The Geometry of Rings and Schemes
Lecture 2: Schemes

Alex Hof

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Last time, we developed an analogy between rings and topological spaces (or other kinds of geometric objects) and used this to construct "the topological space most closely approximating the geometry of a given ring" — that is, its spectrum. This time, we'll start to dig into the structure of these spectra a bit by way of examples; this will naturally lead us to the definition of schemes.

1 A First Look at Affine Space

We begin with a modern take on some very classical objects of study — the collection of polynomials in some finite number of variables and the spaces which can be defined by them.

Definition 1. *Let k be a field (or, really, any ring, but let's keep it simple for now) and $n \geq 0$ an integer. We define* **affine n-space over k** *by $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$.*

To start to get a sense of the geometry of these spaces, we examine a simple example, which was one of the exercises from last week's lecture:

Example 1. *Consider $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[x]$, called the affine line over $\mathbb{C}$.*

To understand this space, we must first figure out what its points are — this is to say that we want to characterize the prime ideals $\mathfrak{p}$ of $\mathbb{C}[x]$. Since $\mathbb{C}[x]$ is a principal ideal domain, we have $\mathfrak{p} = (f)$ for some polynomial $f \in \mathbb{C}[x]$. Moreover, the fact that $\mathbb{C}$ is algebraically closed implies that f can be decomposed as a product of linear factors — that is, $f = a\prod_{i=1}^{d}(x - r_i)$ for some $a, r_1, \ldots, r_d \in \mathbb{C}$ and d the degree of f. If $a \neq 0$ and $d > 1$, we can write f as a product of two polynomials not in $\mathfrak{p}$ — this contradicts the ideal's primality. Hence the points we get for $a \neq 0$ all correspond to prime ideals of the form $\mathfrak{p} = (x - r)$ for $r \in \mathbb{C}$. The map of rings giving the "inclusion" of such a point will be exactly the map $\mathbb{C}[x] \to \mathbb{C}[x]_{(x-r)}/(x - r)\mathbb{C}[x]_{(x-r)} \cong \mathbb{C}$ taking each polynomial $g(x)$ to its value $g(r)$ at r.

Because of the correspondence between these points and elements $r \in \mathbb{C}$, we think of $\mathbb{A}^1_{\mathbb{C}}$ as "the geometric realization of the 1-dimensional vector space $\mathbb{C}$"; we will soon see that a similar intuition applies to affine spaces over algebraically closed fields more generally. However, the Zariski topology will not yield the familiar classical topology on $\mathbb{C}$. Indeed, we can characterize the Zariski-closed subsets of our copy of $\mathbb{C}$ as follows. By definition, they are the collections of such points corresponding to maps which factor through the quotients

$R \to R/I$ for each ideal I. Since $\mathbb{C}[x]$ is a principal ideal domain, these ideals are of the form $I = (h)$ for arbitrary polynomials $f \in \mathbb{C}[x]$; for such an h, we can see that the points $x \mapsto r$ which factor through $\mathbb{C}[x] \to \mathbb{C}[x]/(h)$ are precisely those such that $h(r) = 0$, which is to say the roots of h. Since any nonzero polynomial has finitely many roots and any finite collection of points in $\mathbb{C}$ admits a polynomial with roots exactly at the chosen points, we can see that the closed subsets of our copy of $\mathbb{C}$ will be the whole space and all finite subsets. That is, the Zariski topology on $\mathbb{C}$ is precisely the so-called cofinite topology.

This is almost a full description of $\mathbb{A}^1_{\mathbb{C}}$ — however, there is actually one additional point we've missed so far. Our argument above that $\mathfrak{p} = (f) = \left(a \prod_{i=1}^d (x - r_i)\right)$ must be of the form $(x - r)$ assumed $a \neq 0$; in the case where this does not hold, we obtain the prime ideal $\mathfrak{p} = (0)$, corresponding to the map $\mathbb{C}[x] \to \mathbb{C}[x]_{(0)}/(0)\mathbb{C}[x]_{(0)} \cong \mathbb{C}(x)$ from $\mathbb{C}[x]$ to its field of fractions. To understand how this new point fits into the topology, observe that $\mathbb{C}[x] \to \mathbb{C}(x)$ has kernel (0) and so does not factor through any nontrivial quotient; equivalently, it factors through every localization $\mathbb{C}[x] \to \mathbb{C}[x]_f$ for $f \neq 0$. Hence the point is dense, being contained in every non-empty open subset of $\mathbb{A}^1_{\mathbb{C}}$. This completes our description of the affine line.

The "extra point" in this example illustrates an important distinction between ring spectra and the kinds of topological spaces we are used to dealing with in most branches of geometry and topology: Such spaces are typically Hausdorff (T_2, in Tychonoff's hierarchy), but ring spectra can even have non-closed points (that is, they need not even be T_1). We will see the appropriate algebro-geometric analogue to the Hausdorff property, called *separatedness*, later on in the course.

The presence of non-closed points raises a natural question: Given such a point, what is its closure? We are already in a position to answer this question topologically, on the level of ring spectra — however, as usual, we will prefer to work algebraically, on the level of rings themselves. To this end, note that the idea of asking "whether a point is closed" in our analogy between rings and topological spaces already makes sense — our "point inclusions" are certain maps $R \xrightarrow{\phi} k$, and to ask if a given ring map is a "closed inclusion" is precisely to ask whether it is a quotient map, which is to say surjective. For a given prime $\mathfrak{p}$ in a ring R, the corresponding point inclusion is $R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$; since quotients and localizations commute, this factors as $R \to R/\mathfrak{p} \hookrightarrow (R/\mathfrak{p})_{(0)}$. Hence the point will be closed precisely when the quotient ring $R/\mathfrak{p}$ is already a field — that is, exactly when $\mathfrak{p}$ is maximal.

To understand our non-closed points — that is, those corresponding to non-maximal prime ideals — we must do a bit better than just asking whether a given map is a closed inclusion:

Question. *Let $R \xrightarrow{\phi} R'$ be a map of rings which we think of as being some kind of "inclusion of a subspace" (open, closed, a point, or whatever else). Which quotient of R should we regard as "the closure of R' in R"?*

As usual, we clarify this question by reformulating our topological object in more categorical terms. Specifically, the closure of a subspace Y of a topological space X is the smallest closed subspace containing it, which is to say the smallest closed subspace Z such that the inclusion $Y \hookrightarrow X$ factors through $Z \hookrightarrow X$. This characterization yields:

Question (revised)**.** *Let $R \xrightarrow{\phi} R'$ be a map of rings. What is the furthest quotient (i.e., the quotient by the largest ideal) $R \to R/I$ which ϕ factors through?*

Answer. $R \to R/\ker\phi$.

In the case where ϕ should *not* be thought of as an inclusion of some kind, we should think of $R \to R/\ker\phi$ as instead giving roughly the "closure of the image of ϕ" — with the caveat, of course, that we say this without defining any kind of notion of "the image of ϕ" itself.

From our prior discussion of the maps $R \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, it is now clear that the closure of the point corresponding to $\mathfrak{p}$ is given on the level of rings by the quotient $R \to R/\mathfrak{p}$; on the level of topological spaces, we get the subspace $\operatorname{Spec} R/\mathfrak{p} \subseteq \operatorname{Spec} R$ consisting of points $R \to k$ which factor through this map. In particular, we can see that the point corresponding to a prime $\mathfrak{q}$ lies in the closure of the point corresponding to $\mathfrak{p}$ if and only if $\mathfrak{q} \supseteq \mathfrak{p}$. (In this case we say that the point corresponding to $\mathfrak{p}$ *specializes* to the point corresponding to $\mathfrak{q}$ — observe that the poset of points of $\operatorname{Spec} R$ ordered by specialization exactly recovers the poset of prime ideals of R ordered by inclusion.)

Proposition/Definition 1. *Let R be a domain. Then the point given by the map $R \to R_{(0)}$ from R to its field of fractions has the whole space as its closure — for this reason, we call it the* **generic point** *of* $\operatorname{Spec} R$.

The idea of the terminology is that this point is somehow "everywhere in $\operatorname{Spec} R$" in the sense that it is close enough to every other point that its closure contains them all, while simultaneously being "nowhere in particular" in the sense that it lies in the complement of any particular closed set we might consider. Loosely speaking, we thus think of behavior at the generic point as capturing the "typical" behavior at points of $\operatorname{Spec} R$ — we will see concrete examples of this idea later on in the course.

If $\mathfrak{p}$ is a prime in an arbitrary ring R, we can now understand how the point given by $\mathfrak{p}$ fits into the topology of $\operatorname{Spec} R$ by observing that it is the generic point of the closed subset $\operatorname{Spec} R/\mathfrak{p}$. This allows us to visualize $\operatorname{Spec} R$ by first getting a picture of the closed points, then remembering that we also have a generic point living along every closed subset cut out by a prime ideal. (Such subsets are called *irreducible* — we will explore the topological meaning of this term later.)

In the case of our affine spaces $\mathbb{A}^n_k$, we thus want to begin by understanding the maximal ideals of $k[x_1, \ldots, x_n]$. This is possible, at least in nice cases, by way of the following result:

Theorem 1 (Hilbert's Nullstellensatz)**.** *Let k be a field and $n \geq 0$ an integer. If $\mathfrak{m}$ is a maximal ideal of $k[x_1, \ldots, x_n]$, then the natural map $k \to k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension.*

Since all finite field extensions of an algebraically closed field are trivial, this yields:

Corollary 1 (Hilbert's Weak Nullstellensatz)**.** *Let k be an algebraically closed field, $n \geq 0$ an integer, and $\mathfrak{m}$ a maximal ideal of $k[x_1, \ldots, x_n]$. Then $\mathfrak{m} = (x_1 - r_1, \ldots, x_n - r_n)$ for $(r_1, \ldots, r_n) \in k^n$.*

Hence, as in the case of $\mathbb{A}^1_\mathbb{C}$, we think of $\mathbb{A}^n_k$ as "the geometric realization of the vector space k^n", at least in the case $k = \bar{k}$.

(There is some apparent inconsistency in the literature as to the naming of theorems — some readers may know the result we here call "Hilbert's Nullstellensatz" as "Zariski's

Lemma" and some may have seen a(n at least superficially) different result referred to as "Hilbert's Nullstellensatz". For an explanation of the various flavors of Nullstellensatz and the relationships between them, I recommend Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry.*)

Armed with this knowledge, we turn our attention to a slightly more complicated example than the affine line, which was also an exercise from last week:

Example 2. *Consider* $\mathbb{A}^2_{\mathbb{C}} := \operatorname{Spec}\mathbb{C}[x,y]$. *We wish to find the points of this space and compute the topology.*

By the Nullstellensatz, the closed points will be in bijective correspondence with $\mathbb{C}^2$. *The nontrivial closed subsets of this collection of closed points will be precisely those which can be cut out by bivariate polynomial equations — that is, the points themselves and algebraic plane curves. Other than the generic point of the whole space, the non-closed points turn out to correspond to ideals of the form* (f) *for* f *an irreducible polynomial — these are the generic points of plane curves which are moreover irreducible. (The precise definition of a "curve" in the algebro-geometric setting will be covered later, when we get around to talking about dimension theory — as previously mentioned, we are also deferring our discussion of irreducibility for the time being.)*

This example provides our first instance of a troubling phenomenon, which we will now explore. Observe that the maximal ideal (x,y) defines a closed point, corresponding to the origin in the plane $\mathbb{C}^2$. Since it is closed, its complement U is an open subspace of the topological space $\mathbb{A}^2_{\mathbb{C}} := \operatorname{Spec}\mathbb{C}[x,y]$, the union of open subspaces $\operatorname{Spec}\mathbb{C}[x,y]_x$ and $\operatorname{Spec}\mathbb{C}[x,y]_y$ arising from localizations at single elements.

Our goal in this course so far has been to explore the intrinsic geometry of rings themselves — spectra have been a useful tool for capturing some aspects of their geometry, but as previously mentioned they do not encompass all of the same information. Therefore, we should not be satisfied with U as a topological subspace — we would like to produce an "open inclusion" on the level of rings realizing it. That is, we wish to answer the following:

Question. *Is there a polynomial* $f \in \mathbb{C}[x,y]$ *such that the map* $\operatorname{Spec}\mathbb{C}[x,y]_f \hookrightarrow \operatorname{Spec}\mathbb{C}[x,y]$ *of spectra induced by* $\mathbb{C}[x,y] \to \mathbb{C}[x,y]_f$ *is exactly the inclusion* $U \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$? *Failing this, can we rectify the issue by broadening our notion of open inclusion — that is, is there some other ring map* $\mathbb{C}[x,y] \to S$ *such that* $\operatorname{Spec} S \to \operatorname{Spec}\mathbb{C}[x,y]$ *is exactly* $U \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$?

Answer. *No, on both counts.*

This can be demonstrated by way of the following argument:

Exercise 1. *Let* S *be a ring and* $\phi_x : S \to \mathbb{C}[x,y]_x$ *and* $\phi_y : S \to \mathbb{C}[x,y]_y$ *ring maps such that the compositions* $S \xrightarrow{\phi_x} \mathbb{C}[x,y]_x \to \mathbb{C}[x,y]_{xy}$ *and* $S \xrightarrow{\phi_y} \mathbb{C}[x,y]_y \to \mathbb{C}[x,y]_{xy}$ *with the natural localization maps coincide. Show that there exists a unique ring map* $\phi : S \to \mathbb{C}[x,y]$ *so that the compositions* $S \xrightarrow{\phi} \mathbb{C}[x,y] \to \mathbb{C}[x,y]_x$ *and* $S \xrightarrow{\phi} \mathbb{C}[x,y] \to \mathbb{C}[x,y]_y$ *with the localizations give* ϕ_x *and* ϕ_y *respectively. Conclude that, if we have arbitrary* S, ϕ_x, ϕ_y, *and* ψ *such that the following diagram commutes (that is, every depicted sequence of maps from any of the rings depicted to any other composes to the same thing),* ψ *admits a left inverse* ϕ *(i.e.,* $\phi \circ \psi = \mathrm{id}_{\mathbb{C}[x,y]}$*):*

$$\begin{tikzcd}
 & \mathbb{C}[x,y] \arrow[d, "\psi"] \arrow[ddl] \arrow[ddr] & \\
 & S \arrow[dl, "\phi_x"'] \arrow[dr, "\phi_y"] & \\
\mathbb{C}[x,y]_x \arrow[dr] & & \mathbb{C}[x,y]_y \arrow[dl] \\
 & \mathbb{C}[x,y]_{xy} &
\end{tikzcd}$$

(Here all unlabeled arrows are the appropriate localization maps.)
Conclude that ψ does not induce the map $U \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$ on the level of spectra.

That is, despite not being an open cover in the sense that (x, y) is the unit ideal or in the sense of producing an open cover on the level of spectra, the collection $\{\mathbb{C}[x,y] \to \mathbb{C}[x,y]_x, \mathbb{C}[x,y] \to \mathbb{C}[x,y]_y\}$ "algebraically behaves like an open cover of $\mathbb{C}[x,y]$" to some extent. Unfortunately, this implies that our open subset U cannot be realized on the level of rings, which leads us to try our hand at...

2 Defining Schemes

We've been getting acquainted with the idea that rings have some sort of geometric structure, analogous to that of topological spaces. However, this analogy breaks down somewhat when we consider the idea that we should be able to take unions of open subsets of a space to get another open subset — as the previous example illustrates, we do not have a well-formed way to glue together collections of rings along identifications of "open subspaces", so we cannot even construct a non-pathological "union of open subspaces", much less one which is itself an open subspace. To rectify this issue, we introduce schemes:

Definition 2 (informal)**.** *A* **scheme** *is a geometric object which looks like a ring locally at each point.*

That is, schemes are to rings as manifolds are to open balls in Euclidean space (modulo the usual issue of ring maps "going the wrong direction" — we will think of scheme maps with the directionality which agrees with the conventions for topological spaces and other classical geometric objects).

In practice, this informal definition will often be all we really need — many arguments will be made locally, for example, and so we can make sense of them without developing an explicit, concrete construction of schemes. We will, of course, introduce the most common formalism regardless, but it is best to maintain a clear sense of the informal picture in order to avoid getting lost in the machinery.

How, then, should we actually construct schemes? There are several possible approaches, not all of which we will develop in detail. The first is analogous to the construction of manifolds from atlases:

Idea. *Define a scheme as a collection of rings together with "gluing data" — that is, a collection of compatible identifications between "open subspaces" given by localizations of our rings at single elements. Work out notions of refinement and equivalence for such "atlases" and use this to define maps between schemes locally.*

This is feasible in principle, but I've never seen anyone actually do it — as is the case when defining manifolds via atlases, things quickly become quite cumbersome, despite the conceptual closeness of this approach to our informal picture.

The second possible method comes from category theory — this is the (in)famous "functor of points":

Idea. *Use the Yoneda embedding to regard rings as functors* $\mathrm{Ring} \to \mathrm{Set}$*; concretely, we identify R with the functor* $\mathrm{Hom}_{\mathrm{Ring}}(R, -)$*. We can then, in essence, create new functors which are "unions of rings" by gluing the output sets together along the appropriate identifications — this will give us our notion of schemes.*

Here, as in many cases, the apparent abstruseness and alarming set-theoretic issues endemic to the category-theoretic framing are offset by a genuinely interesting and valuable underlying perspective, which will ultimately be worth the price of admission for the student willing to wade through the machinery. However, it's generally best to approach such topics with some degree of pre-existing intuition, so this course will largely ignore the functor of points in favor of a more concrete viewpoint.

The third possible approach, the one we will actually use, is the sheaf-theoretic one. This begins with the following observation, which is applicable in many contexts beyond the setting of scheme theory:

Idea. *If we have something which is "a topological space together with some extra data or structure", we can often keep track of this by remembering the "good functions" with respect to this extra information on every open subspace.*

Example 3. *If X is a smooth manifold, the underlying topological space need not capture all of the information of the smooth structure on X. However, if we also remember the rings $\mathcal{O}_X(U) := \{$smooth functions $U \to \mathbb{R}\}$ for open sets $U \subseteq X$ (together with the maps between them given by the restriction of functions from open sets to smaller ones), the resulting data will uniquely determine the smooth structure.*

To formalize this notion of a "collection of good functions", recall the following definition (or, if you haven't seen it yet, learn it, from one of our reference texts or elsewhere):

Definition 3. *Let X be a topological space. A* **sheaf** *(of sets) $\mathcal{F}$ on X consists of the following data:*

- *For each open set $U \subseteq X$, a set $\mathcal{F}(U)$, called the set of* **sections** *of $\mathcal{F}$ on U.*
- *For each pair of nested open sets $V \subseteq U \subseteq X$, a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, called the* **restriction map** *from U to V.*

These are subject to the following restrictions:

- *For each open set $U \subseteq X$, $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.*
- *For each sequence $W \subseteq V \subseteq U \subseteq X$ of open subsets, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*
- *If $U \subseteq X$ is open and $\{U_\alpha\}_{\alpha \in A}$ is an open cover of U, then, for each selection $\{f_\alpha \in \mathcal{F}(U_\alpha)\}_{\alpha \in A}$ such that $\rho_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = \rho_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$ for all $\alpha, \beta \in A$, there exists a unique $f \in \mathcal{F}(U)$ such that $\rho_{UU_\alpha}(f) = f_\alpha$ for all $\alpha \in A$.*

A **map of sheaves** *$\mu : \mathcal{F} \to \mathcal{G}$ consists of maps $\mu(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq X$ such that these maps commute with the restrictions.*

In particular, our first motivating proposition from last week can be rephrased as saying that maps to any particular topological space Y form a sheaf.

We can also speak of sheaves of more complicated objects, for example abelian groups or rings, by requiring that all of the sets of sections are such objects and all the maps involved are maps of such objects. (There is also a more streamlined general definition by way of category theory, but we omit it here.)

We formalize the notion of "a space together with a collection of good functions":

Definition 4. *A* **ringed space** *is a pair $(X, \mathcal{O}_X)$ such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A* **map of ringed spaces** *$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a pair $(\phi, \phi^\#)$ such that $\phi : X \to Y$ is a continuous map of topological spaces and $\phi^\# : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ is a map of sheaves of rings on Y. (Here $\phi_* \mathcal{O}_X$ denotes the* **pushforward sheaf** *given by $(\phi_* \mathcal{O}_X)(U) := \mathcal{O}_X(\phi^{-1}(U))$, with the restriction maps inherited from $\mathcal{O}_X$.)*

Since we are keeping our collection of "functions" abstractly in the form of a sheaf, without any particular identification with literal functions on the underlying space, the map $\phi^\#$ is necessary to record the behavior of such "functions" on Y under pullback to X — in the case of actual continuous functions, this would be given by composition with ϕ.

To construct schemes as ringed spaces, we first define a ringed space capturing the geometry of a given ring:

Definition 5. *Let R be a ring. The* **structure sheaf** *$\mathcal{O}_{\operatorname{Spec} R}$ on $\operatorname{Spec} R$ is the sheaf of rings given by setting $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R_f) := R_f$ for $f \in R$, with restrictions given by the localization maps. (Since these open sets form a base for the Zariski topology by definition, and sheaves are defined such that their sections are determined on open covers, this assignment is enough to determine $\mathcal{O}_{\operatorname{Spec} R}$ — that the specified data is compatible with the sheaf axioms follows from our original motivating proposition on the geometry of rings from last week.)*

A ringed space of the form $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is called an **affine scheme***.*

As we have previously mentioned, the topological space $\operatorname{Spec} R$ on its own does not record all of the data of R; however, the ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ does, since we can simply take the ring $\mathcal{O}_{\operatorname{Spec} R}(\operatorname{Spec} R)$ of so-called *global sections* of the structure sheaf to retrieve R.

Exercise 2. *Show that a ring map $R \to S$ functorially induces a map of ringed spaces $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \leftarrow (\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S})$. (Recall that the equivalent result on the level of topological spaces was an exercise last week, in which we introduced the appropriate notion of functoriality.)*

We can now realize schemes in general as ringed spaces which are locally affine schemes:

Definition 6. *A* **scheme** *is a ringed space* $(X, \mathcal{O}_X)$ *which is isomorphic to an affine scheme locally at each point — that is, for every* $x \in X$*, there exists an open set* $U \ni x$*, called an* **affine open neighborhood** *of* x*, and a ring* R *such that we have* $(U, \mathcal{O}_U := \mathcal{O}_X|_U) \cong (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ *as ringed spaces. (In fact, we can then see that* $R \cong \mathcal{O}_X(U)$*.)*

A **map of schemes** $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a map of ringed spaces which is induced by a map of rings locally at each point — that is, for every* $x \in X$*, there exist affine open neighborhoods* $V \ni \phi(x)$ *and* $\phi^{-1}(V) \supseteq U \ni x$ *such that* $\phi|_U : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ *is exactly the map of ringed spaces induced by the ring map* $\mathcal{O}_V(V) \xrightarrow{(\phi|_U)^{\#}(V)} \mathcal{O}_U(U)$*.*

(Equivalently, a map $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *of ringed spaces is a map of schemes if and only if, for every pair of affine opens* $V \subseteq Y$ *and* $U \subseteq \phi^{-1}(V) \subseteq X$*,* $\phi|_U : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ *is induced by the corresponding map of rings.)*

Now that we know what schemes and affine schemes are, we can streamline our terminology somewhat:

Redefinition of Terms. *From now on, when we refer to the* **spectrum** *of a ring* R*, we will implicitly mean the affine scheme* $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ *unless otherwise noted. Moreover, we will typically suppress the structure sheaf from the notation, in these cases and for schemes in general. In particular, when speaking of the* **affine space** $\mathbb{A}^n_k$*, we will now implicitly regard it as a scheme.*

Rather than talking about "ring maps analogous to" **open inclusions***,* **closed inclusions***,* **point inclusions***, or the like, we will now instead simply talk about the corresponding types of map in the context of (affine) schemes. (In the case of open inclusions, where we can have non-affine open subschemes even of affine schemes, we will refer to the maps* $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$ *induced by the localizations as inclusions of* **distinguished affine opens** *instead.)*

A key point is that, thanks to the parenthetical version of the definition of maps of schemes above, any map of affine schemes is induced globally by a map of rings — therefore, affine schemes and their maps are exactly the same as rings and their maps, up to the change in direction. (For those familiar with category theory: The category of affine schemes is the *opposite category* of the category of rings.) To verify that requiring a map to be induced by maps of rings on *some* collection of affine opens of the form above is indeed enough to force the result on *every* pair of affine opens, we make use of the following observation:

Exercise 3 (Vakil's Proposition 5.3.1)**.** *Let* X *be a scheme,* $x \in X$ *a point, and* $U \cong \operatorname{Spec} R$ *and* $V \cong \operatorname{Spec} S$ *affine open neighborhoods of* x*. Then there exist* $f \in R$ *and* $g \in S$ *such that* $\operatorname{Spec} R_f \hookrightarrow U \hookrightarrow X$ *and* $\operatorname{Spec} S_g \hookrightarrow V \hookrightarrow X$ *both give the same open subscheme* W *of* X *and* W *contains* x*.*

From here the proof of the claim more or less follows from the local determinacy of the various types of object involved.

With this dealt with, it is now worth pausing to reflect on our definition. We claimed that it made sense to construct schemes as topological spaces together with some extra data captured by sheaves of "functions", and this appears to have worked, but a question remains:

Question. *Why does it make sense to think of elements of a ring R as "functions" on* $\operatorname{Spec} R$*?*

Answer. *There are at least two possible answers:*

1. *Let R be a $\mathbb{C}$-algebra. Then there is a natural bijection between the set $\{f \in R\}$ and the set of $\mathbb{C}$-algebra maps $\mathbb{C}[x] \to R$ (since such a map is determined by the image of x); moreover, this latter set is precisely the set of maps of $\mathbb{C}$-schemes from* $\operatorname{Spec} R$ *to $\mathbb{A}^1_{\mathbb{C}}$. (By a "map of $\mathbb{C}$-schemes" we mean the following: The inclusions $\mathbb{C} \hookrightarrow R$ and $\mathbb{C} \hookrightarrow \mathbb{C}[x]$ giving the algebra structure induce maps* $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{C}$ *and* $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$*. A map of $\mathbb{C}$-schemes from* $\operatorname{Spec} R$ *to $\mathbb{A}^1_{\mathbb{C}}$ is a map of schemes commuting with these maps to* $\operatorname{Spec} \mathbb{C}$*.) Hence, in this setting, we can think of ring elements as literal maps of schemes to the affine line.*

   More generally, if R is a ring, there is a natural bijection between $\{f \in R\}$ and the set of ring maps $\mathbb{Z}[x] \to R$, for essentially the same reason as before. This latter set is exactly the set of maps of schemes from $\operatorname{Spec} R$ *to* $\mathbb{A}^1_{\mathbb{Z}} := \operatorname{Spec} \mathbb{Z}[x]$.

2. *In the case where $R = k[x_1, \ldots, x_n]$ for k an algebraically closed field, we saw by the Nullstellensatz that the closed points of* $\operatorname{Spec} R$ *correspond to elements of the vector space k^n. We are used to thinking of polynomials $f \in R$ as functions from k^n to k; as we can see by the exact construction of this correspondence, it is possible to interpret the value of f at a point $(r_1, \ldots, r_n) \in k^n$ as the image of f under the quotient map $R \to R/(x_1 - r_1, \ldots, x_n - r_n) \cong k$ giving the point inclusion on the level of rings.*

   Now, for R any ring and $R \xrightarrow{\phi} k$ any point, closed or not, we can analogously think of $f \in R$ as having a "value" at this point given by $\phi(f)$, and so f can be viewed in this way as a "function" on the collection of points of R. Note that, since the "points" in the world of ring theory come in many different flavors, the "values" of f at different points will not all live in the same field; in the case $R = \mathbb{C}[x]$, for example, the "values" of an element of R at the closed points will all be elements of $\mathbb{C}$, but the "value" at the generic point will be an element of $\mathbb{C}(x)$. This is fine.

Pursuant to our second answer, we can offer new interpretations of the inclusions $\operatorname{Spec} R_f \hookrightarrow \operatorname{Spec} R$ and $\operatorname{Spec} R/(f) \hookrightarrow \operatorname{Spec} R$ for $f \in R$ a ring element:

Exercise 4. *Let R be a ring and $f \in R$ an element. Show that the points of* $\operatorname{Spec} R$ *contained in* $\operatorname{Spec} R_f$ *are precisely those where "the value of f" is nonzero. On the other hand, show that the points of* $\operatorname{Spec} R$ *contained in* $\operatorname{Spec} R/(f)$ *are precisely those where "the value of f" is zero. (Note that, although these "values" may live in various fields, asking which ones are zero or nonzero will always make sense.)*

We conclude by touching on the question of non-affine schemes. We have already seen one such object, the complement of the origin in $\mathbb{A}^2_{\mathbb{C}}$. The following example shows that there are things we can do with this machinery beyond just constructing such "missing open subschemes" of affine schemes:

Exercise 5. *Construct a scheme $\mathbb{P}^1_{\mathbb{C}}$, called the projective line over $\mathbb{C}$, as follows. Begin with* $\operatorname{Spec} \mathbb{C}[x]$ *and* $\operatorname{Spec} \mathbb{C}[y]$ *and identify the open subschemes* $\operatorname{Spec} \mathbb{C}[x]_x$ *and* $\operatorname{Spec} \mathbb{C}[y]_y$ *along* *the isomorphism induced by the $\mathbb{C}$-algebra map $\mathbb{C}[x]_x \to \mathbb{C}[y]_y$ given by $x \mapsto y^{-1}$. Describe the underlying topological space of $\mathbb{P}^1_{\mathbb{C}}$ and compute the ring $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\mathbb{P}^1_{\mathbb{C}})$ of global sections of the structure sheaf. Conclude that $\mathbb{P}^1_{\mathbb{C}}$ is not an affine scheme.*