I study the behavior of multivariate holomorphic functions near critical points, as described by the Milnor fibration — in particular, my work investigates the relationship between local changes in the topology of a function's level sets and the invariants of the germ of its critical locus at a given point, providing a promising new connection between the study of the local topology of holomorphic functions and algebraic geometry.

1 Definition of the Milnor Fibration

Let $U \subseteq \mathbb{C}^{n+1}$ be an open neighborhood of the origin, and let $f : U \to \mathbb{C}$ be a nonconstant holomorphic function such that $f(0) = 0$. Suppose that the Jacobian $\left[\frac{\partial f}{\partial x_0} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right]$ of f has full rank at the origin — put another way, 0 is not a critical point of f. Then it is well-known that we can choose local coordinates for U and $\mathbb{C}$ around the origins such that the restriction of f to the corresponding chart is simply the projection onto one coordinate. That is, near any non-critical point, every holomorphic function looks exactly the same — like the projection $B_\varepsilon \times D_\delta \to D_\delta$ for B_ε an open ball of radius $\varepsilon > 0$ in $\mathbb{C}^n$ and D_δ an open disk of radius $\delta > 0$ in $\mathbb{C}$. In particular, the local fiber (i.e., level set) of f over any point of D_δ, including the origin, is simply B_ε.

Now suppose, on the other hand, that the Jacobian has rank zero at the origin, so that 0 is a critical point of f. This is exactly equivalent to saying that 0 is a **singularity** of the fiber $f^{-1}(0)$ — that is, a point at which $f^{-1}(0) \hookrightarrow U$ is not locally the inclusion of a smooth[1] submanifold. It can be shown that, if we make U sufficiently small, $0 \in \mathbb{C}$ will be an isolated critical value of f; that is, the fibers of f over points of $\mathbb{C}$ other than 0 will all be nonsingular. Thus we cannot hope for f to be the projection of a product locally at the origin in these circumstances — the fiber over $0 \in \mathbb{C}$ will necessarily be exceptional.

If we throw away this singular level set, however, we obtain the following result, proven by Milnor for polynomials in [26] and expanded to holomorphic functions by Lê in [16]:

Theorem 1 ([26, 16])**.** *Let f be as described above. Then, for any $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ so small that the restriction*

$$f : B_\varepsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*$$

is a smooth fiber bundle, for $B_\varepsilon \subseteq \mathbb{C}^{n+1}$ the open ball of radius ε at the origin and D_δ^* the punctured open disk of radius δ at the origin. The fiber diffeomorphism type of this fiber bundle (for small enough ε) is independent of the choices made.

That is, if we zoom in far enough on the origin, f will be the projection of a product *locally over D_δ^**, with the shape of the discarded singular fiber somehow reflecting both the shapes of the nonsingular level sets and the way they are twisted around as one moves in a loop around the origin (called the **monodromy** — taking $f(x) = x^2$ as an example, think of the way the two square roots of a small nonzero number are interchanged as it circles the origin once).

This restriction of f is called its **Milnor fibration** at the origin, and encapsulates its local behavior there. Hence, if we wish to understand how holomorphic functions behave in general, understanding the Milnor fibration is a crucial prerequisite. The fiber of the Milnor fibration is called the **Milnor fiber**; because $f^{-1}(0) \cap B_\varepsilon$ will be contractible for small enough ε and $f|_{B_\varepsilon}$ gives a family degenerating to it, the topology of the Milnor fiber is sometimes called the **vanishing topology** of f at the origin.

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[1]To be precise, here I mean smooth as a scheme or complex-analytic space — but such technicalities can be ignored as long as, e.g., f is a square-free polynomial.

2 Background

Despite its importance, there is still much about the Milnor fibration which is not yet understood, and a great deal of work aimed at elucidating its properties has already been done in the decades since Milnor's original work [26]. Here I briefly survey the development of some of the results most relevant to my research; for more comprehensive accounts, see Chapter 3 of [8], Chapter 9 of [23], the 50-year retrospective survey [32], and the handbook entry [17]. As described in [24], the **nearby and vanishing cycles functors** relate the topology of the Milnor fibration to a variety of other areas of pure math; in addition, recent work by Bobadilla and Pełka in [6] relates longstanding questions in the study of the Milnor fibration to the theory of **Floer homology**. On the applied side, information about the Milnor fibration has played a role in **nearest-point problems** and other forms of **optimization** ([21, 20]), including the recent proof of the **multiview conjecture** in the study of **computer vision** ([22]).

2.1 Some Early Results

Even before Milnor, the relationship between a function's smooth level sets and its critical points was an area of active interest, going back to the mid-19th century ([7]). In the case of real-analytic functions, an understanding of the vanishing topology at the simplest possible nontrivial critical points is already quite powerful, and allows us to obtain homotopy cell decompositions of compact smooth manifolds by considering the indices of critical points of appropriately-chosen functions — this is the basis of Morse theory, for which see, e.g., [25]. The complex analogue of this area of study, Picard-Lefschetz theory (e.g., Chapter 1 of [1]), likewise uses knowledge of the vanishing topology at **nondegenerate critical points** (up to coordinate change, $f(x_0, \ldots, x_n) = x_0^2 + \ldots + x_n^2$ at the origin) to derive global consequences for complex manifolds, the local picture being well-understood in this case.

However, polynomials and holomorphic functions in general can have degenerate and even non-isolated critical points, and, if we are interested in a particular function itself rather than the manifold it is defined on, it may not be possible to circumvent this issue. In addition to proving that his fibration exists for all such singularities, Milnor gave an elegant description of the vanishing topology of a function at *any* isolated critical point, degenerate or not:

Theorem 2 ([26])**.** *Let U be a neighborhood of the origin in $\mathbb{C}^{n+1}$ and $f : U \to \mathbb{C}$ a nonconstant holomorphic function with $f(0) = 0$. Suppose that $0 \in \mathbb{C}^{n+1}$ is an isolated point of the critical locus of f. Then the Milnor fiber of f at the origin has the homotopy type of a bouquet of μ_f n-spheres, where*

$$\mu_f := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^{n+1},0}}{\left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n} \right)},$$

for $\mathcal{O}_{\mathbb{C}^{n+1},0} \cong \mathbb{C}\{x_0, \ldots, x_n\}$ the ring of convergent power series at the origin in $\mathbb{C}^{n+1}$, is the **Milnor number** *of f at the origin.*

Hence in this case the reduced homology of the local smooth fiber of f is concentrated in degree n and can be computed straightforwardly as $\mathbb{Z}^{\oplus \mu_f}$. Several years after the publication of Milnor's book, Kato and Matsumoto generalized the first half of this statement to arbitrary singularities:

Theorem 3 ([15])**.** *Let U be a neighborhood of the origin in $\mathbb{C}^{n+1}$ and $f : U \to \mathbb{C}$ a nonconstant holomorphic function with $f(0) = 0$. Let s be the local complex dimension of the critical locus of f at the origin. Then the Milnor fiber of f at the origin is $(n - s - 1)$-connected.*

In particular, since the Milnor fiber of f is a Stein manifold, this implies that its reduced homology can be nonzero only in degrees $[n-s, n]$. (This fact can also be understood in terms of the more modern machinery of perverse sheaves and the **vanishing cycles functor** — see [24] or [19].)

2.2 Variation in Families

However, in the decades following the proof of Theorem 3, a general formula for Milnor fiber homology like the one given by Theorem 2 has remained elusive, even in the case where the critical locus is only 1-dimensional ([3, 28, 33, 34, 35, 36, 37]). A variety of techniques have been developed in hopes of tackling this issue. In particular, there has been substantial interest in ways to control the change in smooth fibers as f varies in a family of holomorphic functions — such results allow us to compute a function's Milnor fiber at the level of homology by, for example, splitting the critical locus into simpler pieces (which gives a corresponding direct sum decomposition in reduced homology per [35]) or transforming the function into a better-understood one with the same Milnor fibration.

In the case of an isolated critical point, such deformations will always preserve the smooth fibers in an appropriate sense, and so one can recover the homological consequences of Theorem 2 by perturbing f to get a function with only nondegenerate critical points and applying the methods of Picard-Lefschetz theory — this technique is called **Morsification** (see, e.g., Chapter 2 of [1]). However, in the non-isolated case, this consistency of smooth fibers does not hold in general, and so additional constraints on the allowable perturbations are needed.

One approach is to restrict to deformations of f through some ideal in $\mathcal{O}_{\mathbb{C}^{n+1},0}$ with respect to which it has **finite extended codimension** ([2]) — essentially, this condition means that such a perturbation will be trivial everywhere except the origin in $\mathbb{C}^{n+1}$, and so the main use of this technique is to simplify the singularity by splitting off nondegenerate critical points, as in the isolated case, while holding the positive-dimensional parts of the critical locus fixed. This area of research arises from the ideas of Siersma in [34]; these were applied to gain information about the Milnor fibration in a number of low-dimensional special cases by various authors in [35, 33, 14, 31, 28, 38, 27, 5, 37]. The theoretical underpinnings of this method were developed by Pellikaan in [30, 29, 31, 28] and taken up much later by Gaffney in [10, 11] and Bobadilla in [4, 2], culminating in the proofs in [2] of the general allowability of such deformations and the possibility of Morsification in this context.

The other main approach is to consider only deformations such that the **Lê numbers** ([18]) at the origin are constant in the family. These quantities, defined by Massey using the machinery of relative polar varieties due to Lê and Teissier, serve in some sense as higher-dimensional analogues to the Milnor number, and track the criticalities introduced by iteratively slicing by increasing numbers of coordinate hyperplanes. Their constancy in a family of holomorphic functions implies the homological consistency of the Milnor fibers in that family, and can also be used to show stronger types of consistency depending on the codimension of the critical locus.

Despite their utility, however, these viewpoints have limitations which make it unlikely that they will provide a sufficient framework to fully describe the Milnor fiber of an arbitrary holomorphic function or polynomial. Approaches based on finite determinacy, as mentioned, are capable of splitting off isolated singularities (corresponding to cycles in the top-degree reduced homology), but cannot manipulate higher-dimensional features of the critical locus. Lê numbers, on the other hand, are designed to be useful for functions with arbitrary-dimensional critical loci, but their constancy is a very strong condition which imposes a great deal of rigidity on the deformation and precludes any splitting behavior of the kind we see in Morsification-type approaches.

3   Present and Future Work

As described above, there is much about the behavior of holomorphic functions near non-isolated critical points which remains mysterious, even at the level of the homology of the Milnor fiber. However, by reinterpreting the known results in the isolated case, we arrive at a new perspective which offers improvements on the existing machinery, and hope of a general solution, by linking the study of the Milnor fibration to concepts from algebraic and complex-analytic geometry.

3.1   Present

Consider a holomorphic function f defined on a neighborhood of the origin in $\mathbb{C}^{n+1}$, with an isolated critical point at the origin. Then Theorem 2 tells us that, although the critical locus of f at 0 consists of only a single point by definition, there is nevertheless a sense in which it determines the Milnor fiber. That is, we can consider the critical locus not just as a set but rather as a **scheme** or **complex-analytic space** with the non-reduced structure given by regarding it as the vanishing of the **Jacobian ideal** $J_f := \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n} \right)$ — in algebraic or complex-analytic geometry this is visualized as a "point with infinitesimal tangent fuzz".

From this perspective, the Milnor number is simply the **length** of this scheme-theoretic critical locus at the origin, and hence Theorem 2 tells us that the structure of the critical point fully determines the Milnor fiber up to homotopy in this case. It is natural to seek to apply this viewpoint to the non-isolated case, especially in light of Theorem 3, and hence we arrive at the following:

Idea. *If f is an arbitrary holomorphic function defined at the origin, its Milnor fibration there should be determined by the scheme structure of its critical locus's embedding there — that is, by the algebro-geometric invariants of the map $\mathbb{C}\{x_0, \ldots, x_n\} \to \mathbb{C}\{x_0, \ldots, x_n\}/J_f$ of convergent power series rings.*

Ideally, we would like to obtain an effective algorithm which computes the homology groups of the Milnor fibration, and their monodromy automorphism, directly from the Jacobian ideal. In [13], I proved that this idea holds in a relative sense by showing that, if a deformation of a holomorphic function germ is also a deformation of the embedding of its critical locus into the ambient space in an appropriate algebraic sense, the Milnor fibration of the function being deformed is preserved by the deformation:

Theorem 4 ([13]). *Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function for integers $n, u \geq 0$ so that the restriction f of F to $(\mathbb{C}^{n+1} \times 0, 0 \times 0)$ is nonzero. Also let $\pi : (\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0) \to (\mathbb{C}^u, 0)$ be the projection, and $J_{F \times \pi}$ the ideal $\left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \right)$ of maximal minors of the Jacobian matrix of $F \times \pi$ in the local ring $\mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}$ of $\mathbb{C}^{n+1} \times \mathbb{C}^u$ at 0×0, so that its vanishing $(C_{F \times \pi}, 0 \times 0) := V(J_{F \times \pi})$ is the germ of an analytic subspace of $\mathbb{C}^{n+1} \times \mathbb{C}^u$ at 0×0. Suppose that the **normal cone** of $(C_{F \times \pi}, 0 \times 0)$ in $(\mathbb{C}^{n+1} \times \mathbb{C}^u, 0 \times 0)$ is flat over $(\mathbb{C}^u, 0)$ — that is, that the **associated graded** $\mathrm{gr}_{J_{F \times \pi}} \mathbb{C}\{x_0, \ldots, x_n, t_1, \ldots, t_u\}$ is flat over $\mathbb{C}\{t_1, \ldots, t_u\}$.*

Then, if we fix a small enough $\varepsilon > 0$, we can choose $\delta > 0$ and $\gamma > 0$ small enough relative to ε so that the following hold: The representatives $F : B_\varepsilon \times B_\gamma \to \mathbb{C}$ and $C_{F \times \pi} \subset B_\varepsilon \times B_\gamma$ exist and, if we let $\Delta := (F \times \pi)(C_{F \times \pi}) \subset \mathbb{C}$ be the discriminant, the restriction

$$F \times \pi : (B_\varepsilon \times B_\gamma) \cap (F \times \pi)^{-1}((D_\delta \times B_\gamma) \setminus \bar{\Delta}) \to (D_\delta \times B_\gamma) \setminus \bar{\Delta}$$

defines a diffeomorphically locally trivial fibration, where $B_\varepsilon \subset \mathbb{C}^{n+1}$, $D_\delta \subset \mathbb{C}$, and $B_\gamma \subset \mathbb{C}^u$ are the open balls of the specified radii.

4

In addition to providing evidence for the validity of the idea discussed above, this result gives a substantial improvement on existing techniques in situations where it applies. In particular, every deformation of a function through an ideal with respect to which it has finite extended codimension, as discussed in Section 2.2, satisfies the hypotheses of Theorem 4 (or, to be precise, of the more general technical version of the theorem in [13]), so this is a generalization of Bobadilla's result in [2] that all such deformations are allowable as it applies to these functions.

As an example application, we consider the case of families of homogeneous polynomials. A homogeneous polynomial admits a **global** or **affine Milnor fibration** given by its restriction to the complement of its vanishing locus, and this can be easily shown to be fiber diffeomorphic to the usual Milnor fibration at the origin. Thus, if $f(x_0, \ldots, x_n)$ is a homogeneous polynomial, its Milnor fiber is simply the set $\{f = 1\}$, and the monodromy is the transformation of this fiber induced by circling the origin in $\mathbb{C}^*$. Theorem 4 gives us:

Corollary 5 ([13]). *Let $H_{n,d} \cong \mathbb{P}^{\binom{n+d}{n}-1}$ be the space of degree-d hypersurfaces in $\mathbb{P}^n$, and let $\Sigma_{n,d}$ be the closed subscheme of $\mathbb{P}^n \times H_{n,d}$ such that the fiber of the natural projection $\Sigma_{n,d} \to H_{n,d}$ over any point is the singular locus of the corresponding hypersurface. Denote by $C\Sigma_{n,d}$ the normal cone to $\Sigma_{n,d}$ in $\mathbb{P}^n \times H_{n,d}$. Then, by iteratively applying generic flatness results to the map $C\Sigma_{n,d} \to H_{n,d}$, we obtain a partition of $H_{n,d}$ into finitely many locally closed algebraic subsets such that the global smooth fibers and monodromies of the polynomials defining the hypersurfaces corresponding to points will be constant along each subset.*

3.2   Future

Despite this progress, much remains to be done. As mentioned, I ultimately hope to obtain effective means of computing the Milnor fibration's homology from the Jacobian ideal — at present, the main challenge in this regard seems to lie in algebraically distinguishing between families in which Theorem 4 deforms the Milnor fibration to that of the deformed germ itself and those in which the deformed smooth fiber is only semilocal, as with Morsifications and other deformations splitting the critical locus.

In the case of a deformation of a holomorphic function by translations of the domain, this amounts to obtaining an algebraic understanding of the minimal **Whitney stratification** of the critical locus (cf. [9, 12]). The proof of Theorem 4 is essentially by borrowing from the theory of $\mathcal{D}$-**modules** to obtain a stratification satisfying the **Thom condition**, a relative version of Whitney's condition (a), using the natural **symplectic structure** of the cotangent bundle; hence there is reason to suspect that more refined techniques along these lines will let us compute a Whitney stratification from the Jacobian ideal.

Once this is done, we can consider such a deformation by translations and use it to relate the Milnor fiber at a point with the Milnor fibers along the adjacent strata, with the hope of providing an algebraically-computable geometric realization and refinement of the **perverse-sheaf-theoretic** results of [19]. Such results would then allow us to inductively determine the desired formula for the homology of the Milnor fibration from the invariants of the Jacobian ideal, starting from the known case of an isolated singularity.

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