The Geometry of Rings and Schemes
Lecture 7: Quasicoherent Sheaves I

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Last week, we discussed how to realize a module over a ring as a scheme over the ring's spectrum with "the fiberwise structure of a vector space" — that is, as a linear fiber space. We also showed that the original module could be retrieved from this construction; in the language of category theory, our results can be summarized as follows:

Proposition 1. *Let R be a ring. Then the functors $M \mapsto \operatorname{Spec} M$ and $V \mapsto L(V)$ give an anti-equivalence between the category of modules over R and a full subcategory of the category of linear fiber spaces over* $\operatorname{Spec} R$.

This is to say that for any module M we have a corresponding linear fiber space $\operatorname{Spec} M$, maps of modules induce maps of linear fiber spaces in the opposite direction, all maps between linear fiber spaces constructed in this way come from module maps, and the operation of taking linear forms retrieves our original modules and maps.

We now seek to extend this machinery to the setting of non-affine schemes. Of course, in this context, it is not clear what the appropriate analogue to a module should be — one approach, suggested by the above, would be to (anti-)identify "modules over S" for a scheme S with linear fiber spaces over S which satisfy some appropriate constraint (say, being affine-locally the spectrum of a module). This approach is feasible but nonstandard and so, to be conversant with the existing literature, we will return to it only after discussing the more widely-used notion of quasicoherent sheaves. To avoid getting bogged down in sheaf-theoretic technicalities already findable in other texts, our exposition will not explore these constructions in detail — rather, we will aim to give a quick overview of the most relevant objects and operations and highlight some expected and unexpected behaviors.

1 Sheaves of Modules

To define an object associated to a given scheme S which is "affine-locally a module", we return to the definition of schemes as spaces endowed with sheaves of "functions".

Definition 1. *Let $(X, \mathcal{O}_X)$ be a ringed space. A* **sheaf of $\mathcal{O}_X$-modules** *is a sheaf of abelian groups $\mathcal{F}$ on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open $U \subseteq X$ and the module actions commute with the restriction maps on $\mathcal{O}_X$ and $\mathcal{F}$ (that is, for open $V \subseteq U \subseteq X$, $f \in \mathcal{O}_X(U)$, and $\lambda \in \mathcal{F}(U)$, $\rho_{UV}(f\lambda) = \rho_{UV}(f)\rho_{UV}(\lambda)$).*

A **map of sheaves of $\mathcal{O}_X$-modules** *$\mathcal{F} \to \mathcal{G}$ is a map of sheaves of abelian groups on X such that the constituent map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a map of $\mathcal{O}_X(U)$-modules for each open $U \subseteq X$.*

Our result last week on the local determinacy of modules lets us turn them into sheaves:

Proposition/Definition 1. *Let R be a ring and M an R-module. Then there is a unique sheaf $\tilde{M}$ of $\mathcal{O}_{\operatorname{Spec} R}$-modules such that $\tilde{M}(\operatorname{Spec} R_f) := R_f \otimes_R M$ for all $f \in R$ and the restriction maps between these distinguished affine opens are the natural ones induced by the localizations. We call this the* **sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules corresponding to** *M.*

Hence we can make sense of the idea of a sheaf of $\mathcal{O}_X$-modules which is "locally a module":

Definition 2. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. $\mathcal{F}$ is said to be* **quasicoherent** *if, for every affine open $\operatorname{Spec} R \subseteq X$, there exists some R-module M such that $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$. (Equivalently: if this is true for every $\operatorname{Spec} R$ in some affine open cover of X.)*

If X is a locally Noetherian scheme, we say that $\mathcal{F}$ is **coherent** *if it is quasicoherent and the modules M mentioned above can moreover be taken to be finitely generated (and hence, by Noetherianity, finitely presented).*

(In the non-Noetherian setting, there is a notion of coherence distinct from the finite generation or finite presentation hypotheses, but we will ignore it, at least for the time being, to keep things simple.)

Example 1. *Let X be a scheme and $n \geq 0$ an integer. Then $\mathcal{O}_X^{\oplus n}$, the free sheaf of rank n on X, is quasicoherent (indeed, if X is locally Noetherian, coherent); on each affine open $\operatorname{Spec} R$, we have $\mathcal{O}_X^{\oplus n}|_{\operatorname{Spec} R} \cong \widetilde{R^{\oplus n}}$.*

More generally, we can carry out many of our typical module operations affine-locally:

Proposition/Definition 2. *Let X be a scheme and $\mathcal{F}$ and $\mathcal{G}$ be quasicoherent sheaves on X, with $\phi : \mathcal{F} \to \mathcal{G}$ a map of sheaves of $\mathcal{O}_X$-modules. Then there exist unique quasicoherent sheaves $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, $\ker \phi$, and $\operatorname{coker} \phi$ of $\mathcal{O}_X$-modules which are given affine-locally by taking the corresponding operations of modules in a consistent fashion. (That is, if $\operatorname{Spec} R \subseteq X$ is an affine open such that $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$, $\mathcal{G}|_{\operatorname{Spec} R} \cong \tilde{N}$, and $\phi|_{\operatorname{Spec} R}$ is induced by $\psi : M \to N$, these sheaves are given by $\widetilde{M \oplus N}$, $\widetilde{M \otimes_R N}$, $\widetilde{\ker \psi}$, and $\widetilde{\operatorname{coker} \psi}$ respectively, and the isomorphisms between different module representations induced by overlaps of affine patches respect these identifications.)*

These objects possess the expected universal properties corresponding to the ones which hold for the corresponding module operations. As a consequence of the kernel and cokernel definitions, we can see moreover that exactness of sequences of quasicoherent sheaves can be defined and checked affine-locally.

Remark 1. *For those more familiar with the theory of sheaves, these definitions are indeed the same as the ones in the general setting of sheaves of $\mathcal{O}_X$-modules; that is, we take the corresponding module operations over every open set and use an operation called "sheafification" to remove any failures of local determinacy which result from this. In particular,* *quasicoherent sheaves inherit the structure of an abelian category from the category of sheaves of $\mathcal{O}_X$-modules.*

The necessity of sheafification means that the things which are true affine-locally need not be true over every open set — for example, $(\operatorname{coker}(\mathcal{F} \to \mathcal{G}))(U) \not\cong \operatorname{coker}(\mathcal{F}(U) \to \mathcal{G}(U))$ *in general if U is a non-affine open set. Hence, although the main takeaway of our present discussion should be that most operations on quasicoherent sheaves do exactly what we expect on affine opens and so do not require us to worry too much about sheaf-theoretic subtleties when we work affine-locally, we can see that more caution is required as soon as we want to say anything about a non-affine open set.*

Standard facts about sheaves now allow us to retrieve a useful result about the exactness of sequences of modules:

Proposition 2. *Let R be a ring and $A \to B \to C$ maps of R-modules. Then this sequence is exact if and only if, for every prime $\mathfrak{p} \subset R$, the induced sequence $A_\mathfrak{p} \to B_\mathfrak{p} \to C_\mathfrak{p}$ is exact.*

This is in keeping with our general philosophy that local rings should reflect behavior in sufficiently small open neighborhoods around the corresponding points.

One might expect from Proposition/Definition 2 that *every* module operation will be affine-locally well-behaved for quasicoherent sheaves; however, this is not the case. For example, there is not in general a quasicoherent sheaf which affine-locally realizes the module of homomorphisms between the modules defining two quasicoherent sheaves. However, this does exist in nice cases:

Proposition/Definition 3. *Let X be a locally Noetherian scheme, $\mathcal{F}$ a coherent sheaf on X, and $\mathcal{G}$ a quasicoherent sheaf on X. Then there is a unique quasicoherent sheaf $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ on X such that, for affine opens* $\operatorname{Spec} R \subseteq X$ *with $\mathcal{F}|_{\operatorname{Spec} R} \cong \tilde{M}$ and $\mathcal{G}|_{\operatorname{Spec} R} \cong \tilde{N}$, we have $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_{\operatorname{Spec} R} \cong \tilde{H}$ for $H = \operatorname{Hom}_R(M, N)$ in a consistent fashion. (I.e., if we also have* $\operatorname{Spec} R' \subseteq X$ *with $\mathcal{F}|_{\operatorname{Spec} R'} \cong \widetilde{M'}$ and $\mathcal{G}|_{\operatorname{Spec} R'} \cong \widetilde{N'}$, $f \in R$, and $f' \in R'$ such that* $\operatorname{Spec} R_f \subseteq X$ *and* $\operatorname{Spec} R'_{f'} \subseteq X$ *agree, the isomorphism* $\operatorname{Hom}_{R_f}(M_f, N_f) \cong (\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))(\operatorname{Spec} R_f) \cong \operatorname{Hom}_{R'_{f'}}(M'_{f'}, N'_{f'})$ *is the one induced under Hom by the corresponding isomorphisms $R_f \cong R'_{f'}$, $M_f \cong \mathcal{F}(\operatorname{Spec} R_f) \cong M'_{f'}$, and $N_f \cong \mathcal{G}(\operatorname{Spec} R_f) \cong N'_{f'}$.)*

If $\mathcal{G}$ is coherent as well, then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent; hence, in particular, the duals of coherent sheaves on a locally Noetherian scheme are coherent. (On the other hand, duals of quasicoherent sheaves will be defined as sheaves of $\mathcal{O}_X$-modules but not necessarily as quasicoherent sheaves.)

As those familiar with sheaf theory may expect, this $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is indeed the usual sheaf of $\mathcal{O}_X$-module homomorphisms — that is, $(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))(U) := \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of maps of sheaves of $\mathcal{O}_U$-modules from the restriction of $\mathcal{F}$ to the restriction of $\mathcal{G}$.

We conclude by giving an explicit example of a sheaf of modules which is not quasicoherent:

Example 2. *Let $X = \operatorname{Spec} \mathbb{C}[x]_{(x)}$, so that the underlying topological space of X is the Sierpinski space (two points, one open and one closed). Then the sheaf $\mathcal{F}$ on X given by $\mathcal{F}(X) := 0$, $\mathcal{F}(\operatorname{Spec} \mathbb{C}(x)) := \mathbb{C}(x)$, and $\mathcal{F}(\emptyset) := 0$, with the natural restriction maps and module structures, is not quasicoherent.*

2 Moving Sheaves Along Maps

So far, we have explored various constructions dealing with quasicoherent sheaves on a given scheme; now we will start to explore how these objects interact with maps between schemes. As usual, we have a well-defined pullback map essentially coming from tensor product — note that this agrees with our prior definition of the "pullback of a module":

Proposition/Definition 4. *Let $\phi : X \to Y$ be a map of schemes and $\mathcal{G}$ a quasicoherent sheaf on Y. Then there is a unique quasicoherent sheaf $\phi^*\mathcal{G}$ on X, called the* **pullback** *of $\mathcal{G}$ along ϕ, which is affine-locally given by pullbacks of modules along ring maps in a consistent fashion. (That is, if we have affine opens $\operatorname{Spec} R \subseteq Y$ and $\operatorname{Spec} S \subseteq \phi^{-1}(\operatorname{Spec} R)$ with $\mathcal{G}|_{\operatorname{Spec} R} \cong \tilde{M}$, then $(\phi^*\mathcal{G})|_{\operatorname{Spec} S} \cong \widetilde{S \otimes_R M}$, and the isomorphisms between module representations induced by overlaps of affine patches respect these identifications.)*

If X and Y are locally Noetherian and $\mathcal{G}$ is coherent, then $\phi^*\mathcal{G}$ is as well.

Observe that, in the case where ϕ is an open inclusion, this pullback agrees with the usual restriction of sheaves.

For the sake of appearances, we give the actual construction of the pullback:

Remark 2. *The pullback sheaf can be defined explicitly using the sheaf-theoretic machinery by $\phi^*\mathcal{G} := \phi^{-1}\mathcal{G} \otimes_{\phi^{-1}\mathcal{O}_Y} \mathcal{O}_X$, where ϕ^{-1} denotes the topological pullback of sheaves given by $(\phi^{-1}\mathcal{F})(U) := \varinjlim_{\phi^{-1}(V) \supseteq U} \mathcal{F}(V)$ and we use the tensor product of sheaves of $\phi^{-1}\mathcal{O}_Y$-modules. In particular, our notion of pullback differs from the usual topological one in that it is not in general left exact; however, by the right-exactness of tensor products, it does remain right exact.*

If you are unfamiliar with or don't remember the details of the topological pullback of sheaves, don't worry too much about this remark — your intuition should mainly be founded on the idea of pullback as being affine-locally given by tensor products. The right-exactness result, however, is worth remembering, so we give it separately:

Proposition 3. *Let $\phi : X \to Y$ be a map of schemes and $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ a short exact sequence of quasicoherent sheaves on Y. Then $\phi^*\mathcal{A} \to \phi^*\mathcal{B} \to \phi^*\mathcal{C} \to 0$ is exact.*

As alluded to, this follows immediately from the fact that exactness of sequences of quasicoherent sheaves can be checked affine-locally and the right-exactness of tensor products.

Having defined the pullback and explored some of its properties, we are now interested in "going the other direction" — that is, in starting with a quasicoherent sheaf on the domain of a map of schemes and somehow inducing a corresponding quasicoherent sheaf on the codomain. We have already seen a way to do this, modulo the quasicoherence hypothesis, via the *pushforward sheaf* introduced in Lecture 2; it remains to ask whether this is quasicoherent. In general, the answer turns out to be no, but there is a nice class of maps for which this difficulty does not occur:

Definition 3. *Let $\phi : X \to Y$ be a map of schemes. We say that ϕ is* **quasicompact** *if every affine open subset of Y has quasicompact preimage under ϕ and* **quasiseparated**

if, for every affine open subset V of Y, $\phi^{-1}(V)$ has the property that finite intersections of quasicompact open subsets are quasicompact. For brevity, we say that ϕ is **qcqs** *if it is both quasicompact and quasiseparated.*

In the Noetherian setting, every map has these properties:

Proposition 4. *Let $\phi : X \to Y$ be a map of schemes and suppose that X is Noetherian. Then ϕ is qcqs.*

Hence, in practice, we do not often need to worry about verifying qcqs hypotheses.

As mentioned, qcqs maps preserve quasicoherence under pushforward:

Proposition 5. *Let $\phi : X \to Y$ a qcqs map of schemes and $\mathcal{F}$ a quasicoherent sheaf on X. Then $\phi_* \mathcal{F}$ is quasicoherent as well.*

As we might expect from the necessity of the qcqs hypothesis, the pushforward sheaf is not as cleanly describable in algebraic terms as the pullback. However, in the special case where $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$ for rings R and S, so that $\mathcal{F} \cong \tilde{M}$ for some S-module M, we can see that $\phi_* \mathcal{F}$ will also be $\tilde{M}$, where M is now considered as an R-module via the ring map $R \to S$ corresponding to ϕ. That is, in the affine case, pushforward corresponds to the so-called *restriction of scalars* from S to R.

More generally, we can see that the pushforward of a quasicoherent sheaf to an affine scheme is given by taking global sections:

Proposition 6. *Let $\phi : X \to Y$ a qcqs map of schemes and $\mathcal{F}$ a quasicoherent sheaf on X. Suppose that $Y = \operatorname{Spec} R$ is affine. Then $\phi_* \mathcal{F} \cong \widetilde{\mathcal{F}(X)}$, where $\mathcal{F}(X)$ is considered as an R-module using the "pullback of functions" map $\phi^\#$ as usual.*

This follows immediately from the definitions and the quasicoherence of the pushforward, but is nevertheless worth noting separately.

We conclude by relating the pullback and pushforward:

Proposition 7. *Let $\phi : X \to Y$ be a qcqs map of schemes, $\mathcal{F}$ a quasicoherent sheaf on X, and $\mathcal{G}$ a quasicoherent sheaf on Y. Then there is a natural identification $\operatorname{Hom}_{\mathcal{O}_X}(\phi^* \mathcal{G}, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, \phi_* \mathcal{F})$.*

(Technically, what we mean here by "natural" is that there is a natural isomorphism of bifunctors $\operatorname{Hom}_{\mathcal{O}_X}(\phi^* -, -) \cong \operatorname{Hom}_{\mathcal{O}_Y}(-, \phi_* -)$; this is to say that ϕ^* and ϕ_* are a pair of adjoint functors between the categories of quasicoherent sheaves on X and Y.)

3 Sheaves of Algebras and Relative Spectra

The notion of a quasicoherent sheaf of modules provides a convenient intermediate step between treating modules as fully algebraic objects and treating them as fully geometric ones — it allows us to pass to the geometric perspective on the ring in question (and hence generalize from rings to schemes) while still treating the module itself in some sense as an algebraic object.

We will now give the equivalent construction for algebras, rather than modules. In this case, the way to create a fully geometric realization over an affine base is more obvious — if R is a ring and A an R-algebra, we can consider the map $R \to A$ defining the algebra structure and take the corresponding map $\operatorname{Spec} A \to \operatorname{Spec} R$ of spectra. To create a "partially geometric" perspective analogous to quasicoherent sheaves of modules, we define:

Definition 4. *Let $(X, \mathcal{O}_X)$ be a ringed space. A* **sheaf of $\mathcal{O}_X$-algebras** *is a sheaf of rings $\mathcal{A}$ on X together with a map $\mathcal{O}_X \to \mathcal{A}$ of sheaves of rings on X. A* **map of sheaves of $\mathcal{O}_X$-algebras** *$\mathcal{A} \to \mathcal{B}$ is a map of sheaves of rings on X such that the composition $\mathcal{O}_X \to \mathcal{A} \to \mathcal{B}$ agrees with the map $\mathcal{O}_X \to \mathcal{B}$.*

If $(X, \mathcal{O}_X)$ is moreover a scheme, then we say that $\mathcal{A}$ is **quasicoherent** *(or, more rarely,* **coherent***) if it is thus as a sheaf of $\mathcal{O}_X$-modules.*

It follows from the definition that a quasicoherent sheaf of algebras is given over each affine open $\operatorname{Spec} R$ by the sheaf $\tilde{A}$ (now endowed with the additional structure of a sheaf of algebras in the obvious way) for some R-algebra A. We can then create a geometric realization of a given quasicoherent sheaf of algebras by patching together the corresponding maps of ring spectra:

Definition 5. *Let X be a scheme and $\mathcal{A}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras. Then the* **relative spectrum** *of $\mathcal{A}$ is defined to be the unique scheme $\operatorname{Spec} \mathcal{A}$ over X such that $\operatorname{Spec} \mathcal{A} \times_X \operatorname{Spec} R \cong \operatorname{Spec}(\mathcal{A}(\operatorname{Spec} R))$ as $\operatorname{Spec} R$-schemes for every affine open $\operatorname{Spec} R \subseteq X$ and these identifications are compatible with $\mathcal{A}$ (that is, for $\operatorname{Spec} S \subseteq \operatorname{Spec} R \subseteq X$, the composition $\operatorname{Spec}(\mathcal{A}(\operatorname{Spec} S)) \cong \operatorname{Spec} \mathcal{A} \times_X \operatorname{Spec} S \hookrightarrow \operatorname{Spec} \mathcal{A} \times_X \operatorname{Spec} R \cong \operatorname{Spec}(\mathcal{A}(\operatorname{Spec} R))$ is precisely the map induced by the restriction $\mathcal{A}(\operatorname{Spec} R) \to \mathcal{A}(\operatorname{Spec} S)$).*

To understand this construction, it may be useful to note that not every scheme over X can arise in this way — in particular, we have the following characterization:

Proposition/Definition 5. *A map $\phi : Y \to X$ of schemes is called* **affine** *if either of the following equivalent conditions holds:*

1. *ϕ is the structure map $\operatorname{Spec} \mathcal{A} \to X$ for some quasicoherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$.*

2. *For each open affine $\operatorname{Spec} R \subseteq X$, the preimage $\phi^{-1}(\operatorname{Spec} R)$ is an affine open subscheme of Y.*

That is, an affine map is one which is given by algebras affine-locally on the target. We can retrieve the sheaf of algebras corresponding to a given affine map by noting the following:

Proposition 8. *Let $\phi : Y \to X$ be an affine map of schemes. Then ϕ is qcqs, so that $\phi_* \mathcal{O}_Y$ is a quasicoherent sheaf of $\mathcal{O}_X$-algebras, and ϕ can be naturally identified with the structure map $\operatorname{Spec} \phi_* \mathcal{O}_Y \to X$.*

Being able to pass between affine maps and the corresponding sheaves will be useful in a number of situations going forward, including the construction of the linear fiber space associated to a quasicoherent sheaf. For now, we will apply this machinery to get a fresh look at closed inclusions. In Lecture 3, we defined a closed inclusion of schemes to be a map given affine-locally on the target by the quotient by some ideal. Now, using quasicoherent sheaves, we can patch these ideals together into a single object:

Definition 6. *Let X be a scheme. A* **quasicoherent sheaf of ideals** *(or, more simply,* **ideal sheaf***) on X is a quasicoherent sheaf $\mathcal{I}$ on X together with an inclusion $\mathcal{I} \hookrightarrow \mathcal{O}_X$ of sheaves of $\mathcal{O}_X$-modules.*

That is, just as an ideal of a ring R is simply an R-submodule of R, we take an "ideal" of the structure sheaf to be a sheaf of submodules. Note that, since the exactness of $0 \to \mathcal{I} \to \mathcal{O}_X$ can be checked affine-locally, an ideal sheaf $\mathcal{I}$ will be given on each affine open $\operatorname{Spec} R$ by $\tilde{I}$ for some ideal I of R. Hence ideal sheaves are a generalization of ideals from the affine case to arbitrary schemes. The relative spectrum construction lets us define the closed subscheme which is the "vanishing" of a given ideal sheaf:

Definition 7. *Let X be a scheme and $\mathcal{I}$ an ideal sheaf on X. Then the* **closed subscheme cut out by** *$\mathcal{I}$, sometimes called the* **vanishing (locus)** *of $\mathcal{I}$ and variously denoted by $V(\mathcal{I})$ or $Z(\mathcal{I})$, is defined to be the scheme over X given by $\operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$.*

It is not difficult to check that this is a closed inclusion and that every closed inclusion arises this way:

Proposition 9. *Let X be a scheme. Then every closed inclusion into X is an affine map, and indeed there is a bijective correspondence between ideal sheaves on X and closed subschemes of X given by the maps $\mathcal{I} \mapsto V(\mathcal{I})$ and $(i : Y \hookrightarrow X) \mapsto \ker(\mathcal{O}_X \to i_*\mathcal{O}_Y)$.*

Describing our closed subschemes using ideal sheaves allows us to perform algebraic manipulations more easily. For example, we can now easily define a notion of the union of two closed subschemes — it has of course always been possible to see what this should be on the level of subsets, but now we can get the scheme structure as well:

Definition 8. *Let X be a scheme and Y_1 and Y_2 be closed subschemes of X. Then the* **union** *of Y_1 and Y_2 in X is the closed subscheme $Y_1 \cup Y_2$ of X cut out by the ideal sheaf $\mathcal{I}_1 \cap \mathcal{I}_2$ on X, where $\mathcal{I}_1$ is the ideal sheaf cutting out Y_1 and $\mathcal{I}_2$ the one cutting out Y_2.*

Affine-locally, as we might expect, this is given by the intersection of ideals.

We conclude by applying this machinery to a non-affine example from Lecture 2:

Example 3. *Let $\mathbb{P}^1_{\mathbb{C}}$ be the projective line over $\mathbb{C}$, given by gluing $\operatorname{Spec} \mathbb{C}[x]$ to $\operatorname{Spec} \mathbb{C}[y]$ along the identification $\operatorname{Spec} \mathbb{C}[x]_x \cong \operatorname{Spec} \mathbb{C}[y]_y$ induced by the map of $\mathbb{C}$-algebras taking y to x^{-1}.*

Recalling that closed points of $\mathbb{A}^1_{\mathbb{C}}$ correspond to values in $\mathbb{C}$, we can see that the inclusion of the point p given by $x = 2$ into $\operatorname{Spec} \mathbb{C}[x]$ can be composed with the inclusion $\operatorname{Spec} \mathbb{C}[x] \hookrightarrow \mathbb{P}^1_{\mathbb{C}}$ to give a closed point of $\mathbb{P}^1_{\mathbb{C}}$. To find the corresponding ideal sheaf $\mathcal{I}$, observe that p is cut out in $\operatorname{Spec} \mathbb{C}[x]$ by the ideal $(x - 2)$; hence we have $\mathcal{I}|_{\operatorname{Spec} \mathbb{C}[x]} \cong (x - 2)$.

On the other hand, since $p \in \operatorname{Spec} \mathbb{C}[x]_x$, we can see that it is also a closed point of $\operatorname{Spec} \mathbb{C}[y]_y$ and hence of $\operatorname{Spec} \mathbb{C}[y]$, given by the coordinate $y = \frac{1}{2}$. As such, $\mathcal{I}|_{\operatorname{Spec} \mathbb{C}[y]} \cong (y - \frac{1}{2})$. These restrictions patch together along the gluing used to define $\mathcal{O}_X$ to give our ideal sheaf $\mathcal{I}$ on $\mathbb{P}^1_{\mathbb{C}}$ in full.

For example, we can now compute the module $\mathcal{I}(\mathbb{P}^1_{\mathbb{C}})$ of global sections. These will be given by pairs of elements (f, g) such that $f \in (x - 2) \subset \mathbb{C}[x]$, $g \in (y - \frac{1}{2}) \subset \mathbb{C}[y]$, and *the image of f in $\mathbb{C}[x]_x$ is identified with the image of g in $\mathbb{C}[y]_y$ under our isomorphism. Writing $f(x) = c(x-2)(x-a_1)\cdots(x-a_r)$ for $c, a_1, \ldots, a_r \in \mathbb{C}$, we can then see that $g(y) = c(y^{-1}-2)(y^{-1}-a_1)\cdots(y^{-1}-a_r) = \frac{c}{y^{r+1}}(2y-1)(a_1y-1)\cdots(a_ry-1)$. However, since $g \in \mathbb{C}[y]$, not just $\mathbb{C}[y]_y$, and $r+1 \geq 1$, it then follows that we must have $c = 0$ to avoid a nontrivial denominator. As such, we can see that in fact $\mathcal{I}(\mathbb{P}^1_{\mathbb{C}}) = 0$.*

Exercise 1. *Let $\mathcal{J}$ be the ideal sheaf cutting out the closed point q given by $y = 0$ in $\mathbb{P}^1_{\mathbb{C}}$ (under the description of $\mathbb{P}^1_{\mathbb{C}}$ by affine patches used in the previous example). Compute $\mathcal{J}(\operatorname{Spec}\mathbb{C}[x])$, $\mathcal{J}(\operatorname{Spec}\mathbb{C}[y])$, and $\mathcal{J}(\mathbb{P}^1_{\mathbb{C}})$.*

Compute each of the corresponding modules of sections for the ideal sheaf cutting out the union $p \cup q$, where p is as in the previous example.