The Geometry of Rings and Schemes
Lecture 11: Separated and Proper Maps

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This week we will step back a bit from the machinery we've been building — linear fiber spaces, tangent schemes, and the like — and develop scheme-theoretic analogues to some more topological notions. (In particular, this lecture could have been Lecture 6 instead of Lecture 11, modulo a reference to affine maps.)

1 Hausdorff Spaces and Separated Maps

Recall the following definition from topology:

Definition 1. *A topological space T is called* **Hausdorff** *if, for all pairs $x, y \in T$ of distinct points, there exist open subsets $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$.*

In the classical setting, most spaces of interest are Hausdorff. However, the Zariski topology is not so well-behaved:

Proposition 1. *Let X be a nonempty locally Noetherian scheme. Then the underlying space of X is Hausdorff if and only if $\dim X = 0$.*

Proof. If the dimension of X is greater than zero (here we include the infinite case), then the characterization of dimension in terms of chains of specializing points ensures that the underlying space of X will not be Hausdorff — in a Hausdorff space, all points must themselves already be closed.

Suppose $\dim X = 0$. Since being Hausdorff is a local condition and the underlying space of X is the same as that of its reduction, we can consider only the reduced affine case — suppose $X = \operatorname{Spec} R$ with $\operatorname{nil}(R) = 0$ and consider a point $x \in X$, with $\mathfrak{p} \subset R$ the corresponding prime ideal. Then the points of $\operatorname{Spec} R_{\mathfrak{p}}$ correspond to prime ideals contained in $\mathfrak{p}$ — since $\dim X = 0$, however, this means that $\operatorname{Spec} R_{\mathfrak{p}}$ is simply the single point x. Since $\operatorname{Spec} R_{\mathfrak{p}}$ is reduced, it is thus reduced and irreducible, and hence $R_{\mathfrak{p}}$ is an integral domain. Therefore, its zero ideal is prime, hence equal to $\mathfrak{p}R_{\mathfrak{p}}$ by the fact that its spectrum contains no other points. As such, $R_{\mathfrak{p}} \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is even a field — we can see also that x is closed by

*First draft of the TeX source provided by Márton Beke.

the zero-dimensionality hypothesis, and so in fact $R_\mathfrak{p} \cong R/\mathfrak{p}$. This is to say that tensoring the short exact sequence

$$0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$$

by $R_\mathfrak{p}$ over R will kill the term $\mathfrak{p}$. Since R is Noetherian by local Noetherianity and so $\mathfrak{p}$ is finitely generated, we can then produce a single $f \notin \mathfrak{p}$ such that $\mathfrak{p} \otimes_R R_f = 0$, which implies $\operatorname{Spec} R_f = \operatorname{Spec} R/\mathfrak{p} = x$ is both open and closed. Hence X is discrete, and so it is Hausdorff. □

Thus, since we should want our "Euclidean spaces" $\mathbb{A}^n_k$ to be "Hausdorff", for example, we return to our usual task of seeking some sort of reformulation of the topological property under consideration which gives a more appropriate analogue on the scheme-theoretic side. To set ourselves up to do so, we introduce, or remind the reader of, the following generalization of the concept of a sequence in a topological space:

Definition 2. *A* **directed set** *is a set A together with a* **preorder** *— that is, a reflexive, transitive binary relation, which we typically denote $\leq$ — such that, for all $\alpha, \beta \in A$, there exists $\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$.*

Given a directed set A and proposition $P(\alpha)$ depending on $\alpha \in A$, we say that P is **eventually true** *if there exists $\beta \in A$ such that $P(\alpha)$ holds for all $\alpha \geq \beta$. We say that P is* **cofinally true** *if it is not eventually false.*

Given a topological space T, we define a **net** *(or* **Moore-Smith sequence***) in T to be a map $A \to T$ of sets for A a directed set; letting x_α denote the image of each $\alpha \in A$, we typically denote such a net by $(x_\alpha)_{\alpha \in A}$ or simply x_α. We say that such a net* **converges** *to $x \in T$ and write $x_\alpha \xrightarrow{\alpha} x$ or $x_\alpha \to x$ if, for every open set $U \ni x$, x_α is eventually in U.*

We verify that this indeed generalizes the notion of a sequence:

Example 1. *Endow $\mathbb{N}$ with the standard ordering. Then a net indexed by $\mathbb{N}$ in a topological space T is simply a sequence in T, and this converges in the sense of nets if and only if it does so in the traditional sense of sequences.*

The utility of nets is as follows. Often it is easier to make a topological argument by reasoning about sequences and their convergence than by working directly with open subsets — likewise, many definitions are more intuitive when stated in these terms. However, sequences are really only enough when we are working with nice classes of spaces — consider, for example, the difference in general between sequential compactness and compactness. Nets resolve this issue in the sense that they allow for convergence-theoretic arguments and definitions which can capture behavior not accounted for by sequences — indeed, the topology of a space can be described completely by specifying which nets converge to which points, subject to appropriate axioms, while the same is not true of sequences.

In particular, returning to the topic of Hausdorff spaces, we have:

Proposition 2. *Let T be a topological space. Then T is Hausdorff if and only if every net in T converges to at most one point.*

For the moment, this is only an intermediate step — which could, in all honesty, have been avoided — on our path to the description of Hausdorff spaces we will actually use to move to the scheme-theoretic setting. However, we will eventually see a scheme-theoretic version of this characterization in Section 3.

We make the following additional observations:

Proposition 3. *Let T_1 and T_2 be topological spaces and $\pi_1 : T_1 \times T_2 \to T_1$ and $\pi_2 : T_1 \times T_2 \to T_2$ be the projections. Then a net x_α in $T_1 \times T_2$ converges to $x \in T_1 \times T_2$ if and only if $\pi_1(x_\alpha) \to \pi_1(x)$ and $\pi_2(x_\alpha) \to \pi_2(x)$.*

Proposition 4. *Let T be a topological space and $S \subseteq T$ a subspace. Then S is closed in T if and only if, for all nets x_α in S and points $x \in T$ such that $x_\alpha \to x$, $x \in S$.*

Together, these facts let us again reformulate our concept of Hausdorff spaces:

Proposition 5. *Let T be a topological space. Then T is Hausdorff if and only if the diagonal embedding $\Delta : T \hookrightarrow T \times T$ is the inclusion of a closed subspace.*

Proof. By Proposition 4, the diagonal in $T \times T$ will fail to be closed if and only if there exists a net (x_α, x_α) in the diagonal converging to a point $(x, y) \in T \times T$ with $x \neq y$. By Proposition 3, the convergence $(x_\alpha, x_\alpha) \to (x, y)$ is equivalence to the convergences $x_\alpha \to x$ and $x_\alpha \to y$. Hence, by Proposition 2, the existence of such a net and pair of points is equivalent to the failure of T to be Hausdorff. $\square$

Since we already know what products and closed inclusions of schemes are, we get the following analogue to Hausdorffness in the algebro-geometric perspective — however, since our products are defined relative to a chosen base scheme, this property is as well:

Definition 3. *Let $X \to Y$ be a map of schemes. The* **diagonal map** *of X over Y is the map $\Delta : X \to X \times_Y X$ induced by the identity map as in the following diagram:*

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

We say that $X \to Y$ is a **separated map** *(or that X is* **separated** *over Y) if Δ is a closed inclusion.*

(In general, Δ is only a locally closed inclusion.)

Hence we have an analogue to being Hausdorff for schemes over a fixed base scheme — in practice this is very often the spectrum of a chosen ground field, but in principle we can ask about separatedness for any map. The corresponding "relative Hausdorff property" in topology is not so widely studied, but it can be characterized in terms of nets:

Proposition 6. *Let $f : X \to Y$ be a map of topological spaces (i.e., a continuous one). Then the diagonal of the fiber product $X \times_Y X$ is closed if and only if, for each net x_α in X and point $y \in Y$ such that $f(x_\alpha) \to y$, there is at most one point $x \in f^{-1}(y)$ such that $x_\alpha \to x$.*

Proof. This is similar to the proof of Proposition 5; the failure of the diagonal to be closed in $X \times_Y X$ is, by Proposition 4, equivalent to the existence of a net (x_α, x_α) converging to a point $(x, x') \in X \times_Y X$ with $x \neq x'$; by the definition of the fiber product, we have $f(x) = f(x')$. Again, by Proposition 3, we can see that this is equivalent to the convergences $x_\alpha \to x$ and $x_\alpha \to x'$. Thus the failure of the diagonal to be closed is equivalent to the existence of a net violating our uniqueness condition. $\square$

Returning to the setting of schemes, we seek our first examples and non-examples of separatedness; to this end, we note the following:

Proposition 7. *Separatedness of maps is a local property on the target.*

Proof. Let $\phi : X \to Y$ be a map of schemes. Then, since fiber products are preserved under pullback, the restriction of $X \times_Y X$ over any open $U \hookrightarrow Y$ is precisely the fiber product $\phi^{-1}(U) \times_U \phi^{-1}(U)$, and the universal property guarantees also that the diagonal map $\phi^{-1}(U) \to \phi^{-1}(U) \times_U \phi^{-1}(U)$ is the restriction over U of the diagonal $\Delta : X \to X \times_Y X$. Since being a closed inclusion is local on the target, we can then see that Δ is a closed inclusion if and only if it is thus locally over Y. $\square$

We can now find many examples of separated maps:

Proposition 8. *Every affine map is separated.*

Proof. Let $X \to Y$ be an affine map. By the locality of separatedness on the target just established, we can reduce to the case where $Y = \operatorname{Spec} R$ is affine and hence $X = \operatorname{Spec} S$ is also thus, with $X \to Y$ induced by a ring map $R \to S$. $X \times_Y X$ is then the spectrum of $S \otimes_R S$, and we can see that the diagonal map corresponds to the ring map $S \otimes_R S \to S$ given by $a \otimes b \mapsto ab$ — hence the separatedness follows by observing that this ring map is surjective, so that the diagonal inclusion is closed. $\square$

On the other hand, we have the following standard example of a non-separated scheme:

Example 2. *We construct a $\mathbb{C}$-scheme X, called the "line with doubled origin", as follows. Begin with two copies $\operatorname{Spec} \mathbb{C}[x]$ and $\operatorname{Spec} \mathbb{C}[y]$ of the affine line and glue them together along the complements $\operatorname{Spec} \mathbb{C}[x]_x$ and $\operatorname{Spec} \mathbb{C}[y]_y$ of the origin in each using the identification induced by the $\mathbb{C}$-algebra isomorphism $\mathbb{C}[x]_x \to \mathbb{C}[y]_y$ with $x \mapsto y$. (Recall: Taking $x \mapsto y^{-1}$ instead is how we got the projective line over $\mathbb{C}$.)*

Then X is not separated over $\mathbb{C}$.

Observe that, if we make the same construction in the classical setting — that is, glue together two copies of $\mathbb{C}$ with the classical topology along $\mathbb{C}^*$ — the result will not be Hausdorff, since any net in $\mathbb{C}^*$ which approaches the origin will converge to two different points.

2 Proper Maps and Proper Maps

We now return to the topological setting to discuss some further properties which we will adapt to apply to schemes. As a first step, we develop the machinery of nets a bit more:

Definition 4. *Let T be a topological space and $(x_\alpha)_{\alpha\in A}$ a net in T. A* **subnet** *of x_α is given by a directed set B together with a preorder-preserving map $B \to A$ (in the sense that $b \leq b'$ implies that the same relation holds between these elements' images) with cofinal image; viewing x_α as a map $A \to T$, we interpret these data as a net according to the composition $B \to A \to T$.*

This gives us a convergence-theoretic reformulation of compactness which works in general:

Proposition 9. *Let T be a topological space. Then T is compact if and only if every net in T has a convergent subnet.*

Before we start to discuss scheme-theoretic versions, we want to develop a relative version of this notion. The most common one is as follows:

Definition 5. *A map $f : X \to Y$ of topological spaces is called* **proper** *if, for every compact subset $K \subseteq Y$, $f^{-1}(K)$ is also compact.*

However, we also have the following:

Definition 6. *A map $f : X \to Y$ of topological spaces is said to be* **closed** *if, for every closed subset $C \subseteq X$, the image $f(C)$ is closed in Y. f is said to be* **universally closed** *if, for every map $Y' \to Y$ of topological spaces, the natural pulled-back map $X \times_Y Y' \to Y'$ is closed.*

Reformulating things in terms of nets gives an indication as to why universal closedness should be considered a relative form of compactness:

Proposition 10. *Let $f : X \to Y$ be a map of topological spaces. Then f is universally closed if and only if, for each net x_α in X and point $y \in Y$ such that $f(x_\alpha) \to y$, x_α has a subnet converging to some point $x \in f^{-1}(y)$.*

Proof. For convenience, we will denote our convergence property by $(*)$.

Then we can see that f having $(*)$ implies that f is closed as follows. Let $C \subseteq X$ be closed and consider a net x_α in C and point $y \in Y$ such that $f(x_\alpha) \to y$. Then $(*)$ implies that there is a subnet of x_α converging to some $x \in f^{-1}(y)$; by the closedness of C and Proposition 4, it then follows that $f(C)$ is closed.

To show that $(*)$ implies universal closedness, it is now enough to verify that it is preserved under pullback. Let $g : Y' \to Y$ be continuous and denote by f' the pulled-back map $X \times_Y Y' \to Y'$. Then, by the construction of the fiber product of topological spaces, a net in $X \times_Y Y'$ will be of the form $((x_\alpha, y'_\alpha))_{\alpha\in A}$ with $(x_\alpha)_{\alpha\in A}$ a net in X and $(y'_\alpha)_{\alpha\in A}$ a net in Y' such that $f(x_\alpha) = g(y'_\alpha)$ for each $\alpha \in A$; moreover, $f'(x_\alpha, y'_\alpha) = y'_\alpha$, so that $f'(x_\alpha, y'_\alpha) \to y' \in Y'$ is equivalent to $y'_\alpha \to y'$.

Hence, for such a choice of net and point, if we let $y = g(y')$, we can see by the fact that f satisfies $(*)$ that $f(x_\alpha) = g(y'_\alpha) \to g(y') = y$ (using the fact that continuous maps preserve convergence) implies the existence of a point $x \in X$ with $f(x) = y$ such that some subnet of x_α converges to x. Then the corresponding subnet of (x_α, y'_α) converges to the point (x, y') by Proposition 3, so that f' satisfies $(*)$ and hence f is universally closed.

On the other hand, suppose f is universally closed to show that it satisfies $(*)$. Consider a net $(x_\alpha)_{\alpha \in A}$ in X and point $y \in Y$ such that $f(x_\alpha) \to y$.

Endow the set $A \sqcup \{\infty\}$ with the topology generated by the base $\{S \subseteq A\} \cup \big\{\{\infty\} \cup \{\beta \in A \mid \beta \geq \alpha\} \mid \alpha \in A\big\}$. Then, for any topological space T, a net in T indexed by A converges to a point if and only if sending ∞ to that point continuously extends the map $A \to T$ defining the net to $A \sqcup \{\infty\}$. (Note that the induced topology on A is discrete, so that any net corresponds to a continuous map.)

Hence, by our supposition that f be universally closed, we can see that the pullback $X \times_Y (A \sqcup \{\infty\}) \to (A \sqcup \{\infty\})$ of f along the continuous map induced by the convergence of the net $f(x_\alpha)$ to y is a closed map. Then, if we let $C = \overline{\{(x_\alpha, \alpha) \mid \alpha \in A\}} \subseteq X \times_Y (A \sqcup \{\infty\})$, we can see that image of C in $A \sqcup \{\infty\}$ is closed — since A is dense in $A \sqcup \{\infty\}$ and the image of C contains A, it follows that C contains some point (x, ∞), which will necessarily satisfy $f(x) = y$ by the definition of the fiber product. By Proposition 4, we then have some directed set B and net $\nu : B \to A$ such that $((x_{\nu(\beta)}, \nu(\beta)))_{\beta \in B}$ converges to (x, ∞).

We will use ν to construct a subnet of our original net converging to x — note, however, that ν itself does not define a subnet since there is no guarantee that it be order-preserving. (However, its image is cofinal by the convergence to ∞ given by Proposition 3.) Consider the product preorder on $A \times B$ — that is, $(\alpha, \beta) \leq (\alpha', \beta')$ if and only if $\alpha \leq \alpha'$ and $\beta \leq \beta'$ — and restrict it to the subset $\Gamma = \{(\alpha, \beta) \in A \times B \mid \nu(\beta) = \alpha\}$. Then the map $\Gamma \to A$ given by the natural projection is order-preserving, and we can see that it has cofinal image since $(\nu(\beta))_{\beta \in B}$ converges to ∞ in $A \sqcup \{\infty\}$. Moreover, the convergence of $(x_{\nu(\beta)})_{\beta \in B}$ to x guarantees that the net $\Gamma \to A \to X$ induced by x_α converges to x. Hence f satisfies $(*)$. □

As a corollary, we can now compare the two different notions of "relative compactness":

Corollary 1. *Let $f : X \to Y$ be a map of topological spaces. Then, if f is universally closed, it is proper as well. On the other hand, if f is proper and Y is a locally compact Hausdorff space, then f is universally closed.*

Proof. Suppose f is universally closed and let $K \subseteq Y$ be a compact subset. Then, if x_α is a net in $f^{-1}(K)$, there is a subnet of $f(x_\alpha)$ converging to some point $y \in K$ by Proposition 9; Proposition 10 then gives a point $x \in f^{-1}(y) \subseteq f^{-1}(K)$ such that some further subnet of this subnet converges to x. Thus $f^{-1}(K)$ is compact by Proposition 9.

On the other hand, suppose that f is proper and Y is a locally compact Hausdorff space. Consider a net x_α in X and point $y \in Y$ such that $f(x_\alpha) \to y$. By the local compactness of Y, we can choose some compact neighborhood $K \ni y$; properness then implies that $f^{-1}(K)$ is compact, and the convergence of $f(x_\alpha)$ implies that it is eventually in the interior of K. Hence x_α is eventually in $f^{-1}(K)$ and so, by compactness, it has a subnet converging to some $x \in f^{-1}(K)$. The continuity of f implies that the corresponding subnet of $f(x_\alpha)$ converges to $f(x)$; since Y is Hausdorff, Proposition 2 then guarantees that $f(x) = y$. As such, f is universally closed by Proposition 10. □

Proposition 10 suggests that, from the perspective of nets and convergence, universal closedness is really the "correct" version of "relative compactness" — by our corollary, however, the two coincide in the classically important setting of locally compact Hausdorff spaces, which perhaps explains why properness is the more widely-used one despite this.

Having set the stage, we now turn our attention to scheme-theoretic versions of these properties. The situation for universal closedness is fairly straightforward:

Definition 7. *Let $\phi : X \to Y$ be a map of schemes. We say that ϕ is* **closed** *if the underlying map of topological spaces is closed. We say that ϕ is* **universally closed** *if, for every map $Y' \to Y$ of schemes, the pulled-back map $X \times_Y Y' \to Y'$ is closed.*

For properness, unfortunately, we have the following definition:

Definition 8. *Let $\phi : X \to Y$ be a map of schemes. We say that ϕ is* **proper** *(or that X is a* **proper** *Y-scheme) if it is separated, finite type, and universally closed.*

This is not really the right analogue to the topological notion of properness on several counts — the finite type hypothesis seems unnecessarily restrictive, for one. Even beyond this, the separatedness requirement seems strange, since there is by no means any requirement for proper maps on the topological side to satisfy the analogous "relative Hausdorff condition" of Proposition 6, and the definition moreover reverses the relationship between universal closedness and properness when compared with the topological setting — in topology, the former implies the latter but not vice versa, while this definition makes the latter imply the former but not vice versa. Nevertheless, this terminology is standard, and so we will reluctantly adhere to it.

In the setting of finite-type separated $\mathbb{C}$-schemes, properness over $\operatorname{Spec}\mathbb{C}$ is the correct analogue to compactness in the classical topology and is sometimes referred to as *completeness*. The projective line $\mathbb{P}^1_{\mathbb{C}}$ we constructed in Lecture 2 is proper over $\mathbb{C}$ — we will see further examples in the weeks to come, when we discuss projective spaces and maps more generally.

3 Valuative Criteria

As promised, we will now develop scheme-theoretic analogues to our characterizations of Hausdorffness and universal closedness, given by Propositions 2 and 10 respectively, in terms of nets and convergence. Our first challenge in this regard is that looking at convergence of sequences and nets more generally in the underlying topological space does not really capture the behavior we're after, as we saw in Proposition 1. We have also the following example:

Example 3. *Consider the affine line $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t]$ and the sequence of closed points $(x_n)_{n\in\mathbb{N}}$ in X given by $x_n = V(t-n)$. Then, for any point $x \in X$, closed or otherwise, the sequence x_n converges to x.*

To see this, observe that our nonempty distinguished affine open subsets of X, being given by the complements of the vanishing loci of univariate polynomials, will be precisely those of the form $X \setminus \{r_1, \ldots, r_\ell\}$ for $r_1, \ldots, r_\ell$ closed points. In particular, for any fixed such subset

U, x_n is eventually in U by the finiteness of the set of points removed. Since the distinguished affine opens form a base for the topology, it thus follows that our sequence converges to every point, as claimed.

(Note that we could just as well have used any sequence of distinct closed points to get the same result.)

Similar examples can be found in more complicated schemes — overall, the point is that convergence in the Zariski topology doesn't really capture the geometry of schemes in the senses that we would like to think of intuitively. To proceed, we examine the following special cases of convergence in the topological setting:

Example 4. *Let T be a topological space and $\gamma : (0,1) \to T$ a continuous map, defining an open path in T. Then, if we endow the open interval $(0,1)$ with the reverse of the usual total ordering on $\mathbb{R}$, we can interpret it as a directed set and thus regard γ as defining a net in T. Under these circumstances, the net so defined will converge to a point $x \in T$ if and only if setting $\gamma(0) = x$ continuously extends γ to a map $[0,1) \to T$.*

Example 5. *Let T be a topological space and $D^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ the punctured unit disk in $\mathbb{C}$. Endow D^* with the preorder given by $z \leq z' \Leftrightarrow |z| \geq |z'|$. If we are given a continuous map $\gamma : D^* \to T$, then the net defined by thus interpreting D^* as a directed set will converge to a point $x \in T$ if and only if setting $\gamma(0) = x$ continuously extends γ to a map from $D := \{z \in \mathbb{C} \mid 0 \leq |z| < 1\}$ to T.*

Thus we can phrase problems about the existence of limit points for real and complex open paths in terms of the convergence of nets. Although we will not pursue the matter in detail here, it turns out that in some circumstances it is enough to consider nets of this form for many practical purposes, just as there nice classes of spaces for which we need only consider convergence of *sequences*, rather than arbitrary nets, to characterize various topological notions. In particular, the *curve selection lemma* from analytic (or, more generally, subanalytic) geometry is of this flavor.

Now, the notion of maps of smooth curves, punctured or otherwise, into a space does translate well to algebraic geometry, since we have already developed theories of dimension and (non)singularity. As in the case of sequences for topological spaces, however, considering these alone will not quite give the results we are looking for in general, and so we will need to consider a more general but less intuitive class of objects. We begin to develop the theory as follows:

Definition 9. *A* **totally ordered abelian group** *is an abelian group $(\Gamma, +)$ with a total ordering $\leq$ such that, for all $a, b, c \in \Gamma$ with $a \leq b$, we have $a + c \leq b + c$.*

Definition 10. *A* **monoid-with-zero** *is a monoid $(M, \cdot)$ with a* **zero element** *$0 \in M$ such that $0 \cdot m = m \cdot 0 = 0$ for all $m \in M$. A* **map of monoids-with-zero** *is a monoid map which takes the zero element to the zero element. The* **kernel** *of such a map of the preimage of the zero element in the target; if this preimage is $\{0\}$, we say that the map has* **trivial kernel**.

Definition 11. *Let K be a field and Γ a totally ordered abelian group. Then, if we regard $\Gamma \sqcup \{\infty\}$ as an abelian monoid-with-zero by setting $\infty + g = g + \infty = \infty$ for all $g \in \Gamma$*

and extend the total ordering by stipulating $g < \infty$ for all $g \in \Gamma$, a **valuation** *on K is a map $v : (K, \cdot) \to (\Gamma \sqcup \{\infty\}, +)$ of monoids-with-zero (i.e., $v(1) = 0$, $v(0) = \infty$, and $v(ab) = v(a) + v(b)$ for $a, b \in K$) such that:*

- *v is surjective.*
- *v has trivial kernel. (I.e., $v(a) = \infty \Rightarrow a = 0$ for all $a \in K$.)*
- *$v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in K$.*

We call a valuation **discrete** *if $\Gamma = \mathbb{Z}$.*

The following example is instructive in developing intuition for valuations:

Example 6. *Let $K = \mathbb{C}(x)$. Then each nonzero $f \in K$ can be written as $cx^r \frac{(x-a_1)\cdots(x-a_n)}{(x-b_1)\cdots(x-b_m)}$ for some $r \in \mathbb{Z}$ and $a_1, \ldots, a_n, b_1, \ldots, b_m, c \in \mathbb{C}^*$; the map $v : K^* \to \mathbb{Z}$ given by $v(f) = r$ results in a well-defined discrete valuation on K (when extended to a map $v : K \to \mathbb{Z} \sqcup \{\infty\}$ by assigning $v(0) = \infty$).*

That is, we think of K as the field of rational functions on the affine line over $\mathbb{C}$ (i.e., the residue field at the generic point) and our valuation as giving the order of vanishing of each such function at the origin, with negative orders of vanishing indicating poles.

Your intuition in general should be similar — think of discrete valuations as defining "orders of vanishing of field elements at specified points", and arbitrary valuations as doing this for some suitable generalization of the notion of an order of vanishing to other totally ordered abelian groups. Of course, since the spectrum of K itself is just a single point at which no nonzero functions vanish, this raises the question: How do we retrieve the space(s) these "points" are points *of*? This can be accomplished for a given valuation as follows:

Definition 12. *Let K be a field, Γ a totally ordered abelian group, and $v : K \to \Gamma \sqcup \{\infty\}$ a valuation. Then the* **valuation ring** *of v is $\mathcal{O}_v := \{f \in K \mid v(f) \geq 0\}$. A ring that can be realized as the valuation ring of a discrete valuation is called a* **discrete valuation ring***, or* **DVR** *for short.*

Proposition 11. *Let K be a field, Γ a totally ordered abelian group, and $v : K \to \Gamma \sqcup \{\infty\}$ a valuation. Then $\mathcal{O}_v$ is a local ring with maximal ideal $\{f \in K \mid v(f) > 0\}$. Moreover, it is a domain with fraction field K.*

Proof. In general, for $f \in K^*$, we have $0 = v(1) = v(ff^{-1}) = v(f) + v(f^{-1})$, so that $v(f^{-1}) = -v(f)$. If $v(f) = 0$, this implies that $v(f^{-1}) = 0$ as well, so all elements outside the set $\{f \in K \mid v(f) > 0\}$ are units. Moreover, the valuation axiom for sums of field elements guarantees that it is closed under addition, and the fact that the valuation is a monoid map guarantees that it is closed under multiplication by elements of $\mathcal{O}_v$, since $v(fg) = v(f) + v(g) \geq v(f)$ for $f, g \in \mathcal{O}_v$. Hence our set is an ideal of $\mathcal{O}_v$ with complement exactly the set of units; this makes it the unique maximal ideal of the ring.

That $\mathcal{O}_v$ is a domain follows by the fact that it is a subring of the field K; moreover, this guarantees that K is an extension of the field of fractions of $\mathcal{O}_v$. However, our equality $v(f^{-1}) = -v(f)$ implies that every element of $K \setminus \mathcal{O}_v$ is the inverse of some element of $\mathcal{O}_v$, so K must be the field of fractions itself. $\square$

Thus we can think of our elements of K as rational functions on the integral scheme $\operatorname{Spec} \mathcal{O}_v$, with the valuation v giving some notion of order of vanishing at the unique closed point. We can see this principle in action in the case of our prior example:

Example 7. *Let $K = \mathbb{C}(x)$ and consider the valuation v defined in Example 6. Then $\mathcal{O}_v = \mathbb{C}[x]_{(x)}$ is the local ring of $\mathbb{A}^1_{\mathbb{C}}$ at the origin.*

The geometric importance of discrete valuation rings in particular is explained by the following result:

Proposition 12. *Let R be a ring. The following are equivalent:*

1. *R is a discrete valuation ring.*
2. *R is a local principal ideal domain, but not a field.*
3. *R is a regular local ring of Krull dimension 1.*

Proof. We prove that each condition implies the next.

Suppose first that $R = \mathcal{O}_v$ for some valuation $v : K \to \mathbb{Z} \sqcup \{\infty\}$ on a field K. We then know already that R is a local domain with field of fractions K by Proposition 11; since valuations are surjective, moreover, there exist negatively-valued elements of K and so R cannot be a field. To show that every ideal of R is principal, consider $I \subseteq R$ an ideal and let $n \in \mathbb{N} \sqcup \{\infty\}$ be the least value of an element $f \in I$. Then, for such a minimal f and $g \in R$ with $v(g) \geq v(f)$, we can see that $v(\frac{g}{f}) = v(g) - v(f) \geq 0$, so $\frac{g}{f} \in R$ and hence $\frac{g}{f} \cdot f = g$ is in the ideal I. Thus $I = \{g \in K \mid v(g) \geq n\} = (f)$, which is principal, as desired.

Now suppose that R is a local principal ideal domain which is not a field. Then, in particular, the maximal ideal $\mathfrak{m}$ is principal, so we can write $\mathfrak{m} = (t)$ for some $t \in R$. Since principal ideal domains are Noetherian, we can see that the Krull dimension of R is bounded above by $\dim_{R/(t)} (t)/(t)^2 = \dim_{R/(t)} R/(t) = 1$. Moreover, since $(0) \subset (t)$ is a chain of prime ideals, $\dim R \geq 1$ as well; as a result, we find that R is regular of Krull dimension 1.

Finally, if R is a regular local ring of Krull dimension 1, we can see that its maximal ideal $\mathfrak{m}$ is generated by one element, say $\mathfrak{m} = (t)$. For $f \in R$, we now set $v(f) := \max\{n \in \mathbb{N} \mid f \in (t^n)\}$, where the maximum is taken to be ∞ if the set is unbounded. By a result from commutative algebra, the *Krull intersection theorem* (for which see, e.g., Section 5.3 of Eisenbud's *Commutative Algebra with a View Toward Algebraic Geometry*), $v(f) = \infty$ if and only if $f = 0$.

Observe now that every nonzero $f \in R$ is equal to $ut^{v(f)}$ for some unit u of R; since $f \in (t^{v(f)})$, we see immediately that this is the case for some u, and $u \notin (t)$ since otherwise we would have $f \in (t^{v(f)+1})$, contradicting the definition of v; since (t) is the maximal ideal of the local ring R, it follows that u is a unit. This implies in particular that R is an integral domain; were it not, it would then follow that t is nilpotent, contradicting the primality of (t). (It is also true more generally that regular local rings of any dimension are domains, as one would expect from the geometric picture.) The field of fractions is thus $K := R_t$, and we can extend v to K by setting $v(f/t^\ell) := v(f) - \ell$ for all $f \in R$ and $\ell \geq 0$. We claim that v now defines a discrete valuation on K with valuation ring R.

As we have already mentioned, $v(0) = \infty$, and we can see that $v(1) = 0$ and $v(ab) = v(a) + v(b)$ for all $a, b \in K$ straightforwardly from the definitions and the description of elements of R. Hence v is a map of monoids-with-zero; this can be seen to be surjective onto $\mathbb{Z} \sqcup \{\infty\}$ by considering appropriate powers of t and, as already noted, the Krull intersection theorem guarantees that its kernel is trivial. Moreover, for $a, b \in K$, if we write $a = f/t^\ell$ and $b = g/t^m$, we have $a + b = \frac{t^m f + t^\ell g}{t^{\ell+m}}$; the numerator is contained in $t^{\min(m+v(f),\ell+v(g))}$, so $v(a+b) \geq \min(m + v(f), \ell + v(g)) - (\ell + m) = \min(v(f) - \ell, v(g) - m) = \min(v(a), v(b))$, as needed. Hence v is a discrete valuation, and we can see from our description of elements of R that $v(a) \geq 0$ for $a \in K$ if and only if $a \in R$, so $R = \mathcal{O}_v$.

The result follows. $\square$

Hence discrete valuation rings are precisely the local rings of nonsingular curves at closed points, and so in particular a map from the spectrum of a DVR to a scheme should be thought of as a kind of scheme-theoretic analogue to the notion of a path in a topological space (or, more accurately, an equivalence class of paths under restrictions to arbitrarily small neighborhoods of the origin in the domain, called a *path-germ*). Moreover, our proof shows for a DVR R that the spectrum of the field of fractions $K = R_t$ is simply the complement the closed point in $\operatorname{Spec} R$; that is, even though it is simply a one-point space, $\operatorname{Spec} K$ should be thought of in some sense as a "punctured curve(-germ)", and maps from it to a given scheme as the scheme-theoretic analogues to "open path(-germ)s" of the sort seen in Examples 4 and 5. Hence, as previously alluded to, we can adapt questions about the convergence of nets, at least those given by paths, to the scheme-theoretic setting by asking instead about the extensibility of maps from $\operatorname{Spec} K$ to a given scheme over the closed point of $\operatorname{Spec} R$.

On the other hand, valuations with values in groups other than $\mathbb{Z}$ produce rings which are not quite so nice in general, and need not be Noetherian or 1-dimensional. It is perhaps best to think of arbitrary valuation rings as being similar to arbitrary directed sets in topology — just as nets are less intuitive generalizations of sequences/paths which retain enough of their features to make sense of convergence questions, maps from spectra of valuation rings will be a less intuitive generalization of maps from smooth curve-germs which are likewise still similar enough to be useful for our desired characterization of properties like separatedness and properness in terms of "scheme-theoretic convergence".

With this in mind, we are now prepared to give our scheme-theoretic version of Proposition 6:

Proposition 13 (valuative criterion of separatedness)**.** *Let $\pi : X \to Y$ be a map of schemes such that X is Noetherian (or, more generally, π is quasiseparated — see Lecture 7). Then π is separated if and only if, for every choice of valuation ring $\mathcal{O}_v$ with field of fractions K and maps $\omega : \operatorname{Spec} K \to X$ and $\mu : \operatorname{Spec} \mathcal{O}_v \to Y$ such that $\mu|_{\operatorname{Spec} K} = \pi \circ \omega$, there is at most one map $\bar{\omega} : \operatorname{Spec} \mathcal{O}_v \to X$ extending ω such that $\pi \circ \bar{\omega} = \mu$. This can be expressed in the following commutative diagram:*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\omega"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec} \mathcal{O}_v \arrow[r, "\mu"] \arrow[ur, dashed, "\nexists\bar{\omega} \text{ or } \exists!\bar{\omega}"] & Y
\end{tikzcd}$$

If Y is locally Noetherian and π is finite type, it suffices to consider only discrete valuation rings.

As in the topological case, we see that working only with our more intuitive class of objects (sequences in topology; DVRs in scheme theory) is possible when working with nice spaces and maps of the sort we are most commonly interested in anyway.

Also as in topology, we find that characterizing things in terms of convergence can make many results more intuitive to prove:

Corollary 2. *Open and closed inclusions are separated, separatedness is preserved under base change and composition, and separatedness is local on the target.*

For a proof sketch (and more properties which follow from the valuative criterion), see Corollary II.4.6 of Hartshorne (although note the Noetherian hypotheses).

Likewise, we have a scheme-theoretic version of Proposition 10:

Proposition 14 (valuative criterion of universal closedness)**.** *Let $\pi : X \to Y$ be a map of schemes such that X is Noetherian (or, more generally, π is quasicompact — see, again, Lecture 7). Then π is universally closed if and only if, for every choice of valuation ring $\mathcal{O}_v$ with field of fractions K and maps $\omega : \operatorname{Spec} K \to X$ and $\mu : \operatorname{Spec} \mathcal{O}_v \to Y$ such that $\mu|_{\operatorname{Spec} K} = \pi \circ \omega$, there is at least one map $\bar{\omega} : \operatorname{Spec} \mathcal{O}_v \to X$ extending ω such that $\pi \circ \bar{\omega} = \mu$. This can be expressed in the following commutative diagram:*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\omega"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec} \mathcal{O}_v \arrow[r, "\mu"] \arrow[ur, dashed, "\exists\bar{\omega}"] & Y
\end{tikzcd}$$

If Y is locally Noetherian and π is finite type, it suffices to consider only discrete valuation rings.

Note that here we have a distinction from the analogous topological result — our scheme-theoretic versions of nets are rigid enough that we do not need any analogue to the notion of passing to a subnet to characterize universal closedness.

The preceding valuative criteria now combine to give us the following result for finite-type maps:

Proposition 15 (valuative criterion of properness)**.** *Let $\pi : X \to Y$ be a finite-type map of schemes such that X is Noetherian (or, more generally, π is quasiseparated). Then π is proper if and only if, for every choice of valuation ring $\mathcal{O}_v$ with field of fractions K and maps $\omega : \operatorname{Spec} K \to X$ and $\mu : \operatorname{Spec} \mathcal{O}_v \to Y$ such that $\mu|_{\operatorname{Spec} K} = \pi \circ \omega$, there is exactly one map $\bar{\omega} : \operatorname{Spec} \mathcal{O}_v \to X$ extending ω such that $\pi \circ \bar{\omega} = \mu$. This can be expressed in the following commutative diagram:*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\omega"] \arrow[d, hook] & X \arrow[d, "\pi"] \\
\operatorname{Spec} \mathcal{O}_v \arrow[r, "\mu"] \arrow[ur, dashed, "\exists!\bar{\omega}"] & Y
\end{tikzcd}$$

If Y is locally Noetherian, it suffices to consider only discrete valuation rings.

The corresponding result for proper maps on the topological side does not hold since, as mentioned, the topological and scheme-theoretic notions of "properness" are not really analogous to one another.

As in the case of separatedness, we again get desirable properties of properness as corollaries to this characterization — see Hartshorne's Corollary II.4.8 for a more fleshed-out version (with, again, some not-strictly-needed Noetherian hypotheses):

Corollary 3. *Closed inclusions are proper, properness is preserved under base change and composition, and properness is local on the target.*