Proxy Variables and Feedback Effects in Decision Making*

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Abstract

When using data, often an analyst only has access to proxies or measurements of the true variables of interest. I propose a framework that models economic decision makers as 'flawed statisticians' who assume potentially noisy proxy variables are perfectly measured. Due to feedback from the decision maker's choices to the distribution over variables, a notion of equilibrium is required to close the model. I illustrate the concept with applications to policing/crime and market entry. In these examples, we see that (1) very small imperfections in the proxy variable can lead to large distortions in beliefs and (2) we can have monotonicity in the quality of decision in the extent of proxy 'noise' even when without equilibrium effects there is non-monotonicity.

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1 Introduction

The analysis of quantitative data in order to inform decisions is increasingly important to organisations and firms. For the case of private companies, McAfee et al. (2012) argue that improvements in the ability of managers to measure, store and collect information about their business can result in large performance gains. However, data used in economic decision making is often an imperfect measurement or proxy of the underlying variables.

Examples of proxy variables that play an important role in driving allocation of economic resources abound. GDP per capita guides entrepreneurs and traders in assessing the relative economic vitality of countries in which they are considering investment, and is used by governments in determining important policy decisions. Yet as a proxy for living standards it has come in for criticism[1]. The use of citation metrics is another case. Governments and academic institutions find these metrics valuable for assessing academic impact, but there is debate around the extent to which they are truly good measures[2].

In this paper I propose a framework for modelling decision makers who naively use possibly mismeasured proxies. The decision makers in this framework are assumed to form expectations of the impact of their actions from the proxy variables they have available, treating the proxy variables as if they were *exactly identical* to the true variables. This follows a long tradition in economics and psychology of modelling agents as 'flawed statisticians', for example early work in behavioural economics on the 'law of small numbers' by Tversky and Kahneman (1971) to more recent work testing the 'What You See Is All There Is' heuristic in experiments (Enke, 2020).

The agents first draw the realisation of a signal variable s. They then choose an action variable x, and both the chosen action and realized signal then affect the realisation of a vector y of outcome variables. Agents are assumed to have a

[1]See Coyle (2015) for an outline of various arguments in this debate.

[2]See Borchardt and Hartings (2018) for discussion and reference to work (Borchardt et al., 2018) providing evidence that for academic chemists there are significant differences between the level of citations a paper receives and perceived academic impact among scientists.

vNM utility function over all of these variables. Finally, a vector of proxy variables $(s^\bullet, x^\bullet, y^\bullet)$ is drawn from a distribution π that depends on (s, x, y), where each of the true variables has a corresponding proxy. The decision makers only have access to data that gives them knowledge of the joint distribution of the proxies. Due to the possible imperfections in the variables available, choices made by these decision makers can affect the data they use to form beliefs. I define a notion of Proxy Equilibrium which ensures consistency between these choices and the the data upon which they are based.

Following Spiegler (2016), we model the relationship between the true variables using *Directed Acyclic Graphs* (henceforth DAGs). Suppose there are $m-2$ outcome variables, so there are m variables in total. Denote the vector of all variables as $v = (s, x, y_1, ..., y_{m-2})$ and the set of variables as M. A DAG $\mathcal{R} = (M, R)$ consists of a set of variables M and a set of directed links $R \subset M \times M$. The set R is an acyclic, irreflexive and asymmetric binary relation which we can use to define $R(i) = \{j \in M | jRi\}$ as the set of 'parents' of the variable i according to the DAG. Figure 1 illustrates different DAG structures for three variables, with jRi meaning that the arrow $\rightarrow$ is pointing from j to i.

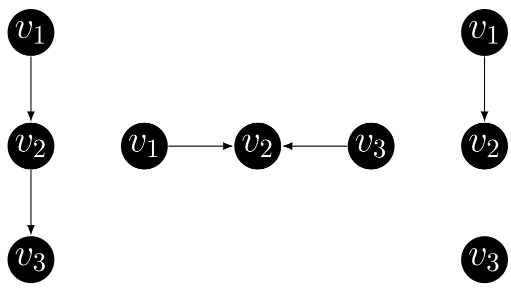

Figure 1: Illustration of three DAGs

To preview an application in the paper, suppose the decision maker is a municipal policymaker who must decide on the number of police officers to employ. Before deciding, they first learn the realisation of a signal variable s that affects the cost of policing. The police number x they choose then affects crime level y in the municipality. The relationship between the three variables can therefore be described by DAG $s \rightarrow x \rightarrow y$, and we can write the joint density of the variables as the product of terms where we condition each variable on its parents according to the DAG.

$$p(y, x, s) = p(y|x)p(x|s)p(s) \tag{1}$$

The two premises of the framework in this paper are that (1) the decision maker only has access to the steady state joint distribution of the proxies, rather than the true variables, and (2) the decision maker understands the true causal model embodied by the DAG given by R, and tries to fit the proxy distribution according to this model.

The following recipe —applied to the policing example— illustrates what this implies for belief formation. First, the decision maker's strategy generates a joint density over the true variables $p(y, s, x)$. The mapping π, takes any realization of the true variables (y, x, s) and maps them into the distribution over the proxy variables $(y^\bullet, x^\bullet, s^\bullet)$. This induces a joint density over the proxies $p_\pi(y^\bullet, x^\bullet, s^\bullet)$. Finally, the decision maker applies the conditional independence relations for the true variables to the joint density of the proxies. This generates a distorted factorised joint density of the proxies.

$$q_\pi(y^\bullet, x^\bullet, s^\bullet) = p_\pi(y^\bullet|x^\bullet)p_\pi(x^\bullet|s^\bullet)p_\pi(s^\bullet) \tag{2}$$

The decision maker then uses this distorted joint distribution over proxies to form expectations. We distinguish between the agent's knowledge of the joint distribution of the observed variables and the fact they learn the true realisation of the signal and action variables. This means that, for example, the municipal leader knows the level of police staffing that they choose but not how police numbers covary with crime levels. As a result, given the signal-action combination (s, x) the agent forms conditional beliefs over the other variables by substituting the value of the realisations into $q_\pi(y^\bullet|s^\bullet = s, x^\bullet = x)$, essentially treating the measured variables as if they were *identical* to the true variables.

The story I have in mind is that the joint distribution over proxies is generated as a long run steady state of some learning process. The learning process is not that of a long lived agent who repeats the same decision problem enough times to generate an asymptotic sample, but instead a short-lived agent who does not generate enough experiences of the effect of their own action and signals and has to rely on a large public dataset of potentially mismeasured proxies. For concreteness, in the policing example we can imagine a sequence of short lived municipal leaders. The data generated from each municipal leader's tenure is too sparse to apply the law of large numbers, so the leaders have to draw inference from the experiences of other municipalities in other time periods by using a national dataset designed for social scientists researching crime.

After defining the concept in Section 2, I present applications to decision making in organisations. I first develop the application to policing in Section 3, and show that arbitrarily small imperfections in the recorded measurements of police staffing numbers can lead to there being equilibria with very large distortions in beliefs. After the policing example, I build an application to overoptimism in market entry in Section 4. In this application more 'noise' in the proxy variables always results in worse decisions in equilibrium, something that is not true without equilibrium effects. In Section 5, I present two results exploring what happens when the proxy mapping is close to perfect measurement in the total variation norm. The first result shows that proximity to perfect measurement is often unrestrictive. For many strategies we can construct a proxy mapping that is very close to perfect measurement and sustains the proposed strategy as a Proxy Equilibrium. The second result give general conditions under which close proximity to perfect measurement results in beliefs that are close to rational expectations. The result makes clear why this does not hold in the policing example.

The contribution of this paper is twofold. First, it contributes to the literature on solution concepts with bounded rational expectations by considering issues of measurement and proxies in an equilibrium framework. The concept generally does not fall neatly in others in the literature, and I explore these connections in Section 6. Secondly, it develops applications of the concept to a variety of settings in which organisations use data to form beliefs.

2 Modelling Set Up

There is a product space of m variables $V_1 \times ... \times V_m \subseteq \mathbb{R}^m$. We label the variables according to the set $M \equiv \{1, ..., m\}$ and let $v_i \in V_i$ be a realisation of variable $i \in M$, and $v_N = (v_j)_{j \in N} \in \prod_{j \in N} V_j$ be a realisation of the subset of variables $N \subseteq M$. We can further divide and order the set M into the signal variable $\mathbf{s} = \{1\}$, an action variable $\mathbf{x} = \{2\}$ and a set of outcomes $\mathbf{Y} = \{3, ..., m\}$. Let the spaces of the signal, action and outcome variables be $S = V_1$, $X = V_2$ and $Y = \prod_{j \in \mathbf{Y}} V_j$ respectively. In examples, sometimes we denote the label of a variable w as $\mathbf{w}$ when it is easier to do so rather than enumerate all the variables. Any subset of the outcome variables $N \subseteq \mathbf{Y}$ is denoted by $y_N = (v_j)_{j \in N} \in \prod_{j \in N \subseteq \mathbf{Y}} V_j$. Let the notation $v_{-i} \in V_{-i} \equiv \prod_{j \neq i} V_j$ denote the realisation and space of all true variables except i. We label the Borel Sets[3] of V as $W \in \mathcal{W}$ and for any subset variables N and its complement $-N \equiv M/N$ we label the Borel Sets $W_N \in \mathcal{W}_N$ and $W_{-N} \in \mathcal{W}_{-N}$.

The idea will be that our decision maker learns the realization of a signal $s \in S$, before choosing an action $x \in X$ resulting in a distribution over outcome variables $y \in Y$. The payoff of the decision maker (henceforth DM) is defined over all the variables by utility function $u : S \times X \times Y \to \mathbb{R}$. The utility function is bounded in absolute value by an integrable function $h : S \times X \times Y \to \mathbb{R}$.

As discussed in the introduction, we assume that the relationship between the variables can be described by a Directed Acyclic Graph (henceforth DAG) $\mathcal{R} = (M, R)$. We postpone the discussion of the use of DAGs in the formalism until Section 6.0.2. For expositional ease, we mostly refer to the DAG by its set of directed links R throughout the paper, as it is normally clear what the variable space is. The only restrictions we place on the DAG structure is that for any $\mathbf{x}$, $R(\mathbf{x}) = \{\mathbf{s}\}$ and $R(\mathbf{s}) = \emptyset$, so the DM learns the realisation of the signal variable before acting, this is the only variable which causally precedes the action and no variables causally precede the realisation of the signal. Let $P \in \Delta(V)$

[3] In our examples, we assume the topology on all variable spaces is the discrete topology when variables are finite and the Euclidean topology otherwise.

be the objective distribution over the variables. We assume throughout that this distribution admits a density p with respect to some σ-finite measure μ[4]. We can then factorise this density as follows using the DAG structure:

$$q(v) = \prod_{i=1}^{m} p(v_i | v_{R(i)}) = \prod_{i=3}^{m} p(y_i | v_{R(i)}) p(x|s) p(s) \tag{3}$$

Since the DAG R is assumed to be correct, the factorisation formula above does not distort the true conditional independence relationships between the true variables and thus $q(v) = p(v)$, the true joint density, for all $v \in V$.

2.1 Proxy variables

We define a set of m proxy variables in the space $V^\bullet = V_1^\bullet \times ... \times V_m^\bullet$, such that $V_i = V_i^\bullet$ for each $i \in M$. We label the proxy variables using the set $M^\bullet = \{1^\bullet, ..., m^\bullet\}$, so that the proxy for variable i is $i^\bullet$. Henceforth, we call the variables M the *true* variables and the variables $M^\bullet$ the *proxy* variables. The spaces of proxy signal, action and outcome variables are labelled analogously to their true variables[5], while the Borel Sets of $V^\bullet$, $V_N^\bullet$ and $V_{-N}^\bullet$ are $W^\bullet \in \mathcal{W}^\bullet$, $W_N^\bullet \in \mathcal{W}_N^\bullet$ and $W_{-N}^\bullet \in \mathcal{W}_{-N}^\bullet$ respectively. When there is no ambiguity, we use i instead of $i^\bullet$ to refer to the matching proxy variable for the true variable i. In order to allow ourselves to talk about applying the DAG R to the proxy variables $M^\bullet$ we extend the DAG R so it also contains directed links between the proxy variables, so that if $j \to k$ is a link in R then so is $j^\bullet \to k^\bullet$.

We define a *proxy mapping* $\pi : V \to \Delta(V^\bullet)$ which defines a distribution over the measured proxies for any realisation of the true variables.

$$P_\pi(W^\bullet) = \int_V \pi(W^\bullet | v) [\prod_{i=1}^{m} p(v_i | v_{R(i)})] d\mu(v) = \int_V \pi(W^\bullet | v) p(v) d\mu(v) \tag{4}$$

[4]In all examples in this paper this will either be the counting measure for the finite case or Lebesgue measure for the continuum case.

[5]This means that $\mathbf{s}^\bullet = \{1^\bullet\}$, $\mathbf{x}^\bullet = \{2^\bullet\}$ and $\mathbf{y}^\bullet = \{3^\bullet, ..., m^\bullet\}$ and the variable spaces can be denoted $s^\bullet \in S^\bullet = V_1^\bullet$, $x^\bullet \in X^\bullet = V_2^\bullet$ and $y^\bullet \in Y^\bullet = \prod_{j \in \mathbf{Y}^\bullet} V_j^\bullet$ for the signal, action and outcome proxies respectively.

We assume that the proxy mapping is such that the induced distribution over proxies P_π admits a density p_π and that for all variables $i \in M$, $p_\pi(v^\bullet_{R(i)}) > 0$ for any realisation $v^\bullet_{R(i)} \in V^\bullet_{R(i)}$, so that the parents of any variable according to the true DAG have full support. The DM is modelled as fitting the true DAG R over the measured variables $M^\bullet$, leading to potentially distorted beliefs even though the DM does understand the causal relationship between the variables that matter for their utility. For each variable i, the agent can take the distribution of the proxy variable conditional on the proxy of its parents according to the true DAG : $p_\pi(v^\bullet_i | v^\bullet_{R(i)})$. These conditional distributions are well defined by our assumptions on the proxy mapping. The DAG structure R ensures that these conditional distributions can be pasted together to form a belief over the joint distribution of the variables. Formally, the DM factorises the density over the observed variables p_π into the distorted belief density q_π as follows.

$$q_\pi(v^\bullet) = \prod_{i=1}^{m} p_\pi(v^\bullet_i | v^\bullet_{R(i)}) = \prod_{i=3}^{m} p_\pi(y^\bullet_i | v^\bullet_{R(i)}) p_\pi(x^\bullet | s^\bullet) p_\pi(s^\bullet) \tag{5}$$

The DM can then form conditional beliefs:

$$q_\pi(y^\bullet | s^\bullet, x^\bullet) = \frac{q_\pi(y^\bullet, x^\bullet, s^\bullet)}{\int_{Y^\bullet} q_\pi(\tilde{y}^\bullet, x^\bullet, s^\bullet) d\mu(\tilde{y}^\bullet)} \tag{6}$$

Given this distorted belief distribution, the agent with signal s chooses an action vector x to maximize the perceived utility given below.

$$U(s, x; q_\pi) = \int_{Y^\bullet} u(y = y^\bullet, x, s) q_\pi(y^\bullet | s^\bullet = s, x^\bullet = x) d\mu(y^\bullet) \tag{7}$$

2.1.1 The proxy mapping

It is possible that a proxy is a perfect measure for the underlying true variable, $v^\bullet_i = v_i$ almost everywhere for some variable(s) i. Indeed, for all the applications in this paper some of the variables are perfectly observed. In the case where $v^\bullet_i \neq v_i$ with nonzero probability, we say that $i^\bullet$ is a *mismeasurement* of i. In examples, for simplicity we generally avoid drawing a distinction between the true and proxy variable when the proxy is a perfect measurement.

A proxy mapping that induces an identical joint distribution over the proxy variables and the true variables —for any initial distribution of the true variables— is called the *perfect measurement* mapping. Throughout the paper, we say that the beliefs induced by the perfect measurement mapping comprise the *rational expectations benchmark* or induce *correct beliefs*. One could argue that what comprises rational expectations is ambiguous, given the DM only has knowledge of the proxies. However, what it means to correctly form beliefs under this ambiguity is not the focus of this paper.

Given the possibility of perfect measurement, it is sometimes useful to illustrate the dependencies of the proxy variables with an auxiliary DAG structure to the underlying DAG R. The following two diagrams give different examples of the form that the proxy mapping can take. The black nodes and arrows indicate the true variables while the blue nodes indicate the proxy variables and the red arrows indicate auxiliary links.

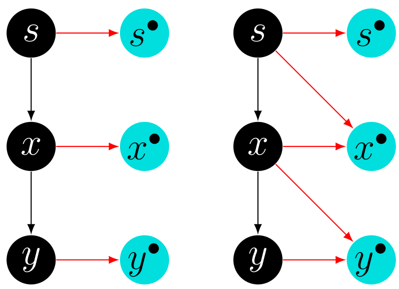

Figure 2: DAG structures for different proxy mappings

In the first diagram, assuming finite variables, the proxy mapping can be factorised into three separate mappings.

$$\pi(y^\bullet, x^\bullet, s^\bullet | y, x, s) = \pi_y(y^\bullet | y)\pi_x(x^\bullet | x)\pi_s(s^\bullet | s)$$

The second diagram indicates more dependency in the measurements, with the action variable having an effect on the divergence between the outcome proxy and the true outcome variable and the signal variable doing the same for the action and its proxy. Again, we can factorise into separate mappings.

$$\pi(y^\bullet, x^\bullet, s^\bullet|y, x, s) = \pi_y(y^\bullet|y, x)\pi_x(x^\bullet|x, s)\pi_s(s^\bullet|s)$$

In both cases, the auxiliary DAG structure is consistent with perfect measurements of some variables. For example, for both diagrams it could be the case that $\pi_s(s^\bullet = s|s) = 1$ for all $s \in S$, so we have perfectly measured signal variables.

2.2 Equilibrium

We can see how the decision maker's strategy can affect expectations by considering the first graph in Figure 2. Here we have potentially imperfect measurements of all the variables, but no links between proxies and true variables that are not being proxied for. However, even in this case we can have endogeneity of beliefs. This can be seen by calculating the perceived conditional distribution of the outcome variable, given the factorization given by the auxiliary DAG.

$$q_\pi(y^\bullet|x^\bullet) = \frac{\sum_{y,x,s} \pi_y(y^\bullet|y)\pi_x(x^\bullet|x)p(y|x)p(x|s)p(s)}{\sum_{x,s} \pi_x(x^\bullet|x)p(x|s)p(s)} \tag{8}$$

It is clear that this expression is not invariant to the DM's choice of strategy $p(x|s)$. We will later give conditions under which there are no equilibrium effects, but in general to characterize the DM's choices we need to define an equilibrium concept in order to establish consistency between strategies and beliefs. To ensure that conditional distributions are well defined we first define an equilibrium with a small trembling probability and then define an equilibrium as the limit when this probability goes to zero.

Definition 1. *Given true DAG R and a proxy mapping π, a strategy mapping that admits a full-support density $\sigma^*_\epsilon(x|s)$ for every $s \in S$, that induces a density over observed variables q is an ϵ-**Proxy Equilibrium** if the following two conditions hold:*

1. *Belief density q_π satisfies $q_\pi(v^\bullet) = \prod_{i=1}^n p_\pi(v_i^\bullet|v_{R(i)}^\bullet)$, where p_π is the density of the proxy variables induced by the proxy mapping π and the true distribu-*

tion over variables P. The true distribution admits a density p that is such that $p(x|s) = \sigma^*_\epsilon(x|s)$.

2. *For every $s \in S$, define the following set:*

$$X(s;\sigma^*_\epsilon) \equiv$$
$$\{x \in X : x \notin \arg\max \int_{Y^\bullet} u(y = y^\bullet, x, s) q_\pi(y^\bullet | s^\bullet = s, x^\bullet = x) d\mu(y^\bullet)\}$$

Then for every interval[6] *$I \subseteq X(s;\sigma^*_\epsilon)$, the strategy mapping is such that $\sigma^*_\epsilon(I|s) < \epsilon$*

Definition 2. *A strategy σ^* is an* ***Proxy Equilibrium*** *if there exists a sequence $\{\sigma^*_l\}_{l=1}^\infty$ converging in distribution to σ^**[7] *as well as a sequence $\epsilon^l \to 0$, such that for every l, σ^*_l is an ϵ^l-Proxy Equilibrium.*

The first part of the definition describes how the beliefs of the DM are induced by proposed equilibrium strategy. The second part ensures that action-signal combinations that are not in the best response correspondence must have vanishing probability under the equilibrium strategy.

When the variable product space $V = V^\bullet$ is a finite set, we can show the existence of at least one Proxy Equilibrium using conventional methods.

Proposition 1. *Assume the set $V = V^\bullet$ is finite. Then a Proxy Equilibrium exists.*

Proof. In Appendix □

3 An illustrative example: Police and Thieves

In this application we consider the leader of a municipal authority, who has to make a decision on the number of police officers to hire. The municipal leader

[6] An interval here is defined on $X \subset \mathbb{R}$, they can be closed: $\{x \in X : a \leq x \leq b\}$, open: $\{x \in X : a < x < b\}$ or half open: $\{x \in X : a < x \leq b\}$, $\{x \in X : a \leq x < b\}$.

[7] More precisely, for every $s \in S$ the sequence of measures over X induced by the sequence of strategy mappings evaluated at s converges in distribution to the measure over X induced by the strategy map σ^* evaluated at s.

wants to hire police officers in order to reduce crime. We assume that there is noise in the measured variable for change in police numbers. Concern about measurement error in police staffing figures is not unprecedented. It is argued by Chalfin and McCrary (2018) that based on discrepancies between official data and administrative and census information there is significant measurement error in police staffing numbers in the literature estimating the effect of police numbers on crime. For expositional purposes, we assume that the crime variable is measured perfectly. This turns out not to make a difference to the equilibria that we characterise.

The structure of the problem facing the municipal leader is as follows, first they learn the realisation of a variable affecting the cost of changing police numbers s. This is assumed to be distributed normally in the population of municipalities used in the dataset under consideration, $s \sim \mathcal{N}(0, \sigma_s^2)$. The municipal leader then chooses the change in the number of police officers. This affects the change in crime observed under their leadership via the relationship $y = \alpha + \beta x + u$, where $u \sim \mathcal{N}(0, \sigma_u^2)$ and $\beta < 0$. In the available data, it is assumed that changes in police numbers are measured as $x^\bullet = x + \epsilon$, where ϵ is normally distributed measurement error $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. The relationship between the variables can be characterized by the following DAG.

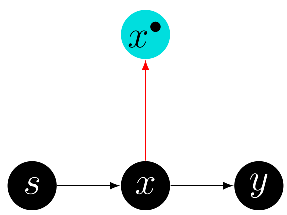

Figure 3: Crime and Policing

The utility function of the municipal leader trades off crime and policing costs.[8]. Higher s is assumed to reflect lower costs of altering police numbers[9].

[8]The piecewise form is assumed to avoid the problem of dividing by zero, since this occurs with measure zero it has no effect on the equilibria.

[9]If the mean of the distribution of μ_s is high enough, the probability of negative costs of changing police numbers is small.

$$u(y,x,s) = \begin{cases} -y - \frac{1}{s}\frac{1}{2}x^2 & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases} \tag{9}$$

Denote the rational expectations benchmark for how policing affects crime levels in expectation by $\mathbb{E}[y|x] = f(x) = \alpha + \beta x$. We can see that by plugging this in to the utility function and calculating the best response that the optimal strategy under rational expectations for the municipal leader is to set police numbers such that $x^*(s) = -\beta s$. Thus the rational expectations benchmark is for police numbers to be increased by more when the costs of doing so are low (higher s) and the effect of police numbers on crime is greater (higher $|\beta|$). Define a *linear equilibrium* as an equilibrium in which the strategy of the policy maker can be expressed as a linear function of the cost variable, $x(s) = \theta_0 + \theta_1 s$ for some $(\theta_0, \theta_1) \in \mathbb{R}^2$. We can characterize all the linear equilibria of the model as follows[10].

Proposition 2. *There is always a linear equilibrium in which the municipal leader never changes police numbers, with best response $x^{nv}(s) = 0$.*

In addition, if $|\beta| \geq 2\frac{\sigma_\epsilon}{\sigma_s}$, then there exist two additional linear equilibria, with best response $x^-(s) = -(-\frac{1}{2}\beta - \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$ and $x^+(s) = -(-\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2}})s$.

Proof. In Appendix □

Due to the measurement error in the police numbers proxy, there is generally downward attenuation bias in the municipal leaders estimate of the expected change in the level of crime for any given change in police numbers. However, when there is more variation in police staffing numbers the data the measurement error is a smaller fraction of the total variance of the proxy. This means the downward attenuation effect is lessened compared to when there is little or no variation in true police staffing numbers. This effect generates potential multiplicity of equilibria.

[10]Due to the difficulty in characterizing non-linear equilibria in this setting, we do not attempt to do so.

Adding normally distribution measurement error to the crime variable, so that $y^{\bullet} = y + v$ with $v \sim \mathcal{N}(0, \sigma_v^2)$, does not change either the set of Proxy Equilibria nor does it change the rational expectations benchmark. That the rational expectations benchmark is unchanged is easy to see due to linearity of expectations. The proxy-equilibrium case is due to both the linearity of the conditional expectation and the fact that the additional variance in $y^{\bullet}$ does not affect the marginal perceived incentive over the policing variable.

Another interesting feature is that one of the equilibrium has the municipalities not varying police numbers at all, regardless of how great the effect of policing on crime is. This extreme, no variation equilibrium exists for any small amount of measurement error in the proxy $\sigma_\epsilon^2 > 0$ no matter how close to zero. For σ_ϵ^2 close to zero, the case when the proxy is close to a perfect measurement, the equilibrium with the smaller positive amount of variation in policing converges to the zero variation equilibrium while the equilibrium with the larger amount of variation converges to the rational expectations benchmark.

4 Endogenous Hubris and Market Entry

Businesses entering into new markets have high rates of failure. Using data from the US Census Bureau Haltiwanger (2015) calculates that half of new firms exit the market within 5 years. In UK data, 38 percent of enterprises newly born in 2016 survived 5 years[11]. A literature in business and economics attributes these seemingly excessive levels of market entry to overoptimism on the part of the potential market entrants, see Hayward et al. (2006), Cooper et al. (1988), Malmendier and Tate (2005).

We build an application of our solution concept that generates firms that have an upwardly biased assessment of the payoffs from entering new markets as a feature of equilibrium. Firms draw on noisily recorded data drawn from past entrants. There is a variable $s \in [0, 1] \equiv S$, representing the location of markets in some space, which could be geographical or based on demographic information.

[11] This statistic is from Office for National Statistics (2022).

This variable is distributed such that it admits a continuous full-support density function $p(s)$. After learning the realisation of this variable, the potential market entrant has to make a binary decision on whether to enter $x = 1$ or not $x = 0$. The payoff of the entrant is measured via an outcome variable $y \in \mathbb{R} \equiv Y$ representing the profitability of the enterprise, so that $u(y, x, s) = y$. The outcome variable is determined by both the entry decision and the market location variable by the following relationship.

$$\mathbb{E}[y|x, s] = \int_Y yp(y|x, s)d\mu(y) = \begin{cases} m(s) \text{ if } x = 1 \\ 0 \text{ if } x = 0 \end{cases} \tag{10}$$

We assume that the function $m : S \to \mathbb{R}$ is strictly increasing, bounded and right-continuous, with a single point of crossing $\alpha \in [0, 1]$ such that $m(s) < 0$ for all $s \in [0, \alpha)$ and $m(s) \geq 0$ for $s \in [\alpha, 1]$. Thus for high enough realizations of the market location variable, the expected profitability of entry is always greater than the payoff of zero from not entering. We assume the potential entrant does not have data on how the market location s varies with the outcome and action variable, but instead has access to a noisy recorded proxy variable $s^\bullet$. The idea is that each market has a very granular definition, and in data it can only be recorded in an imprecise fashion. This could be due to data protection reasons or due to the categories available to the data recorder not being as fine-grained as the data itself.

Thus we assume the proxy variable is generated by a mapping that has the following 'window' form. There is some parameter $h \in (0, \frac{1}{2})$ such that for every $s \in [h, 1 - h]$ we have that $s^\bullet$ is uniformly distributed on $[s - h, s + h]$. For all $s \in [0, h)$ we have that $s^\bullet$ is distributed uniformly on $[0, 2h)$ and for all $s \in (1-h, 1]$ we have that $s^\bullet$ is distributed uniformly on $s \in (1 - 2h, 1]$. This window form of proxy noise is similar to the notion of similarity used in Steiner and Stewart's (2008) model of learning in games. The conditional independence relationships between the true variables and the proxy are illustrated in the DAG below.

Under rational expectations the best response of the potential entrant is clear;

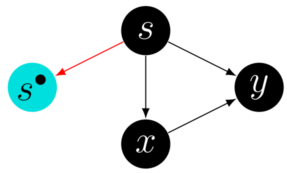

Figure 4: Market Entry

when $s \in [0, \alpha)$ $x = 0$ is optimal while for $s \in [\alpha, 1]$ the payoff from entering is above zero and therefore optimal. We are going to see that in equilibrium, there is over-entry. Since in the equilibrium data there is negligible observations of entrants below a certain market location, the proxy observations for these markets are disproportionately higher location markets that have been misclassified as lower ones. This leads to an overestimate of the payoff from entering at these lower levels. However, enough over-entry reduces the extent of this proxy bias and in equilibrium the DM is indifferent between entering or not at some cut-off strength level below the cut-off they would enter at under rational expectations.

Given an induced perceived distribution over the outcome variable $q_\pi(y|x, s^\bullet)$, we give an expression for the perceived expected utility below.

$$\begin{aligned} U(s, x = 1; q_\pi) &= \int_{Y^\bullet} y q_\pi(y|x = 1, s^\bullet = s) d\mu(y) \\ &= \int_{Y^\bullet} y [\int_S p(y|x = 1, \tilde{s}) p_\pi(\tilde{s}|s^\bullet = s, x = 1) d\mu(\tilde{s})] d\mu(y) \\ &= \int_S [\int_Y y p(y|x = 1, \tilde{s}) d\mu(y)] p_\pi(\tilde{s}|s^\bullet = s, x = 1) d\mu(\tilde{s}) \\ &= \int_S m(\tilde{s}) p_\pi(\tilde{s}|s^\bullet = s, x = 1) d\mu(\tilde{s}) \end{aligned} \tag{11}$$

The perceived utility of $x = 0$ at s is always zero; $U(s, x = 0; q_\pi) = 0$. We can see that the perceived utility depends on the distribution $p_\pi(\tilde{s}|s^\bullet = s, x = 1)$ induced by the strategy σ. Given the distribution over proxies, this can be calculated as follows.

$$p_\pi(s|s^\bullet, x=1) = \frac{p_\pi(s, s^\bullet, x=1)}{\int_S p_\pi(\hat{s}, s^\bullet, x=1) d\mu(\hat{s})}$$

$$= \begin{cases} \frac{1_{[s\in[s^\bullet-h, s^\bullet+h]} p(s)\sigma(x=1|s)}{\int_{s^\bullet-h}^{s^\bullet+h} p(\hat{s})\sigma(x=1|\hat{s})d\mu(\hat{s})} & \text{if } s^\bullet \in [h, 1-h] \\ \frac{1_{[s\in[0,h]} p(s)\sigma(x=1|s)}{\int_0^h p(\hat{s})\sigma(x=1|\hat{s})d\mu(\hat{s})} & \text{if } s^\bullet \in [0, h) \\ \frac{1_{[s\in[1-h,1]} p(s)\sigma(x=1|s)}{\int_{1-h}^1 p(\hat{s})\sigma(x=1|\hat{s})d\mu(\hat{s})} & \text{if } s^\bullet \in (1-h, 1] \end{cases} \tag{12}$$

We make the following endpoint assumptions, which ensure that the firm will not enter at low enough of s and there is always an equilibrium in which the firm will enter at high values of s.

Assumption 1. *We say that $m(.)$, $p(.)$ and h satisfy the **endpoint assumptions** if $\int_h^{2h} m(s)p(s)d\mu(s) < 0$ and $\int_{1-h}^1 m(s)p(s)d\mu(s) > 0$.*

We can then show existence and characterise the Proxy Equilibria.

Proposition 3. *Assume the primitives $m(.)$, $p(.)$ and h satisfy the **endpoint assumptions**. Then there is a cut-off $\bar{s} \in [h, 1-h]$ such that there is a Proxy Equilibrium with strategy $\sigma(x=1|s) = 0$ for $s \in [0, \bar{s}]$ and $\sigma(x=1|s) = 1$ for $s \in (\bar{s}, 1]$.*

In addition, there is always a Proxy Equilibrium where $\sigma(x=1|s) = 0$ for all $s \in [0, 1]$.

There are no other Proxy Equilibria.

Proof. In Appendix □

We then have the following result, which gives us that in cases where there is entry in equilibrium there is excessive entry. In addition, we see that noisier proxies —in the form of higher h— always lead to greater levels of excessive entry.

Proposition 4. *The cut-off in the Proxy Equilibrium with positive probability of entry, $\bar{s}$, is always strictly less than α.*

Moreover, consider two noise parameters $1 > h_2 > h_1 > 0$, such that given $m(.)$, $p(.)$ the endpoint assumptions are satisfied for both. We have that the cut-off $\bar{s}(h_2)$ under the positive entry Proxy Equilibrium with noise parameter h_2 is strictly less that the cut-off $\bar{s}(h_1)$ under h_1.

Proof. In Appendix □

The intuition for this result is as follows. In general —for a fixed belief distribution— it is ambiguous whether larger h increases or decreases the payoff from entry at any given s. However, in equilibrium what matters is the beliefs at the pivotal cut-off $\bar{s}$. At the cut-off the DM must be indifferent between entering and not. An increase in h will always lead to greater weight on the part of the function above the cut-off, in particular greater weight on the positive part of $m(s)$. This pushes up the expected payoff from entering strictly above zero at this cut-off, and the new cut-off at the larger h must be below in order to restore indifference.

In contrast if there are no equilibrium effects and the belief distribution is fixed, we can construct cases where entry is excessive and increasing h results in a best response with less entry.

Example 1. *Let the function $m(.)$ take the following form.*

$$m(s) = \begin{cases} s - \frac{5}{2} & \text{if } s \in [0, \frac{1}{4}) \\ s - \frac{1}{2} & \text{if } s \in [\frac{1}{4}, \frac{1}{2}) \\ s + \frac{1}{2} & \text{if } s \in [\frac{1}{2}, 1] \end{cases} \tag{13}$$

Assume the distribution over signals $p(s)$ is uniform and the conditional distribution $p(.|s^\bullet, x = 1)$ is set exogenously to that which would be induced by the strategy $\sigma(x = 1|s) = 1$ for all s. This means that $p(.|s^\bullet, x = 1)$ is uniform in $s \in [s^\bullet - h, s^\bullet + h]$ for each $s^\bullet \in [h, 1-h]$ and uniform on $s \in [0, h)$ and $s \in (1-h, 1]$ for $s^\bullet \in [0, h)$ and $s^\bullet \in (1-h, 1]$ respectively.

If $\frac{1}{4} < h < \frac{1}{2}$, then the cut-off signal at which the DM switches from $x = 0$ to $x = 1$ is $\bar{s}(h) = \frac{1+2h}{3+2h} \in [0, \frac{1}{2})$. The parameters are such that $0 < \bar{s}(h) - h < \frac{1}{4}$ and $\frac{1}{2} < \bar{s}(h) + h < 1$. Since $\frac{d\bar{s}(h)}{dh} = \frac{4}{(2h+3)^2} > 0$ we can then see that as h increases by *a small amount, the cut-off increases. This reduces the extent of entry at signals in* $[0, \frac{1}{2})$ *where it gives a negative true expected payoff.*

4.0.1 Differences with Jehiel (2018)

There are similarities between this application and the model of selection bias induced overconfidence of Jehiel (2018). In his model as in ours, there is an upwardly bias distortion in beliefs in equilibrium which induces an entry action to occur at a threshold earlier than under rational expectations. However, there are subtle differences between the two models, and the result that greater 'noise' always induces greater over-entry that holds in our example under Proxy Equilibrium does not necessarily hold in Jehiel (2018).

The result that increases in the noise in the proxy increases over-entry does not have an analogue in Jehiel (2018). In that paper, the reason why the effect of noise on entry is ambiguous is that a more accurate signal mechanically means that any fixed strategy is more selective, which increases the extent of upward bias at each signal. There is a countervailing effect in that for a fixed level of selection, a more accurate signal increases the expected payoff at higher signals and reduces the expected payoff at lower signals. In our application, only the latter effect is operating and thus we get the result that more noise increases over-entry, while in Jehiel (2018) which effect is larger depends on the parameterisation.

5 Almost Perfect Proxies

The policing example shows that even when we have arbitrarily close to perfect measurements for all the variables, under Proxy Equilibrium beliefs can be very far from the rational expectations benchmark. In this section we present two results. Our first result shows that even under a strong notion of distance from perfect measurement, by choosing a particular proxy mapping π it is possible sustain many strategies as Proxy Equilibria that would not be in the best response correspondence under correct beliefs. The proxy mapping is tailored to the particular strategy we are trying to sustain. The result holds when all variables are finite and the DAG over the true variables is fully connected, under a fairly weak condition on the proposed equilibrium strategy. The constructed proxy mapping has very close to perfect measurement and has perfect measurement of all the outcome variables. Although we only show the result in a restricted setting it is likely the proof strategy can extend more generally.

The second result gives conditions under which we have convergence to rational expectations. It shows that if the joint density over variables satisfies a full support assumption, then beliefs become arbitrarily close to rational expectations as the proxy variables become close to perfect measurements. This result concerns potentially out of equilibrium beliefs, and can be used as a diagnostic when considering equilibria in which the full-support assumption does not hold. For example, in our policing application the full-support assumption does not always hold and therefore we can have large belief distortions even as the proxy noise is close to zero.

The total variation distance between probability measures Q_1 and Q_2 on measure space $(\Omega, \mathcal{A})$ is:

$$TV(Q_1, Q_2) = \sup_{A \in \mathcal{A}} |Q_1(A) - Q_2(A)| \tag{14}$$

We use this to define a notion of proximity of the proxy variables to perfect measurement. Define the *perfect measurement proxy mapping* as $\pi_\delta : V \to \Delta(V^\bullet)$ such that $P_\pi(W^\bullet = W) = \int_V \pi_\delta(W^\bullet = W|v)p(v)d\mu(v) = P(W)$ for every $W \in \mathcal{W}$ and any P. We then have the following definition.

Definition 3. *We say the proxy distortion mapping π is strongly η-close to perfect if for $\eta > 0$ we have that:*

$$\sup_{v \in V} TV(\pi(.|v), \pi_\delta(.|v)) < \eta \tag{15}$$

In our first result, we restrict to finite variable spaces. The construction is possible if the proposed strategy satisfies the following condition. The condition requires that for every action not in the support of the strategy, we can find a outcome realization that results in strictly lower utility than the rational expectations benchmark utility from any action in the support.

Definition 4. *A strategy $\sigma^* : S \to \Delta(X)$ is* ***potentially implementable*** *if at every $s \in S$, the following two conditions hold.*

1. *For any two actions $x, x' \in supp\{\sigma^*(.|s)\}$, we have that:*

$$\sum_{y \in Y} u(y, x, s)p(y|x, s) = \sum_{y \in Y} u(y, x', s)p(y|x', s) \tag{16}$$

2. *For every action $x^{ns} \notin supp\{\sigma^*(.|s)\}$, given any action $x^s \in supp\{\sigma^*(.|s)\}$ we have that:*

$$\sum_{y \in Y} u(y, x^s, s)p(y|x^s, s) > u(\hat{y}, x^{ns}, s) \tag{17}$$

   For some $\hat{y}$ that is realized with positive probability under σ^*: $p(\hat{y}|\tilde{x}, \tilde{s})\sigma^*(\tilde{x}|\tilde{s})p(\tilde{s}) > 0$ for some $(\tilde{x}, \tilde{s})$.

We then have the following result.

Proposition 5. *Let $V = V^\bullet$ be finite, the objective DAG R be fully connected and $\sigma^* : S \to \Delta(X)$ be a* ***potentially implementable strategy****.*

Then there exists a proxy mapping under which σ^* is a Proxy Equilibrium. Moreover, we can construct this proxy mapping such that it is strongly η-close to perfect for any $\eta > 0$ and the outcome variables $\boldsymbol{Y}$ are perfectly measured.

Proof. In Appendix □

Our result can be proved by constructing a proxy mapping that has a small probability of randomly allocating a particular realization of the true outcome vector to the proxy of an action-signal combination that has zero probability under the proposed equilibrium strategy. The particular outcome vector is chosen so as to deter the DM from choosing that particular zero-probability action-signal combination.

For our second result, we can weaken our notion of distance to perfect measurement so that it is specific to a particular distribution P. This weaker definition is sufficient for our result.

Definition 5. *We say the proxy mapping π is η-close to perfect given the distribution over true variables P if for $\eta > 0$ we have that:*

$$TV(P_\delta, P_\pi) < \eta \tag{18}$$

Under the following assumptions, we can always ensure the perceived belief of the DM induced by the proxy mapping is close to rational expectations with a proxy mapping that is close enough to perfect.

Assumption 2. *The distribution F is said to satisfy the* ***full support assumption*** *with respect to DAG R if it admits a density $f(v)$, and that for every variable i this density gives non-zero probability to every realisation of the parent variables; $f(v_{R(i)}) > 0$ for all $v_{R(i)} \in V_{R(i)}$.*

Assumption 3. *The distribution F is said to satisfy the* ***boundedness assumption*** *if there exists a constant $G > 0$ such that for every variable i the conditional density $f(v_i|v_{R(i)}) \leq G$ for every $v_i \in V_i$ and $v_{R(i)} \in V_{R(i)}$.*

These assumptions rule out zero probability events in the denominator of conditional probabilities, which then ensures the convergence of the joint distribution is passed through into the factorised conditional density. We define the following notion of *perceived ex-ante expected indirect utility*, which depends on the true distribution over variables p and the proxy mapping π. This allows us to compare the similarity of different beliefs by the extent to which they lead to different perceived utility. If beliefs differ by a measure zero event, that does not matter if we are comparing using this concept.

$$\mathcal{V}(p,\pi) = \int_{Y^\bullet \times X^\bullet \times S^\bullet} u(y^\bullet, x^\bullet, s^\bullet) q_\pi(y^\bullet | x^\bullet, s^\bullet) q_\pi(x^\bullet | s^\bullet) q_\pi(s^\bullet) d\mu \tag{19}$$

$$= \int_{V^\bullet} u(v^\bullet) q_\pi(v^\bullet) d\mu \tag{20}$$

We can then show that a continuity property holds for perceived ex-ante indirect utility under the full support assumptions.

Proposition 6. *Assume the* ***full support assumption*** *and the* ***boundedness assumption*** *holds for the true distribution P.*

Then, for any $\epsilon > 0$, there exists an $\eta > 0$ such that if the proxy mapping π is ***η-close to perfect*** *given true distribution P and induces a distribution over the proxy variables that satisfies the* ***full support assumption*** *and the* ***boundedness assumption****, then $\left|\mathcal{V}(p,\pi_\delta) - \mathcal{V}(p,\pi)\right| < \epsilon$.*

Proof. In Appendix $\square$

It is clear that the policing example does not satisfy the full support assumption when the no variation equilibrium strategy is played. However, in the cases where the full support assumption is satisfied then Proposition 6 holds and we have beliefs that are close to rational expectations for proxies that are close enough to perfect measurements. We can use the Hellinger distance, convergence in which implies convergence in the Total Variation distance, to derive an expression for the distance between any joint gaussian for proxies and true variables. For example, we can obtain an expression for the square of the Hellinger distance between the distribution for the policing variable $\mathbf{x}$ and its proxy, which are distributed $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $x^\bullet \sim \mathcal{N}(\mu_x, \sigma_x^2 + \sigma_\epsilon^2)$ respectively.

$$H^2(P_\pi^{\mathbf{x}}, P^{\mathbf{x}}) = 1 - \sqrt{\frac{2\sigma_x\sqrt{\sigma_x^2 + \sigma_\epsilon^2}}{2\sigma_x^2 + \sigma_\epsilon^2}} \tag{21}$$

We can then see that as $\sigma_\epsilon \to 0$ we can make the two distributions arbitrarily close and thus satisfy our requirements for Proposition 6.

6 Relationship to other concepts

6.0.1 Bayesian Network Equilibrium

In this section I present a result showing that under certain conditions Proxy Equilibrium can be nested as a special case of the Bayesian Network Equilibrium (henceforth BNE) of Spiegler (2016). The conditions are that (1) all the variables spaces are finite, (2) the proxy mapping can be factorised by an 'auxiliary DAG' and (3) is such that actions and signals are perfectly measured. We first define what it means for the density induced by the proxy mapping π and the distribution over true variables P to be factorised using an auxiliary DAG. We denote the joint space of true and proxy variables by $W = V \times V^\bullet$ and let $w \in W$ be a joint realisation of these variables.

Definition 6. *An* ***auxiliary DAG*** R^{aux} *is an acyclic, irreflexive and asymmetric binary relation that consists of a set of directed links between the true and proxy variables;* $R^{aux} \subset M \times M^\bullet$*. It must satisfy the following two conditions:*

1. *The parents of any proxy variable* $i^\bullet \in M^\bullet$ *must be in the set of true variables* M*;* $R^{aux}(i^\bullet) = \{j \in M \times M^\bullet | jR^{aux}i\} \subset M$*.*

2. *The parent's of any proxy variable* $i^\bullet \in M^\bullet$ *must contain the true variable it is a proxy for;* $i \in R^{aux}(i^\bullet)$*.*

If we can factorise the proxy mapping according to the auxiliary DAG, we can write the proxy mapping as the product of individual terms for each true variable-proxy variable pair.

$$\pi(v_i^\bullet \times ... \times v_m^\bullet | v_i \times ... \times v_m) = \prod_{i=1}^{m} \pi_i(v_i^\bullet | v_{R^{aux}(i^\bullet)}) \tag{22}$$

As before, in the finite variable case we can say that a variable i is *perfectly measured* if its only parent is the true variable it is a proxy for, $R^{aux}(i^\bullet) = \{i\}$, and we have that $\pi_i(v_i^\bullet = v_i | v_i) = 1$, $\pi_i(v_i^\bullet \neq v_i | v_i) = 0$.

The joint density of the proxies and the true variables can be factorised as follows. This factorization applies the links in the auxiliary DAG R^{aux} to the proxy variables and the true objective DAG R to the true variables.

$$\begin{aligned} f(v^\bullet, v) =& p(v) \prod_{j=1}^{m} \pi_j(v_i^\bullet | v_{R^{aux}(i)}) \\ =& \prod_{i=1}^{m} p(v_i | v_{R(i)}) \prod_{j=1}^{m} \pi_j(v_i^\bullet | v_{R^{aux}(i)}) \end{aligned} \tag{23}$$

Define a DAG R^{act} which takes any link from $\mathbf{x}$ or $\mathbf{s}$ into an outcome variable $\mathbf{y}_j$ and instead links it to the proxy for that outcome $\mathbf{y}_j^\bullet$. We then combine this with the links between outcomes applied to the proxy variables and the link into the true action variable to obtain DAG $R^s = [R^{act}] \cup [\cup_{j \in M^\bullet} R(j) / \{\mathbf{x}^\bullet, \mathbf{s}^\bullet\}] \cup [R(\mathbf{x})]$. Using R^s to factorize the joint density obtained in (23) gives the following.

$$f_{R^s}(v^\bullet, v) = p(s)p(x|s) \prod_{k \in \mathbf{Y}} p(v_k) \prod_{i \in Y^\bullet} f(v_i^\bullet | w_{R^s(i)})$$

Where $w_{R^s(i)}$ is a member of the set $W_{R^s(i)} \equiv \prod_{j \in R^s(i)} V_j \times \prod_{k \in R^s(i)} V_k^\bullet$.

We illustrate how these factorisations work in Figure 5. There are three variables, an action x that is perfectly measured, a mediator m and a final outcome o. The two outcome variables are m and o. In the first graph, the links between the true variables R are in black and the auxiliary DAG links R^{aux} are in red. There is a link from the action into the proxy for the mediator, while the outcome proxy is only influenced by the true outcome. The diagram on the right shows the construction of the equivalent BNE factorisation. The DAG R^{act} is illustrated in green, creating a link from x into $m^\bullet$ that parallels the link in the true DAG R from x into m. Finally there is a link in blue from $m^\bullet$ into $o^\bullet$, which is the link in the DAG R applied to the proxies. The BNE equivalent to the Proxy equilibrium involves applying the both the links in the right hand graph to a factorisation over all 5 variables.

This can then be used to then relate the solution concept of this paper to the

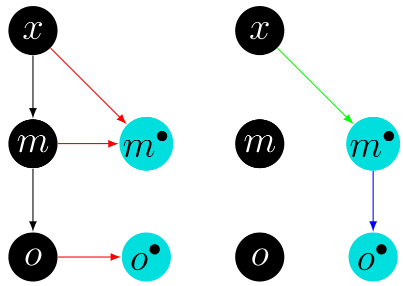

Figure 5: An illustration of how Proxy Equilibrium relates to Bayesian Network Equilibrium

BNE of Spiegler (2016).

Proposition 7. *Let $\hat{u}(y^\bullet, x, s) \equiv u(y^\bullet = y, x, s)$ be the proxy utility function defined over the proxy outcome variables $\boldsymbol{Y}^\bullet$ rather than the true outcome variables $\boldsymbol{Y}$. Assume that the following two conditions hold:*

1. *The proxy mapping π factorises according to the auxiliary DAG R^{aux}.*

2. *Both the action and signal variables are perfectly measured.*

Then the Proxy Equilibrium is also a BNE in which:

i The utility function of the DM is $\hat{u}$

ii The objective joint density over the variables $V^\bullet \times V$ can be objectively factorised using the DAG $R^* \equiv [\cup_{i \in \boldsymbol{M}^\bullet} R^{aux}(i)] \cup [\cup_{j \in N} R(j)]$

iii The DM instead forms beliefs by factorising using the subjective DAG $R^s \equiv [R^{act}] \cup [\cup_{j \in M^\bullet} R(j)/\{\boldsymbol{x}^\bullet, \boldsymbol{s}^\bullet\}] \cup [R(\boldsymbol{x})]$.

Proof. In Appendix □

A simple consequence of this result is that we also obtain for free conditions under which there are no equilibrium effects, in the sense that the beliefs of the DM are invariant to the strategy. Let $p_{\tilde{R}}$ be the density obtained by the factorisation of density p by a DAG $\tilde{R}$.

$$p_{\tilde{R}}(w) = \prod_{i \in M} p(w_i | w_{\tilde{R}(i)}) \tag{24}$$

We can then take the following definition and result from Spiegler (2016).

Definition 7. *A DAG $\hat{R}$ is consequentially rational with respect to a true DAG R^* if for any two objective distributions that are consistent with R^*, p^1 and p^2; we have that if $p^1(y^\bullet|x,s) = p^2(y^\bullet|x,s)$ and $p^1(s) = p^2(s)$ for all $(y^\bullet, x, s)$, then it must be that $p^1_{\hat{R}}(y^\bullet|x,s) = p^2_{\hat{R}}(y^\bullet|x,s)$ for all $(y^\bullet, x, s)$.*

Proposition 8. *If a Proxy Equilibrium satisfies the conditions of Proposition 7, we have that it is also a BNE with objective DAG R^* , subjective DAG R^s and utility function v. Accordingly, there are no equilibrium effects due to consequential rationality of the DAG R^s with respect to R^* if and only if the following conditions hold*[12].

1. *For every $i \in \boldsymbol{Y}^\bullet$, if $\boldsymbol{x} \notin R^s(i)$, then $y^\bullet_i \perp_{R^*} x|y^\bullet_{R^s(i)}, s$*

2. *For every $i \in \boldsymbol{Y}^\bullet$, if $\boldsymbol{s} \notin R^s(i)$, then $y^\bullet_i \perp_{R^*} s|y^\bullet_{R^s(i)}, x$*

6.0.2 Use of DAGs in the formalism

The use of DAGs allows us to both define a notion of variables and capture causal relationships between these variables in a relatively straightforward way. It is also possible to conceive of a modification of the Proxy Equilibrium concept that does not make use of the DAG language or structures. For simplicity, assume again all the variable spaces are finite. Then we can define the density of the signal variable $p(s)$ and the conditional density of the outcome variables $p(y|x,s)$, while placing the minimal causal structure that the action variable must have the signal as a parent. We can then write the joint density over the variables in the following form.

$$p(y,x,s) = p(y|x,s)p(x|s)p(s) \tag{25}$$

The joint density over the proxy variables and the true variables can then be expressed using the proxy mapping π.

[12]The third condition in Proposition 13 of Spiegler (2016) does not matter as by assumption in the proxy solution concept $R(0) = \{\emptyset\}$.

$$p(v, v^\bullet) = \pi(v^\bullet|v)p(v) \tag{26}$$

$$p(v^\bullet) = \sum_{v \in V} p(v, v^\bullet) \tag{27}$$

We can then form a conditional density over the outcome proxy variables given the action and signal proxies. Unlike in the main Proxy Equilibrium concept of this paper, we do not first factorise the joint density of the proxy variables but instead just take the conditional distribution directly from the joint distribution.

$$q_\pi(y^\bullet|x^\bullet, s^\bullet) = \frac{p(y^\bullet, x^\bullet, s^\bullet)}{\sum_{\tilde{y}^\bullet \in Y} p(\tilde{y}^\bullet, x^\bullet, s^\bullet)} \tag{28}$$

The definition of the concept then proceeds exactly as the definition of Proxy Equilibrium with (28) replacing the pre-factorised conditional density. An easy way to see the difference between Proxy Equilibrium and the DAGless variant is to compare the expression (8) that we derived from Figure 2 to the expression we would get if we conditioned on both the action and the signal proxy. For this example, this would be how beliefs are formed in a DAGless variant of the concept. We can see that these expressions are generally distinct.

$$q_\pi(y^\bullet|x^\bullet, s^\bullet) = \frac{\sum_{y,x,s} \pi_y(y^\bullet|y)\pi_x(x^\bullet|x)\pi_s(s^\bullet|s)p(y|x)p(x|s)p(s)}{\sum_{x,s} \pi_x(x^\bullet|x)\pi_s(s^\bullet|s)p(x|s)p(s)}$$
$$q_\pi(y^\bullet|x^\bullet) = \frac{\sum_{y,x,s} \pi_y(y^\bullet|y)\pi_x(x^\bullet|x)p(y|x)p(x|s)p(s)}{\sum_{x,s} \pi_x(x^\bullet|x)p(x|s)p(s)}$$

The DAGless variant result in beliefs for the decision maker that are hard to interpret. For example, consider that the true DAG is $x \to m \to y$, where m is a mediator between the action and outcome variable. Without using DAGs or a similar structure, it would not be possible to capture that the agent has knowledge that the action only affects the outcome through the mediating variable. In the DAGless variant, we would take the conditional distribution over proxies $y^\bullet, m^\bullet|x^\bullet$. This imposes that the agents have an additional layer of bounded rationality, in that they not only mistake the proxies for the true variables but also have an incorrect causal understanding of the relationship between the true variables.

7 Related Literature

The strand of literature that this paper is most clearly related to is that on solution concepts with bounded rational expectations. In particular, the work on using Bayesian Networks as a formalism to model causal misperceptions originating from Spiegler (2016) and developed to explore interactive beliefs in games (Spiegler, 2021); political narratives (Eliaz and Spiegler, 2020), (Eliaz et al., 2022); persuasion (Eliaz et al., 2021); contract theory (Schumacher and Thysen, 2022) and deception (Spiegler, 2020). Proxy Equilibrium is not nested as a special case of the Berk-Nash equilibrium of Esponda and Pouzo (2016) in an obvious way. This is due to the latter concept requiring that actions and signals are perfectly observed. Other solution concepts in this tradition include the Cursed Equilibrium of Eyster and Rabin (2005), the Behavioural Equilibrium of Esponda (2008), the Analogy Based Expectation Equilibrium of Jehiel (2005), Jehiel and Koessler (2008) and the Bias-Belief Equilibrium of Heller and Winter (2020).

We can see this solution concept literature as modelling players whose actions contribute to an long-run steady state distribution of the outcomes of past decisions in the same or similar situations. In contrast, there is a literature modelling players in games as extrapolating from small samples of the equilibrium behaviour of other players, the seminal work being Osborne and Rubinstein (1998) and Osborne and Rubinstein (2003). Several recent papers developing similar ideas include Salant and Cherry (2020), Patil and Salant (2020) and Gonçalves (2022).

This paper also connects to a body of work on naive inference from selected observations as a form of decision making bias. This models of sampling investors in Jehiel (2018) and elections with retrospective voters in Esponda and Pouzo (2017). Spiegler (2017) explores a procedure in which an analyst extrapolates from a dataset with partially missing information. Fudenberg et al. (2022) presents an equilibrium concept in which agents have selective recollection of their past experience. In all of these works, agents are considering a partially missing distribution. Under Proxy Equilibrium, data does not have to be fully missing but can be distorted by measurement error instead.

Finally, there is a link between this paper and the literature on overconfidence in the sense of over-precision as discussed in Moore and Healy (2008). As in the case of over-precision, in Proxy Equilibrium agents underestimate the extent of the divergence of observable variables from true variables. The size of the overconfidence literature makes it impossible to cover fully here, but examples modelling over-precision specifically include applications to political ideology (Ortoleva and Snowberg, 2015), speculative bubbles in finance (Scheinkman and Xiong, 2003) and volatility in securities markets (Daniel et al., 1998).

A Appendix: Proofs

Proof of Proposition 1

Proof. Denote the set of all strategies conditional on signal s as $\Sigma(s)$. Unlike the main body of the paper, we make the dependence of the perceived conditional distribution on the proposed equilibrium strategy $\tilde{\sigma}$ explicit, so use notation $q_\pi(y^\bullet|s^\bullet = s, x^\bullet = x; \tilde{\sigma})$. Define the best response correspondence, given strategy $\hat{\sigma}_\xi$ and $\xi > 0$ as

$$
\begin{aligned}
&BR_\xi(\hat{\sigma}_\xi, s)\\
&= \{\underset{\sigma(.|s)\in\Sigma(s)}{\arg\max} \int_X \sigma(x|s)[\int_{Y^\bullet} u(y = y^\bullet, x, s) q_\pi(y^\bullet|s^\bullet = s, x^\bullet = x; \hat{\sigma}_\xi) d\mu(y^\bullet)] d\mu(x)\\
&\text{s.t } \sigma(x'|s) \geq \xi \ \ \forall x' \in X\}
\end{aligned}
$$

Stack the best response correspondences for each signal $s \in S$ into a vector $BR_\xi(\hat{\sigma}) = \prod_{s\in S} BR_\xi(\hat{\sigma}, s)$. Since $q_\pi(y^\bullet|s^\bullet = s, x^\bullet = x; \tilde{\sigma})$ is continuous in σ and the best response correspondence is the set of maximizers over a compact set defined by a finite set of inequalities, $BR_\xi(\hat{\sigma})$ is nonempty for any $\hat{\sigma}$. Moreover due to linearity in $\sigma(x|s)$, $BR_\xi(.)$ convex valued and continuity of $q_\pi(y^\bullet|s^\bullet = s, x^\bullet = x; \tilde{\sigma})$ implies continuity of the maximand, meaning $BR_\xi(.)$ has closed graph. We therefore have met all the requirements of Kakutani's fixed point theorem and a fixed point exists for any $\xi > 0$, $\sigma^*_\xi \in BR_\xi(\sigma^*_\xi)$.

For any $\epsilon > 0$, we can choose $\xi > 0$ in such a way that ensures that our ξ-fixed point is an ϵ-Proxy Equilibrium. In the finite case, the largest interval subset of $X(s; \sigma^*_\epsilon)$ is itself. We have that $\sigma^*_\xi(x|s) = \xi$ for all $x \in X(s; \sigma^*_\epsilon)$ and $s \in S$. Therefore, we can choose $\xi > 0$ to ensure that $\sum_{x\in X(s;\sigma^*_\epsilon)} \sigma^*_\xi(x|s) = |X(s; \sigma^*_\epsilon)|\xi < \epsilon$ for all $s \in S$. This ensures our fixed point, which we denote σ^*_ϵ, meets the definition of ϵ-Proxy Equilibrium.

Since finiteness ensures the space of strategies Σ is compact, we can find a convergent sequence of ϵ-equilibria as $\epsilon \to 0$, $\sigma^*_\epsilon \to \sigma^*$. □

Proof of Proposition 2

Proof. We first propose a generic linear solution $x(s) = \theta_0 + \theta_1 s$, which is then used to calculate the perceived expectation $\mathbb{E}[y|x^\bullet = x]$ using the properties of the normal distribution. Under the proposed best response function, the joint normal distribution of $(y, x^\bullet)$ is:

$$\begin{pmatrix} y \\ x^\bullet \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \alpha + \beta(\theta_0 + \theta_1\mu_s) \\ \theta_0 + \theta_1\mu_s \end{pmatrix}, \begin{pmatrix} \beta^2\theta_1^2\sigma_s^2 + \sigma_u^2 & \beta\theta_1^2\sigma_s^2 \\ \beta\theta_1^2\sigma_s^2 & \theta_1^2\sigma_s^2 + \sigma_\epsilon^2 \end{pmatrix}\right)$$

Using this we can calculate the conditional expectation of y given $x^\bullet$.

$$\mathbb{E}[y|x^\bullet] = \alpha + \beta(\theta_0 + \theta_1\mu_s) + \frac{\beta\theta_1^2\sigma_s^2}{\theta_1^2\sigma_s^2 + \sigma_\epsilon^2}(x^\bullet - \theta_0 - \theta_1\mu_s)$$

Using the utility function we then get perceived expected utility $U(x, s; q) = -\mathbb{E}[y|x^\bullet = x] - \frac{1}{2}\frac{1}{s}x^2$. Solving for a maximum then gives us $x(s) = \frac{-\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2} \cdot s$. In order to have a linear equilibria, we must therefore have $\theta_0 = 0$ and $\theta_1 = \frac{-\beta\sigma_s^2\theta_1^2}{\theta_1^2\sigma_s^2+\sigma_\epsilon^2}$. We can solve the latter cubic equation to get the equilibria in the statement of the proposition.

To show that these equilibria are proxy equilibria, we must show that there is a sequence of proxy equilibria that converges in distribution to them. Let $x^\epsilon(s) = \theta_0^\epsilon + \theta_1^\epsilon s + \sigma_u^2$, where $u \sim \mathcal{N}(0, \sigma_u^2)$. Then the induced distribution over $(y, x^\bullet)$ is almost identical to above except $Var(x^\bullet) = \theta_1^2\sigma_s^2 + \sigma_\epsilon^2 + \sigma_u^2$. As such the cubic equation that characterizes the proposed ϵ-Proxy Equilibria is now:

$$\theta_1^\epsilon = \frac{-\beta(\sigma_s^2(\theta_1^\epsilon)^2 + \sigma_u^2)}{(\theta_1^\epsilon)^2\sigma_s^2 + \sigma_\epsilon^2 + \sigma_u^2}$$

For each of the three proposed equilibria we can find θ_1^η and $\sigma_u^{2,\eta} > 0$ parameterised by $\eta > 0$, where the former converges to the equilibrium value and the latter converges to zero. For the no variation equilibrium, we can set:

$$\theta_1^{nv,\eta} = \frac{(-\beta\sigma_s^2 + \frac{\sigma_\epsilon^2}{\beta} - \eta) + \sqrt{(\beta\sigma_s^2 - \frac{\sigma_\epsilon^2}{\beta} + \eta)^2 - 4\sigma_s^2\eta\beta}}{2\sigma_s^2}$$
$$\sigma_u^{2,nv,\eta} = (\frac{-\sigma_\epsilon^2}{\beta} + \eta)\theta_1^{nv,\eta}$$

These parameters solve the equilibrium quadratic and $\theta_1^{nv,\eta} \to 0$, $\sigma_u^{2,\eta} \to 0$ as $\eta \to 0$. Since $\theta_1^{nv,\eta} > 0$ and $\frac{-\sigma_\epsilon^2}{\beta} + \eta > 0$ for small enough $\eta > 0$, we have that $\sigma_u^{2,\eta} > 0$ for small enough η.

Similarly, assuming $|\beta| \geq 2\frac{\sigma_\epsilon}{\sigma_s}$ holds, for the θ_1^- and θ_1^+ equilibria, we can define:

$$\theta_1^{-,\eta} = -\frac{1}{2}\beta - \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2 + \eta}}$$
$$\sigma_u^{2,-,\eta} = \theta_1^{2,-,\eta}\eta$$
$$\theta_1^{+,\eta} = -\frac{1}{2}\beta + \frac{1}{2}\sqrt{\beta^2 - 4\frac{\sigma_\epsilon^2}{\sigma_s^2 + \eta}}$$
$$\sigma_u^{2,+,\eta} = \theta_1^{2,+,\eta}\eta$$

As $\eta \to 0$, we have $(\theta_1^{-,\eta}, \theta_1^{+,\eta}) \to (\theta_1^-, \theta_1^+)$, $(\sigma_u^{2,-,\eta}, \sigma_u^{2,+,\eta}) \to (0,0)$. These parameters all solve the equilibrium quadratic, we can thus use them to define a sequence of ϵ-Proxy Equilibria.

For any $s \in S$, define $\sigma^\eta(.|s)$ as the strategy mapping implied by any of the perturbed $(\theta_1^\eta, \sigma_u^{2,\eta})$ parameters above, let $\sigma^{\eta,\delta}(.|s)$ be the Dirac measure at $x(s) = \theta_1^\eta \cdot s$ and $\sigma^*(.|s)$ be the Dirac measure at $x(s) = \theta_1^* \cdot s$ where θ_1^* is the parameter of any of the equilibria stated in the proposition. Since we have defined convergent sequence of parameters above, and the normal distribution converges to the degenerate Dirac measure as $\sigma_u^{2,\eta} \to 0$, we have that $\sigma^\eta(.|s)$ converges in distribution to $\sigma^*(.|s)$ as $(\theta_1^\eta, \sigma_u^{2,\eta}) \to (\theta_1^*, 0)$. Likewise, as $\theta_1^\eta \to \theta_1^*$ we have that $\sigma^{\eta,\delta}(.|s)$ converges in distribution to $\sigma^*(.|s)$.

Using the triangle inequality, we have that for any interval $I \subseteq X$:

$$|\sigma^{\eta,\delta}(I|s) - \sigma^{\eta}(I|s)| \leq |\sigma^{\eta,\delta}(I|s) - \sigma^{*}(I|s)| + |\sigma^{*}(I|s) - \sigma^{\eta}(I|s)|$$

For any $\epsilon > 0$, due to the fact any interval I in this application is a continuity set, the portmanteau lemma[13] and convergence in distribution of both the terms on the right hand side of the above equation means we can find an $\eta > 0$ such that $|\sigma^{\eta,\delta}(I|s) - \sigma^{\eta}(I|s)| < \epsilon$. Let $X(s;\sigma^*_\eta)$ be the subset of actions that are not a best response against the beliefs induced by the linear-normal strategy given by parameters $(\theta^\eta_1, \sigma^{2,\eta}_u)$. Since for any interval $\tilde{I} \subseteq X(s;\sigma^*_\eta)$, we have that $\sigma^{\eta,\delta}(\tilde{I}|s) = 0$, this means that $\sigma^{\eta}(\tilde{I}|s) < \epsilon$. Thus we have an ϵ-Proxy Equilibrium. This means that for any of the proposed Proxy Equilibria in the statement of the proposition we can find a sequence of ϵ-Proxy Equilibria that converge in distribution to that Proxy Equilibrium.

□

Proof of Proposition 3

We first show the following fact which is used several times in our proof.

Lemma A.1. *Let* $I_{[a_1,b_1]} = [a_1, b_1]$, $I_{[a_2,b_2]} = [a_2, b_2]$ *be intervals in* $[0, 1]$, *with* $a_1 > a_2$ *and* $b_1 > b_2$. *Then for any* $s_1 > s_2$ *we have that:*

$$1[s_1 \in [a_1, b_1]] \cdot 1[s_2 \in [a_2, b_2]] \geq 1[s_2 \in [a_1, b_1]] \cdot 1[s_1 \in [a_2, b_2]] \tag{29}$$

Moreover, this inequality holds strictly if $s_1 \in I_{[a_1,b_1]} \setminus I_{[a_2,b_2]}$ *or* $s_2 \in I_{[a_2,b_2]} \setminus I_{[a_1,b_1]}$.

These facts also hold for half-open intervals $I_{[a_1,b_1)} = [a_1, b_1)$ *and* $I_{[a_2,b_2)} = [a_2, b_2)$.

Proof. For the right hand side of the inequality to be equal to one requires $s_2 < s_1 \leq b_2$, $s_2 \geq a_1 > a_2$, $s_1 > s_2 \geq a_1$ and $s_1 \leq b_2 < b_1$ so $s_2 \in [a_2, b_2]$ and $s_1 \in [a_1, b_1]$ and the left hand side is also equal to one.

[13] See Billingsley (2012) Theorem 25.8 on page 358 for a proof.

The second part of the result holds by definition and the third part is clear by applying the arguments above again. □

We can then show an increasing best response property.

Lemma A.2. *Given a perceived distribution over outcomes induced by a full-support strategy σ, we have that $U(s, x = 1; q_\pi)$ is strictly increasing in $s \in [h, 1-h]$.*

Proof. For a given induced distribution p_π, we can use integration by parts to write the perceived utility of the DM as follows.

$$\begin{aligned} U(s, x = 1; q_\pi) &= \int_0^1 m(\tilde{s}) p_\pi(\tilde{s}|s^\bullet = s, x = 1) d\mu(\tilde{s}) \\ &= m(1) - \int_0^1 P_\pi(\tilde{s}|s^\bullet = s, x = 1) dM(\tilde{s}) \end{aligned}$$

Where $P_\pi(\tilde{s}|s^\bullet, x = 1)$ is the cdf of the induced distribution and M is the Lebesgue-Stieltjes measure satisfying $M((s_l, s_h]) = m(s_h) - m(s_l)$ for any $0 \leq s_l < s_h \leq 1$. Since $m(.)$ is strictly increasing and right continuous, this measure exists. Therefore, to show the result it is enough to show $P_\pi(s|s_1^\bullet, x = 1) \leq P_\pi(s|s_2^\bullet, x = 1)$ for any $1 - h \geq s_1^\bullet > s_2^\bullet \geq h$ and all s, with strict inequality for all s in some interval $[a, b] \subseteq [0, 1]$. Our assumptions about the conditional distribution of the proxies gives us the following sequence of claims.

By Lemma A.1, we have that for any $s_1 > s_2$ and $1 - h \geq s_1^\bullet > s_2^\bullet \geq h$.

$$\begin{aligned} &1[s_1 \in [s_1^\bullet - h, s_1^\bullet + h]] \cdot 1[s_2 \in [s_2^\bullet - h, s_2^\bullet + h]] \\ &\qquad \geq 1[s_2 \in [s_1^\bullet - h, s_1^\bullet + h]] \cdot 1[s_1 \in [s_2^\bullet - h, s_2^\bullet + h]] \end{aligned}$$

With strict inequality if $s_1 \in [s_1^\bullet - h, s_1^\bullet + h] \setminus [s_2^\bullet - h, s_2^\bullet + h]$ or $s_2 \in [s_2^\bullet - h, s_2^\bullet + h] \setminus [s_1^\bullet - h, s_1^\bullet + h]$, given any $1 - h \geq s_1^\bullet > s_2^\bullet > h$.

Multiplying both sides by $\frac{p(s_1)\sigma(x=1|s_1)}{\int_S p(s)\sigma(x=1|s)p(s_1^\bullet|s)d\mu(s)} \cdot \frac{p(s_2)\sigma(x=1|s_2)}{\int_S p(s)\sigma(x=1|s)p(s_2^\bullet|s)d\mu(s)}$, we can then write:

$$p_\pi(s_1|s_1^\bullet)p_\pi(s_2|s_2^\bullet) \geq p_\pi(s_1|s_2^\bullet)p_\pi(s_2|s_1^\bullet)$$

Integrating over both sides then gives us that $P_\pi(s|s_1^\bullet, x = 1) \leq P_\pi(s|s_2^\bullet, x = 1)$ for any $s_1^\bullet > s_2^\bullet$ and s. By the strict inequality case above, we have that $P_\pi(s|s_1^\bullet, x = 1) < P_\pi(s|s_2^\bullet, x = 1)$ for any $s \in [s_1^\bullet - h, s_1^\bullet + h] \cup [s_2^\bullet - h, s_2^\bullet + h]$. This completes the proof. $\square$

An outline of the proof is as follows. We first show why any Proxy Equilibrium with partial entry must have a cut-off structure, and give a condition that the cut-off must satisfy in terms of perceived expected utility. We then show how we can construct a sequence of ϵ-Proxy Equilibria that converge to this cut-off structure. Finally, we combine this with the endpoint assumptions to show that any equilibria with entry at some signal has this cut-off structure.

We consider partial entry Proxy Equilibrium in which $\sigma(x = 1|s) > 0$ at a strict subset of $s \in S^> \subset [0, 1]$. For such an equilibrium to exist, we must have a sequence of ϵ^l-Proxy Equilibria, $\{\sigma^*_{\epsilon^l}\}_{l=1}^\infty$ that converge in distribution to it. Given we are considering partial entry Proxy Equilibria, for large enough l we must have that the induced belief q_π^l in the ϵ^l-Proxy Equilibrium in the sequence is such that $U(s, x = 1; q_\pi^l) \leq 0$ for all $s \in [0, h]$, and that $x = 1$ is a best response to q_π^l for some s. By the fact perceived utility is increasing in $s \in [h, 1 - h]$ from Lemma A.2 there must be a cut-off $\bar{s}^{\epsilon^l} \in [h, 1]$ such that a best response is $x = 0$ for $s \in [0, \bar{s}^{\epsilon^l}]$ and $x = 1$ for $s \in (\bar{s}^{\epsilon^l}, 1]$.

By the definition ϵ^l-Proxy Equilibrium, we must have that $\sigma^*_{\epsilon^l}(x = 1|s) < \epsilon^l$ for all $s \in [0, \bar{s}^{\epsilon^l}]$ and $\sigma^*_{\epsilon^l}(x = 1|s) \geq 1 - \epsilon^l$ for all $s \in (\bar{s}^{\epsilon^l}, 1]$. The perceived utility of the DM at this cut-off $\bar{s}^{\epsilon^l}$ is then:

$$\int_{\bar{s}^{\epsilon^l} - h}^{\bar{s}^{\epsilon^l}} m(\tilde{s}) \frac{\sigma_\epsilon(x = 1|\tilde{s})p(\tilde{s})}{\int_{\bar{s}^{\epsilon^l} - h}^{\bar{s}^{\epsilon^l} + h} \sigma_\epsilon(x = 1|\hat{s})p(\hat{s})d\mu(\hat{s})} d\mu(\tilde{s}) +$$
$$\int_{\bar{s}^{\epsilon^l}}^{\bar{s}^{\epsilon^l} + h} m(\tilde{s}) \frac{\sigma_\epsilon(x = 1|\tilde{s})p(\tilde{s})}{\int_{\bar{s}^{\epsilon^l} - h}^{\bar{s}^{\epsilon^l} + h} \sigma_\epsilon(x = 1|\hat{s})p(\hat{s})d\mu(\hat{s})} d\mu(\tilde{s}) = 0$$

Thus as $l \to \infty$, if our sequence of ϵ-Proxy Equilibria converges it will converge to a Proxy Equilibrium with cut-off $\bar{s}^*$ such that $\sigma^*(x = 1|s) = 0$ for $s \in [0, \bar{s}^*)$ and $\sigma^*(x = 1|s) = 1$ for $s \in [\bar{s}^*, 1]$. The perceived utility at the cut-off $\bar{s}^*$ will then be:

$$\int_{\bar{s}}^{\bar{s}+h} m(\tilde{s}) \frac{p(\tilde{s})}{\int_{\bar{s}}^{\bar{s}+h} p(\hat{s}) d\mu(\hat{s})} d\mu(\tilde{s}) = 0 \tag{30}$$

We then construct strategies that can form a sequence of ϵ-Proxy Equilibria that converge to a partial entry Proxy Equilibria. These strategies have a cut-off form where $\sigma_\xi(x|s) = \xi \in (0, \frac{1}{2})$ for $s \in [0, \bar{s})$ and $\sigma_\xi(x|s) = 1 - \xi \in (\frac{1}{2}, 1)$ for $s \in [\bar{s}, 1]$, with $\bar{s} \in [0, 1]$ as the cut-off. We can then define the following conditional density over $s \in [h, 1-h]$ given $s^\bullet = \bar{s}$.

$$g_\xi(s|s^\bullet = \bar{s}) = \frac{(1-\xi)1[\tilde{s} \in [\bar{s}, \bar{s}+h]] + \xi 1[\tilde{s} \in [\bar{s}-h, \bar{s})]}{(1-\xi)\int_{\bar{s}}^{\bar{s}+h} p(\hat{s}) d\mu(\hat{s}) + \xi \int_{\bar{s}-h}^{\bar{s}} p(\hat{s}) d\mu(\hat{s})} p(\tilde{s}) \tag{31}$$

For any cut-off $\bar{s} \in [h, 1-h]$ and k, we can choose:

$$\xi(\bar{s}, k) = \frac{k \int_{\bar{s}}^{\bar{s}+h} p(\tilde{s}) d\mu(\tilde{s})}{k \int_{\bar{s}}^{\bar{s}+h} p(\tilde{s}) d\mu(\tilde{s}) + (1-k) \int_{\bar{s}-h}^{\bar{s}} p(\tilde{s}) d\mu(\tilde{s})} \tag{32}$$

Which is arbitrarily small for small enough $1 > k > 0$. This ensures that:

$$\int_{\bar{s}-h}^{\bar{s}} g_\xi(\tilde{s}|s^\bullet = \bar{s}) d\mu(\tilde{s}) = \frac{\xi(\bar{s}, k) \int_{\bar{s}-h}^{\bar{s}} p(\tilde{s}) d\mu(\tilde{s})}{(1 - \xi(\bar{s}, k)) \int_{\bar{s}}^{\bar{s}+h} p(\tilde{s}) d\mu(\tilde{s}) + \xi(\bar{s}, k) \int_{\bar{s}-h}^{\bar{s}} p(\tilde{s}) d\mu(\tilde{s})} = k$$

We can then write the perceived utility at $\bar{s} \in [h, 1-h]$ against the beliefs induced by strategy $\sigma_{\xi(\bar{s},k)}$ with cut-off $\bar{s} \in [h, 1-h]$ in the following way:

$$\int_0^1 m(\tilde{s}) g_{\xi(\bar{s},k)}(\tilde{s}|s^\bullet = \bar{s}) d\mu(\tilde{s}) = (1-k)\overline{U}(\bar{s}, x = 1; \bar{s}) + k\underline{U}(\bar{s}, x = 1; \bar{s}) \tag{33}$$

Which is a linear combination of the terms:

$$\overline{U}(\bar{s}, x=1; \bar{s}) = \int_{\bar{s}}^{\bar{s}+h} m(\tilde{s}) \frac{p(\tilde{s})}{\int_{\bar{s}}^{\bar{s}+h} p(\hat{s}) d\mu(\hat{s})} d\mu(\tilde{s}) \tag{34}$$

$$\underline{U}(\bar{s}, x=1; \bar{s}) = \int_{\bar{s}-h}^{\bar{s}} m(\tilde{s}) \frac{p(\tilde{s})}{\int_{\bar{s}-h}^{\bar{s}} p(\hat{s}) d\mu(\hat{s})} d\mu(\tilde{s}) \tag{35}$$

We show that (33) is strictly increasing in $\bar{s} \in [h, 1-h]$ by showing (34) and (35) are strictly increasing in $\bar{s}$.

Lemma A.3. *The expressions $\overline{U}(\bar{s}, x = 1; \bar{s})$ and $\underline{U}(\bar{s}, x = 1; \bar{s})$ are strictly increasing for all $\bar{s} \in [h, 1-h]$.*

Proof. We define densities $\overline{g}(s; \bar{s}) = \frac{1_{[s \in [\bar{s}, \bar{s}+h]]} p(s)}{\int_{\bar{s}}^{\bar{s}+h} p(\tilde{s}) d\mu(\tilde{s})}$ and $\underline{g}(s; \bar{s}) = \frac{1_{[s \in [\bar{s}-h, \bar{s})]} p(s)}{\int_{\bar{s}-h}^{\bar{s}} p(\hat{s}) d\mu(\hat{s})}$. We show that for $s_1 > s_2$ and $1 - h \geq \bar{s}_1 > \bar{s}_2 \geq h$ we have the following inequalities:

$$\overline{g}(s_1; \bar{s}_1)\overline{g}(s_2; \bar{s}_2) \geq \overline{g}(s_1; \bar{s}_2)\overline{g}(s_2; \bar{s}_1) \tag{36}$$

$$\underline{g}(s_1; \bar{s}_1)\underline{g}(s_2; \bar{s}_2) \geq \underline{g}(s_1; \bar{s}_2)\underline{g}(s_2; \bar{s}_1) \tag{37}$$

Where for any $h \leq \bar{s}_2 < \bar{s}_1 \leq 1 - h$, we can find intervals $I_1, I_2 \subset [0, 1]$ such that if for $s_1 > s_2$, $s_1 \in I_1$, $s_2 \in I_2$ the inequality holds strictly. The above inequalities reduce to:

$$1_{[s_1 \in [\hat{s}_1, \hat{s}_1+h], s_2 \in [\bar{s}_2, \bar{s}_2+h]]} \geq 1_{[s_1 \in [\bar{s}_2, \bar{s}_2+h], s_2 \in [\bar{s}_1, \bar{s}_1+h]]}$$

$$1_{[s_1 \in [\hat{s}_1-h, \hat{s}_1), s_2 \in [\bar{s}_2-h, \bar{s}_2)]} \geq 1_{[s_1 \in [\bar{s}_2-h, \bar{s}_2), s_2 \in [\bar{s}_1-h, \bar{s}_1)]}$$

Which proves our inequality result by Lemma A.1. We can then use the same steps as in the proof of Lemma A.2 to prove the result. $\square$

With these results in hand, we can then both show existence of and characterize the equilibria for this application.

Proposition 3

Proof. At any ϵ-Proxy Equilibrium, the perceived utility of the DM is increasing strictly for $s \in [h, 1-h]$ by Lemma A.2. The structure of the window form of proxy mapping means that the beliefs of the DM are identical on $s \in [0, h]$. If the DM is mixing $\sigma_\epsilon(x = 1|s) > \epsilon$ on $s \in [0, h]$, then due to increasing expected payoff on $s \in [h, 1-h]$, they must be playing $\sigma_\epsilon(x = 1|s) \geq 1 - \epsilon$ on $s \in (h, 1]$. As $\epsilon \to 0$ and $\sigma_\epsilon \to \sigma$ their perceived utility at any $s \in [0, h]$ given potential equilibrium σ is:

$$\int_0^h m(s) \frac{\sigma(x = 1|s)p(s)}{\int_0^1 \sigma(x = 1|\tilde{s})p(\tilde{s})d\mu(\tilde{s})} d\mu(s) + \int_h^{2h} m(s) \frac{p(s)}{\int_0^1 \sigma(x = 1|\tilde{s})p(\tilde{s})d\mu(\tilde{s})} d\mu(s)$$

By the endpoint assumption and the fact that $m(.)$ is increasing this is strictly negative. Thus for small enough ϵ at any ϵ-Proxy Equilibrium we must have $\sigma(x = 1|s) < \epsilon$ for $s \in [0, h]$.

From this argument and Lemma A.2, any ϵ-Proxy Equilibria in which $\sigma(x = 1|s) \geq \epsilon$ for some s must have some cut-off $\bar{s} \in (h, 1]$ such that $\sigma(x = 1|s) \geq \epsilon$ only if $s > \bar{s}$. The endpoint assumption $\int_{1-h}^1 m(s)p(s)d\mu(s) > 0$ ensures that any potential equilibrium strategy with cut-off $\bar{s} \geq 1 - h$ will have $\sigma(x = 1|s) = 1$ as a best response for all $s \in [1 - h, 1]$, and thus $\sigma(x = 0|s) < \epsilon$ for $s \in [1 - h, 1]$ in any ϵ-Proxy Equilibrium.

We construct the following cut-off ϵ-Proxy Equilibrium strategy. For any cut-off $\bar{s} \in (h, 1]$, $\epsilon > 0$ and $k_\epsilon \in (0, 1)$, define $\xi(\bar{s}, k_\epsilon)$ as in (32). Let $\sigma_\epsilon(x = 1|s) = \xi(\bar{s}, k_\epsilon)$ on $s \in [0, \bar{s}]$ and $\sigma_\epsilon(x = 1|s) = 1 - \xi(\bar{s}, k_\epsilon)$ on $s \in (\bar{s}, 1]$. We choose $k_\epsilon > 0$ small enough such that $\epsilon > \sup_{\bar{s} \in [h, 1-h]} \xi(\bar{s}, k_\epsilon)$. Then if we can find a $\bar{s}^*$ such that a best response to the beliefs induced by σ_ϵ is $\sigma(x = 1|s) = 0$ on $s \in [0, \bar{s}^*]$ and $\sigma(x = 1|s) = 1$ on $s \in (\bar{s}^*, 1]$ we have an ϵ- Proxy Equilibrium.

We have shown that the constructed strategy induces the beliefs at the cut-off $\bar{s}$ according to equation (33). We have also shown in Lemma A.3 that this expression is strictly increasing in the cut-off $\bar{s} \in [h, 1 - h]$. Moreover, we have that as $\epsilon \to 0$, $k_\epsilon \to 0$, so this expression converges to that in equation (30). By the endpoint assumptions (30) is strictly negative at $\bar{s} = h$ and strictly positive at $\bar{s} = 1 - h$. Thus we can find a small enough $\epsilon > 0$ and hence $k_\epsilon > 0$ such that $\int_0^1 m(\tilde{s}) g_{\xi(\bar{s},k_\epsilon)}(\tilde{s}|s^\bullet = h) d\mu(\tilde{s}) < 0$ and $\int_0^1 m(\tilde{s}) g_{\xi(\bar{s},k_\epsilon)}(\tilde{s}|s^\bullet = 1-h) d\mu(\tilde{s}) > 0$. Since $\int_0^1 m(\tilde{s}) g_{\xi(\bar{s},k_\epsilon)}(\tilde{s}|s^\bullet = \bar{s}) d\mu(\tilde{s})$ is continuous and increasing in $\bar{s} \in [h, 1-h]$, we can find a $\bar{s} = \bar{s}^*$ at which it is equal to zero by the intermediate value theorem. This $\bar{s}^*$ then gives us our ϵ-Proxy Equilibrium cut-off as stated above. As $\epsilon \to 0$, we can find a sequence of ϵ-Proxy Equilibria of this form that converge to that in the statement of the proposition.

For the final part of the proposition, we can always find a sequence of ϵ-Proxy Equilibria that converges to a Proxy Equilibrium with $x = 0$ for all $s \in [0, 1]$. For example, with small enough $\epsilon > 0$ we can have an ϵ-Proxy Equilibrium such that the DM plays $x = 1$ with probability $\epsilon > 0$ on $[0, \alpha)$ and probability ϵ^2 on $[\alpha, 1]$. This induces beliefs to which $x = 0$ is a best response for all s. □

Proof of Proposition 4

Proof. As shown in Proposition 3, there is a cut-off $\bar{s}^* \in [h, 1-h]$ that characterizes the cut-off equilibrium where $\sigma(x = 1|s) = 0$ for $s \in [0, \bar{s}^*]$ and $\sigma(x = 1|s) = 1$ for $s \in (\bar{s}^*, 1]$. We have shown the cut-off must solve the following equation.

$$\overline{U}(\bar{s}^*, x = 1; \bar{s}^*) = \int_{\bar{s}^*}^{\bar{s}^*+h} m(\tilde{s}) \frac{p(\tilde{s})}{\int_{\bar{s}^*}^{\bar{s}^*+h} p(\hat{s}) d\mu(\hat{s})} d\mu(\tilde{s}) = 0$$

For the first part, consider that the statement is not true and we have that $\bar{s}^* \geq \alpha$. Then we have $\overline{U}(\bar{s}^*, x = 1; \bar{s}^*) > 0$ as all the probability weight in the distribution is in $s \in [\alpha, 1]$, a contradiction. Thus the cut-off must be such that $\bar{s} < \alpha$.

For the second part, the end point assumptions being satisfied mean we are comparing the cut-off equilibrium at h_1 with the cut-off equilibrium at h_2. Consider the perceived utility at the cut-off under the equilibrium with noise h_1.

$$\overline{U}(\bar{s}(h_1), x = 1; h_1) = \int_{\bar{s}(h_1)}^{\bar{s}(h_1)+h_2} m(\tilde{s}) \frac{p(\tilde{s})}{\int_{\bar{s}}^{\bar{s}(h_1)+h} p(\hat{s}) d\mu(\hat{s})} d\mu(\tilde{s})$$

As this is an equilibrium cut-off, we must have that:

$$\int_{\alpha}^{\bar{s}(h_1)+h_1} m(\tilde{s})p(\tilde{s})d\mu(\tilde{s}) + \int_{\bar{s}(h_1)}^{\alpha} m(\tilde{s})p(\tilde{s})d\mu(\tilde{s}) = 0$$

If $\bar{s}(h_1)$ is fixed, then as h_1 increases to h_2, the first part of this expression that has weight on the positive part of the function $m(.)$ increases while the second part stays fixed. Thus the perceived utility at cut-off $\bar{s}(h_1)$ when the perceived distribution is induced by a strategy with cut-off $\bar{s}(h_1)$, must become positive at noise parameter $h_2 > h_1$. We have that $\overline{U}(\bar{s}(h_2), x = 1; h_1) > 0$, $\overline{U}(h_1, x = 1; h_1) < 0$ by the endpoint assumptions and $\overline{U}(\bar{s}, x = 1; h_1)$ is continuous in $\bar{s} \in [h_1, 1-h_1]$. Therefore by the intermediate value theorem we can find a new cut-off $h_1 < \bar{s}(h_1) < \bar{s}(h_2)$ that characterizes the positive entry equilibrium under noise parameter h_1. □

Proof of Proposition 5

Proof. We construct a sequence of full support equilibria that converges to σ^*. Let σ^*_ϵ be such that for any $(\hat{x}, \hat{s})$ if $\hat{x} \in supp\{\sigma^*(.|\hat{s})\}$ then $\sigma^*_\epsilon(\hat{x}|\hat{s}) = \sigma^*(\hat{x}|\hat{s}) > \epsilon$ and if $\hat{x} \notin supp\{\sigma^*(.|\hat{s})\}$ then $\sigma^*_\epsilon(\hat{x}|\hat{s}) = \epsilon$.

Let $XS^{ns}(\sigma^*) \equiv \{(x, s) \in X \times S : x \notin supp\{\sigma^*(.|s)\}\}$. By the assumption of the proposition, we can construct a function $k : XS^{ns}(\sigma^*) \to Y$ by selecting $\hat{y} \in Y$ that satisfies the following condition for every (x^{ns}, s) such that $x^{ns} \notin supp\{\sigma^*(.|\hat{s})\}$, given any $x^s \in supp\{\sigma^*(.|\hat{s})\}$.

$$\sum_{y \in Y} u(y, x^s, s)p(y|x^s, s) > u(\hat{y}, x^{ns}, s) \tag{38}$$

If the inequality holds for multiple potential y then we select one arbitrarily.

Denote $K(y) \equiv \{(x^\bullet, s^\bullet) \in X^\bullet \times S^\bullet : y = k(x, s) \text{ and } (x^\bullet, s^\bullet) = (x, s)\}$ as the set of all actions and signals mapped to y by k projected onto the proxy variables for actions and signals. We construct a proxy mapping π_c that randomizes uniformly over all elements of $K(y)$. That is:

$$\pi_c(y^\bullet, x^\bullet, s^\bullet|y,x,s) = \begin{cases} \frac{1}{|K(y)|} \text{ if } y^\bullet = y, \;\; K(y) \neq \emptyset \text{ and } (x^\bullet, s^\bullet) \in K(y) \\ 0 \text{ if } y^\bullet = y, \;\; K(y) \neq \emptyset \text{ and } (x^\bullet, s^\bullet) \notin K(y) \\ \pi_\delta(y^\bullet, x^\bullet, s^\bullet|y,x,s) \text{ otherwise} \end{cases}$$

From this we can then form another proxy mapping π_η that draws the perfect measurement mapping with probability $1 - \frac{\eta}{2}$ and mapping π_c with probability $\frac{\eta}{2}$. Denote by

$$L(\sigma^*) \equiv \{(x^\bullet, s^\bullet) \in X^\bullet \times S^\bullet : x \in supp\{\sigma^*(.|s)\} \text{ and } (x^\bullet, s^\bullet) = (x,s)\}$$

the set of actions and signal combinations such that the action is in the support of the signal under strategy σ^*, projected onto the proxy variable space. We can then write:

$$\pi_\eta(v^\bullet|v) = (1 - \frac{\eta}{2})\pi_\delta(v^\bullet|v) + \frac{\eta}{2}\pi_c(v^\bullet|v)$$

This mapping is clearly strongly η-close to perfect. For any $(x^\bullet, s^\bullet) \in L(\sigma^*)$ we have perfect measurement under π_η and the conditional distribution over y given $(x^\bullet, s^\bullet)$ is equal to the rational expectations benchmark independently of the strategy; $q_{\pi_\eta}(y|x^\bullet, s^\bullet) = p(y|x^\bullet, s^\bullet)$. For any $(x^\bullet, s^\bullet) \notin L(\sigma^*)$, we can write the conditional distribution as:

$$q_{\pi_\eta}(\hat{y} = k(x^\bullet, s^\bullet)|x^\bullet, s^\bullet) = \\ \frac{\epsilon(1-\frac{\eta}{2})p(s^\bullet)p(\hat{y}|x^\bullet, s^\bullet) + \frac{\eta}{2|K(y)|}\sum_{x,s} p(s)\sigma^*_\epsilon(x|s)p(\hat{y}|x,s)}{\epsilon(1-\frac{\eta}{2})p(s^\bullet) + \frac{\eta}{2|K(y)|}\sum_{x,s} p(s)\sigma^*_\epsilon(x|s)p(\hat{y}|x,s)}$$

For small enough $\epsilon > 0$, this is arbitrarily close to one for any $\eta > 0$ due to our assumption in the statement that that $p(\hat{y}|\tilde{x},\tilde{s})\sigma^*(\tilde{x}|\tilde{s})p(\tilde{s}) > 0$ for some $(\tilde{x},\tilde{s})$. Therefore, we can push the perceived expected utility of any action taken at a signal that is not in the support of the proposed strategy; $(x^\bullet, s^\bullet) \notin L(\sigma^*)$ arbitrarily close to the right hand side of the inequality (38). Any action-signal combination in the support of σ^*; $(x^\bullet, s^\bullet) \in L(\sigma^*)$, has perceived expected utility equal to the rational expectations benchmark. Therefore we have that the inequality in (38) allows us to support σ^*_ϵ as an ϵ-Proxy Equilibrium for $\epsilon > 0$ close enough to zero. Since $\sigma^*_\epsilon \to \sigma^*$ as $\epsilon \to 0$, we have our result. $\square$

Proof of Proposition 6

First we define some notation. Denote the measure over the proxies induced by the perfect measurement mapping as P_δ and the measures over any subset of the proxy variables $N^\bullet \subset M^\bullet$ induced by the proxy mapping and the perfect measurement mapping as P^N_π and P^N_δ respectively. These are related to the measure over all the variables.

$$P^N_\pi(W^\bullet_N) = P_\pi(W^\bullet_N \times \{V^\bullet_{-N}\}) = \int_V \pi(W^\bullet_N \times \{V^\bullet_{-N}\}|v)p(v)d\mu(v) \tag{39}$$

Similarly, $P^N_\delta(W^\bullet_N) = P_\delta(W^\bullet_N \times \{V^\bullet_{-N}\})$. Note also that $P^N_\delta(W^\bullet_N = W_N) = P(W_N)$ for all subsets $N \subseteq M$, so that with perfect measurement the distribution over the proxies is identical to that of the true variables. We first prove a sequence of lemmas.

Lemma A.4. *For any subset of the true variables N and their respective proxy variables $N^\bullet$, if the proxy mapping is η-close to perfect then we have that:*

$$TV(P^N_\delta, P^N_\pi) < \eta \tag{40}$$

Proof. Denote the sub σ-algebra over the variables $N^\bullet$ as $\mathcal{W}^\bullet_N$, then we can show that the distance for the marginal distribution over the subset of variables N is smaller than the distance for all the variables.

$$
\begin{aligned}
TV(P_\delta^N, P_\pi^N) &= \sup_{A \in \mathcal{W}_N^\bullet} |P_\delta^N(A) - P_\pi^N(A)| \\
&= \sup_{A \in \mathcal{W}_N^\bullet \times \{V_{-N}^\bullet\}} |P_\delta(A) - P_\pi(A)| \\
&\leq \sup_{A \in \mathcal{W}^\bullet} |P_\delta(A) - P_\pi(A)| = TV(P_\delta, P_\pi) < \eta
\end{aligned}
$$

Where the last line follows as $\mathcal{W}_N^\bullet \times \{V_{-N}^\bullet\} \subset W^\bullet$. This completes the proof. □

Lemma A.5. *Given a distribution over true variables P, if the sequence $\{\pi_n\}_{n=1}^\infty$ induces a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^\infty$ that converges to the true distribution P in the total variation distance, then there exists a subsequence $\{\pi_{n_k}\}_{k=1}^\infty$ that for every subset $N \subseteq M$ induces a subsequence of densities over the proxy variables $\{p_{\pi_{n_k}}^N\}_{k=1}^\infty$ that converges pointwise almost everywhere to the true density, $p_{\pi_{n_k}}^N(v_N^\bullet) \to p_\delta^N(v_N^\bullet)$ for almost all $v_N^\bullet \in V_N^\bullet$.*

Proof. An alternative expression for the total variation distance, given that the distributions Q_1 and Q_2 over some measure space $(\Omega, \mathcal{A})$ admit densities q_1 and q_2 with respect to some dominating measure μ, is as follows[14].

$$
TV(Q_1, Q_2) = \frac{1}{2}\int_\Omega |q_1 - q_2| d\mu = \frac{1}{2}\|Q_1 - Q_2\|_1 \tag{41}
$$

Therefore, convergence in the total variation distance is equivalent to convergence in the L1 norm. By Lemma A.4 we have that the sequence $\{\pi_n\}_{n=1}^\infty$ induces a sequence of distributions over the proxy variables in N $\{P_{\pi_n}^N\}_{n=1}^\infty$ which converges to the true distribution $P_{\pi_\delta}^N$ in the total variation distance, and thus the L1 norm.

As each distribution in this sequence as well as the limit is assumed to admit a density function, we have that $\|p_{\pi_n}^N - p_\delta^N\|_1 \to 0$. Thus, by Theorem 13.6 (pp 465) of Charalambos and Aliprantis (2006) we have that there is a subsequence $\{p_{\pi_{n_k}}^N\}_{k=1}^\infty$ which converges pointwise to the true density p_δ^N almost everywhere. □

We then extend the full support and boundedsness assumptions in the main body to account for sequences.

[14] See Tsybakov (2008) page 84 in Chapter 2.4 for a proof of this fact.

Definition 8. *The sequence* $\{\pi_n\}_{n=1}^{\infty}$ *and distribution over the true variables* P*, are said to satisfy the* ***sequential assumptions*** *with respect to DAG* R *if*

1. *Each* P_{π_n} *admits a density function* p_{π_n} *with respect to some* σ*-finite measure.*

2. *The distribution over the true variables admits a density* $p(v)$*, and that for every variable* i *this density gives non-zero probability to every realisation of the parent variables* $p(v_{R(i)}) > 0$*.*

3. *Each density in the sequence* $\{p_{\pi_n}\}_{n=1}^{\infty}$ *is such that* $p_{\pi_n}(v^{\bullet}_{R(i)}) > 0$ *for every variable* $i \in N^{\bullet}$ *and every realisation* $v^{\bullet}_{R(i)} \in V^{\bullet}_{R(i)}$*.*

4. *There exists some constant* $G > 0$ *such that for every variable, each conditional density in the sequence induced by* $\{\pi_n\}_{n=1}^{\infty}$ *and* P *is such that* $p_{\pi_n}(v^{\bullet}_i|v^{\bullet}_{R(i)}) \leq G$*.*

We then use these assumptions and Lemma A.5 to extend this to the induced conditional distributions over proxy variables that form the agent's beliefs under Proxy Equilibrium.

Lemma A.6. *If the sequence* $\{\pi_n\}_{n=1}^{\infty}$ *and distribution over the true variables* P*, satisfy the* ***sequential assumptions*** *with respect to DAG* R *and if this sequence induces a sequence of distributions over the proxy variables* $\{P_{\pi_n}\}_{n=1}^{\infty}$ *that converges to the true distribution* P *in the total variation distance, then there exists a subsequence* $\{\pi_{n_k}\}_{k=1}^{\infty}$ *such that the induced subsequence of perceived joint densities over proxy variables* $\{q_{\pi_{n_k}}\}_{k=1}^{\infty}$ *converges pointwise to the true joint density* q_{δ} *almost everywhere.*

Proof. The perceived joint distribution over proxies given $p(v)$ and mapping π_k can be written as:

$$q_{\pi_k}(y^{\bullet}, x^{\bullet}, s^{\bullet}) = \prod_{i=1}^{m} p_{\pi^k}(v^{\bullet}_i|v^{\bullet}_{R(i)}) \tag{42}$$

This is the product of expressions of the form $p_{\pi^k}(v^{\bullet}_i|v^{\bullet}_{R(i)})$ for all $i \in M$. By Lemma A.5 and the full support assumption, both the numerator and denominator of $p_{\pi^k}(v^{\bullet}_i|v^{\bullet}_{R(i)})$ converge pointwise almost everywhere to the true marginal distribution over the variables as the subsequence converges, thus as $\pi_{n_k} \to \pi_\delta$ we have that $p_{\pi^k}(v_i^\bullet | v_{R(i)}^\bullet)$ converges pointwise almost everywhere. The algebra of limits then ensures the product of these terms (42) converges to q_δ pointwise almost everywhere. $\square$

Since we only have almost everywhere convergence of our sequence of perceived densities, it could be the cases that there are measure zero realisations at which we do not have convergence. However, the use of perceived ex-ante indirect utility as the object of convergence circumvents this problem. We extend our sequence result to convergence of the perceived ex-ante expected indirect utility.

Lemma A.7. *If the sequence $\{\pi_n\}_{n=1}^\infty$ and distribution over the true variables P, satisfy the* ***sequential assumptions*** *with respect to DAG R, they induce a sequence of distributions over the proxy variables $\{P_{\pi_n}\}_{n=1}^\infty$ that converges to the true distribution P, then there exists a subsequence $\{\pi_{n_k}\}_{k=1}^\infty$ such that the induced subsequence of perceived ex-ante expected indirect utilities $\{\mathcal{V}(p, \pi_{n_k})\}_{k=1}^\infty$ converges to the true ex-ante expected indirect utility $\mathcal{V}(p, \pi_\delta)$.*

Proof. By Lemma's A.6 and the sequential assumptions, there exists a subsequence of perceived densities over proxy variables $\{q_{\pi_{n_k}}\}_{k=1}^\infty$ such that $\left|q_{\pi_{n_k}}\right| \leq G^m \equiv H$ for all n_k, and which converges almost everywhere to q_δ. Then the product of the utility function and this sequence $\{u \cdot q_{\pi_{n_k}}\}_{k=1}^\infty$ converges almost everywhere to $u \cdot q_\delta$ and is bounded above by some integrable function $\left|u \cdot q_{\pi_{n_k}}\right| \leq H \cdot h$ for all n_k.

We can then apply the dominated convergence theorem to prove the result. $\square$

This result can then be used to show the following continuity property for the perceived ex-ante expected indirect utility, under the full support assumption.

Proof of Proposition 7

Proof. Let $\epsilon > 0$. We will show any ϵ-Proxy Equilibrium is an ϵ-perturbed personal equilibrium from Spiegler (2016). Take an ϵ-Proxy Equilibrium σ_ϵ^*. Remember that $X(s; \sigma_\epsilon^*)$ is the set of actions that are not a best response to the beliefs induced by σ_ϵ^* at s. We have that $\sigma_\epsilon^*(x|s) < \epsilon$ for any $x \in X(s; \sigma_\epsilon^*)$, since we are in the finite variable case and therefore any x is an interval. Therefore, if $\sigma(x|s) > \epsilon$, x must be a best response at s to the beliefs induced by σ_ϵ^*, matching the definition of an ϵ-perturbed personal equilibrium. Showing that under the assumptions of the proposition, the beliefs induced in the ϵ-Proxy Equilibrium are the same as those in the proposed BNE construction then completes the proof, as the limit of any sequence of ϵ-Proxy Equilibria is also the limit of any sequence of ϵ-perturbed personal equilibria.

As stated, the above assumptions mean we can write the joint density of the true and proxy variables as:

$$f(v^\bullet, v) = \prod_{i=1}^{m} p(v_i|v_{R(i)}) \prod_{j=1}^{m} \pi_j(v_i^\bullet|v_{R^{aux}(i)})$$

This density clearly can be factorised by objective DAG R^*. If we factorise this according to subjective DAG R^s we get the following joint density:

$$f_{R^s}(v^\bullet, v) = p(s)p(x|s) \prod_{k \in \mathbf{Y}} p(v_k) \prod_{i \in Y^\bullet} f(v_i^\bullet|w_{R^s(i)})$$

Since the true outcome variables are irrelevant for utility and independent from all other variables according to the subjective factorisation, it is sufficient for our purposes to take the joint distribution over the proxy outcome variables and the action and signal variables, and then condition on the action and signal variables.

$$q_{\hat{R}}(y^\bullet, x, s) = p(s)p(x|s) \prod_{i \in Y^\bullet} f(v_i^\bullet|w_{R^s(i)})$$

$$q_{\hat{R}}(y^\bullet|x, s) = \frac{q_{\hat{R}}(y^\bullet, x, s)}{\sum_{\tilde{y}^\bullet \in Y^\bullet} q_{\hat{R}}(\tilde{y}^\bullet, x, s)}$$

Using this gives us an expression for perceived expected utility.

$$\sum_{y^\bullet \in Y^\bullet} v(y^\bullet, x, s) q_{\hat{R}}(y^\bullet|x, s)$$

Under the assumptions given, the expression for the conditional distribution of the imperfectly measured proxies is identical to the conditional distribution we would obtain over these variables under the Proxy Equilibrium. It also meets all the requirements for BNE, and as the equilibrium definitions are otherwise identical any Proxy Equilibrium is also a BNE. □

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