_1)\overline{g}(z_2; \bar{z}_2) \geq \overline{g}(z_1; \bar{z}_2)\overline{g}(z_2; \bar{z}_1) \tag{32}$$

$$\underline{g}(z_1; \bar{z}_1)\underline{g}(z_2; \bar{z}_2) \geq \underline{g}(z_1; \bar{z}_2)\underline{g}(z_2; \bar{z}_1) \tag{33}$$

Where for any $h \leq \bar{z}_2 < \bar{z}_1 \leq 1-h$, we can find intervals $I_1, I_2 \subset [0, 1]$ such that if for $z_1 > z_2$, $z_1 \in I_1$, $z_2 \in I_2$ the inequality holds strictly. The above inequalities reduce to:

$$1_{[z_1 \in [\hat{z}_1, \hat{z}_1+h], z_2 \in [\bar{z}_2, \bar{z}_2+h]]} \geq 1_{[z_1 \in [\bar{z}_2, \bar{z}_2+h], z_2 \in [\bar{z}_1, \bar{z}_1+h]]}$$

$$1_{[z_1 \in [\hat{z}_1-h, \hat{z}_1), z_2 \in [\bar{z}_2-h, \bar{z}_2)]} \geq 1_{[z_1 \in [\bar{z}_2-h, \bar{z}_2), z_2 \in [\bar{z}_1-h, \bar{z}_1)]}$$

Which proves our inequality result by Lemma A.1. We can then use the same steps as in the proof of Lemma A.2 to prove the result. $\square$

With these results in hand, we can then both show existence of and characterize the equilibria for this application.

Proposition 3

Proof. At any ϵ-Proxy Equilibrium, the perceived utility of the DM is increasing strictly for $z \in [h, 1-h]$ by Lemma A.2. The structure of the window form of proxy mapping means that the beliefs of the DM are identical on $z \in [0, h]$. If the DM is mixing $\sigma_\epsilon(x = 1|z) > \epsilon$ on $z \in [0, h]$, then due to increasing expected payoff on $z \in [h, 1-h]$, they must be playing $\sigma_\epsilon(x = 1|z) \geq 1 - \epsilon$ on $z \in (h, 1]$. As $\epsilon \to 0$ and $\sigma_\epsilon \to \sigma$ their perceived utility at any $z \in [0, h]$ given potential equilibrium σ is:

$$\int_0^h m(z) \frac{\sigma(x = 1|z)p(z)}{\int_0^1 \sigma(x = 1|\tilde{z})p(\tilde{z})d\mu(\tilde{z})} d\mu(z) + \int_h^{2h} m(z) \frac{p(z)}{\int_0^1 \sigma(x = 1|\tilde{z})p(\tilde{z})d\mu(\tilde{z})} d\mu(z)$$

By the endpoint assumption and the fact that $m(.)$ is increasing this is strictly negative. Thus for small enough ϵ at any ϵ-Proxy Equilibrium we must have $\sigma(x = 1|z) < \epsilon$ for $z \in [0, h]$.

From this argument and Lemma A.2, any ϵ-Proxy Equilibria in which $\sigma(x = 1|z) \geq \epsilon$ for some z must have some cut-off $\bar{z} \in (h, 1]$ such that $\sigma(x = 1|z) \geq \epsilon$ only if $z > \bar{z}$. The endpoint assumption $\int_{1-h}^1 m(z)p(z)d\mu(z) > 0$ ensures that any potential equilibrium strategy with cut-off $\bar{z} \geq 1 - h$ will have $\sigma(x = 1|z) = 1$ as a best response for all $z \in [1 - h, 1]$, and thus $\sigma(x = 0|z) < \epsilon$ for $z \in [1 - h, 1]$ in any ϵ-Proxy Equilibrium.

We construct the following cut-off ϵ-Proxy Equilibrium strategy. For any cut-off $\bar{z} \in (h, 1]$, $\epsilon > 0$ and $k_\epsilon \in (0, 1)$, define $\xi(\bar{z}, k_\epsilon)$ as in (28). Let $\sigma_\epsilon(x = 1|z) = \xi(\bar{z}, k_\epsilon)$ on $z \in [0, \bar{z}]$ and $\sigma_\epsilon(x = 1|z) = 1 - \xi(\bar{z}, k_\epsilon)$ on $z \in (\bar{z}, 1]$. We choose $k_\epsilon > 0$ small enough such that $\epsilon > \sup_{\bar{z} \in [h, 1-h]} \xi(\bar{z}, k_\epsilon)$. Then if we can find a $\bar{z}^*$ such that a best response to the beliefs induced by σ_ϵ is $\sigma(x = 1|z) = 0$ on $z \in [0, \bar{z}^*]$ and $\sigma(x = 1|z) = 1$ on $z \in (\bar{z}^*, 1]$ we have an ϵ- Proxy Equilibrium.

We have shown that the constructed strategy induces the beliefs at the cut-off $\bar{z}$ according to equation (29). We have also shown in Lemma A.3 that this expression is strictly increasing in the cut-off $\bar{z} \in [h, 1 - h]$. Moreover, we have that as $\epsilon \to 0$, $k_\epsilon \to 0$, so this expression converges to that in equation (26). By the endpoint assumptions (26) is strictly negative at $\bar{z} = h$ and strictly positive at $\bar{z} = 1 - h$. Thus we can find a small enough $\epsilon > 0$ and hence $k_\epsilon > 0$ such that $\int_0^1 m(\tilde{z}) g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet = h) d\mu(\tilde{z}) < 0$ and $\int_0^1 m(\tilde{z}) g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet = 1-h) d\mu(\tilde{z}) > 0$. Since $\int_0^1 m(\tilde{s}) g_{\xi(\bar{z},k_\epsilon)}(\tilde{z}|z^\bullet = \bar{z}) d\mu(\tilde{z})$ is continuous and increasing in $\bar{z} \in [h, 1-h]$, we can find a $\bar{z} = \bar{z}^*$ at which it is equal to zero by the intermediate value theorem. This $\bar{z}^*$ then gives us our ϵ-Proxy Equilibrium cut-off as stated above. As $\epsilon \to 0$, we can find a sequence of ϵ-Proxy Equilibria of this form that converge to that in the statement of the proposition.

For the final part of the proposition, we can always find a sequence of ϵ-Proxy Equilibria that converges to a Proxy Equilibrium with $x = 0$ for all $z \in [0, 1]$. For example, with small enough $\epsilon > 0$ we can have an ϵ-Proxy Equilibrium such that the DM plays $x = 1$ with probability $\epsilon > 0$ on $[0, \alpha)$ and probability ϵ^2 on $[\alpha, 1]$. This induces beliefs to which $x = 0$ is a best response for all z. □

Proof of Proposition 4

Proof. As shown in Proposition 3, there is a cut-off $\bar{z}^* \in [h, 1-h]$ that characterizes the cut-off equilibrium where $\sigma(x = 1|z) = 0$ for $z \in [0, \bar{z}^*]$ and $\sigma(x = 1|z) = 1$ for $z \in (\bar{z}^*, 1]$. We have shown the cut-off must solve the following equation.

$$\overline{U}(\bar{z}^*, x = 1; \bar{z}^*) = \int_{\bar{z}^*}^{\bar{z}^*+h} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}^*}^{\bar{z}^*+h} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z}) = 0$$

For the first part, consider that the statement is not true and we have that $\bar{z}^* \geq \alpha$. Then we have $\overline{U}(\bar{z}^*, x = 1; \bar{z}^*) > 0$ as all the probability weight in the distribution is in $z \in [\alpha, 1]$, a contradiction. Thus the cut-off must be such that $\bar{z} < \alpha$.

For the second part, the end point assumptions being satisfied mean we are comparing the cut-off equilibrium at h_1 with the cut-off equilibrium at h_2. Consider the perceived utility at the cut-off under the equilibrium with noise h_1.

$$\overline{U}(\bar{z}(h_1), x = 1; h_1) = \int_{\bar{z}(h_1)}^{\bar{z}(h_1)+h_1} m(\tilde{z}) \frac{p(\tilde{z})}{\int_{\bar{z}}^{\bar{z}(h_1)+h_1} p(\hat{z}) d\mu(\hat{z})} d\mu(\tilde{z})$$

As this is an equilibrium cut-off, we must have that:

$$\int_{\alpha}^{\bar{z}(h_1)+h_1} m(\tilde{z})p(\tilde{z})d\mu(\tilde{z}) + \int_{\bar{z}(h_1)}^{\alpha} m(\tilde{z})p(\tilde{z})d\mu(\tilde{z}) = 0$$

If $\bar{z}(h_1)$ is fixed, then as h_1 increases to h_2, the first part of this expression that has weight on the positive part of the function $m(.)$ increases while the second part stays fixed. Thus the perceived utility at cut-off $\bar{z}(h_1)$ when the perceived distribution is induced by a strategy with cut-off $\bar{z}(h_1)$, must become positive at noise parameter $h_2 > h_1$. We have that $\overline{U}(\bar{z}(h_1), x = 1; h_2) > 0$, $\overline{U}(h_1, x = 1; h_2) < 0$ by the endpoint assumptions and $\overline{U}(\bar{z}, x = 1; h_2)$ is continuous in $\bar{z} \in [h_1, 1-h_1]$. Therefore by the intermediate value theorem we can find a new cut-off $\bar{z}(h_2) < \bar{z}(h_1)$ that characterizes the positive entry equilibrium under noise parameter h_2. □

Proof of Proposition 6

We prove Proposition 6 before Proposition 5 as we will use results in this section in the proof of the latter.

We first prove a sequence of lemmas. Remember that we denote $w = (y, x, z)$, the perfect measurement mapping is denoted π_δ and that $\mathcal{W}$ is the set of all the Borel sets of $Y \times X \times Z$. Enumerate the signal, action and circumstance variables as 1, 2 and 3 respectively. Then we can denote any subset of the variable space $\{1, 2, 3\}$ by $N \subseteq 2^3$. We write the measure over the subset of true variables in N as P^N and the subset of the proxy variables in N as P^N_π. These are related to the measure over all the variables.

$$P^N_\pi(W_N) = P_\pi(W_N \times W_{-N}) = \int_{Y\times X\times Z} \pi(W_N \times W_{-N}|w)p(w)d\mu(w) \quad (34)$$

We denote the set of Borel sets of the variable space only containing variables in N by $\mathcal{W}_N$, and $W_N \in \mathcal{W}_N$.

Lemma A.4. *For any $\eta > 0$, if the proxy mapping is strongly η-close to perfect then it is also η-close to perfect given any distribution over the true variables P.*

In addition, for any subset of the variables N, if the proxy mapping is η-close to perfect given distribution P then we have that:

$$TV(P^N, P_\pi^N) < \eta \tag{35}$$

Proof. For the first part, we have that:

$$\begin{aligned}
TV(P, P_\pi) &= \sup_{A\in\mathcal{W}} |P(A) - P_\pi(A)| \\
&= \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} \pi_\delta(A|w)p(w)d\mu(w) - \int_{Y\times X\times S} \pi(A|w)p(w)d\mu(w)| \\
&= \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} (\pi_\delta(A|w) - \pi(A|w))p(w)d\mu(w)| \\
&\leq \sup_{A\in\mathcal{W}} |\int_{Y\times X\times Z} \left|\pi_\delta(A|w) - \pi(A|w)\right| p(w)d\mu(w)| \\
&<| \int_{Y\times X\times Z} \eta p(w)d\mu(w)| = \eta
\end{aligned}$$

For the second part, we can show that the distance for the marginal distribution over the subset of variables N is smaller than the distance for all the variables.

$$\begin{aligned}
TV(P^N, P_\pi^N) &= \sup_{A\in\mathcal{W}_N} |P^N(A) - P_\pi^N(A)| \\
&= \sup_{A\in\mathcal{W}_N\times\{V_{-N}\}} |P(A) - P_\pi(A)| \\
&\leq \sup_{A\in\mathcal{W}} |P(A) - P_\pi(A)| = TV(P, P_\pi) < \eta
\end{aligned}$$

Where the last line follows as $\mathcal{W}_N \times \{V_{-N}\} \subset \mathcal{W}$. This completes the proof. $\square$

Lemma A.5. *Given a distribution over true variables P, let the sequence $\{\pi_n\}_{n=1}^\infty$ induce a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^\infty$ that converges to the true distribution P in the total variation distance.*

Then there exists a subsequence $\{\pi_{n_k}\}_{k=1}^{\infty}$ that induces, for any subset of the variables N,a subsequence of densities over the proxy variables $\{p_{\pi_{n_k}}^N\}_{k=1}^{\infty}$ that converges pointwise μ almost everywhere to the true density, $p_{\pi_{n_k}}^N(w) \to p^N(w)$ for almost all $w \in Y \times X \times Z$.

Proof. An alternative expression for the total variation distance, given that the distributions Q_1 and Q_2 over some measure space $(\Omega, \mathcal{A})$ admit densities q_1 and q_2 with respect to some dominating measure μ, is as follows[7].

$$TV(Q_1, Q_2) = \frac{1}{2}\int_{\Omega} |q_1 - q_2| d\mu = \frac{1}{2}\|Q_1 - Q_2\|_1 \tag{36}$$

Therefore, convergence in the total variation distance is equivalent to convergence in the L1 norm. By Lemma A.4 we have that the sequence $\{\pi_n\}_{n=1}^{\infty}$ induces a sequence of distributions over the variables in N, $\{P_{\pi_n}^N\}_{n=1}^{\infty}$ which converges to the true distribution P^N in the total variation distance, and thus the L1 norm.

As each distribution in this sequence as well as the limit is assumed to admit a density function, we have that $\|p_{\pi_n}^N - p^N\|_1 \to 0$. Thus, by Theorem 13.6 (pp 465) of Charalambos and Aliprantis (2006) we have that there is a subsequence $\{p_{\pi_{n_k}}^N\}_{k=1}^{\infty}$ which converges pointwise to the true density p^N almost everywhere. □

We then extend this to the induced conditional distributions over proxy variables that form the agent's beliefs under Proxy Equilibrium. For any distribution over the true variables P, let $XZ^+ = \{(x, z) \in X \times Z : p(x, z) > 0\}$.

Lemma A.6. *Given a distribution over true variables P, let the sequence $\{\pi_n\}_{n=1}^{\infty}$ induce a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^{\infty}$ that converges to the true distribution P in the total variation distance.*

Then for μ-almost every $(y, x, z) \in Y \times XZ^+$ there exists a subsequence $\{\pi_{n_k}\}_{k=1}^{\infty}$ such that the induced subsequence of perceived conditional densities $\{p_{\pi_{n_k}}(y^\bullet = y | x^\bullet = x, z^\bullet = z)\}_{k=1}^{\infty}$ converges pointwise to the true conditional density $p(y|x, z)$ almost everywhere.

[7] See Tsybakov (2008) page 84 in Chapter 2.4 for a proof of this fact.

Proof. By Lemma A.5 and the fact $(x, z) \in XZ^+$, for μ-almost every $(y, x, z) \in Y \times XZ^+$ both the numerator and denominator of $p_{\pi^k}(y^\bullet = y|x^\bullet = x, z^\bullet = z)$ converge pointwise to the true joint density as $k \to \infty$. $\square$

This result can then be used to show the following continuity property for the perceived ex-ante expected indirect utility, under the full support assumption. Remember that under the full-support assumption $XZ^+ = X \times Z$.

Proposition 6

We prove a stronger result.

Proposition 7. *For μ almost every $(y, x, z) \in Y \times XZ^+$, for any $\epsilon > 0$ there exists an $\eta > 0$ such that if the proxy mapping π is η-**close to perfect** given true distribution P and induces a distribution over the proxy variables that satisfies the **full support assumption**, then $|p_\pi(y^\bullet = y|x^\bullet = x, z^\bullet = z) - p(y|x, z)| < \epsilon$.*

Proof. Assume for contradiction that there exists an $\epsilon > 0$ and (y, x, z) at which $p(x, z) > 0$ holds such that for any $\eta > 0$, we can find a π such that the induced distribution over proxies satisfies the full support assumption given P, $TV(P, P_\pi) < \eta$ and $|p_\pi(y^\bullet = y|x^\bullet = x, z^\bullet = z) - p(y|x, z)| \geq \epsilon$. Then by setting $\eta = \frac{1}{n}$, we can define a sequence $\{\pi_n\}_{n=1}^\infty$ that induces a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^\infty$ converging to the true distribution P in the total variation distance. Then by Lemma A.6 we have that $p_{\pi_{n_k}}(y^\bullet = y|x^\bullet = x, z^\bullet = z) \to p(y|x, z)$, a contradiction.

Since Lemma A.6 holds for μ-almost every (y, x, z), we have that the result holds μ almost everywhere. $\square$

Since the full support assumption on P implies that $XZ^+ = X \times Z$, this also proves Proposition 6.

Proof of Proposition 5

Proof. We can write the perceived conditional distribution induced by proxy mapping π as:

$$p_\pi(y^\bullet|x^\bullet,z^\bullet) = \frac{\sum_{y,x,s,z}\pi(y^\bullet,x^\bullet,z^\bullet|y,x,z)p(y|x,z)\sigma(x|z)p(z)}{\sum_{y^\bullet,y,x,s,z}\pi(y^\bullet,x^\bullet,z^\bullet|y,x,z)p(y|x,z)\sigma(x|z)p(z)} \tag{37}$$

Where

$$\sigma(x|z) = \sum_{s\in S}\sigma(x|s,z)p(s|z) \tag{38}$$

If we define the probability of perfect measurement at realization (y,x,z):

$$\pi^{perf}(y,x,z) = \pi(y^\bullet = y, x^\bullet = x, z^\bullet = z|y,x,z) \tag{39}$$

And the set of all true variable realizations that are imperfectly measured as $(y^\bullet,x^\bullet,z^\bullet)$ along at least one dimension:

$$IMP(y^\bullet,x^\bullet,z^\bullet) = \{(y,x,z)\in Y\times X\times Z : (y\neq y^\bullet)\vee(x\neq x^\bullet)\vee(z\neq z^\bullet)\} \tag{40}$$

Then can then write out (37) in a way that splits it into a part coming from perfect measured realizations of the true variables and a mismeasured part:

$$\begin{aligned} p_\pi(y^\bullet|x^\bullet,z^\bullet) &= \pi^{perf}(y^\bullet,x^\bullet,z^\bullet)\frac{p(y^\bullet|x^\bullet,z^\bullet)\sigma(x^\bullet|z^\bullet)p(z^\bullet)}{\sum_{y^\bullet,y,x,s,z}\pi(y^\bullet,x^\bullet,z^\bullet|y,x,z)p(y|x,z)\sigma(x|z)p(z)} \\ &+ (1-\pi^{perf}(y^\bullet,x^\bullet,z^\bullet))\frac{\sum_{(y,x,z)\in IMP(y^\bullet,x^\bullet,z^\bullet)}\frac{\pi(y^\bullet,x^\bullet,z^\bullet|y,x,z)}{(1-\pi^{perf}(y^\bullet,x^\bullet,z^\bullet))}p(y|x,z)\sigma(x|z)p(z)}{\sum_{y^\bullet,y,x,s,z}\pi(y^\bullet,x^\bullet,z^\bullet|y,x,z)p(y|x,z)\sigma(x|z)p(z)} \end{aligned} \tag{41}$$

$\Rightarrow$ (sufficiency): Define $X^{ns}(\sigma^*,s,z) \equiv \{x\in X : x\notin supp\{\sigma^*(.|s,z)\}\}$ and $XZ^{ns}(\sigma^*) \equiv \{(x,z)\in X\times Z : x\notin supp\{\sigma^*(.|z)\}\}$. We can split the second condition in the proposition into two cases:

Case 1: For each z, for every action $x^{ns}\notin supp\{\sigma^*(.|z)\}$, either there exists an outcome $\hat{y}\in Y$ such that for any $s\in S$ and $x^s\in supp\{\sigma^*(.|s,z)\}$ we have that:

$$\sum_{y\in Y}u(y,x^s,s)p(y|x^s,z) > u(\hat{y},x^{ns},s) \tag{42}$$

Case 2: We have that for all $\hat{y} \in supp\{p(y|x^{ns}, z)\}$:

$$\sum_{y \in Y} u(y, x^s, s) p(y|x^s, z) = u(\hat{y}, x^{ns}, s) \tag{43}$$

By the second condition of Potentially Implementability, one of these two cases must hold. Using this, we can construct a function $k : XZ^{ns}(\sigma^*) \to Y$ by selecting $\hat{y} \in Y$ such that x^s gives at least as high expected utility as action x^s at signal-circumstance combination (x, z), given particular beliefs. These beliefs are such that x^s results in the distribution over y that occurs under perfect measurement and x^{ns} results in $\hat{y}$ with certainty. If several outcomes satisfy either condition then pick one arbitrarily.

We construct a sequence of full support equilibria that converges to σ^*. Let σ^*_ϵ be such that for any (x, s, z) if $x \in supp\{\sigma^*(.|s, z)\}$ then $\sigma^*_\epsilon(x|s, z) = \sigma^*(x|s, z) - \epsilon$, and if $x \notin supp\{\sigma^*(.|s, z)\}$ then $\sigma^*_\epsilon(x|s, z) = \frac{\epsilon}{|X^{ns}(\sigma^*, s, z)|}$.

Denote $K(y) \equiv \{(x^\bullet, z^\bullet) \in X \times Z : y = k(x, z) \text{ and } (x^\bullet, z^\bullet) = (x, z)\}$ as the set of all actions and signals mapped to y by k. We construct a proxy mapping π_c that randomizes uniformly over all elements of $K(y)$. That is:

$$\pi_c(y^\bullet, x^\bullet, z^\bullet | y, x, z) = \begin{cases} \frac{1}{|K(y)|} \text{ if } y^\bullet = y, \ K(y) \neq \emptyset \text{ and } (x^\bullet, z^\bullet) \in K(y) \\ 0 \text{ if } y^\bullet = y, \ K(y) \neq \emptyset \text{ and } (x^\bullet, z^\bullet) \notin K(y) \\ \pi_\delta(y^\bullet, x^\bullet, z^\bullet | y, x, z) \text{ otherwise} \end{cases}$$

From this we can then form another proxy mapping π_η that draws the perfect measurement mapping with probability $1 - \frac{\eta}{2}$ and mapping π_c with probability $\frac{\eta}{2}$. Denote by

$$XZ^s(\sigma^*) \equiv \{(x, z) \in X \times Z : x \in supp\{\sigma^*(.|z)\}\}$$

the set of actions and signal combinations such that the action is in the support of the signal under strategy σ^*. We can then write:

$$\pi_\eta(y^\bullet, x^\bullet, z^\bullet|y, x, z) = (1 - \frac{\eta}{2})\pi_\delta(y^\bullet, x^\bullet, z^\bullet|y, x, z) + \frac{\eta}{2}\pi_c(y^\bullet, x^\bullet, z^\bullet|y, x, z)$$

This mapping is clearly strongly η-close to perfect. For any $(x^\bullet, z^\bullet) \in XZ^s(\sigma^*)$ we have perfect measurement under π_η and the conditional distribution over y given $(x^\bullet, z^\bullet)$ is equal to the rational expectations benchmark independently of the strategy; $p_{\pi_\eta}(y|x^\bullet, z^\bullet) = p(y|x^\bullet, z^\bullet)$. For any $(x^\bullet, z^\bullet) \notin XZ^s(\sigma^*)$, we can write the conditional distribution for any $\hat{y} = k(x^\bullet, z^\bullet)$ as:

$$p_{\pi_\eta}(\hat{y}|x^\bullet, z^\bullet) = \\ \frac{\epsilon(1 - \frac{\eta}{2})p(z^\bullet)p(\hat{y}|x^\bullet, z^\bullet) + \frac{\eta}{2|K(\hat{y})|}\sum_{x,s} p(z)\sigma^*_\epsilon(x|z)p(\hat{y}|x, z)}{\epsilon(1 - \frac{\eta}{2})p(z^\bullet) + \frac{\eta}{2|K(y)|}\sum_{x,z} p(z)\sigma^*_\epsilon(x|z)p(y = k(x, z)|x, z)} \tag{44}$$

While for any $\tilde{y} \neq k(x^\bullet, z^\bullet)$

$$p_{\pi_\eta}(\tilde{y}|x^\bullet, z^\bullet) = \\ \frac{\epsilon(1 - \frac{\eta}{2})p(z^\bullet)p(\tilde{y}|x^\bullet, z^\bullet)}{\epsilon(1 - \frac{\eta}{2})p(z^\bullet) + \frac{\eta}{2|K(y)|}\sum_{x,z} p(z)\sigma^*_\epsilon(x|z)p(y = k(x, z)|x, z)} \tag{45}$$

For small enough $\epsilon > 0$, (44) is arbitrarily close to one for any $\eta > 0$ due to our assumption in the statement that $p(y|x, z)$ has full support. Therefore, we can push the perceived expected utility of any action taken at a signal that is not in the support of the proposed strategy; $(x^\bullet, z^\bullet) \notin XZ^s(\sigma^*)$ arbitrarily close to putting probability one on $y = k(x^\bullet, z^\bullet)$, with the remaining weight assigned to $p(y|x^\bullet, z^\bullet)$.

Any action-signal combination in the support of σ^*; $(x^\bullet, z^\bullet) \in XZ^s(\sigma^*)$, has perceived expected utility equal to the rational expectations benchmark. Therefore for either of the two cases derived from the second condition of potential implementability, (42) and (43), for small enough $\eta > 0$ the constructed proxy mapping allows us to support σ^*_ϵ as an ϵ-Proxy Equilibrium for $\epsilon > 0$ close enough to zero. Since $\sigma^*_\epsilon \to \sigma^*$ as $\epsilon \to 0$, we have our result.

$\Leftarrow$ (necessity): If we first consider the expression (37), we can write the numerator and denominator as:

$$p_\pi(y^\bullet, x^\bullet, z^\bullet; \sigma) = \sum_{y,x,s,z} \pi(y^\bullet, x^\bullet, z^\bullet | y, x, z) p(y|x, z) \sigma(x|z) p(z) \tag{46}$$

$$p_\pi(x^\bullet, z^\bullet; \sigma) = \sum_{y^\bullet, y,x,s,z} \pi(y^\bullet, x^\bullet, z^\bullet | y, x, z) p(y|x, z) \sigma(x|z) p(z) \tag{47}$$

Both are continuous in the strategy σ, meaning $p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma)$ is well defined for any full support strategy σ. Thus if we take a potentially implementable strategy σ^*, for any $\frac{\delta}{2} > 0$ we can find an $\xi > 0$ such that for any full support strategy σ^*_ϵ satisfying $\max_{(x,z,s)\in X\times Z\times S} |\sigma^*_\epsilon(x|s, z) - \sigma^*(x|s, z)| < \xi$ we have that $|p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*_\epsilon) - p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*)| < \frac{\delta}{2}$ for any $y^\bullet \in Y$ if $x^\bullet \in supp\{\sigma^*(.|z^\bullet)\}$.

From Lemma A.4 and Proposition 6, we have for any $\frac{\delta}{2} > 0$ we can find a $\eta > 0$ such that if π is strongly η-close to perfect then $|p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*) - p(y^\bullet|x^\bullet, z^\bullet)| < \frac{\delta}{2}$ for any $y^\bullet \in Y$ if $(x^\bullet, z^\bullet)$ such that $x^\bullet \in supp\{\sigma^*(.|z^\bullet)\}$.

Combining these observations, we have that for any $\delta > 0$, there exists $\xi > 0$ and $\eta > 0$ such that if for any full support strategy σ^*_ϵ, $\max_{(x,z,s)\in X\times Z\times S} |\sigma^*_\epsilon(x|s, z) - \sigma^*(x|s, z)| < \xi$ and π is strongly η-close to perfect, then if $(x^\bullet, z^\bullet)$ such that $x^\bullet \in supp\{\sigma^*(.|z^\bullet)\}$:

$$\begin{aligned}
&|p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*_\epsilon) - p(y^\bullet|x^\bullet, z^\bullet)| \\
&\leq |p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*_\epsilon) - p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*)| + |p_\pi(y^\bullet|x^\bullet, z^\bullet; \sigma^*) - p(y^\bullet|x^\bullet, z^\bullet)| \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned} \tag{48}$$

For all $y^\bullet \in Y$.

Thus we can find an $\eta > 0$ such that for any full-support strategy that is close to the proposed strategy the perceived conditional distribution over outcomes is arbitrarily close to the true conditional distribution for action-signal combinations that are in the support of that strategy.

If the proposed strategy violates the first condition of potential implementability for some (x, z, s) at which $x \in supp\{\sigma^*(.|s, z)\}$, then the above fact means that we cannot find any convergent sequence of full-support strategies in which x is a best response at this (s, z) combination, meaning we cannot support the proposed strategy as a Proxy Equilibrium. Similarly, if the second condition of potential implementability is violated, then for some (s, z), $x^{ns} \notin supp\{\sigma^*(.|z)\}$ and $x^s \in supp\{\sigma^*(.|s, z)\}$ we have:

$$\sum_{y \in Y} u(y, x^s, s)p(y|x^s, z) < \sum_{y \in Y} u(y, x^{ns}, s)q(y|x^{ns}, z) \tag{49}$$

For any full-support conditional distribution $q : X \times Z \to \Delta(Y)$. Since we have convergence of the conditional distribution at $x^s \in supp\{\sigma^*(.|s, z)\}$ for any π that is η-close to perfect as $\pi \to 0$, and by (41) any π that has some possibility of perfect measurement must induce a full-support conditional distribution over Y, we have that any proposed strategy that violates the second condition cannot be implemented as a Proxy Equilibrium.

□

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