The Geometry of Rings and Schemes
Lecture 6: The Geometry of Modules

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October 15, 2024

At this point we will shift gears a little and attempt to gain a geometric understanding of a new kind of algebraic object: modules over a given ring. In the setting of algebraic geometry, these are typically dealt with using the machinery of *(quasi)coherent sheaves*, which we will discuss next week. However, as we will see, this approach to the subject is in some sense not fully geometric — ideally, we would like to be able to realize each module itself as a "space" of some kind, which the sheaf-theoretic machinery on its own does not do.

The constructions which allow us to accomplish this goal have been known to the algebro-geometric community as a whole more or less since the beginning of scheme theory — in Grothendieck's *Éléments de Géométrie Algébrique*, the central object goes by the name "fibré vectoriel". However, for various reasons — many of them good — these objects are not typically treated as an essential component in introductions to the subject, and hence are not widely used in discussing and working with quasicoherent sheaves — to the best of my knowledge, there isn't even a particularly well-accepted way to refer to them in English. (The term we will use, "linear fiber space" follows the convention of Gerd Fischer's *Complex Analytic Geometry*, which discusses the corresponding setup in the complex-analytic context.)

As I say, there *are* good reasons to leave these objects alone. The language we will be compelled to frame them in is unfortunately somewhat categorical, and in particular it makes heavy use of fiber products, analogues of a topological notion which is itself not typically completely familiar or intuitive to the first-time student of scheme theory. However, my sense is that once one manages to ferret out the picture lurking behind the formalisms, the payoff is worth it, and, personally speaking, I do not feel that I really understood coherent sheaves in a satisfactory way until I learned to conceptualize them in these terms.

1 Motivation

Let R be a ring and M an R-module. We have already seen that we can interpret R geometrically by identifying it (up to a reversal in the directions of maps) with the corresponding affine scheme $\operatorname{Spec} R$. We now seek to do the same thing with M — that is, we would like to produce some kind of geometric space that captures the information algebraically encoded in M.

We will begin by considering how M interacts with the geometry of $\operatorname{Spec} R$:

Definition 1. *Let $\phi : R \to S$ be a map of rings and M an R-module. We define the* **pullback** *of M along ϕ (or, speaking more strictly, along the induced map of spectra) to be the S-module $S \otimes_R M$.*

In particular, if ϕ corresponds to a map that we think of as an "inclusion" of some kind — distinguished affine open, closed, point, etc. — we call $S \otimes_R M$ the **restriction** *of M to S (strictly speaking, to* $\operatorname{Spec} S$*).*

That is, if we want to realize M as a space of some kind, we should think of it as living "over $\operatorname{Spec} R$" and, as in the case of rings, we can define pullbacks (and restrictions in particular) by way of the tensor product.

The basic constraint we should expect modules to satisfy if we are to regard them as having some geometry compatible with that of $\operatorname{Spec} R$ is the same "local determinacy" condition we've seen in the context of ring maps, sheaves, and so forth:

Proposition 1. *Let R be a ring, M and N R-modules, and $\{f_\alpha\}_{\alpha \in A}$ a collection of elements of R such that $(f_\alpha \mid \alpha \in A)$ is the unit ideal (i.e., such that the subschemes* $\operatorname{Spec} R_{f_\alpha}$ *give an open cover of* $\operatorname{Spec} R$*). For convenience, write $M_\alpha = R_{f_\alpha} \otimes_R M$ and $N_\alpha = R_{f_\alpha} \otimes_R N$ for each $\alpha \in A$, and $M_{\alpha\beta} = R_{f_\alpha f_\beta} \otimes_R M$ and $N_{\alpha\beta} = R_{f_\alpha f_\beta} \otimes_R N$ for each $\alpha, \beta \in A$.*

Suppose that, for each $\alpha \in A$, we are given an R_{f_α}-module map $\phi_\alpha : M_\alpha \to N_\alpha$ and that, for all $\alpha, \beta \in A$, the maps $M_{\alpha\beta} \to N_{\alpha\beta}$ induced by ϕ_α and ϕ_β are the same. Then there is a unique R-module map $\phi : M \to N$ such that the induced map $M_\alpha \to N_\alpha$ is ϕ_α for each $\alpha \in A$.

That is, maps of R-modules are locally determined on $\operatorname{Spec} R$. As a consequence, we can see that most other aspects of module theory carry the same property:

Corollary 1. *Modules themselves and individual module elements are locally determined. Exactness of sequences of modules can also be checked locally on* $\operatorname{Spec} R$*.*

As such, the outlook for treating modules geometrically seems good so far. To start to get a sense of what the "shape" of a module should be, we observe the result of restricting it over each point:

Proposition 2. *Let R be a ring, M an R-module, and $\mathfrak{p} \subset R$ a prime, with $R \to \kappa(\mathfrak{p})$ the natural map to the residue field. Then $\kappa(\mathfrak{p}) \otimes_R M$ is a vector space over $\kappa(\mathfrak{p})$ (and, if M is finitely generated, this vector space will be finite-dimensional).*

Example 1. *Let $R = \mathbb{C}[x, y]$. Set $M = R^{\oplus 2}$ and let N be the ideal (x, y), regarded as an R-module — note that this can be written as $\frac{Re_1 \oplus Re_2}{(ye_1 - xe_2)}$, where e_1 is the generator corresponding to x and e_2 the one corresponding to y.*

For each prime $\mathfrak{p} \subset R$, we can compute $\kappa(\mathfrak{p}) \otimes_R M$ by recalling that tensor products distribute over direct sums — hence we obtain the 2-dimensional vector space $\kappa(\mathfrak{p})^{\oplus 2}$ over the residue field at each point.

Likewise, for each prime $\mathfrak{p} \subset R$, we see that $\kappa(\mathfrak{p}) \otimes_R N \cong \frac{\kappa(\mathfrak{p})e_1 \oplus \kappa(\mathfrak{p})e_2}{(ye_1 - xe_2)}$ since tensor products preserve quotients as well (for those familiar with the terminology, this is to say that the tensor product is right exact). The dimension of this vector space over $\kappa(\mathfrak{p})$ now depends on our choice of $\mathfrak{p}$ — if $\mathfrak{p} = (x, y)$, it is isomorphic to $\kappa(\mathfrak{p})^{\oplus 2} \cong \mathbb{C}^{\oplus 2}$, but otherwise the relation is nontrivial and so it is isomorphic to the the one-dimensional vector space $\kappa(\mathfrak{p})^{\oplus 1}$.

Thus the "algebraic fibers" of M over the points of R are vector spaces (potentially of varying dimension), which are perhaps more readily understandable than arbitrary modules. In particular, we have already discussed the idea that, for a field k and integer $n \geq 0$, the vector space $k^{\oplus n}$ can be "geometrically realized" as the affine space $\mathbb{A}_k^n$. Our goal for today will be to generalize this idea of geometric realization from the "algebraic fibers" of M over points of $\operatorname{Spec} R$ to module itself — this is where the notion of *linear fiber spaces* will come into play. Of course, we will then want to generalize from the affine setting to arbitrary schemes — to see what the analogue to a module should be in this broader context, we'll discuss *quasicoherent sheaves* next week.

2 Linear Fiber Spaces in Topology

We begin with a topic which perhaps has somewhat limited interest in its own right — the analogues to the objects we seek to construct in the topological setting. In this context, linear fiber spaces qua linear fiber spaces are not a particularly prominent area of past or present research, and we will not need to draw on any of what we construct here explicitly in the scheme-theoretic world. Rather, the present discussion is mainly for the sake of intuition; for the moment, we can deal with the new ideas we're interested in in the more familiar topological context before we layer in the additional complication of working scheme-theoretically.

Let T be a topological space. We want to think about the idea of "a space over T whose fibers are vector spaces" (here, as in the case of schemes, a "space over T" is just another way to say "a space with a chosen map to T"). Taken literally, this requirement is fairly weak — for example, we could consider the map $\bigsqcup_{x \in T} \mathbb{R}^n \to T$ taking each copy of $\mathbb{R}^n$ to the corresponding point of T, or we could build a space which is not the disjoint union of its fibers which has the property that the vector space structures of nearby fibers have nothing to do with each other. Such maps technically satisfy the hypothesis as stated, but don't really get at what we want to mean by it.

One typical way to solve this problem is through the idea of a *vector bundle* over T, which you may have encountered before. Essentially, the idea is to avoid these kinds of pathologies in the relationships between the fibers from point to point by enforcing a local triviality condition — that is, each point of T has a neighborhood U over which our map can be expressed as the projection $\mathbb{R}^n \times U \to U$ compatibly with our vector space structures on fibers.

This solution works quite well, and the theory of vector bundles is the basis for, e.g., much of differential geometry. However, it suffers from an important limitation: Vector bundles are required to have (locally) constant fiber dimension. If A and B are vector bundles over T, for example, and we have a fiberwise-linear map $A \to B$ commuting with the projections to T, we might like to be able to talk about the (fiberwise) kernel of this map — but this object will not be a vector bundle unless we also enforce that our map have (locally) constant rank on T, and many maps of vector bundles arising naturally in practice do not satisfy this hypothesis.

Hence we need some weaker notion stringent enough to exclude the most pathological cases but permissive enough to allow variations in fiber dimension. We will accomplish this using the machinery of fiber products discussed in Lecture 3 — these allow us to make sense of the idea of "vector space operations on fibers" in some reasonably cohesive, consistent way.

Since the definition of a vector space involves being able to multiply by scalars, we will start our discussion with the trivial rank-1 vector bundle $\mathbb{R} \times T \xrightarrow{\pi} T$. $\mathbb{R}$, considered with the classical topology, has the structure of a *topological ring* (i.e., a ring where all the operations are continuous), and this naturally induces on $\mathbb{R} \times T$ the structure of a "fiberwise ring over T". That is, we have the following data:

- The *zero section* $z : T \to \mathbb{R} \times T$ given by $z(x) = (0, x)$.
- The *unit section* $u : T \to \mathbb{R} \times T$ given by $u(x) = (1, x)$.
- The *(fiberwise) addition map* $(\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \xrightarrow{+} \mathbb{R} \times T$ given by $+(r_1, r_2, x) = (r_1 + r_2, x)$. (Here we are using the fact that $(\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \cong \mathbb{R} \times \mathbb{R} \times T$.)
- The *(fiberwise) additive inverse* $\mathbb{R} \times T \xrightarrow{-} \mathbb{R} \times T$ given by $-(r, x) = (-r, x)$.
- The *(fiberwise) multiplication map* $(\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \xrightarrow{\cdot} \mathbb{R} \times T$ given by $\cdot(r_1, r_2, x) = (r_1 r_2, x)$ (again using $(\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \cong \mathbb{R} \times \mathbb{R} \times T$).

These maps are indeed sections and fiberwise operators, respectively, because they commute with the natural projections of all spaces involved to T (e.g., $\pi \circ z = \mathrm{id}_T$, so that, $z(x)$ is a point of the fiber over x for all $x \in T$, and $\pi \circ +$ gives the natural projection of $(\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \cong \mathbb{R} \times \mathbb{R} \times T$ to T, so that the sum of two points in the same fiber remains in that fiber). Moreover, they do actually give us a ring structure on each fiber, precisely because they satisfy the appropriate analogues to the ring axioms in the setting of topological spaces over T. For example, we can express the idea that the unit of a ring R acts as the identity by requiring that the composition $R \xrightarrow{1 \times \mathrm{id}} R \times R \xrightarrow{\cdot} R$ be the identity map on R; the corresponding requirement that the unit section give the point in each fiber acting as the multiplicative identity is precisely that $\mathbb{R} \times T \xrightarrow{(u \circ \pi) \times_T \mathrm{id}} (\mathbb{R} \times T) \times_T (\mathbb{R} \times T) \xrightarrow{\cdot} \mathbb{R} \times T$ be the identity map on $\mathbb{R} \times T$.

A complete description of rings in these terms is given in Section 5; to adapt them for our present purposes, we replace the natural maps to the one-point space everywhere by the projections to T and the cartesian product $\times$ everywhere by the fiber product $\times_T$.

Exercise 1. *Write down the adapted ring axioms and verify that* $\mathbb{R} \times T$ *satisfies them.*

(In category-theoretic terms, all of this has been to say that $\mathbb{R} \times T$ is a "ring in the category of topological spaces over T".)

It is easy to get lost in the formalisms here — as you go, make sure you understand why the diagram you write down for each ring axiom is just expressing the requirement that the axiom hold in every fiber.

Now that we understand our fiberwise scalars, we can get at our idea of "a space over T whose fibers are vector spaces":

Definition 2. *Let T be a topological space. A* **linear fiber space** *over T is a topological space $X \xrightarrow{\pi} T$ over T with the structure of a "fiberwise vector space over $\mathbb{R} \times T$" — that is, we have the following maps, all commuting everywhere with the natural projections to T:*

- *A* **zero section** $z : T \to X$.
- *A* **(fiberwise) addition map** $X \times_T X \xrightarrow{+} X$.
- *A* **(fiberwise) additive inverse** $X \xrightarrow{-} X$.
- *A* **(fiberwise) scalar multiplication map** $(\mathbb{R} \times T) \times_T X \xrightarrow{\cdot} X$.

Moreover, these maps satisfy the axioms for a module over $\mathbb{R} \times T$ given in Section 5 when we again replace the maps to the one-point space by the projections to T and the cartesian product $\times$ by the fiber product $\times_T$ everywhere.

A **map of linear fiber spaces** Φ *from X to Y is a continuous map commuting with the projections to T which is compatible with the zero section and all operations — that is, $\Phi \circ z_X = z_Y$, $\Phi \circ +_X = +_Y \circ (\Phi \times_T \Phi)$, $\Phi \circ -_X = -_Y \circ \Phi$, and $\Phi \circ \cdot_X = \cdot_Y \circ (\mathrm{id} \times_T \Phi)$.*

(More briefly: A linear fiber space over T is an $(\mathbb{R} \times T)$-module in the category of topological spaces over T.)

Exercise 2. *Show that every vector bundle is a linear fiber space over its base.*

Observe that, since fiber products are preserved under pullbacks, we can pull back the module axioms over each point of T to show that our operations indeed induce the structure of a (topological) vector space on each fiber. Moreover, nothing forces these vector spaces to have the same dimension.

Remark 1. *A word of caution: One might expect from what we have said so far that every linear fiber space with constant fiber dimension will be a vector bundle. This is not true — consider, for example, the subspace of $\mathbb{R}^3$ given by the union of the z-axis with the complement in the xy-plane of the y-axis, and take the projection to $\mathbb{R}$ given by the x-coordinate. Taken with the natural operations, this is indeed a linear fiber space over $\mathbb{R}$, but it is not a vector bundle. Hence, in the topological setting, this concept does not exclude all of the pathologies, and we need some extra hypothesis — for example, we could require that our linear fiber spaces be locally given by closed subsets of $V \times T$ for topological vector spaces V.*

However, in the scheme-theoretic setting there is already enough rigidity to exclude examples of this sort, and, since we are mainly interested in the topological situation to make our discussion of schemes more intuitive, we will not pursue the matter further.

Remark 2. *Speaking more precisely, the objects we have been discussing in this section are real linear fiber spaces. It is, of course, possible to make the same constructions with $\mathbb{C}$ (or any other topological field) in place of $\mathbb{R}$.*

Having established some level of intuition in the topological setting, we now turn our attention to...

3 Linear Fiber Spaces over Schemes

We now make the same constructions in the setting of schemes. The analogy is fairly direct, building on our previously-established parallels between fiber products in each context — the main challenge will be to remain grounded and connect the geometric picture to the affine-local algebraic implications.

Let S be a scheme. Recall that $\mathbb{A}^1_S := S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[t]$, considered together with the natural fiber product projection $\mathbb{A}^1_S \to S$, is the "trivial rank-1 vector bundle over S"; over each affine open $\operatorname{Spec} R$ of S, this is simply the map on ring spectra induced by the natural map $R \to R[t]$. In particular, if we pull back over any point $\operatorname{Spec} k$ of S, we get the map $\mathbb{A}^1_k \to \operatorname{Spec} k$ induced by the natural map $k \to k[t]$; that is, the fiber over any point is our "geometric realization" of the one-dimensional vector space over the residue field, as we should expect from a "rank-1 vector bundle". We can think of the "triviality" part as coming from the fact that this "bundle" is pulled back from $\operatorname{Spec}\mathbb{Z}$ — in particular, this means that, if there is a map $S \to \operatorname{Spec} K$ with K a field, $\mathbb{A}^1_S$ will simply be the product (over K) of S with the "1-dimensional vector space" $\mathbb{A}^1_K$.

Now, as in the case of topological spaces, we can verify that the "trivial rank-1 vector bundle" has a "ring structure in each fiber":

Proposition/Definition 1. *Let S be a scheme. Then we define the following maps of schemes over S:*

- *The* **zero section** $z : S \to \mathbb{A}^1_S$ *given on each affine open* $\operatorname{Spec} R$ *by the R-algebra map $R \to R[t]$ taking t to 0.*
- *The* **unit section** $u : S \to \mathbb{A}^1_S$ *given on each affine open* $\operatorname{Spec} R$ *by the R-algebra map $R \to R[t]$ taking t to 1.*
- *The* **(fiberwise) addition map** $\mathbb{A}^1_S \times_S \mathbb{A}^1_S \xrightarrow{+} \mathbb{A}^1_S$ *given on each affine open* $\operatorname{Spec} R$ *by the R-algebra map $R[t] \to R[t_1, t_2]$ taking t to $t_1 + t_2$.*
- *The* **(fiberwise) additive inverse** $\mathbb{A}^1_S \xrightarrow{-} \mathbb{A}^1_S$ *given on each affine open* $\operatorname{Spec} R$ *by the R-algebra map $R[t] \to R[t]$ taking t to $-t$.*
- *The* **(fiberwise) multiplication map** $\mathbb{A}^1_S \times_S \mathbb{A}^1_S \xrightarrow{\cdot} \mathbb{A}^1_S$ *given on each affine open* $\operatorname{Spec} R$ *by the R-algebra map $R[t] \to R[t_1, t_2]$ taking t to $t_1 t_2$.*

These maps make $\mathbb{A}^1_S$ a **(fiberwise) ring over** S *— that is, these maps satisfy the ring axioms described in Section 5 when we replace the maps to the one-point space by the structure maps to S and the cartesian product $\times$ by the fiber product $\times_S$ everywhere.*

Here we have implicitly used the identification $\mathbb{A}^1_S \times_S \mathbb{A}^1_S \cong \mathbb{A}^2_S$ — that is, on the level of affine opens, $R[t_1] \otimes_R R[t_2] \cong R[t_1, t_2]$. The intuition you should have for our affine-local descriptions of these maps is again in terms of ring elements as functions — in particular, we view $t \in k[t]$ as the "coordinate function" on the "one-dimensional vector space" $\mathbb{A}^1_k$, and $t \in R[t]$ as giving the "coordinate function" on each fiber of the "trivial rank-1 vector bundle" over $\operatorname{Spec} R$. Then, for example, we can understand the addition map $\mathbb{A}^2_S \to \mathbb{A}^1_S$ by saying that the coordinate function on the target should be (more precisely, pull back to) the sum of the coordinates on the source.

Exercise 3. *Explicitly write down the diagrams giving the ring axioms in this case and verify that they commute.*

Exercise 4. *Let k be an algebraically closed field, so that the closed points of $\mathbb{A}^1_k$ correspond to elements of k and the closed points of $\mathbb{A}^2_k \cong \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ correspond to pairs of such elements. Verify that the maps given here, when restricted to closed points, induce the usual ring operations on k.*

Exactly as in the topological setting, we can now define the notion of a "scheme over S which is a vector space in every fiber":

Definition 3. *Let S be a scheme. A* **linear fiber space** *over S is an S-scheme X together with maps of schemes over S:*

- *A* **zero section** $z : S \to X$.
- *A* **(fiberwise) addition map** $X \times_S X \xrightarrow{+} X$.
- *A* **(fiberwise) additive inverse** $X \xrightarrow{-} X$.
- *A* **(fiberwise) scalar multiplication map** $\mathbb{A}^1_S \times_S X \xrightarrow{\cdot} X$.

We require that these maps satisfy the axioms for a module over $\mathbb{A}^1_S$ given in Section 5 when we replace the maps to the one-point space by the structure maps to S and and the cartesian product $\times$ by the fiber product $\times_S$ everywhere.

A **map of linear fiber spaces** Φ *from X to Y is a map of schemes over S commuting with the zero section and all operations, as in the topological case — that is, $\Phi \circ z_X = z_Y$, $\Phi \circ +_X = +_Y \circ (\Phi \times_S \Phi)$, $\Phi \circ -_X = -_Y \circ \Phi$, and $\Phi \circ \cdot_X = \cdot_Y \circ (\mathrm{id} \times_S \Phi)$.*

Exercise 5. *Let k be a field and $n \geq 0$ an integer. Show that $\mathbb{A}^n_k := k[x_1, \ldots, x_n]$ is a linear fiber space over $\operatorname{Spec} k$ with the zero section and operations given respectively by the k-algebra maps:*

- *$k[x_1, \ldots, x_n] \to k$ taking each x_i to 0.*
- *$k[x_1, \ldots, x_n] \to k[x'_1, \ldots, x'_n, x''_1, \ldots, x''_n]$ taking each x_i to $x'_i + x''_i$.*
- *$k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ taking each x_i to $-x_i$.*
- *$k[x_1, \ldots, x_n] \to k[t, x_1, \ldots, x_n]$ taking each x_i to tx_i.*

Verify that, if k is algebraically closed, the restrictions of these maps to closed points give the usual vector space operations on k^n.

Exercise 6. *Use adaptations of the arguments of the previous exercise and the local determinacy of schemes and their maps to show that, for any scheme S and integer $n \geq 0$, $\mathbb{A}^n_S$ is a linear fiber space over S.*

4 Modules as Linear Fiber Spaces

Now that we understand both the vector space structure on affine spaces and the idea of a scheme over a given scheme whose fibers are vector spaces in the same sense, let us return to the question of a modules, with the goal of finding such a space which in some sense realizes a given module.

The simplest case, of course, is when our ring is a field:

Question. *Let k be a field, and V a k-module (i.e., a vector space). How do we get "the affine space geometrically realizing V"?*

If $V \cong k^{\oplus n}$, of course, our answer should be $\mathbb{A}_k^n$ — but how should we get at this space without already knowing the dimension, or in the case of an infinite-dimensional vector space? To start toward an answer, we note that, when we *are* given an explicit choice of basis, say $v_1, \ldots, v_n \in V$, we can realize V by $\operatorname{Spec} k[v_1, \ldots, v_n]$ (at least up to a dual — we will discuss this issue in more detail shortly). The following object provides a coordinate-free generalization of this construction:

Definition 4. *Let R be a ring and M an R-module. Then the* **symmetric algebra** *of M is the commutative ring*

$$\operatorname{Sym}(M) := \frac{\bigoplus_{\ell=0}^{\infty} M^{\otimes \ell}}{(a \otimes b - b \otimes a \mid a, b \in M)},$$

where all tensor products are over R, we take $M^{\otimes 0}$ to be R, and the multiplication operation is given by $\otimes$. (The denominator above is a two-sided ideal in the noncommutative ring given by the numerator — modding out by it is precisely enough to make the resulting ring commutative.)

The definition may not seem intuitive at first, but in practice the symmetric algebra is often easy to compute:

Example 2. *If $M \cong Re_1 \oplus \ldots \oplus Re_n$ is a (finite-rank) free module, then* $\operatorname{Sym}(M) \cong R[e_1, \ldots, e_n]$.

Example 3. *If $R = \mathbb{C}[x, y]$ and $N = (x, y) \cong \frac{Re_1 \oplus Re_2}{(ye_1 - xe_2)}$, as in Example 1, then* $\operatorname{Sym}(N) \cong R[e_1, e_2]/(ye_1 - xe_2)$.

More generally, we have:

Proposition 3. *Let R be a ring and M an R-module. Suppose we have a presentation $R^{\oplus q} \to R^{\oplus p} \to M \to 0$ — that is, M is generated by elements $e_1, \ldots, e_p$, modulo relations $r_{1j}e_1 + \ldots + r_{pj}e_p$ for $1 \leq j \leq q$. Then $\operatorname{Sym}(M) \cong R[e_1, \ldots, e_p]/(r_{1j}e_1 + \ldots + r_{pj}e_p \mid 1 \leq j \leq q)$.*

The analogous result holds for infinite presentations as well.

That is, if we can write down M in terms of generators and relations, then $\operatorname{Sym}(M)$ will be the R-algebra with the same generators and relations. (We could have defined the symmetric algebra this way, but the construction we used makes it more apparent that this is independent of the chosen presentation.)

We can now answer our question without needing an explicit choice of basis:

Answer (up to a dual)**.** *Take* $\operatorname{Spec}\operatorname{Sym}(V)$.

By analogy, we can now create the "geometric realization" for a module over an arbitrary ring:

Definition 5. *Let R be a ring and M an R-module. We define the* **spectrum** *of M to be the* $\operatorname{Spec} R$*-scheme* $\operatorname{Spec} M := \operatorname{Spec}\operatorname{Sym}(M)$*, endowed with the structure of a linear fiber space over* $\operatorname{Spec} R$ *by the following maps of R-algebras:*

- $\operatorname{Sym}(M) \to R$ *taking each $m \in M^{\otimes 1}$ to* 0.
- $\operatorname{Sym}(M) \to \operatorname{Sym}(M) \otimes_R \operatorname{Sym}(M)$ *taking each $m \in M^{\otimes 1}$ to $m \otimes 1 + 1 \otimes m$.*
- $\operatorname{Sym}(M) \to \operatorname{Sym}(M)$ *taking each $m \in M^{\otimes 1}$ to $-m$.*
- $\operatorname{Sym}(M) \to \operatorname{Sym}(M)[t]$ *taking each $m \in M^{\otimes 1}$ to tm.*

Exercise 7. *Show that, when M is a free module of finite rank n, these maps are the same as the ones used to define the linear fiber space structure on $\mathbb{A}^n_R$ in Exercises 5 and 6.*

Exercise 8. *Verify that these operations make* $\operatorname{Sym}(M)$ *into a linear fiber space in general.*

Exercise 9. *Let R be a ring and $\phi : M \to N$ a map of R-modules. Show that the spectrum construction functorially (and contravariantly!) induces a map* $\operatorname{Spec} N \to \operatorname{Spec} M$ *of linear fiber spaces over* $\operatorname{Spec} R$.

Bonus: Show that every map $\operatorname{Spec} N \to \operatorname{Spec} M$ *of linear fiber spaces over* $\operatorname{Spec} R$ *is induced by some such ϕ.*

Of course, for this to be a good analogue to the notion of a spectrum of a ring, it should be possible to retrieve M from $\operatorname{Spec} M$ somehow. To this end, we define:

Definition 6. *Let S be a scheme and X a linear fiber space over S. A* **linear form** *on X is a map $X \to \mathbb{A}^1_S$ of linear fiber spaces over S. We denote the set of all linear forms on X by $L(X)$. If $\Phi : X \to Y$ is a map of linear fiber spaces over S, we call the induced map $\Phi^* : L(Y) \to L(X)$ given by composition with Φ* **pullback of linear forms**.

If we are working over an affine scheme $\operatorname{Spec} R$, the collection of linear forms can be viewed as an R-module:

Proposition/Definition 2. *Let S be a scheme and X a linear fiber space over S. Then the binary operation $L(X) \times L(X) \to L(X)$ sending (ϕ, ψ) to the composition $X \xrightarrow{\phi \times_S \psi} \mathbb{A}^1_S \times_S \mathbb{A}^1_S \xrightarrow{+} \mathbb{A}^1_S$ makes $L(X)$ an abelian group. For any map of linear fiber spaces over S, moreover, pullback of linear forms respects this group structure.*

If $S = \operatorname{Spec} R$ is affine, we can also define a multiplication map $R \times L(X) \to L(X)$ which sends (r, ϕ) to the composition of ϕ with the map $\mathbb{A}^1_R \to \mathbb{A}^1_R$ given by the R-algebra map $R[t] \to R[t]$ taking t to rt. This makes $L(X)$ an R-module, and pullback along any map of linear fiber spaces is an R-module homomorphism.

As promised, we can now retrieve M:

Exercise 10. *Let R be a ring and M an R-module. Show that $M \cong L(\operatorname{Spec} M)$. (Hint: For a given module element $m \in M$, what should the corresponding map $R[t] \to \operatorname{Sym}(M)$ be?)*

Let N be another R-module, and $\phi : M \to N$ a map of R-modules. Show that the map $L(\operatorname{Spec} M) \to L(\operatorname{Spec} N)$ given by pullback along the map of linear fiber spaces defined in Exercise 9 is identified with ϕ under these isomorphisms.

Hence, just as we can view R as "the collection of functions on $\operatorname{Spec} R$", we can interpret M as "the collection of linear forms on $\operatorname{Spec} M$". This brings us back to our earlier note that, for a vector space V, $\operatorname{Spec} V$ should really be thought of as geometrically realizing the dual of V rather than V itself — Exercise 9 shows that the map induced on spectra by a map of vector spaces goes in the opposite of the original direction, as is the case with duals of vector spaces, and, indeed, we have just explicitly identified V with the collection of linear forms on $\operatorname{Spec} V$ in the appropriate sense. For this reason, many algebraic geometers prefer to think that the affine space "corresponding to" V should be the spectrum $\operatorname{Spec} V^\vee$ of its dual instead. However, if we consider the situation between rings and schemes, this begins to look like a less natural choice — to be consistent, we should *want* to view an algebra-geometry correspondence as a dualizing operation, so that the algebraic object can be interpreted as the collection of functions on the corresponding geometric space.

Indeed, this gut feeling is borne out by the result of Exercise 10, and by the absence of an analogous way to retrieve an R-module M from $\operatorname{Spec} \operatorname{Sym}(M^\vee)$; when we are working over an arbitrary ring, taking the dual of a module can destroy information, so the analogue to the correspondence $V \leftrightarrow \operatorname{Spec} V^\vee$ is not actually a correspondence in this setting. (Issues also arise if we attempt to include infinite-dimensional vector spaces in our "correspondence" $V \leftrightarrow \operatorname{Spec} V^\vee$.)

To conclude, we return to a definition from last week. We declared the "cotangent module" of a scheme at a point to be the vector space $\mathfrak{m}/\mathfrak{m}^2$ over its residue field, where $\mathfrak{m}$ is the maximal ideal of the local ring. As we mentioned at that point, the typical convention is to dualize this (as an algebraic vector space) to get the definition of "the tangent space at the point". However, we should generally want to think of a tangent space geometrically, rather than algebraically, and hence we should instead take this "dualization" to be exactly the passage from algebra to geometry:

Definition 7. *Let X be a scheme, $x \in X$ a point, and $\mathfrak{m} \subset \mathcal{O}_{X,x}$ the maximal ideal of the local ring. We define the* **Zariski tangent space** *of X at x to be the affine space $\operatorname{Spec}(\mathfrak{m}/\mathfrak{m}^2)$ over the spectrum of the residue field.*

Later on, when we discuss differentials, we will see how to knit these tangent spaces together into a single linear fiber space over the scheme.

5 Appendix: Algebraic Objects via Diagrams

From algebra, we are familiar with various kinds of objects — groups, rings, modules, and so forth — as sets endowed with some additional operations and distinguished elements, subject to some axioms about the ways these data interact with each other. In settings like topology or differential geometry, where our objects and maps are closely tied to underlying sets, we can often apply these descriptions with extra compatibility hypotheses to define the analogous notions — for example, a Lie group is just a smooth manifold together with group operations on the underlying set, subject to the additional hypothesis that all of the maps involved be smooth.

In the setting of schemes, of course, this kind of approach will not be enough. Instead, we need to express things like group axioms in forms which lend themselves well to generalization — essentially, in category-theoretic terms. To this end, we will now give full descriptions of some common algebraic objects in such language, as promised earlier in today's notes.

In what follows, we will denote the one-point set by P, and, for any set S, the natural map $S \to P$ by π. If $A_1 \times \cdots \times A_n$ is a product of sets, we will denote the projection onto the ith factor by π_i; if B is another set and $f_i : B \to A_i$ are maps, we will denote the induced map $B \to A_1 \times \cdots \times A_n$ by $f_1 \times \cdots \times f_n$. To save space, we will often omit composition symbols — that is, for maps f and g, we write gf instead of $g \circ f$. We will also write binary operations using infix notation — e.g., $f \cdot g$ in place of the composition $\cdot(f \times g)$.

First recall the notion of a *monoid*, a set with an associative binary operation which has an identity element:

Definition 8. *A* **monoid** *consists of a set M, together with maps $M \times M \xrightarrow{\cdot} M$ and $P \xrightarrow{u} M$, such that the following diagrams commute:*

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\pi_1 \times (\pi_2 \cdot \pi_3)"] \arrow[d, "(\pi_1 \cdot \pi_2) \times \pi_3"'] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "(u\pi) \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times (u\pi)"'] \arrow[dr, "\mathrm{id}"] & M \times M \arrow[d, "\cdot"] \\
M \times M \arrow[r, "\cdot"] & M
\end{tikzcd}$$

A monoid is said to be **commutative** *if the following diagram also commutes:*

$$\begin{tikzcd}
M \times M \arrow[r, "\cdot"] \arrow[d, "\pi_2 \times \pi_1"'] & M \\
M \times M \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

The map $\cdot$ is of course the binary operation, while u is the map taking the unique point of P to the identity element. The first diagram expresses the associativity requirement, while the second tells us that our identity element actually acts as the identity under the monoid's operation; we use the constant map $u\pi$ given by the composition $M \xrightarrow{\pi} P \xrightarrow{u} M$ to pick this element out.

We now define a more familiar object, a group, by requiring that each element have an inverse under the operation:

Definition 9. *A* **group** *is a set G together with maps $G \times G \xrightarrow{\cdot} G$, $P \xrightarrow{e} G$, and $G \xrightarrow{i} G$ such that $(G, \cdot, e)$ is a monoid and the following diagram commutes:*

$$
\begin{tikzcd}
G \arrow[r, "i\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times i"'] \arrow[dr, "e\pi"] & G\times G \arrow[d, "\cdot"] \\
G\times G \arrow[r, "\cdot"] & G
\end{tikzcd}
$$

G is called an **abelian group** *if, moreover, the monoid $(G,\cdot,e)$ is commutative.*

The map $i : G \to G$ is the one taking each element to its inverse; the diagram simply says that multiplying an element by its inverse on either said will always result in the identity element.

We can now combine our notions of commutative monoids and abelian groups to get rings — remember that, for us, rings are always commutative with identity elements. Thus we have:

Definition 10. *A* **ring** *is a set R together with maps $R\times R \xrightarrow{+} R$, $P \xrightarrow{z} R$, $R \xrightarrow{-} R$, $R\times R \xrightarrow{\cdot} R$, and $P \xrightarrow{u} R$ such that $(R,+,z,-)$ is an abelian group, $(R,\cdot,u)$ is a commutative monoid, and the following diagram commutes:*

$$
\begin{tikzcd}
R\times R\times R \arrow[r, "\pi_1\times(\pi_2+\pi_3)"] \arrow[d, "\pi_1\times\pi_2\times\pi_1\times\pi_3"'] & R\times R \arrow[r, "\cdot"] & R \\
R\times R\times R\times R \arrow[rr, "(\pi_1\cdot\pi_2)\times(\pi_3\cdot\pi_4)"] & & R\times R \arrow[u, "+"]
\end{tikzcd}
$$

This new diagram expresses the requirement that multiplication distribute over addition — note that this actually gives us distributivity from both sides by the commutativity requirements.

Finally, we have the notion of a module over a given ring:

Definition 11. *A* **module** *over a ring $(R,+,z_R,-,\cdot,u)$ is an abelian group $(M,+,z_M,-)$ together with a map $R\times M \xrightarrow{\cdot} M$ such that the following diagrams commute:*

$$
\begin{tikzcd}
R\times M\times M \arrow[r, "\pi_1\times(\pi_2+\pi_3)"] \arrow[d, "\pi_1\times\pi_2\times\pi_1\times\pi_3"'] & R\times M \arrow[r, "\cdot"] & M \\
R\times M\times R\times M \arrow[rr, "(\pi_1\cdot\pi_2)\times(\pi_3\cdot\pi_4)"] & & M\times M \arrow[u, "+"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
R\times R\times M \arrow[r, "(\pi_1+\pi_2)\times\pi_3"] \arrow[d, "\pi_1\times\pi_3\times\pi_2\times\pi_3"'] & R\times M \arrow[r, "\cdot"] & M \\
R\times M\times R\times M \arrow[rr, "(\pi_1\cdot\pi_2)\times(\pi_3\cdot\pi_4)"] & & M\times M \arrow[u, "+"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
R\times R\times M \arrow[r, "(\pi_1\cdot\pi_2)\times\pi_3"] \arrow[d, "\pi_1\times(\pi_2\cdot\pi_3)"'] & R\times M \arrow[d, "\cdot"] \\
R\times M \arrow[r, "\cdot"] & M
\end{tikzcd}
$$

$$\begin{tikzcd} M \arrow[r, "(u\pi)\times\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & R \times M \arrow[d, "\cdot"] \\ & M \end{tikzcd}$$

The new map $\cdot$ gives the ring action; the new diagrams express, respectively, distribution of the action over the module's addition, distribution over the ring's addition, compatibility with the ring's multiplication, and compatibility with the identity element of the ring.

Exercise 11. *Convince yourself that these definitions are the same as the ones you already know from algebra.*