The Geometry of Rings and Schemes
Lecture 12: Projectivization I

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So far, most of our motivation and examples have come from the study of affine space and its subschemes — that is, of the vanishing loci of collections of polynomials on Euclidean space and the ways we can understand their behavior through algebra. Questions about such objects, of course, comprised many of the central concerns of classical algebraic geometry — however, there is also another setting, called *projective space*, a sort of compactified version of Euclidean space which has been of interest both historically and in contemporary mathematics. Projective space is perhaps less immediately intuitive than Euclidean space, but it arises naturally in mathematical formalizations of questions of visual perspective and offers some theoretical advantages over the affine setting.

We will now build up the machinery needed for the scheme-theoretic perspective on projective space. Perhaps unsurprisingly, our approach to this will actually be relative — that is, we deal with maps of schemes rather than just schemes and work everywhere with "fiberwise versions of the constructions we are interested in".

1 Classical Motivation

We begin, as usual, by providing a sketch of the phenomena we are trying to replicate in more traditional topological terms. Classically speaking, projective space is defined to be the "space of lines through the origin in Euclidean space" — that is, every point corresponds to such a line, and the topology reflects our intuitive notion of what it should mean for a sequence (or, more generally, family indexed by a directed set) of lines to approach another. (If you want: The topology can be induced by metrizing according to the angles between different lines in the planes they span.) We realize this by means of a quotient:

Definition 1. *Let $n \geq 0$ be an integer. Then* **real projective n-space** *is*

$$\mathbb{RP}^n := \frac{\mathbb{R}^{n+1} \setminus 0}{(p \sim q \mid p \textit{ and } q \textit{ lie on the same line through the origin})}$$

and **complex projective n-space** *is*

$$\mathbb{CP}^n := \frac{\mathbb{C}^{n+1} \setminus 0}{(p \sim q \mid p \textit{ and } q \textit{ lie on the same (complex) line through the origin})}.$$

*First draft of the TeX source provided by Márton Beke.

If $p = (p_0, \dots, p_n)$ is a point in $\mathbb{R}^{n+1} \setminus 0$ or $\mathbb{C}^{n+1} \setminus 0$, we write $[p_0 : \cdots : p_n]$ for the corresponding point in the quotient space $\mathbb{RP}^n$ or $\mathbb{CP}^n$ respectively.

As it turns out, $\mathbb{R}^n$ is a real manifold of real dimension n, and $\mathbb{C}^n$ is a complex manifold of complex dimension n. To demonstrate the idea behind the argument for this in general, and to illustrate our prior claim that projective space is a "compactified version of Euclidean space", we consider the following example:

Example 1 ($\mathbb{RP}^2$ as a compactification of $\mathbb{R}^2$)**.** *Consider $\mathbb{R}^3$ with coordinates x, y, and z, and let P_0 be the xy-plane, given by the equation $z = 0$. Then we have a closed subspace of $\mathbb{RP}^2$, the space of lines through the origin in $\mathbb{R}^3$, consisting of those lines which lie entirely in P_0; since $P_0 \cong \mathbb{R}^2$, this subspace is in fact a copy of $\mathbb{RP}^1$.*

Now consider its complement $\mathbb{RP}^2 \setminus \mathbb{RP}^1$, the space of lines which are not entirely horizontal. Let P_1 be the plane in $\mathbb{R}^3$ parallel to P_0 at height 1; that is, the one cut out by the equation $z = 1$. Then any line through the origin in $\mathbb{R}^3$ which is not contained in P_0 — that is, which is not completely horizontal — will meet P_1 at a unique point, and we can see that both the map $\mathbb{RP}^2 \setminus \mathbb{RP}^1 \to P_1$ taking each line to this intersection point is a homeomorphism. Hence $\mathbb{RP}^2 \setminus \mathbb{RP}^1 \cong P_1 \cong \mathbb{R}^2$.

Thus we can think of $\mathbb{RP}^2$ as "$\mathbb{R}^2$, compactified by adding a copy of $\mathbb{RP}^1$ at infinity". To see how this works in practice, consider a sequence of points in $\mathbb{R}^2$ running off to infinity, say $(x_n, y_n) = (n, 0)$. Then we have the corresponding sequence $[x_n : y_n : 1]$ of points in $\mathbb{RP}^2 \setminus \mathbb{RP}^1$ given by taking the lines from the origin in $\mathbb{R}^3$ through the corresponding points in P_1; as we let $n \to \infty$, we can see that these lines (which, in this case, lie in the xz-plane, with equations $y = 0$ and $z = \frac{1}{n}x$) get closer and closer to being horizontal. In particular, they approach the x-axis, which corresponds to the point $[1 : 0 : 0]$ in $\mathbb{RP}^2$. That is, we started with a divergent sequence in $\mathbb{R}^2$ and, by identifying it with a sequence of non-horizontal lines through the origin in $\mathbb{R}^3$, have been able to produce a point of the "$\mathbb{RP}^1$ at infinity" — that is, a horizontal line — to which it converges.

Of course, there is nothing special about z — we can make the same sort of identification using any of our coordinates, or indeed any linear form. This gives a cover of $\mathbb{RP}^2$ by open patches homeomorphic to $\mathbb{R}^2$, and it is not difficult to see that the transition maps between coordinate patches are smooth, so that real projective 2-space is a smooth 2-dimensional real manifold.

Similar remarks apply to $\mathbb{RP}^n$ for any n, and indeed to $\mathbb{CP}^n$ as well — in this level of generality, we take our P_0 to be the vanishing of a single linear form on $(n+1)$-dimensional Euclidean space, so that it has dimension n over $\mathbb{R}$ or $\mathbb{C}$ respectively.

As mentioned, working in projective space makes many algebro-geometric questions better-behaved and easier to answer. For example, if we have two distinct lines in $\mathbb{R}^2$ or $\mathbb{C}^2$, the common vanishing locus of their defining equations may consist of a single point, the point of intersection between the two lines, or it may be empty, in the case where the lines are parallel and so there is no such intersection point. However, in projective space, this latter possibility is eliminated — that is, if we embed Euclidean 2-space in projective 2-space as described above, the closures of our chosen parallel lines will meet at a single point at infinity, the one corresponding to the line through the origin parallel to both. Hence, in projective 2-space, the intersection of two distinct (projective) lines will always consist of exactly one point. More generally, a result called *Bézout's theorem*, which we will probably not get around to discussing in detail, guarantees that in an appropriate sense the number of intersection points of a collection of polynomials in $\mathbb{CP}^n$ is entirely predictable from the polynomials' degrees, whereas the same is not true if we work only in $\mathbb{C}^n$.

2 Projective Space via Actions

As a first step toward adapting projective space to the scheme-theoretic context, we examine the equivalence relation used to define it in more detail. We will discuss the real case; the complex one is entirely analogous.

Recall from our discussion of topological linear fiber spaces in Lecture 6 that $\mathbb{R}$ is a topological ring, and $\mathbb{R}^{n+1}$'s structure as a topological vector space means in particular that there is a scalar multiplication map $\mathbb{R} \times \mathbb{R}^{n+1} \dot{\to} \mathbb{R}^{n+1}$. Ignoring, for the moment, the additive structures on our spaces, we can see that this map gives a *topological monoid action*, or more specifically, a *topological monoid-with-zero action*, of $\mathbb{R}$ on $\mathbb{R}^{n+1}$; we now review these definitions for the sake of completeness.

Definition 2. *A* **monoid** *is a set M together with an associative binary operation $M \times M \dot{\to} M$ and element $1 \in M$ such that $1 \cdot m = m \cdot 1 = m$ for all $m \in M$. A* **monoid-with-zero** *is a monoid with an element $0 \in M$ such that $0 \cdot m = m \cdot 0 = 0$ for all $m \in M$. A* **group** *is a monoid such that, for each $m \in M$, there is an element $m^{-1} \in M$ such that $m \cdot m^{-1} = m^{-1} \cdot m = 1$.*

A monoid, monoid-with-zero, or group is said to be **commutative**, *or* **abelian**, *if $m \cdot m' = m' \cdot m$ for all $m \in M$.*

Definition 3. *Let $(M, \cdot, 1)$ be a monoid and X a set. A* **(left) monoid action** *of M on X is a map $M \times X \dot{\to} X$ such that $m \cdot (m' \cdot x) = (m \cdot m') \cdot x$ and $1 \cdot x = x$ for all $m, m' \in M$ and $x \in X$. If M is a monoid-with-zero, this is called a* **monoid-with-zero action** *if moreover there is an element $0 \in X$ such that $0 \cdot x = 0$ for all $x \in X$. On the other hand, if M is a group, any monoid action is called a* **group action**.

As in Lecture 6, our method for formulating the appropriate topological versions (and, eventually, scheme-theoretic versions) of these objects will be to first rephrase them in terms of maps of sets and commutative diagrams, then replace these by corresponding maps and diagrams in the category of topological spaces — i.e., require all maps involved to be continuous. We have done this already in Lecture 6 for monoids, groups, and commutativity — for the remaining definitions, see Section 5.

Now, as mentioned, our scaling action of $\mathbb{R}$ on $\mathbb{R}^{n+1}$ is a monoid-with-zero action, and we can see that this restricts to a group action $\mathbb{R}^* \times (\mathbb{R}^{n+1} \setminus 0) \dot{\to} \mathbb{R}^{n+1} \setminus 0$. That is, if we limit ourselves to scaling by nonzero values, the result is still a continuous monoid action and hence, since $\mathbb{R}^* := \mathbb{R} \setminus 0$ is a group, a group action — moreover, the origin is its own orbit, so the restriction $\mathbb{R}^* \times (\mathbb{R}^{n+1} \setminus 0) \dot{\to} \mathbb{R}^{n+1}$ has image $\mathbb{R}^{n+1} \setminus 0$.

The orbits of this group action on $\mathbb{R}^{n+1} \setminus 0$ are precisely the equivalence classes of the relation used to define the quotient in Definition 1 — that is, the (punctured) lines through the origin in $\mathbb{R}^{n+1} \setminus 0$. Thus the projective space $\mathbb{RP}^n$ is precisely the quotient of $\mathbb{R}^{n+1} \setminus 0$

by the usual $\mathbb{R}^*$-action, and, for $(p_0, \ldots, p_n), (q_0, \ldots, q_n) \in \mathbb{R}^{n+1} \setminus 0$, we have $[p_0 : \cdots : p_n] = [q_0 : \cdots : q_n]$ if and only if there exists $\lambda \in \mathbb{R}^*$ with $p_i = \lambda q_i$ for all $0 \leq i \leq n$.

Taking this perspective allows us to extend our notion of projective space to vector bundles, and even (appropriate subsets of) arbitrary linear fiber spaces:

Definition 4. *Let $\pi : E \to B$ be a real linear fiber space in the topological setting (for example, a vector bundle), with $z : B \to E$ the zero section. Then, by definition, the fiberwise scaling action of $\mathbb{R} \times B$ on E (given by a map $(\mathbb{R} \times B) \times_B E \overset{\cdot}{\to} E$) is a monoid-with-zero action in the setting of topological spaces over B — that is, a map that satisfies the definitions of Section 5 with the one-point space P replaced by B, all Cartesian products replaced by the fiber product $\times_B$ over B, and all topological spaces and maps replaced by spaces and maps over B. This restricts to a fiberwise group action $(\mathbb{R}^* \times B) \times_B (E \setminus z(B)) \to E \setminus z(B)$ on the complement of the zero section in E.*

The **projectivization** *of E is the topological space $\mathbb{P}(E)$ over B given by taking the fiberwise quotient of $E \setminus z(B)$ by this group action. If $X \subseteq E$ is a subspace which is invariant under the scaling action — that is, $(\mathbb{R} \times B) \times_B E \overset{\cdot}{\to} E$ restricts to a fiberwise monoid-with-zero action $(\mathbb{R} \times B) \times_B X \overset{\cdot}{\to} X$ — then we can define the* **projectivization** *of X similarly as the quotient $\mathbb{P}(X)$ of $X \setminus z(B)$ by the induced fiberwise group action of $\mathbb{R}^* \times B$; this is exactly the subspace of $\mathbb{P}(E)$ whose points correspond to lines lying in X.*

That is, just as every fiber of $\pi : E \to B$ is a vector space, every fiber of the structure map $\mathbb{P}(E) \to B$ is a projective space, the space of lines through the origin in the corresponding fiber of π. Our definition of projectivization for subsets of E says essentially that, if we have a subset which is fiberwise a union of lines through the origin, we can consider the collection of all points corresponding to those lines as a subspace of $\mathbb{P}(E)$. (In particular, through the case where B is the one-point space, this definition encompasses projectivizations of unions of lines through the origin in Euclidean space.)

As mentioned, everything discussed in this section applies, mutatis mutandis, to complex projective space and projectivizations of complex vector bundles and linear fiber spaces.

3 $\mathbb{N}$-Graded Rings and Conical Fiber Spaces

We now seek to replicate our construction in the world of schemes. If k is a field and $n \geq 0$ is an integer, recall from our realization of $\mathbb{A}_k^{n+1}$ as a linear fiber space over $\operatorname{Spec} k$ in Lecture 6 that the scaling action is given by the map $\mathbb{A}_k^1 \times_k \mathbb{A}_k^{n+1} \to \mathbb{A}_k^{n+1}$ corresponding to the k-algebra map $k[x_0, \ldots, x_n] \to k[x_0, \ldots, x_n, t]$ taking x_i to tx_i for each $0 \leq i \leq n$. More generally, if R is a ring, the fiberwise scaling action on $\mathbb{A}_R^{n+1}$, the trivial rank-$(n+1)$ vector bundle over $\operatorname{Spec} R$, is given by the map $\mathbb{A}_R^1 \times_R \mathbb{A}_R^{n+1} \to \mathbb{A}_R^{n+1}$ corresponding to the R-algebra map $R[x_0, \ldots, x_n] \to R[x_0, \ldots, x_n, t]$ taking x_i to tx_i for each $0 \leq i \leq n$; if X is a scheme, the scaling action on any vector bundle of rank $(n+1)$ will by definition be given affine-locally by the preceding map under the trivializations.

Our goal will be to do scheme-theoretically what we did in the classical setting — that is, puncture $\mathbb{A}_k^{n+1}$ by throwing away the origin (or $\mathbb{A}_R^{n+1}$ by throwing away the zero section) and take a "quotient by the scaling action", whatever that means in this context. In doing so, we will actually work with a more general class of objects which includes, for example, the scheme-theoretic analogues to the "unions of lines through the origin in a linear fiber space" discussed above, since we will need the full generality and doing so doesn't really change the algebraic side of the picture that much — however, be sure to keep these basic examples in mind as you follow along and try to build out your geometric intuition for the things we're doing.

To begin, we note that the information of a monoid action by $\mathbb{A}^1$ of the sort discussed above is traditionally encapsulated in the following slightly different form:

Definition 5. *Let $R \to S$ be a map of rings. An* $\mathbb{N}$**-grading** *of S over R is a decomposition of S as a direct sum of R-modules $S = \bigoplus_{d\in\mathbb{N}} S_d$ such that $S_iS_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$ — that is, the product of an element in S_i with an element in S_j under the ring's multiplication will always lie in S_{i+j}. The choice of a ring S together with an $\mathbb{N}$-grading over R on S is called an* $\mathbb{N}$**-graded** R**-algebra***; often, we drop $\mathbb{N}$ from the terminology when it can be assumed from context. When we speak of* $\mathbb{N}$**-gradings** *of a ring S without reference to R, or* $\mathbb{N}$**-graded rings***, we will mean implicitly that the gradings involved are over $\mathbb{Z}$, so that the direct sum decomposition is simply as abelian groups.*

An element $h \in S$ is called **homogeneous** *if $h \in S_d$ for some $d \in \mathbb{N}$; this d is called its* **degree***. For an arbitrary $f \in S$, the direct sum decomposition guarantees that we can write $f = f_0 + f_1 + \ldots + f_N$ for some $N \in \mathbb{N}$ such that f_i is homogeneous of degree i for all $0 \leq i \leq N$; we call these f_is the* **homogeneous parts** *of f. If $I \subseteq S$ is an ideal, we say that I is* **homogeneous** *if any of the following equivalent conditions hold:*

- *I can be generated by homogeneous elements (not necessarily all of the same degree).*
- *For each $f \in S$, $f \in I$ if and only if every homogeneous part of f is in I.*
- *S/I has an induced $\mathbb{N}$-grading such that the natural surjection $S \twoheadrightarrow S/I$ preserves degrees.*

The homogeneous ideal $S_+ := S_1 \oplus S_2 \oplus S_3 \oplus \ldots$ generated by the positive-degree elements is called the **irrelevant ideal***. We say that S is* **generated in degree** 1 *if any of the following equivalent conditions holds:*

- *S_+ is the ideal generated by the elements of S_1.*
- *S is generated as an S_0-algebra by the elements of S_1.*
- *If we consider S_1 as an S_0-module, then the natural map $\mathrm{Sym}(S_1) \to S$ is a surjection.*

The prototypical example of a graded ring is as follows:

Example 2. *Let R be a ring, $n \geq 0$ an integer, and $S := R[x_1, \ldots, x_n]$ the polynomial ring in n variables over R. Then S is an $\mathbb{N}$-graded R-algebra with the usual notion of degree; that is, $S_0 = R$, $S_1 = \bigoplus_{i=1}^n Rx_i$, $S_2 = \bigoplus_{1\leq i\leq j\leq n} Rx_ix_j$, and so forth, so that the degree-d part S_d is the free R-module generated by the degree-d monomials in the variables $x_1, \ldots, x_n$. The irrelevant ideal is $(x_1, \ldots, x_n)$, and we can see that S is generated in degree 1 using any of our equivalent descriptions of this property — note in particular that the natural map $\mathrm{Sym}(S_1) = \mathrm{Sym}\left(\bigoplus_{i=1}^n Rx_i\right) \to S$ is an isomorphism.*

More generally, we have:

Example 3. *Let R be a ring and M an R-module. Then, if we set*

$$S := \operatorname{Sym} M = \frac{\bigoplus_{\ell=0}^{\infty} M^{\otimes \ell}}{(a \otimes b - b \otimes a \mid a, b \in M)},$$

S is an $\mathbb{N}$-graded R-algebra when considered with the grading such that, for each $d \in \mathbb{N}$, S_d is the image of $M^{\otimes d}$ under the quotient map $\bigoplus_{\ell=0}^{\infty} M^{\otimes \ell} \to S$. Then we can see that the irrelevant ideal is generated by $S_1 \cong M^{\otimes 1} \cong M$, and indeed that the natural map $\operatorname{Sym}(S_1) \to S$ is an isomorphism.

(This is a generalization of Example 2 since the polynomial ring in n variables is the symmetric algebra of a free module of rank n.)

As mentioned, specifying an $\mathbb{N}$-grading on a ring is the same as giving a monoid action of $\mathbb{A}^1$:

Proposition 1. *Let $R \to S$ be a ring map. Then specifying an $\mathbb{N}$-grading of S over R is the same as giving a map $\mathbb{A}^1_R \times_R \operatorname{Spec} S \to \operatorname{Spec} S$ of R-schemes which satisfies the axioms for a monoid action of $\mathbb{A}^1_R$ from Section 5 when we replace the maps to the one-point space by the structure maps to $\operatorname{Spec} R$ and the cartesian product $\times$ by the fiber product $\times_R$ everywhere.*

(Strictly speaking, we have a stronger result: An anti-equivalence of categories between R-algebras with such gradings and affine schemes over $\operatorname{Spec} R$ carrying such $\mathbb{A}^1_R$-actions.)

Proof. If we have an $\mathbb{N}$-grading $S = \bigoplus_{d \in \mathbb{N}} S_d$ over R, our monoid action will be given by the R-algebra map $S \to S \otimes_R R[t] \cong S[t]$ taking each homogeneous element $h \in S_d$ (for any $d \in \mathbb{N}$) to $t^d h$; this is indeed an R-algebra map by virtue of the facts that the direct sum decomposition is as R-modules and that degree is multiplicative. To verify that it gives a monoid action, we must check the following, per Definition 10:

- The two maps $S \to S[t_1, t_2]$ given by, respectively, applying our map $S \to S[t]$ twice (with a relabeling of variables) and applying our map $S \to S[t]$ followed by the map $S[t] \to S[t_1, t_2]$ taking t to $t_1 t_2$ (induced by the monoid structure of $\mathbb{A}^1_R$) are the same.
- The composition of $S \to S[t]$ with the quotient map $S[t] \to S[t]/(t-1) \cong S$ is the identity.

The first of these claims is clear since both maps in question take each $h \in S_d$ to ${t_1}^d {t_2}^d h$. Likewise, we can verify the second by noting that this composition takes $h \in S_d$ to $1^d h = h$. Hence each grading over R gives us an $\mathbb{A}^1_R$-action.

On the other hand, if we have an $\mathbb{A}^1_R$-action given by an R-algebra map $\alpha : S \to S[t]$, we construct an $\mathbb{N}$-grading over R as follows. For each $d \in \mathbb{N}$, set $S_d := \{h \in S \mid \alpha(h) = t^d h\}$. That each S_d is an R-module follows from the fact that α is an R-algebra map, and we can see that the R-module map $\bigoplus_{d \in \mathbb{N}} S_d \to S$ induced by the inclusions $S_d \hookrightarrow S$ is itself injective since $t^d h = t^{d'} h$ for $d \neq d'$ implies $h = 0$. To show that it is surjective, observe for arbitrary $f \in S$ that $\alpha(f) = f_0 + t f_1 + \ldots + t^n f_n$ for some $f_0, \ldots, f_n \in S$. Since the monoid axioms guarantee that the composition of α with the quotient map $S[t] \to S[t]/(t-1) \cong S$ is the identity, we can see that $f = f_0 + 1 \cdot f_1 + \ldots + 1^n \cdot f_n = f_0 + \ldots + f_n$, so the surjectivity will follow if we can show $f_d \in S_d$ for each $0 \le d \le n$.

If we apply α twice to f, treating it first as a map $S \to S[t_1]$ and then as a map $S[t_1] \to S[t_1, t_2]$ induced by a map $S \to S[t_2]$, we obtain the element $\alpha(f_0) + t_1\alpha(f_1) + \ldots + {t_1}^n\alpha(f_n)$. On the other hand, the monoid axioms guarantee that this will be equal to the result $f_0 + t_1t_2f_1 + \ldots + {t_1}^n{t_2}^nf_n$ of composing α with the map $t \mapsto t_1t_2$ induced by the monoid action. Hence, for each $0 \le d \le n$, we find that $\alpha(f_d) = {t_2}^d f_d$ (when we treat α as a map $S \to S[t_2]$), so $f_d \in S_d$ as desired. Thus $S \cong \bigoplus_{d\in\mathbb{N}} S_d$ as R-modules; that $S_iS_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{N}$ follows from the fact that α is a ring map.

It is not difficult to show that these operations are inverse to one another and moreover give us a contravariant identification of grading-preserving R-algebra maps with maps of affine schemes over R with $\mathbb{A}^1_R$-monoid actions. $\square$

Note that here R is really not so important, affecting only the nature of the direct sum decomposition, and, if we have maps $R' \to R \to \mathbb{Z}$, then an $\mathbb{N}$-grading over R will already be an $\mathbb{N}$-grading over R' as well. In particular, all $\mathbb{N}$-gradings are $\mathbb{N}$-gradings over $\mathbb{Z}$ — that is, gradings where we decompose S as a direct sum of abelian groups — and when we note that they are also $\mathbb{N}$-gradings over R we are simply observing that the corresponding $\mathbb{A}^1$-action happens to "preserve the fibers of $\operatorname{Spec} S \to \operatorname{Spec} R$" (where, as usual, this statement is slightly nonliteral due to the possibility of non-reduced behavior).

Now observe that, for any $\mathbb{N}$-grading of a ring S, there is a distinguished "final" ring map $R \to S$ such that this is an $\mathbb{N}$-grading over R — that is, the inclusion $S_0 \hookrightarrow S$. (For the category-heads: This is indeed final in the sense that every other such map $R \to S$ factors through it uniquely.)

Geometrically, this inclusion corresponds to the natural projection of $\operatorname{Spec} S$ onto its image in $\operatorname{Spec} S$ under multiplication by zero — that is, if we compose the ring map $S \to S[t]$ given by the scaling action with the quotient map $S[t] \to S[t]/(t)$ given by the zero section of $\mathbb{A}^1$ and take the "closure of the image" of this composed map by factoring through the quotient $S/S_+ \cong S_0$ by its kernel S_+, as in Lecture 2, we find that the result is precisely our inclusion $S_0 \hookrightarrow S$:

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r, hook, "z\times\mathrm{id}"] \arrow[drr, "\exists!"'] & \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} S \arrow[r, "\cdot"] & \operatorname{Spec} S \\ & & \operatorname{Spec} S_0 \arrow[u, hook] \end{tikzcd}$$

Hence the monoid-with-zero structure of $\mathbb{A}^1$ gives a distinguished projection of $\operatorname{Spec} S$ with "fibers preserved by the scaling action". As a sanity check:

Exercise 1. *Let T be a topological space and $\mathbb{R} \times T \to T$ a continuous monoid action of $(\mathbb{R}, \cdot, 1)$ on T (i.e., a continuous map satisfying the axioms from Definition 10 when we work in the setting of topological spaces). Describe the projection of T onto a closed subspace induced by multiplication by zero as above, and use the monoid axioms to show that each of its fibers is closed under the scalar multiplication.*

Indeed, we can use the monoid-with-zero structure to give the following refinement to Proposition 1:

Proposition 2. *Let $R \hookrightarrow S$ be an inclusion of rings. Then specifying an $\mathbb{N}$-grading of S with $S_0 = R$ is the same as giving a map $\mathbb{A}^1_R \times_R \operatorname{Spec} S \to \operatorname{Spec} S$ of R-schemes which satisfies the axioms for a monoid-with-zero action of $\mathbb{A}^1_R$ from Section 5 when we replace the maps to the one-point space by the structure maps to $\operatorname{Spec} R$ and the cartesian product $\times$ by the fiber product $\times_R$ everywhere.*

Proof. Proposition 1 gives the correspondence between gradings and actions — we will discuss the remaining details.

If we have an $\mathbb{N}$-grading such that $S_0 = R$, the quotient map $S \to S/S_+ \cong R$ gives our zero section $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S$, and we can see that this is indeed a section — i.e., a map of $\operatorname{Spec} R$-schemes — since $R \to S \to S/S_+ \cong R$ is the identity. That it satisfies the necessary condition from Definition 10 follows by observing that this quotient map can be written as the composition $S \to S[t] \to S[t]/(t)$ of the maps corresponding to the monoid action and the zero section of $\mathbb{A}^1_R$ respectively.

On the other hand, if our monoid action of $\mathbb{A}^1_R$ is moreover a monoid-with-zero action, then we have a zero section $\operatorname{Spec} R \to \operatorname{Spec} S$, given by a ring map $S \to R$ with $R \to S \to R$ the identity — in particular, this implies that $S \to R$ is a surjection. Moreover, from the commutativity requirement in Definition 10, we find that the composition $S \to R \hookrightarrow S$ agrees with the map $S \to S[t] \to S[t]/(t) \cong S$, which takes S_0 identically to itself and sends all elements of S_+ to 0. Hence, as the image of this map, S_0 must be contained in R. However, we know that our action corresponds to a grading over R, and from this it follows that R is contained in S_0 as well. Thus $S_0 = R$, as desired. □

Hence, in the setting of affine schemes, all of the monoid action information we want to consider can be encapsulated in the more algebraically familiar notion of an $\mathbb{N}$-graded ring, and conversely we can understand the various aspects of an $\mathbb{N}$-grading geometrically in terms of monoid and monoid-with-zero actions of the affine line on the corresponding ring spectrum. We now expand our attention to the setting of arbitrary schemes:

Definition 6. *Let X be a scheme. A* **conical fiber space** *over X is a scheme C over X together with a map $X \xrightarrow{z} C$ of X-schemes, called the* **zero section***, and a map $\mathbb{A}^1_X \times_X C \xrightarrow{\cdot} C$, called the* **scalar multiplication***, such that $\cdot$ and z define a monoid-with-zero action of $\mathbb{A}^1_X$ on C in the sense of the axioms given in Section 5 (where we adapt these by working with maps of X-schemes instead of maps of sets, replacing the maps to the point by the structure maps to X, and replacing the cartesian product by the fiber product over X everywhere).*

This is to say that a conical fiber space over X is a scheme over X with a “fiberwise scaling action of the affine line as a monoid-with-zero” — for each $x \in X$, $\mathbb{A}^1_{\kappa(x)}$ acts on the fiber of C over x by multiplication so that multiplication by zero sends everything to the point $z(x)$.

Remark 1. *The term “conical” comes from the following observation: If we have a topological space T admitting a continuous monoid-with-zero action of $\mathbb{R}_{\geq 0}$ (considered as a monoid-with-zero using the usual multiplication of real numbers), then this restricts to a group action of $\mathbb{R}_{>0}$. If this action is in particular free on the complement of the basepoint x in T, we can see that $T \setminus x$ is a union of open rays given by the orbits of the action, so* *that T itself is the union of rays with the endpoints all identified — that is, an open cone. (Specifically, this is the cone over the quotient of $T \setminus x$ by the $\mathbb{R}_{>0}$-action.)*

Likewise, if we have an appropriate monoid-with-zero action on a space T by all of $\mathbb{R}$, we can follow the same reasoning to interpret T as a "two-sided cone" over the corresponding quotient space. By analogy, we think of a space with a well-behaved monoid-with-zero action of $\mathbb{C}$ as a "complex cone" — that is, a union of complex lines with the origin identified — and so a scheme with a monoid-with-zero action of $\mathbb{A}^1_k$ for k some field is likewise interpretable as a cone in some broad sense (although it is not necessarily true that such an action will, e.g., restrict to a free group action in the way described above). Hence a conical fiber space over a scheme is one such that "each fiber is a cone".

Example 4. *Let X be a scheme. Then any linear fiber space over X is also a conical fiber space over X simply by forgetting the additive structure; this follows immediately by comparing the two collections of axioms defining linear and conical fiber spaces respectively.*

This is to say that any scheme which is "fiberwise a vector space" over X in particular has a "fiberwise $\mathbb{A}^1_X$-action" by the scalar multiplications of the vector spaces, and each vector space can be thought of as "the union of its lines through the origin" (although this is not literally true on the level of underlying sets because of the presence of non-closed points).

In order to describe conical fiber spaces algebraically, we make the following relative version of Definition 5:

Definition 7. *Let X be a scheme and $\mathcal{A}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras. Then an* **$\mathbb{N}$-grading** *of $\mathcal{A}$ is a direct sum decomposition $\mathcal{A} = \bigoplus_{d\in\mathbb{N}} \mathcal{A}_d$ as sheaves of $\mathcal{O}_X$-modules such that $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{N}$; in general, we adapt the terminology of Definition 5 to this setting by replacing ring elements by sections, ideals by sheaves of ideals, and so forth.*

We now have the following generalization of Proposition 2:

Proposition 3. *Let X be a scheme. Then we have an anti-equivalence between the category of conical fiber spaces $C \xrightarrow{\pi} X$ over X such that π is an affine map and the category of $\mathbb{N}$-graded quasicoherent sheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras such that $\mathcal{A}_0 = \mathcal{O}_X$; this is given by the functors taking $\mathcal{A}$ to $\operatorname{Spec}\mathcal{A}$ and C to $\pi_*\mathcal{O}_C$.*

Proof idea. If we work affine-locally on X, this is essentially the correspondence of Proposition 2 together with the usual correspondence between affine maps and quasicoherent sheaves of algebras. □

Remark 2. *The condition $\mathcal{A}_0 = \mathcal{O}_X$ is needed to ensure that the grading gives an honest monoid-with-zero action, per Proposition 2; if we consider an $\mathbb{N}$-graded algebra not satisfying this condition, we find instead that $\operatorname{Spec}\mathcal{A}$ is a conical fiber space over the relative spectrum $\operatorname{Spec}\mathcal{A}_0$.*

4 $\mathbb{Z}$-Graded Rings and Projectivization

With the introduction of conical fiber spaces, we now have the objects we will projectivize, and it is clear how to take the complement of the zero section in such a scheme — it remains to define the quotient by the group action. We begin by formalizing the group in question:

Proposition/Definition 1. *Let X be a scheme. Then $(\mathbb{A}^1_X)^* := X \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}[t]_t$, the* **punctured affine line** *over X, is an abelian group over X (i.e., one that satisfies the diagrammatic versions of the abelian group actions from Lecture 6 in the setting of schemes over X) when considered with the group structure induced by the multiplication on $\mathbb{A}^1_X$ and the inverse map $\mathbb{Z}[t]_t \to \mathbb{Z}[t]_t$ taking t to t^{-1}.*

As the name would suggest, the punctured affine line is simply the trivial line bundle over X with the zero section removed. As in the case of the affine line, we can describe the rings and algebras admitting actions of the punctured affine line:

Definition 8. *Let $R \to S$ be a map of rings. A* **$\mathbb{Z}$-grading** *of S over R is a decomposition $S = \bigoplus_{d \in \mathbb{Z}} S_d$ as a direct sum of R-modules such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \in \mathbb{Z}$. If X is a scheme and $\mathcal{A}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras, a* **$\mathbb{Z}$-grading** *of $\mathcal{A}$ is defined analogously as a decomposition into a direct sum of sheaves of $\mathcal{O}_X$-modules. The terminology we will use for $\mathbb{Z}$-graded rings and sheaves will generally follow that introduced in Definition 5.*

We now have the following analogue to Proposition 1:

Proposition 4. *Let $R \to S$ be a ring map. Then specifying a $\mathbb{Z}$-grading of S over R is the same as giving a map $(\mathbb{A}^1_R)^* \times_R \operatorname{Spec} S \to \operatorname{Spec} S$ of R-schemes which defines a group action over* $\operatorname{Spec} R$.

Proof idea. Entirely analogous to the proof of Proposition 1. □

(More broadly, we can see that there will be a correspondence between affine schemes over a scheme X admitting an action of the punctured line and $\mathbb{Z}$-graded algebra sheaves.)

Now, it is clear that any $\mathbb{N}$-grading is also a $\mathbb{Z}$-grading with the negatively-indexed homogeneous parts equal to zero; similarly, any $\mathbb{A}^1$-action induces a $(\mathbb{A}^1)^*$-action simply by restricting over the complement of the zero section in $\mathbb{A}^1$. The advantage of viewing an $\mathbb{N}$-graded ring as $\mathbb{Z}$-graded can be seen through the following:

Proposition 5. *Let $R \to S$ be a ring map fix a $\mathbb{Z}$-grading of S over R. Then, for each degree $d \in \mathbb{Z}$ and homogeneous degree-d element $h \in S_d$, the localization S_h admits a unique $\mathbb{Z}$-grading over R such that the natural map $S \to S_h$ is degree-preserving, given by taking $\frac{1}{h} \in (S_h)_{-d}$.*

We can now construct the desired quotient of the complement of the zero section in a given (relatively affine) conical fiber space. We begin with the case where the base scheme is affine: Let R be a ring and observe that the affine conical fiber spaces over $\operatorname{Spec} R$ are precisely the spectra of $\mathbb{N}$-graded algebras S over R with $S_0 = R$. For such an algebra, the inclusion of the zero section corresponds to the quotient map $S \to S/S_+$, and so its complement is the union of the open subschemes $\operatorname{Spec} S_f$ as f runs over any generating set of the ideal S_+. In particular, since S_+ is a homogeneous ideal, we can restrict our attention to a homogeneous generating set; explicitly, our complement is covered by the open subschemes S_h for $h \in S_d$ where $d \geq 1$ is a positive integer.

By our prior proposition, each such S_h is $\mathbb{Z}$-graded; that is, while it no longer necessarily admits an action by $\mathbb{A}^1_R$, it does still have the restricted action of $(\mathbb{A}^1_R)^*$. Geometrically, the point is that the vanishing locus $V(h)$ consists of the zero section, since the degree of h

is positive, together with some "union of lines through the origin" — that is, orbits of the $(\mathbb{A}^1_R)^*$-action. The complement $\operatorname{Spec} S_h$ is then the "union of all remaining $(\mathbb{A}^1_R)^*$-orbits", with the zero section excluded. We must now ask:

Question. *What should the "quotient of* $\operatorname{Spec} S_h$ *by its* $(\mathbb{A}^1_R)^*$*-action" be algebraically?*

The answer arises by thinking of S_h, as usual, as the collection of "functions on $\operatorname{Spec} S_h$" — if one wants, the collection of maps $\operatorname{Spec} S_h \to \mathbb{A}^1_R$ of schemes over $\operatorname{Spec} R$. We should then ask which functions f should factor as follows:

$$\begin{tikzcd} \operatorname{Spec} S_h \arrow[r, "f"] \arrow[d, two heads] & \mathbb{A}^1_R \\ \text{“}\operatorname{Spec} S_h/(\mathbb{A}^1_R)^*\text{”} \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

From the analogous situation in the topological setting, we should expect that the answer is "functions which are invariant under the group action" — that is, those functions such that $f(gx) = f(x)$ for all g in the group and x in the space under consideration. Hence:

Answer. *The spectrum of the ring* $(S_h)_0$ *which is the degree-zero part of* S_h*; under our correspondence between gradings and actions, this is precisely the collection of* $f \in S_h$ *which are fixed by the map* $S_h \to S_h[t]_t$ *giving the group action.*

This operation — taking the spectrum of the invariant subring under the group action — gives us the quotient in the sense of *geometric invariant theory*, and may not always give nice results of the form we want, although it works in this case. (As one example, trying to take the "quotient" of $\operatorname{Spec} S$ by $\mathbb{A}^1_R$ in the same way leads to difficulties.) In general, the quotient by a group action may not exist in the world of schemes, and may need to be defined as a more general object called a *stack* — we won't get into the details.

We are now, at long last, ready to define our analogue to the classical projectivization of Definition 4:

Proposition/Definition 2. *Let R be a ring and S an $\mathbb{N}$-graded R-algebra with $S_0 = R$. Then the* $\operatorname{Spec} R$*-schemes* $\operatorname{Spec}(S_h)_0$ *as h runs over the homogeneous elements of S_+ naturally patch together to form a single* $\operatorname{Spec} R$*-scheme* $\operatorname{Proj} S$ *(or* $\mathbb{P}(\operatorname{Spec} S)$*), called the* **homogeneous spectrum** *of S (or the* **projectivization** *of* $\operatorname{Spec} S$*). The maps induced by the ring inclusions* $(S_h)_0 \hookrightarrow S_h$ *glue together into a map* $\operatorname{Spec} S \setminus \operatorname{Spec} R \to \mathbb{P}(\operatorname{Spec} S)$*, called the* **quotient map** *(where* $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S$ *is the zero section).*

If X is a scheme and $C \to X$ is a conical fiber space over X which is affine over X, with $\mathcal{A}$ the corresponding $\mathbb{N}$-graded quasicoherent algebra sheaf, applying this construction over affine opens gives a well-defined X-scheme $\operatorname{Proj} \mathcal{A}$ *(also called* $\mathbb{P}(C)$*), which we call the* **(relative) homogeneous spectrum** *of $\mathcal{A}$ (or the* **projectivization** *of C). As in the affine case, we have the* **quotient map** $C \setminus X \to \mathbb{P}(C)$*, where $X \hookrightarrow C$ is the zero section.*

We will explore this construction in more detail, and introduce the most typical examples, in next week's lecture.

5 Appendix: More Algebraic Objects via Diagrams

As in Lecture 6, we will denote the one-point set by P, and, for any set S, the natural map $S \to P$ by π. If $A_1 \times \cdots \times A_n$ is a product of sets, we will denote the projection onto the ith factor by π_i; if B is another set and $f_i : B \to A_i$ are maps, we will denote the induced map $B \to A_1 \times \cdots \times A_n$ by $f_1 \times \cdots \times f_n$. To save space, we will often omit composition symbols — that is, for maps f and g, we write gf instead of $g \circ f$. We will also write binary operations using infix notation — e.g., $f \cdot g$ in place of the composition $\cdot(f \times g)$.

Definition 9. *A* **monoid-with-zero** *consists of a set M together with maps $M \times M \xrightarrow{\cdot} M$, $P \xrightarrow{u} M$, and $P \xrightarrow{z} M$ such that $(M, \cdot, u)$ is a monoid and, in addition, the following diagram commutes:*

$$\begin{tikzcd}
M \arrow[r, "(z\pi)\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times(z\pi)"'] \arrow[dr, "z\pi"] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"] & M
\end{tikzcd}$$

Definition 10. *Let $(M, \cdot, u)$ be a monoid and X a set. Then a* **(left) monoid action** *of M on X is a map $M \times X \xrightarrow{\cdot} X$ such that the following diagrams commute:*

$$\begin{tikzcd}
M \times M \times X \arrow[r, "(\pi_1\cdot\pi_2)\times\pi_3"] \arrow[d, "\pi_1\times(\pi_2\cdot\pi_3)"'] & M \times X \arrow[d, "\cdot"] \\
M \times X \arrow[r, "\cdot"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "(u\pi)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & M \times X \arrow[d, "\cdot"] \\
 & X
\end{tikzcd}$$

If we also have a map $P \xrightarrow{z} M$ such that $(M, \cdot, u, z)$ is a monoid-with-zero, we say that our monoid action is a **monoid-with-zero action** *when we have a map $P \xrightarrow{x} X$, called the* **basepoint** *or* **zero section** *of X, such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "(z\pi)\times\mathrm{id}"] \arrow[dr, "x\pi"'] & M \times X \arrow[d, "\cdot"] \\
 & X
\end{tikzcd}$$