Computational optimal transport

Lecture 1: Semidiscrete optimal transport and applications to non-imaging optics

Quentin Mérigot

Joint works with J. Kitagawa, P. Machado, J. Meyron, B. Thibert

CEA/EDF/Inria School on Computational Optimal transport, Paris, 2019

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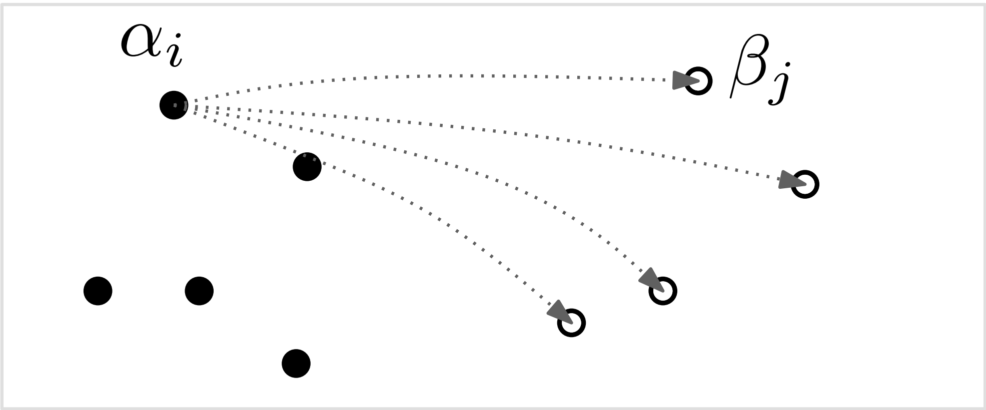

Discrete source and target

linear programming

Hungarian algorithm

Sinkhorn/IPFP

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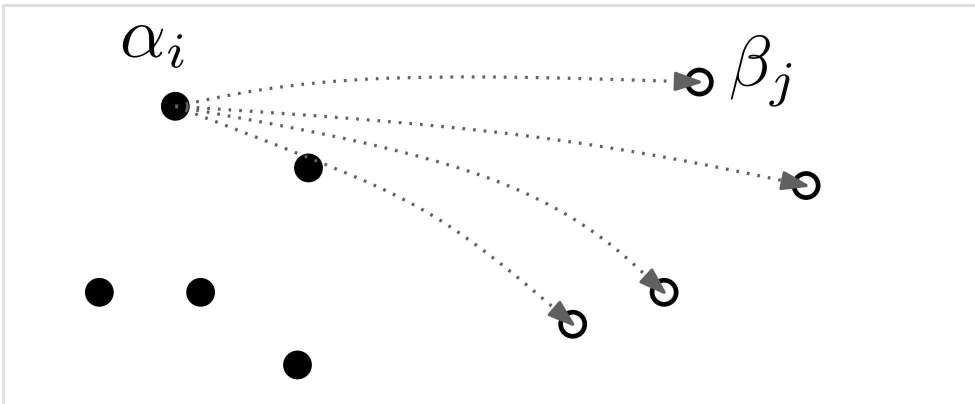

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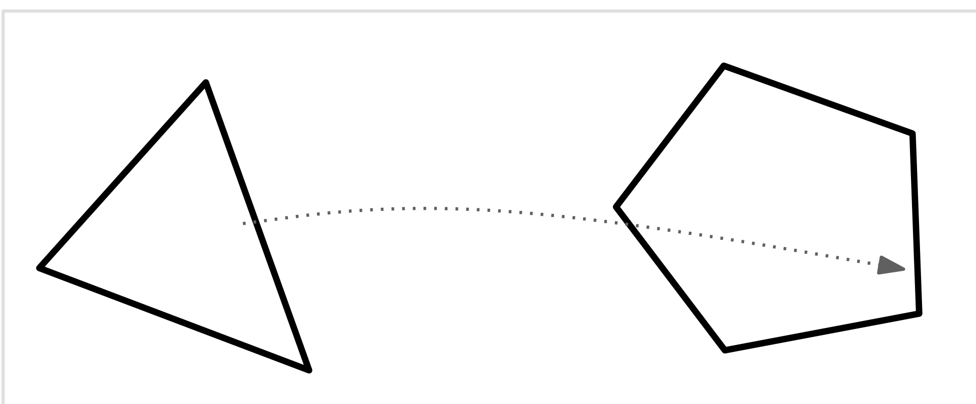

Source and target with density:

Benamou-Brenier formulation

finite-differences for Monge-Ampère

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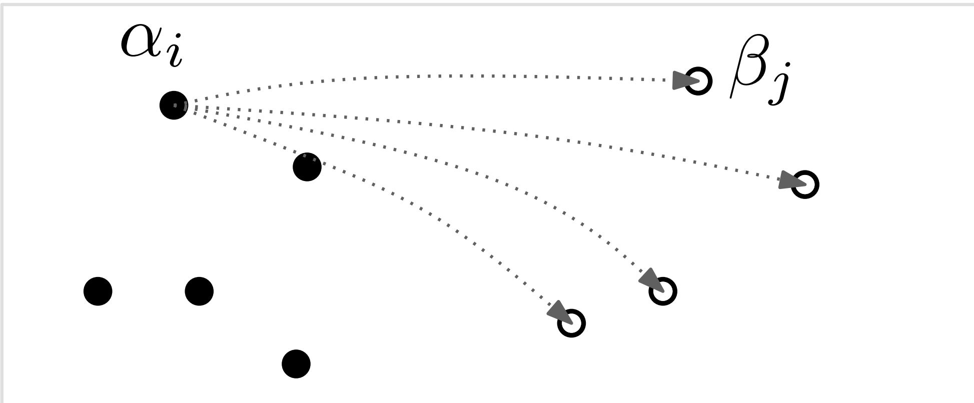

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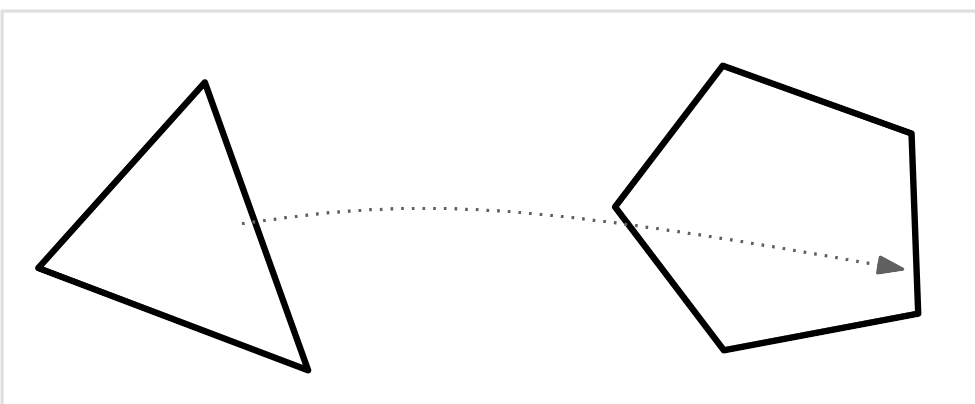

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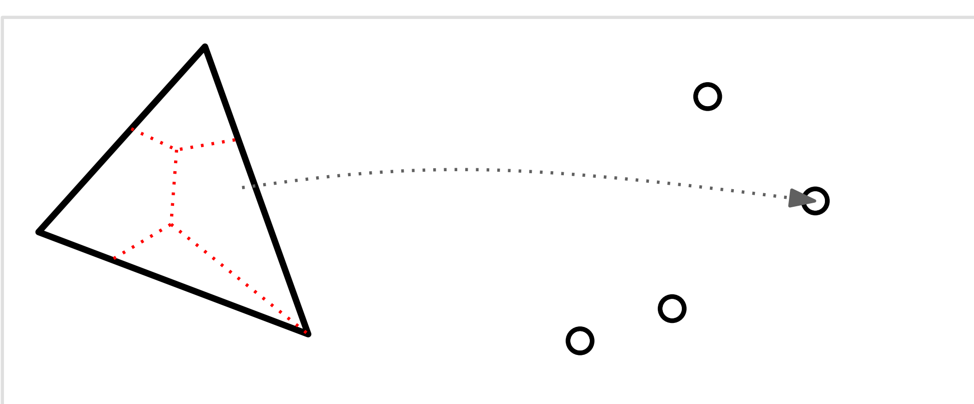
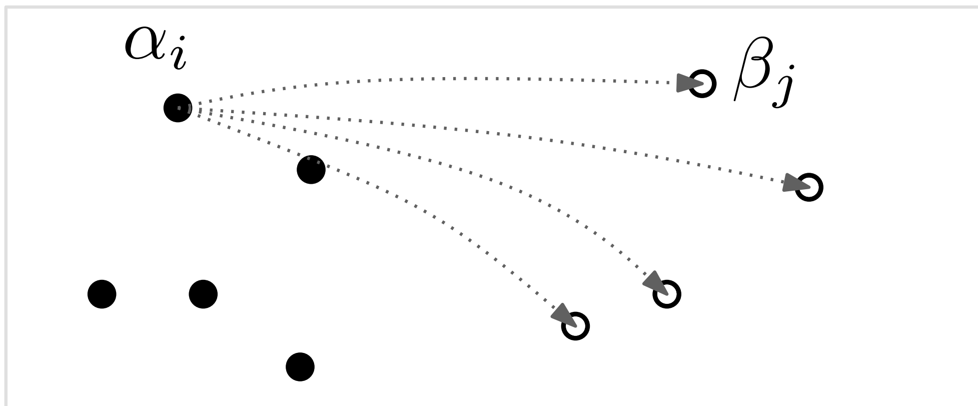


Source with density, discrete target:

Minkowski, Alexandrov, etc.

Computational optimal transport

Flexibility for the cost function **but** computationally expensive



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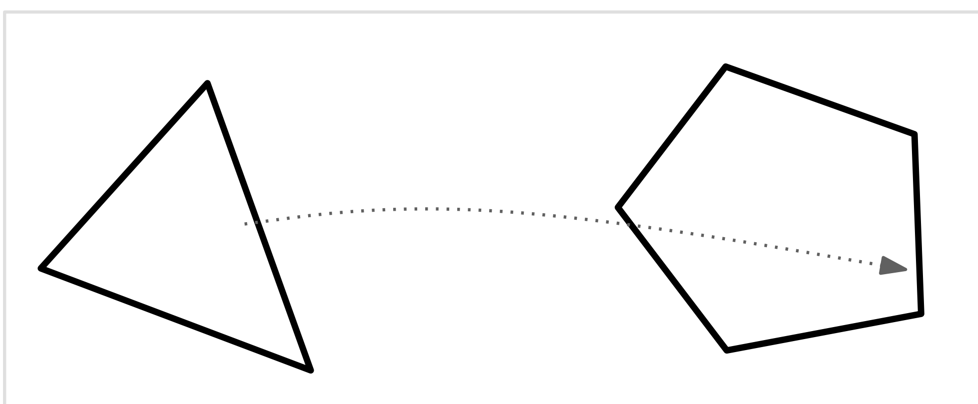

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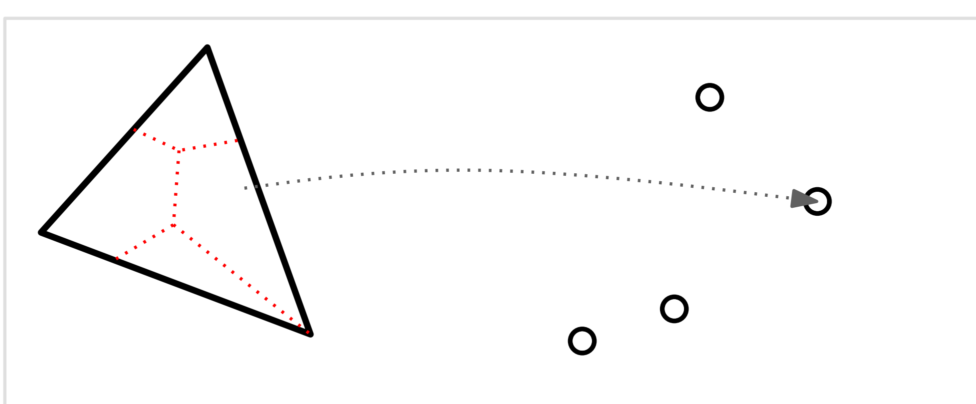

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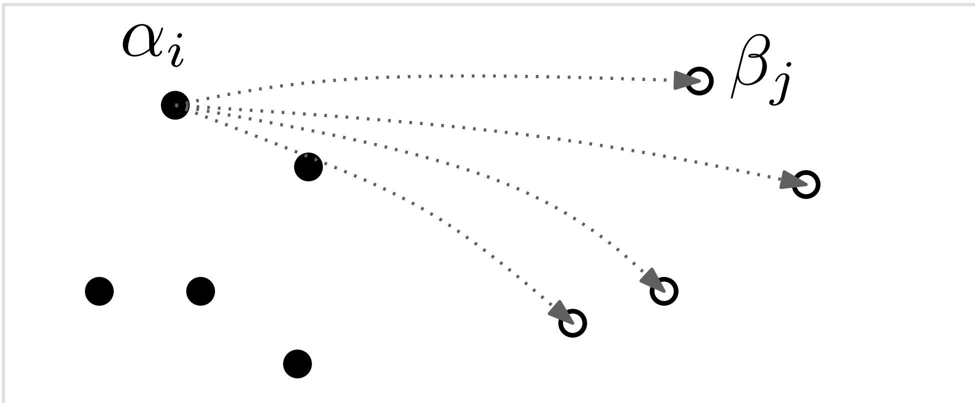
Computationally efficient **but** restricted to "structured" cost functions.

Computational optimal transport



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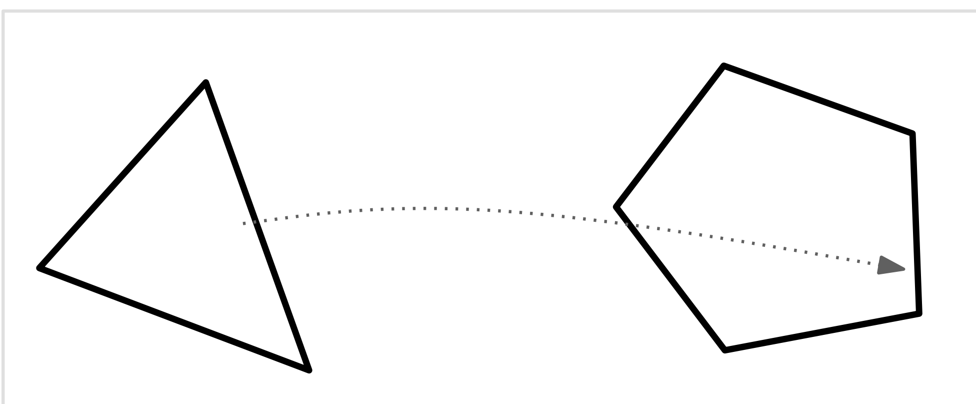

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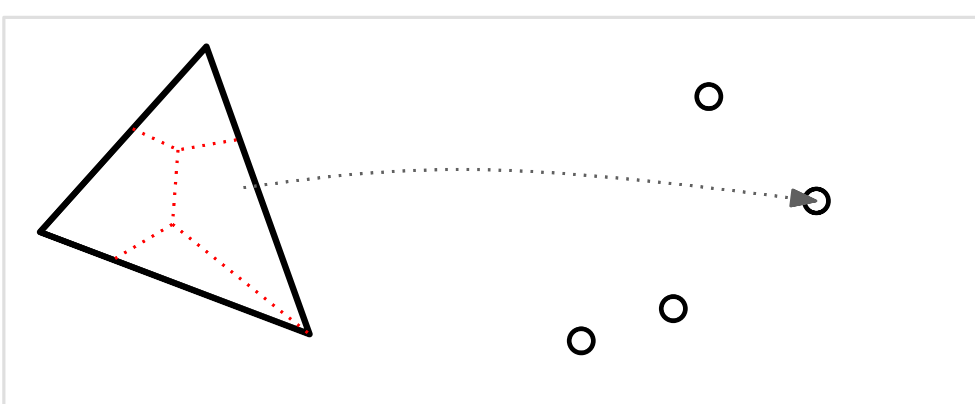

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"semi-discrete optimal transport"

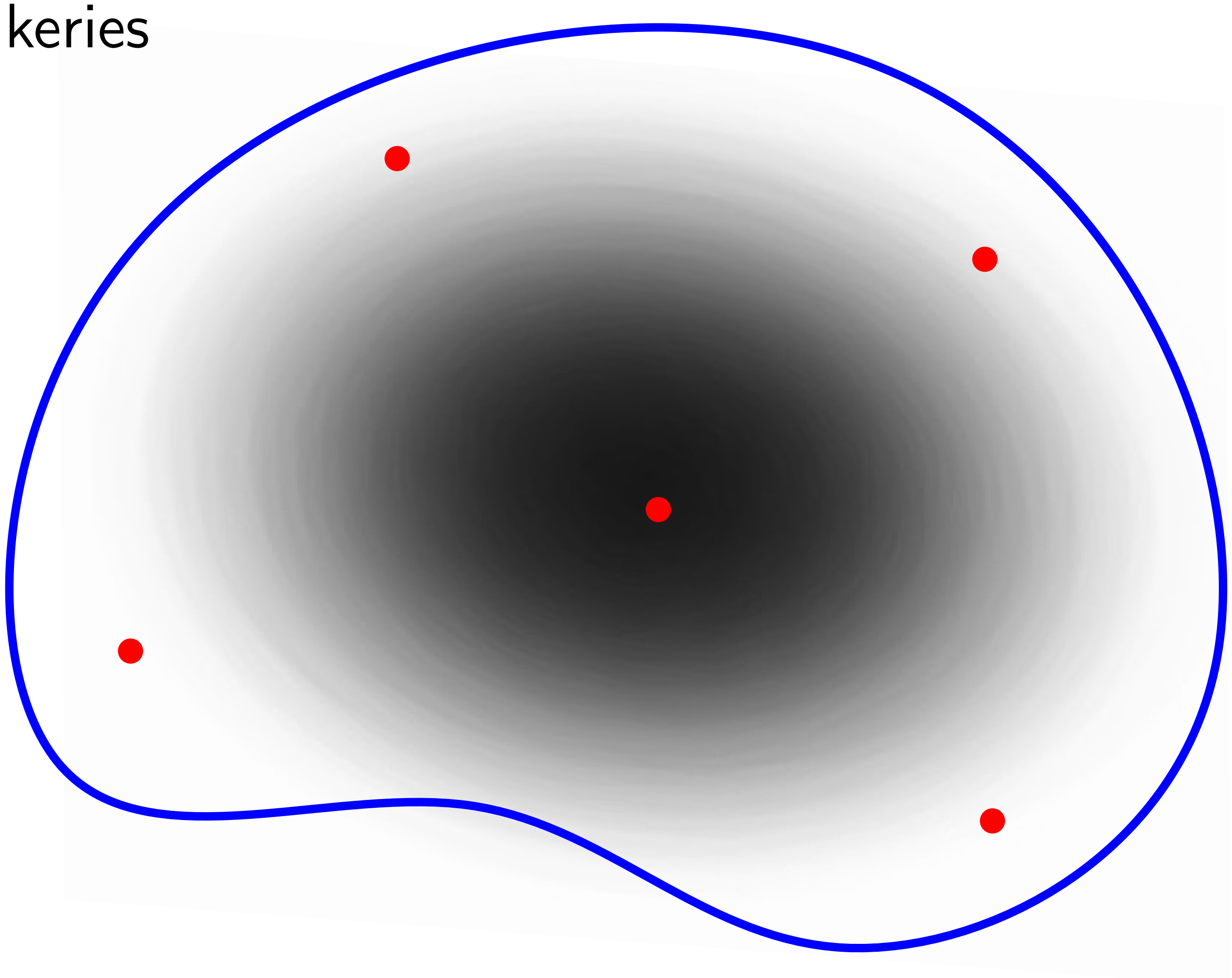
1. Semi-discrete optimal transport

An economic interpretation

$\rho : X \to \mathbb{R}$ density of population

$c(x, y) =$ cost of walking from x to y

$Y =$ location of bakeries



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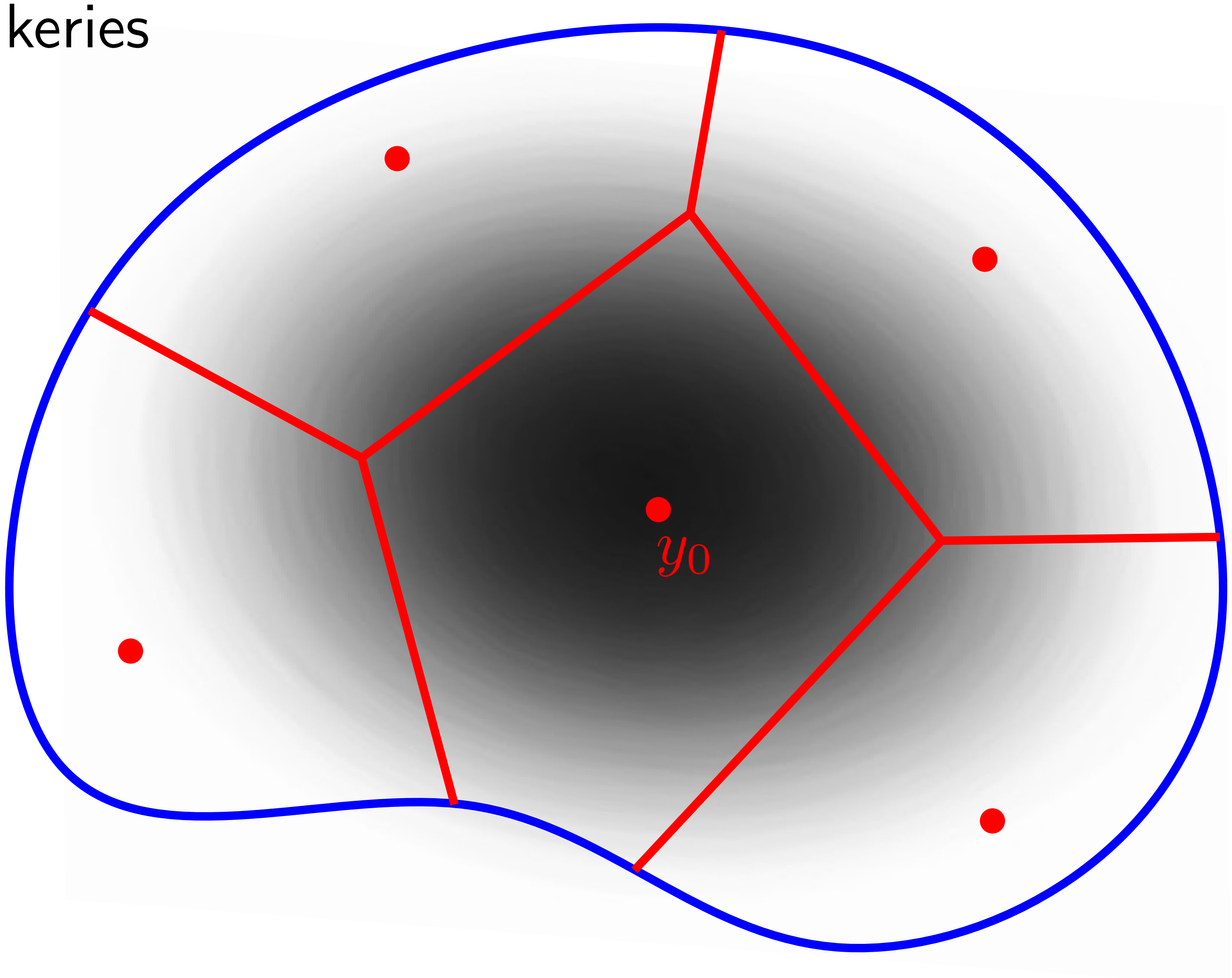

- If the price of bread is uniform, people go the closest bakery:

$$\boxed{\mathrm{Vor}(y) = \{x \in X; \forall z \in Y,\ c(x, y) \le c(x, z)\}}$$

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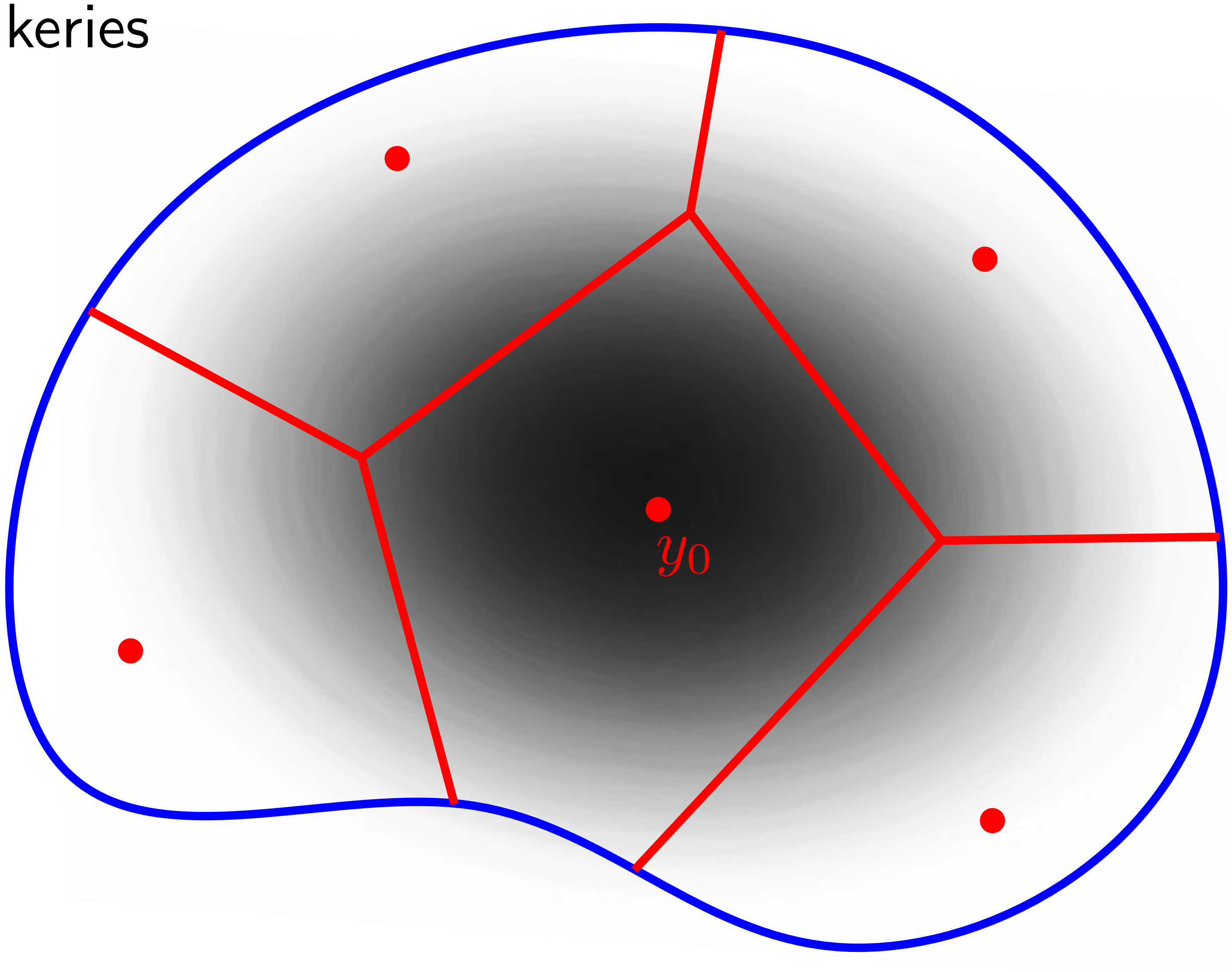

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Minimizes total distance walked ... **but** might exceed the capacity of bakery y_0!

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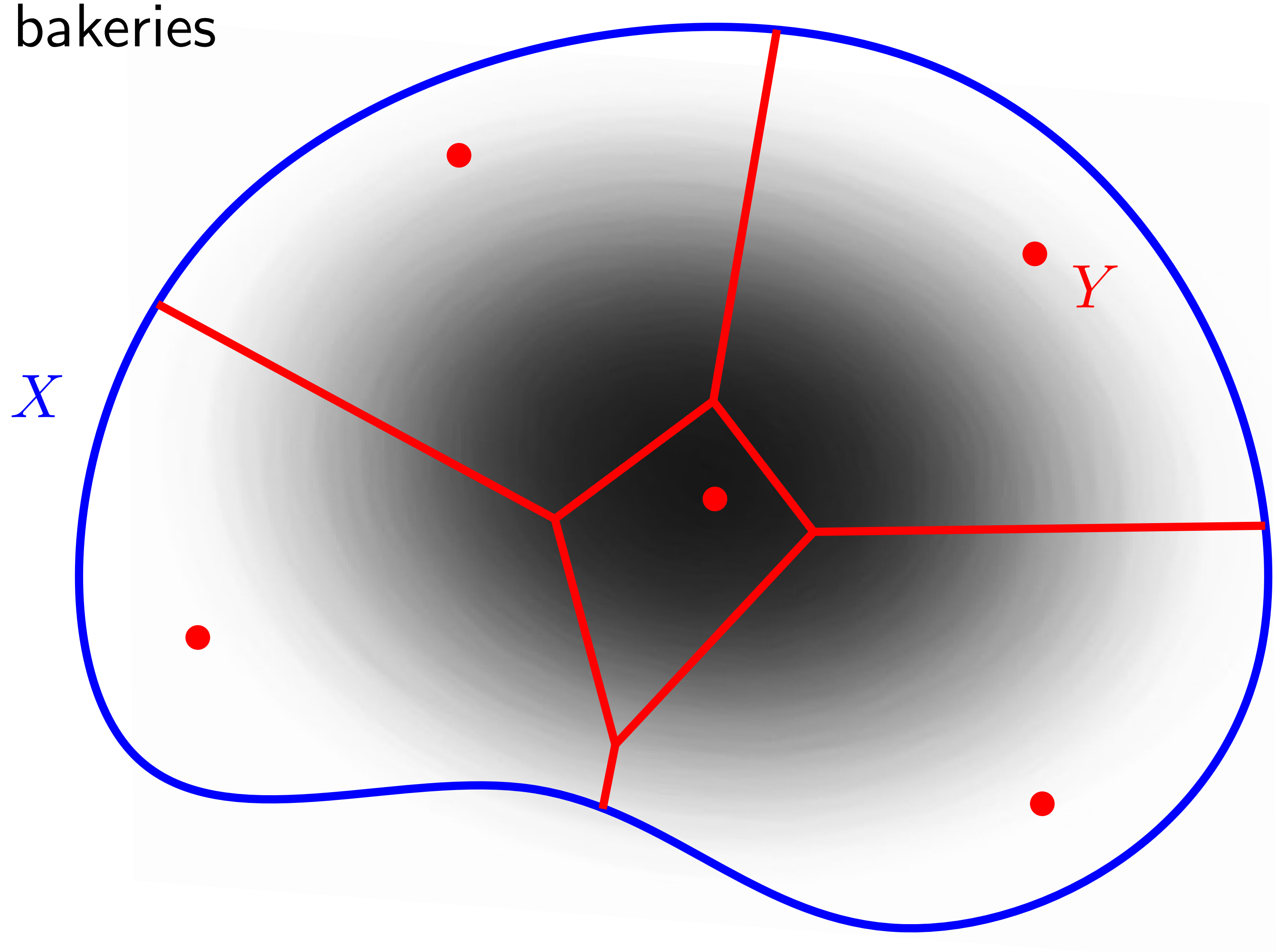
Y = location of bakeries



- If prices are given by $\psi : Y \to \mathbb{R}$, people make a compromise:

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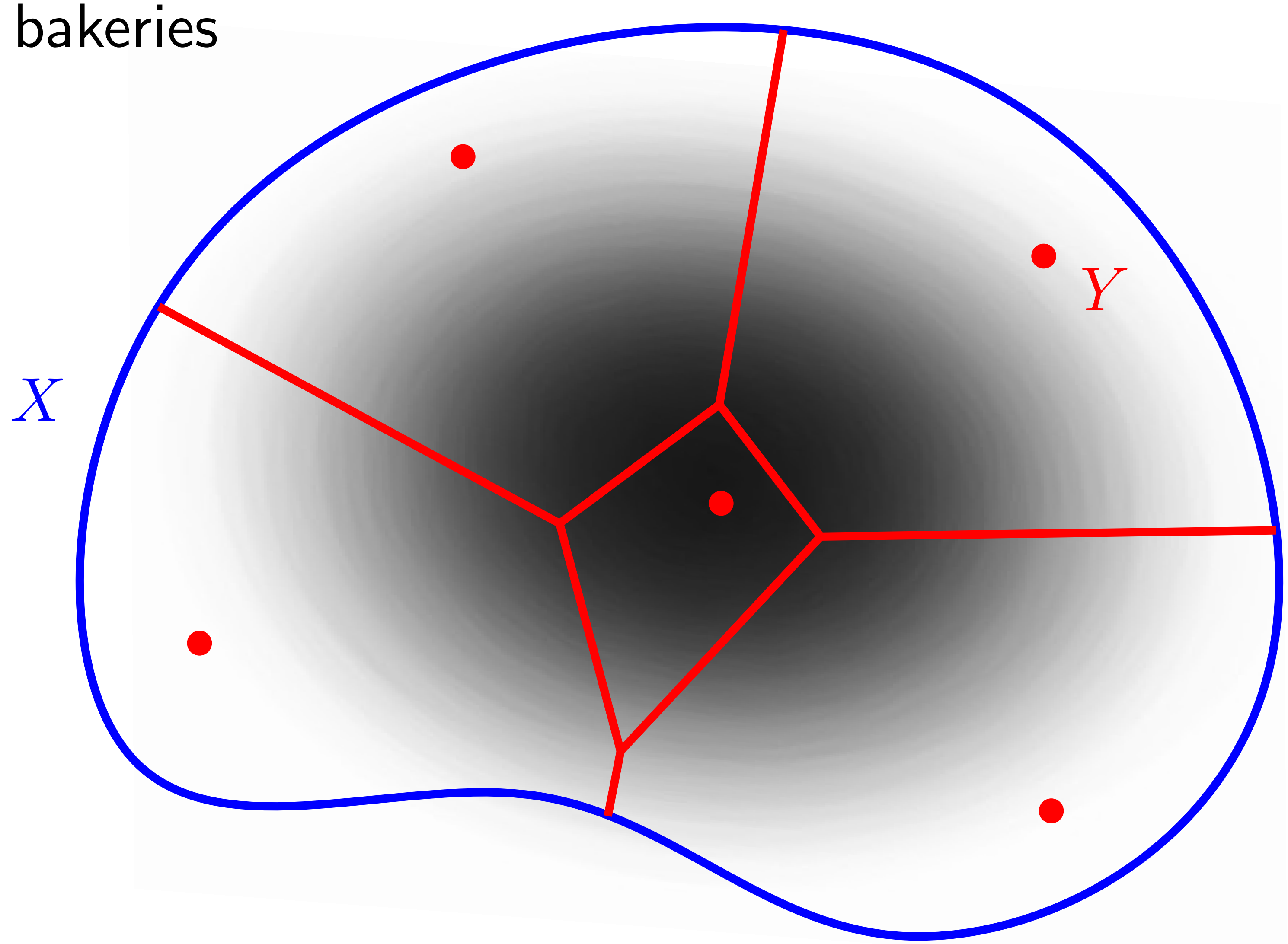

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Still minimizes total distance walked **under capacity constraints**.

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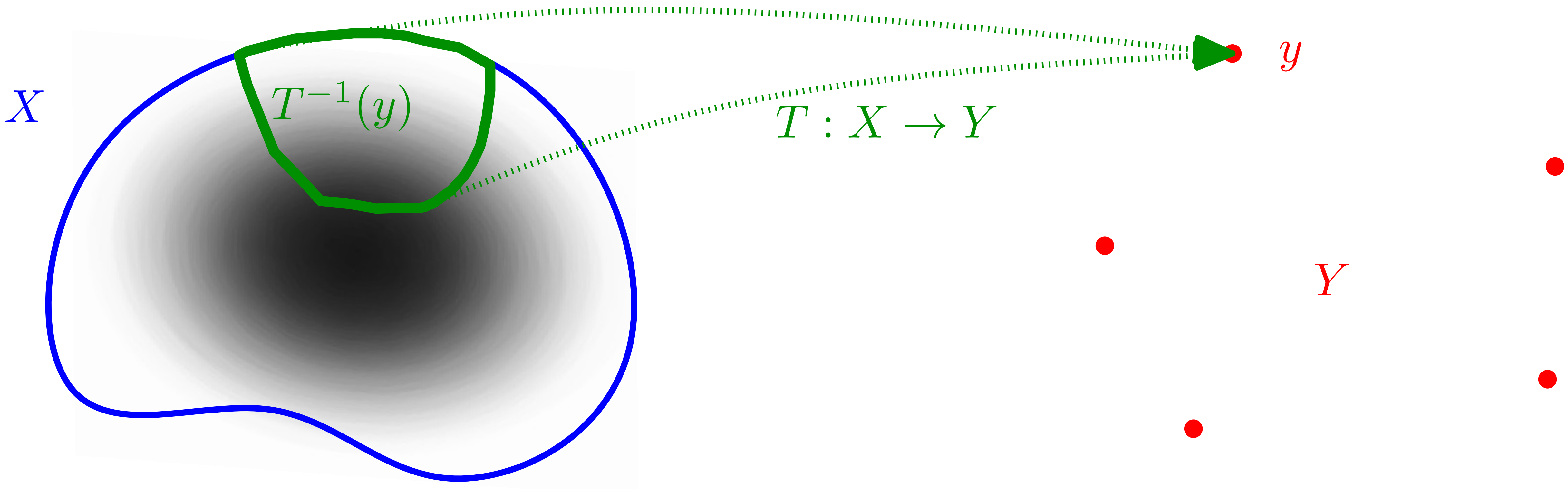

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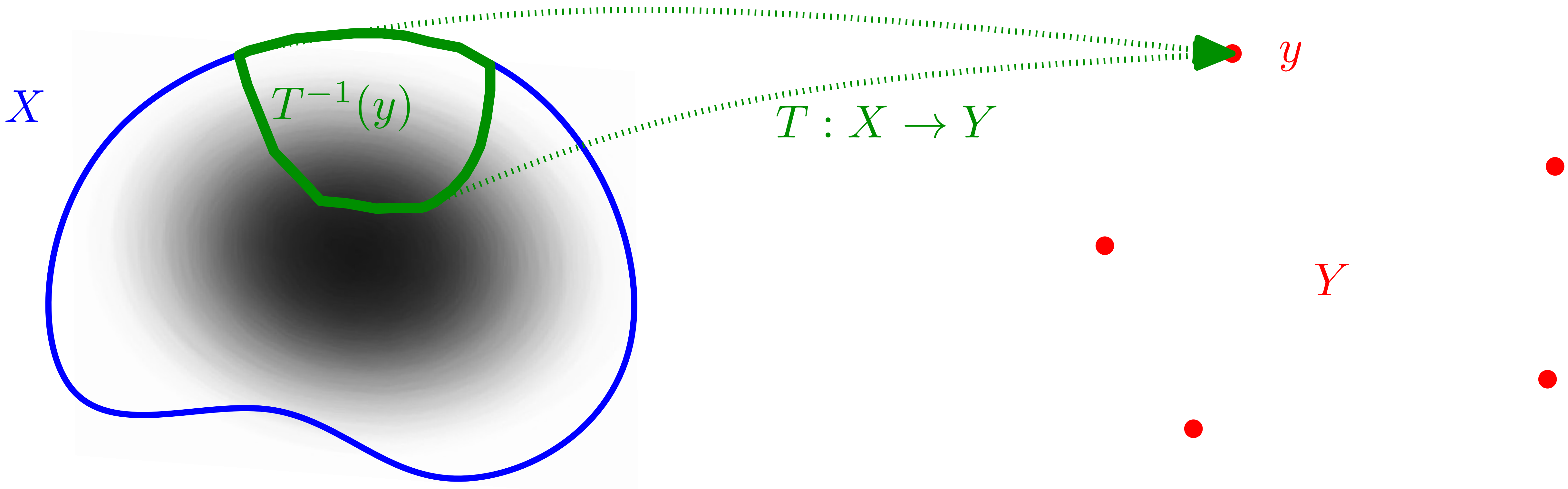

Transport map: $T : X \to Y$ that satisfies **capacity constraints**: $\rho(T^{-1}(y)) = \nu_y$

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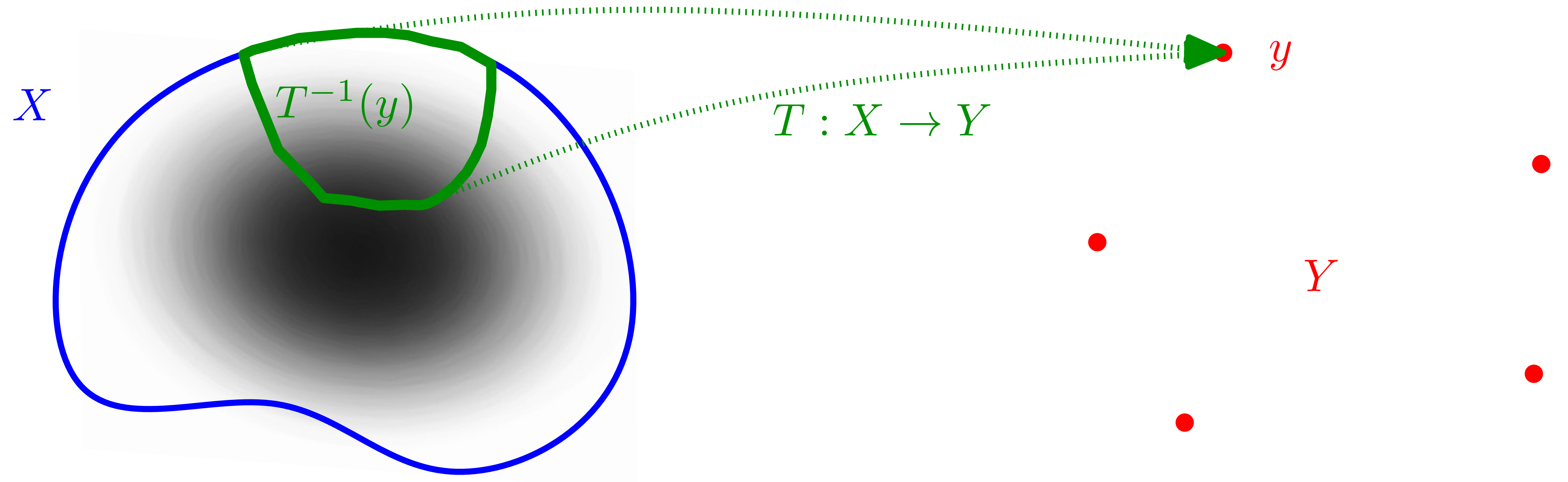

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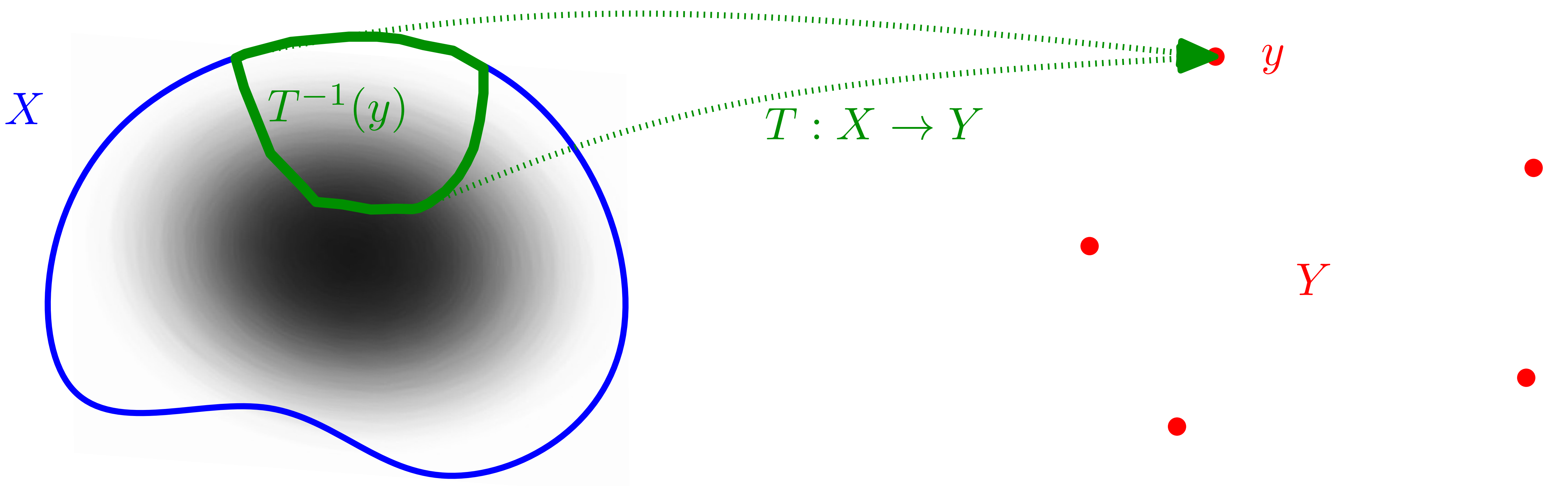

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$\longrightarrow$ With $c = \frac{1}{2}\|.\|^2$, $\mathcal{C}^{1/2}$ is the quadratic Wasserstein distance.

Optimal transport and Laguerre Diagrams

We assume **(Twist):** $\forall x \in X$, the map $y \in Y \mapsto \nabla_x c(x, y)$ is injective.

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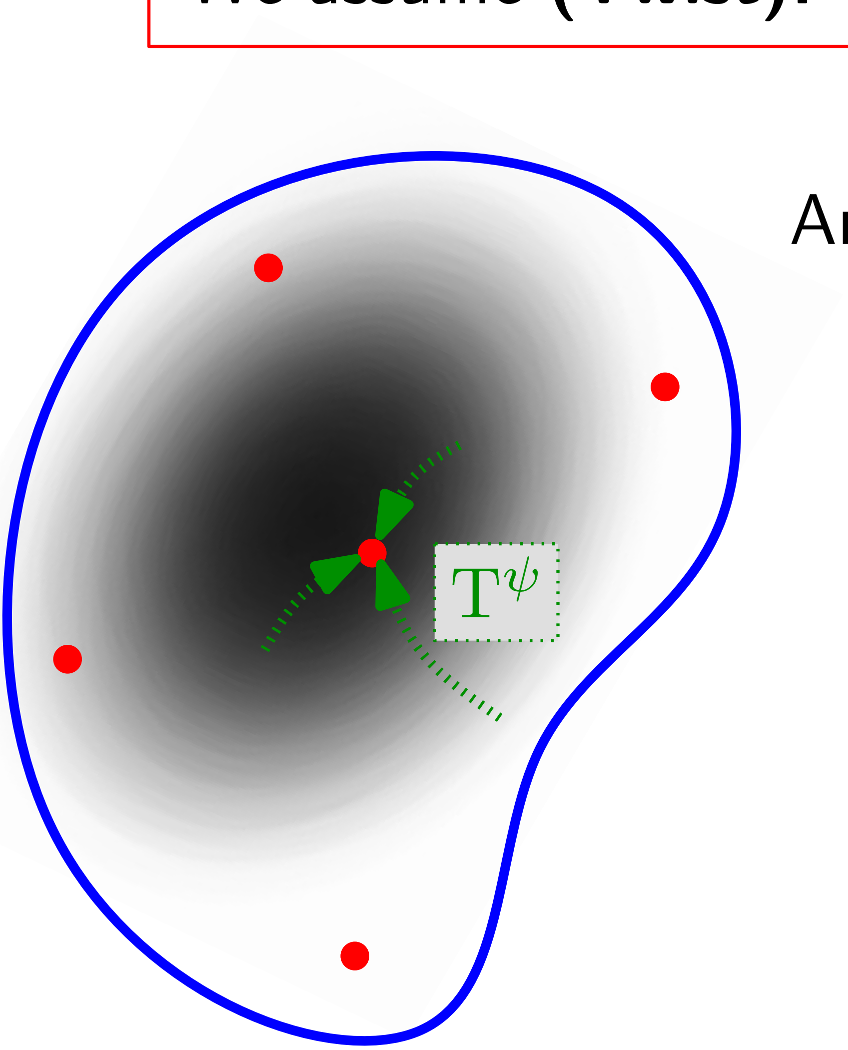

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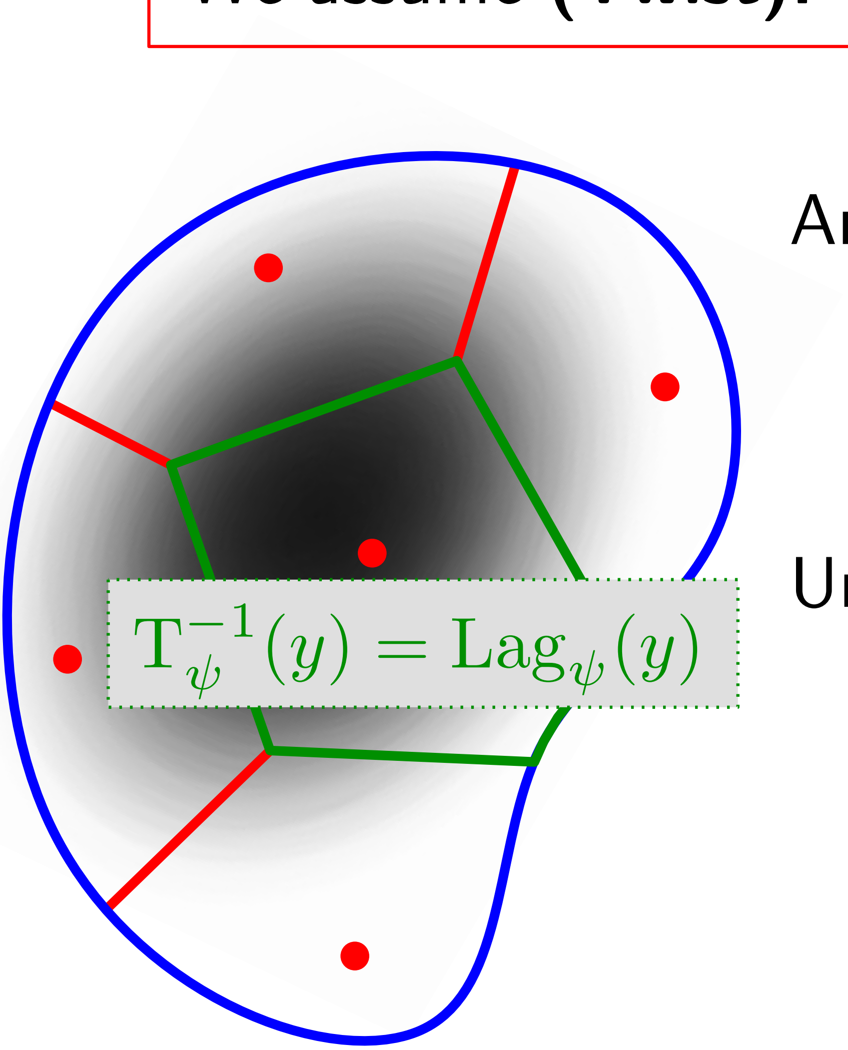

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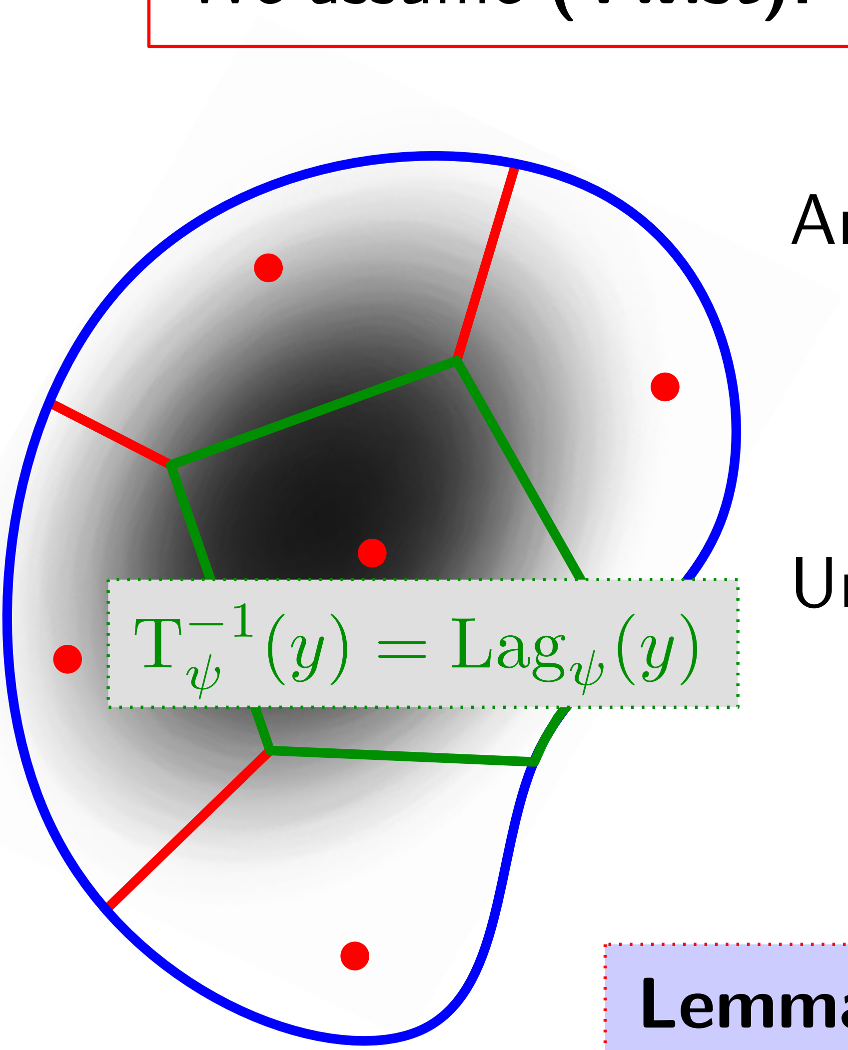

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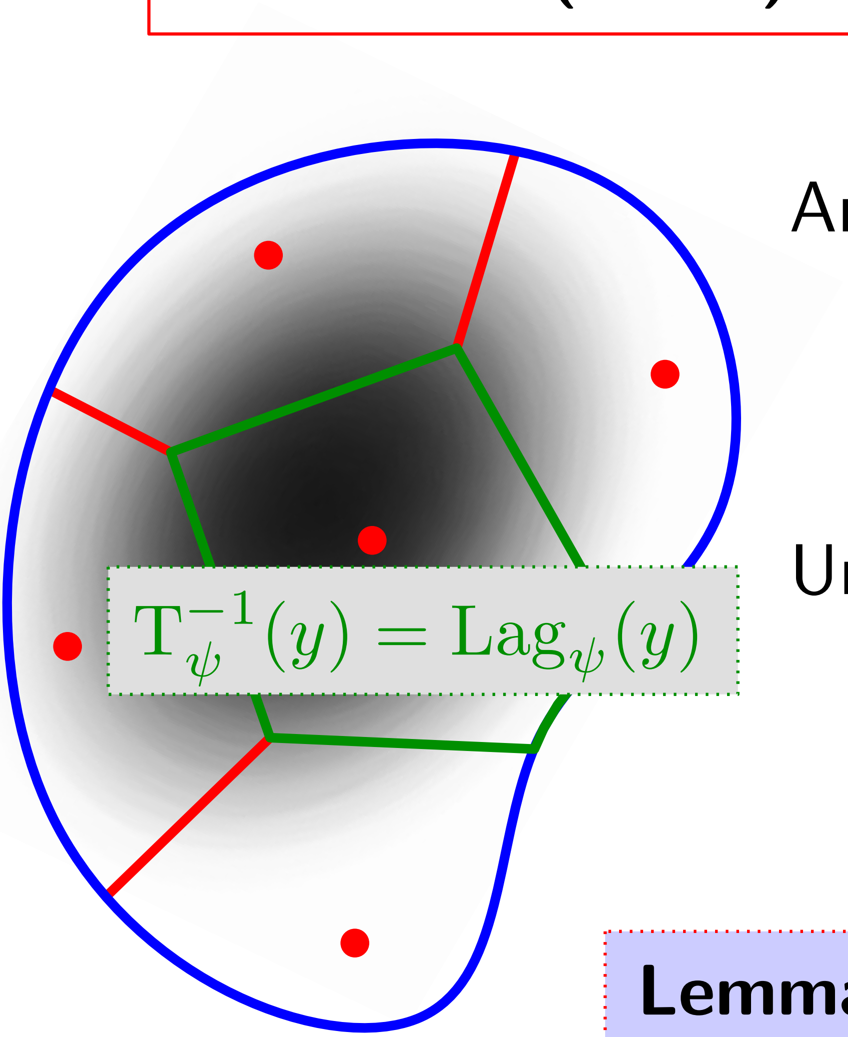

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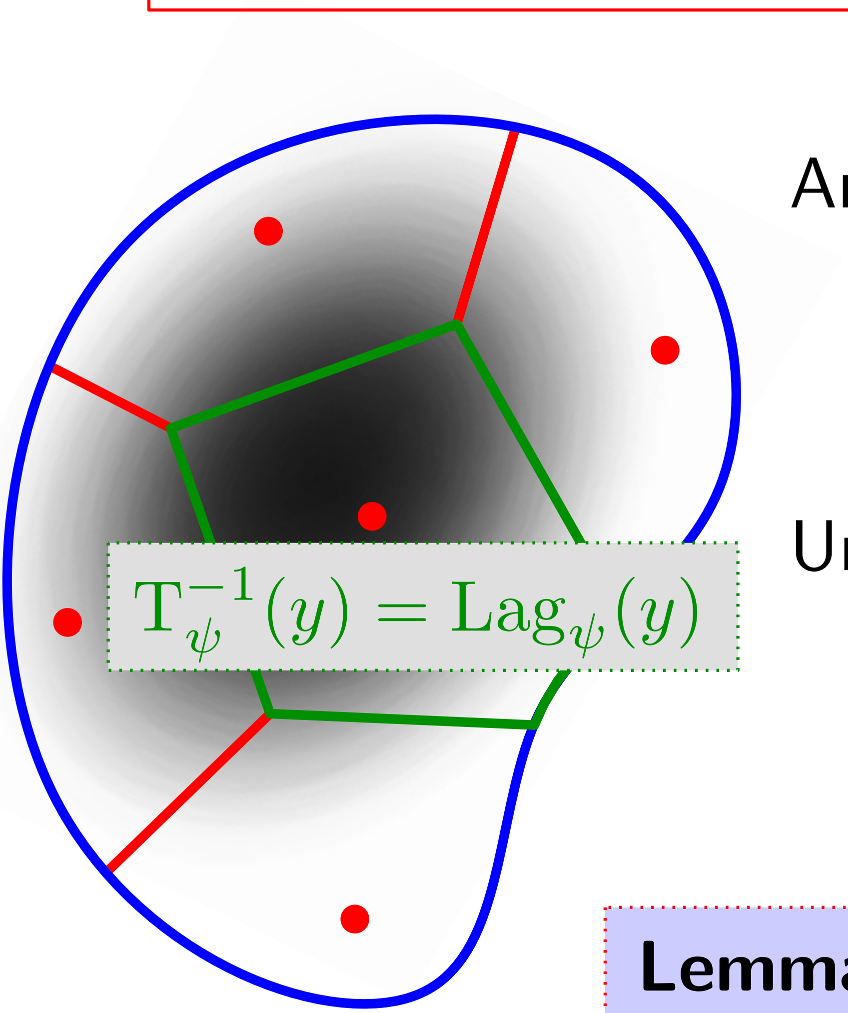

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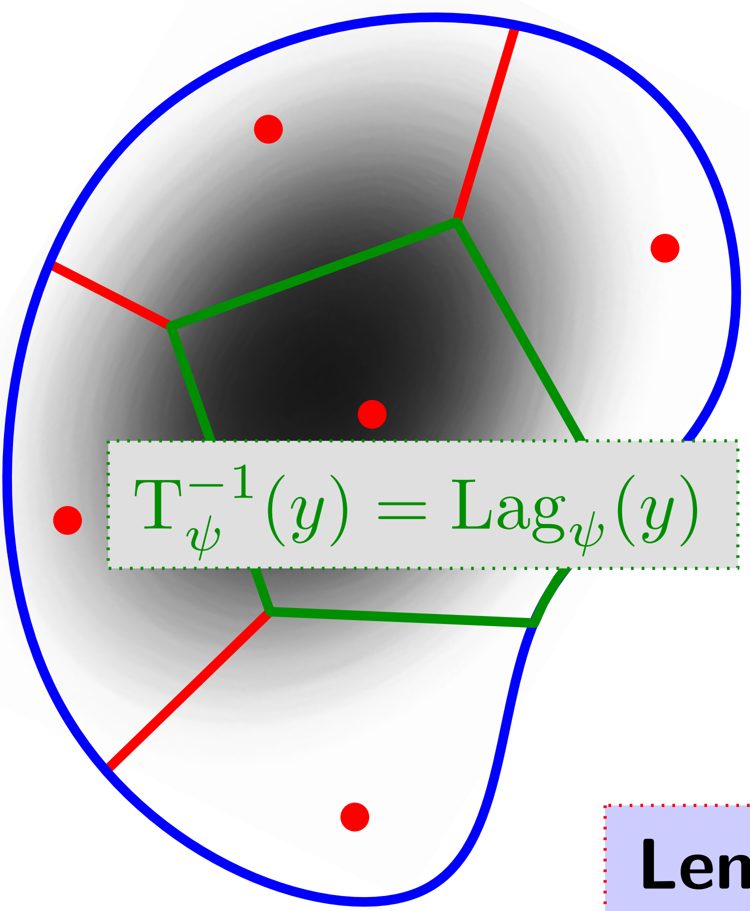

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Semi-discrete OT as Concave Maximization

Theorem: Finding an **optimal transport** between ρ and $\nu = \sum_Y \nu_y \delta_y$

$\iff$ finding **prices** ψ on Y such that $\nu_\psi = \nu$ [Gangbo McCann '96]

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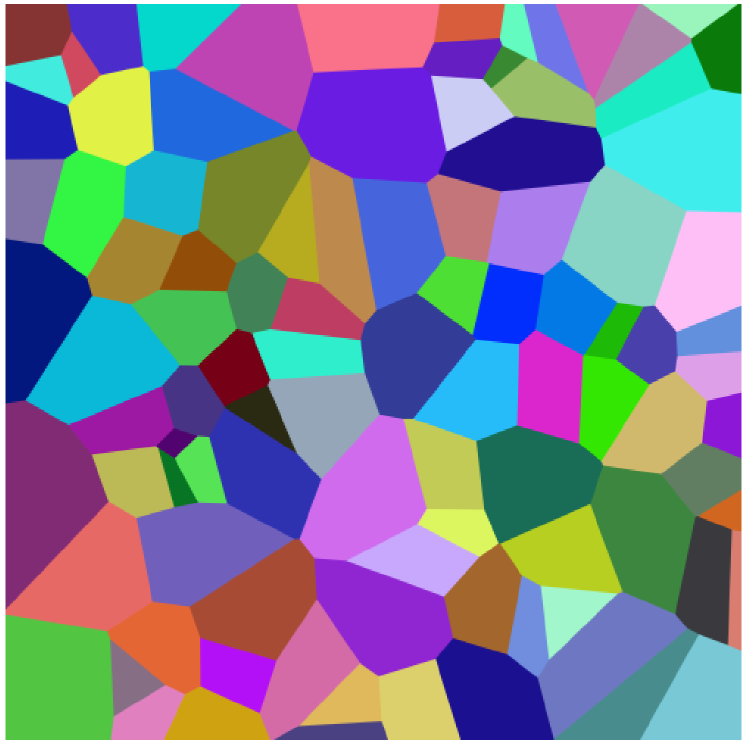
- Fast and accurate solvers, suitable for PDE discretization.

Numerical example 1

Source: $\rho =$ uniform on $[0,1]^2$, **Cost:** $c(x,y) = \|x-y\|^2$

Target: $\nu = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{y_i}$ with y_i uniform i.i.d. in $[0,1]^2$

Voronoi diagram



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$\varepsilon_0 \simeq 0.05$

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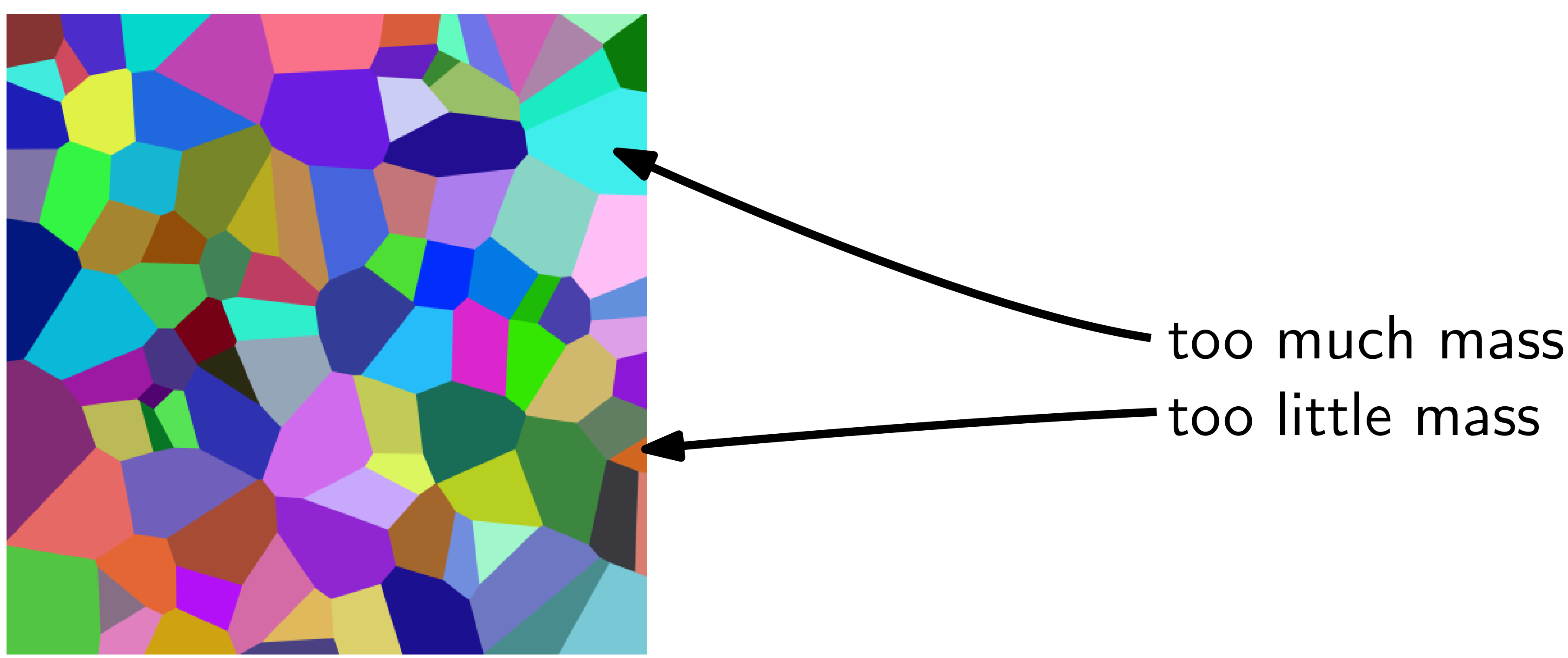

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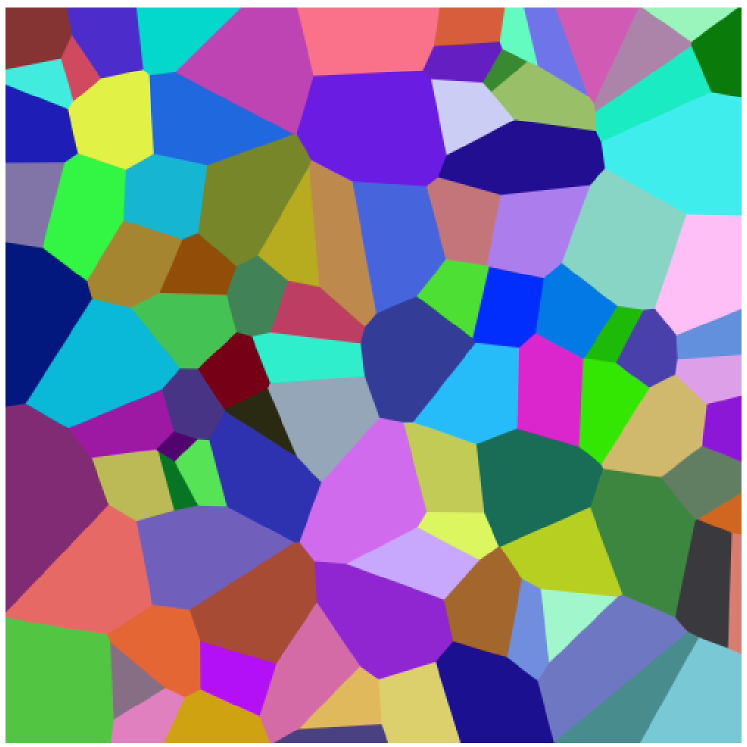

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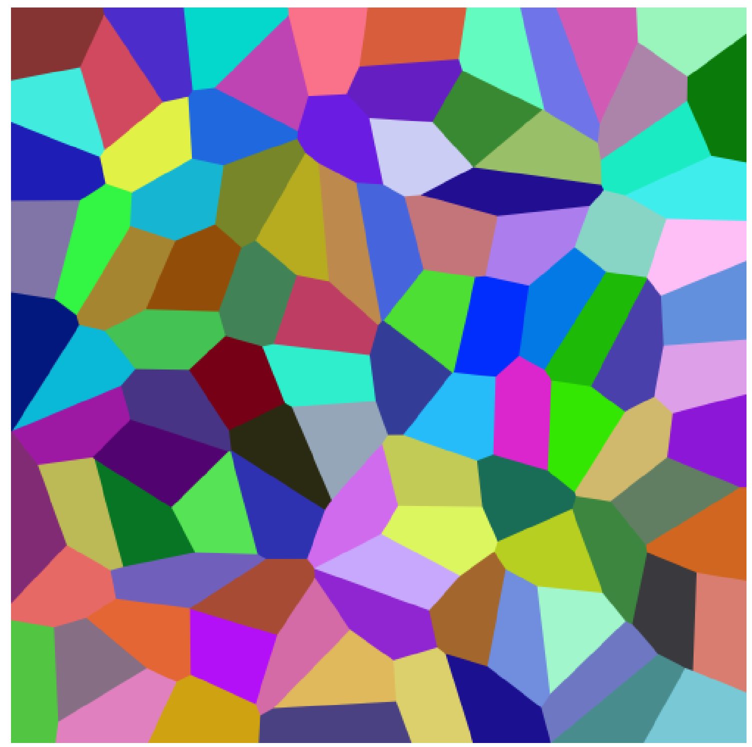
Laguerre diagram



$\psi_1 = \mathrm{Newt}(\psi_0)$

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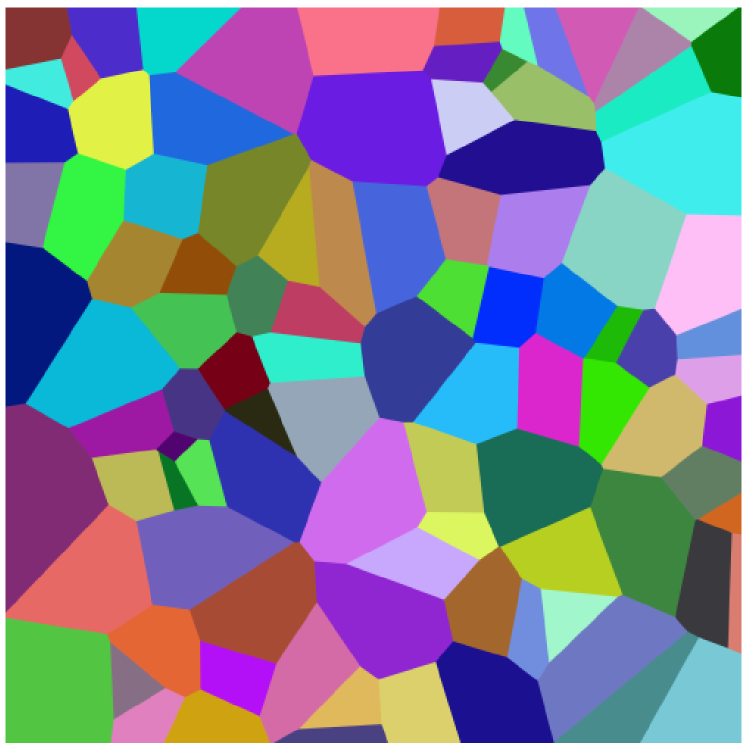
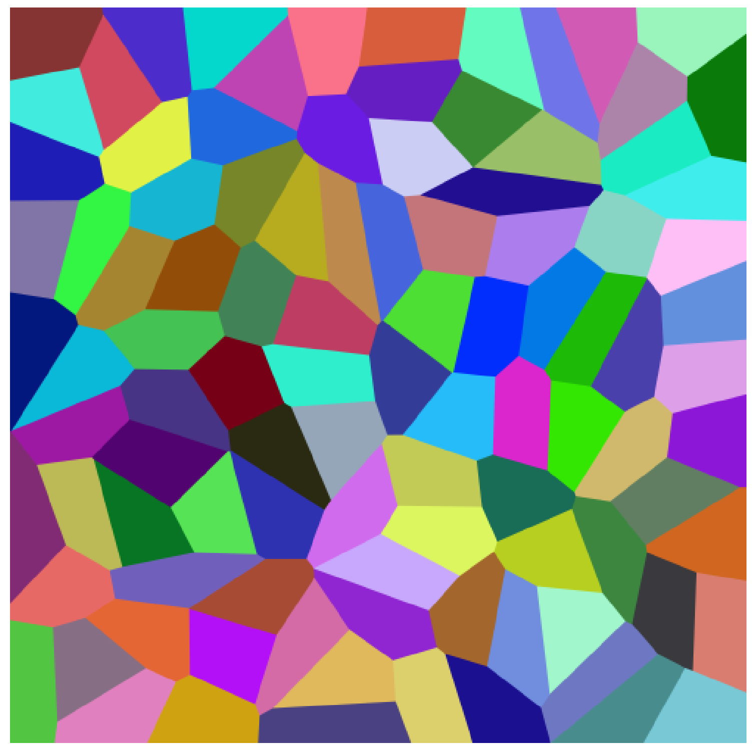


$\psi_0 = 0$

$\varepsilon_0 \simeq 0.05$

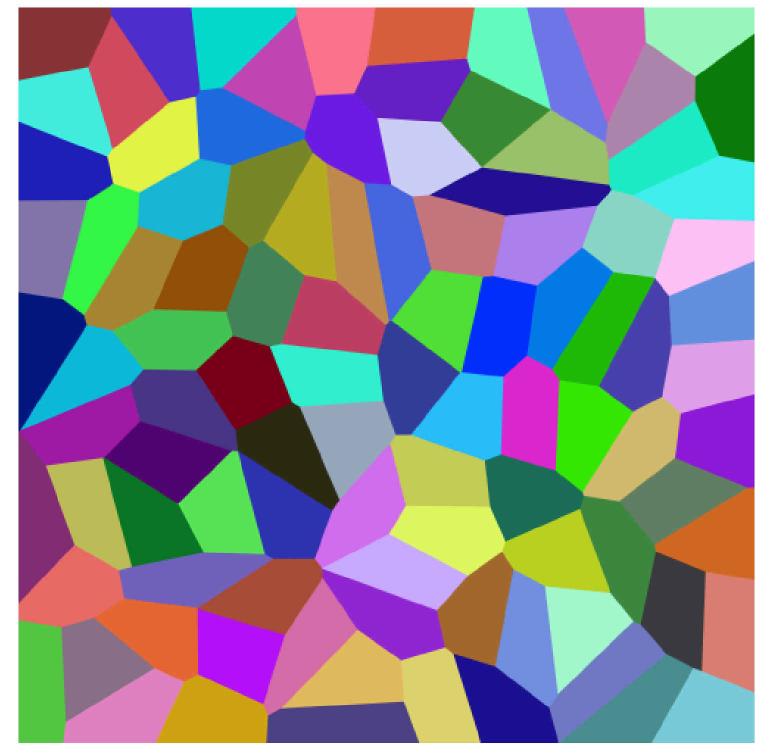
Laguerre diagram



$\psi_1 = \mathrm{Newt}(\psi_0)$

$\varepsilon_1 \simeq 0.007$

Laguerre diagram



$\psi_2 = \mathrm{Newt}(\psi_1)$

$\varepsilon_2 \simeq 10^{-9}$

Where $\varepsilon_k := \sum_i |\rho(\mathrm{Lag}_i(\psi_k)) - \frac{1}{N}|$ is the amount of misallocated mass.

Numerical example 2

Source: $\rho =$ uniform on $[0,1]^2$, **Cost:** $c(x,y) = \|x-y\|^2$

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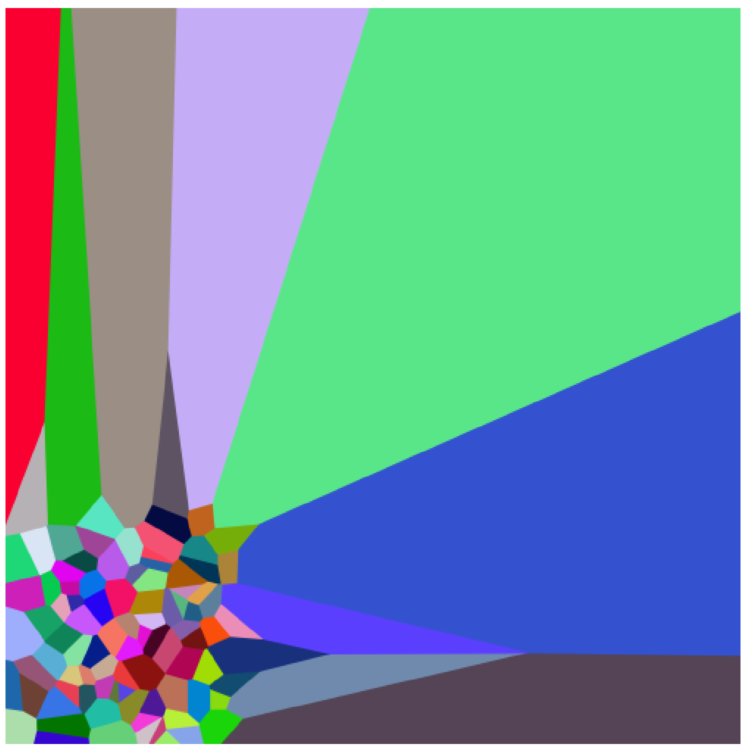

$\psi_0 = 0$

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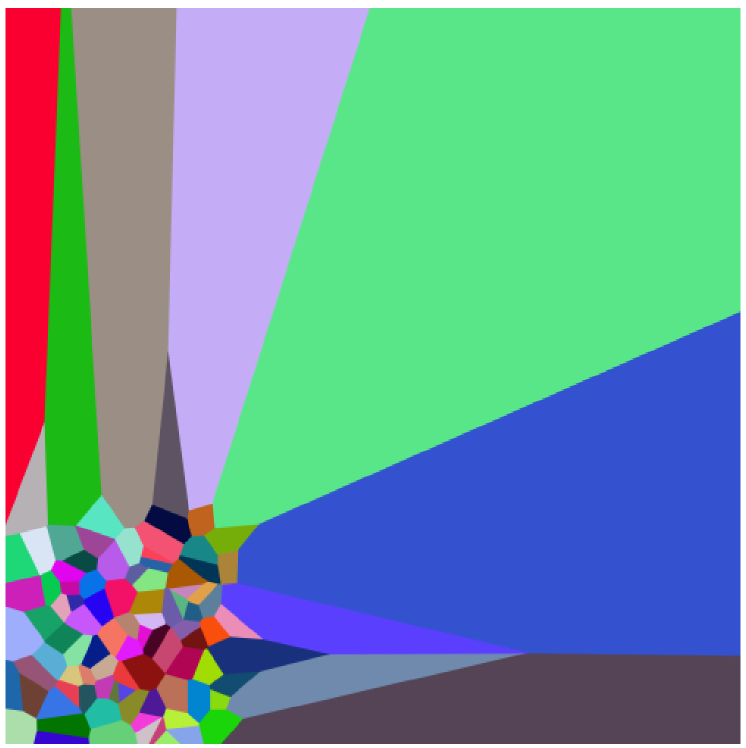

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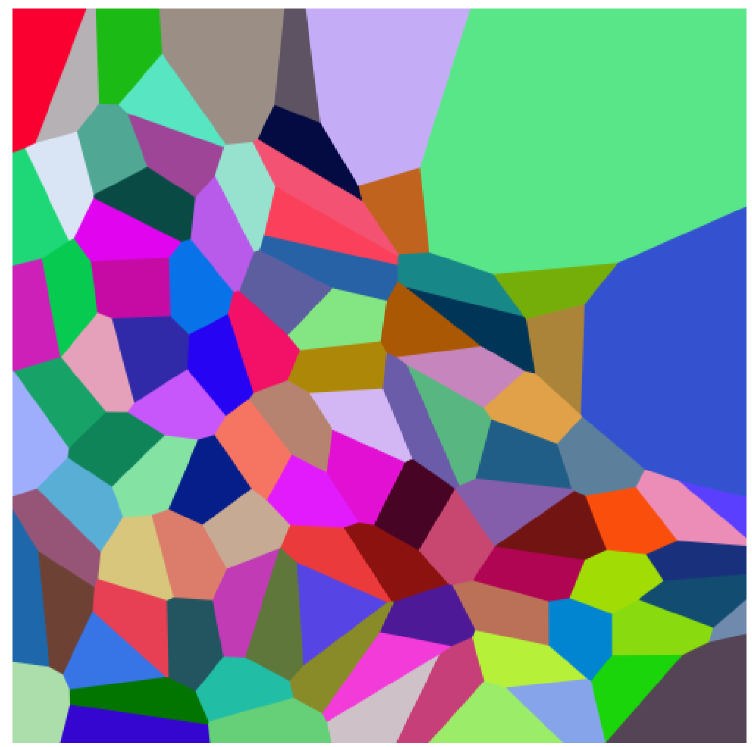

$\psi_1 = \mathrm{Newt}(\psi_0)$

$\varepsilon_1 \simeq 0.024$

NB: The points do **not** move.

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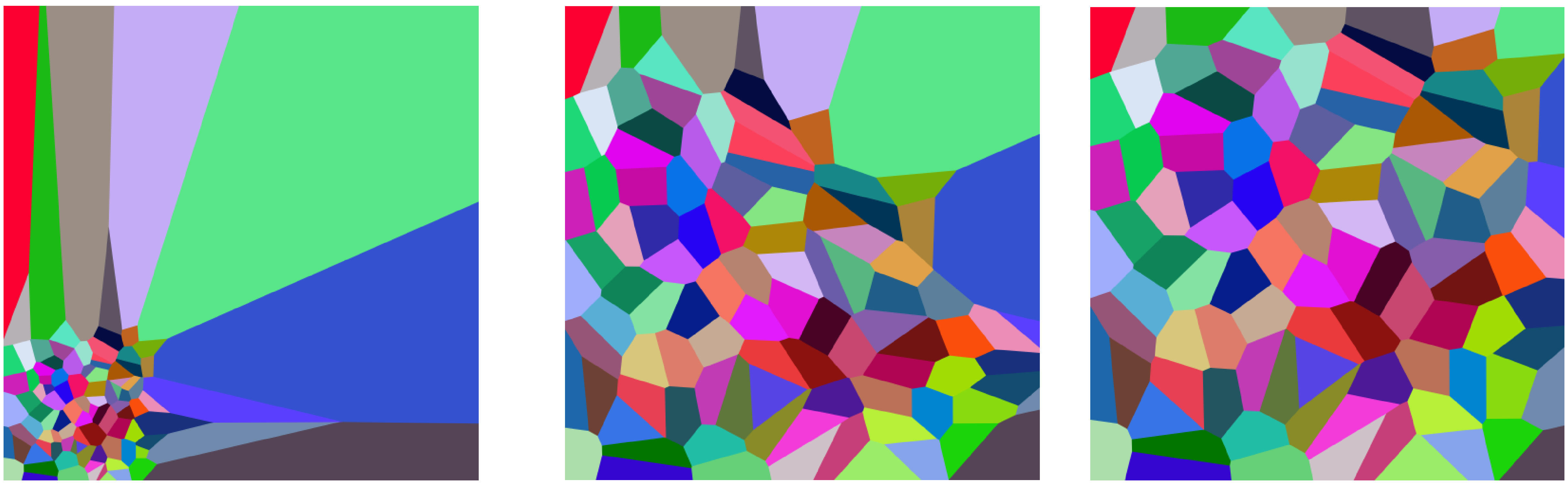
Voronoi diagram | Laguerre diagram | Laguerre diagram



$\psi_0 = 0$ | $\psi_1 = \mathrm{Newt}(\psi_0)$ | $\psi_2 = \mathrm{Newt}(\psi_1)$

$\varepsilon_0 \simeq 0.48$ | $\varepsilon_1 \simeq 0.024$ | $\varepsilon_2 \simeq 10^{-6}$

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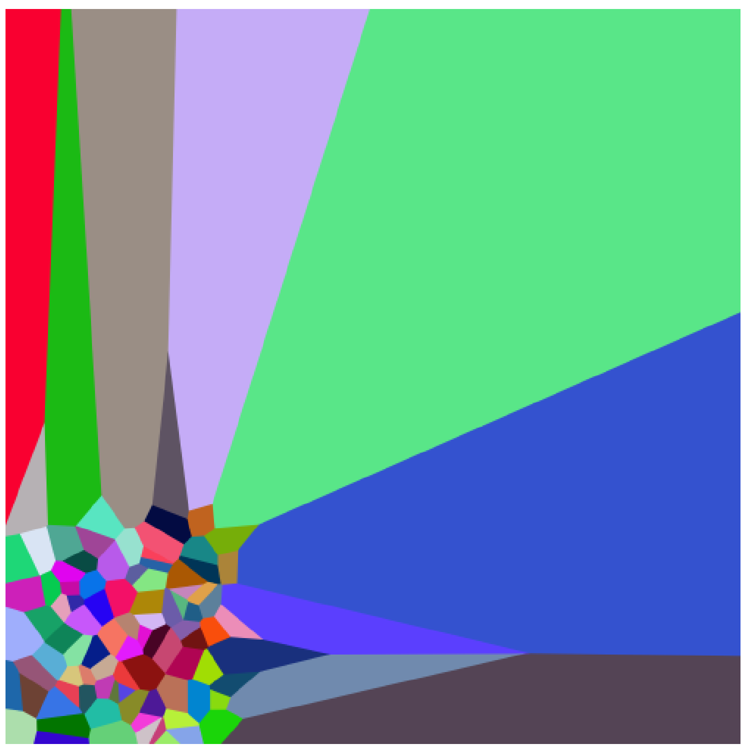

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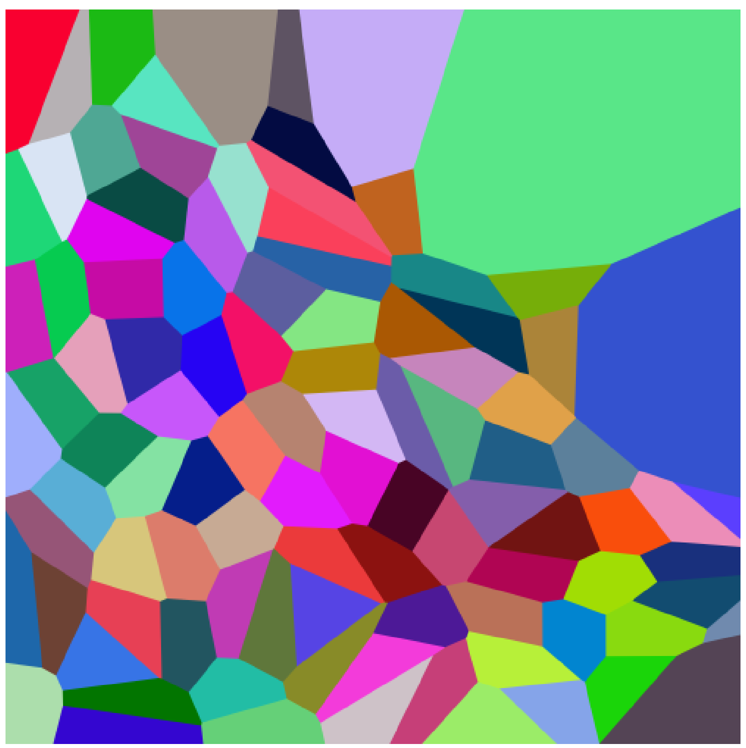

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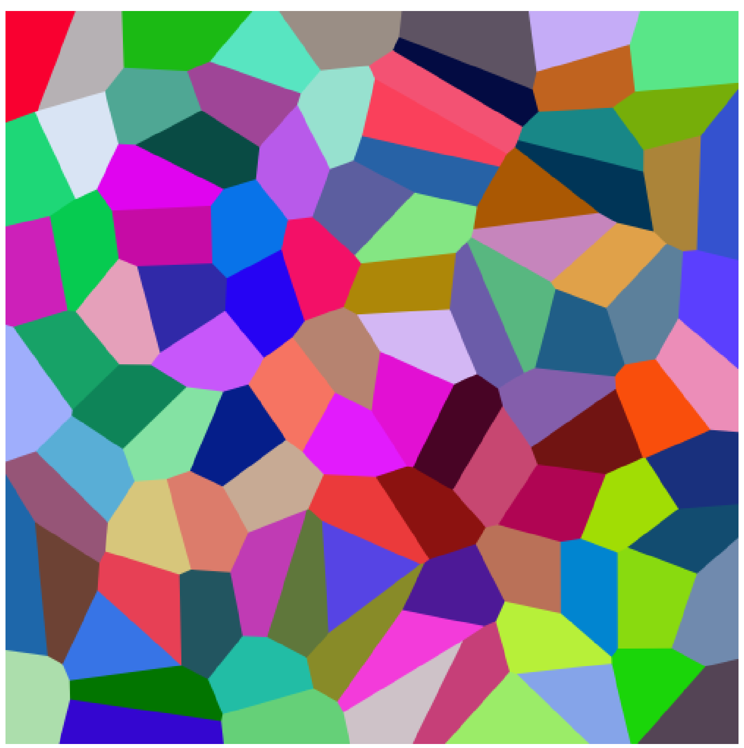

$\psi_2 = \mathrm{Newt}(\psi_1)$

$\varepsilon_2 \simeq 10^{-6}$

NB: The points do **not** move.

Laguerre diagrams are able to encode an actual *transport* of mass (large movement).

2. Quadratic case, $c(x,y) = \|x-y\|^2$

Newton for Optimal transport/Monge-Ampère

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Goal: Identify sufficient conditions for the **global** convergence of Newton's method :
$\longrightarrow$ the cost function needs some geometric properties (satisfied by $\|.\|^2$)
$\longrightarrow$ the support of the density must be connected in a strong sense

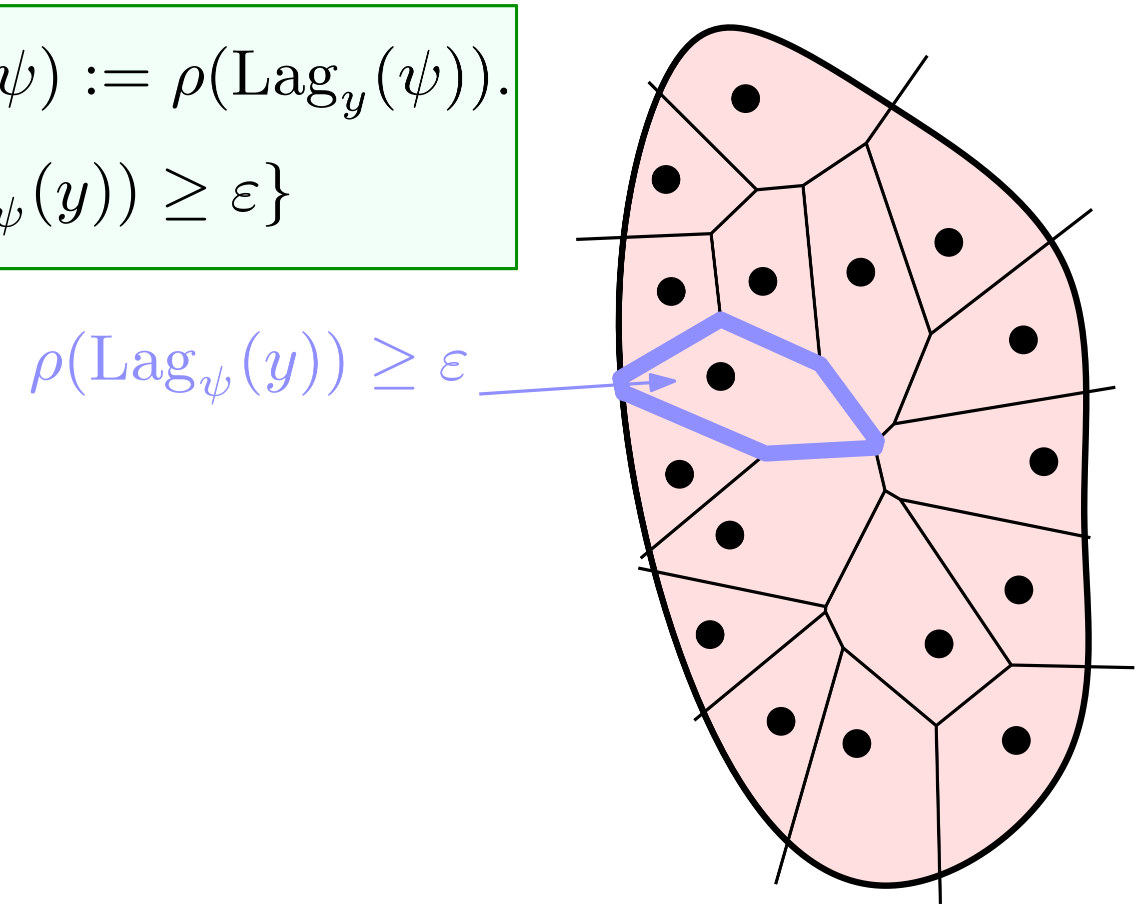
Specificity of $c = \|.\|^2$: Laguerre cells are polyhedra.

Damped Newton's Algorithm

cf [Mirebeau '15]

Discrete Monge-Ampère op.: $G : \mathbb{R}^Y \to \mathbb{R}^Y, \quad G_y(\psi) := \rho(\mathrm{Lag}_y(\psi)).$

Admissible domain: $E_\varepsilon := \{\psi \in \mathbb{R}^Y ; \forall y \in Y, \rho(\mathrm{Lag}_\psi(y)) \geq \varepsilon\}$



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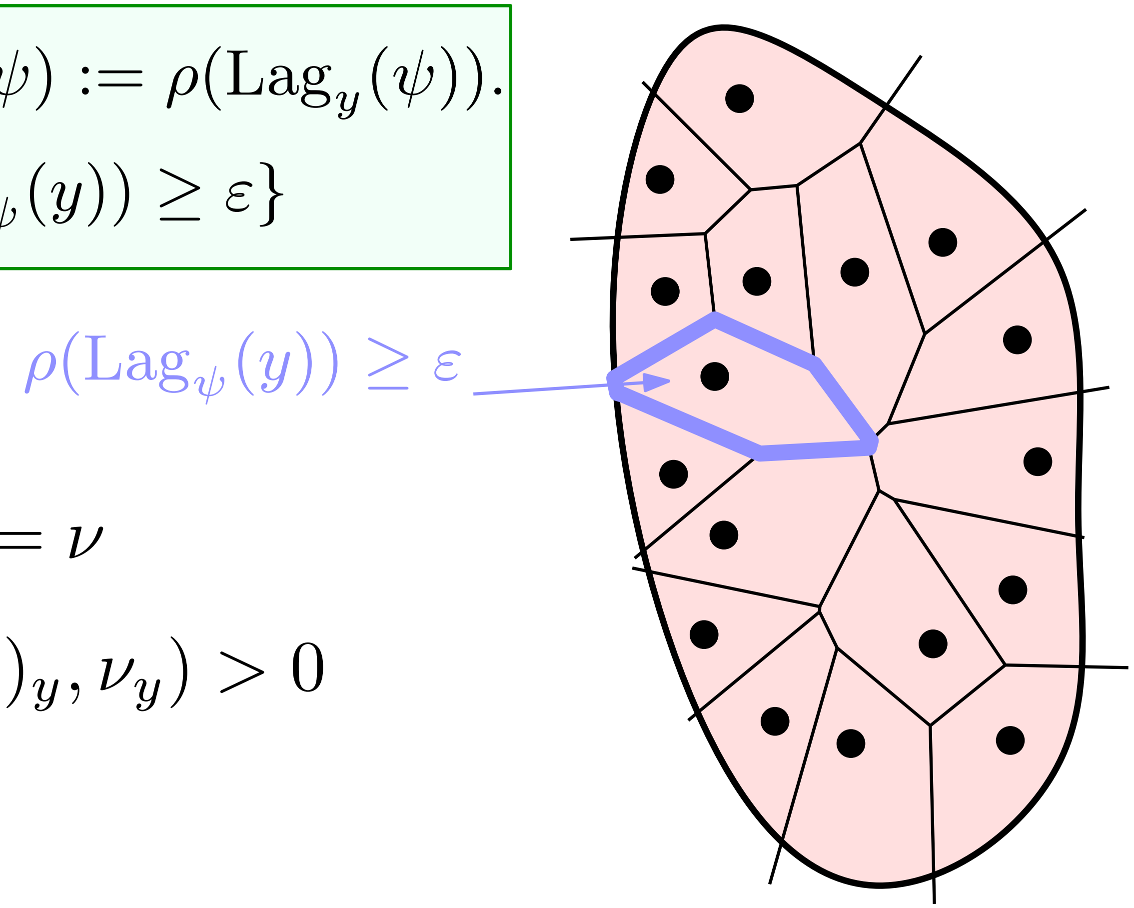

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Input: $\psi_0 \in \mathbb{R}^Y$ s.t. $\varepsilon := \frac{1}{2} \min_{y \in Y} \min(G(\psi_0)_y, \nu_y) > 0$

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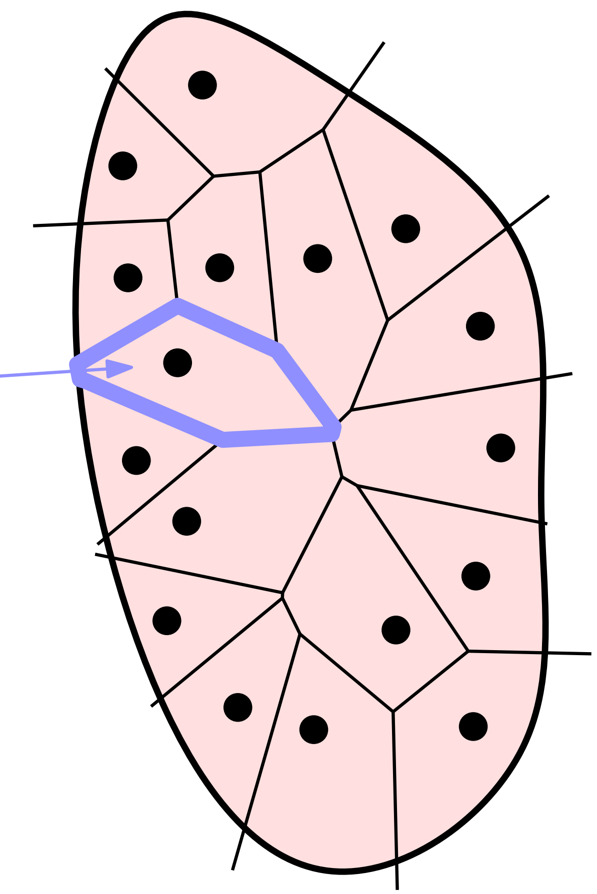

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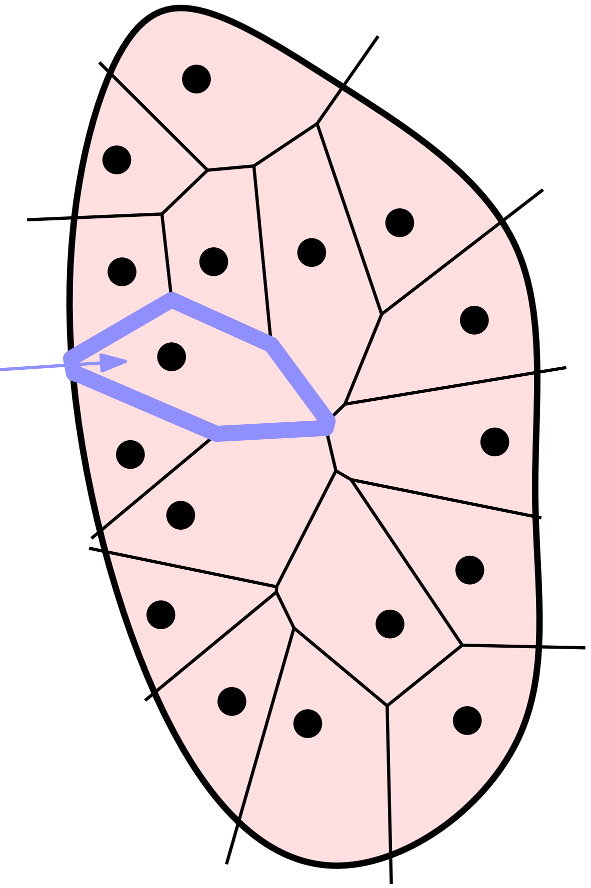

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constructing such ψ_0 is a difficult practical problem...
(possible answers: ad-hoc constructions, multi-scale strategies, etc.)

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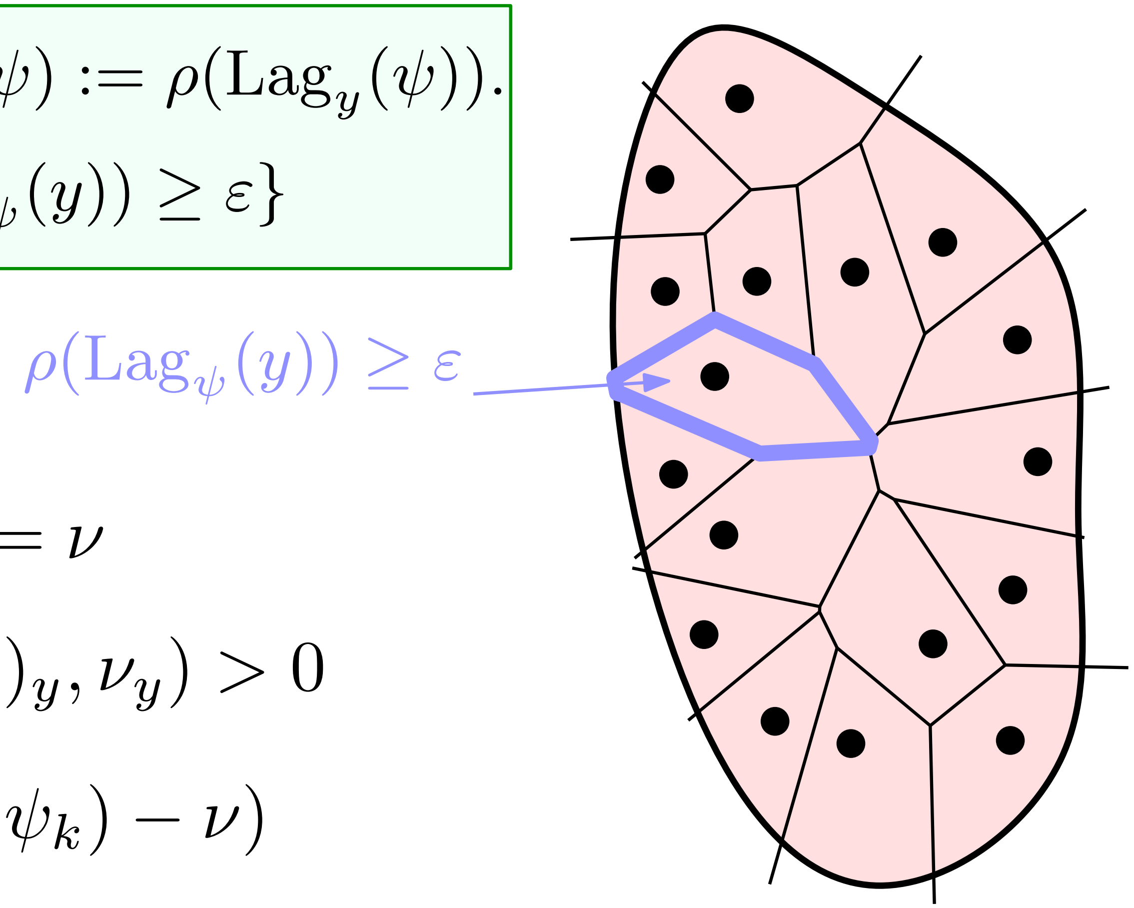

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$\longrightarrow \psi_{k+1} := \psi_k^{\tau_k}$

Proposition: The damped Newton's algorithm converges **globally** provided that:

(Smoothness): $\nabla \mathcal{K} = G - \nu$ is $\mathcal{C}^1$ on E_ε.

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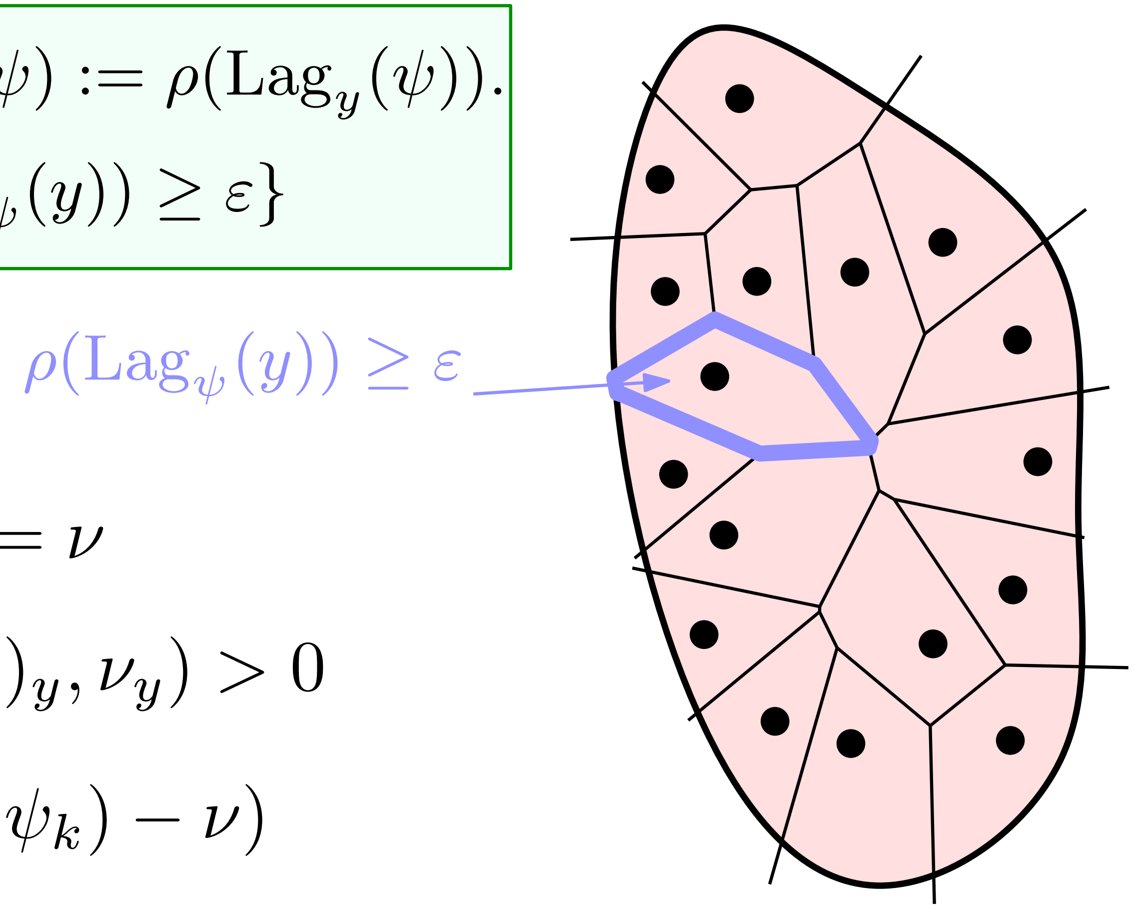

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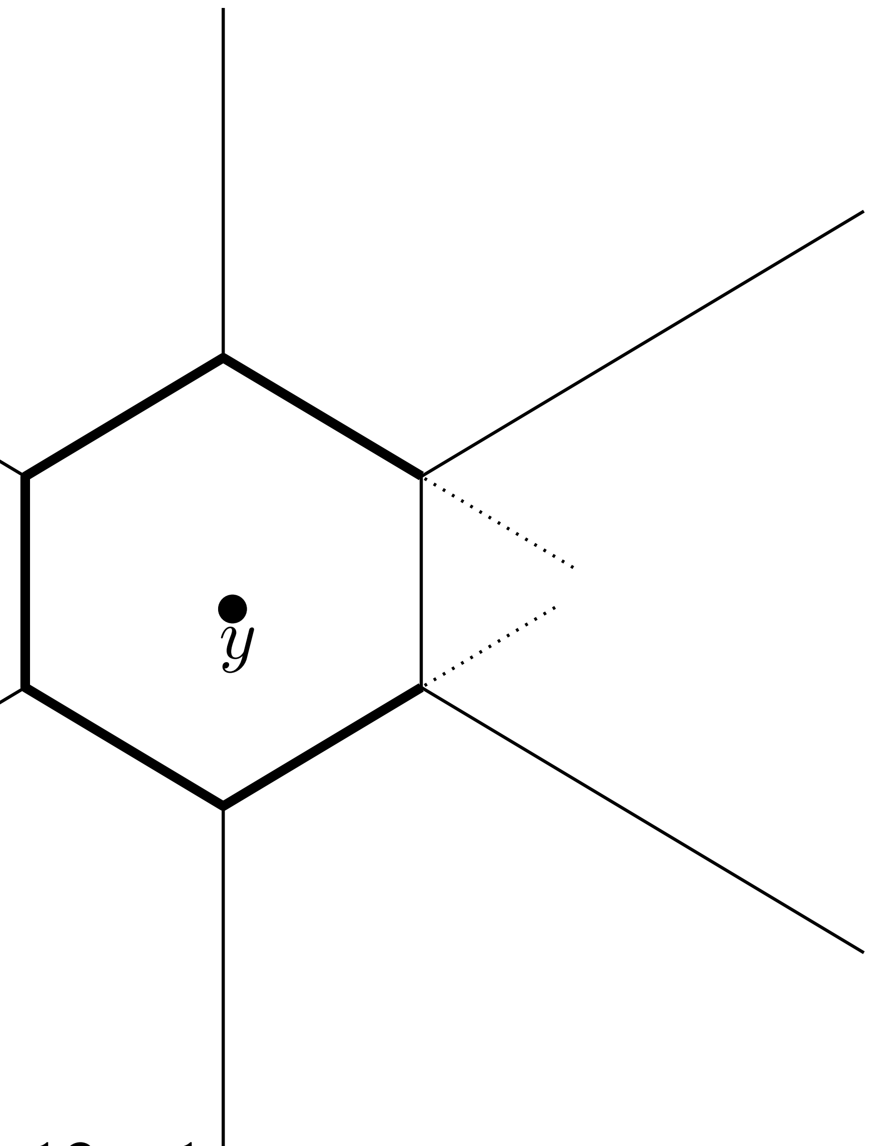
(Strict concavity): $\forall \psi \in E_\varepsilon, \ \mathrm{D}^2\mathcal{K}(\psi) = DG(\psi)$ is neg. definite on $E_\varepsilon \cap \{cst\}^\perp$

(Smoothness) Computation of $\mathrm{D}^2\mathcal{K} = \mathrm{D}G$

Def: $G : \mathbb{R}^Y \to \mathbb{R}^Y, \quad G_y(\psi) = \rho(\mathrm{Lag}_\psi(y))$.



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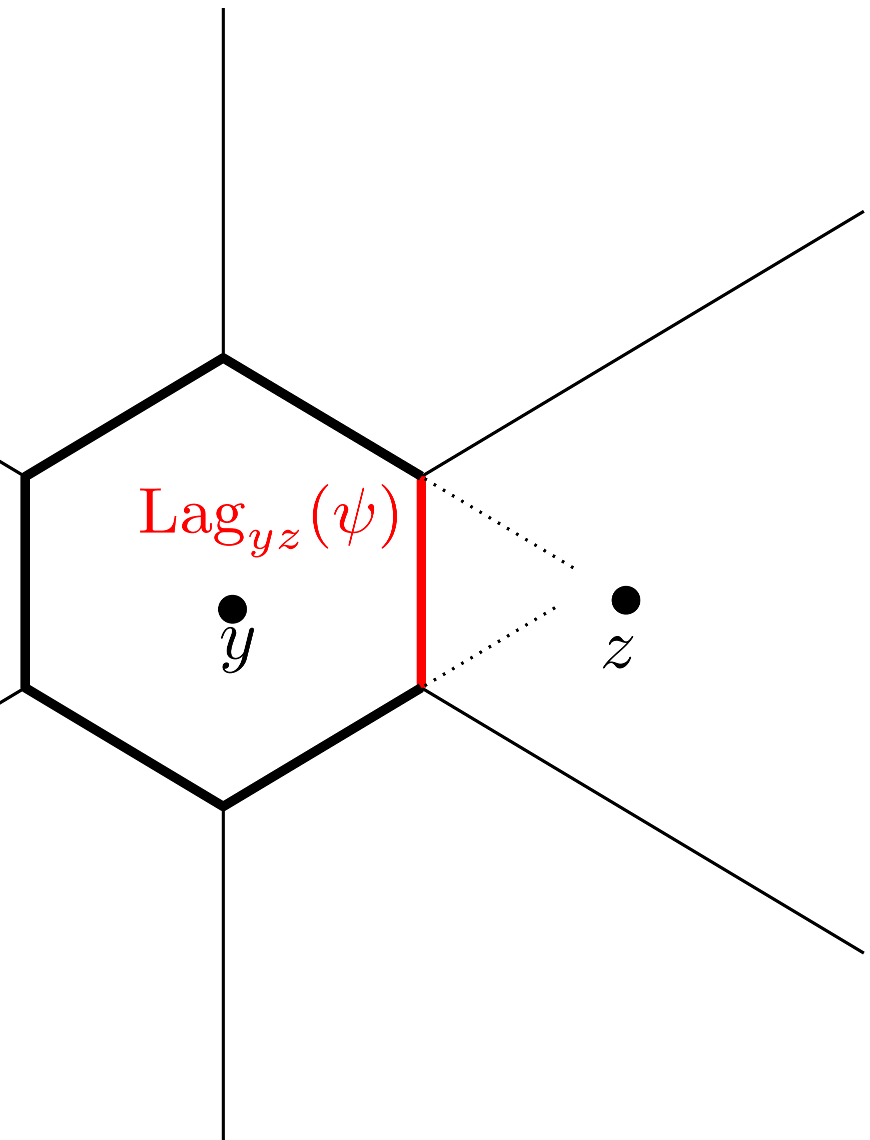

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$d-1$-dimensional measure

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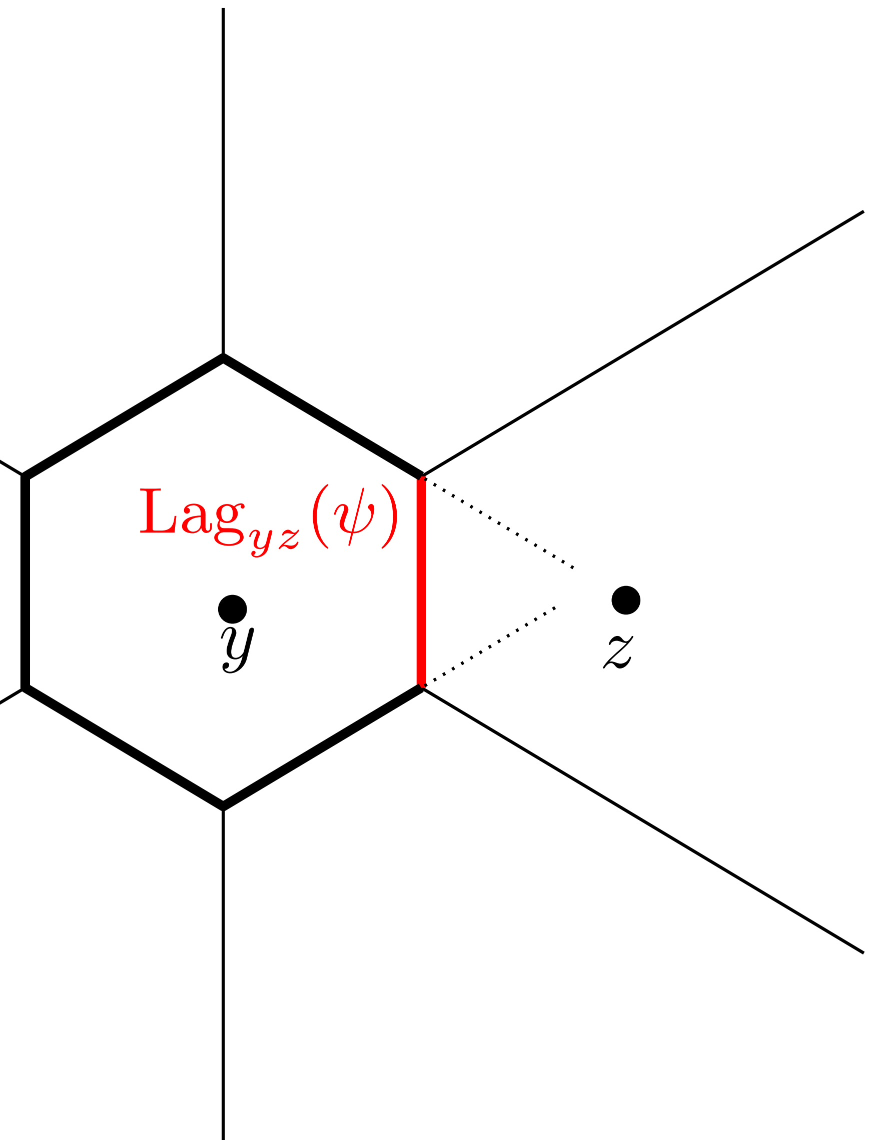

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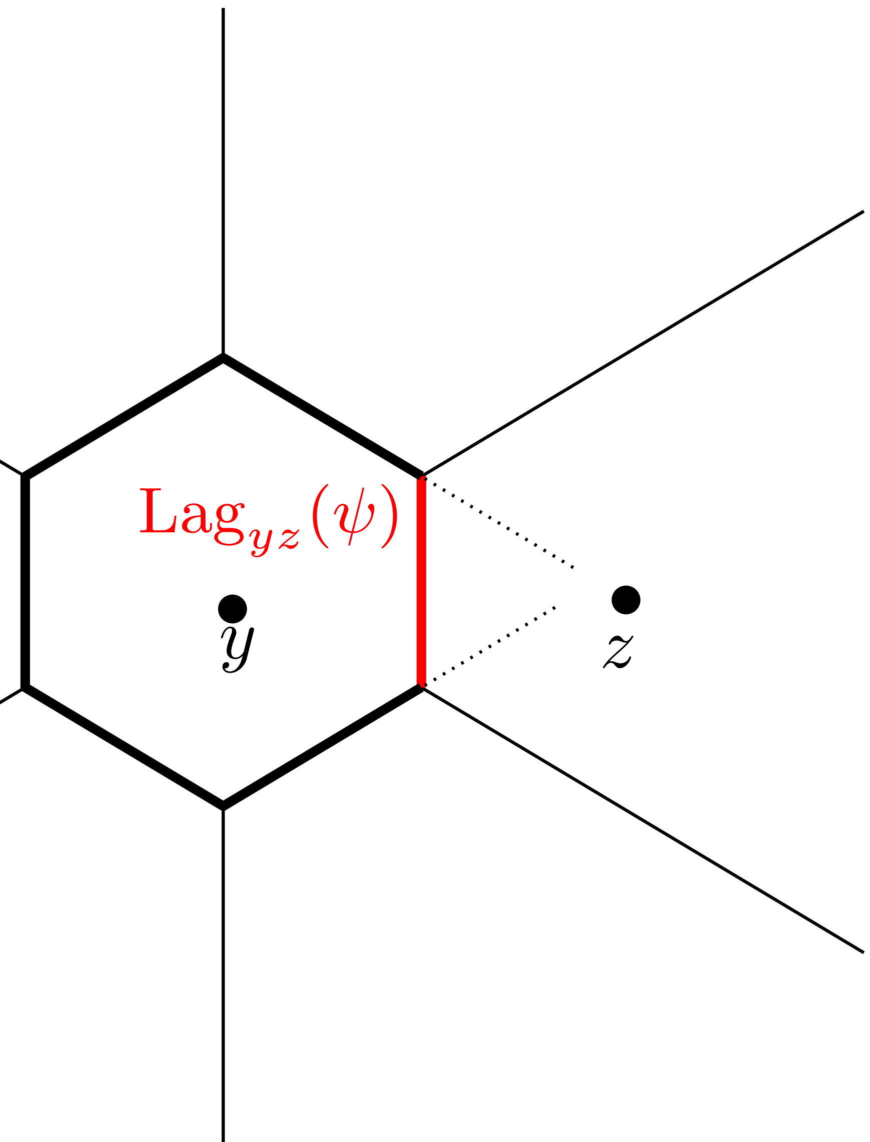

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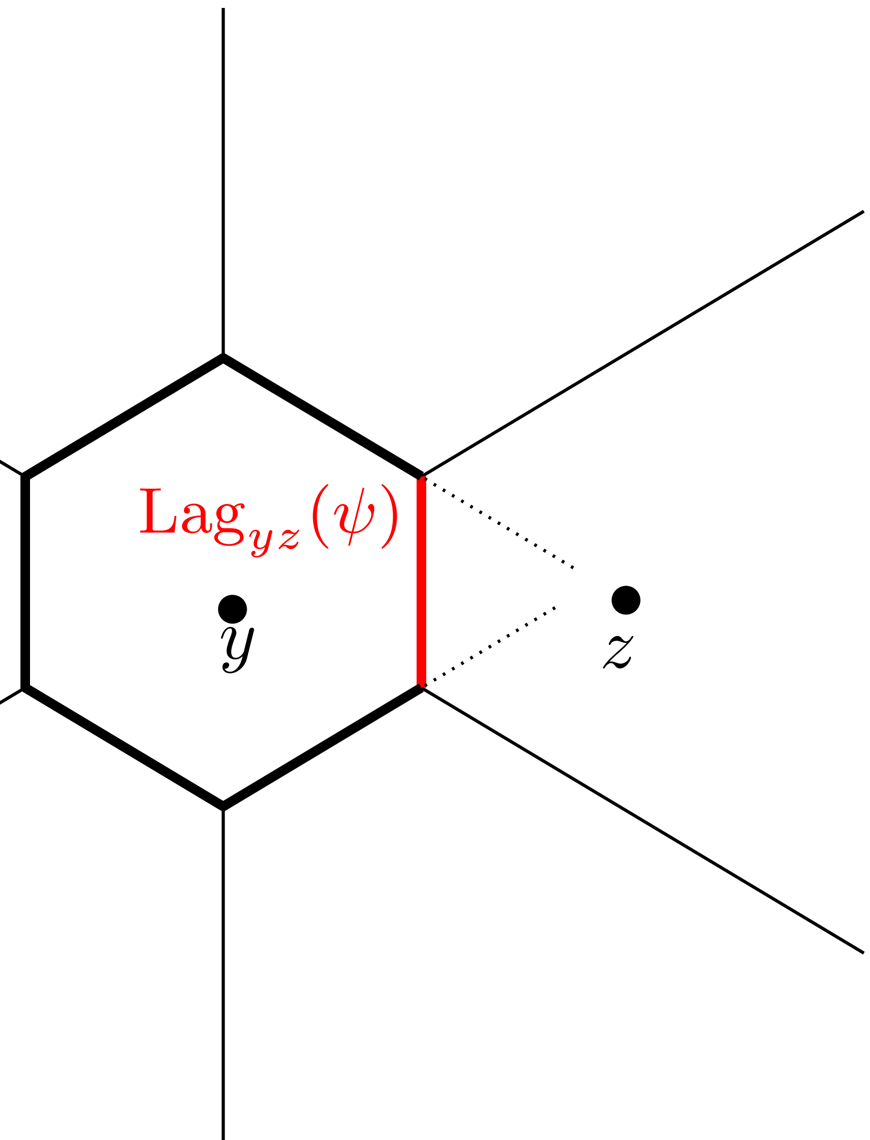

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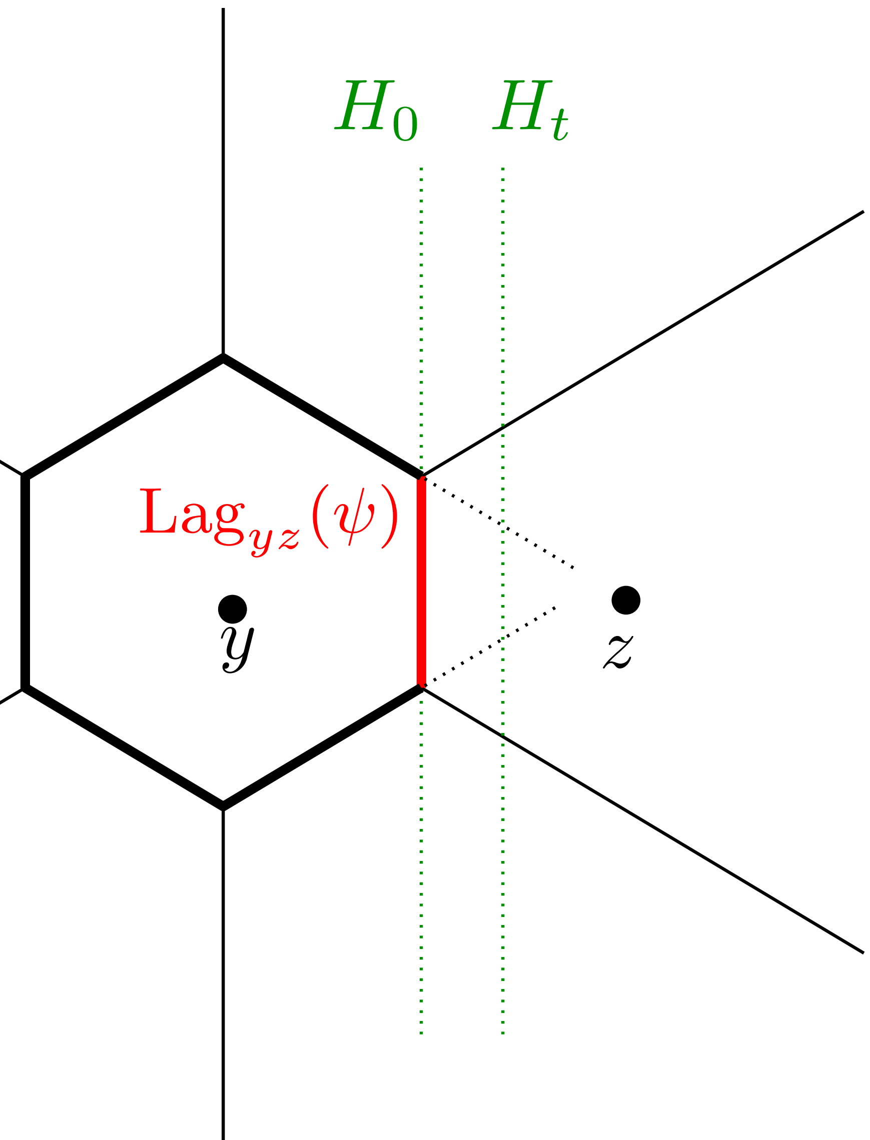

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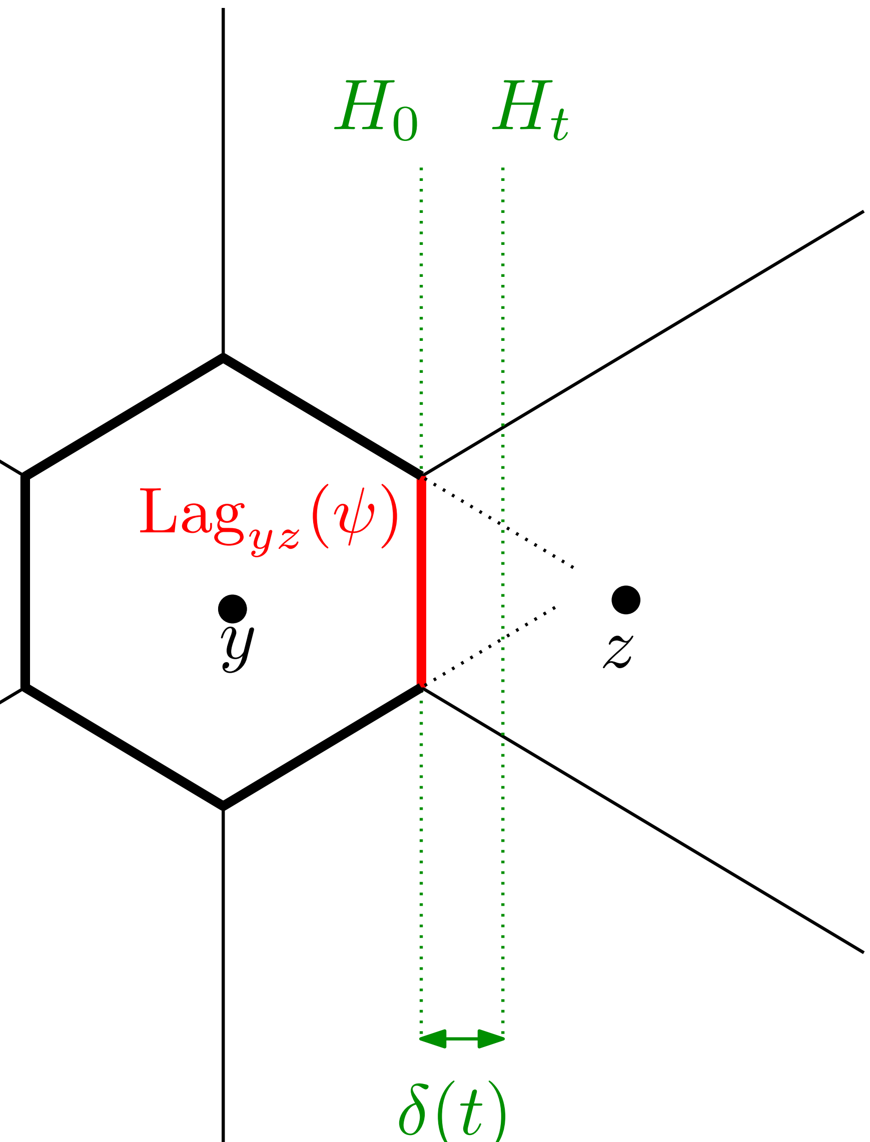

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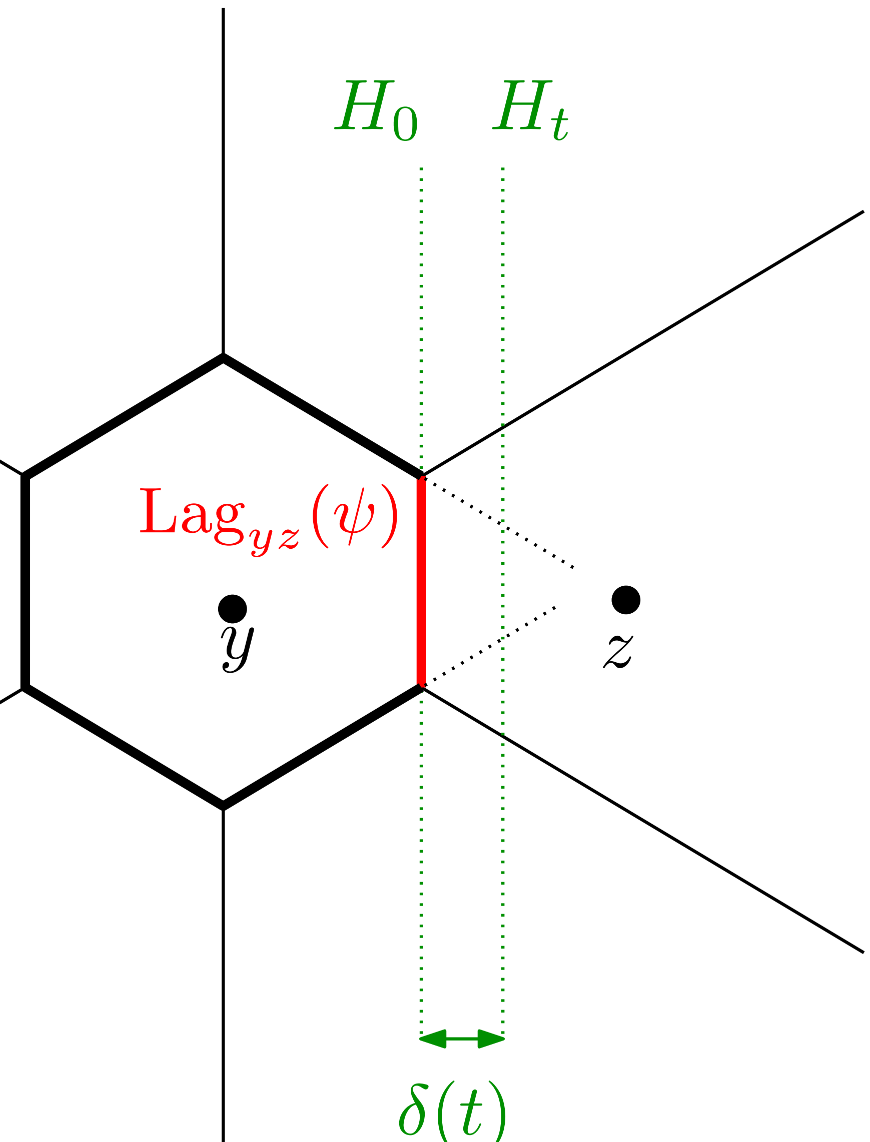

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$\longrightarrow$ Proof of (B): Relies on $G_y(\psi + cst) = G_y(\psi)$.

non-(Smoothness) of Kantorovich's functional

recall: $G_y(\psi) := \rho(\mathrm{Lag}_\psi(y))$

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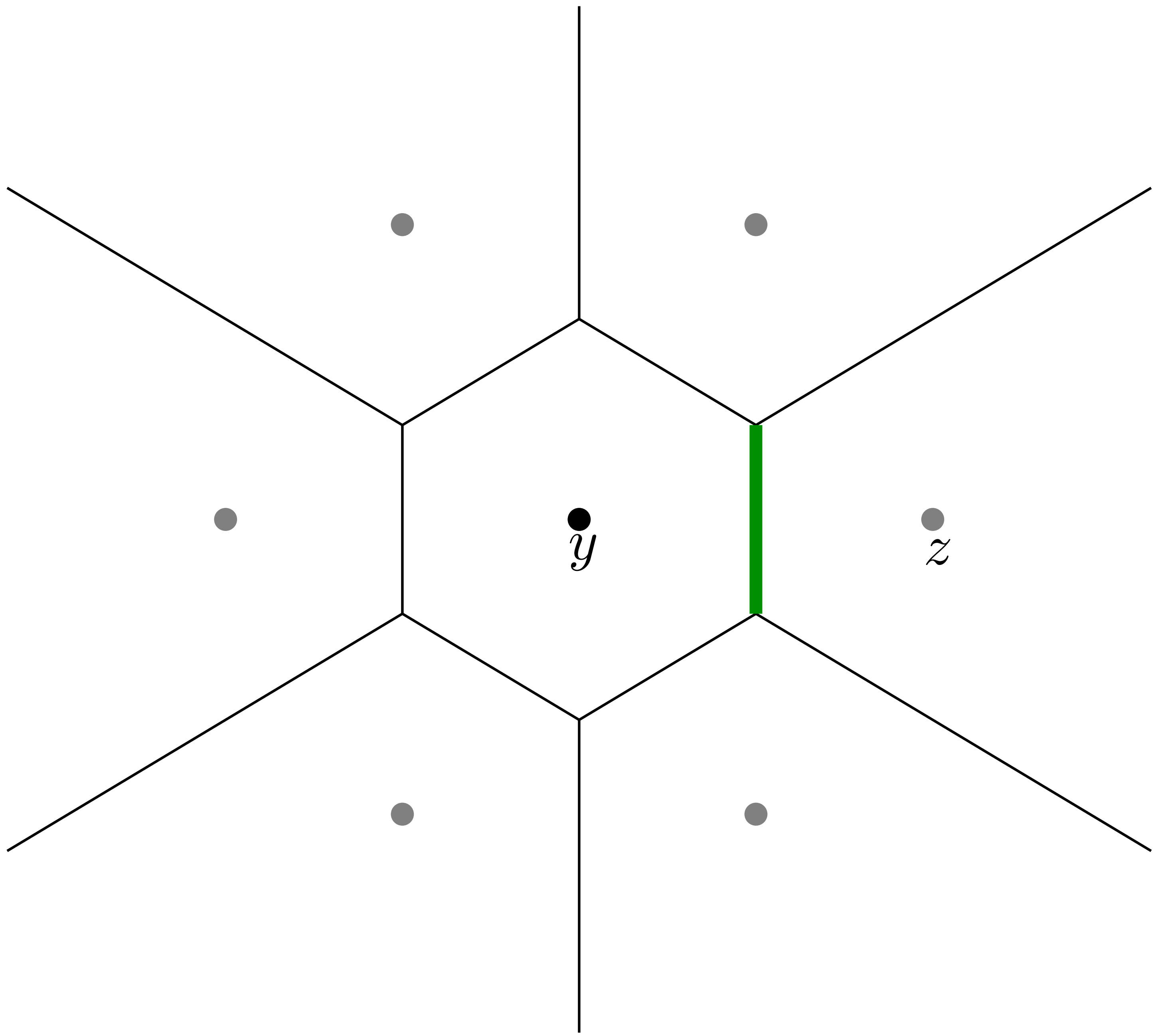

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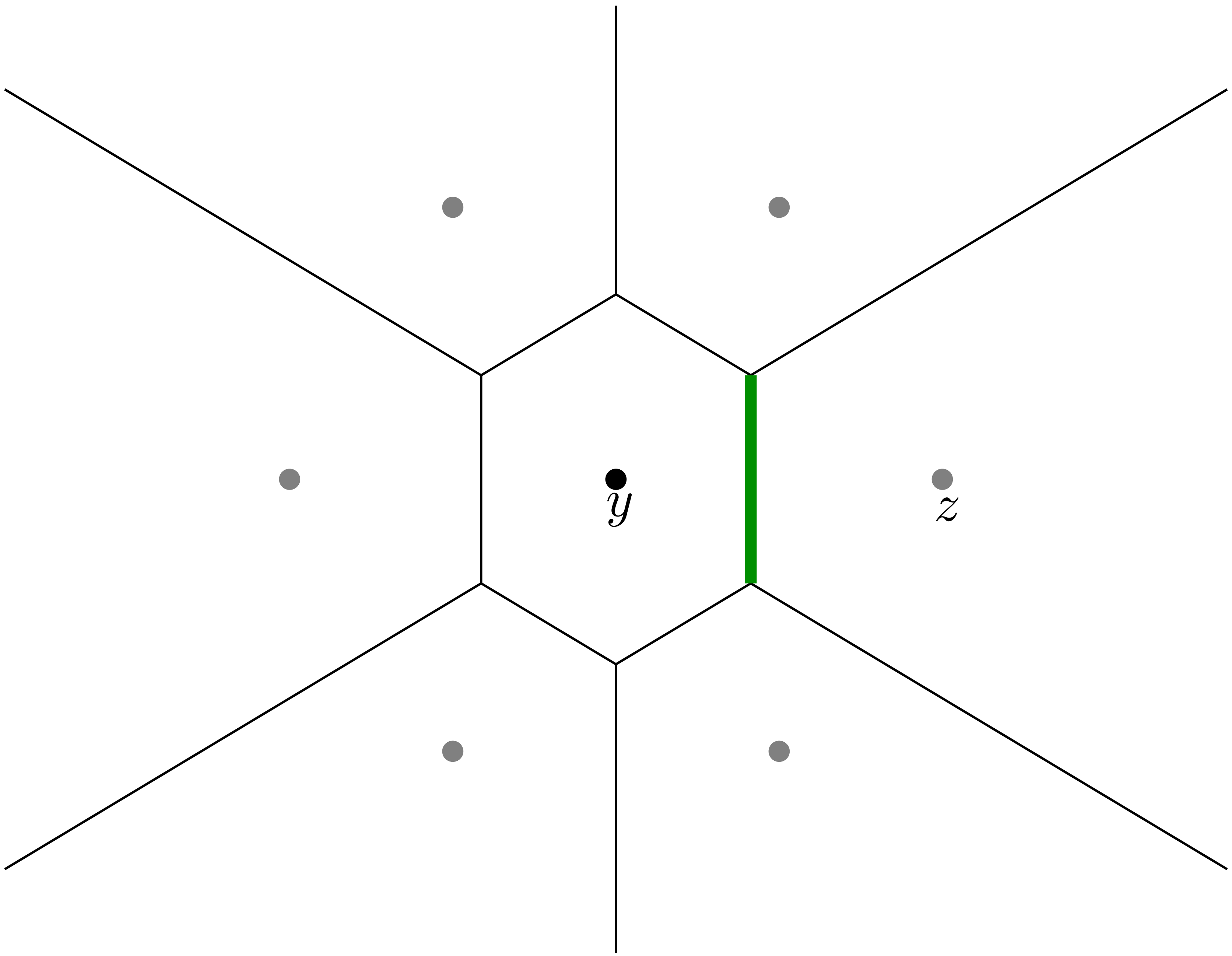

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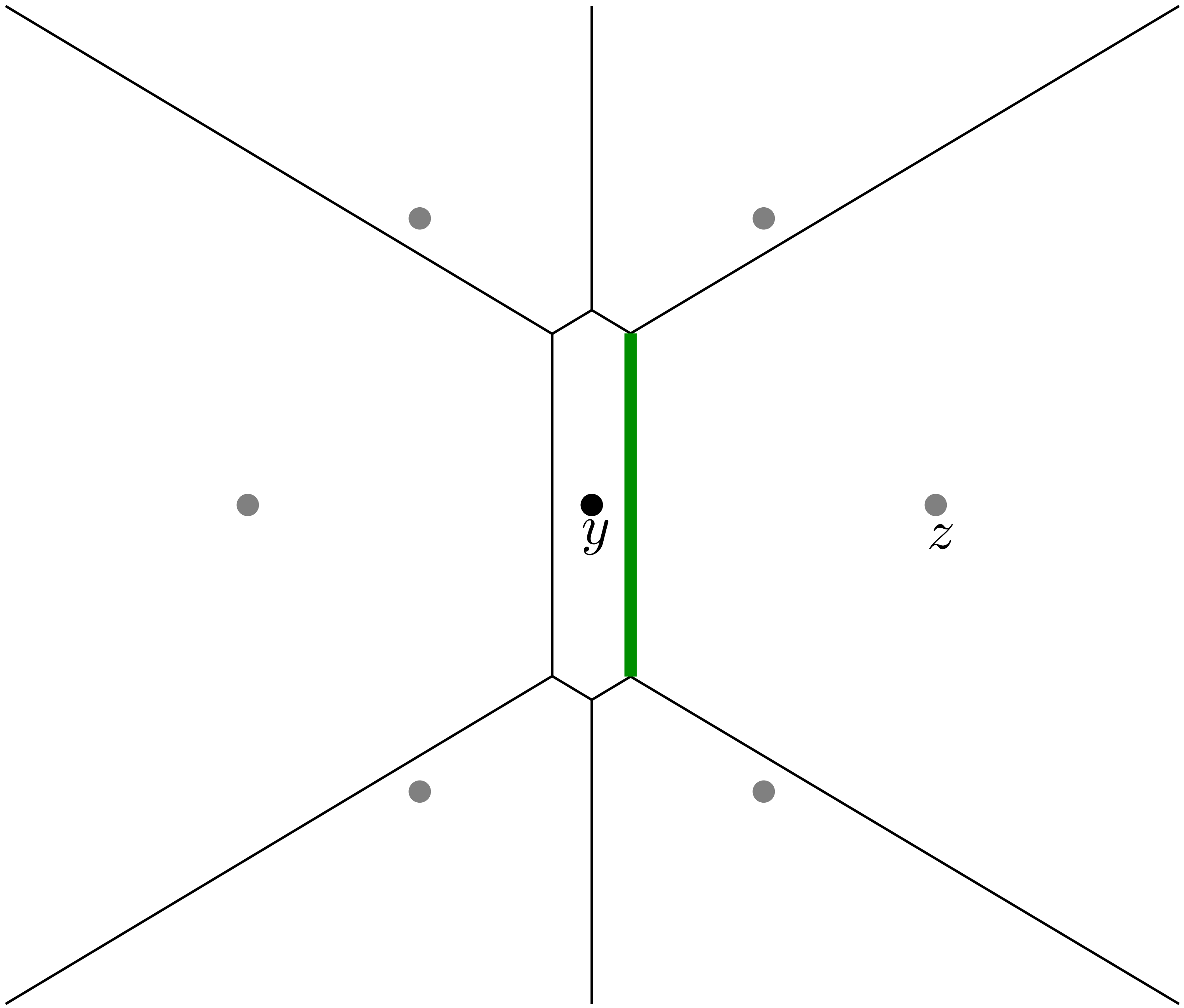

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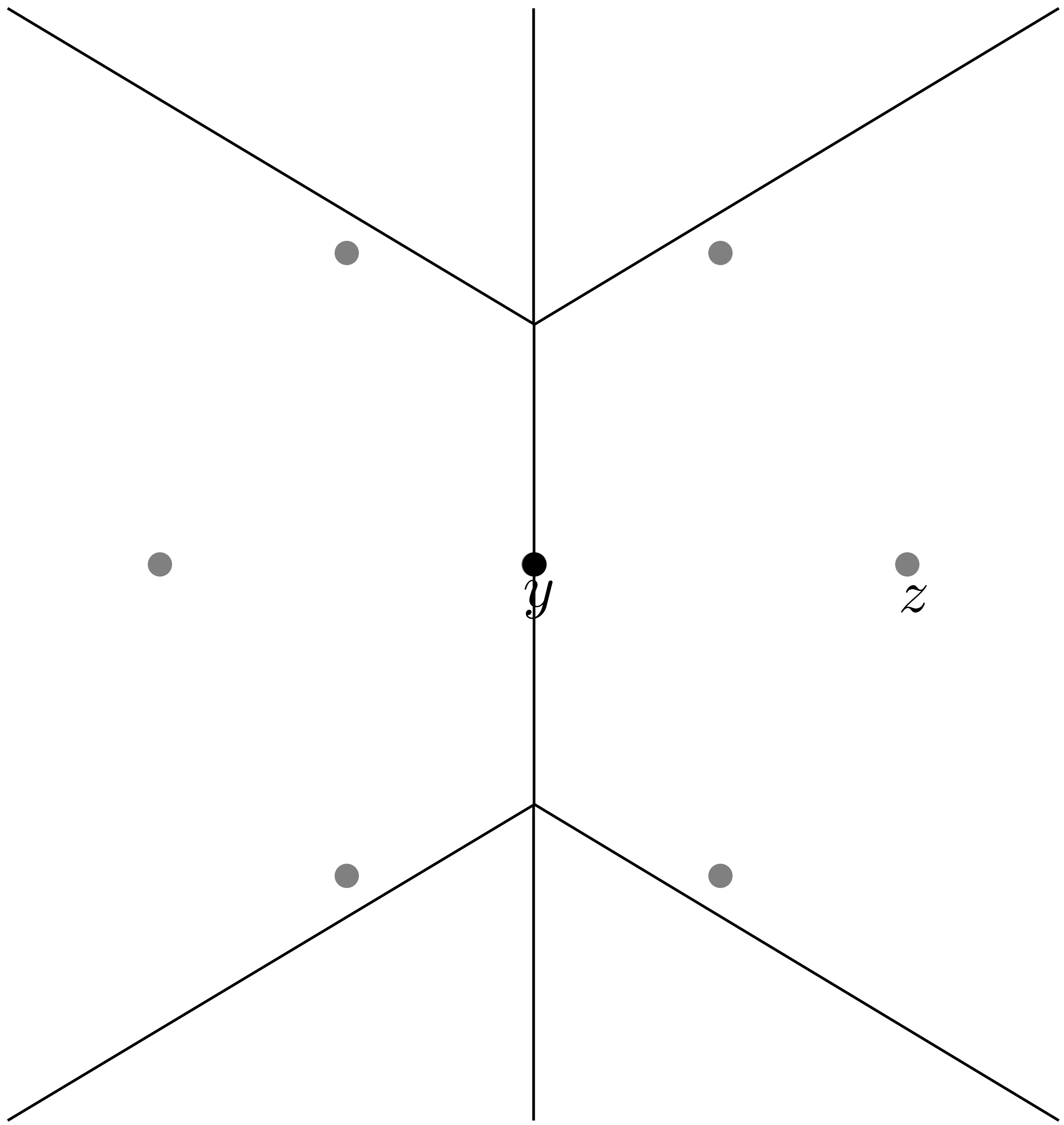

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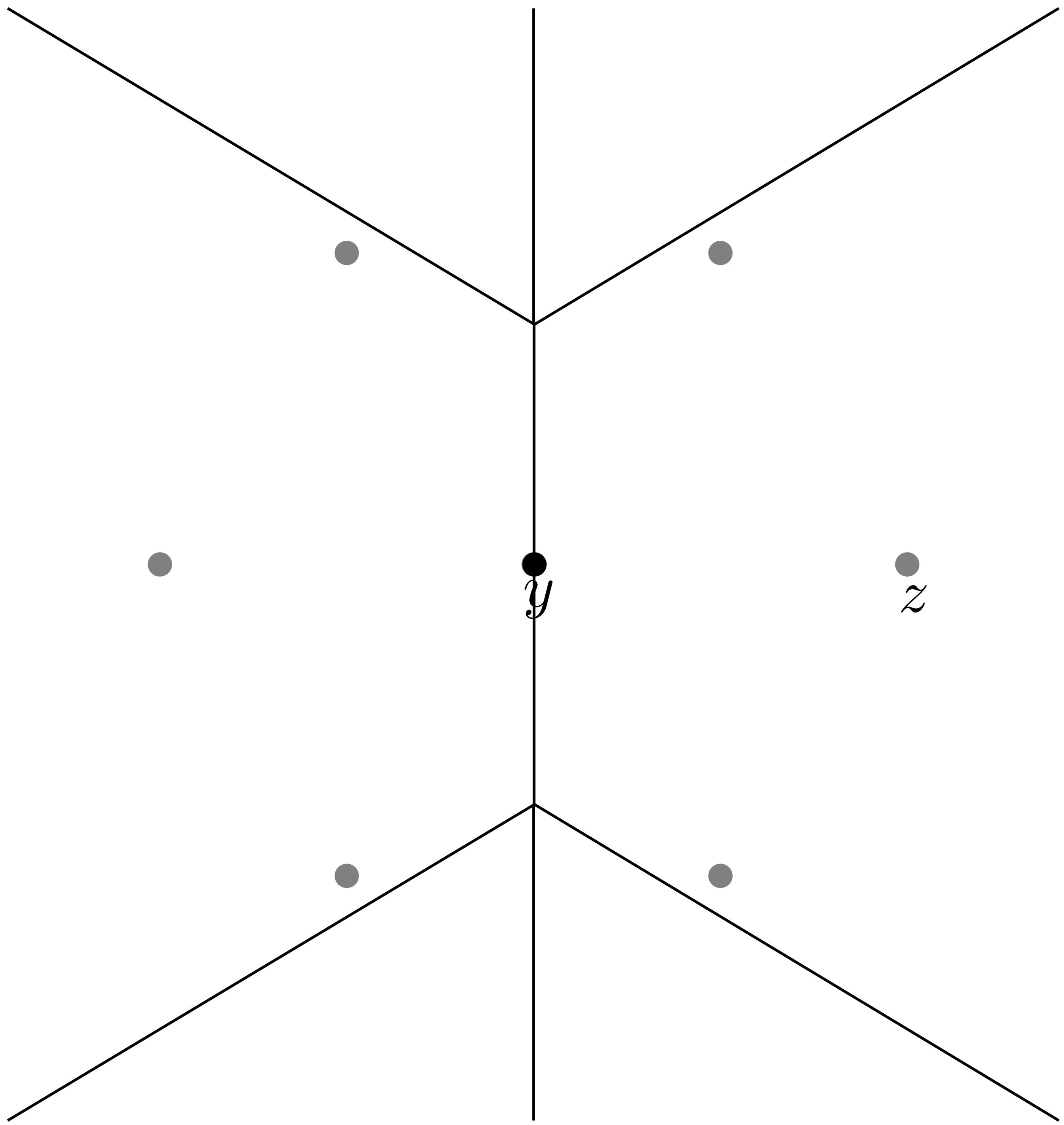

$\frac{\partial G_y}{\partial z}(\psi_t)$ increases ... and then suddenly vanishes.

$\Longrightarrow$ **we require** $\rho(\mathrm{Lag}_\psi(y)) > 0$ **at all times**

(Strong concavity) of Kantorovich's functional

Def: $G : Y^{\mathbb{R}} \to Y^{\mathbb{R}}, \quad G_y(\psi) = \rho(\mathrm{Lag}_\psi(y)).$

Recall: $\frac{\partial G_y}{\partial z}(\psi) = \frac{1}{2\|y-z\|} \int_{\boxed{\mathrm{Lag}_{yz}(\psi)}} \rho(x)\,\mathrm{d}\,x \qquad \frac{\partial G_y}{\partial y}(\psi) = -\sum_{z \neq y} \frac{\partial G_y}{\partial z}(\psi)$

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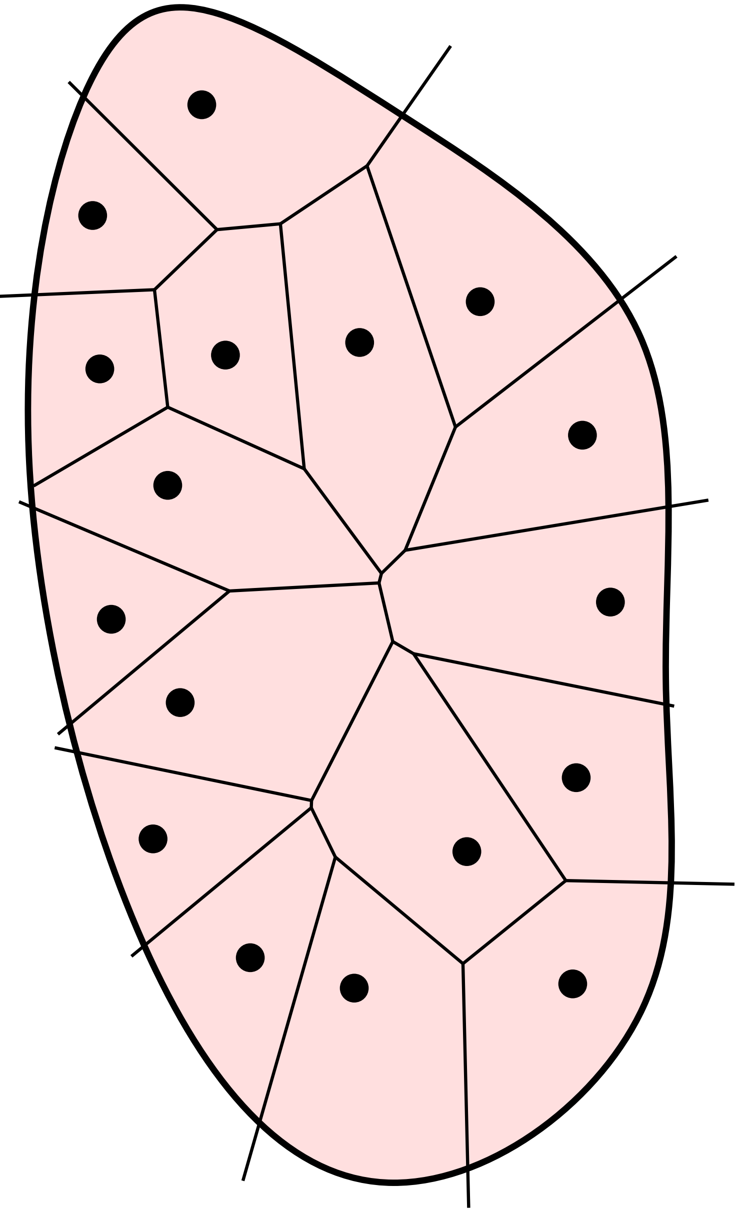

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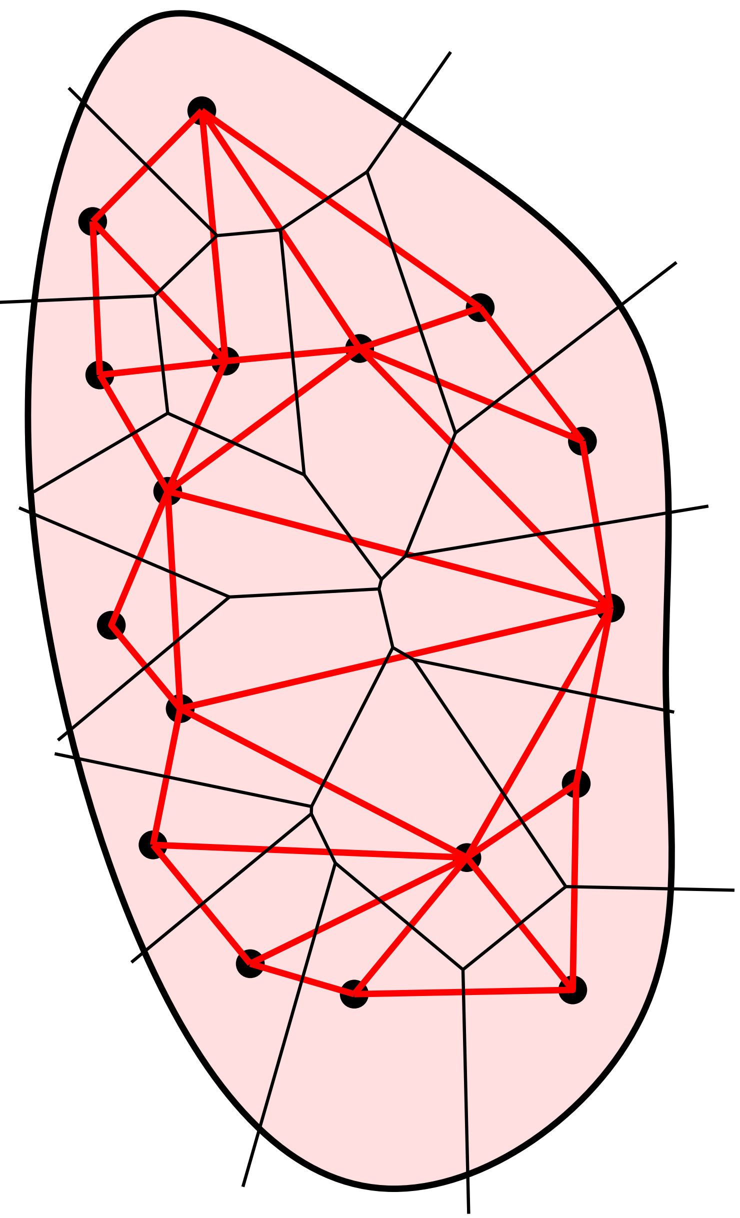

- Consider the matrix $(L_{yz}) := \frac{\partial G_y}{\partial z}(\psi)$ and the graph H:

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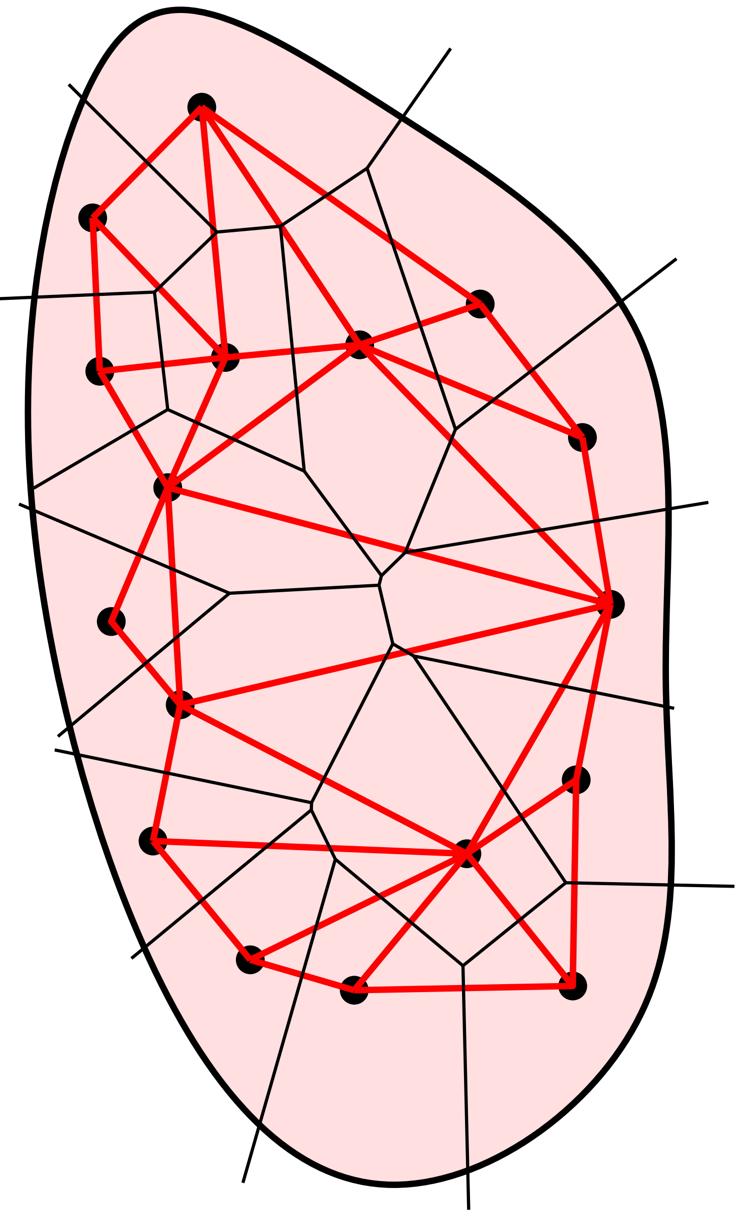

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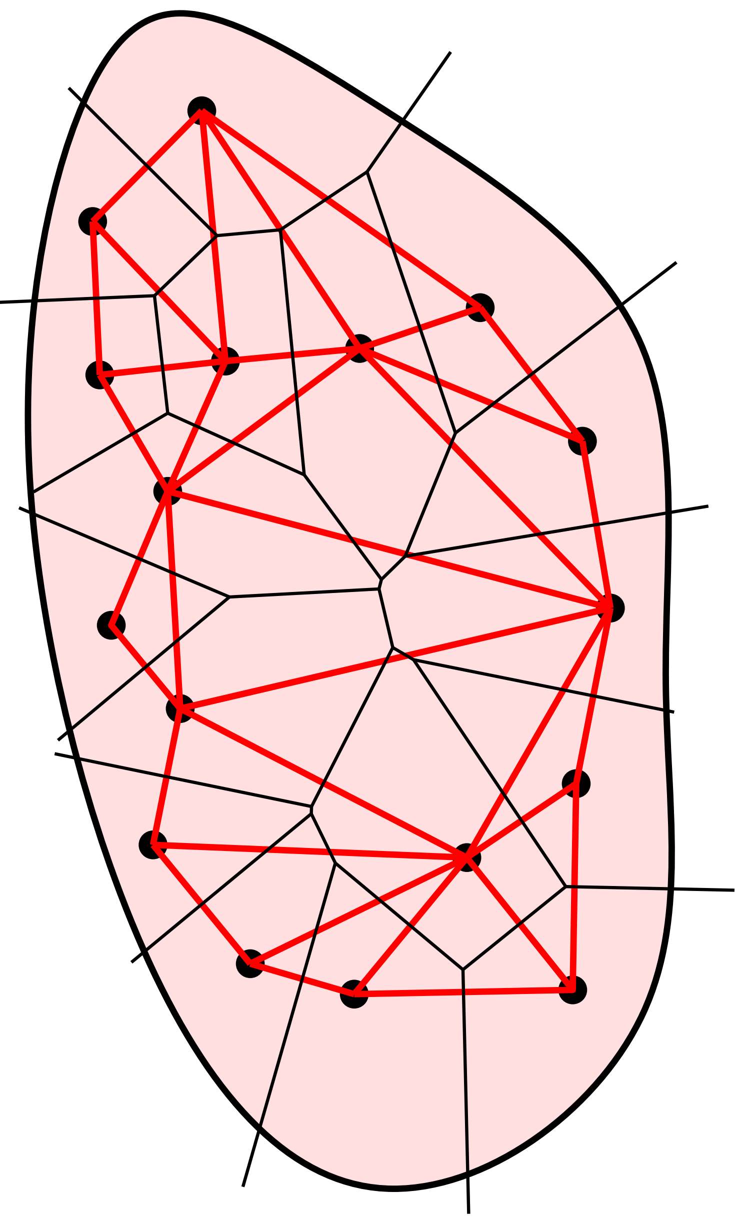

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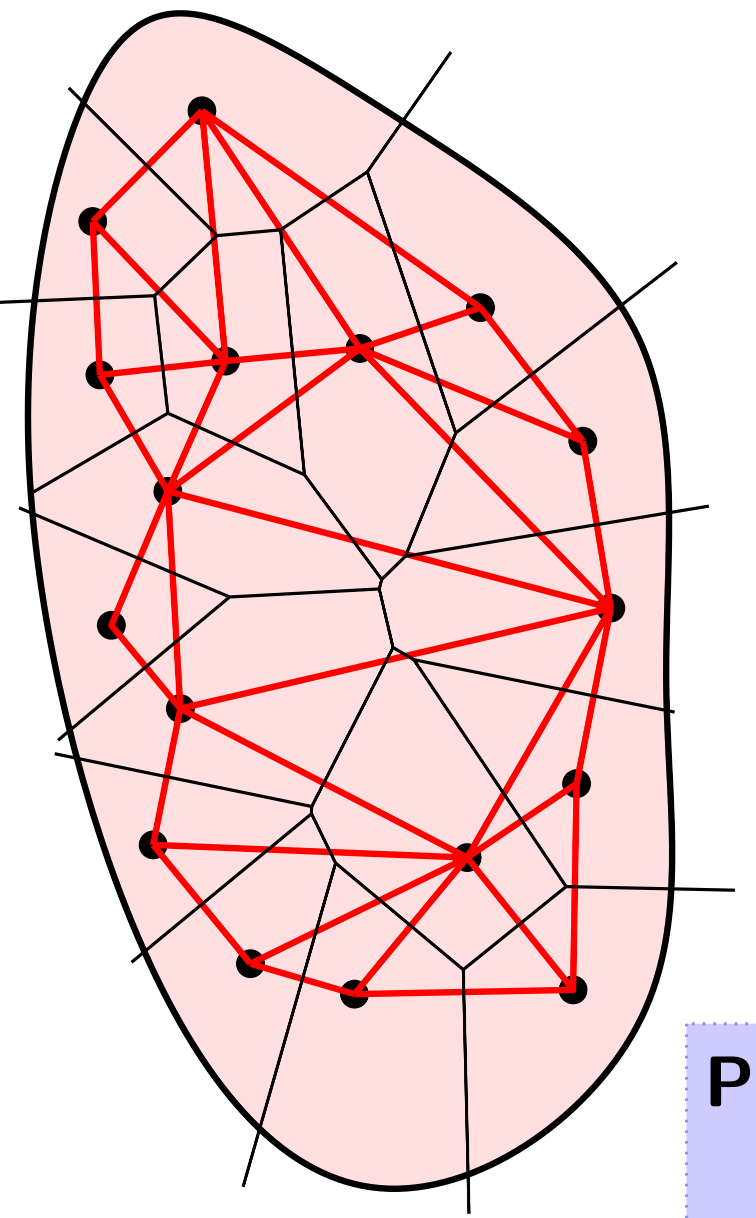

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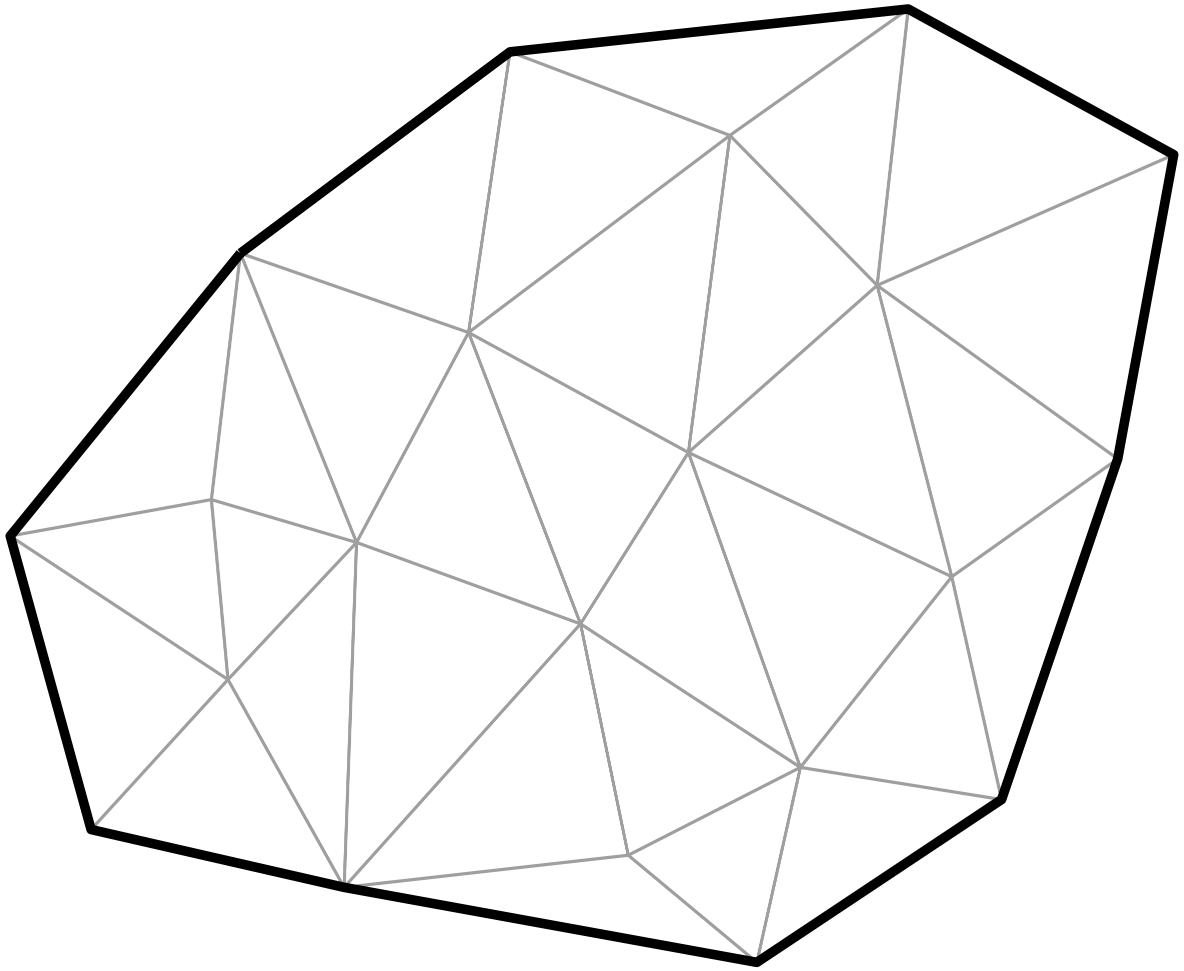
Proposition: Assume $\rho \in \mathcal{C}^0_c(\mathbb{R}^d)$ and $\{\rho > 0\}$ connected. Then, $\forall \psi \in E_\varepsilon, \; \mathrm{D}^2\mathcal{K}(\psi) = DG(\psi)$ is neg. definite on $E_\varepsilon \cap \{cst\}^\perp$

Numerics: implementation details



$\longrightarrow$ Convex decomposition of domain X, ρ is piecewise-constant/linear

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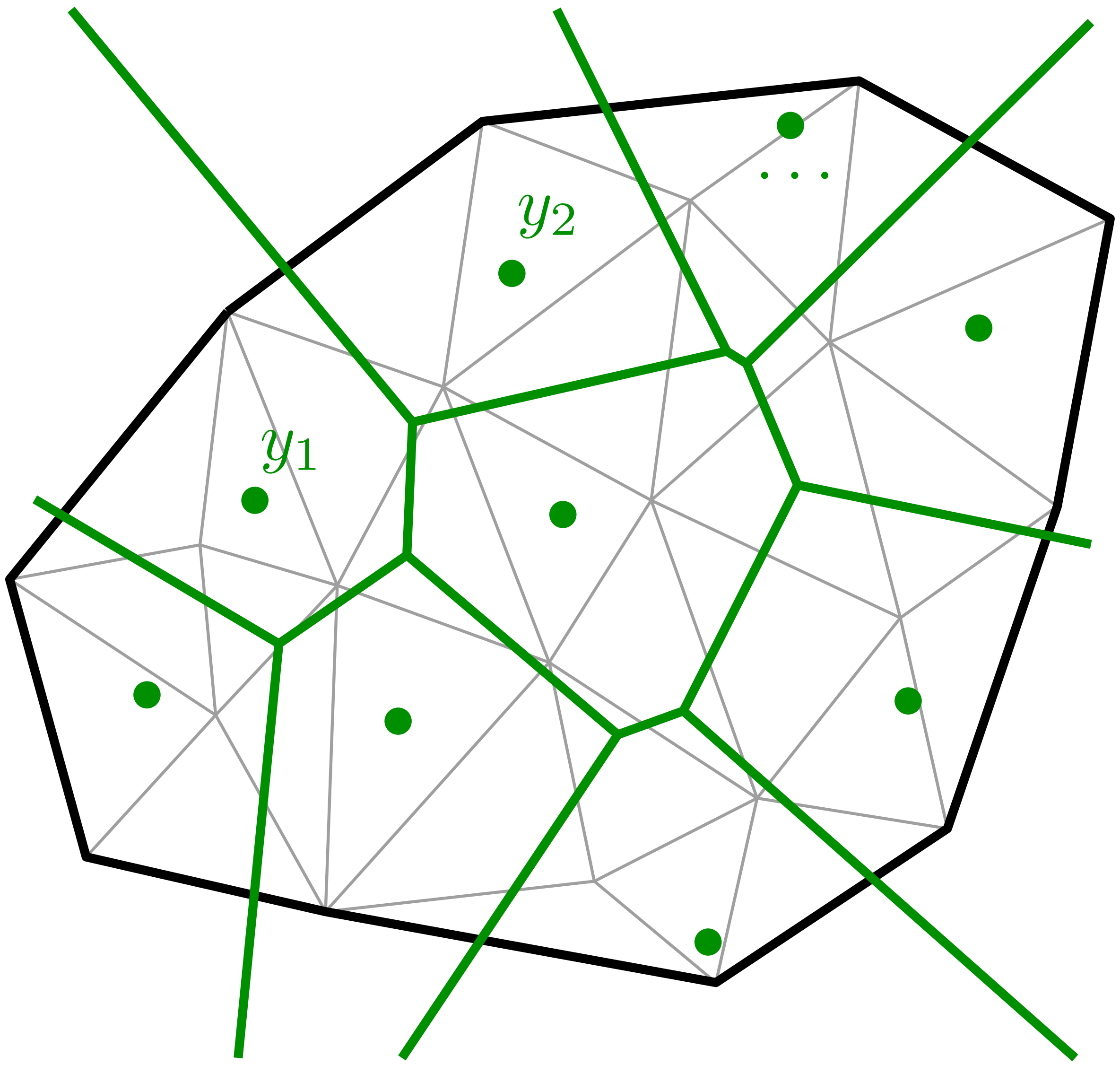

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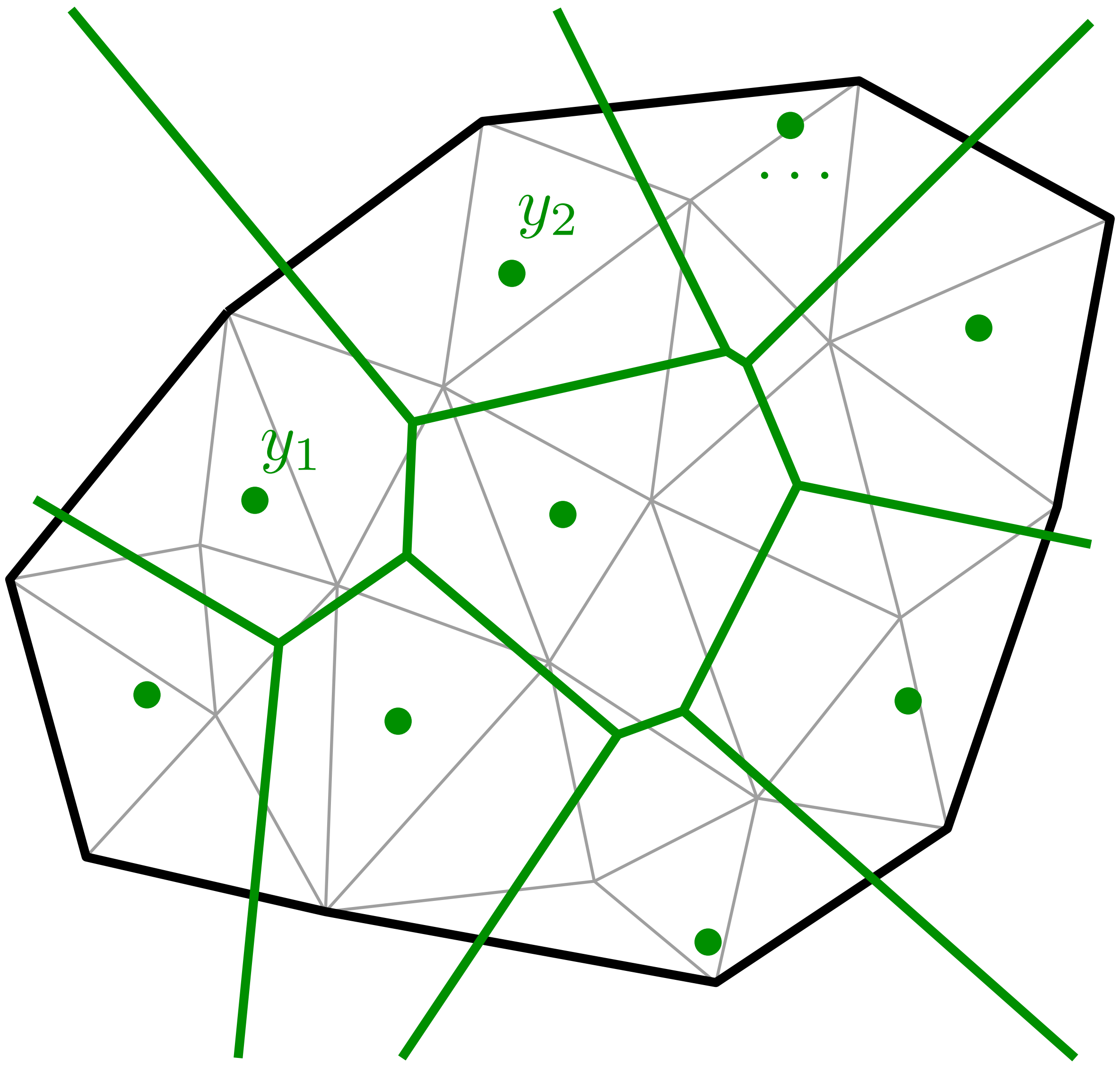

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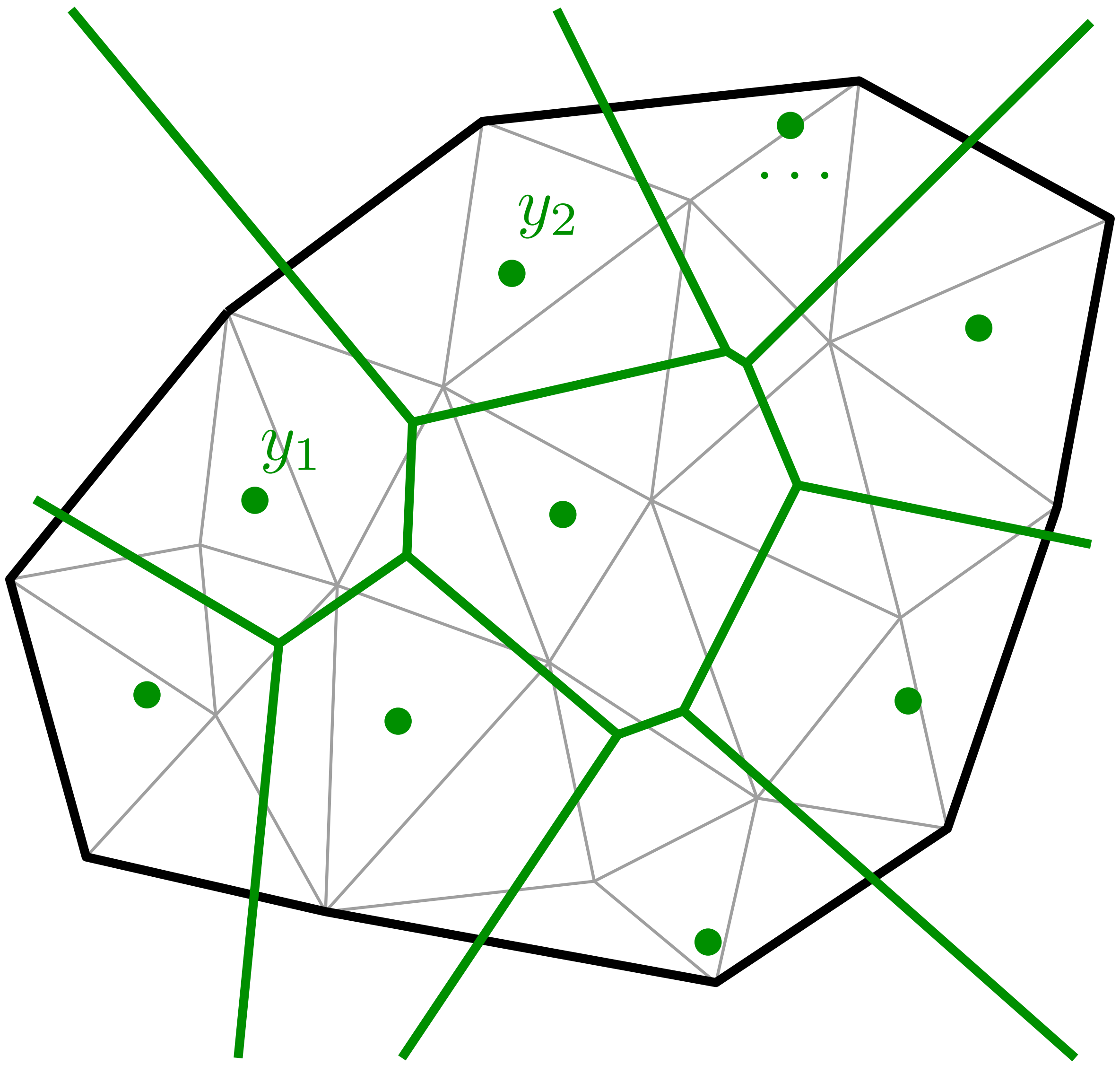

Public implementations:

PyMongeAmpere (2D, Python/C++)
`http://github.com/mrgt/PyMongeAmpere`

Graphite (2D/3D, C++)
`http://alice.loria.fr/software/graphite`

PySDOT (2D/3D, C++)
`https://github.com/sd-ot/pysdot`

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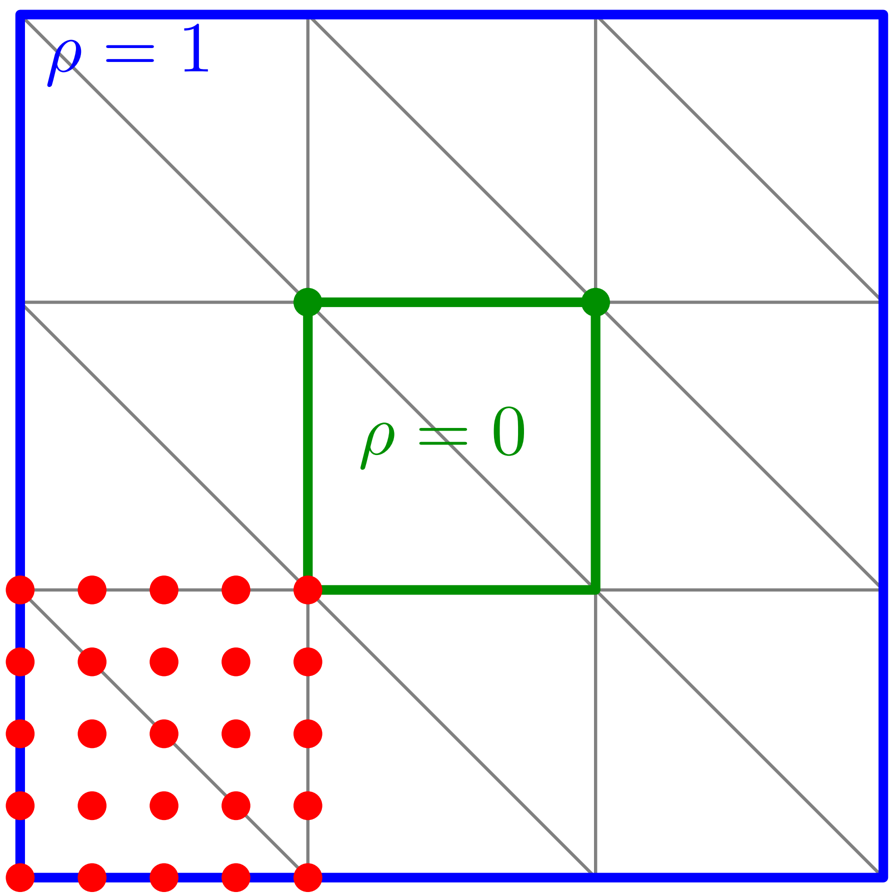
Numerics: vanishing density



Source: PL density on $X = [0, 3]^2$

Target: Uniform grid Y in $[0, 1]^2$.

`Demo`

- The damped Newton's algorithm converges even when ρ vanishes.
- Computational cost seems nearly linear in number of Diracs.

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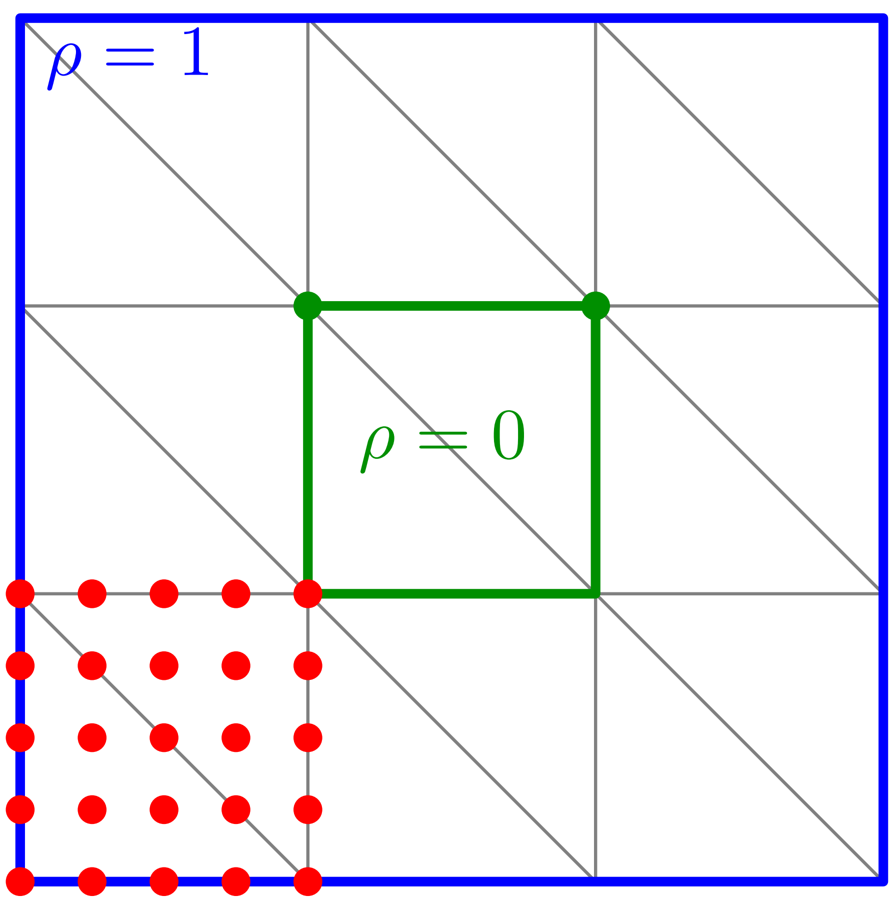

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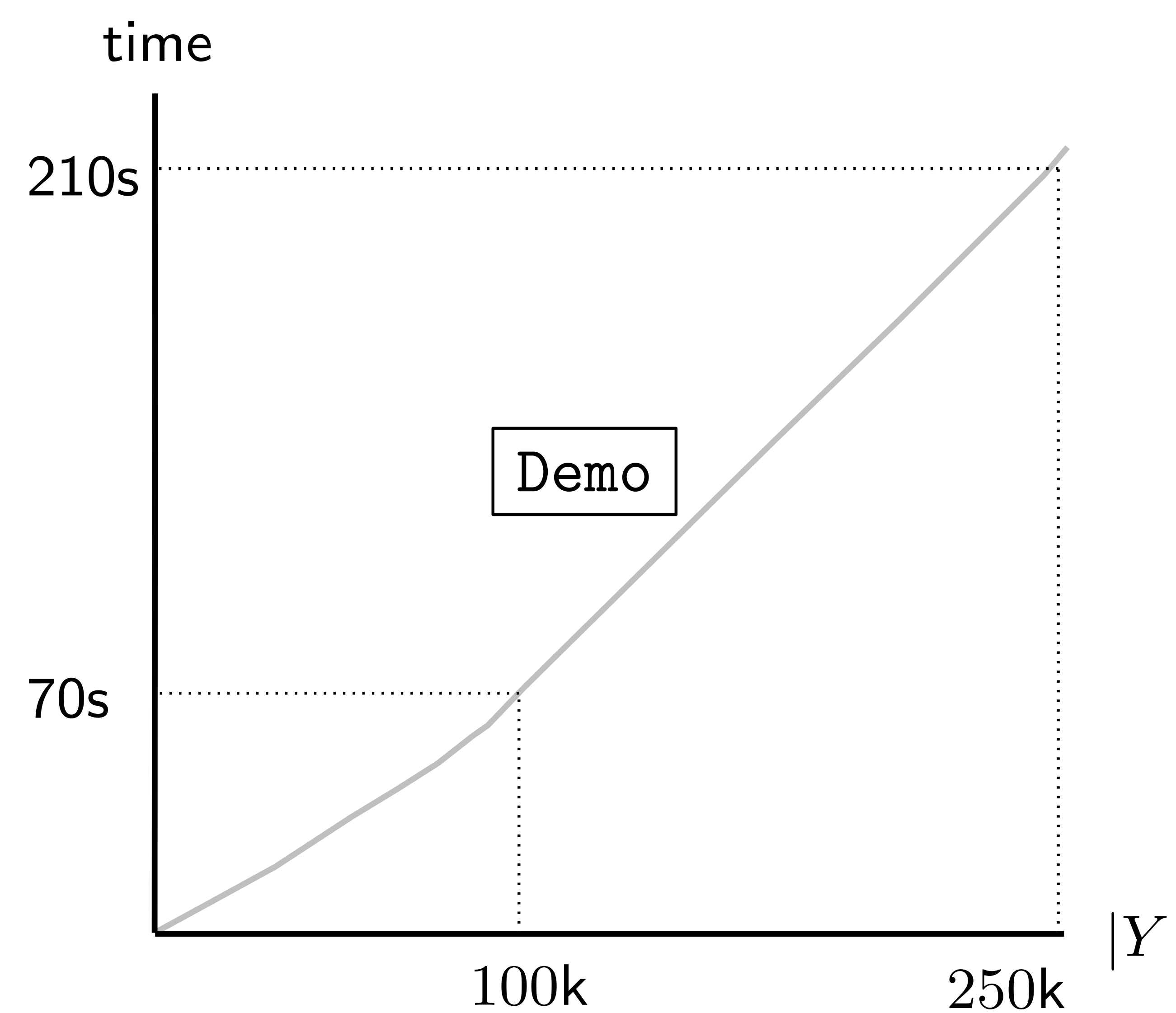

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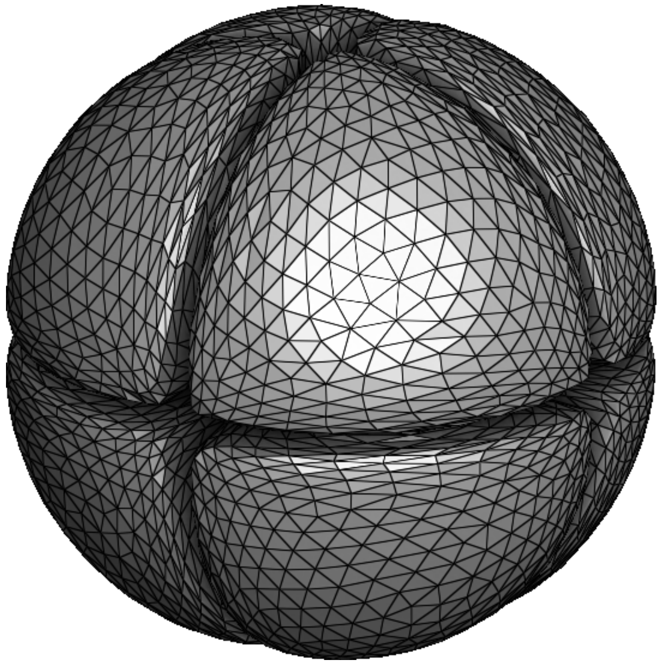
Numerics: mesh interpolation

[Lévy '15]



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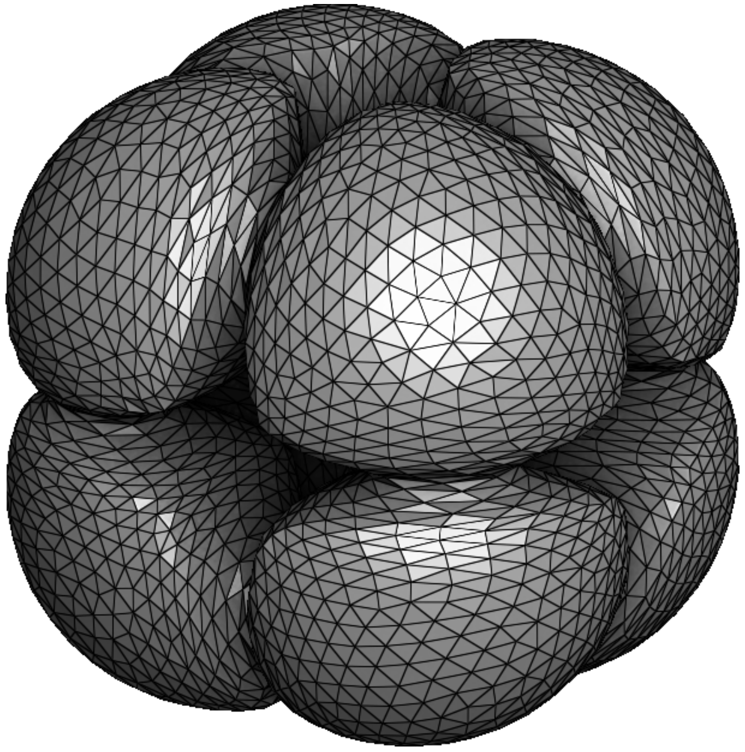

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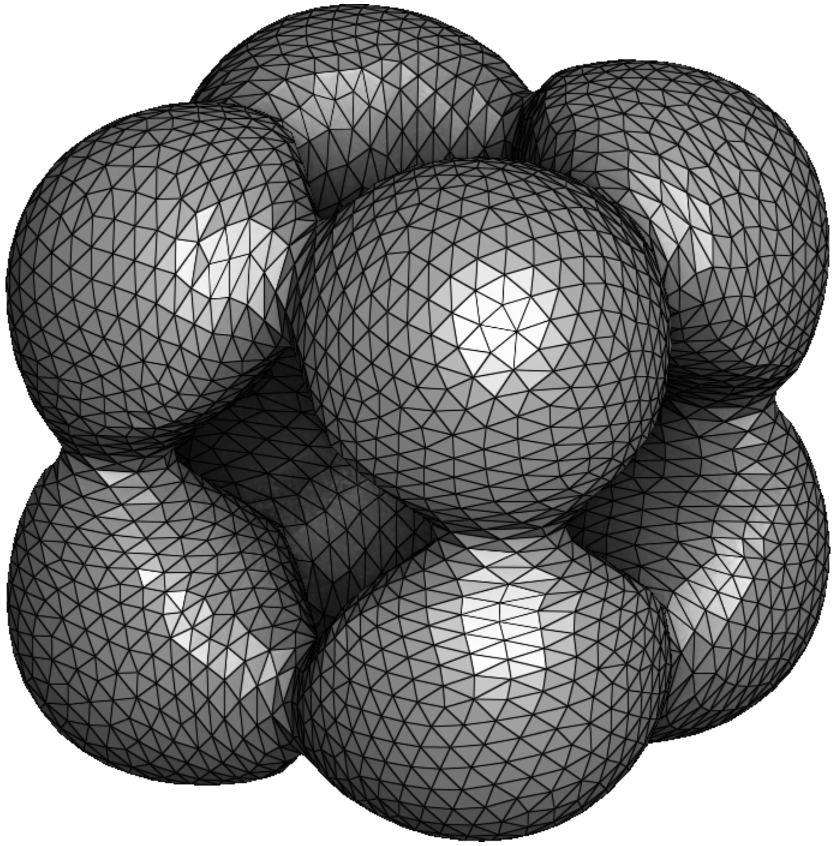

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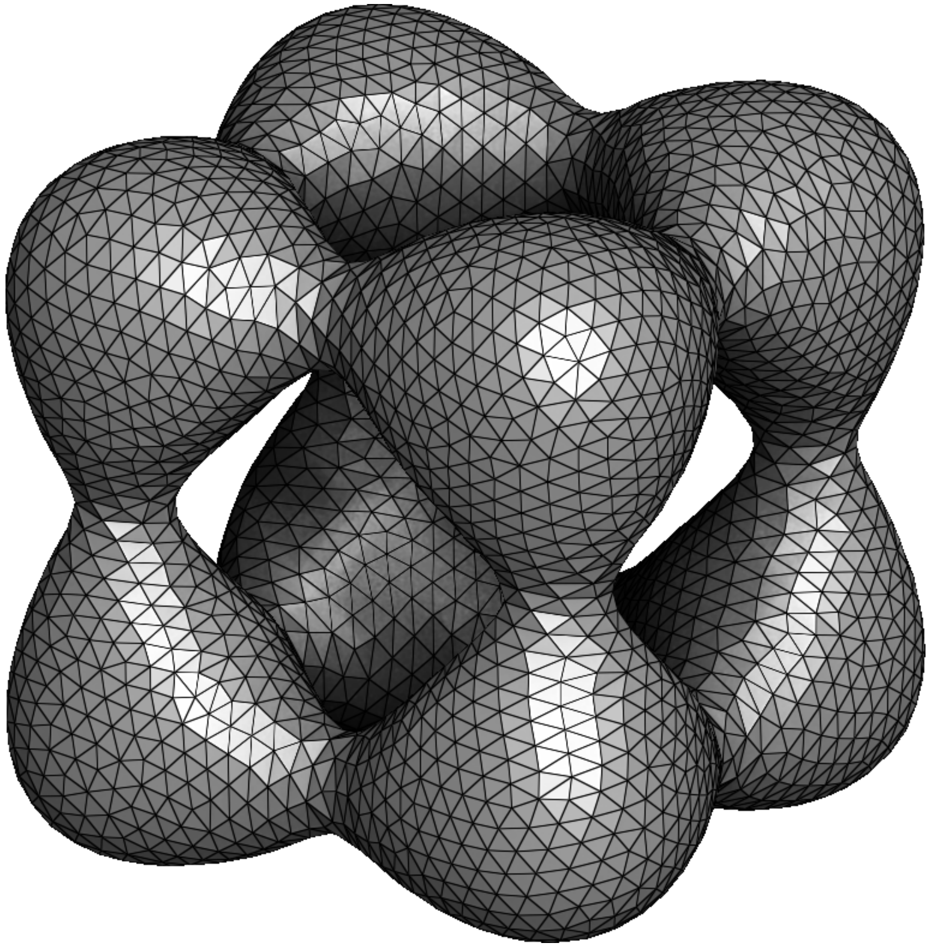

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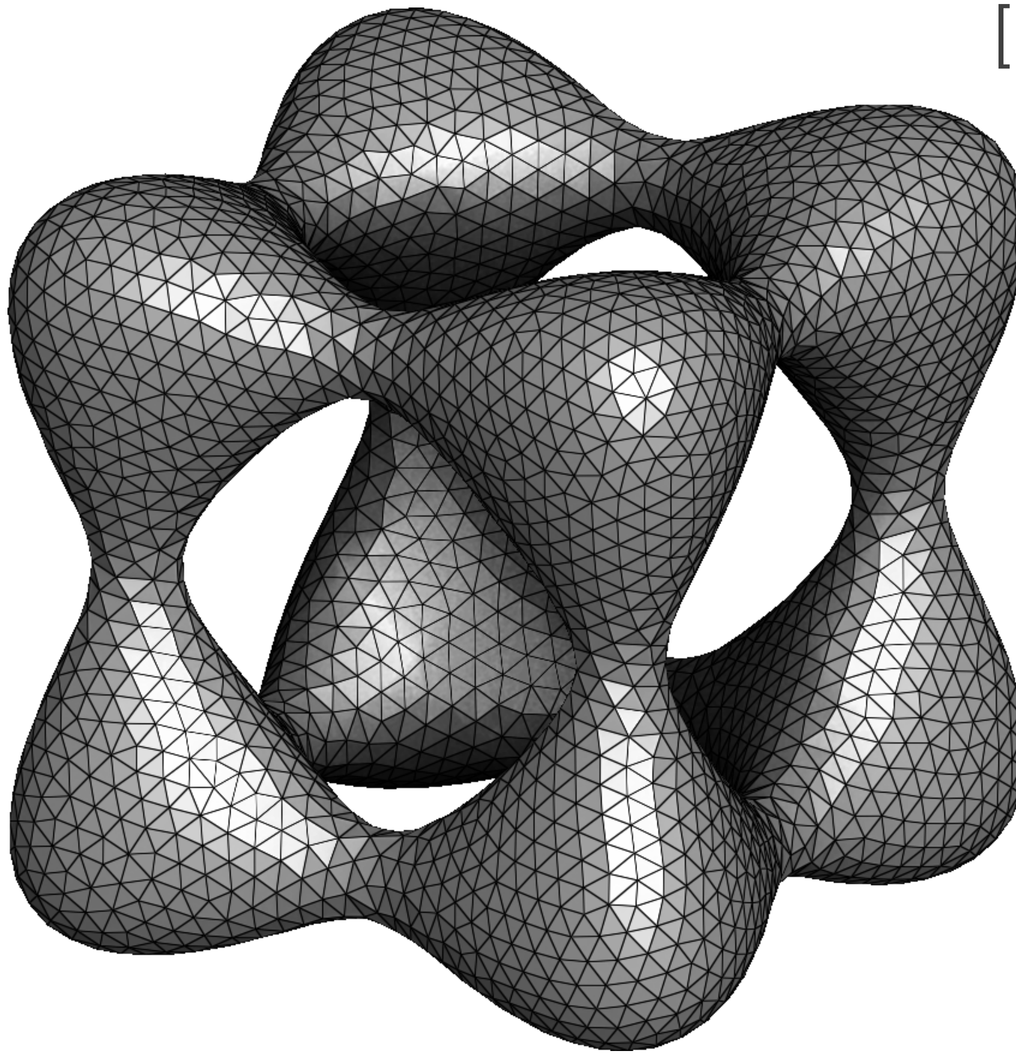

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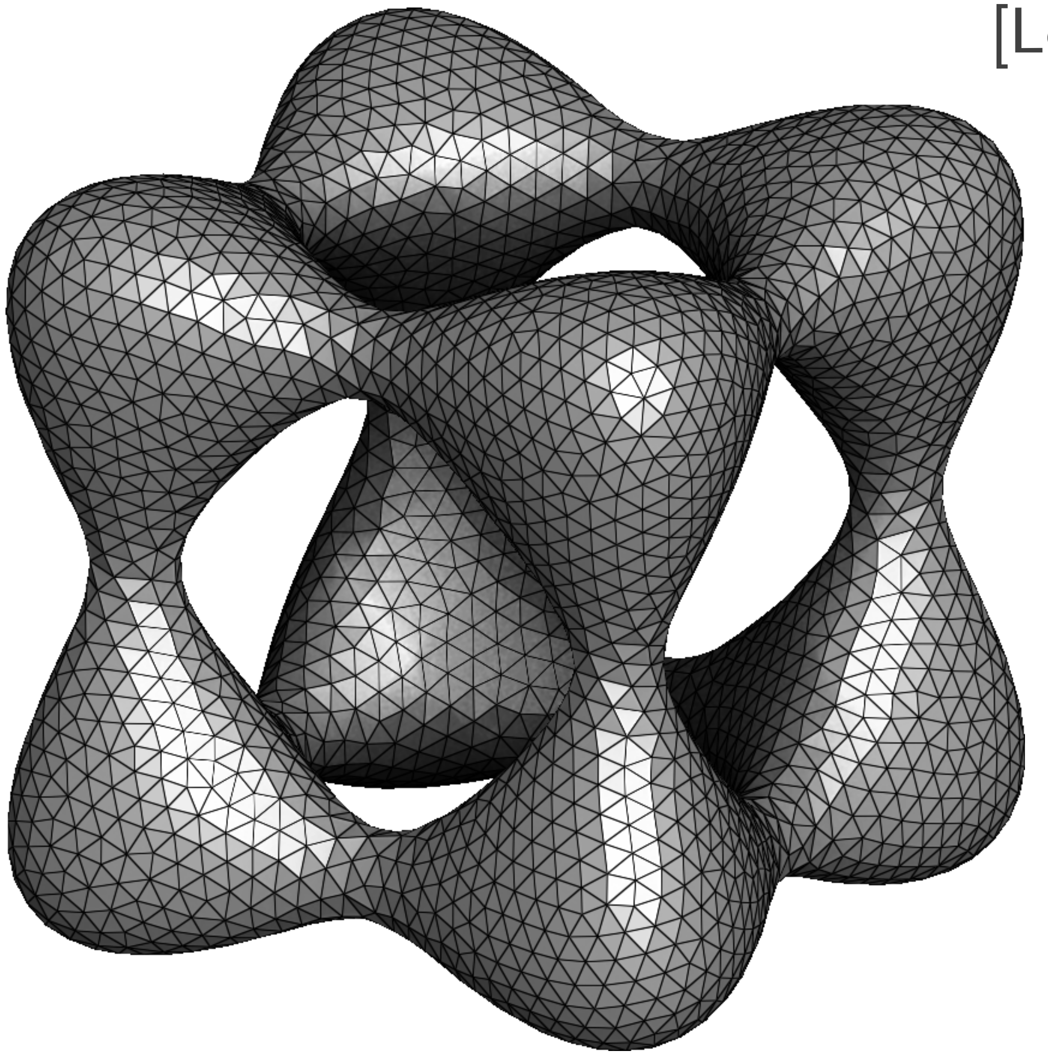
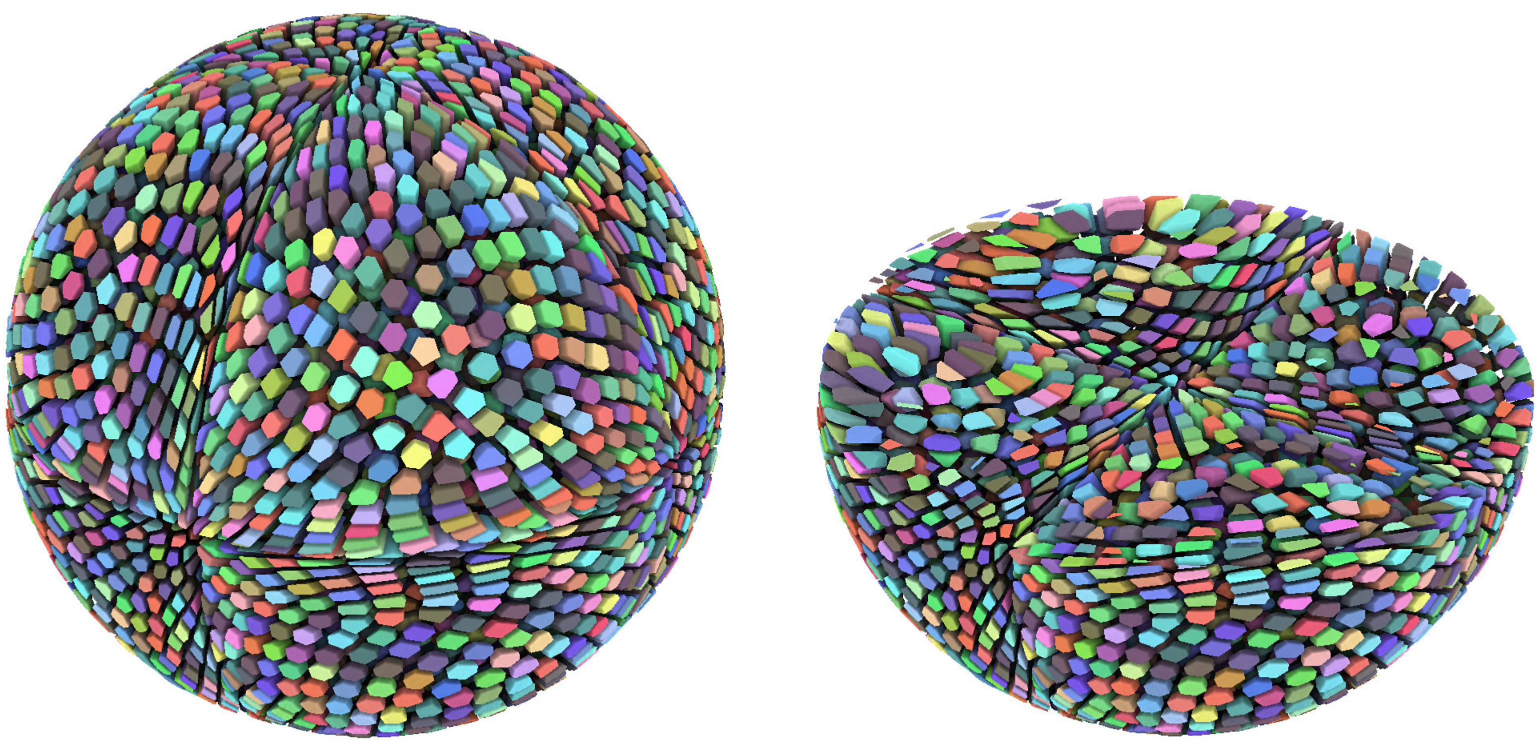
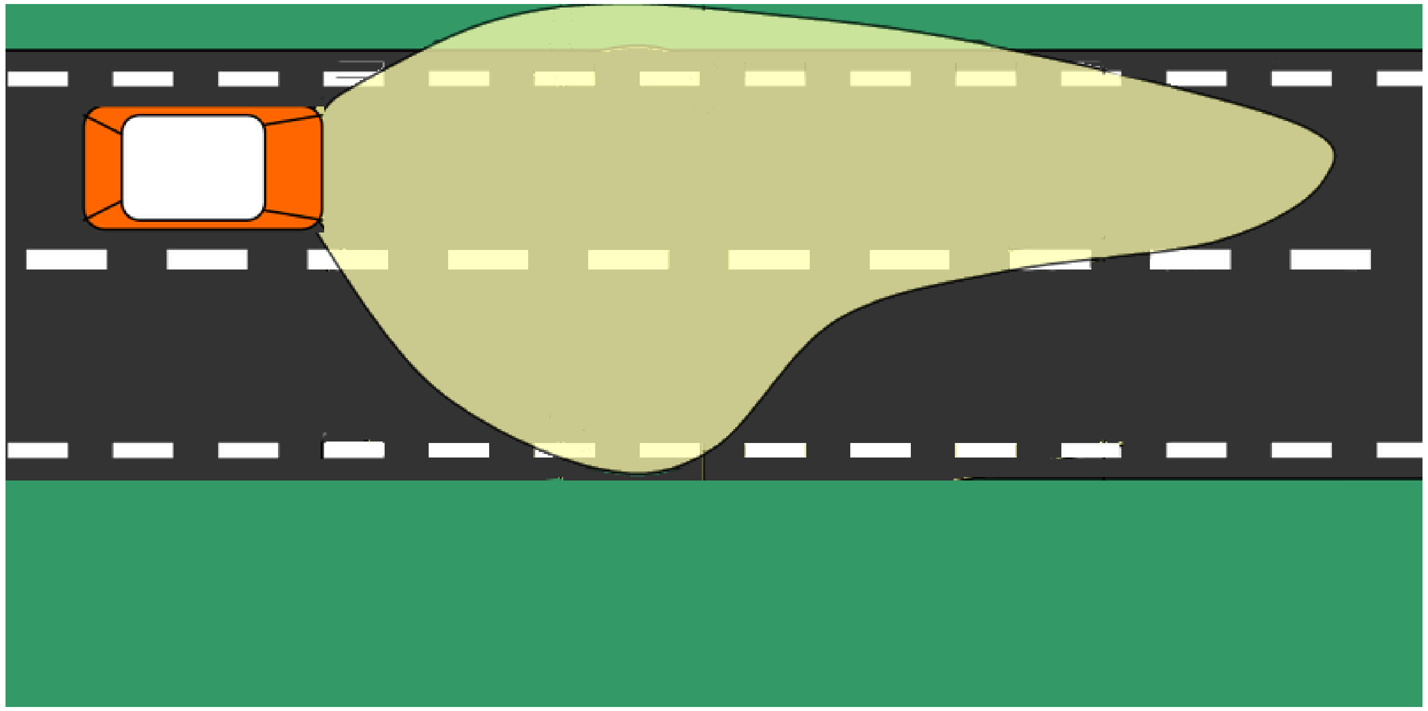


Laguerre cells



3. Application: reflector design in optics

Motivation



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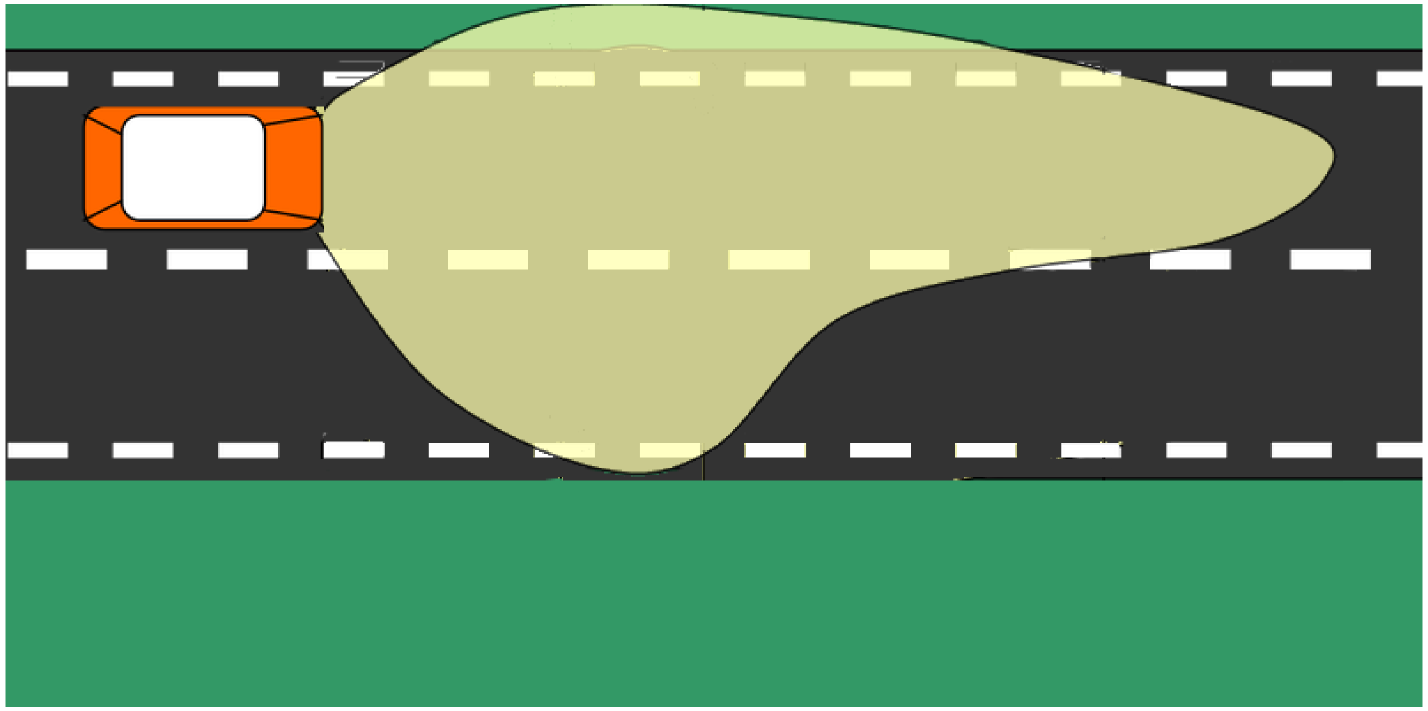

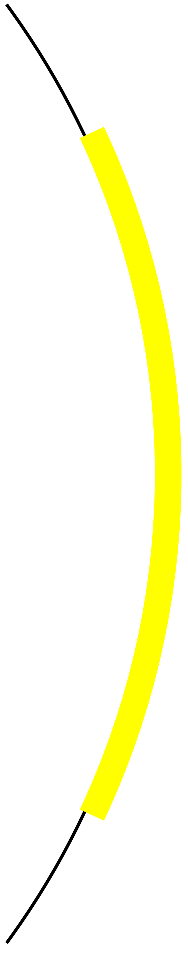

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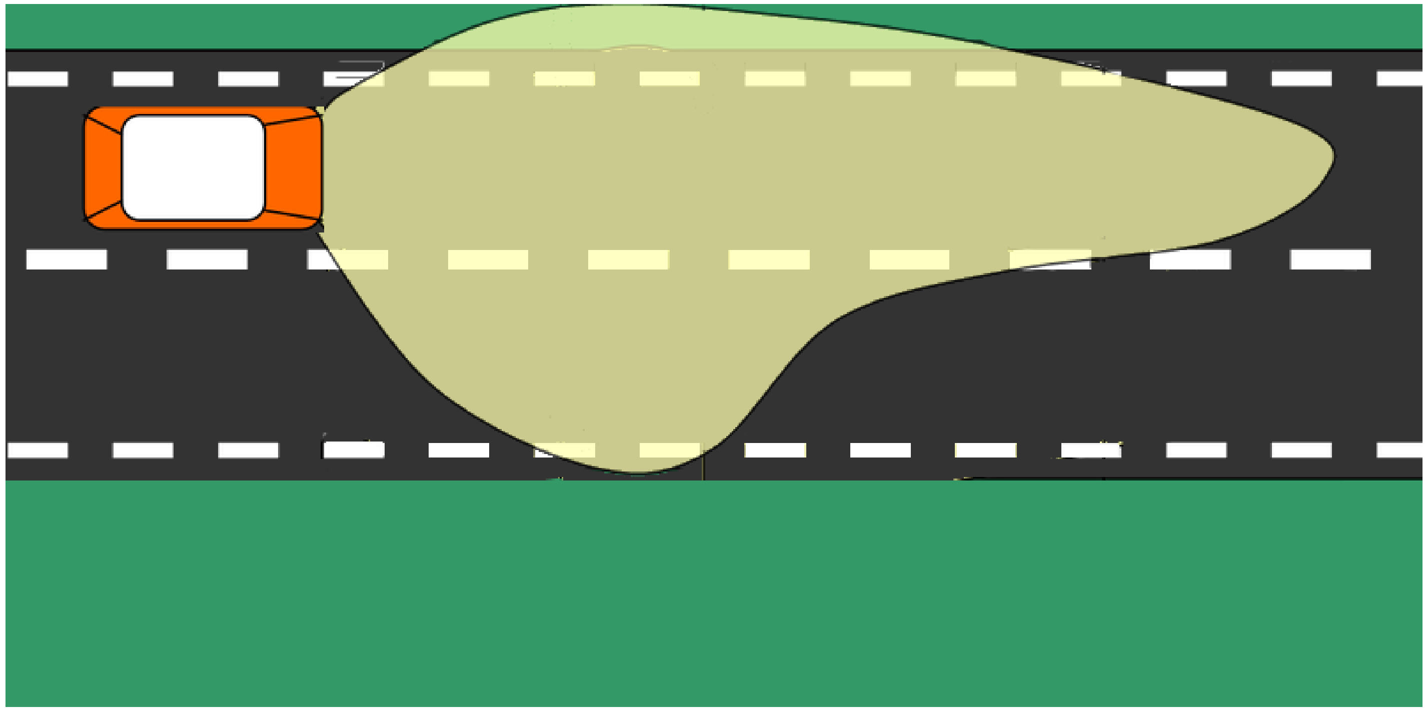
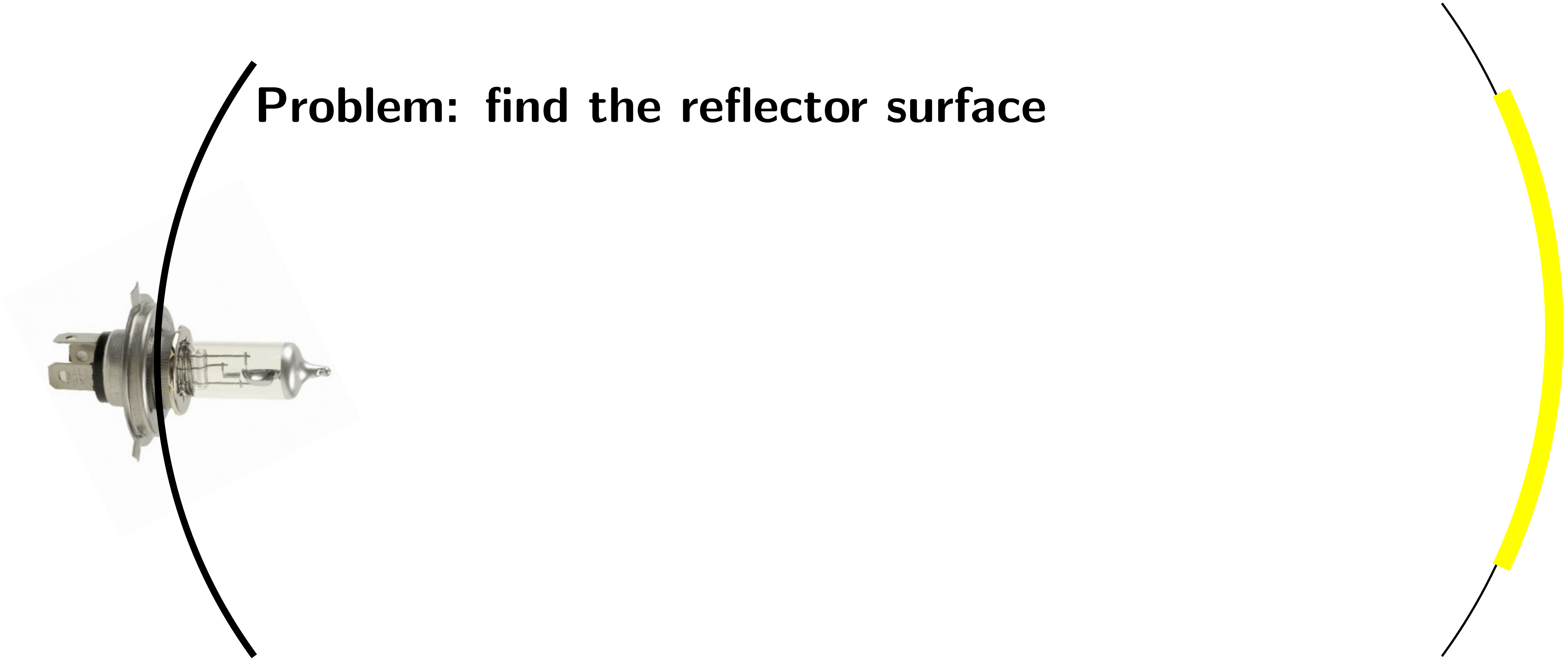
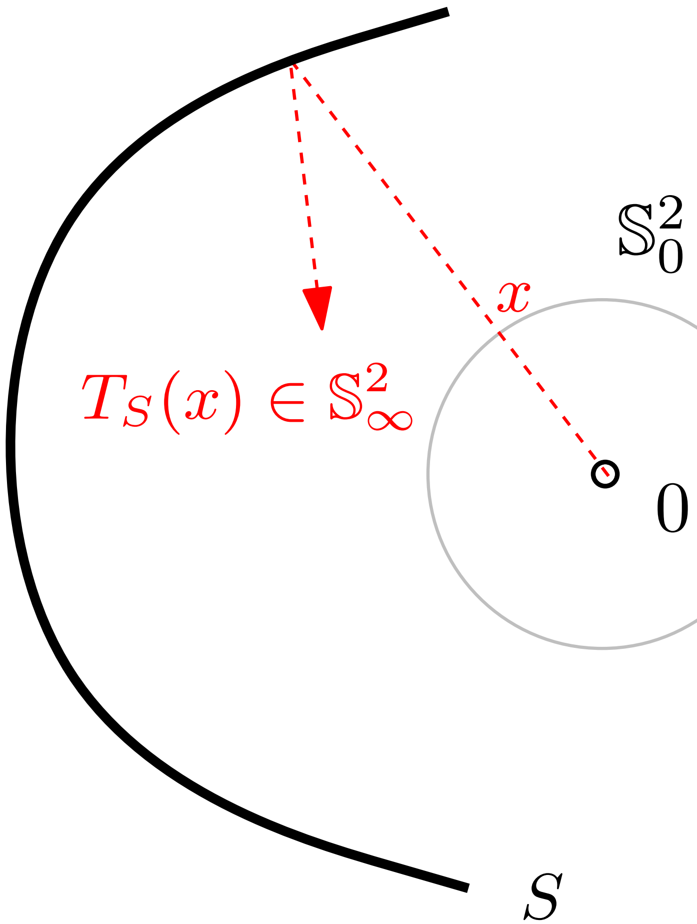


Problem: find the reflector surface



(Point source) Inverse Reflector Problem



Forward problem:

point light source $:= \rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathbb{S}_0^2)$
surface S

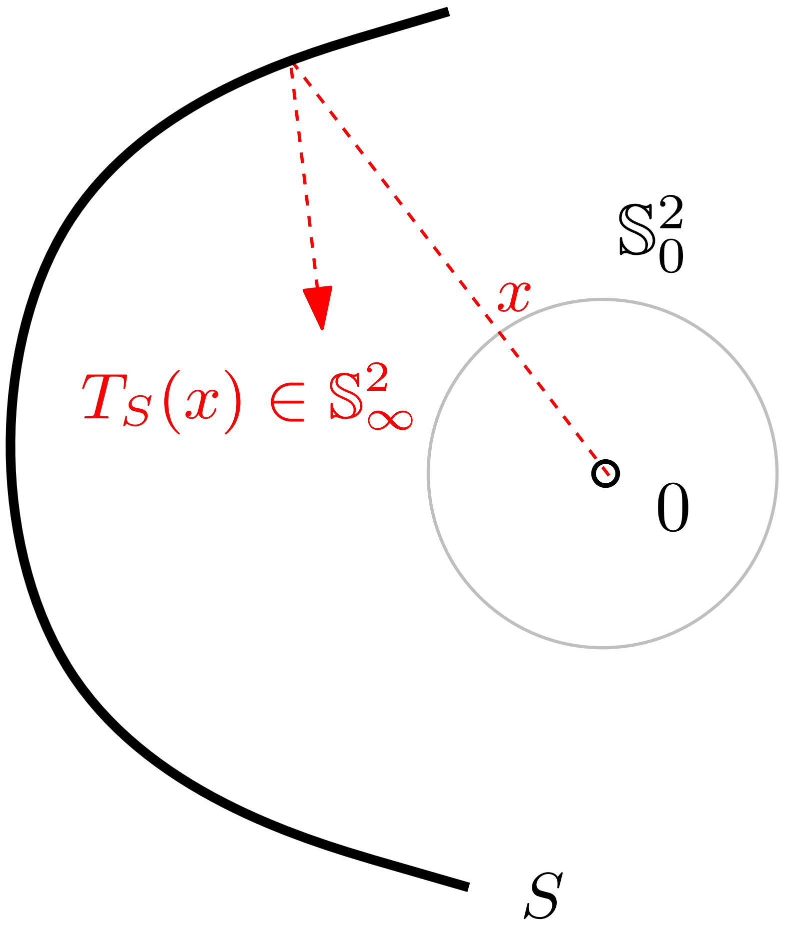
INPUT

raytracing

light distribution after reflection : $T_{S\#}\rho \in \mathrm{Prob}(\mathbb{S}^2_\infty)$

OUTPUT

(Point source) Inverse Reflector Problem



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Inverse problem:

INPUT

source: $\rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathbb{S}^2_0)$
target: $\nu \in \mathrm{Prob}(\mathbb{S}^2_\infty)$

??

surface S s.t. $T_{S\#}\rho = \nu$

OUTPUT

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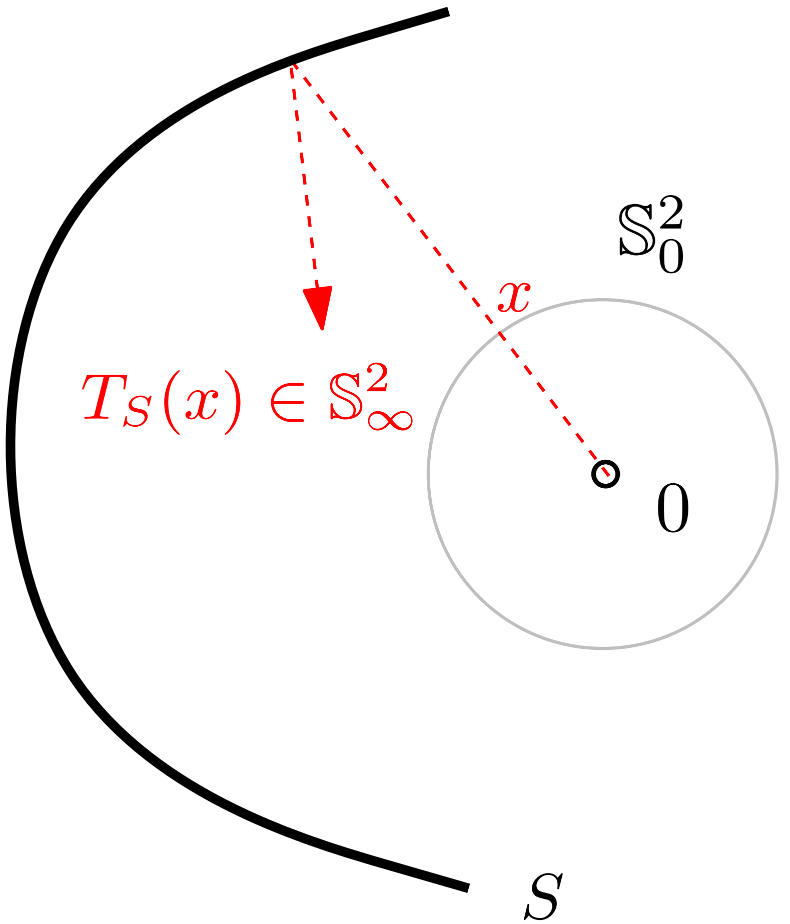

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$\longrightarrow$ Creating optical components for **distributing light energy**

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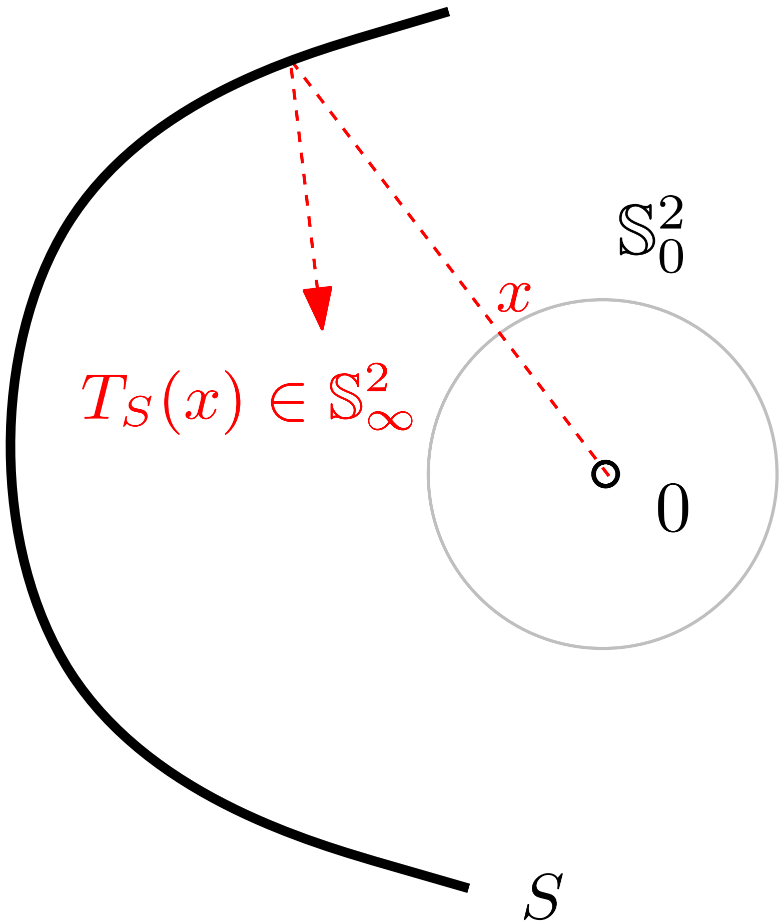

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OUTPUT

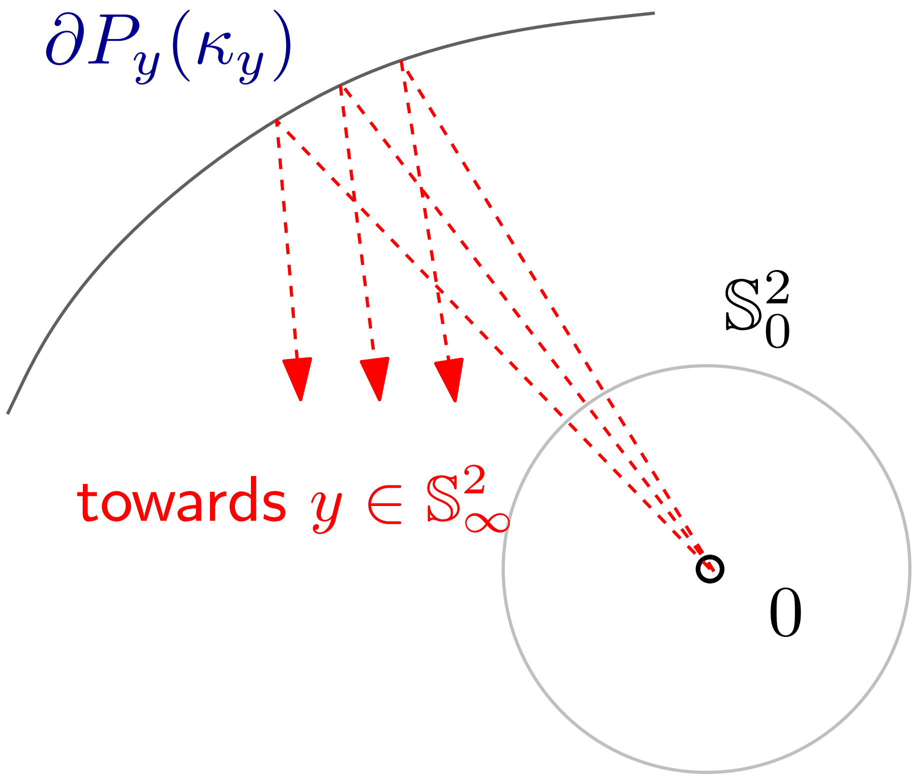
$\longrightarrow$ Creating optical components for **distributing light energy**

$\longrightarrow$ Equivalent to Monge-Ampère eq. / OT on $\mathbb{S}^2$

[Caffarelli & Oliker 94]
[Guan & Wang 98] [Wang 96]
[Glimm-Oliker '03] [Wang '04]

Semidiscrete Inverse Reflector Problem

Assume $\nu := \sum_{y \in Y} \nu_y \delta_y$, and let $P_y(\kappa_y) :=$ solid paraboloid of revolution with focal 0, direction y and focal distance κ_y



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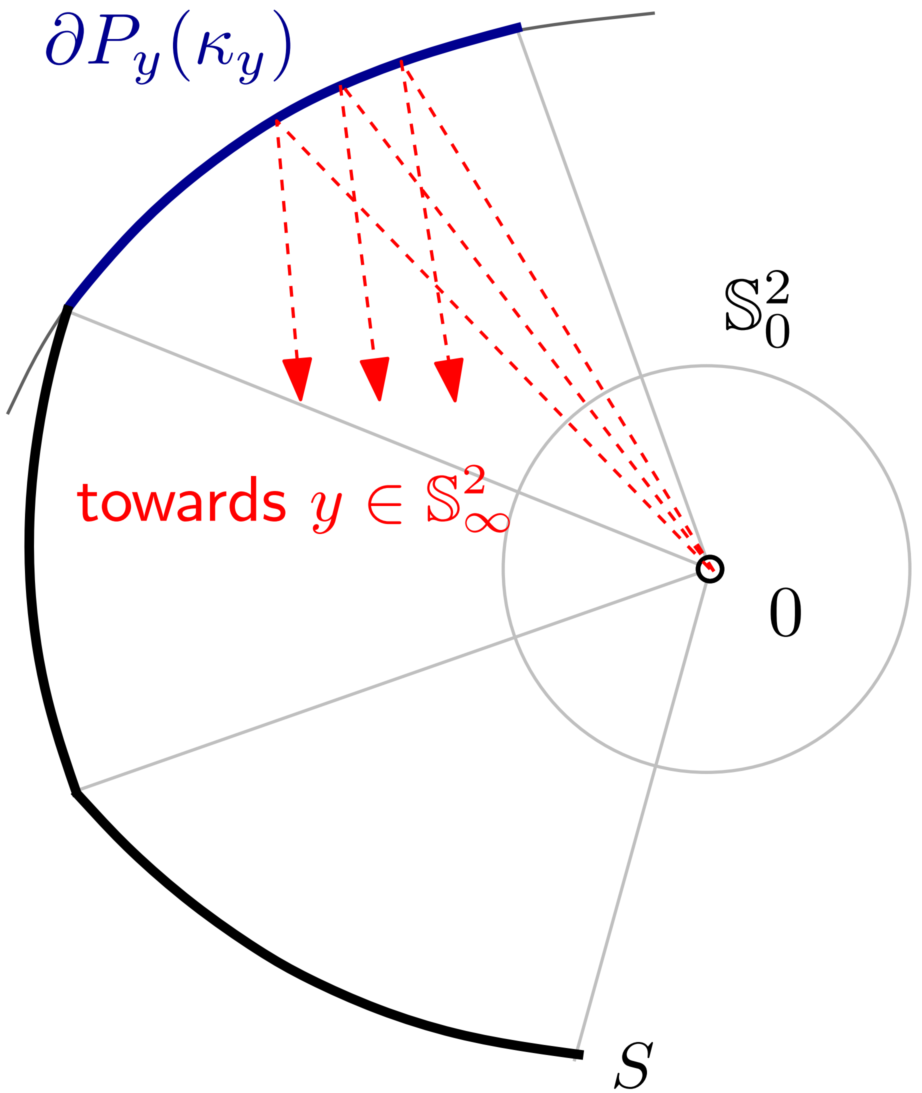
$S :=$ surface $= \partial \left(\cap_y P_y(\kappa_y) \right)$



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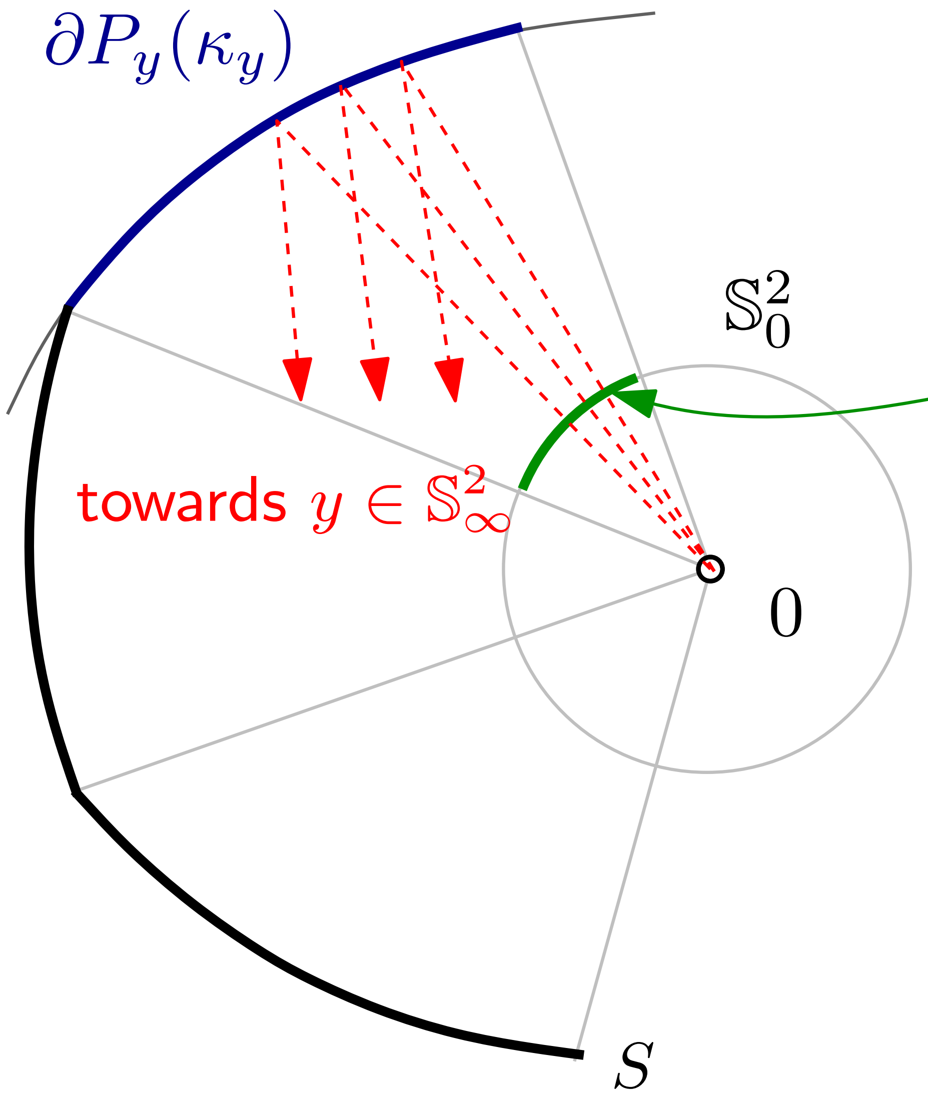

$\rho(V_\kappa(y)) =$ amount of light reflected towards $y \in \mathbb{S}^2_\infty$.

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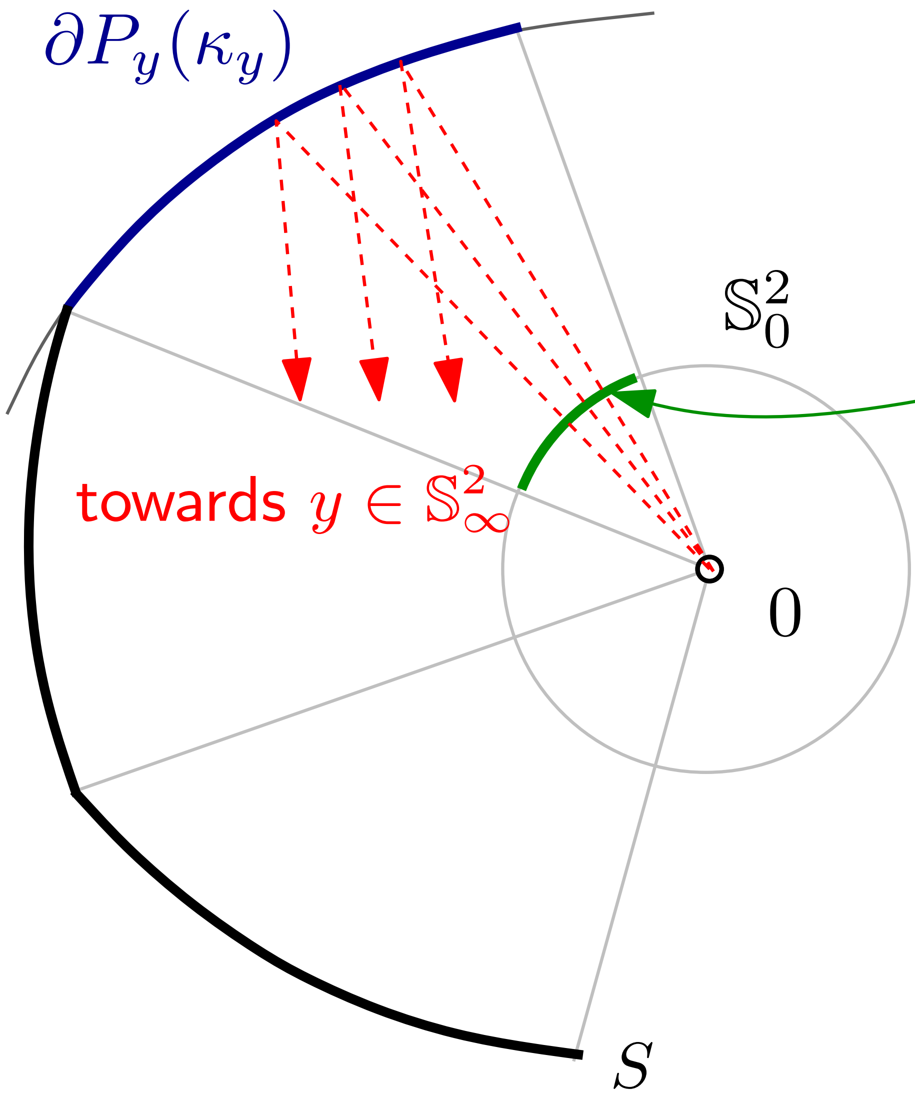

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$\longrightarrow$ Can be adjusted by playing with focal distance κ_y

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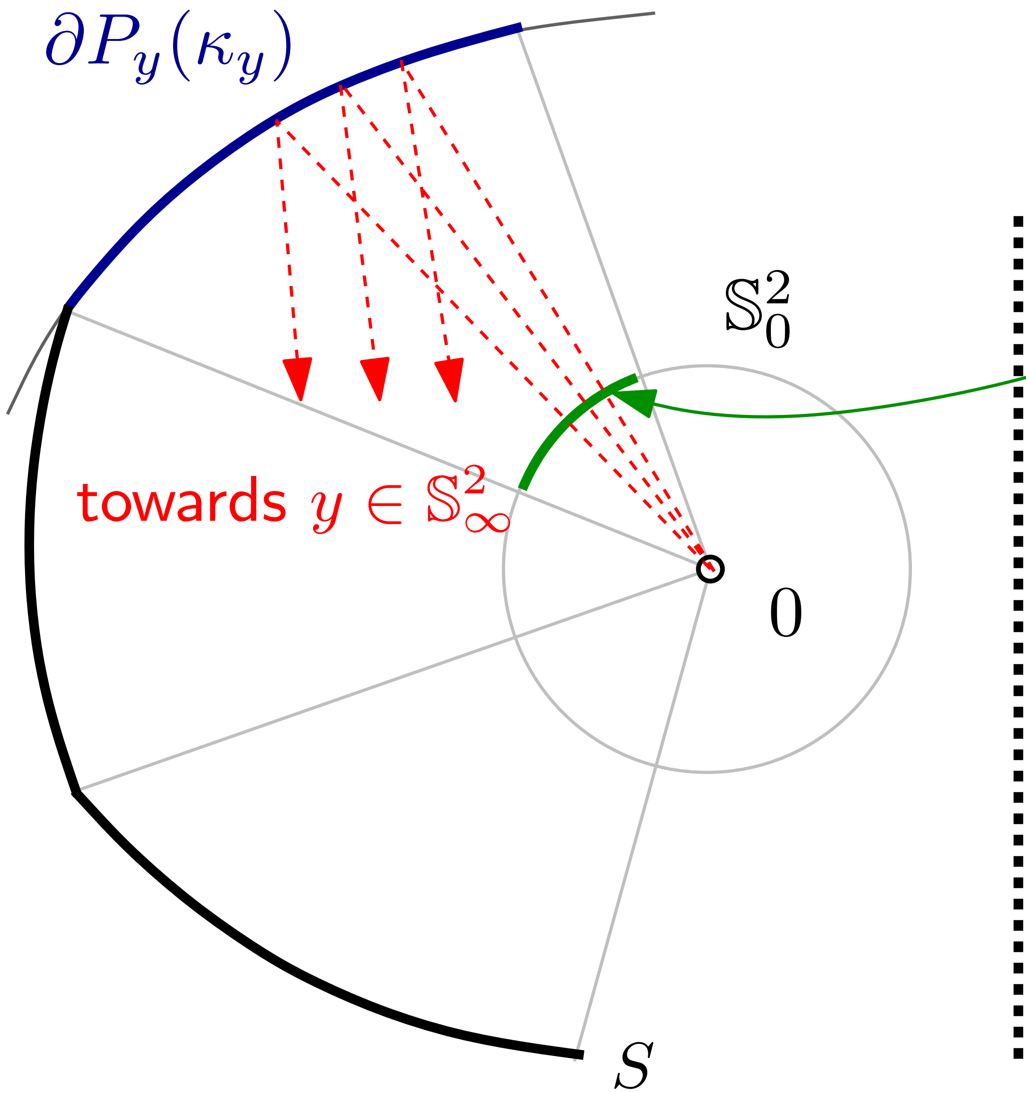

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$\longrightarrow$ Can be adjusted by playing with focal distance κ_y

$\longrightarrow$ Focal distance $\simeq$ prices in the economic example

Semidiscrete Inverse Reflector Problem

Assume $\nu := \sum_{y \in Y} \nu_y \delta_y$, and let $P_y(\kappa_y) :=$ solid paraboloid of revolution with focal 0, direction y and focal distance κ_y

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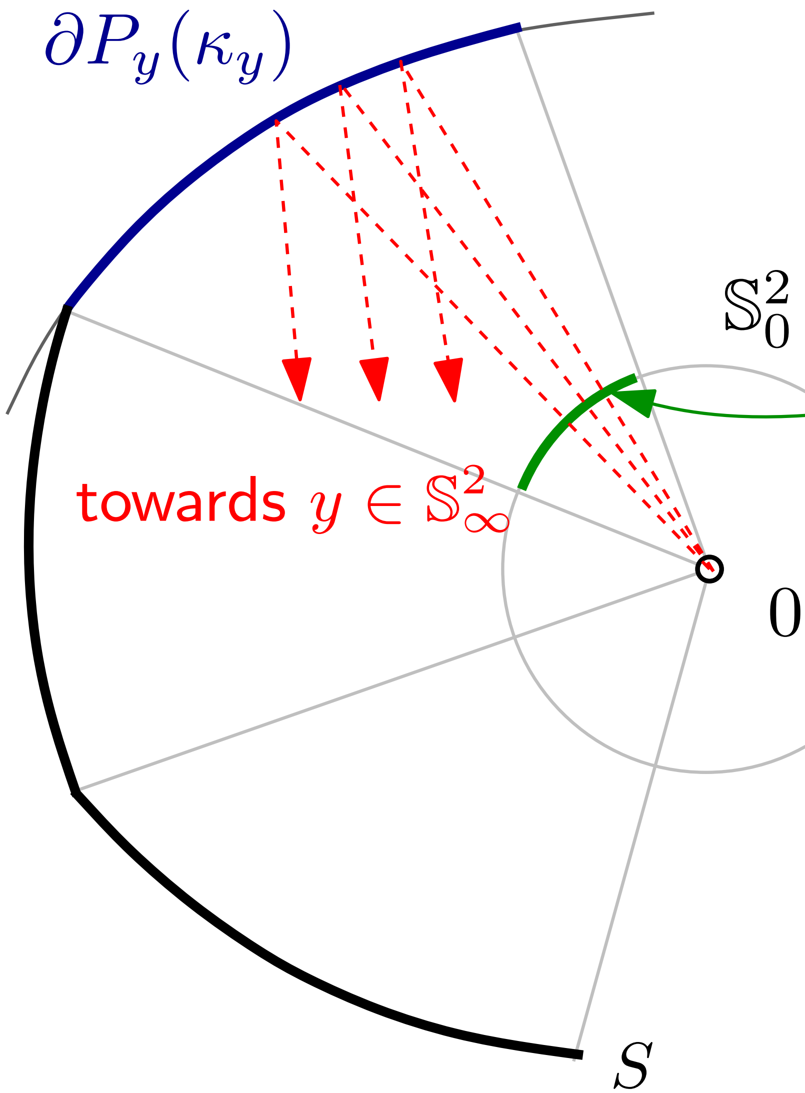

$\rho(V_\kappa(y)) =$ amount of light reflected towards $y \in \mathbb{S}^2_\infty$.

$\longrightarrow$ Can be adjusted by playing with focal distance κ_y

$\longrightarrow$ Focal distance $\simeq$ prices in the economic example

and, indeed, $V_y(\kappa)$ is a **Laguerre cell** !

$V_y(\kappa) = \mathrm{Lag}_y(\psi)$ for $\psi(y) = \log(\kappa_y)$
and $c(x, y) = -\log(1 - \langle x|y \rangle)$

Semidiscrete Inverse Reflector Problem

Assume $\nu := \sum_{y \in Y} \nu_y \delta_y$, and let $P_y(\kappa_y) :=$ solid paraboloid of revolution with focal 0, direction y and focal distance κ_y

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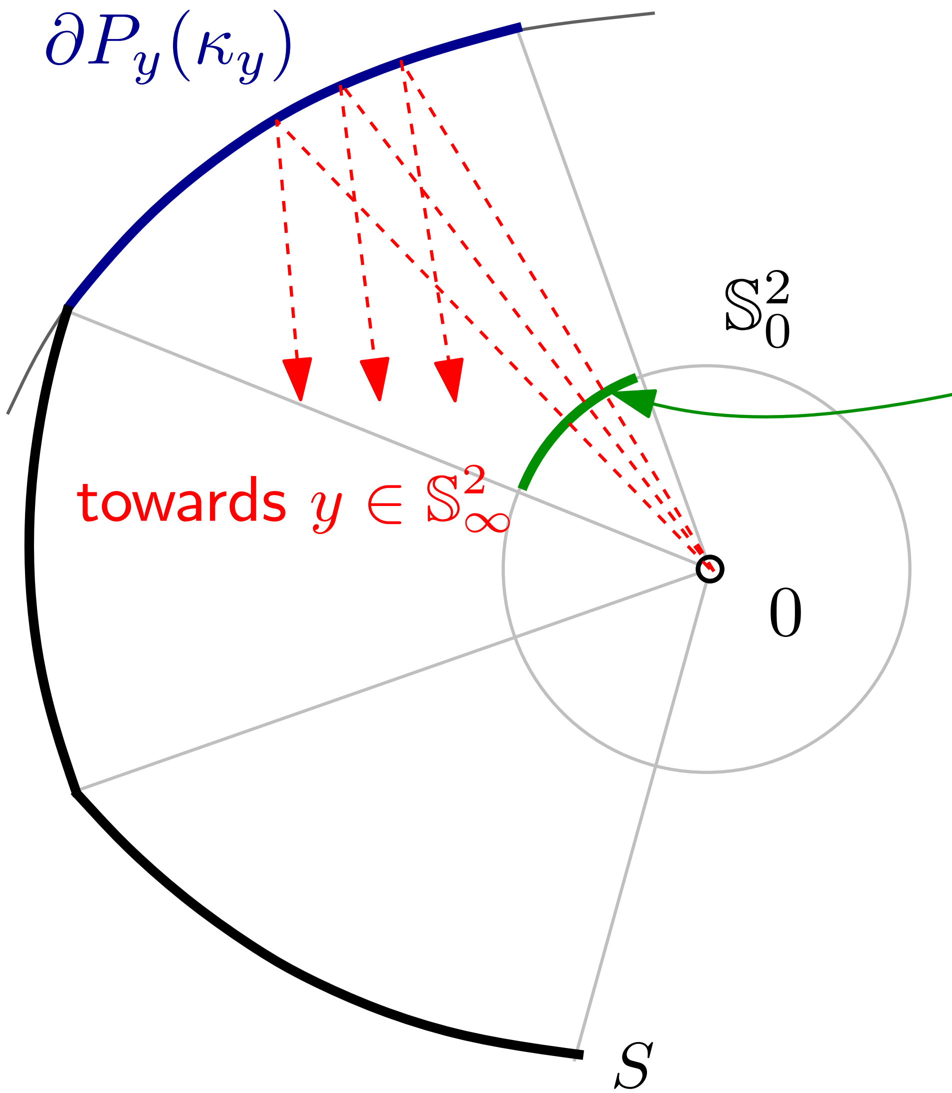

$\rho(V_\kappa(y))$ = amount of light reflected towards $y \in \mathbb{S}^2_\infty$.

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$$\boxed{V_y(\kappa) = \mathrm{Lag}_y(\psi) \text{ for } \psi(y) = \log(\kappa_y) \text{ and } c(x,y) = -\log(1 - \langle x|y\rangle)}$$

Theorem: Semidiscrete Inverse Reflector Problem

$\Longleftrightarrow$ semidiscrete OT problem on $\mathbb{S}^2$ for $c(x,y) = -\log(1 - \langle x|y\rangle)$

[Glimm-Oliker '03] [Wang '04]

Numerics: point source/far-field target I

Difficulty: Given $Y \subseteq \mathbb{S}^2$ and $\psi : Y \to \mathbb{R}$, computation the Laguerre cells

$$\mathrm{Lag}_y(\psi) = \{x \in \mathbb{S}^2 \mid \forall z \in Y, \ -\log(1 - \langle x|y\rangle) + \psi(y) \leq -\log(1 - \langle x|z\rangle) + \psi(z)\}$$

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There is fortunately a trick:

$$x \in \mathrm{Lag}_y(\psi) \Longleftrightarrow y = \arg\min_{z \in Y} \tfrac{\kappa_z}{1 - \langle x|z\rangle}, \quad \text{where} \;\; \kappa_y := \psi(y).$$

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There is fortunately a trick:

$$\begin{aligned} x \in \mathrm{Lag}_y(\psi) &\iff y = \arg\min_{z\in Y} \tfrac{\kappa_z}{1-\langle x|z\rangle}, \quad \text{where } \ \kappa_y := \psi(y). \\ &\iff y = \arg\min_{z\in Y} \tfrac{\langle x|z\rangle - 1}{\kappa_z} \end{aligned}$$

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$$\Longleftrightarrow y = \arg\min_{z \in Y} \tfrac{\langle x|z\rangle - 1}{\kappa_z}$$

$$\Longleftrightarrow y \in \arg\min_{z \in Y} \|x + \underbrace{\boxed{\tfrac{z}{2\kappa_z}}}_{-p_z}\|^2 \underbrace{\boxed{- \|\tfrac{z}{2\kappa_z}\|^2 - \tfrac{1}{\kappa_z}}}_{w_z}$$

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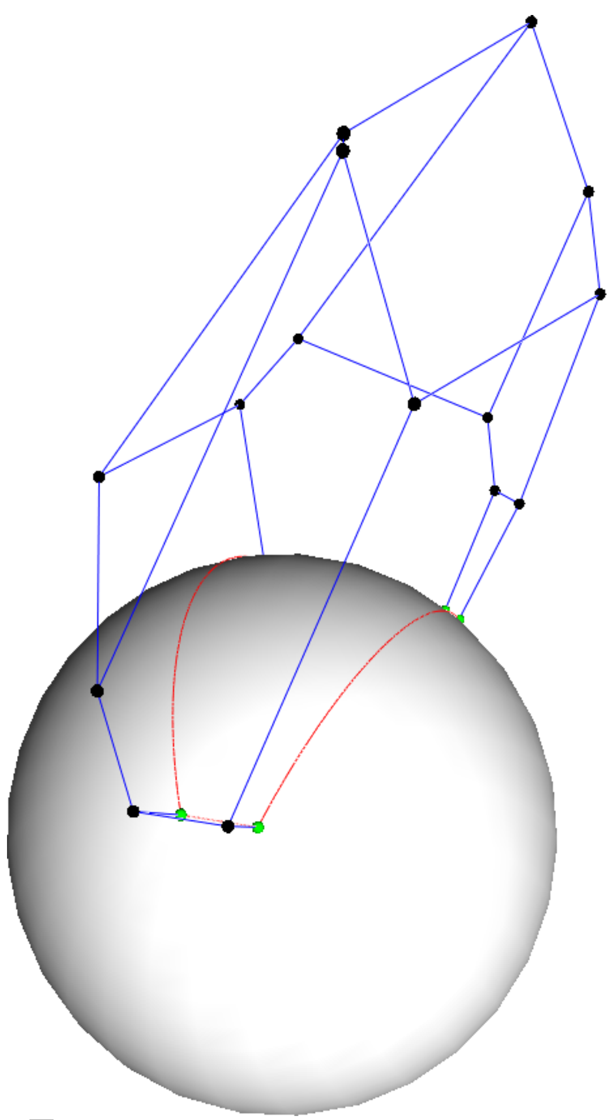
$$\iff x \in \mathbb{S}^2 \cap \text{ Laguerre cell for the squared Euclidean cost!}$$



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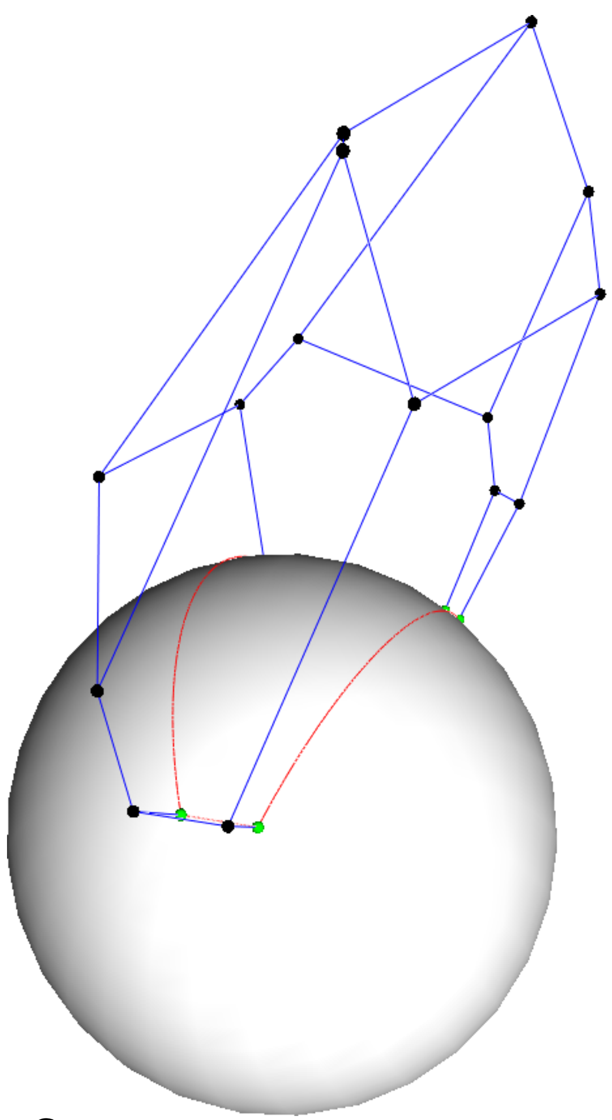


$\longrightarrow$ can be computed using 3D power diagram + intersection

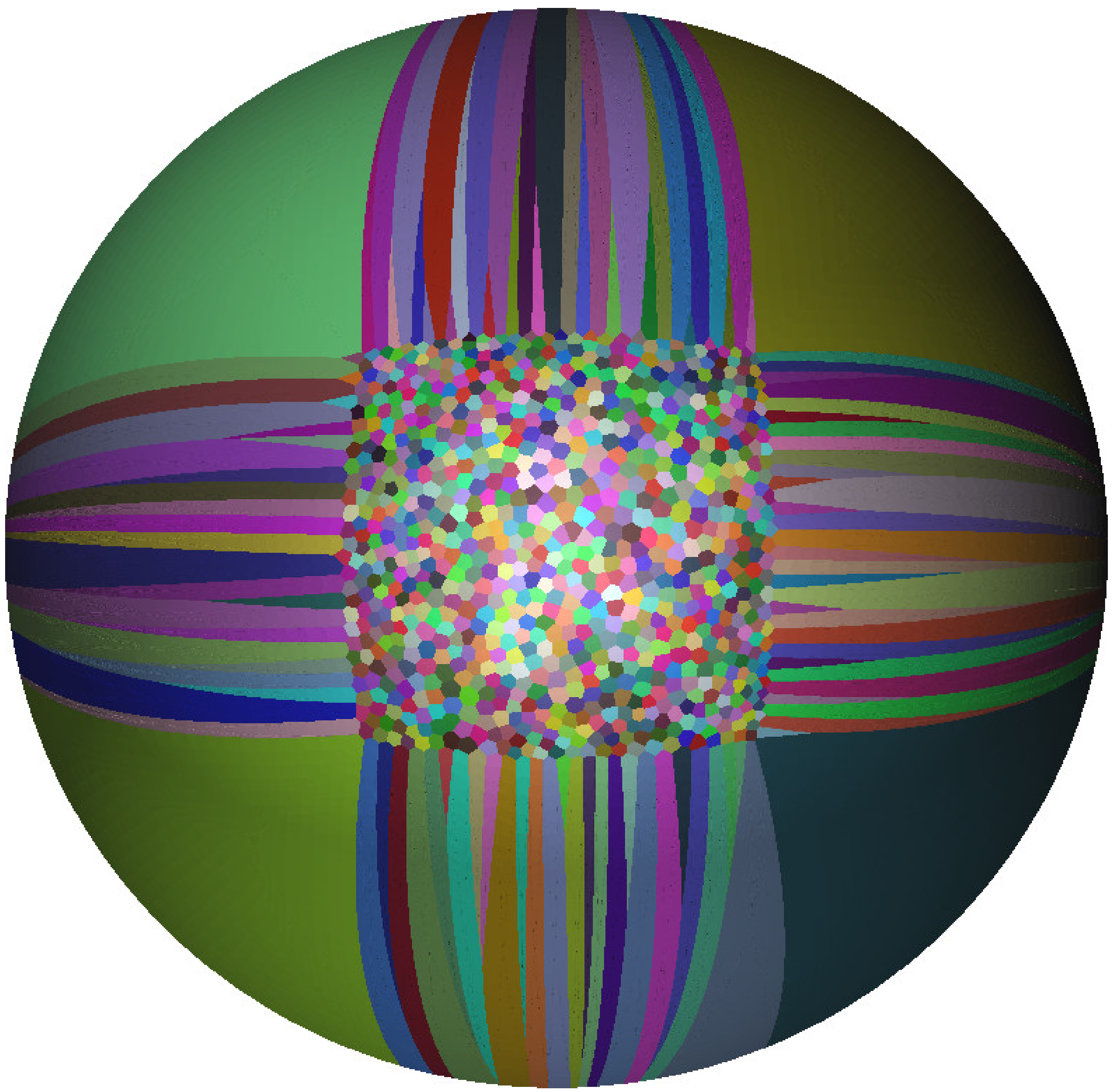
$\longrightarrow$ similar tricks allow to deal with most cost from optics

$\longrightarrow$ computing efficiently Laguerre cells for general costs is however unfeasible in practice.

Numerics: point source/far-field target 1

$\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ obtained by discretizing a picture of G. Monge.

$\rho =$ uniform measure on half-sphere $X := \mathbb{S}^2_+$ $N = 1000$



drawing of $(\mathrm{Lag}_\psi(y_i))$ on $\mathbb{S}^2_+$ for $\psi = 0$ [Machado, M., Thibert '14]

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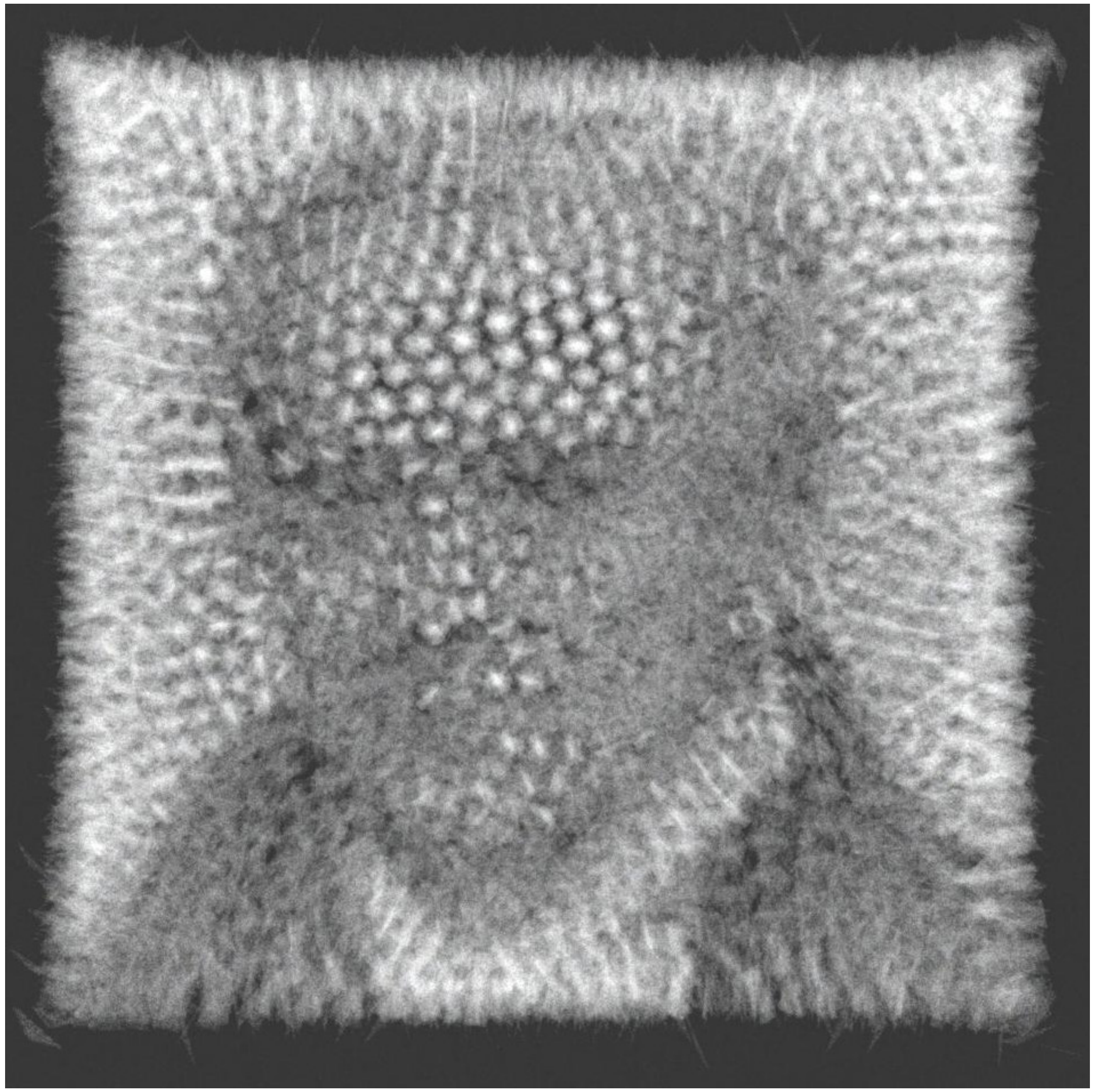

reflected image (using LuxRender) [Machado, M., Thibert '14]

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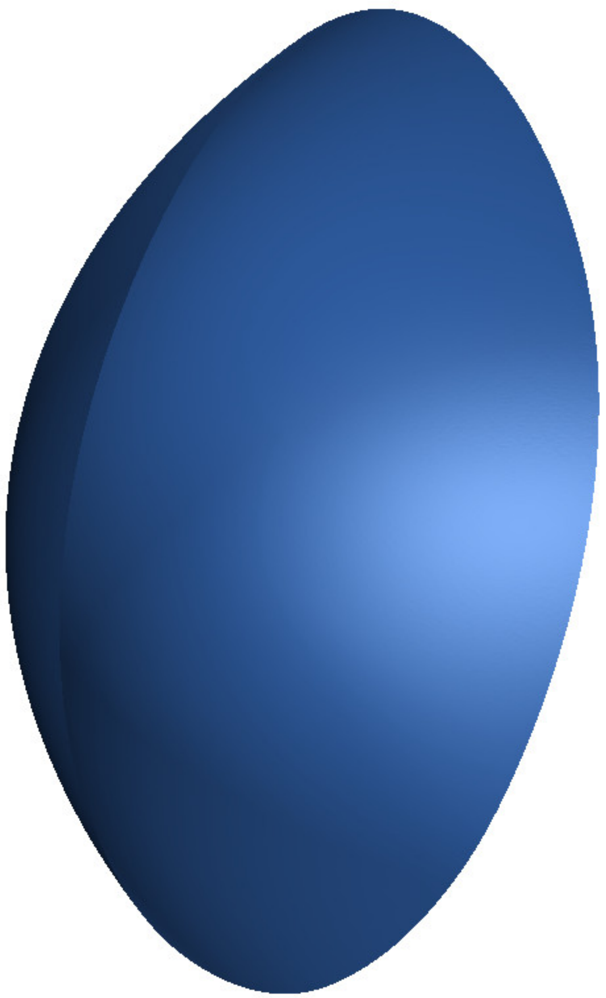


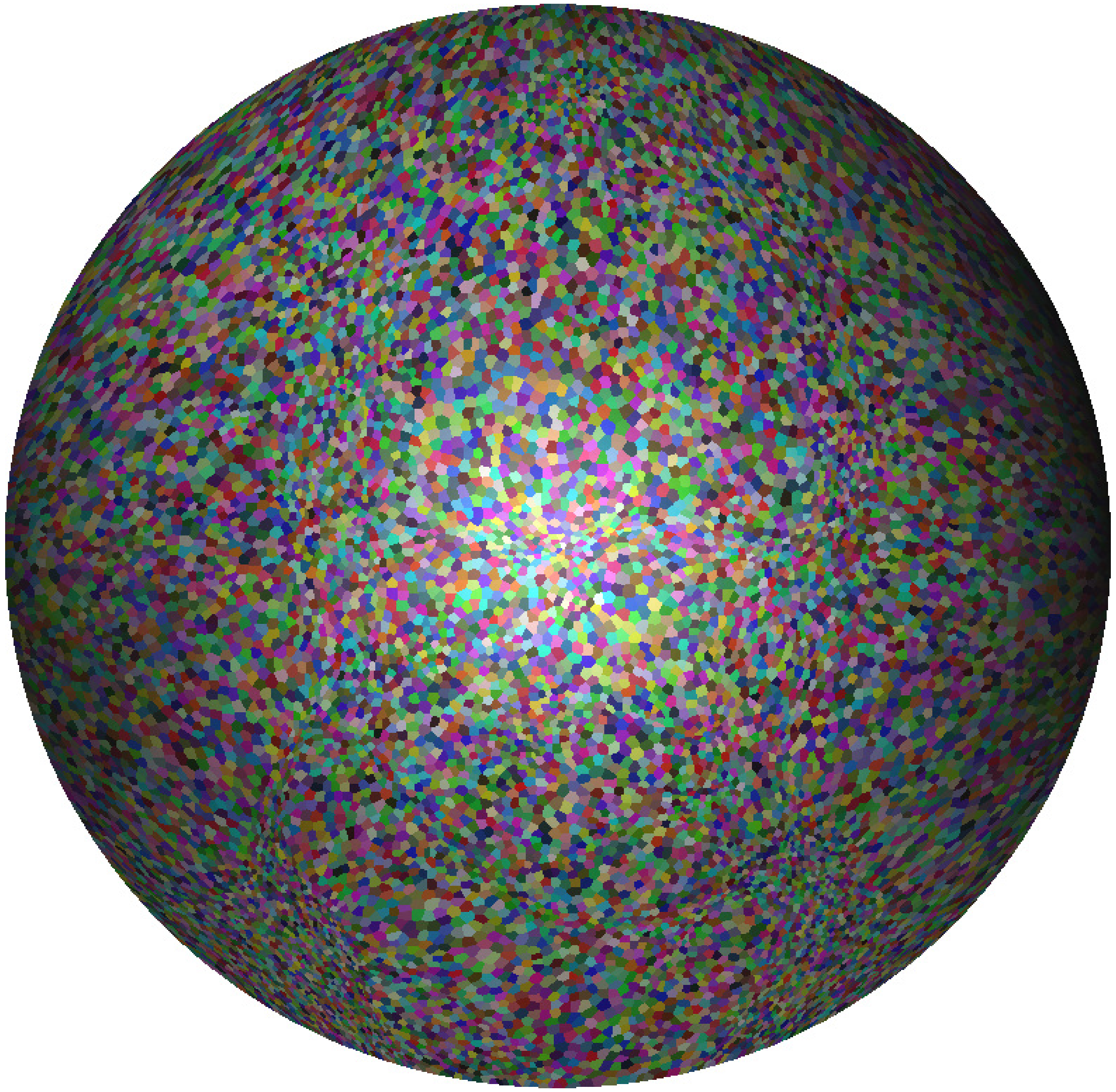
constructed reflector

[Machado, M., Thibert '14]

Numerics: point source/far-field target 2

$\nu = \sum_{i=1}^N \nu_i \delta_{y_i}$ obtained by discretizing a picture of G. Monge.

$\rho =$ uniform measure on half-sphere $X := \mathbb{S}^2_+$ $\qquad N = 15000$



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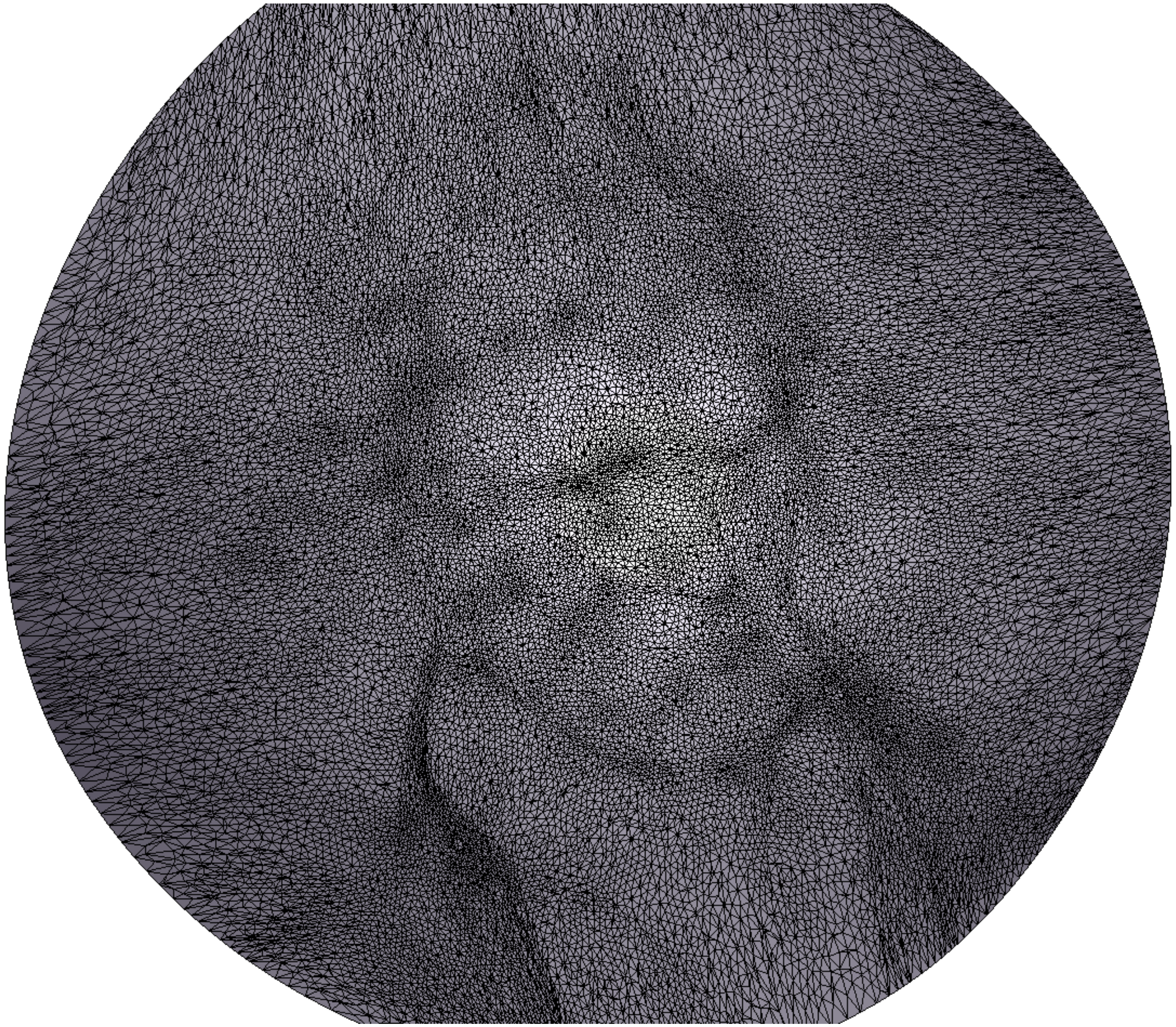

triangulated reflector

[Machado, M., Thibert '14]

Numerics: point source/far-field target 2

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reflected image (using LuxRender) [Machado, M., Thibert '14]

Numerics: point source/far-field target 3

$\nu = \sum_{i=1}^{N} \nu_i \delta_{y_i}$ = discretization of the "Cameraman" picture [Meyron, M., Thibert '17]

ρ = non-uniform measure on half-sphere $X := \mathbb{S}^2_+$ $N = 250k$

Desired target ν

Numerics: point source/far-field target 3

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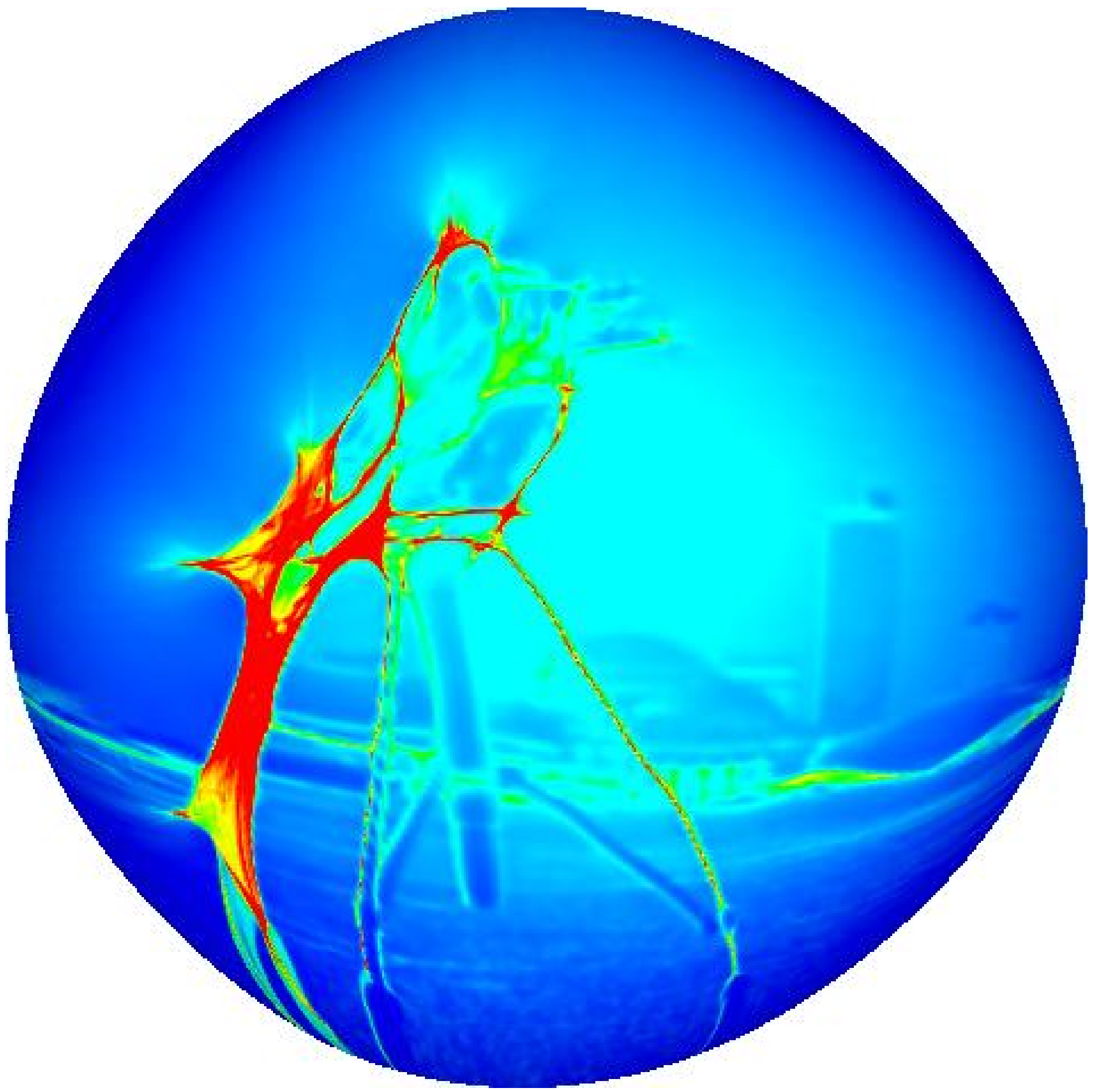

Constructed reflector
color = mean curvature

Numerics: point source/far-field target 3

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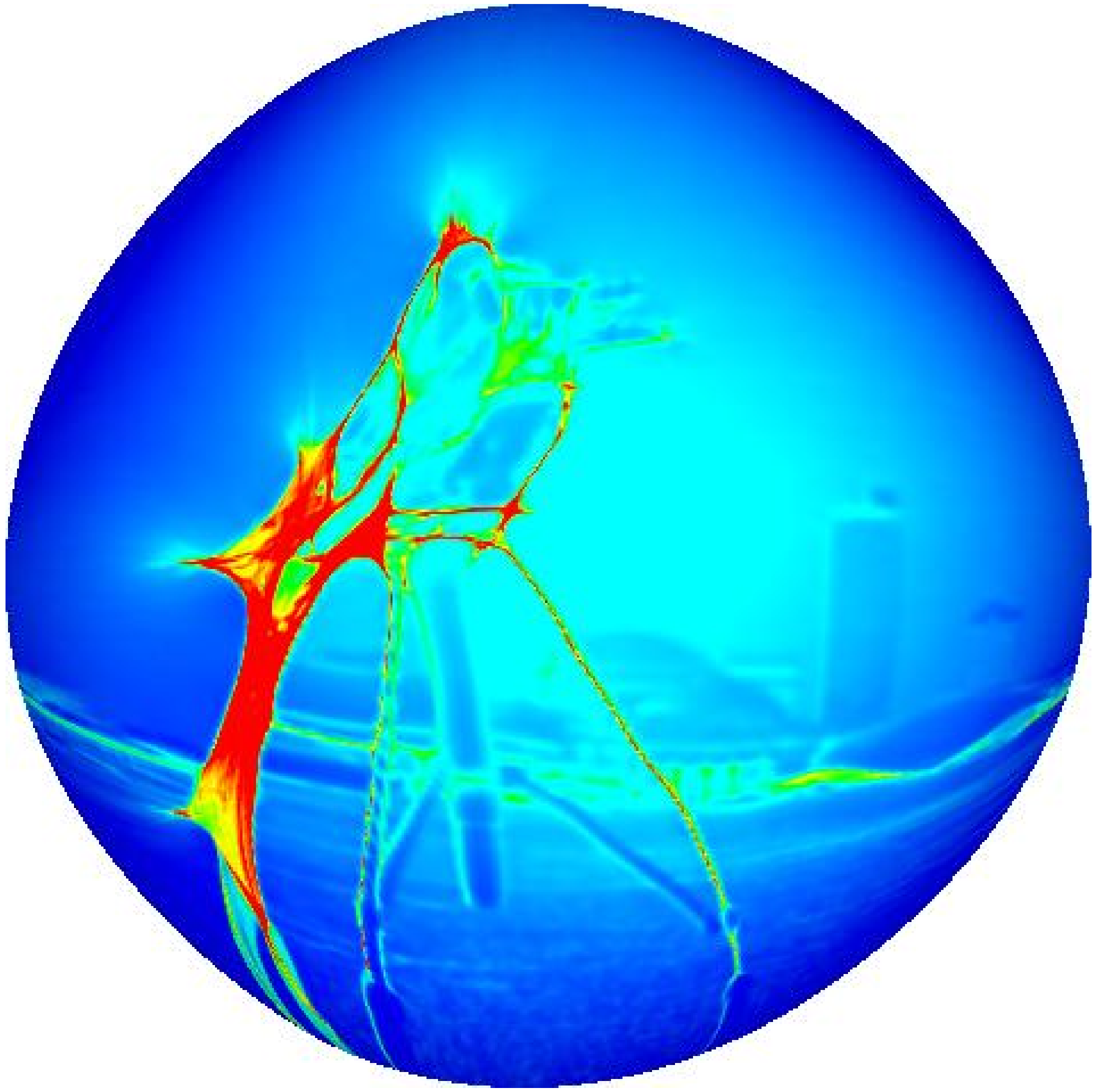
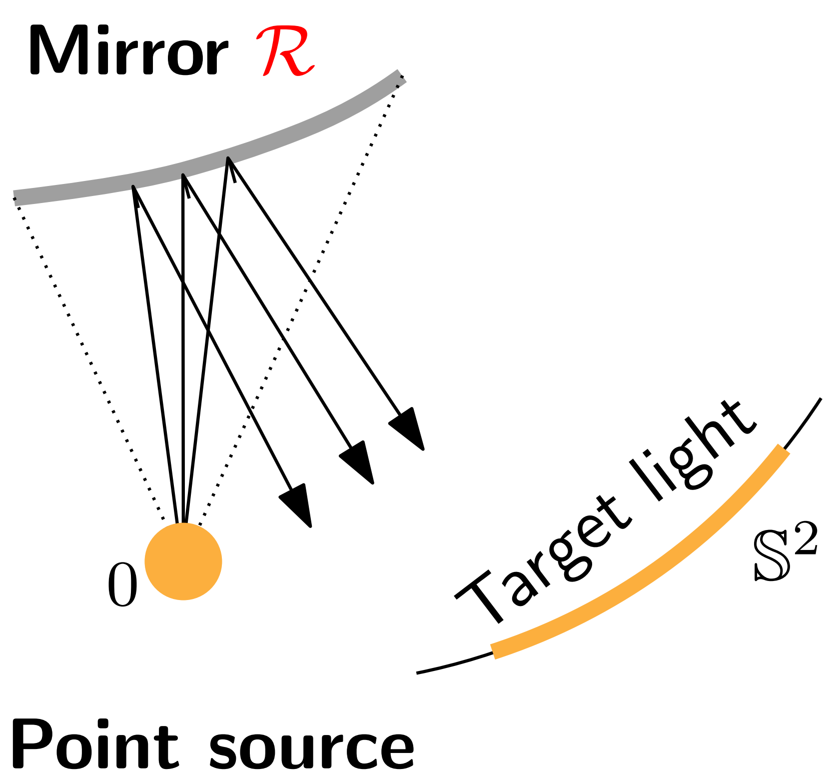


Constructed reflector
color = mean curvature

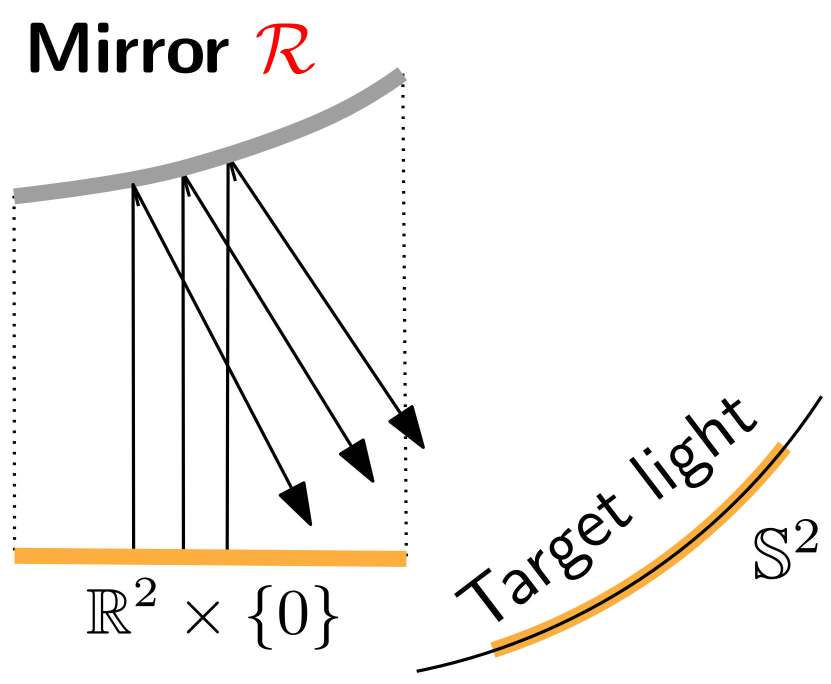
Resimulated image

3. Far-field problems in non-imaging optics

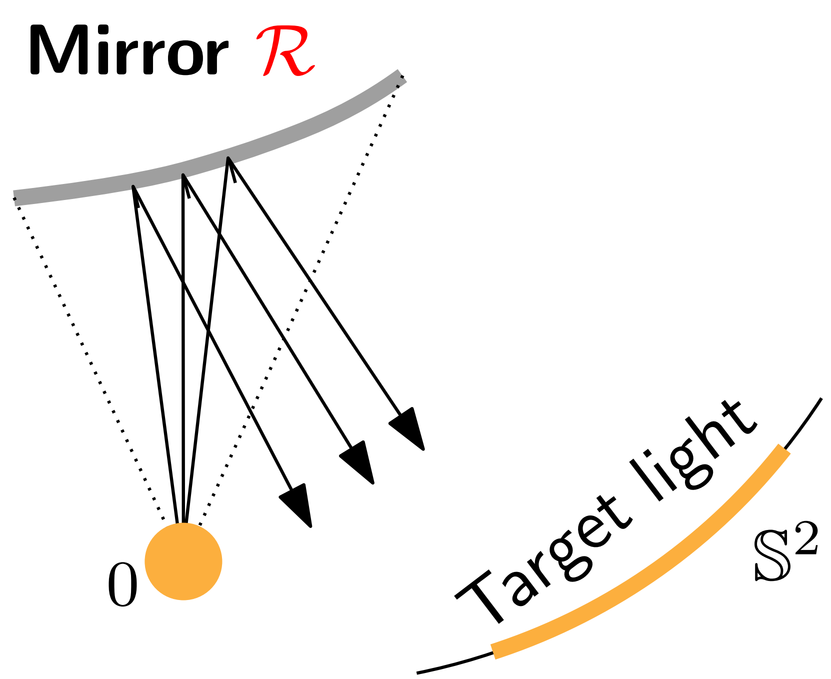
Four non-imaging optics problems



Four non-imaging optics problems

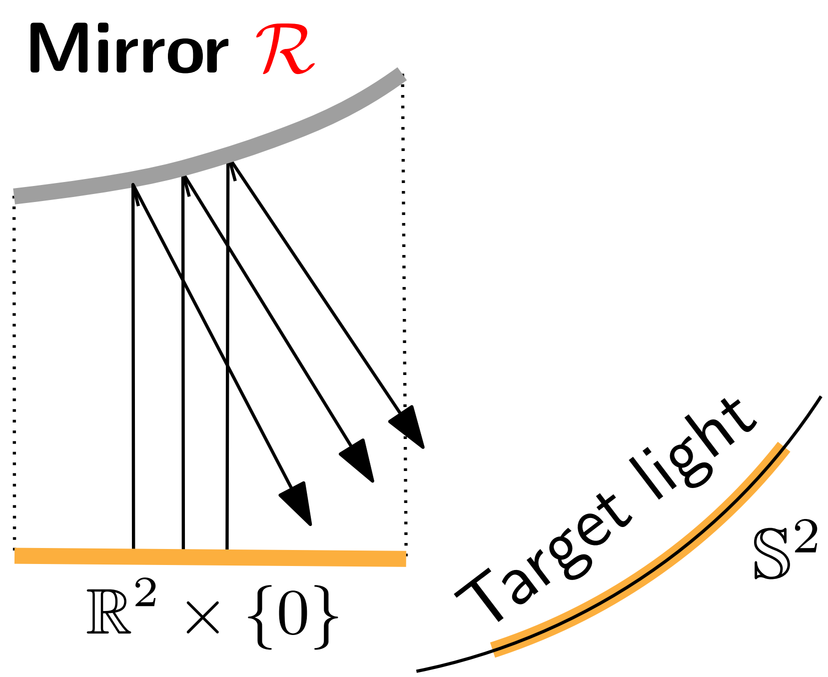


Collimated source

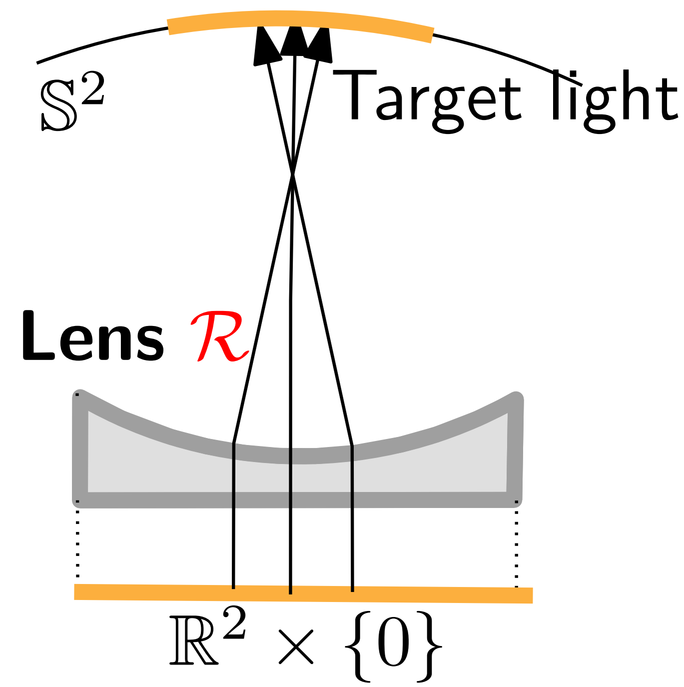


Point source

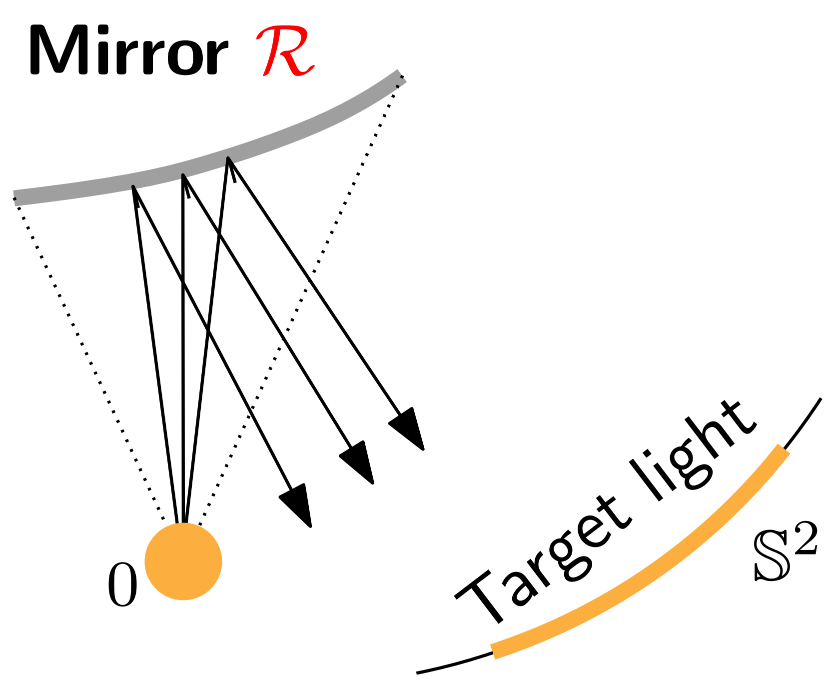
Four non-imaging optics problems



Collimated source



Collimated source



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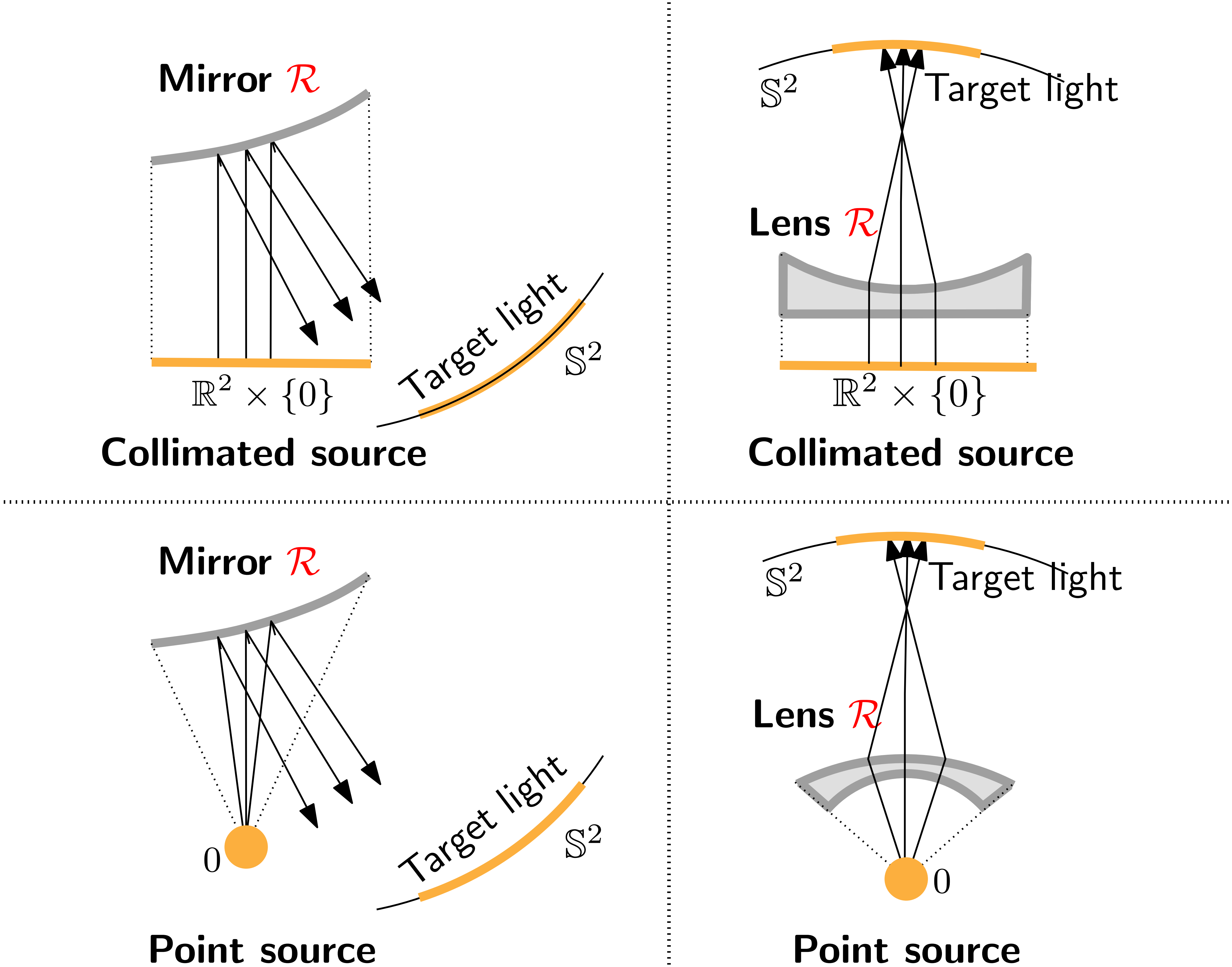
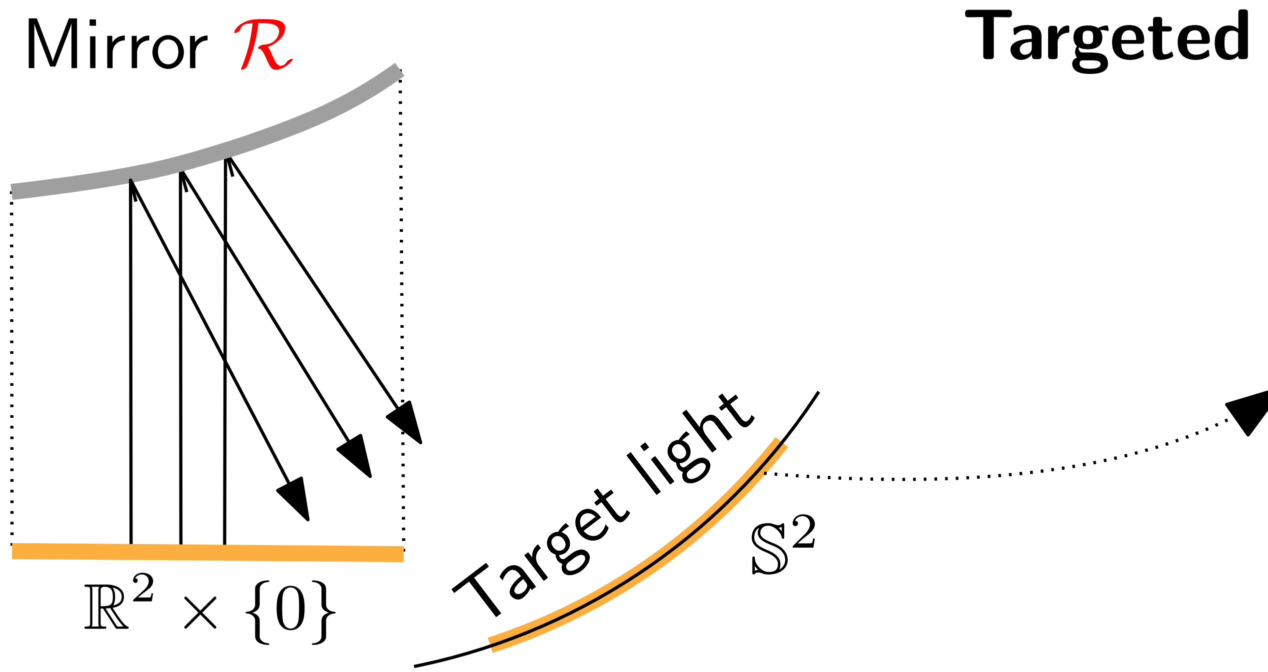
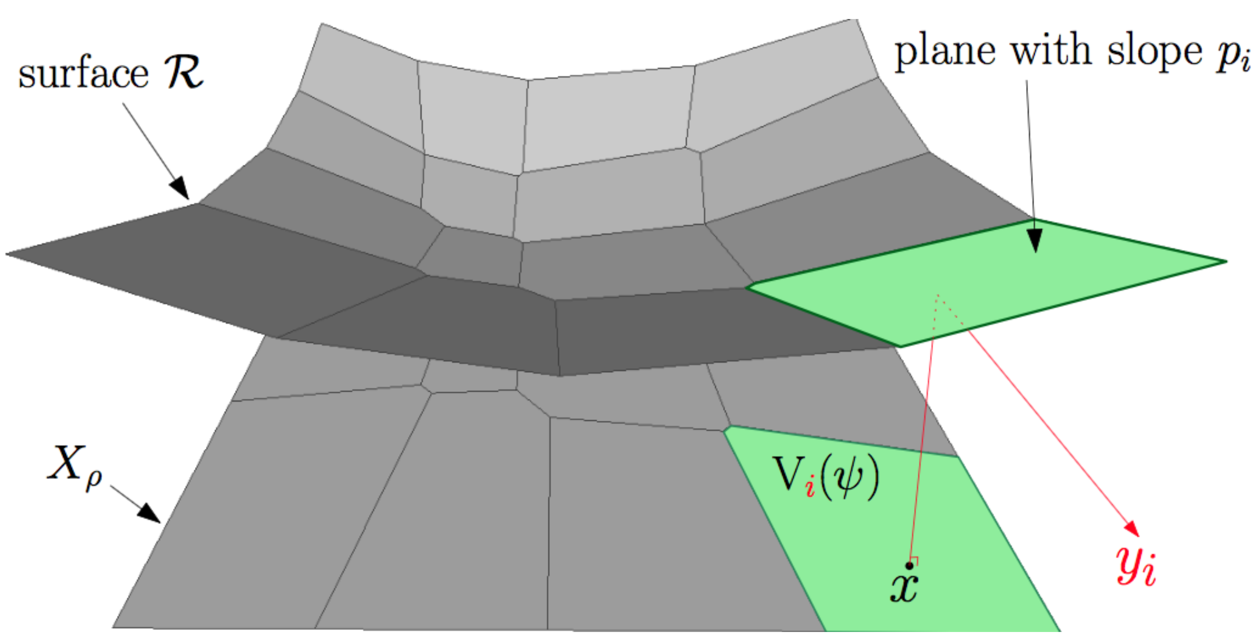


Example 1: mirror for collimated source



Collimated source

Targeted image: $N = 400 \times 480$ Diracs



- Computation of $V_i(\psi)$ = intersection of a power diagram with a $\mathbb{R}^2 \times \{0\}$
- A single algorithm handles the 4 cases.

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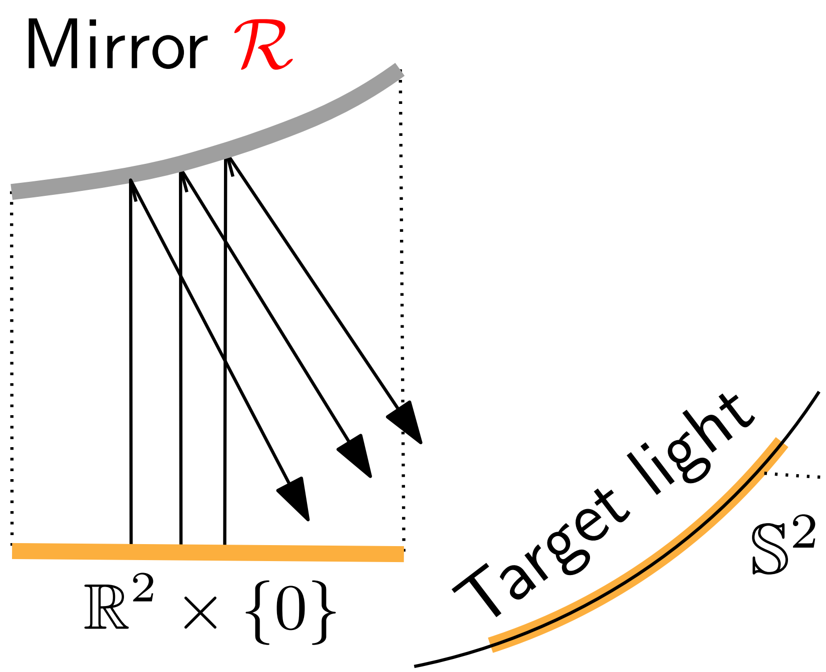

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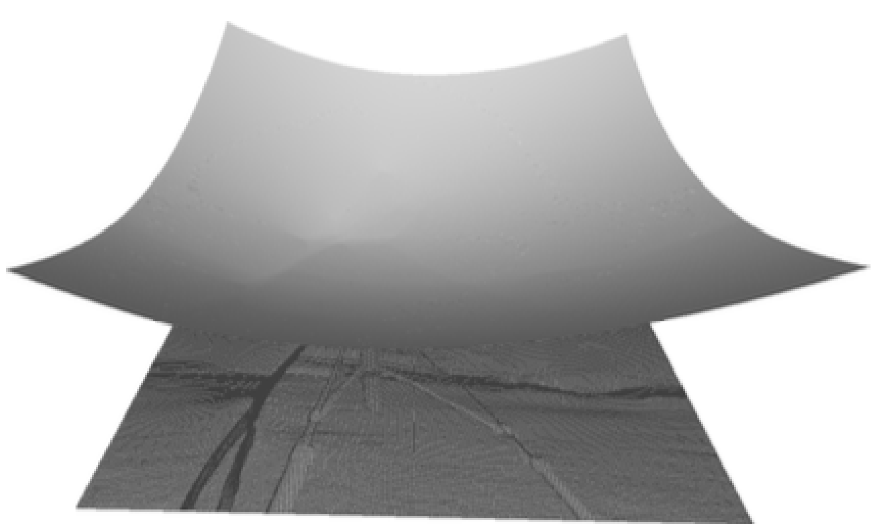
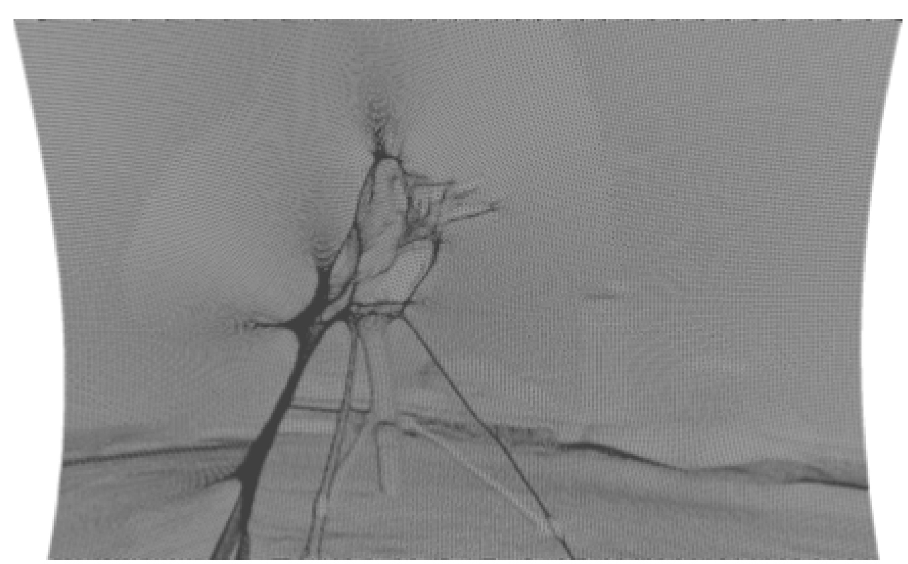
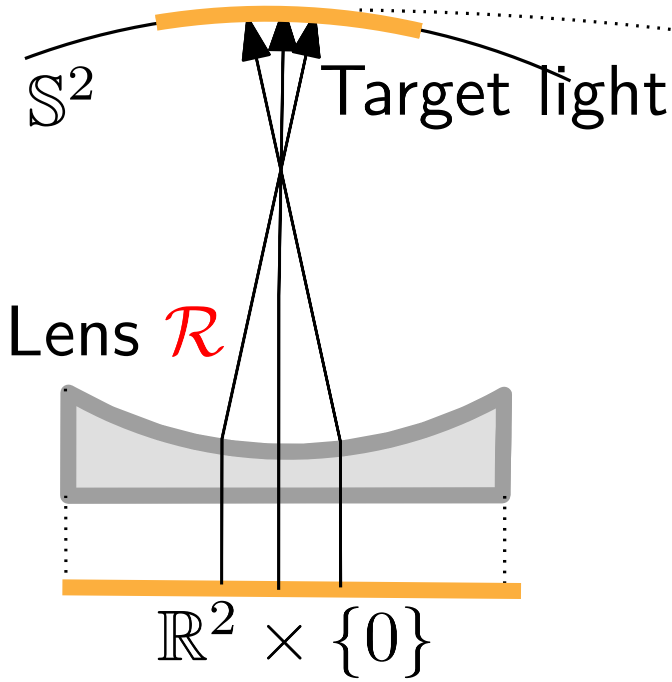




Reflected image

Example 2: lens for collimated source



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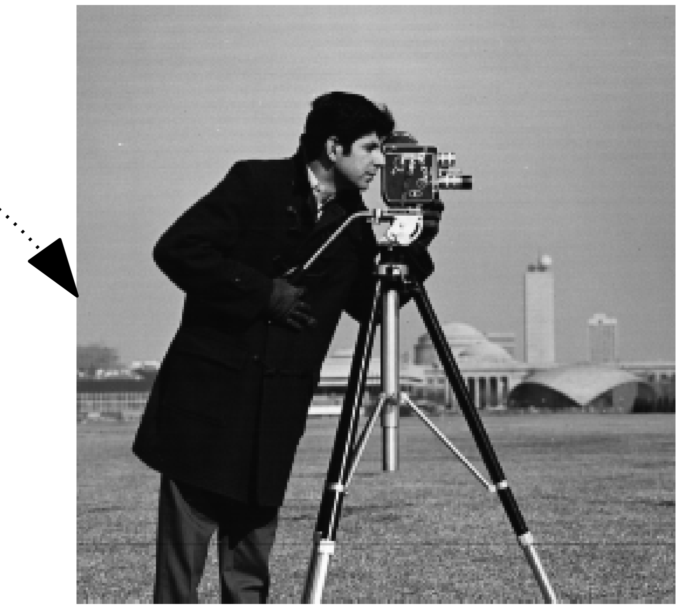

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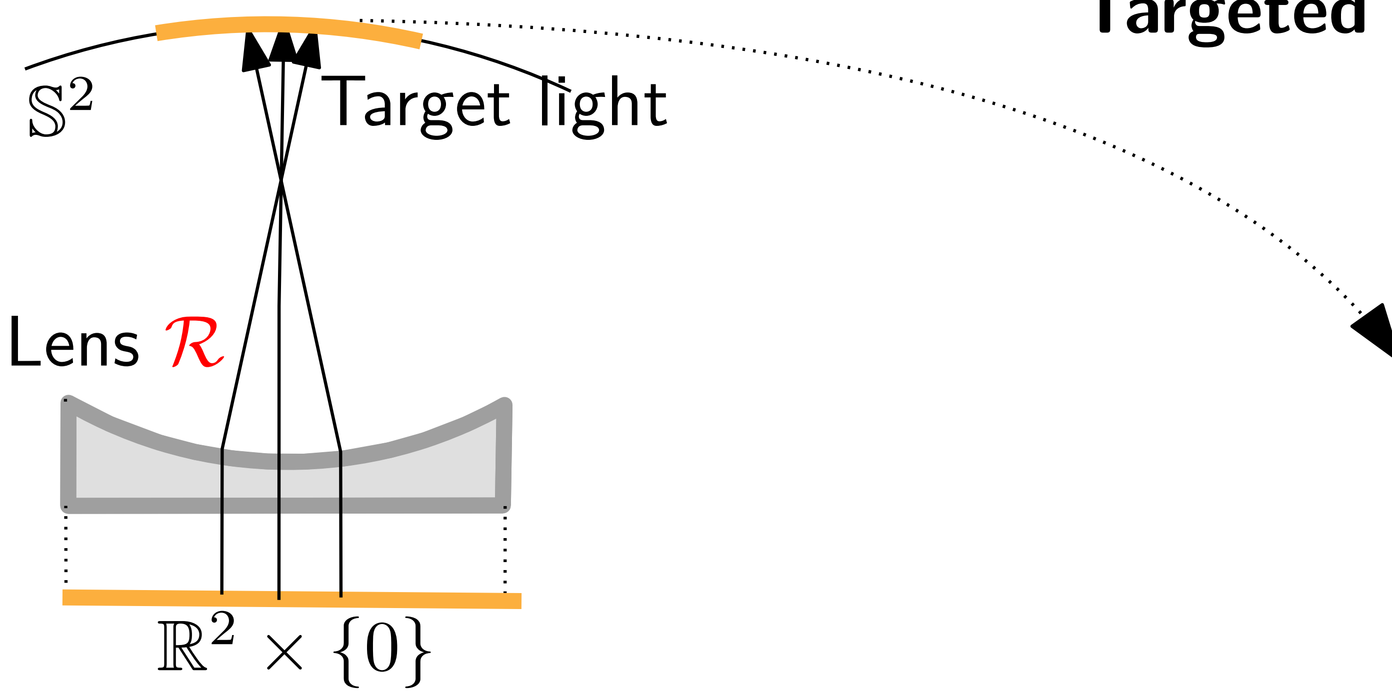

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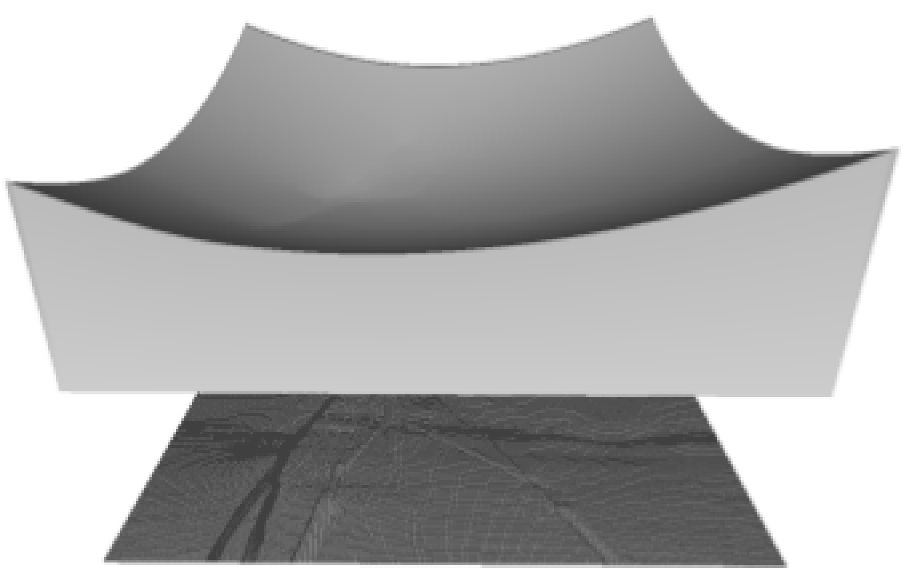

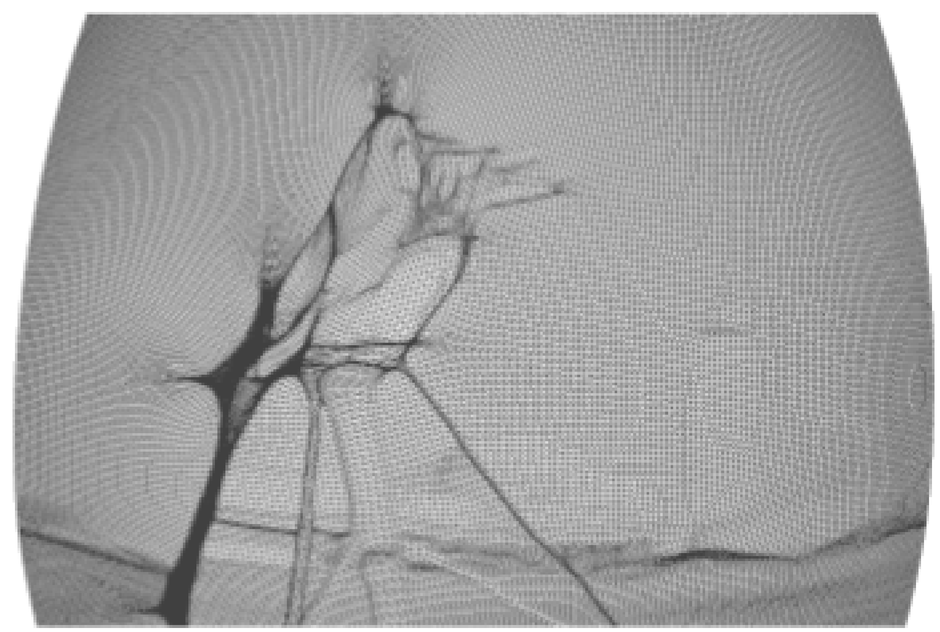

Reflected image

Example 3: physical prototype

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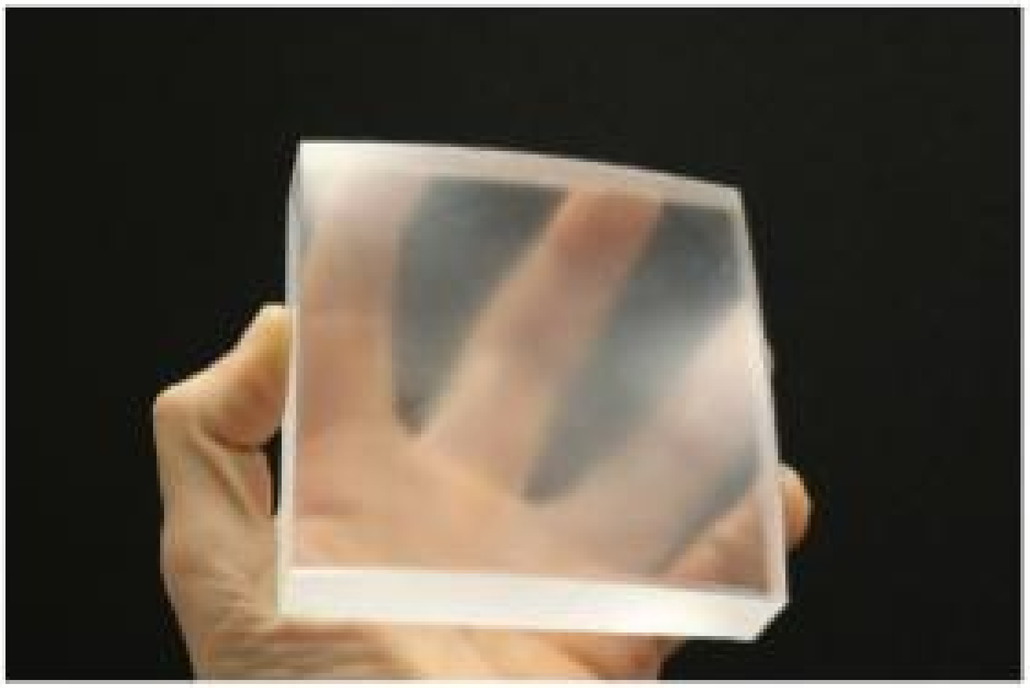

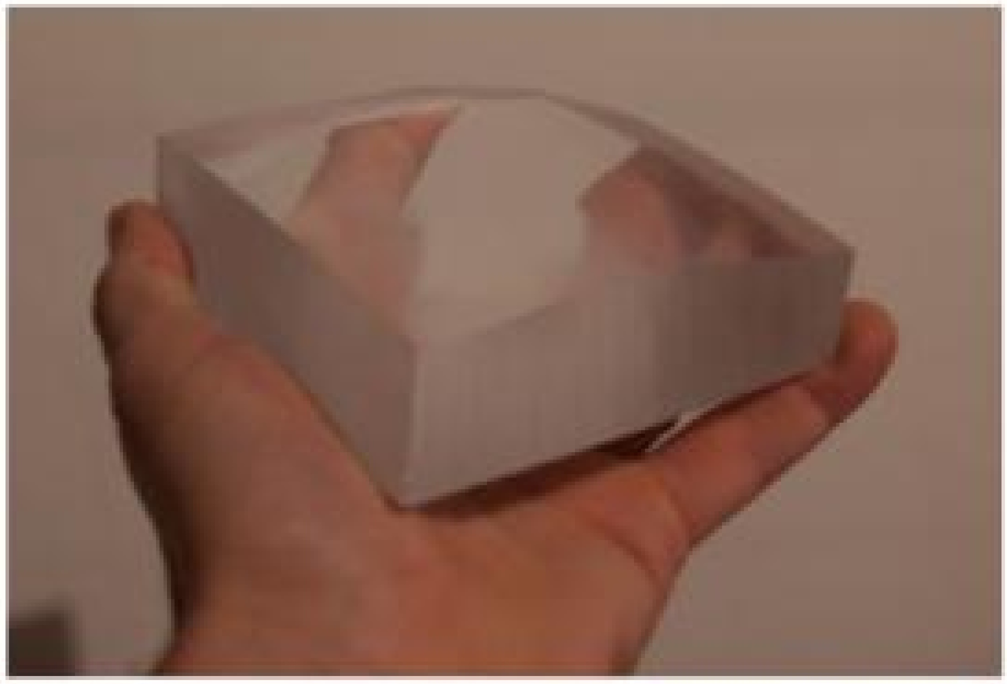

5. Perspectives

Prescribed generated jacobian equations?

Example: Near-field reflector problems: paraboloids $\longrightarrow$ ellipsoids

Prescribed generated jacobian equations?

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- **Paraboloids** of revolution are parameterized radially by $x \in \mathbb{S}^2 \mapsto \frac{\kappa_y}{1-\langle x|y\rangle}$

$$\longrightarrow \arg\min_y \frac{\kappa_y}{1-\langle x|y\rangle} = \arg\min_y \log(\kappa_y) - \log(1 - \langle x|y\rangle)$$

"potential" + "cost"

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$\longrightarrow$ one needs to consider generalized Laguerre cells of the form

$$\widetilde{\mathrm{Lag}}_y(\psi) = \{x \in X; \forall z \in Y,\ c(x,y,\psi(y)) \leq c(x,z,\psi(z))\}$$

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Prescribed generated jacobian eqn = generalization of OT with additional non-linearity

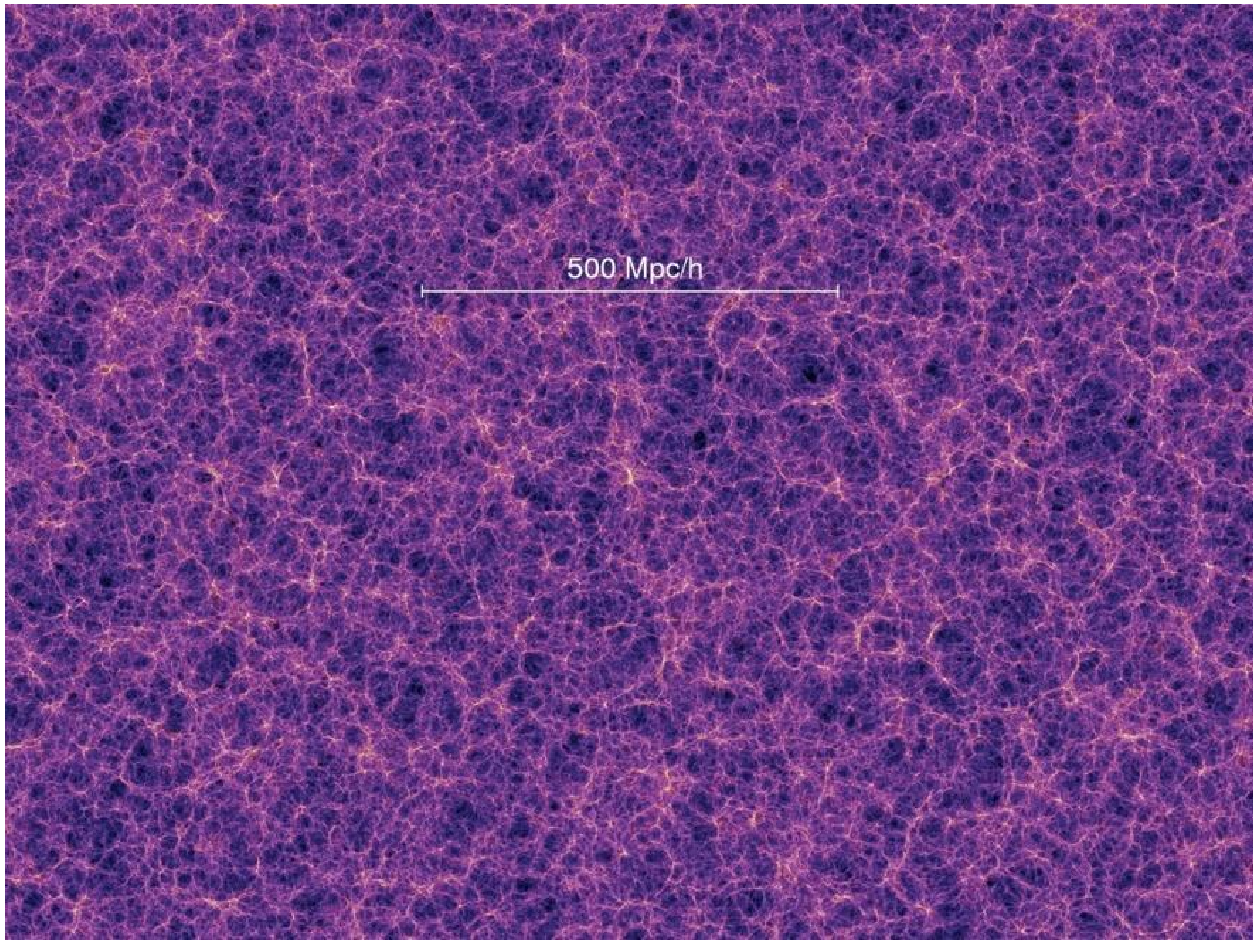
$\longrightarrow$ **also occur naturally in economics** $\longrightarrow$ no variational formulation

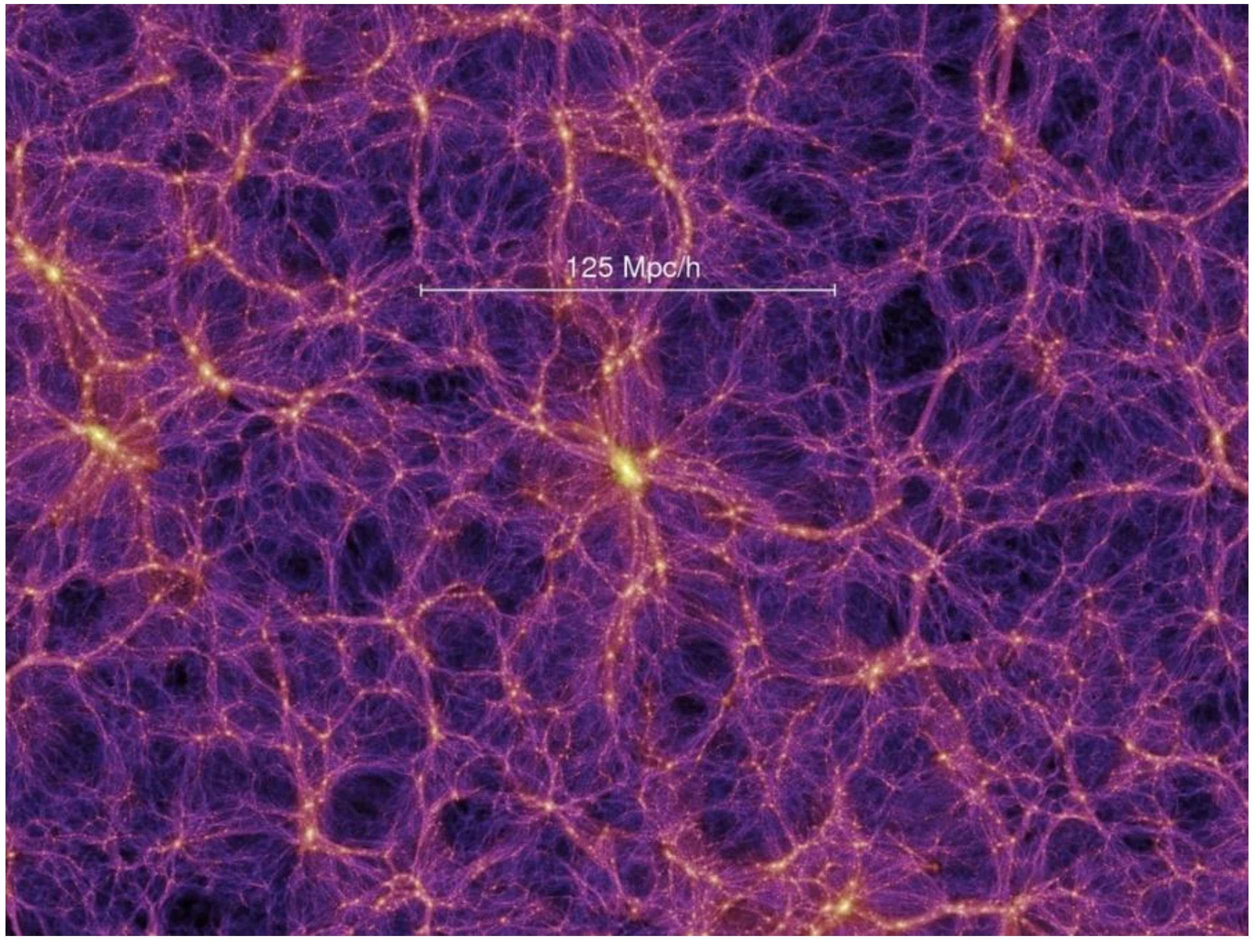
$\longrightarrow$ no satisfying numerical methods

Large-scale computations?

Under Zeldovich's approximation, early universe reconstruction problem
$\iff$ OT problem between Lebesgue on $\mathbb{R}^3/\mathbb{Z}^3$ and a probability measure μ describing the current distribution of mass in the universe.

[Brenier, Frisch, Hénon, Loeper, Matarrese, Mohayaee and A. Sobolevskii, 2003]





Millenial simulation project, MPI Astrophysics

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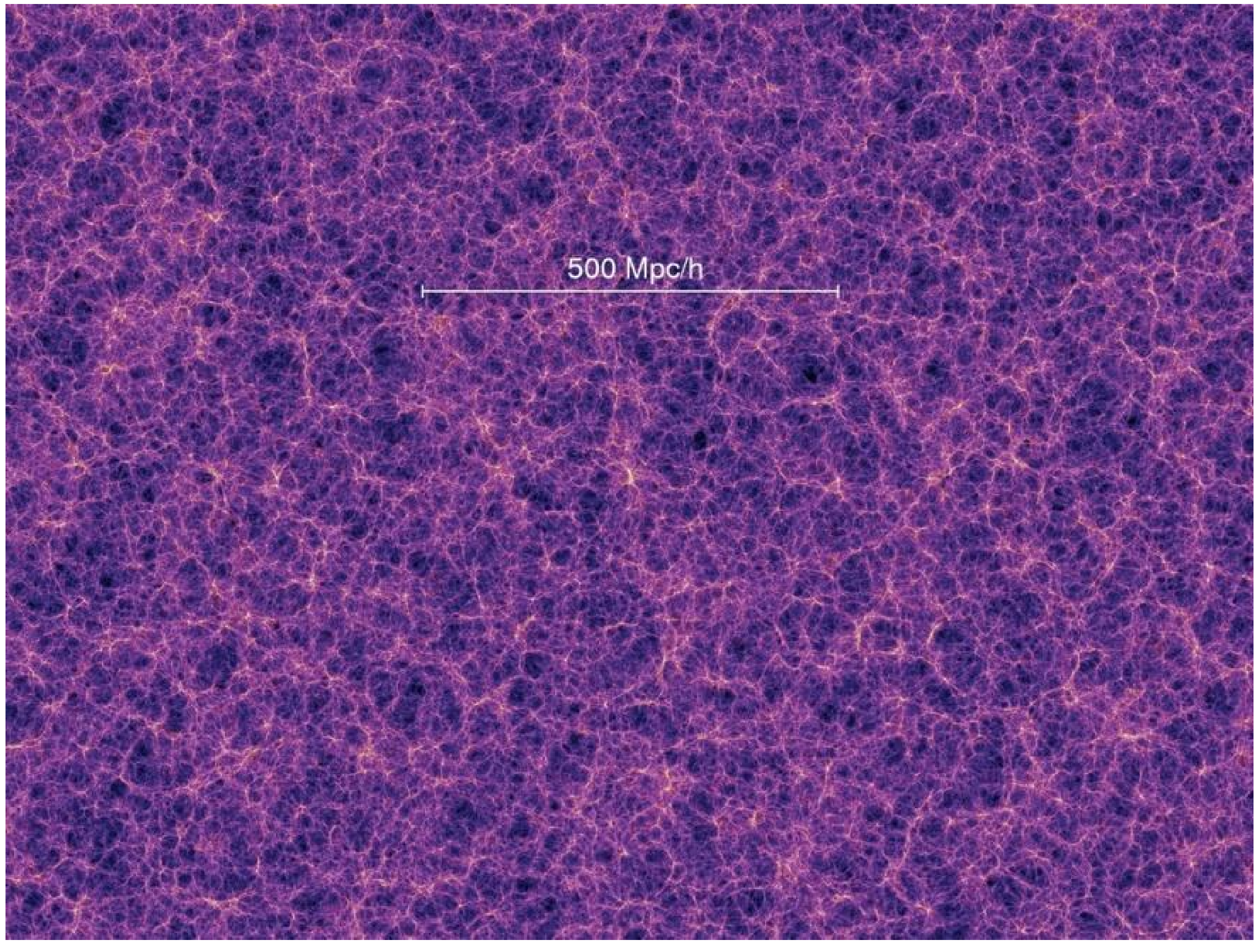
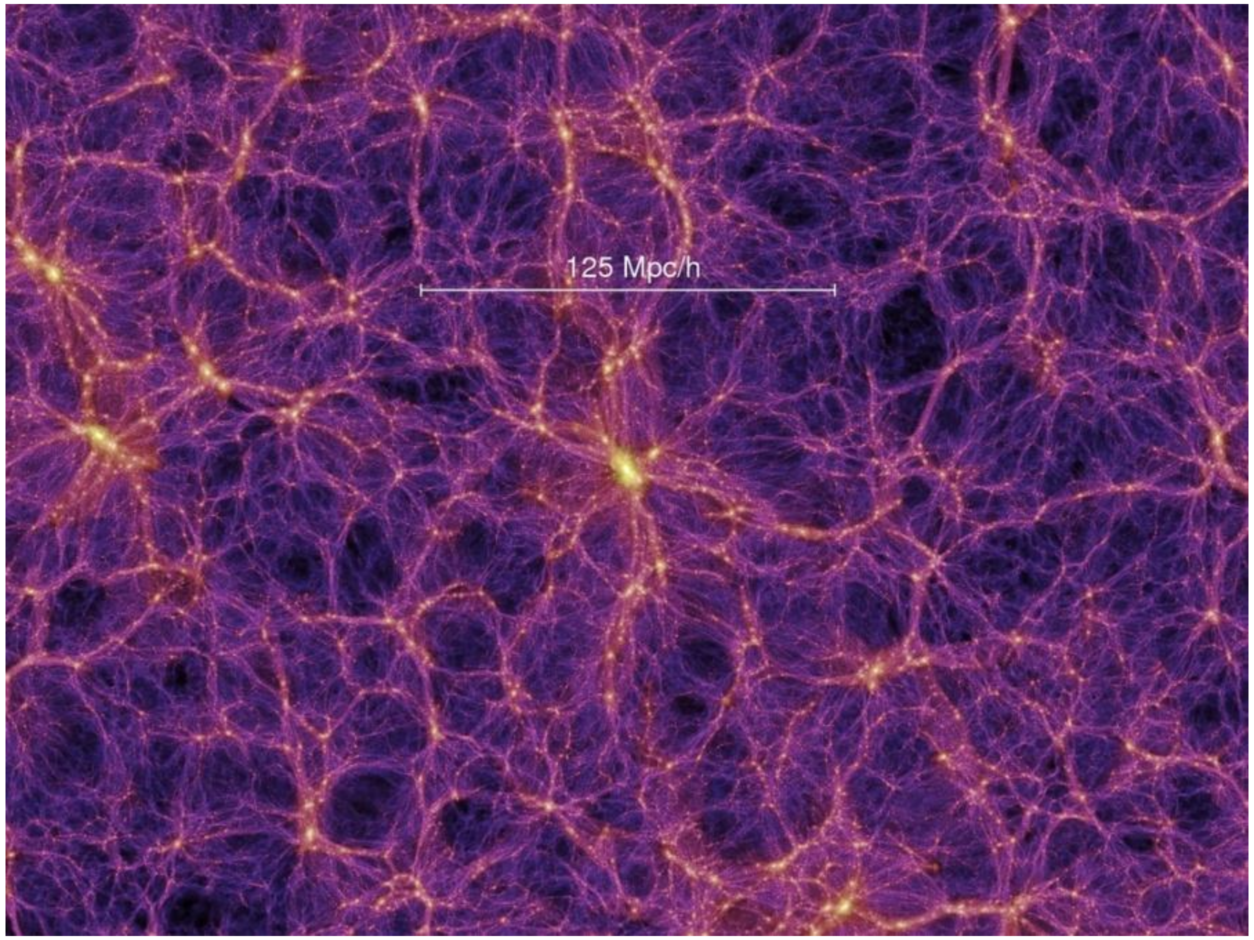




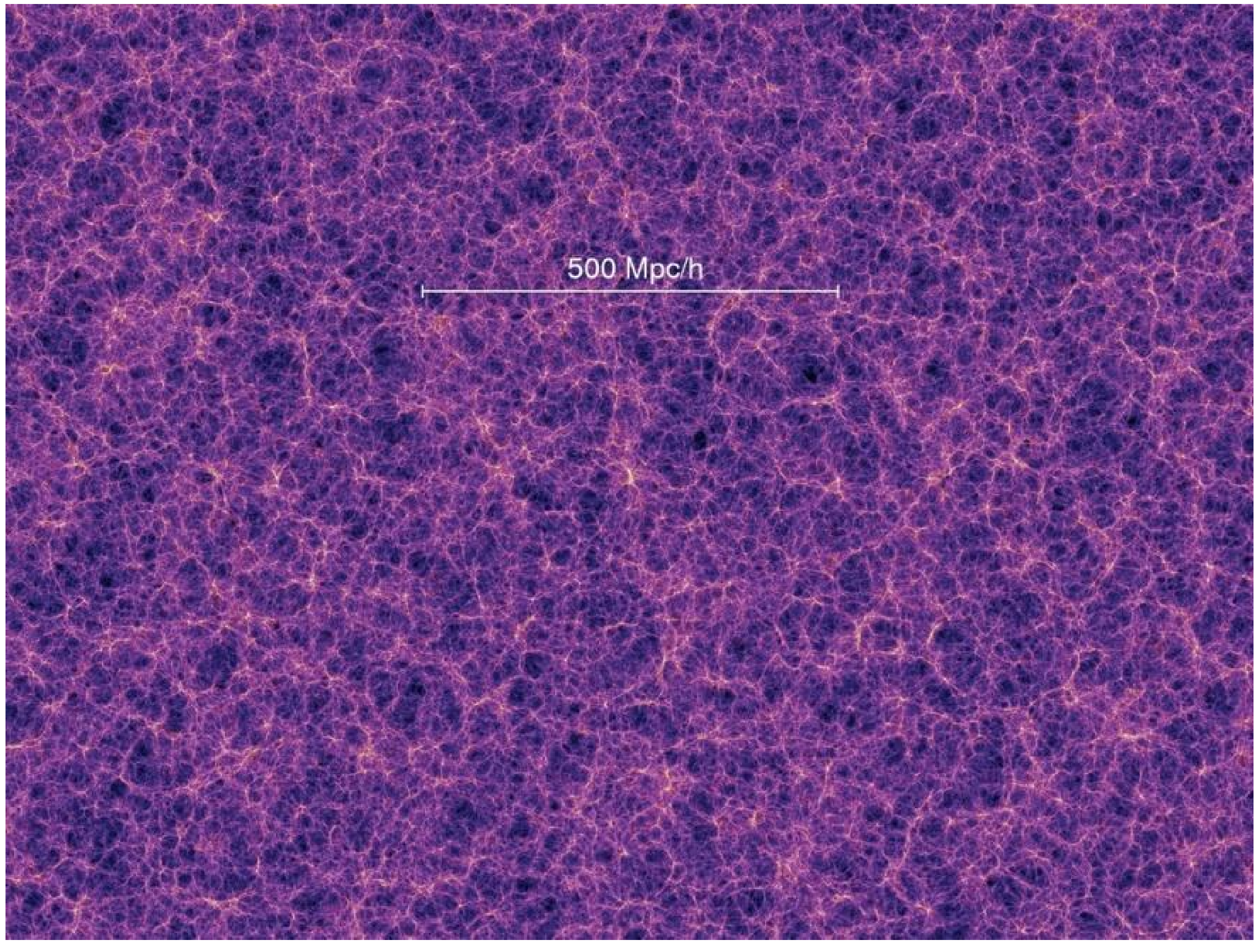
Millenial simulation project, MPI Astrophysics

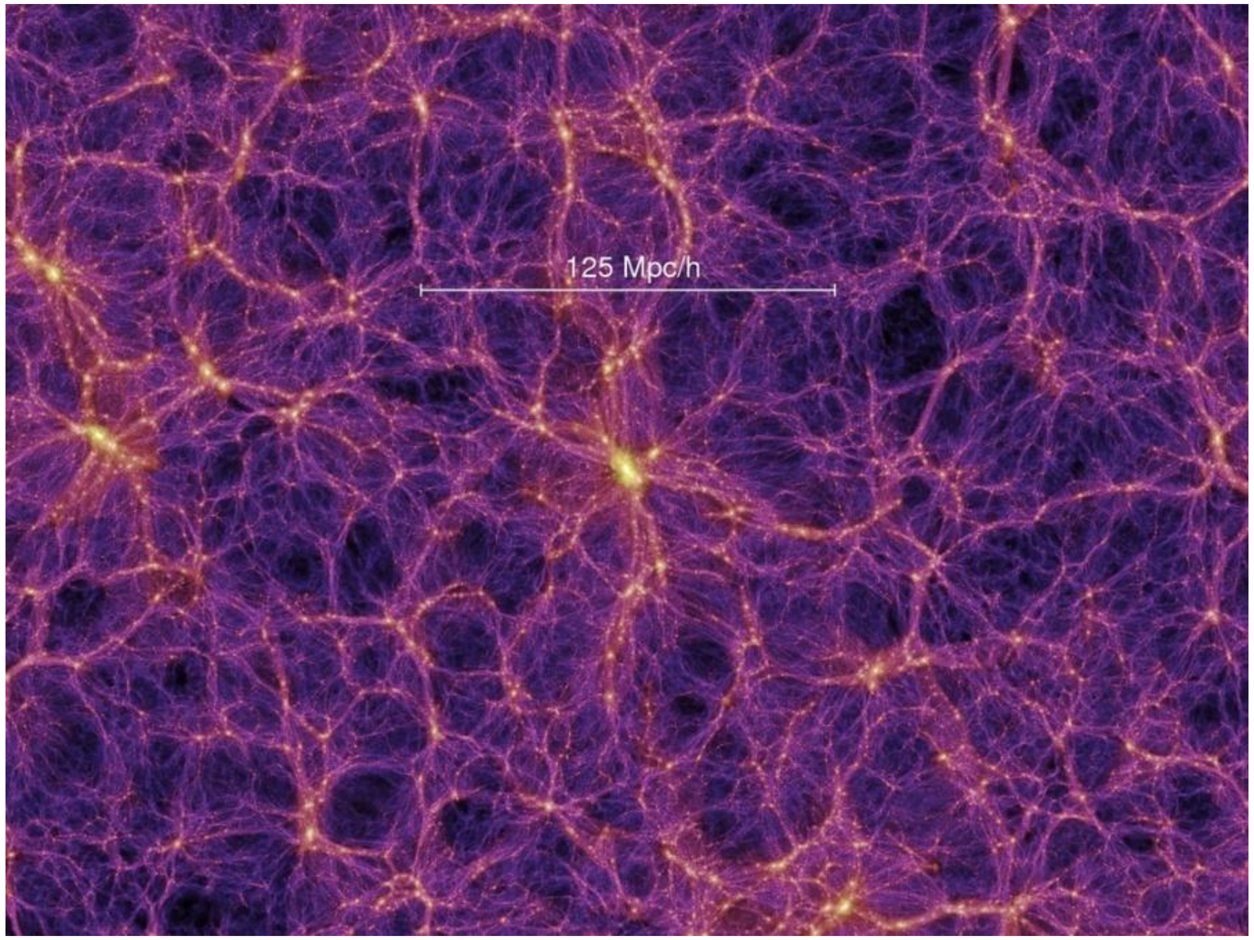
$\longrightarrow$ μ is approximated using databases of *clusters of galaxies*, with up to 10^8 entries.

Large-scale computations?

Under Zeldovich's approximation, early universe reconstruction problem
$\iff$ OT problem between Lebesgue on $\mathbb{R}^3/\mathbb{Z}^3$ and a probability measure μ describing the current distribution of mass in the universe.

[Brenier, Frisch, Hénon, Loeper, Matarrese, Mohayaee and A. Sobolevskii, 2003]





Millenial simulation project, MPI Astrophysics

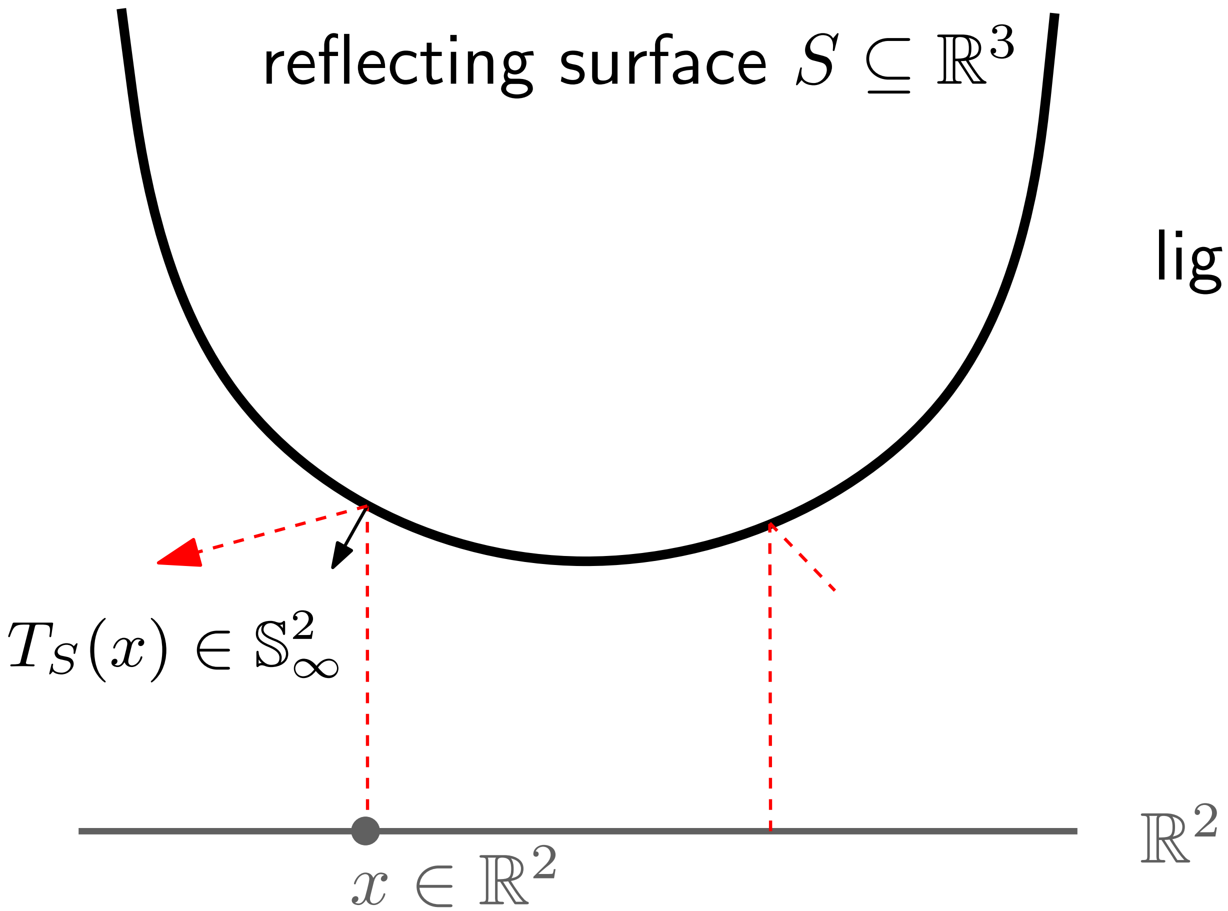
$\longrightarrow$ μ is approximated using databases of *clusters of galaxies*, with up to 10^8 entries.

$\longrightarrow$ state-of-the art semi-discrete solvers: 10^6 Diracs in $\simeq$10 min in 3D...

Appendix

4. Collimated source / Far-field target

Collimated source / far-field target



collimated light source $:= \rho \in \mathrm{Prob}^{\mathrm{ac}}(\mathbb{R}^2)$

light distribution after reflection $T_{S\#}\rho \in \mathrm{Prob}(\mathbb{S}^2_\infty)$

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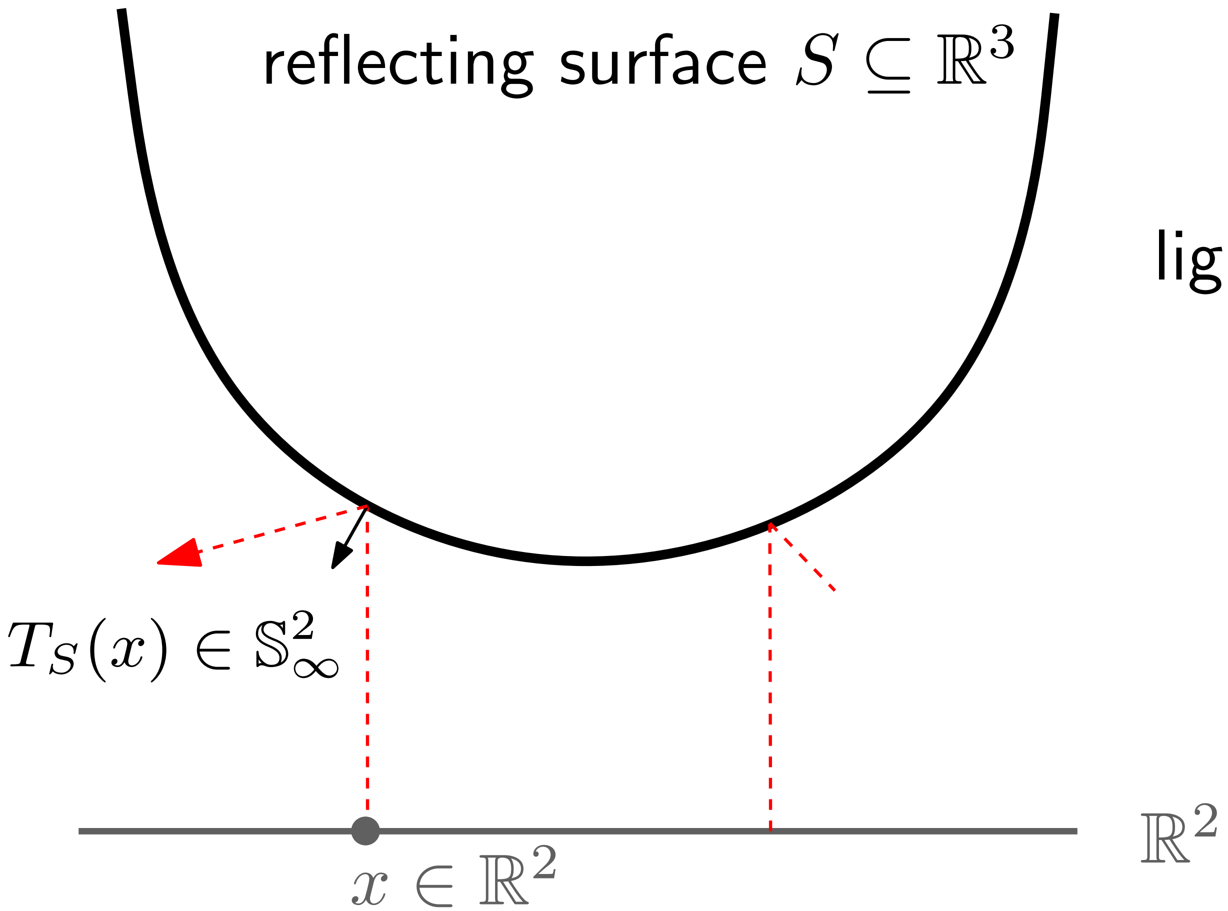

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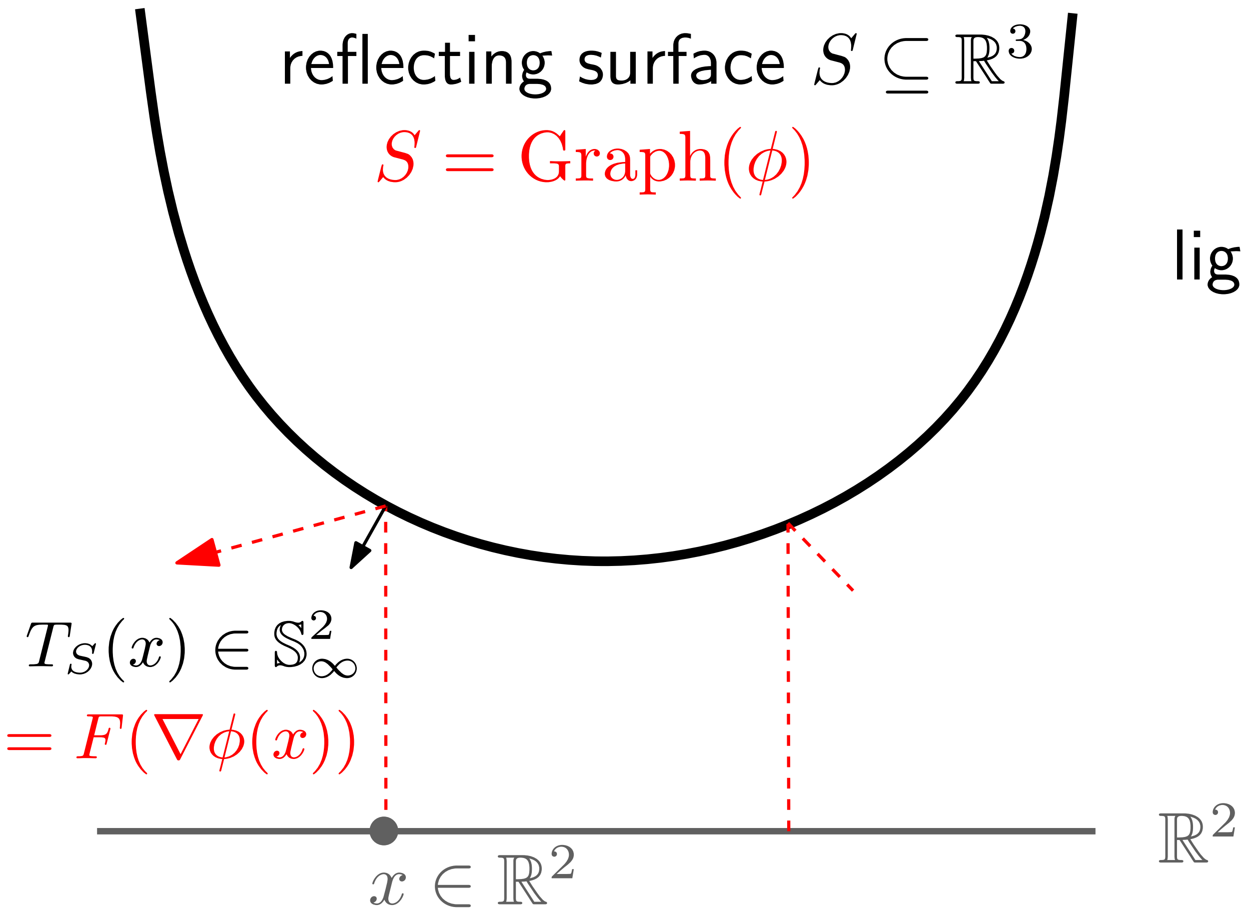

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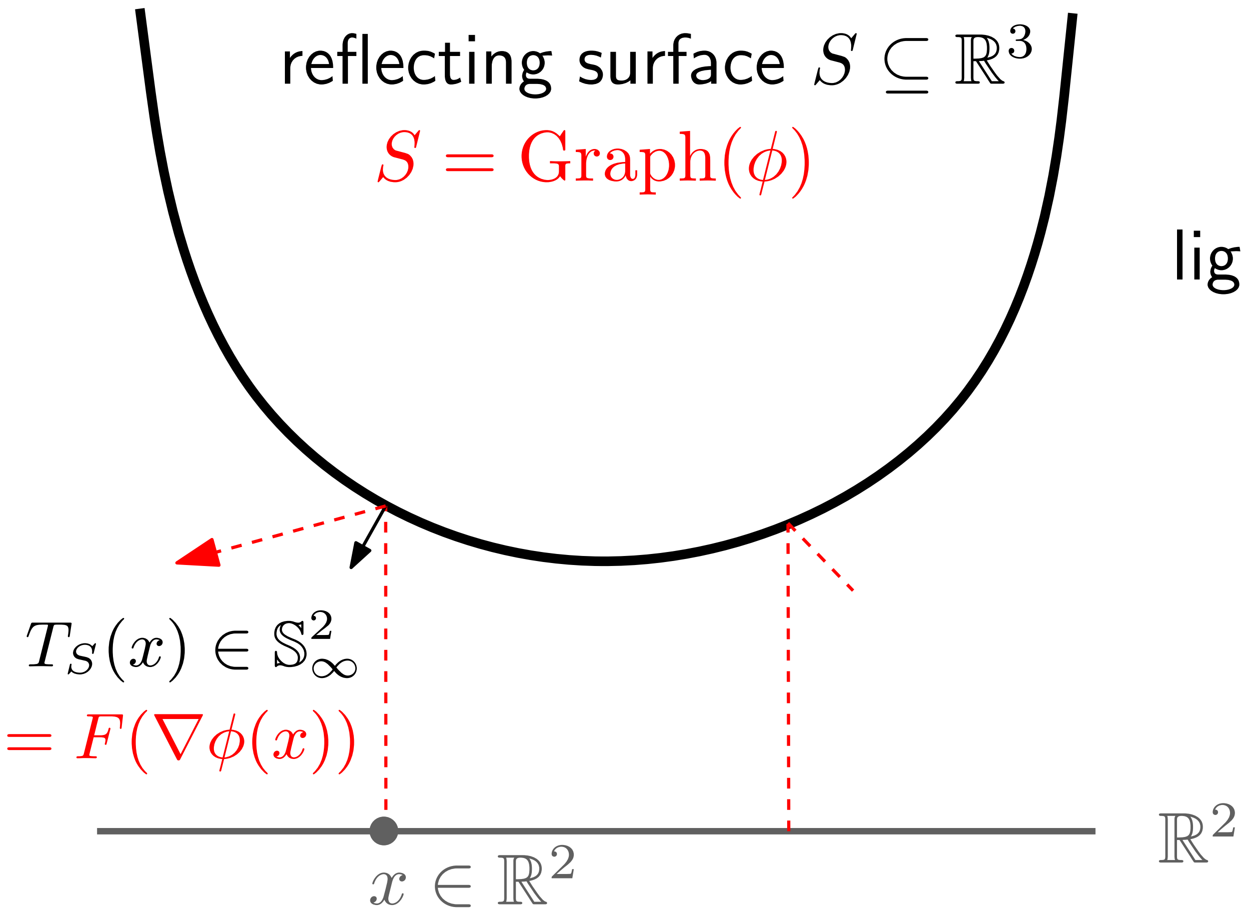

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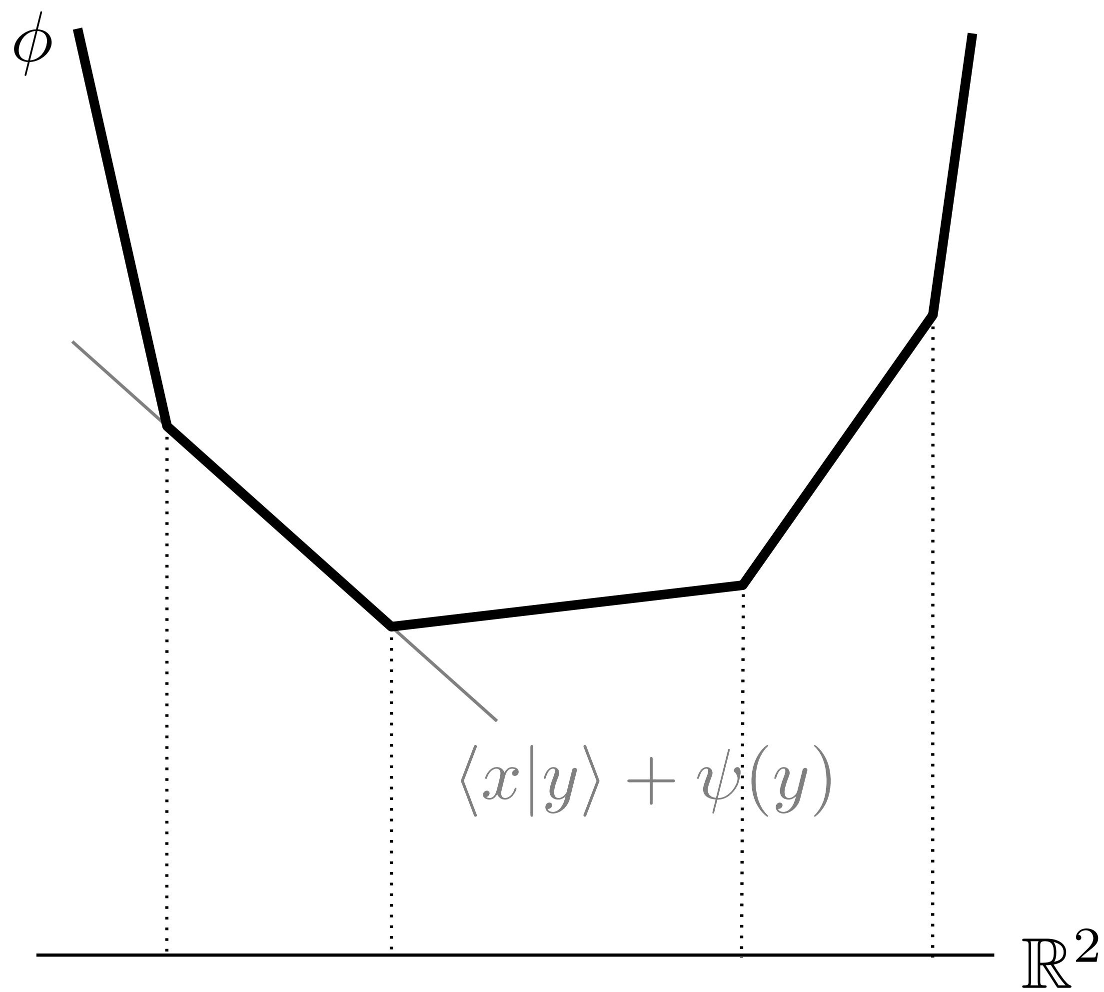
$$\iff \nabla\phi_\#\rho = \nu \text{ where } \nu = F^{-1}_\#\mu \in \mathrm{Prob}(\mathbb{R}^2)$$

Collimated source/far field target: semidiscrete case

Assuming $\nu := \sum_{y \in Y} \nu_y \delta_y$ and $\phi(x) = \max_{y \in Y} \langle x|y \rangle - \psi(y)$ for some $\psi : Y \to \mathbb{R}$.



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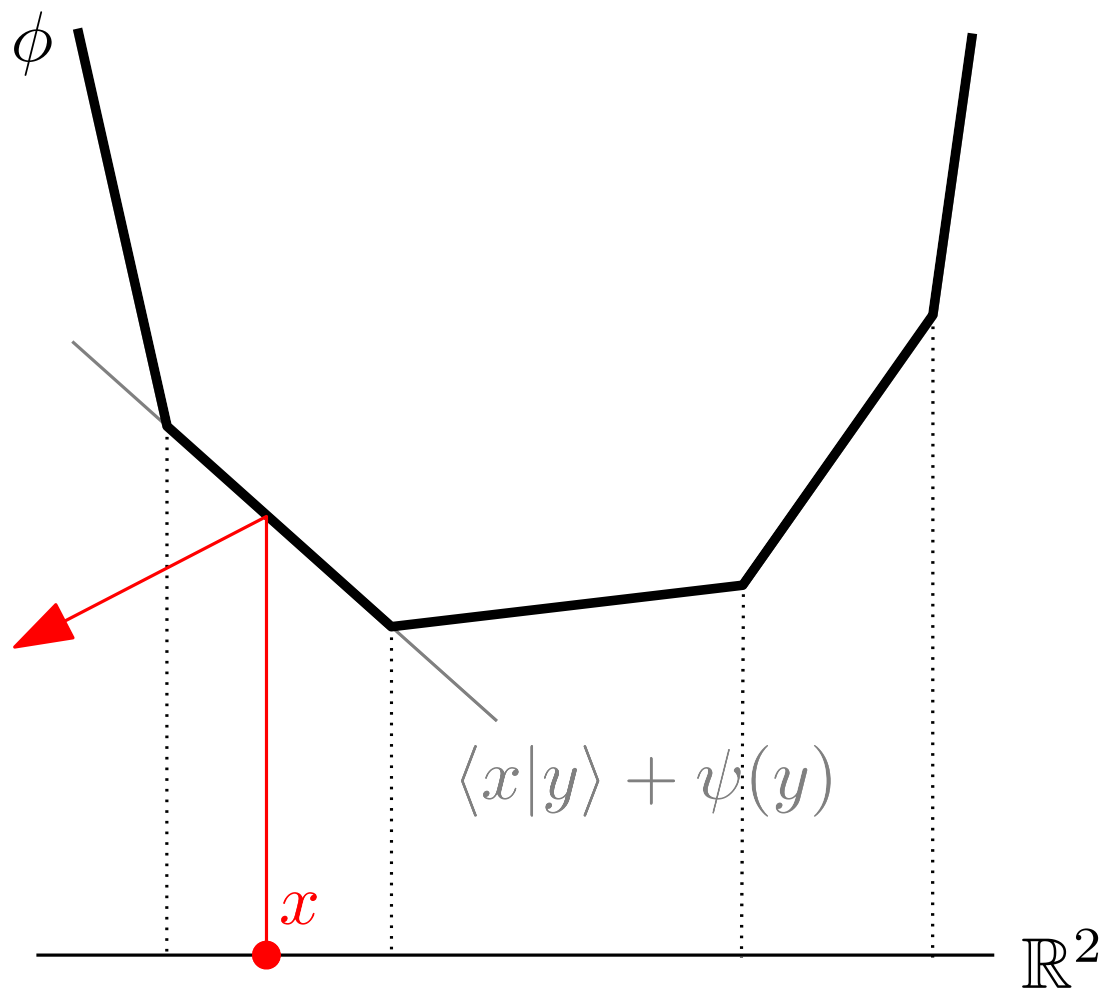

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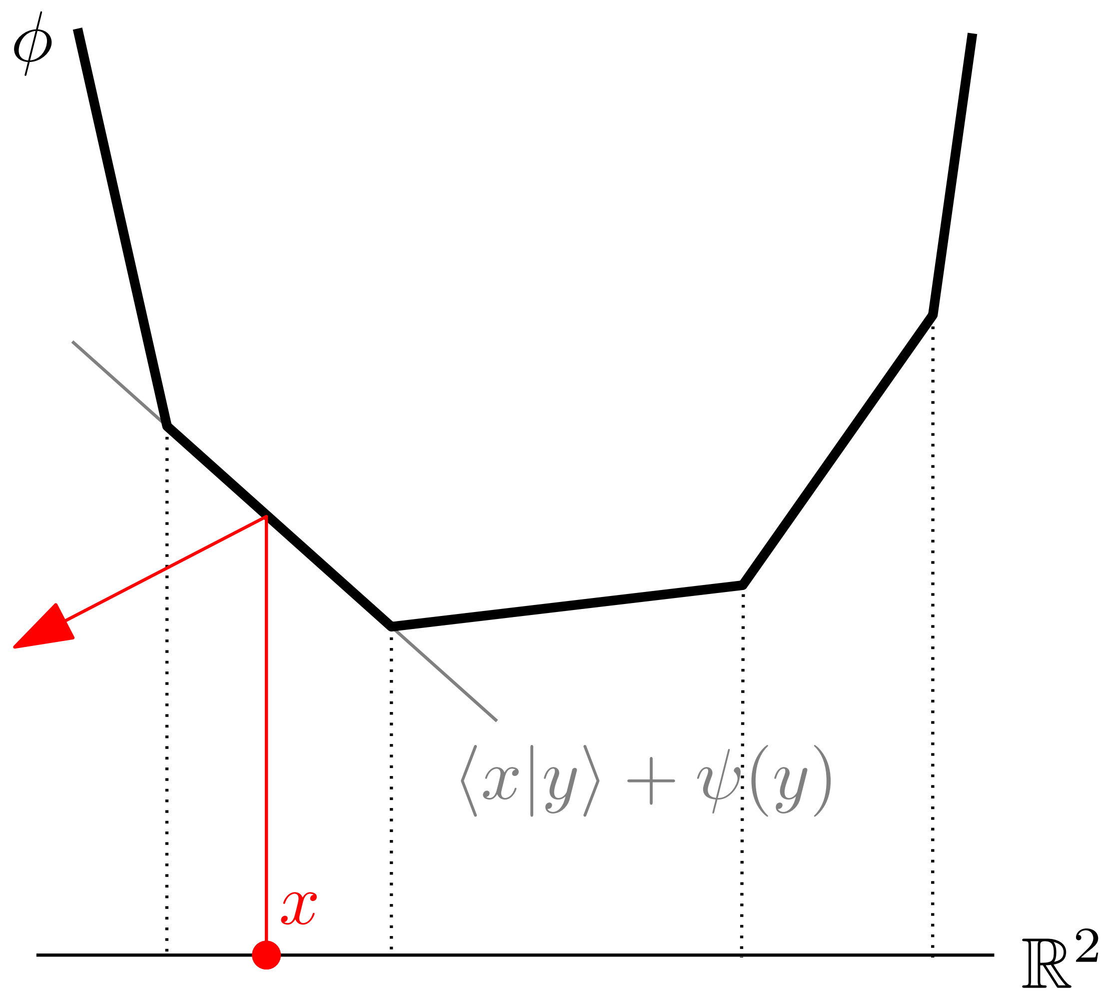

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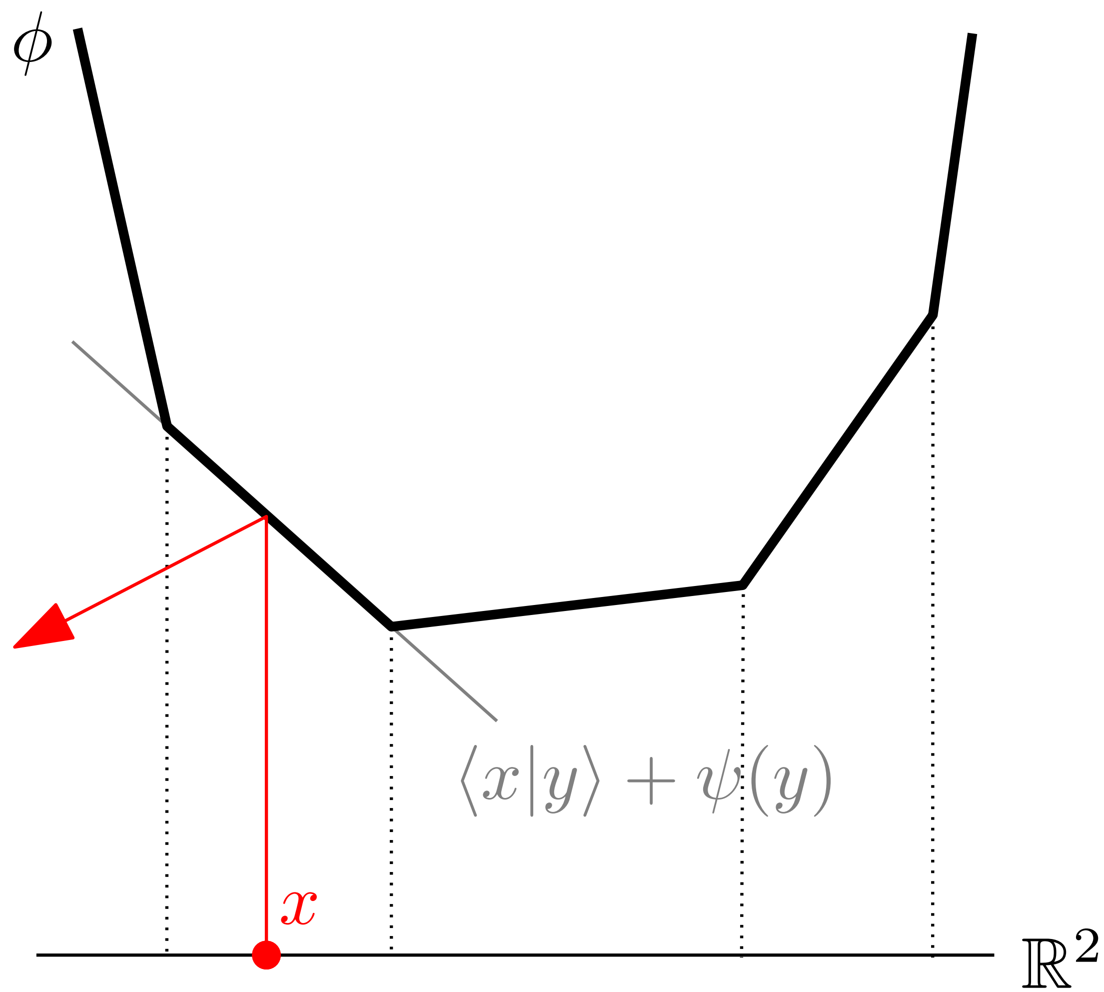

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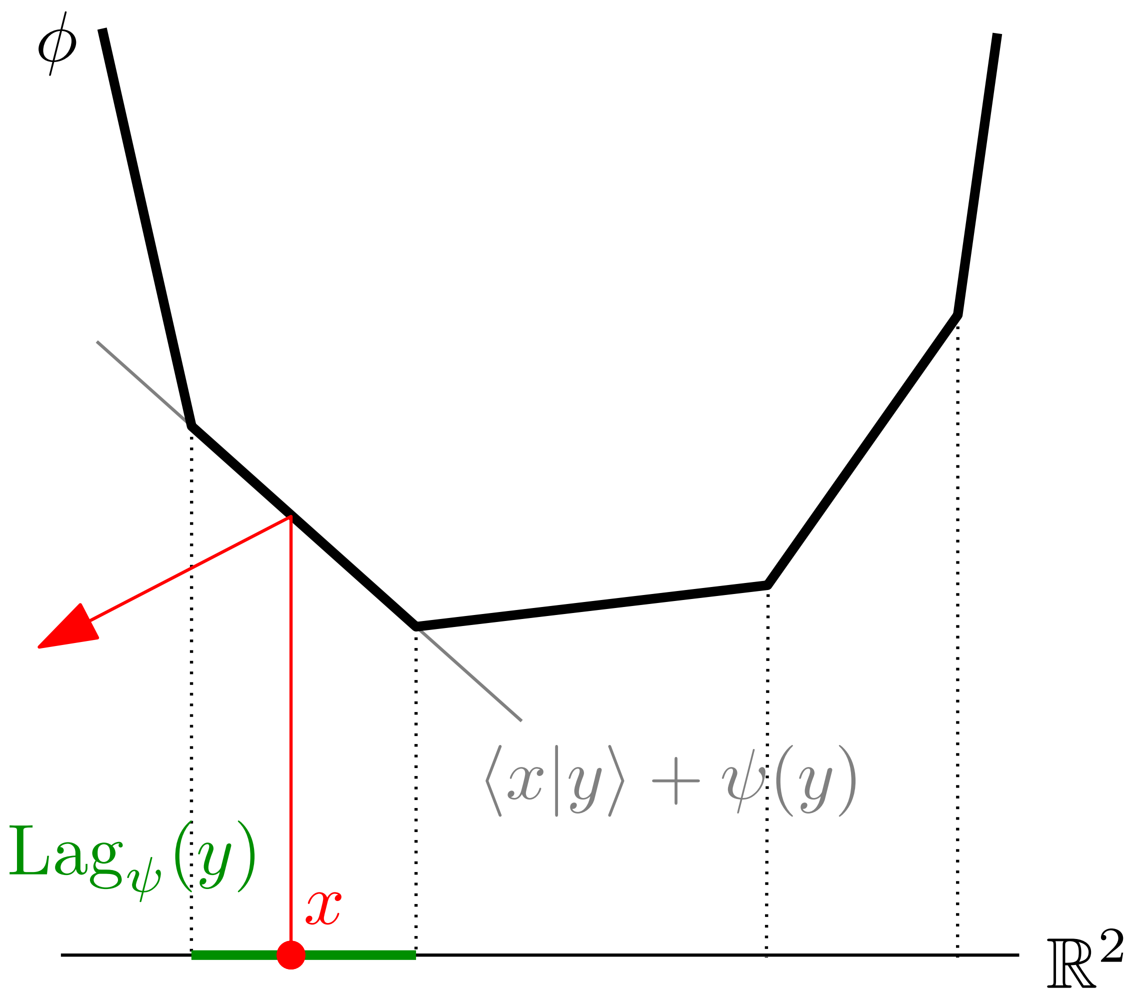

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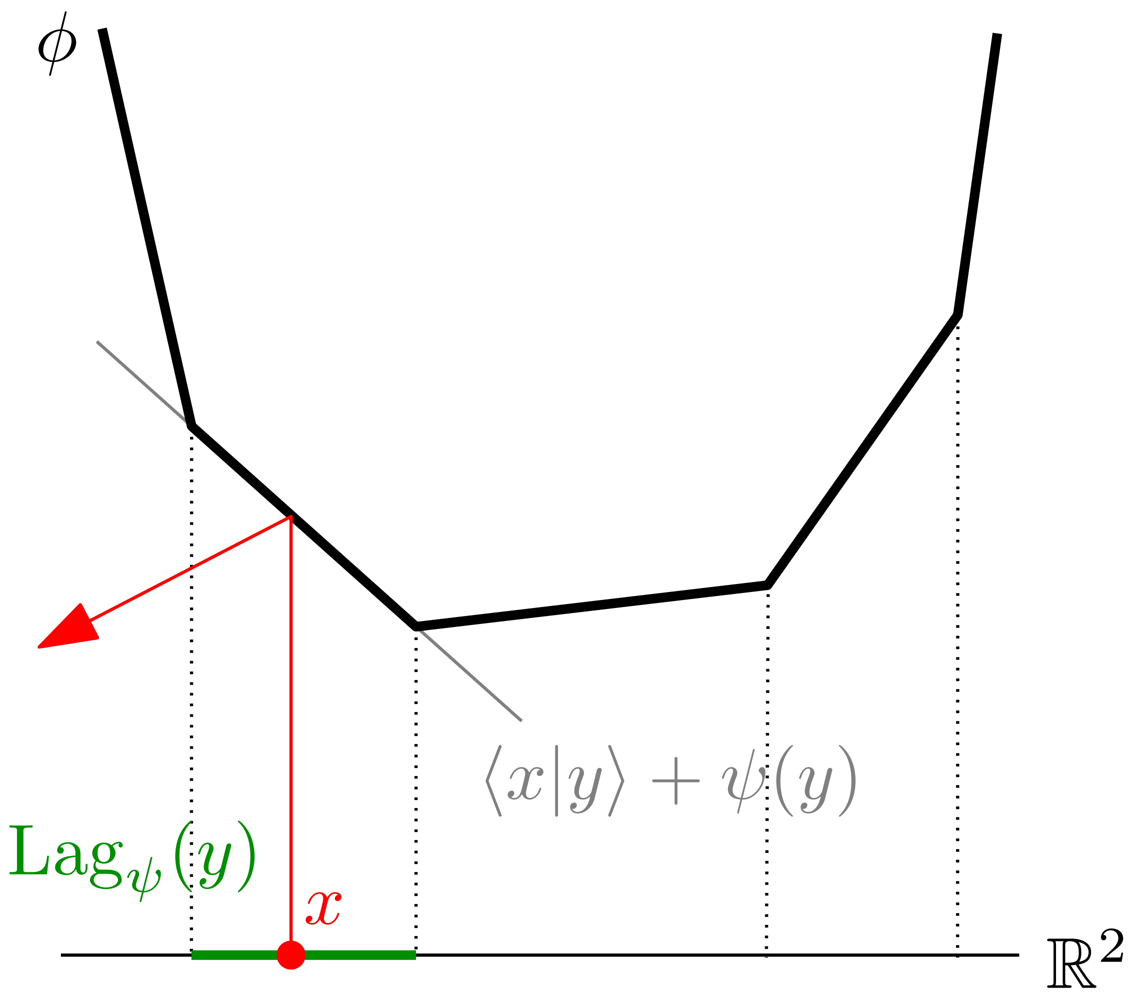

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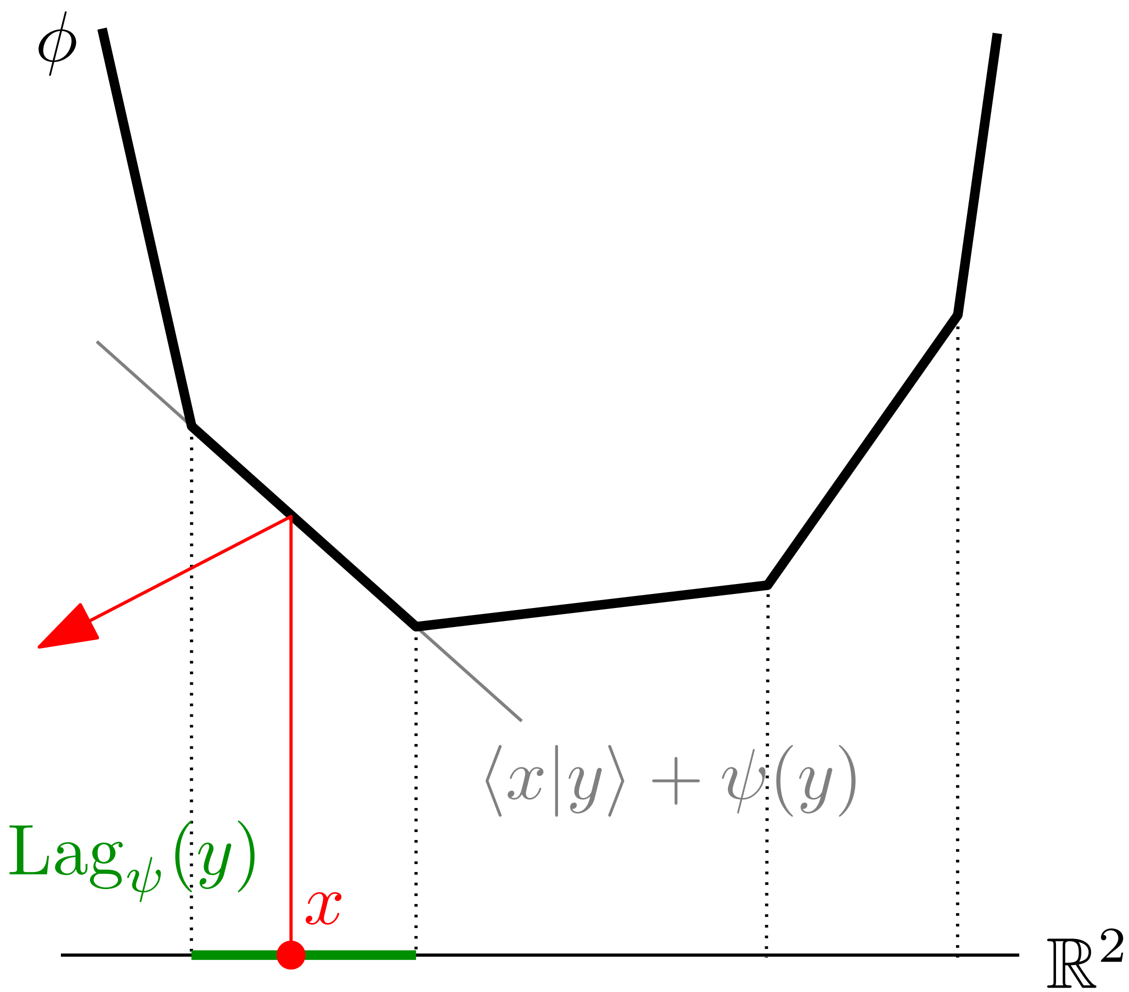

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$\longrightarrow$ Equivalent to a semidiscrete optimal transport problem for $c(x,y) = -\langle x|y\rangle$

$\longrightarrow$ $\simeq$ Minkowski problem (reconstructing a polyhedron from facet areas/directions)

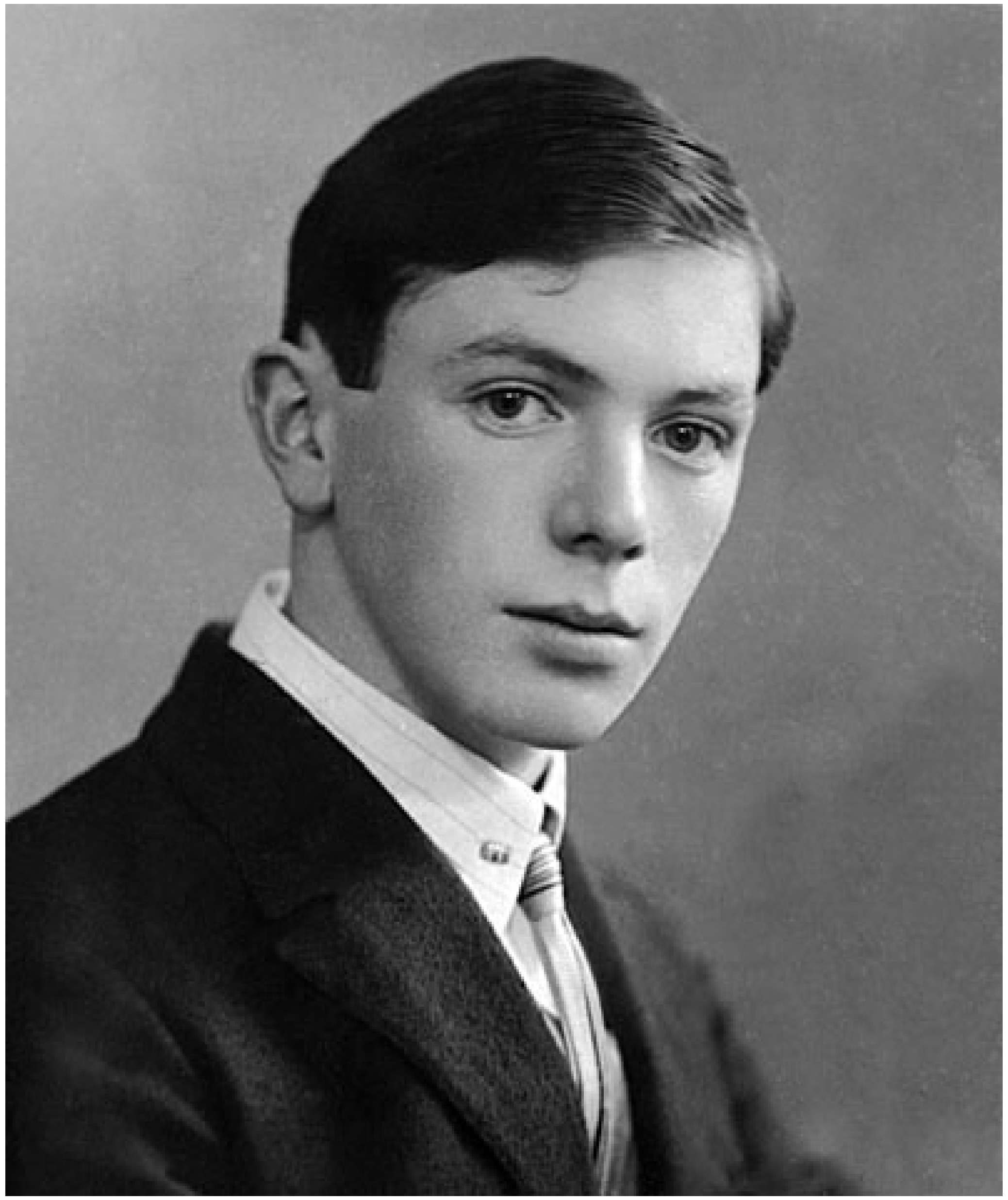
Numerics: collimated source/far-field target

[M. '15]

$\longrightarrow$ $c = \|.\|^2$ on $\mathbb{R}^2$

$\longrightarrow$ implementation using Python/C++: `github.com/mrgt/PyMongeAmpere`

$\longrightarrow$ post-processing (spline interpolation, raytracing) by S. Legrand (Inria engineer)



targeted image $N = 400 \times 480$

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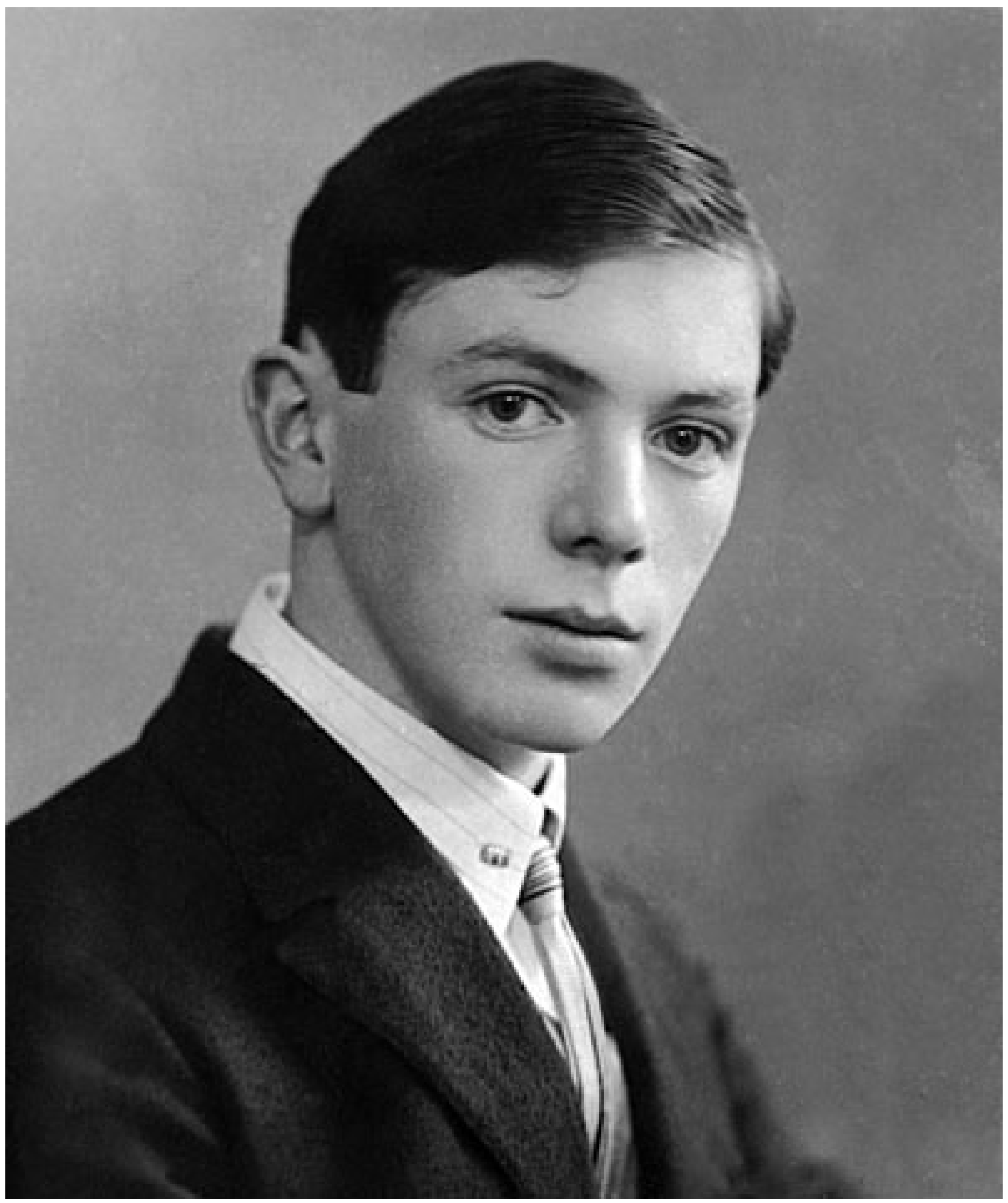

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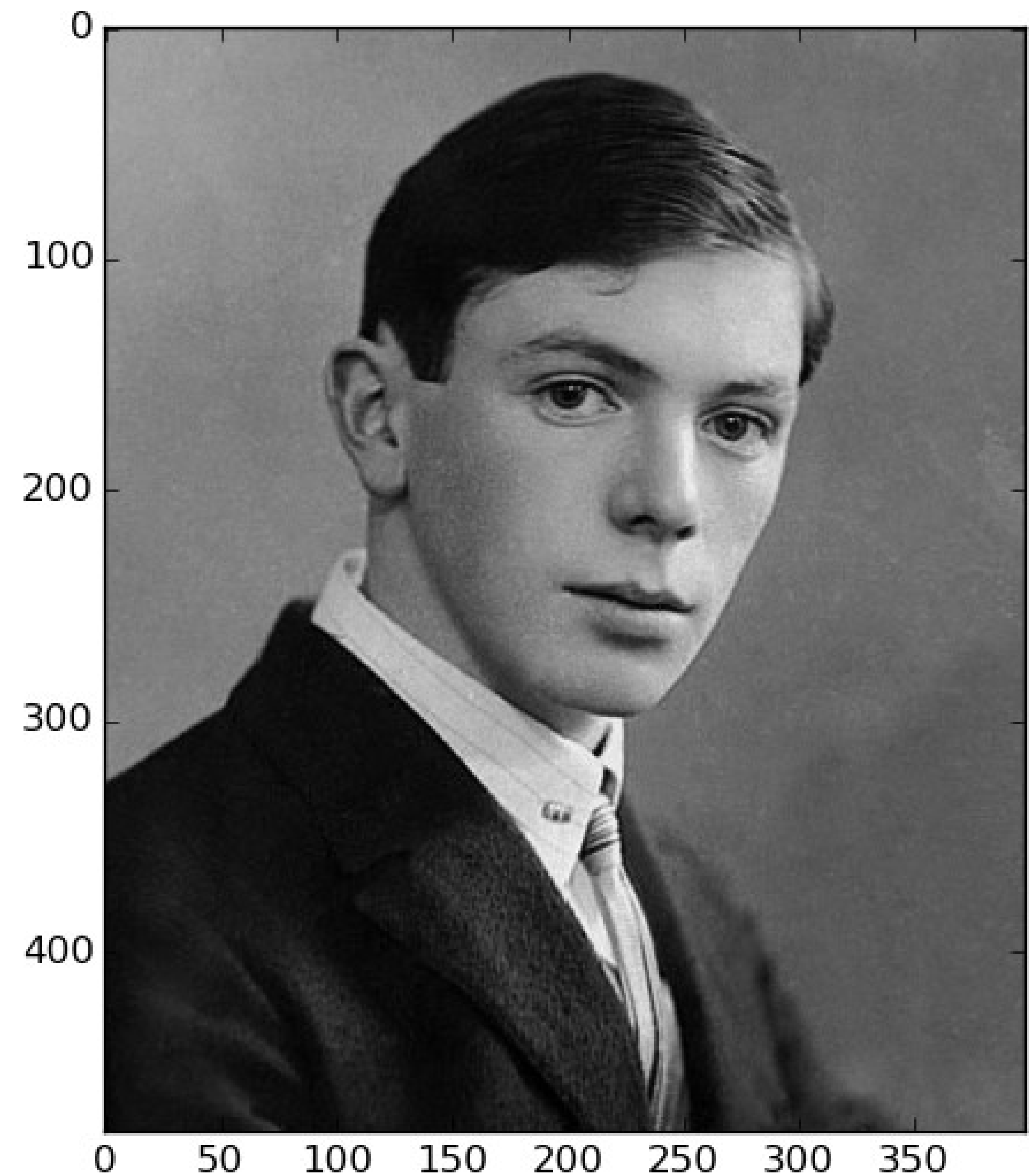

reflected image

3. Convergence of the damped Newton's algorithm for **(MTW)** costs

[Kitagawa, M., Thibert 2016]

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$\longrightarrow$ Discrete analogue of Ma-Trudinger-Wang / Loeper's condition

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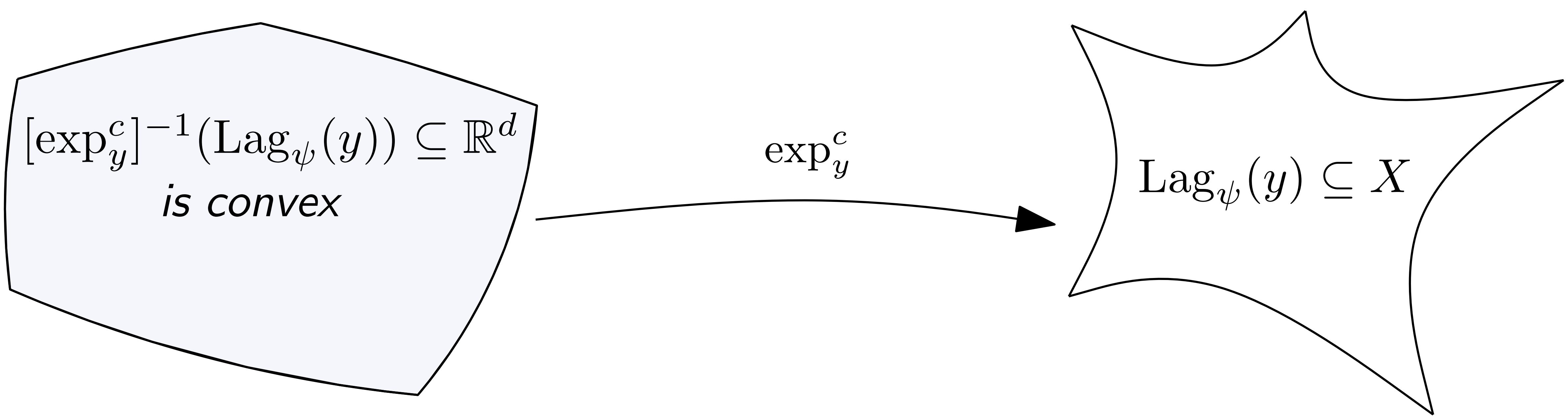

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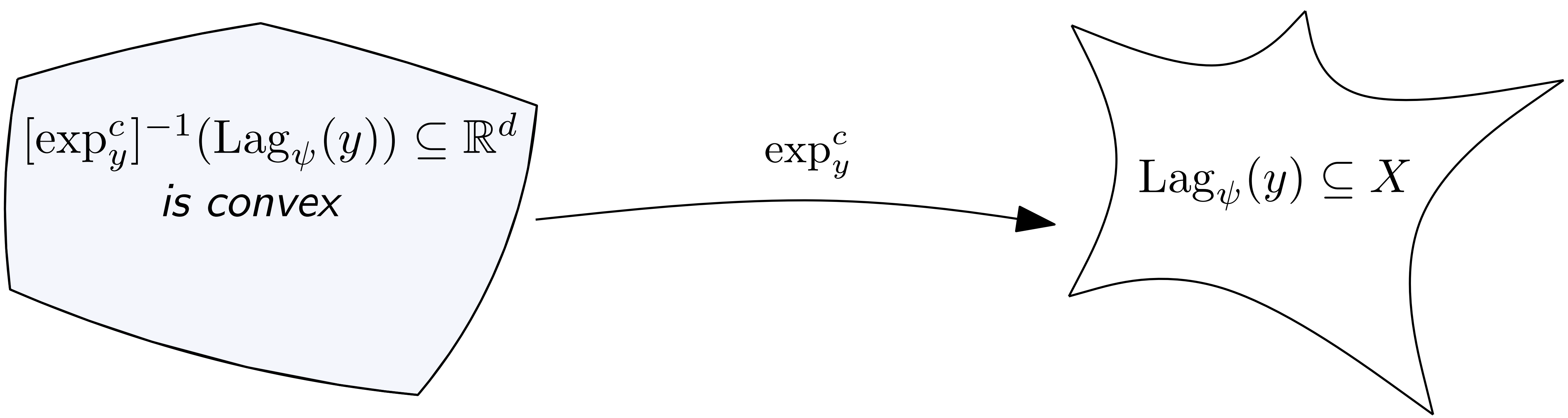

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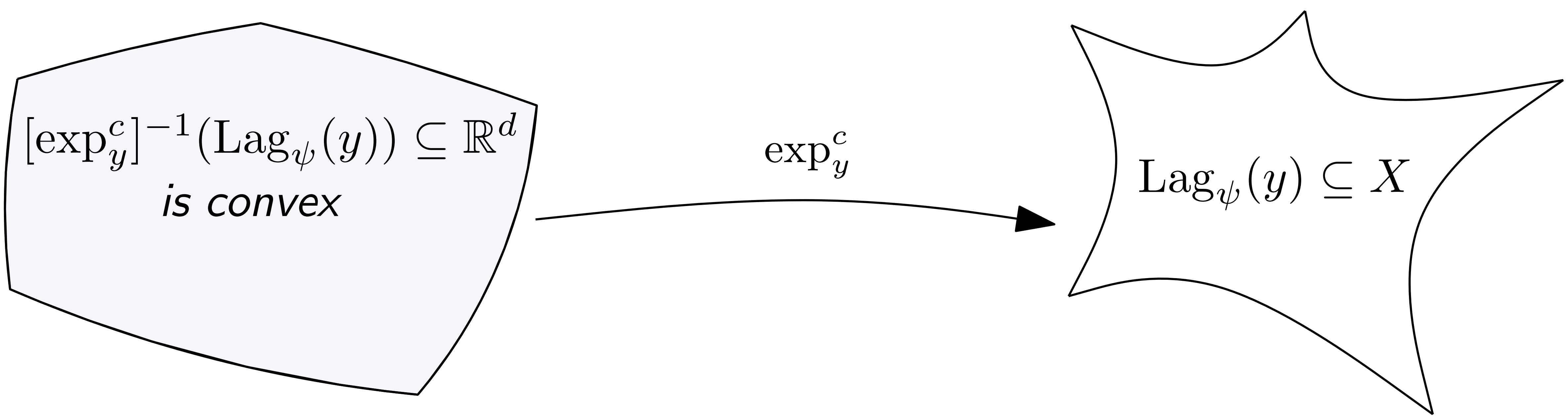

$\longrightarrow$ Restrictive condition, which fortunately is satisfied for the inverse reflector problem.

Examples of costs satisfying **(MTW)**

- **(CS/FF)** $c(x, y) = -\langle x|y\rangle$ on $\mathbb{R}^d$, Laguerre cells are polyhedra $\implies$ **(Conv)**

$$\boxed{\mathrm{Lag}_\psi(y) := \{x \in X; \forall z \in Y, \ -\langle x|y\rangle + \psi(y) \leq -\langle x|z\rangle + \psi(z)\}}$$

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$\longrightarrow$ as a consequence, $c(x,y) = \log(1 - \langle x|y\rangle)$ does not satisfy **(Conv)** !

(Smoothness) of Kantorovich's functional

Def: X is called c-convex if for all $y \in Y$, $[\exp_y^c]^{-1}(X)$ is convex.

recall: $G_y(\psi) := \rho(\mathrm{Lag}_\psi(y))$ $\quad E_\varepsilon := \{\psi \in Y^{\mathbb{R}}; \forall y \in Y,\ \rho(\mathrm{Lag}_y(\psi)) \geq \varepsilon\}$

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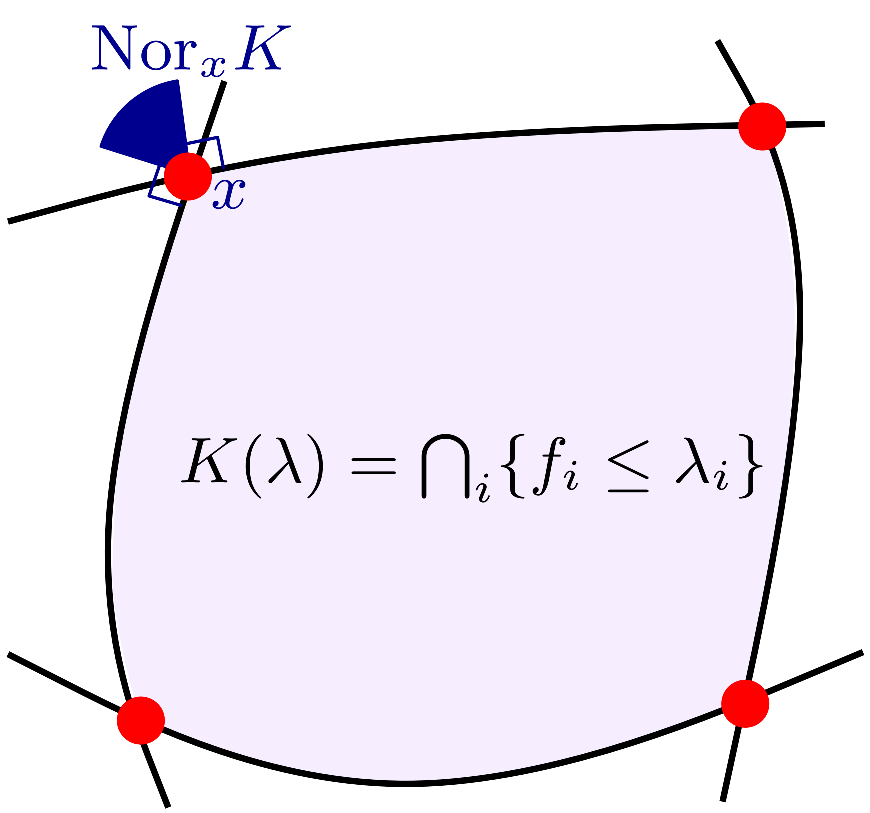
[Kitagawa, M., Thibert '15]



Proof steps:

$\longrightarrow$ Localization to c-exponential chart:

$Y = \{y_0, y_1, \ldots, y_N\} \qquad \hat{X} = \exp_{y_0}^{-1}(X)$ convex

$\hat{\rho} =$ density of $\exp_{y_0}^{-1}\big|_{\#} \rho$

$f_i : p \in \hat{X} \mapsto c(\exp_{y_0} p, y_0) - c(\exp_{y_0} p, y_i)$ quasi-convex

$\lambda_i := \psi(y_i) - \psi(y_0)$

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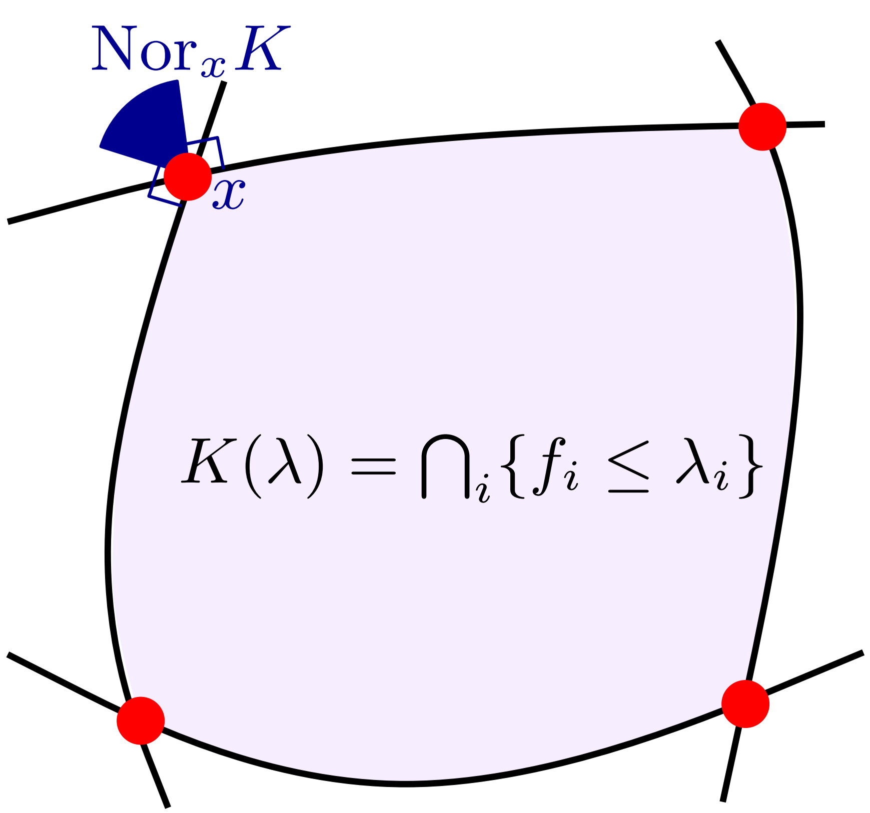

Proof steps:

$\longrightarrow$ Localization to c-exponential chart:

$Y = \{y_0, y_1, \dots, y_N\}$ $\qquad \hat{X} = \exp_{y_0}^{-1}(X)$ convex

$\hat{\rho} =$ density of $\exp_{y_0}^{-1}\big|_{\#} \rho$

$f_i : p \in \hat{X} \mapsto c(\exp_{y_0} p, y_0) - c(\exp_{y_0} p, y_i)$ quasi-convex

$\lambda_i := \psi(y_i) - \psi(y_0)$

$\longrightarrow$ Local regularity result under transversality condition

(Smoothness) of Kantorovich's functional

Def: X is called c-convex if for all $y \in Y$, $[\exp_y^c]^{-1}(X)$ is convex.

recall: $G_y(\psi) := \rho(\mathrm{Lag}_\psi(y))$ $\quad E_\varepsilon := \{\psi \in Y^{\mathbb{R}}; \forall y \in Y,\ \rho(\mathrm{Lag}_y(\psi)) \geq \varepsilon\}$

Theorem: Assume **(Twist)**, **(Twist')**, **(Conv)**, X is c-convex and $\rho \in \mathcal{C}^\alpha(X)$.

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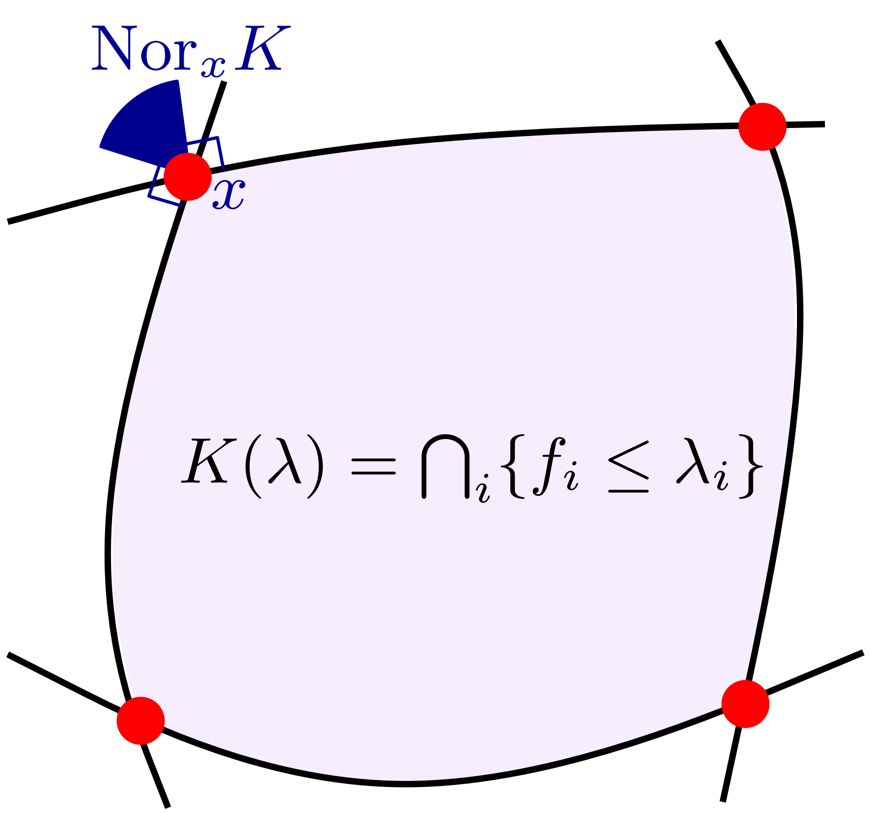

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$\longrightarrow$ $\psi \in E_\varepsilon \implies$ transversality condition

Convergence Theorem

Theorem: Let X be a (closed) bounded domain with smooth boundary, Y be a finite set and $c \in \mathcal{C}^2(X \times Y)$. Assume:

(A) c satisfies **(Twist)**, **(Twist')**, **(Conv)** and X is c-convex

(B) $\rho \in \mathcal{C}^\alpha(X)$ and satisfies a weighted L^1-Poincaré inequality, i.e.

$$\forall f \in \mathcal{C}^1(X), \quad \|f - \mathbb{E}_\rho(f)\|_{\mathrm{L}^1(\rho)} \leq \mathrm{cst} \cdot \|\nabla f\|_{\mathrm{L}^1(\rho)}$$

Then, the damped Newton algorithm for SD-OT converges **globally** with linear rate and locally with rate $(1+\alpha)$.

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  Near field: OT is replaced by **generated Jacobian equations** satisfying $\simeq$**(Conv)**

[Trudinger '12]