The Geometry of Rings and Schemes
Lecture 9: Calculus on Schemes I

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In previous lectures, we've touched on the ideas of "first-order derivative information" and tangent spaces to schemes at particular points. Now, using our newly-developed machinery of quasicoherent sheaves and linear fiber spaces, we will define an object capturing all of these tangent spaces at once, analogous to the tangent bundle of a smooth manifold — a sort of "tangent linear fiber space" over a given scheme. In the setting of nonsingular finite-type schemes over $\mathbb{C}$, our prior knowledge of differential geometry makes it easy to see what this should be in any individual case:

Example 1. *Consider the affine plane $\mathbb{A}^2_{\mathbb{C}}$. Then the tangent bundle should be the "trivial rank-2 vector bundle over $\mathbb{C}$" — that is, it should be $\mathbb{A}^2_{\mathbb{A}^2_{\mathbb{C}}} \cong \mathbb{A}^4_{\mathbb{C}}$ together with the projection $\mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ onto the first two coordinates, corresponding to the natural inclusion $\mathbb{C}[x,y] \to \mathbb{C}[x,y,dx,dy]$. Observe here that the fiber over any closed point of $\mathbb{A}^2_{\mathbb{C}}$ — which will be cut out by a maximal ideal of the form $(x-a, x-b)$ for $a, b \in \mathbb{C}$ — is a plane $\mathbb{A}^2_{\mathbb{C}} \cong \operatorname{Spec} \mathbb{C}[dx, dy]$; as expected, the tangent space to 2-dimensional Euclidean space at any point is itself a plane.*

Now consider the closed subscheme $V(y-x^2) := \operatorname{Spec} \mathbb{C}[x,y]/(y-x^2)$, which gives a parabola in our affine plane. The fiber product $V(y-x^2) \times_{\mathbb{A}^2_{\mathbb{C}}} \mathbb{A}^4_{\mathbb{C}} = \operatorname{Spec}[x,y,dx,dy]/(y-x^2)$ gives the restriction of the tangent bundle of the plane to this curve; again, the fiber over any closed point is a plane with coordinates dx and dy. To see what the tangent bundle to the parabola itself should be, recall from calculus that the tangent line to the curve $y = x^2$ at any point should have slope $2x$; treating this as a line through the origin in the tangent plane at the given point, we can then see that the equation should be $dy = 2xdx$ in our coordinates dx and dy.

Hence the tangent bundle to the parabola is given by $\operatorname{Spec} \mathbb{C}[x,y,dx,dy]/(y-x^2, dy - 2xdx)$.

Of course, this construction is all very ad-hoc, relying on our external knowledge of facts from calculus; in this lecture and the next, we'll try to rectify this issue by appropriately reformulating the notion of a tangent bundle from differential geometry so that it admits a natural scheme-theoretic analogue.

[*]First draft of the TeX source provided by Márton Beke.

1 Motivation from Differential Geometry

In differential geometry, the tangent bundle is usually defined locally — that is, by constructing it as $\mathbb{R}^{2n} \to \mathbb{R}^n$ for $\mathbb{R}^n$ and then gluing together along charts. As mentioned, we want to define an analogous concept for schemes — however, this definition isn't particularly useful for such a purpose, since schemes don't all admit local isomorphisms to affine space.

To find a better definition, recall first the following related concept:

Definition 1. *Let X be a smooth manifold. Then a* **smooth vector field** *on X is a smooth section of the tangent bundle projection $\pi : TX \to X$ — that is, a smooth map $\sigma : X \to TX$ such that $\pi \circ \sigma = \mathrm{id}_X$.*

This is to say that a smooth vector field on X is choice of tangent vector at every point of X such that the chosen vectors vary smoothly as we move from point to point.

In order to be able to define smooth vector fields without already knowing what a tangent bundle is, we observe first that they act naturally on smooth functions by taking directional derivatives:

Proposition 1. *Let X be a smooth manifold, V a smooth vector field on X, $U \subseteq X$ an open subset, and $f : U \to \mathbb{R}$ a smooth function. Then the function $Vf : U \to \mathbb{R}$ which gives the directional derivative of f in the direction of $V(x)$ at each point $x \in U$ is also smooth.*

If we let $\mathcal{O}_X$ denote the sheaf of smooth real-valued functions on X, V then defines an $\mathbb{R}$-linear map $V : \mathcal{O}_X \to \mathcal{O}_X$ of sheaves by the assignment $f \mapsto Vf$. Moreover, this map satisfies the usual product rule $V(fg) = fVg + gVf$ for derivatives.

This action is important for our purposes because of the following fact from differential geometry:

Proposition 2. *In the setting of the prior proposition, the function taking each vector field to its corresponding map $\mathcal{O}_X \to \mathcal{O}_X$ gives an isomorphism between the real vector space of smooth vector fields on X and the real vector space of $\mathbb{R}$-linear sheaf maps $\mathcal{O}_X \to \mathcal{O}_X$ satisfying the product rule.*

Proof. See, e.g., Proposition 8.15 of John M. Lee's *Introduction to Smooth Manifolds* (although a little bit of extra work with bump functions is needed to see that such maps $\mathcal{O}_X \to \mathcal{O}_X$ are in fact determined by the corresponding maps $\mathcal{O}_X(X) \to \mathcal{O}_X(X)$ on global sections which Lee considers). □

Since we have analogues to "smooth real-valued functions" in the scheme setting in the form of ring elements (more broadly, sections of the structure sheaf), the alternate algebraic definition of vector fields provided by this proposition can be translated to the world of schemes — we will then be able to retrieve the analogue of the notion of a tangent bundle from this using the machinery we have been developing for passing back and forth between sheaves and linear fiber spaces.

Before doing so, we note also the following fact on the behavior of vector fields with respect to certain kinds of smooth maps:

Proposition 3. *Let $\Phi : X \to Y$ be a constant-rank map of smooth manifolds (so that every fiber of Φ) is itself a smooth manifold. Then a vector field V on X will be everywhere tangent to the fibers if Φ if and only if, for every smooth function $g : Y \to \mathbb{R}$, $V(g \circ \Phi)$ is the zero function.*

This is to say that the vectors given by V point along the fibers of Φ exactly when every function constant along each fiber of Φ has zero derivative in the direction of V. This will turn out to be important in the scheme-theoretic setting because of the necessity of defining things in relative terms as we did with products of schemes.

2 Differentials and Tangent Schemes

As mentioned, it seems that it will be easiest for us to define "the (sheaf of) vector fields on a scheme" and then reverse-engineer our analogue to the tangent bundle from this. However, our correspondence between vector bundles (or linear fiber spaces more generally) and sheaves is contravariant, rather than covariant — that is, rather than the tangent bundle corresponding to its sheaf of sections, the vector fields, we should think of it as corresponding to its sheaf of linear forms, the "cotangent vector fields" or "covector fields", which differ from vector fields by a dual.

To begin defining the objects we need, we formalize the notion of "a map satisfying the product rule":

Definition 2. *Let $\phi : R \to S$ be a map of rings and M an S-module. An R-linear* **derivation** *of S into M is a map $d : S \to M$ of abelian groups satisfying the following hypotheses:*

- *For all $f, g \in S$, $d(fg) = fdg + gdf$. (This is called the* **Leibniz rule** *or* **product rule***.)*
- *For all $r \in R$, $d\phi(r) = 0$. (Equivalently, given the Leibniz rule: d is a map of R-modules.)*

Let $\mathrm{Der}_R(S, M)$ denote the set of all such derivations, endowed with the structure of an S-module using the natural S-module operations on maps into M.

(In the case of sheaves of rings and modules, we make the corresponding definitions by requiring our conditions on the map of sections over every open set.)

For a given ring map $R \to S$, we can of course consider many different S-modules M and, for any given M, many different R-derivations of S into M. However, it turns out that all information about such derivations can essentially be encapsulated in a single S-module:

Definition 3. *Let $\phi : R \to S$ be a map of rings. The* **module of Kähler differentials** *of S over R is the S-module*

$$\Omega_{S/R} := \frac{\bigoplus_{f \in S} Sdf}{\substack{(d(fg) - fdg - gdf \mid f, g \in S) \\ + \\ (d(\phi(a)f + \phi(b)g) - \phi(a)df - \phi(b)dg \mid a, b \in R, f, g \in S)}}.$$

The map $d : S \to \Omega_{S/R}$ given by $f \mapsto df$ is called the **universal derivation** *of S over R.*

That is, we construct the module of differentials by considering the module freely generated by the symbols $df, f \in S$, and then modding out by precisely the relations needed to make $f \mapsto df$ a derivation over R. Indeed, this construction makes the map d live up to the name "universal derivation":

Proposition 4. *Let $\phi : R \to S$ be a map of rings. Then the universal derivation of S over R is an R-linear derivation of S into $\Omega_{S/R}$. Moreover, for any S-module M and R-linear derivation $d_M : S \to M$, there exists a unique map $\psi : \Omega_{S/R} \to M$ of S-modules such that $d_M = \psi \circ d$.*

This universal property gives rise to a natural identification $\operatorname{Der}_R(S, M) \cong \operatorname{Hom}_S(\Omega_{S/R}, M)$ between R-derivations of S into M and S-module homomorphisms $\Omega_{S/R} \to M$. (Here "natural" means that, for an S-module map $M \to M'$, the identifications respect the maps $\operatorname{Der}_R(S, M) \to \operatorname{Der}_R(S, M')$ and $\operatorname{Hom}_S(\Omega_{S/R}, M) \to \operatorname{Hom}_S(\Omega_{S/R}, M')$ given by composition.)

In particular, we can see that $\operatorname{Der}_R(S, S) \cong \operatorname{Hom}_S(\Omega_{S/R}, S) =: \Omega^\vee_{S/R}$; that is, if we take the dual of our module of differentials as an S-module, we get the module of derivations of S into itself. This is our first inkling of the idea that the Kähler differentials should be viewed as covector fields on $\operatorname{Spec} S$, since the derivations from S to itself are "vector fields"; we will return to this later.

For now, as a prelude to taking this more geometric perspective, we want to move from the setting of modules to that of sheaves. Of course, the thing to verify here is that the formation of differentials commutes with localization:

Proposition 5. *Let $R \to S$ be a map of rings and $U \subseteq S$ a multiplicatively closed set. Then $\Omega_{U^{-1}S/R} \cong U^{-1}S \otimes_S \Omega_{S/R}$. In particular, $\Omega_{S_f/R} \cong S_f \otimes_S \Omega_{S/R}$ for any $f \in S$ and $\Omega_{S_\mathfrak{p}/R} \cong S_\mathfrak{p} \otimes_S \Omega_{S/R}$ for any prime ideal $\mathfrak{p} \subset S$.*

Proof sketch. Essentially, what we need to verify is that $U^{-1}S \otimes_S \Omega_{S/R}$ already contains an element playing the role of $d(1/u)$ for each $u \in U$; this can be done using the usual quotient rule $du^{-1} = -1u^{-2}du$, which follows by observing that $0 = d1 = d(uu^{-1}) = udu^{-1} + u^{-1}du$. □

This having been established, we find that there is a well-defined sheaf on the domain of a given map of schemes affine-locally agreeing with the Kähler differentials:

Proposition/Definition 1. *Let $\Phi : X \to Y$ be a map of schemes. Then there exists a unique quasicoherent sheaf $\Omega_{X/Y}$ of $\mathcal{O}_X$-modules such that, for all affine opens $\operatorname{Spec} R \subseteq Y$ and $\operatorname{Spec} S \subseteq \Phi^{-1}(\operatorname{Spec} R)$, $\Omega_{X/Y}|_{\operatorname{Spec} S} \cong \widetilde{\Omega}_{S/R}$ and the gluings are compatible with these identifications. (That is, the following holds: Suppose we have affine opens $\operatorname{Spec} R, \operatorname{Spec} R' \subseteq Y$, $\operatorname{Spec} S \subseteq \Phi^{-1}(\operatorname{Spec} R)$, and $\operatorname{Spec} S' \subseteq \Phi^{-1}(\operatorname{Spec} R')$ with elements $f \in R$, $f' \in R'$, $g \in S$, and $g' \in S'$ such that $\operatorname{Spec} S_g \subseteq \Phi^{-1}(\operatorname{Spec} R_f)$, $\operatorname{Spec} S'_{g'} \subseteq \operatorname{Spec} \Phi^{-1}(R'_{f'})$, $\operatorname{Spec} R_f$ and $\operatorname{Spec} R'_{f'}$ agree as open subschemes of Y, and $\operatorname{Spec} S_g$ and $\operatorname{Spec} S'_{g'}$ agree as open subschemes of X. Then the isomorphism $\Omega_{S_g/R_f} \cong S_g \otimes_S \Omega_{S/R} \cong \Omega_{X/Y}(\operatorname{Spec} S_g) \cong \Omega_{X/Y}(\operatorname{Spec} S'_{g'}) \cong S'_{g'} \otimes_{S'} \Omega_{S'/R'} \cong \Omega_{S'_{g'}/R'_{f'}}$ induced by our local identifications of sheaves $\Omega_{X/Y}|_{\operatorname{Spec} S} \cong \widetilde{\Omega}_{S/R}$ and $\Omega_{X/Y}|_{\operatorname{Spec} S'} \cong \widetilde{\Omega}_{S'/R'}$ agrees with the isomorphism $\Omega_{S_g/R_f} \cong \Omega_{S'_{g'}/R'_{f'}}$ naturally induced by the commutative square*

$$\begin{array}{ccc} S_g & \xrightarrow{\sim} & S'_{g'} \\ \uparrow & & \uparrow \\ R_f & \xrightarrow{\sim} & R'_{f'} \end{array}$$

of ring maps.)

We call $\Omega_{X/Y}$ *the* **sheaf of (relative) Kähler differentials** *of* X *over* Y *and may also denote it by* Ω_Φ*. Moreover, we define the* **(relative, or fiberwise) tangent scheme** *of* X *over* Y *to be the linear fiber space* $T_{X/Y} := \operatorname{Spec}_+ \Omega_{X/Y}$ *over* X *(which we may also denote* T_Φ*).*

If we are understood to be working in the context of schemes over Y *(as, for example, when* $Y = \operatorname{Spec} \mathbb{C}$ *and we are in the setting of* $\mathbb{C}$*-schemes), we may drop* Y *from the notations and speak only of* Ω_X *and* T_X*.*

The concept is that, just as was the case with products of schemes, it is most productive to define tangent schemes of when working over a given base scheme Y; in that setting, $T_{X/Y}$ is "the" tangent scheme to a Y-scheme X, just as $X \times_Y X'$ is "the" product of Y-schemes X and X'. As mentioned in the definition, a very common use case is $Y = \operatorname{Spec} \mathbb{C}$ or, more generally, $Y = \operatorname{Spec} k$ for k a field, so that for a k-scheme X the tangent scheme T_X is defined relative to the map to the one-point space $\operatorname{Spec} k$ — this gives the most direct analogue to the tangent bundle in the setting of smooth manifolds.

The sheaf of Kähler differentials, being the sheaf of *linear forms* on our tangent bundle analogue, should be considered as the "sheaf of (relative) covector fields" on our scheme. To verify that our analogy holds up, we must check that the *sections* of the tangent scheme of correspond to derivations from the structure sheaf to itself; in light of Proposition 4, this is a special case of the more general observation about quasicoherent sheaves:

Proposition 6. *Let* X *be a scheme and* $\mathcal{F}$ *a quasicoherent sheaf of* $\mathcal{O}_X$*-modules. Then* $\{$*sections of* $\operatorname{Spec}_+ \mathcal{F} \to X\} \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ *as* $\mathcal{O}_X(X)$*-modules.*

Proof sketch (affine case). Suppose $X = \operatorname{Spec} R$ for some ring R, so that $\mathcal{F} = \tilde{M}$ for M some R-module. Then sections of $\operatorname{Spec}_+ \mathcal{F} \to X$, which are defined to be maps $X \to \operatorname{Spec}_+ \mathcal{F}$ of schemes which yield id_X when composed with the projection, can be exactly identified with R-algebra maps $\operatorname{Sym} M \to R$ by the usual correspondence between maps of affine schemes and maps of rings (the requirement to compose to the identity becomes the requirement that $R \to \operatorname{Sym} M \to R$ be the identity, which is precisely encapsulated by the requirement that $\operatorname{Sym} M \to R$ be an R-algebra map).

Since $\operatorname{Sym} M$ is generated as an R-algebra by its degree-1 part $M^{\otimes 1} = M$, we can see that the R-algebra maps $\operatorname{Sym} M \to R$ are naturally in bijective correspondence with R-module maps $M \to R$. The collection of all such maps is $\operatorname{Hom}_R(M, R)$, which is exactly $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ by the correspondence between modules and quasicoherent sheaves in the affine case. $\square$

We elide the verification that these identifications preserve the R-module structure and the gluing arguments necessary for the proof in general. In the case we are concerned with, this result has the following consequence when combined with Proposition 4:

Corollary 1. *Let $\Phi : X \to Y$ be a map of schemes. Then the sections of $T_{X/Y}$ can be canonically identified with the $\Phi^{-1}\mathcal{O}_Y$-linear derivations $\mathcal{O}_X \to \mathcal{O}_X$.*

Affine-locally, where Φ is induced by a ring map $R \to S$, this is just to say that the sections of the tangent scheme are given by $\mathrm{Der}_R(S, S)$. If $Y = \operatorname{Spec} k$ for k a field, we have $\Phi^{-1}\mathcal{O}_Y = \underline{k}$ (the *constant sheaf* which is given by k on every connected open set) and so we can see that our "vector fields" in the sense of sections of the tangent scheme are, indeed, simply the k-linear derivations from $\mathcal{O}_X$ to itself, as in differential geometry.

More generally, the $\Phi^{-1}\mathcal{O}_Y$-linearity says that our vector fields on $T_{X/Y}$ are "tangent to the fibers of Φ", as in Proposition 3; that is, $T_{X/Y}$ should really be thought of as giving the *fiberwise* tangent scheme of X over Y. As a consequence, we should expect that tangent schemes play well with pullback from the base. This turns out to be true:

Proposition 7. *Let $R \to S$ be a map of rings, $R \to R'$ another, and set $S' := S \otimes_R R'$. Then $\Omega_{S'/R'} \cong R' \otimes_R \Omega_{S/R}$.*

Consequently, if $X \to Y$ and $Y' \to Y$ are maps of schemes and we set $X' := X \times_Y Y'$, then $T_{X'/Y'} \cong T_{X/Y} \times_Y Y'$ as linear fiber spaces over X'.

That is, we have the following diagram of pullback squares:

$$\begin{array}{ccc} T_{X'/Y'} & \longrightarrow & T_{X/Y} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

In particular, taking $Y' \to Y$ to be the inclusion of a point $y \hookrightarrow Y$, we see that, indeed, the restriction of $T_{X/Y}$ over the fiber X_y of X over y is precisely the tangent scheme $T_{X_y/y}$ of that fiber as a $\kappa(y)$-scheme (where $\kappa(y)$, as always, is the residue field of Y at y).

3 First Computations

So far, our definitions have all been rather abstract — for practical purposes, of course, we would like to be able to actually construct the objects in question. Here we will begin to establish some basic results letting us do so, to be continued in next week's lecture.

Our first result tells us that the tangent scheme to affine space is as expected:

Proposition 8. *Let R be a ring and $n \geq 0$ an integer. Set $S := R[x_1, \ldots, x_n]$, considered as an R-algebra in the usual way. Then $\Omega_{S/R} \cong \bigoplus_{i=1}^n S dx_i$.*

As a result, if Y is a scheme and $X = \mathbb{A}^n_Y$, then $T_{X/Y} \cong \mathbb{A}^n_X = \mathbb{A}^n_{\mathbb{A}^n_Y} \cong \mathbb{A}^{2n}_Y$.

This is to say that the fiberwise tangent bundle to a trivial rank-n vector bundle is, indeed, a trivial rank-n vector bundle over the total space, and hence a trivial rank-$2n$ fiber bundle over the original base space. Taking the base space to be a point, we find our analogue to the observation that the tangent bundle of $\mathbb{R}^n$ is $\mathbb{R}^{2n}$ in the differential-geometric setting; for example, we have:

Example 2. *Let* $Y = \operatorname{Spec}\mathbb{C}$ *and* $X = \mathbb{A}^n_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x_1,\ldots,x_n]$. *Then* $T_{X/Y} \cong \mathbb{A}^n_X \cong \mathbb{A}^{2n}_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x_1,\ldots,x_n,dx_1,\ldots,dx_n]$.

In practice, since much of algebraic geometry is concerned with finite-type schemes over fields, this initial computation will be enough to allow us to compute most of the tangent schemes we care about in practice when applied together with a few standard results relating different sheaves of Kähler differentials. To start off, we observe that maps of schemes induce maps on tangent schemes:

Proposition/Definition 2. *Let* $R \to S \to T$ *be ring maps. Then the composition* $S \to T \xrightarrow{d} \Omega_{T/R}$ *is an* R*-linear derivation of* S *into* $\Omega_{T/R}$*, considered as an* S*-module using restriction of scalars along the map* $S \to T$*. By the universal property of Kähler differentials, this factors as the universal derivation followed by an* S*-module map* $\Omega_{S/R} \to \Omega_{T/R}$*; by standard properties of modules and tensor products, this corresponds to a* T*-module map* $T \otimes_S \Omega_{S/R} \to \Omega_{T/R}$*.*

Hence, if X *and* Y *are schemes over* Z *and* $\Phi : X \to Y$ *is a map of* Z*-schemes, there is a natural map* $\Phi^*\Omega_{Y/Z} \to \Omega_{X/Z}$ *of quasicoherent sheaves on* X*. The induced map*

$$D_Z\Phi : T_{X/Z} \to \Phi^*T_{Y/Z}$$

of linear fiber spaces over X *is called the* **(relative, or fiberwise) differential** *of* Φ *over* Z*. (When the base scheme* Z *is understood, we may denote this simply by* $D\Phi$*.)*

This is exactly analogous to the differential-geometric setting, where a smooth map induces a corresponding map from the tangent bundle of the source to the pullback of the tangent bundle of the target. (Also as in differential geometry, we can of course compose this with the natural fiber product projection $\Phi^*T_{Y/Z} \to T_{Y/Z}$ to get a map $T_{X/Z} \to T_{Y/Z}$ induced by Φ.) On the level of sheaves, the map $\Phi^*\Omega_{Y/Z} \to \Omega_{X/Z}$ corresponds precisely to the usual pullback of differential forms.

Example 3. *Let* $X = \mathbb{A}^2_{\mathbb{C}}$ *and* $Y = \mathbb{A}^1_{\mathbb{C}}$*, with* $\Phi : X \to Y$ *a coordinate projection, induced by the natural inclusion* $\mathbb{C}[x] \to \mathbb{C}[x,y]$ *of rings. Working over* $\operatorname{Spec}\mathbb{C}$*, we wish to compute* $D\Phi : T_X \to \Phi^*T_Y$*.*

This corresponds on the level of modules to the map $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \Omega_{\mathbb{C}[x]/\mathbb{C}} \to \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$ *induced by the ring maps* $\mathbb{C} \to \mathbb{C}[x] \to \mathbb{C}[x,y]$*. By Proposition 8,* $\Omega_{\mathbb{C}[x]/\mathbb{C}} \cong \mathbb{C}[x]dx$*, so* $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \Omega_{\mathbb{C}[x]/\mathbb{C}} \cong \mathbb{C}[x,y]dx$*, and* $\Omega_{\mathbb{C}[x,y]/\mathbb{C}} \cong \mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy$*. Fairly clearly, our map should then be the inclusion* $\mathbb{C}[x,y]dx \hookrightarrow \mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy$*, if there is any justice in the world, but let's follow through the steps just to be sure.*

The composition $\mathbb{C}[x] \to \mathbb{C}[x,y] \xrightarrow{d} \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$ *is determined by the assignment* $x \mapsto x \mapsto dx$ *and the fact that this map is a derivation. The* $\mathbb{C}[x]$*-module map* $\Omega_{\mathbb{C}[x]/\mathbb{C}} \to \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$*, which is to say* $\mathbb{C}[x]dx \to \mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy$*, is then given by taking the abstract differential* $dx \in \Omega_{\mathbb{C}[x]/\mathbb{C}}$ *to the differential* $dx \in \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$ *of* x *under our composed map, so it is determined by the assignment* $dx \mapsto dx$*, as we expect. Of course, the corresponding map* $\mathbb{C}[x,y]dx \to \mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy$ *is then the expected one as well.*

As such, the corresponding map of symmetric algebras is the natural inclusion $\mathbb{C}[x,y,dx] \to \mathbb{C}[x,y,dx,dy]$*, and so* $D\Phi$ *is the projection* $\mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ *onto the first three coordinates, where both these spaces are viewed as vector bundles over* $X = \mathbb{A}^2_{\mathbb{C}}$ *using the projection to the first two coordinates.*

As a more complicated example, we have:

Example 4. *Let $X = \mathbb{A}^2_{\mathbb{C}}$ and $Y = \mathbb{A}^1_{\mathbb{C}}$, with $\Phi : X \to Y$ a the map induced by the $\mathbb{C}$-algebra map $\mathbb{C}[t] \to \mathbb{C}[x, y]$ given by $t \mapsto xy$. Working over $\operatorname{Spec}\mathbb{C}$ as in the prior example, we again wish to compute $D\Phi : T_X \to \Phi^* T_Y$.*

Here $\Phi^* T_Y$ is the trivial rank-1 vector bundle over $X = \operatorname{Spec}\mathbb{C}[x, y]$, given by $\operatorname{Spec}\mathbb{C}[x, y, dt]$, and T_X is the trivial rank-2 vector bundle, given by $\operatorname{Spec}\mathbb{C}[x, y, dx, dy]$. To determine $D\Phi$, we must figure out where dt goes under the map $\mathbb{C}[x, y, dt] \to \mathbb{C}[x, y, dx, dy]$ induced by $\mathbb{C}[x, y] \otimes_{\mathbb{C}[t]} \Omega_{\mathbb{C}[t]/\mathbb{C}} \to \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$, which is to say $\mathbb{C}[x, y]dt \to \mathbb{C}[x, y]dx \oplus \mathbb{C}[x, y]dy$. As before, this map takes dt to the image of t under the composition $\mathbb{C}[t] \to \mathbb{C}[x, y] \xrightarrow{d} \Omega_{\mathbb{C}[x,y]/\mathbb{C}}$, which is precisely $d(xy) = ydx + xdy$.

In the classical setting, our Φ corresponds to the polynomial map $\mathbb{C}^2 \to \mathbb{C}$ given by $f(x, y) = xy$; we can then see that the induced pullback of differentials is as given by our formula above, and the map on tangent bundles, being given by the Jacobian matrix $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \end{bmatrix}$, also agrees.

As a first general computational result, we observe that the tangent spaces to fibers of a map are given precisely by the kernel of its differential, as in the case of a constant-rank map of smooth manifolds as considered in Proposition 3:

Proposition 9 (relative (co)tangent sequence)**.** *Let $R \to S \to T$ be maps of rings. Since $d : T \to \Omega_{T/S}$ is R-linear by virtue of the fact that it satisfies the stronger condition of S-linearity, we obtain a map of T-modules $\Omega_{T/R} \to \Omega_{T/S}$ from the universal property of $\Omega_{T/R}$. Then, recalling the map of Proposition/Definition 2, we have the following exact sequence of T-modules:*

$$T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0.$$

Hence, if X and Y are schemes over Z and $\Phi : X \to Y$ a map of Z-schemes, we have an exact sequence

$$\Phi^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of quasicoherent sheaves on X, and thus an exact sequence

$$0 \to T_{X/Y} \to T_{X/Z} \xrightarrow{D_Z\Phi} \Phi^* T_{Y/Z}$$

of linear fiber spaces over X.

That is, $T_{X/Y} = T_\Phi$ is naturally identified with $\ker D_Z\Phi$, no matter the base scheme Z.

Example 5. *In the setting of Example 3, we find by Proposition 9 that $\Omega_{\mathbb{C}[x,y]/\mathbb{C}[x]} \cong \frac{\mathbb{C}[x,y]dx \oplus \mathbb{C}[x,y]dy}{\mathbb{C}[x,y]dx} \cong \mathbb{C}[x, y]dy$. That is, the relative tangent scheme $T_{X/Y}$ is precisely the vertical bundle $\operatorname{Spec}\mathbb{C}[x, y, dy]$ of X over Y.*

Example 6. *In the setting of Example 4, Proposition 9 now tells us that*

$$\Omega_{\mathbb{C}[x,y]/\mathbb{C}[t]} \cong \frac{\mathbb{C}[x, y]dx \oplus \mathbb{C}[x, y]dy}{(ydx + xdy)}$$

and so $T_{X/Y} = \operatorname{Spec} \mathbb{C}[x, y, dx, dy]/(ydx + xdy)$. We can use this together with our base change result (Proposition 7) to compute tangent schemes to fibers of Φ.

Let p be the closed point in the affine line Y cut out by the equation $t = 1$, with inclusion $i : p \hookrightarrow Y$. Then $\Phi^{-1}(p) = \operatorname{Spec} \mathbb{C}[x, y]/(xy - 1)$ is the scheme corresponding to the graph in the xy-plane of the function $y = \frac{1}{x}$. By Proposition 7 and the fact that p has residue field $\mathbb{C}$, we find that $T_{\Phi^{-1}(p)} = T_{\Phi^{-1}(p)/p} = i^* T_{X/Y} = (\operatorname{Spec} \mathbb{C}[x, y, dx, dy]/(ydx+xdy) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t-1))$. This is the scheme $\operatorname{Spec} \mathbb{C}[x, y, dx, dy]/(xy - 1, ydx + xdy)$ over $\Phi^{-1}(p)$; since x is now invertible in the corresponding ring, with inverse given by y, we have

$$T_{\Phi^{-1}(p)} = \operatorname{Spec} \frac{\mathbb{C}[x, y, dx, dy]}{(xy - 1, dy + \frac{y}{x} dx)} = \operatorname{Spec} \frac{\mathbb{C}[x, y, dx, dy]}{(xy - 1, dy + \frac{1}{x^2} dx)}.$$

That is, the tangent line to the graph of $y = \frac{1}{x}$ at each point is given by $dy = -\frac{1}{x^2} dx$, which has exactly the slope we expect from our prior calculus knowledge.

Exercise 1. *In the setting of the prior example, let o be the origin in the affine line $Y = \mathbb{A}^1_{\mathbb{C}}$, cut out by the equation $t = 0$. Compute $T_{\Phi^{-1}(o)}$ — what are the dimensions of its fibers over the closed points of $\Phi^{-1}(o) = \operatorname{Spec} \mathbb{C}[x, y]/(xy)$?*

Exercise 2. *Verify that our formal definitions agree with the ad-hoc constructions of Example 1.*