Intermediate Statistics

Hypothesis Testing

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Introduction

We've now discussed estimation in two ways:

1. **Point Estimation**: A single value
2. **Confidence Intervals**: A range of values

Typically, we care more about the estimates **in reference to** a given value:

- Is the effect of a policy greater than 0?
- Does something affect women more than men?
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- Do people prefer Popeye's chicken sandwiches over Chick-Fil-A?

All of these questions can be answered under the framework of **hypothesis testing**

Why Hypothesis Test?

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Hypothesis testing **automates** (at least a little bit) **that process**.

HYPOTHESIS AND TEST PROCEDURES

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- If the estimated results are **drastically different** from the world of the null hypothesis, we **reject** it in favor of the alternative

As per the scientific method, a test has two outcomes: we either **reject** $\mathcal{H}_0$ or **fail to reject** (never accept $\mathcal{H}_0$)

The Null Hypothesis, $\mathcal{H}_0$

The null hypothesis is representative of the **status quo**

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Typically, we write $\mathcal{H}_0$ as an **equality**: $\mathcal{H}_0 : \theta = \theta_0$

The Alternative Hypothesis, $\mathcal{H}_a$

There are three ways to assert that a null hypothesis is **misspecified**:

1. $\theta > \theta_0$
2. $\theta < \theta_0$
3. $\theta \neq \theta_0$

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The first two are **one-sided** alternatives, the second is **two-sided**.

- The alternative hypothesis should be **contextually-motivated**
  - Like any good hypothesis, it needs a story
- When properly motivated, one-sided tests have more **power** because they put less strain on the testing procedure

A Generalized Testing Outline

To perform a hypothesis test properly, each of these elements must be **explicitly stated**:

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2. A **test statistic**: the estimated value $\hat{\theta}$
3. A **rejection region** that denotes what values of the test statistic will lead to a rejection of $\mathcal{H}_0$
   - Typically a **tail** of a distribution
   - Can therefore be denoted by a single parameter α, which we call the **significance level** (just like in CIs)
   - Must be **pre-specified**

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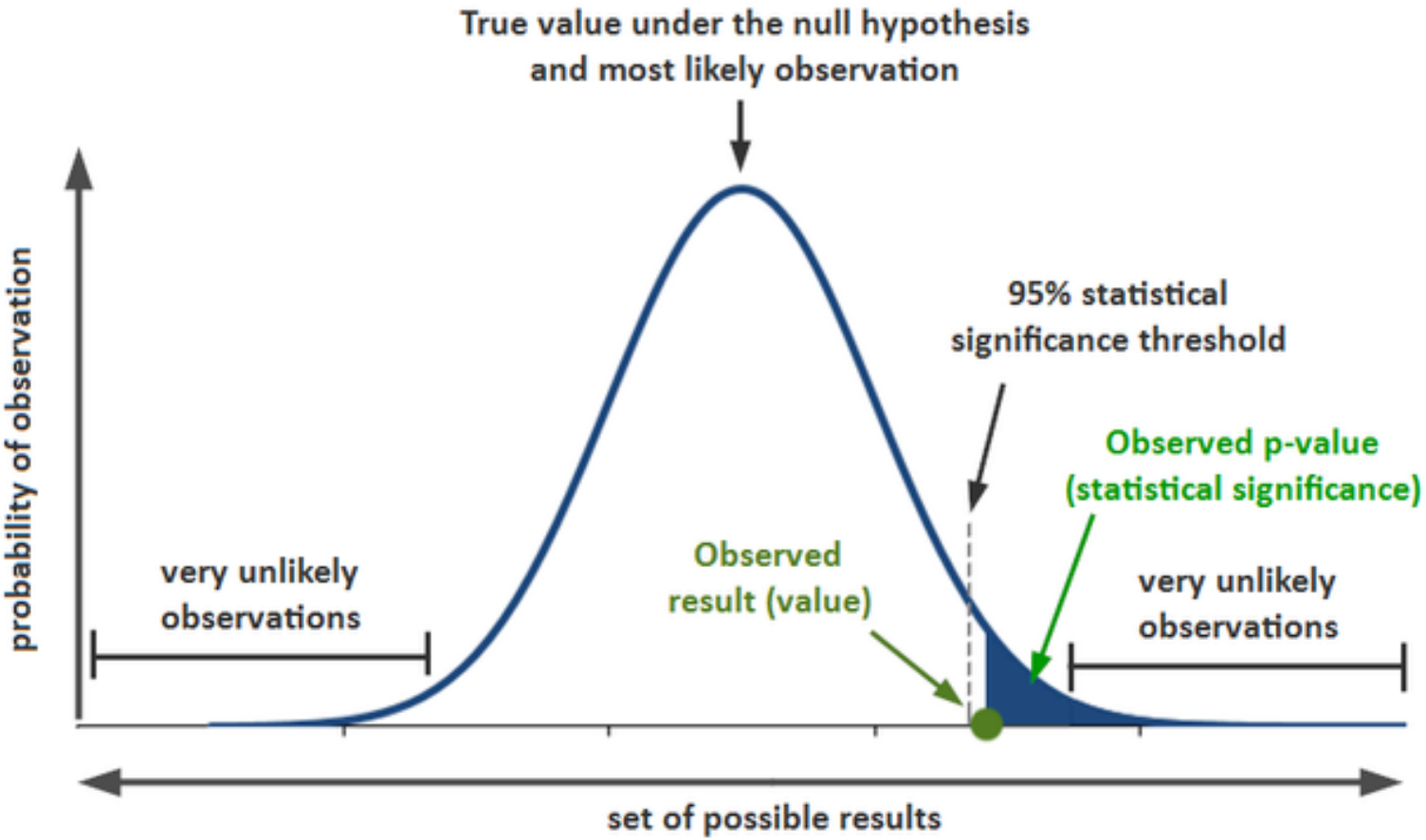
3. Test statistic: $\overline{x}_m - \overline{x}_f$, an unbiased and consistent estimator of $\mu_m - \mu_f$
4. Rejection region: $\alpha = 0.05$, so that we reject if the test statistic is in the 95th percentile or higher of its distribution *under the null*

A Testing Example



Source: Steemit post

Errors in Hypothesis Testing

The testing procedure should correctly identify when to reject/fail to reject most of the time. However, there is always potential for errors, to which we give specific names:

	Null hypothesis is TRUE	**Null hypothesis is FALSE**
Reject null hypothesis	Type I Error (False positive)	Correct outcome! (True positive)
Fail to reject null hypothesis	Correct outcome! (True negative)	Type II Error (False negative)

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- Since we've fixed a θ_0 under the null, this α is fixed (and is the same as α in previous slides!)
- β varies depending on the true parameter value θ) (which we can't know!)
- There is a **tradeoff** in minimizing these errors—decreasing α usually requires increasing β for all θ

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Q: Which error should we be more worried about?

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Consider a (grim) example of investigating a report for child maltreatment:

- What's the (binary) hypothesis we are testing?
- What's the Type 1 Error? Why is it bad?
- What's the Type 2 Error? Why is it bad?
- In what ways might policy be set up to prioritize one of these errors? Research? What should we do about it?

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Unfortunately, research tends to bias us **towards Type II errors** (we'll revisit this when we get to power)

TESTS FOR MEANS AND PROPORTIONS

Tests for a Population Mean

So let's do our first **hypothesis tests**!

$$\mathcal{H}_0 : \mu = \theta_0$$
$$\mathcal{H}_a : \mu \neq \theta_0$$
$$\text{Test Statistic : a function of } \overline{x}$$
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We will cover **three cases** for this test

1. A normal population, σ known
2. Large sample tests (non-normal popln)
3. A normal population, σ unknown

Motivating Example

Throughout, we will use the example of Card and Kreuger (1994)

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- Our goal: measure its effects on employment
- $\theta_0 =$ Avg employment rate **before** policy change
- $\overline{x} =$ Avg employment rate **after** change
  - **Q:** If we reject $\mathcal{H}_0$, what are we concluding?

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  is a good measure of **how far** estimator is from truth **when the null hypothesis is true**

- Hence, if $|Z|$ is too large, we can reject $\mathcal{H}_0$

Calculating the Test Statistic

Suppose that $\theta_0 = 0.20$ with $\sigma = .10$.[1] We randomly sample 100 stores after the change and obtain a mean of 0.16 as the employment rate. We are interested in performing this test with 95% confidence, so $\alpha = 0.05$.

Q: What is the test statistic?

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$$\begin{aligned} Z &= \frac{0.16 - 0.20}{0.10/10} \\ &= \frac{-0.04}{0.01} \\ &= -4 \end{aligned}$$

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Okay, but **what does this mean?**

- This is where we need our **rejection region**

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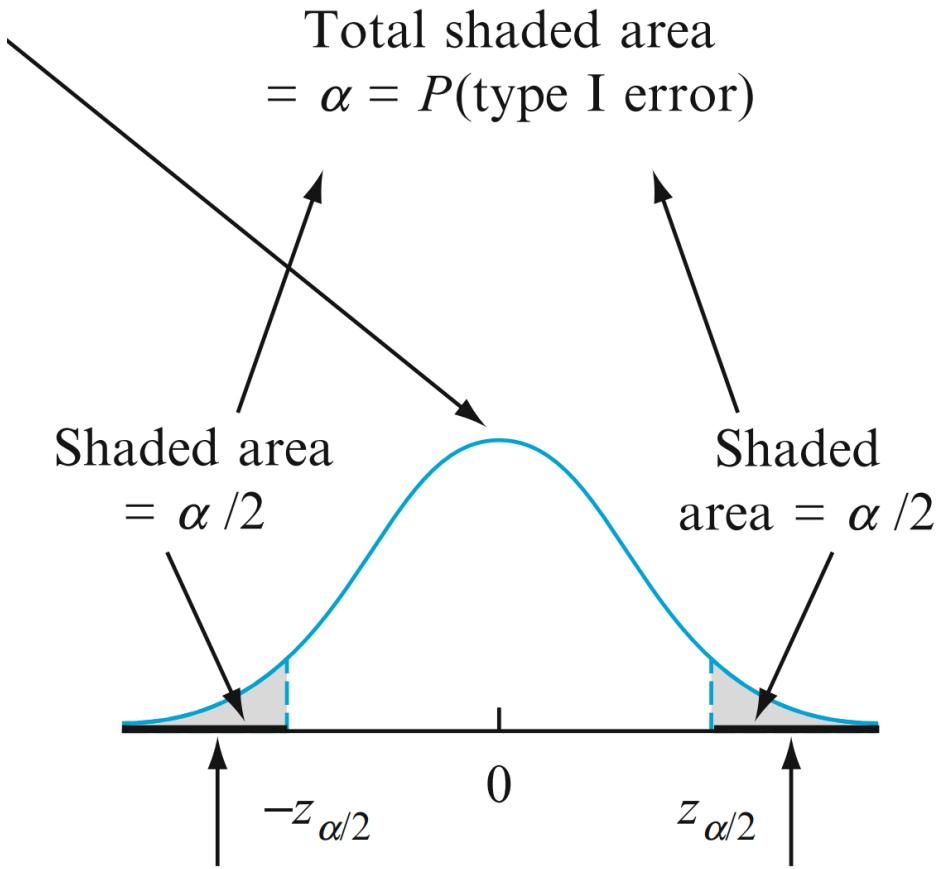
Rejection Regions

Our rejection region gives us **cutoff values** for the test stat:



Rejection region: either

$z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$

Rejection Regions

Our rejection region gives us **cutoff values** for the test stat:

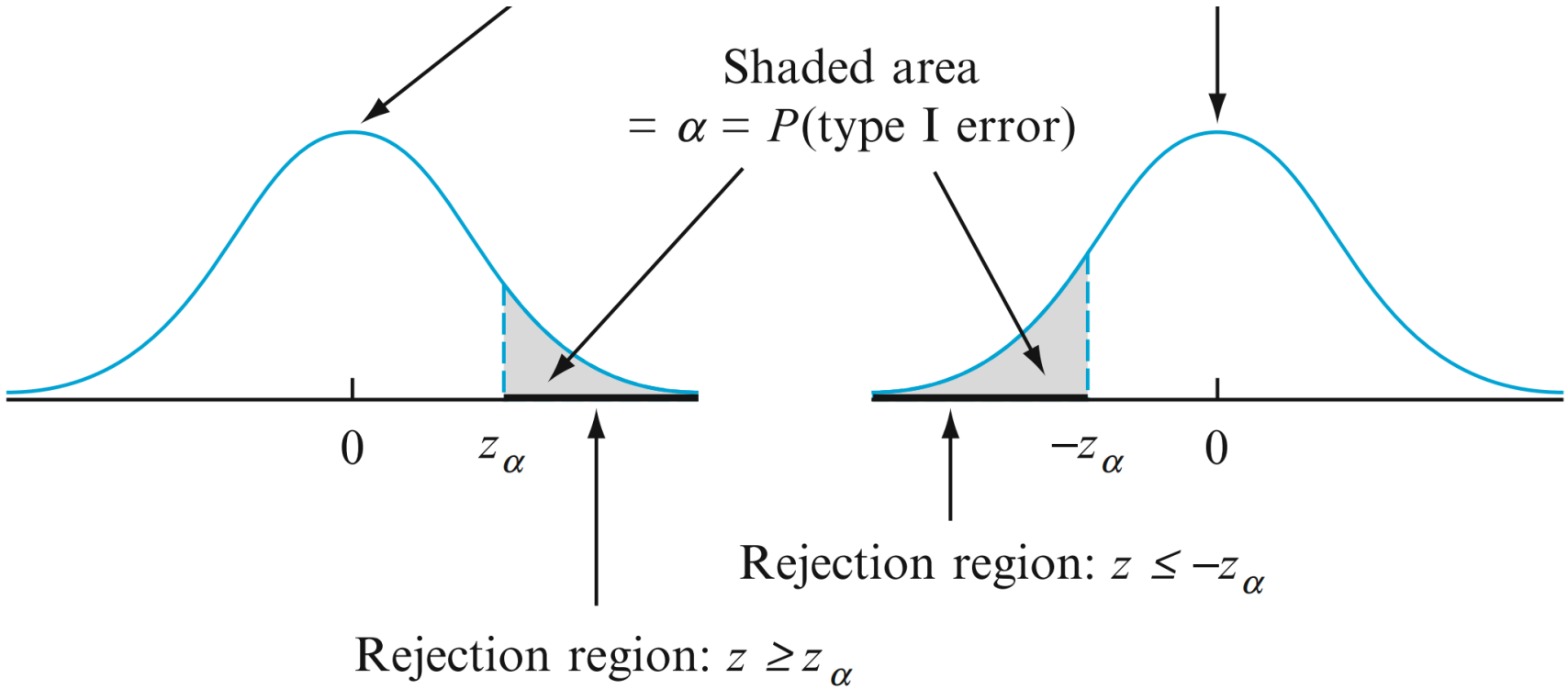
In this case, if $\alpha = 0.05$, then $Z_{0.05/2} = Z_{0.025} = \pm 1.96$.

- Since the test stat is **larger in absolute value** than the cutoff, we **reject the null hypothesis**
- In this case[2], what is the interpretation?

[2]Which is not exactly what Card and Krueger found.

One-Sided vs. Two-Sided Rejection Regions

Earlier, we performed a **two-sided** test. But what if we only want to test for a **decrease** in employment?



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- Greater probability of rejecting the null hypothesis
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- If your results suggested the opposite direction, you would attribute that to random sampling!

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- If your results suggested the opposite direction, you would attribute that to random sampling!
- **Q:** Does a one-sided test change our results for our example? (now, $Z_\alpha = -1.645$)

HYPOTHESIS TEST PROCEDURES

1. State the null and alternative hypotheses (context $\Rightarrow$ math)
2. Choose the correct test statistic and give its formula under $\mathcal{H}_0$
3. State the significance level α and the associated rejection region (a picture may help)
4. Compute the test statistic under $\mathcal{H}_0$ and determine the results of the test (reject or fail to reject)
5. Interpret the test results (math $\Rightarrow$ context)

Case 2: Large Sample Tests

When n is large enough, we can invoke the Central Limit Theorem to deal with potentially non-normal populations

- Same logic as for confidence intervals
- Since $s \approx \sigma$,

$$Z = \frac{\overline{X} - \theta_0}{S/\sqrt{n}}$$

has an **approximately** normal distribution. We can then use **the same** test procedure!

- Recall that $n > 30$ is our rule of thumb for "large samples"

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We can use the same **test procedure**, but our **rejection regions** will change slightly.

Example: Tipping

One restaurant manager is very angry that he isn't receiving a lot of tips. He wants to test whether his customers are systematically leaving tips below the customary 15%. He doesn't want to get too angry without cause, so he sets $\alpha = 0.01$.

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Q: What is the testing procedure?

$$\mathcal{H}_0 : \mu = 15$$
$$\mathcal{H}_a : \mu < 15$$
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$$T = \frac{-3}{4.8/4} = \frac{-12}{4.8} = -2.5$$

$$t_{0.01,15} = 2.602$$

We therefore **fail to reject** in this small sample.

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The test statistic becomes:

$$Z = \frac{-3}{12/10} = \frac{-30}{12} = -2.5$$

$$Z_{0.01} = 2.326$$

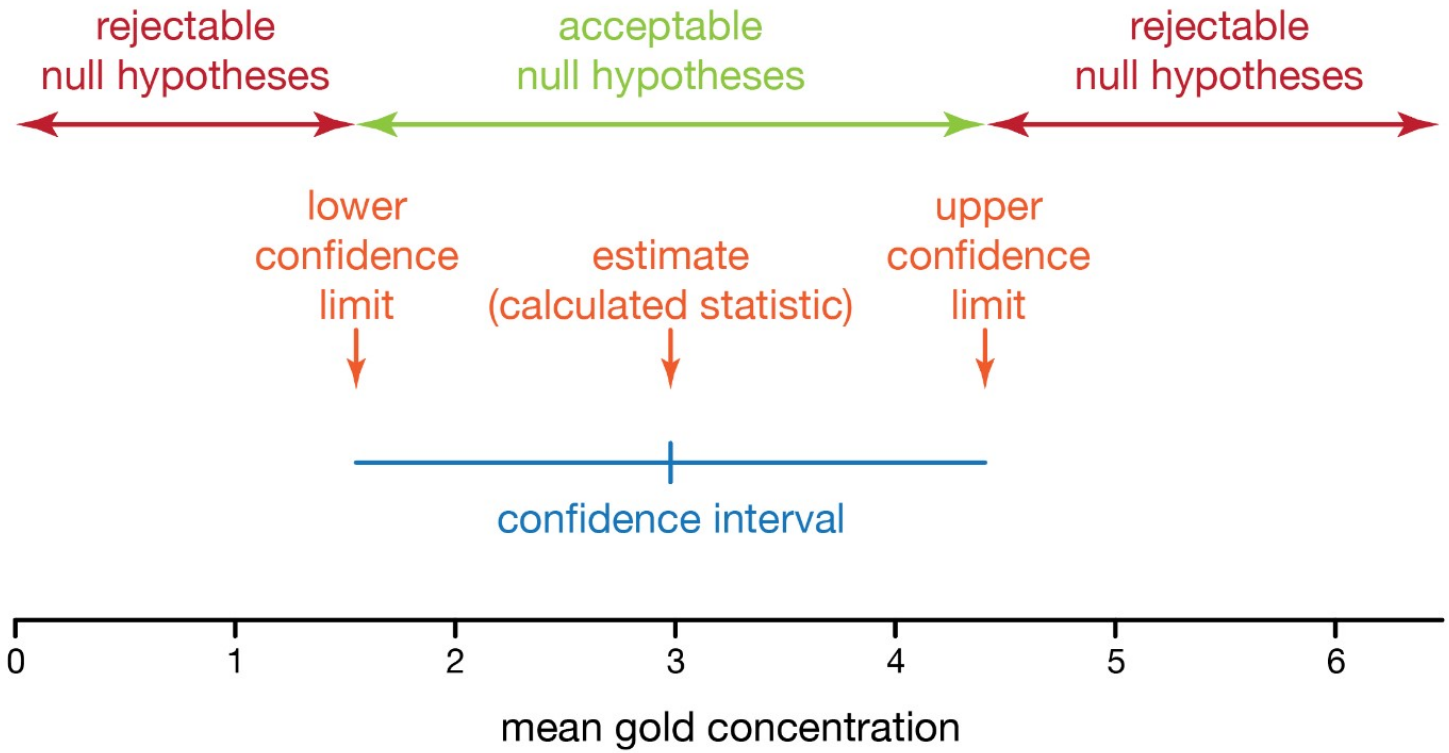
We therefore **reject the null hypothesis** given the additional data.

Relationship between $\mathcal{H}_0$ and confidence intervals

Do you notice a relationship between **test statistics** and **confidence intervals**?

- The margin of error in a CI is the same as the test statistic!
- That is, a test's critical values *define* a CI around $\mathcal{H}_0$

How does α relate to the level of confidence?



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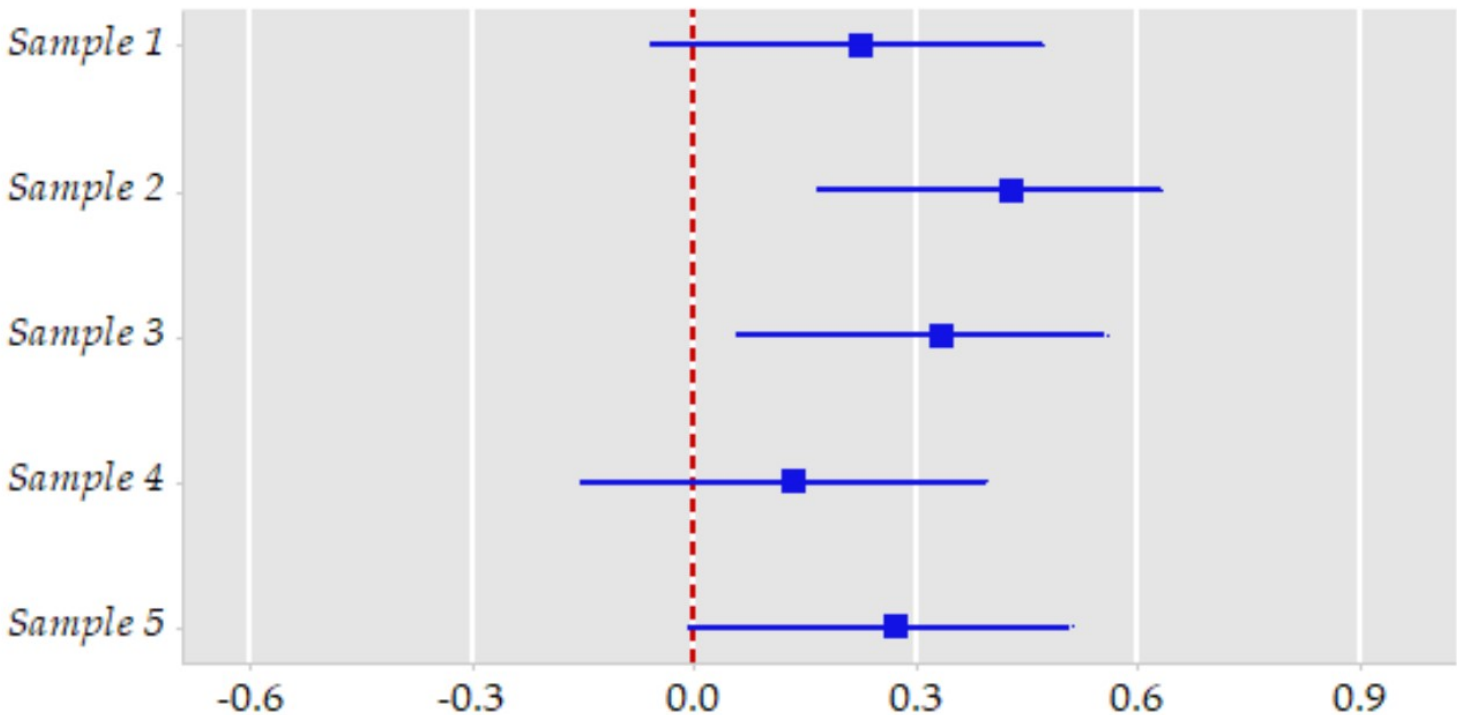
What is significant here? What is significantly different from other samples?



A General Version of the Test Statistic

For large samples, we can write the test statistic for any parameter θ against a null $\mathcal{H}_0 : \theta = \theta_0$ in a general way:

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When $\sigma_{\hat{\theta}}$ involves unknown parameters, we replace it with its **estimator** $\hat{\sigma}_{\hat{\theta}}$

Example: Large Sample Tests for a Proportion

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- We test whether p is different from some p_0.
- Based on the assumption that $\hat{p} = X/n$ is approximately normally distributed when n is large enough

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- In this special case, we **do not** need to estimate $\sigma_{\hat{p}}$
- For small samples, would need to use the binomial distribution directly $\Rightarrow$ can be quite tedious

P-VALUES

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- We compare a **test statistic** T to a **hypothetical distribution**
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- We compare a **test statistic** T to a **hypothetical distribution**
  - If the test stat is "unlikely" given distribution for $\mathcal{H}_0$, we reject
- We can therefore define the **probability** $\Pr(T|\mathcal{H}_0)$.
  - What is probability we see results as extreme as T given $\mathcal{H}_0$?

p-values

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p-values

Definition. The ***p*-value** is the probability **conditional on the null being true** of observing a test statistic at least as contradictory as T.

- The smaller the *p*-value, the more likely we should **reject** $\mathcal{H}_0$.
- Now, α defines a "**sufficiently small**" probability for rejection

Example: Tipping

Last class, we had an angry manager concerned about tips. We tested a stat of $Z = -2.5$ against a null hypothesis of $\mu = 15\%$. The p-value is thus

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There is a **0.6% chance** of getting Z under $\mathcal{H}_0$!

- Hence, for any $\alpha >= 0.006$, we would **reject** $\mathcal{H}_0$.
- Since p defines the smallest α for which we reject, we occasionally call it the **observed significance level** of the data

Significance & R

When $p < \alpha$ (so that we **reject** the null), we generally refer to p as **significant**

- Otherwise, p is **insignificant**

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R can perform all the steps of a hypothesis test for you!

- It even calculates p-values for you!
- Let's check this out in R.

Interpreting p-values

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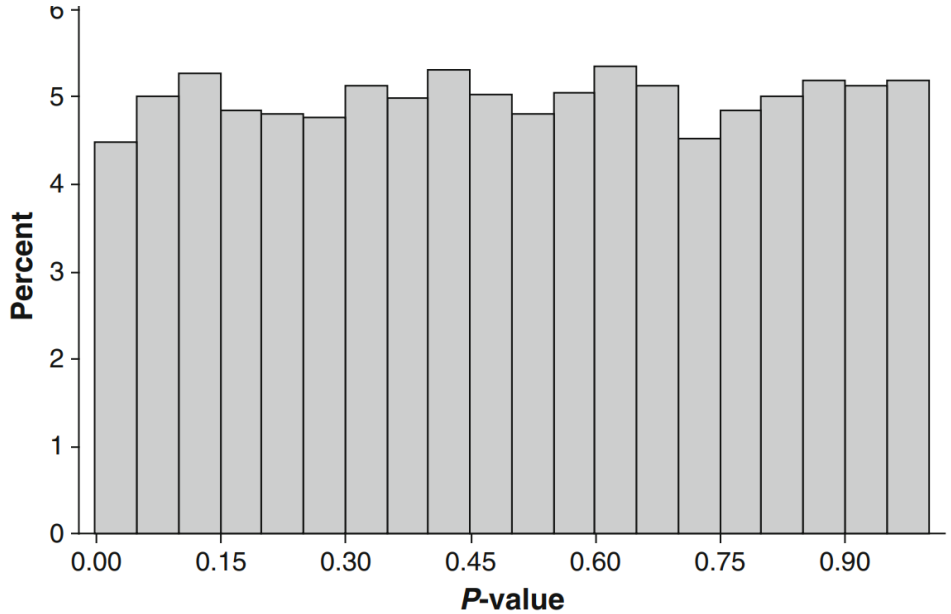
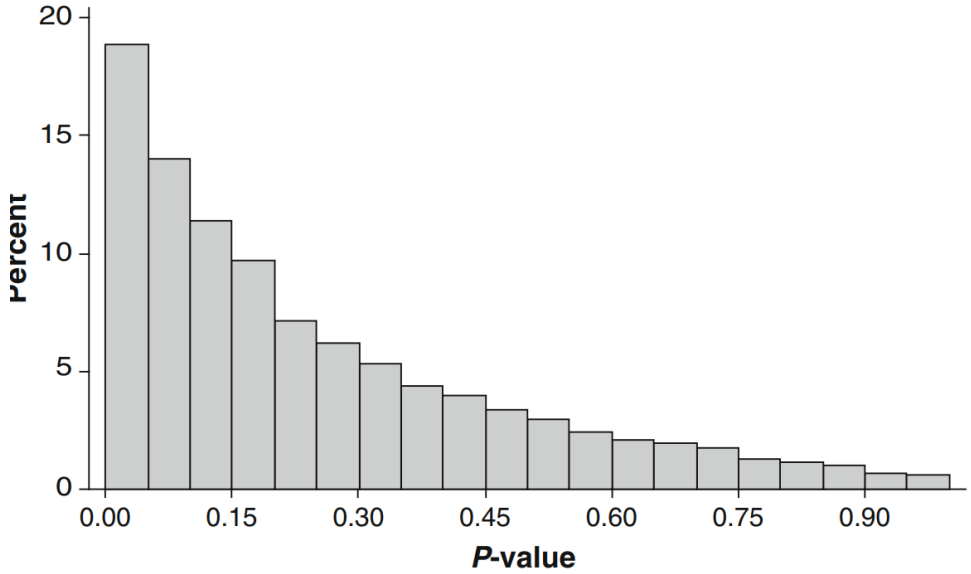
- The conditional probability that we obtain T if $\mathcal{H}_0$ is true
- It is therefore **hypothesis dependent**
- The p-value is also **sample dependent**, so it is a random variable
  - We talk about its **distribution** under two cases:
  - $\mathcal{H}_0$ is **true**
  - $\mathcal{H}_0$ is **false**

The Distribution of p-values



When $\mathcal{H}_0$ is true, the p-value has a **uniform** distribution

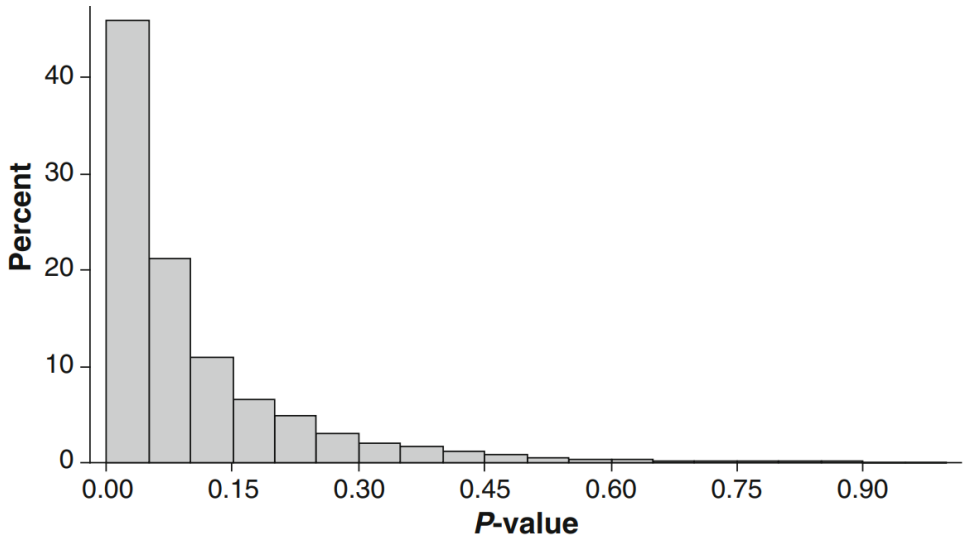
The Distribution of p-values



When $\mathcal{H}_0$ is "slightly" false, the p-value is skewed towards lower values

Q: How likely are we to reject the null hypothesis here (for $\alpha = 0.05$)? What type of error are we likely to commit?

The Distribution of p-values



When $\mathcal{H}_0$ is "more" false, the p-value becomes more skewed

Q: What is the probability of a Type II error now?

HOW TO CHOOSE A TEST

Interpreting and Expanding Hypothesis Testing

Now that we've covered the algorithm for testing, let's talk **interpretation**

- What does it mean for my project when $p < \alpha$?
- Can I rig the system to get what I want?
- What happens to my tests if its assumptions are violated?

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These are very important questions in academic research. Today:

1. Statistical vs. economic significance
2. p-hacking and publication bias
3. Likelihood ratio tests and general test statistics

What does "significance" mean?

So you've rejected your first null hypothesis. Does this make you a big shot?

- **Strong** rejection of $\mathcal{H}_0$ is associated with a **small** p-value
- However, p-values decrease with sample size!

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Suppose we test $\mathcal{H}_0 : \overline{x} = 100$ against $\mathcal{H}_a : \overline{x} > 100$

n	P-value when $\bar{x} = 101$
25	.3085
100	.1587
400	.0228
900	.0013
1600	.0000335
2500	.000000297
10,000	7.69×10^{-24}

With $n > 400$, we've rejected, but the rejection looks **stronger** as $n \to \infty$

Significant, Very Significant, or *Extremely* Significant?

How much stock do we put in the p-value?

Symbol	Phrase	p
+	Approaching Significance	$p < .1$
*	Significant	$p < 0.05$
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- Remember, hypothesis testing is a *binary* decision—can't make it continuous now!
- What does this mean in the context of big data?
- What does it mean in the context of your project?

Economic Significance & Big Data

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Economic Significance & Big Data

So the difference may be **(even extremely) statistically** significant as $n \to \infty$. But is going from 100 to 101 **a lot**?
The **context** here determines the **economic significance** of a test:

- We care not only if the data detect a difference, but also what that difference means **in the real world**
- In the world of big data ($n > 10^6$), this is an especially large concern!

Example ideas:

- Women's labor force participation increases their probability of divorce...by 0.33% (Dong, 2018)

POWER AND *P*-HACKING

What does "insignificant" mean?

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For example:

> *The other day I shot baskets with Michael Jordan. He shot 7 straight free throws; I hit 3 and missed 4 and then rushed to the sideline, grabbed my laptop, and calculated a P value of .07.*

Does this mean there is no difference? Of course not!

- What is one big statistical issue here?

Sample Size and Power

What if the two had thrown many more free throws?

- $\mathcal{H}_{\text{Alex}}$: Differences would have started to emerge
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- Depends on your question (1-sided or 2? What type of estimate? How large is economically meaningful?)
- Rule of thumb: if $N \uparrow 4x$, CIs $\downarrow 2x$

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- Rule of thumb: if $N \uparrow 4x$, CIs $\downarrow 2x$
- Formally, the **power** of a test, $1 - \beta$, is $\Pr(\text{reject } \mathcal{H}_0 | \mathcal{H}_0 \text{ is false}, N)$

Thinking about Sample Size

In a lot of cases, we can't choose N. But when can we?

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For a simple hypothesis test (one mean), look at the **margin of error** of a CI:

$$ME = \frac{CV * SD}{\sqrt{N}}$$

$$\Rightarrow N = 4\left(\frac{SD}{CV}\right)^2$$

Power Calculations

β (remember what that is?) depends on sample size (N) but also:

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Suppose we run a lot of experiments. How do we calculate power given these?

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If you specify N, SDs, and α, you can calculate power yourself!

Notes on Power

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Q: Why does power decrease with α and increase with N? (Can you answer this one?)

Hunting for Stars

Another issue related to hypothesis testing is ***p*-hacking**: pushing your research design until you get a significant result.

There are **two** main types of *p*-hacking:

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Both types forget that p has a uniform distribution under $\mathcal{H}_0$! We are more likely to get **false positives** if we repeatedly sample

Example 1: The Sad Grad Student

Suppose that you are a grad student who desperately wants a good research project (to get a good job). You run an experiment to look at an effect of X on Y:

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So **what have you proven?**

- Likely **nothing**.
- You changed the test 4 times, so you drew from the distribution of p 4 times. Hence, p is no longer interpretable.

Example 2: Wild Heterogeneity Search

You've recognized the error of your ways and moved on to a new research question: how job training affects employees.

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Lots of research does this!

- Let's check out this example

Publication Bias and Scrutinizing Research

Unfortunately, we like to publish significant results

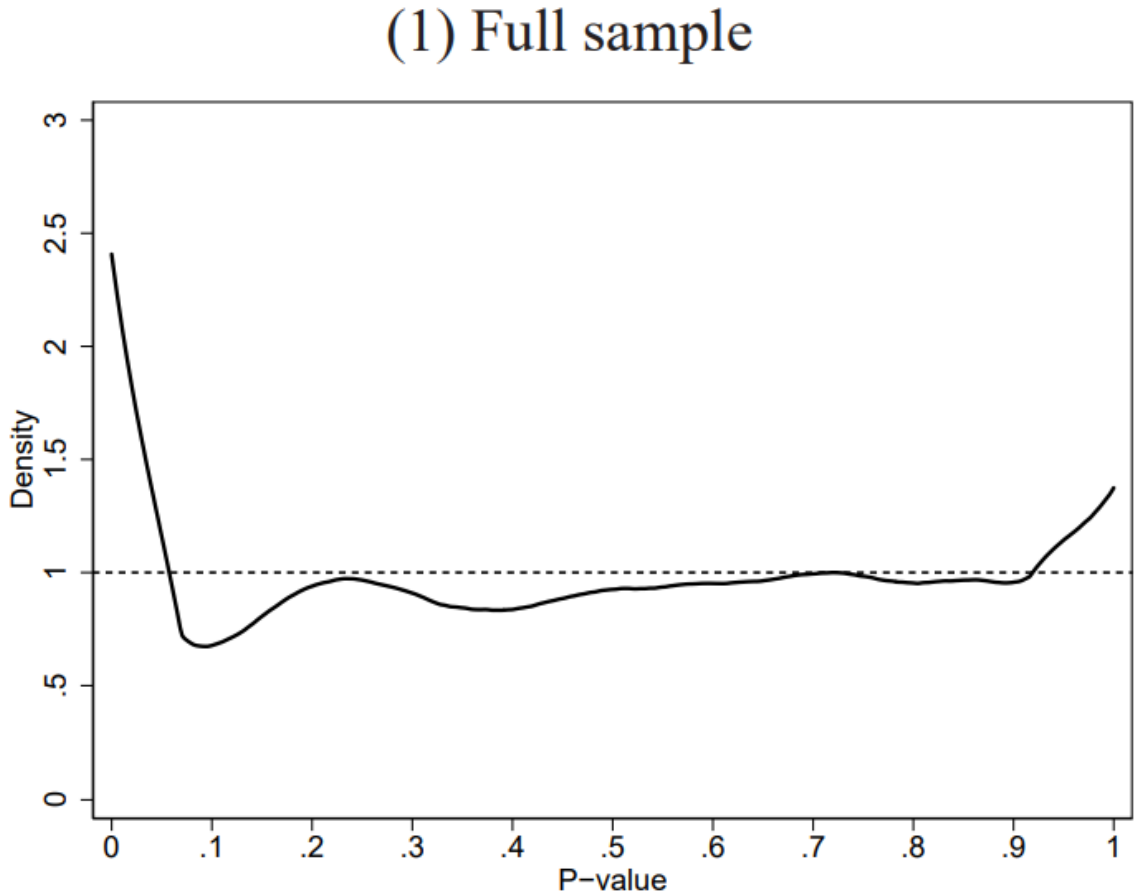

Figure: Source: Snyder & Zhuo, 2018

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Publication Bias and Scrutinizing Research

There have been lots of discussions lately about **publication bias** and **replication crises** in academic research.

So **what** should you do about it?

1. Play around with this interactive tool to practice your own *p*-hacking!
2. Use your testing power wisely!
3. Read research carefully!

TESTING TAKEAWAYS

1. There is a pre-specified way to do a test
2. Need to know what you're assuming to interpret test correctly
3. Need to be careful of performing too many tests
4. Need to know what you're actually finding with your results.

QUESTIONS?