New families of strongly regular graphs

S.G. Barwick, Wen-Ai Jackson and Tim Penttila

November 23, 2015

Abstract

In this article we construct a series of new infinite families of strongly regular graphs with the same parameters as the point-graphs of non-singular quadrics in $\mathrm{PG}(n, 2)$.

1 Introduction

A strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$, is a graph with v vertices such that each vertex lies on k edges; any two adjacent vertices have exactly λ common neighbours; and any two non-adjacent vertices have exactly μ common neighbours.

In [4], Godsil-McKay take a graph Γ, and use a vertex partition to construct a new graph Γ' that has the same spectrum as Γ. It is well-known (see for example [3]) that if a graph Γ' has the same spectrum as a strongly regular graph Γ, then Γ' is also strongly regular with the same parameters as Γ. Specialising the Godsil-McKay construction to a partition of size two in a strongly regular graph gives the following result.

Result 1.1 The Godsil-McKay construction. *Let Γ be a strongly regular graph and partition the vertices into two sets $\mathcal{X}, \mathcal{Y}$. Then $\{\mathcal{X}, \mathcal{Y}\}$ is called a* Godsil-McKay partition *if the following two conditions are satisfied.*

I. *The set $\mathcal{X}$ induces a regular subgraph.*
II. *Each vertex in $\mathcal{Y}$ is adjacent to 0, $\frac{1}{2}|\mathcal{X}|$ or $|\mathcal{X}|$ vertices in $\mathcal{X}$.*

Construct a new graph Γ' from Γ by: for each vertex R in $\mathcal{Y}$ with $\frac{1}{2}|\mathcal{X}|$ neighbours in $\mathcal{X}$, delete these $\frac{1}{2}|\mathcal{X}|$ edges and join R to the other $\frac{1}{2}|\mathcal{X}|$ vertices in $\mathcal{X}$. Then the new graph Γ' is strongly regular with the same parameters as Γ.

We consider the strongly regular graphs constructed from a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n,q)$, see [6, Chapter 22] for more information on quadrics. The *projective index* g of $\mathcal{Q}_n$ is the dimension of the largest subspace contained in $\mathcal{Q}_n$. Note that $\mathcal{Q}_n$ is a polar space of rank $g+1$. If $n=2r$ is even, then a non-singular quadric is a parabolic quadric, denoted $\mathcal{P}=Q(2r,q)$, which has projective index $g=r-1$. If $n=2r+1$ is odd, then there are two types of non-singular quadrics. The elliptic quadric denoted $\mathcal{E}=Q^-(2r+1,q)$ has projective index $g=r-1$. The hyperbolic quadric denoted $\mathcal{H}=Q^+(2r+1,q)$ has projective index $g=r$.

The *point-graph* of the non-singular quadric $\mathcal{Q}_n$ is denoted by Γ or $\Gamma_{\mathcal{Q}_n}$, and is defined as follows. The vertices of Γ are the points of $\mathcal{Q}_n$, and two vertices are adjacent in Γ if the corresponding points of $\mathcal{Q}_n$ lie on a line contained in $\mathcal{Q}_n$. It is well known (see for example [2]) that Γ is a strongly regular graph.

The article proceeds as follows. Section 2 describes our construction of a series of infinite families of strongly regular graphs, the proof of the construction is given in Section 3. In Section 4, we classify and count the maximal cliques in these constructed graphs. Section 5 looks at isomorphism, and shows that our construction yields new families of strongly regular graphs. and the last auto section

2 Our construction

We begin with a small example to illustrate the general technique.

Example 2.1 Let ℓ be a line of the elliptic quadric $\mathcal{E}=Q^-(2r+1,q)$ in $\mathrm{PG}(2r+1,q)$. Define a new graph Γ_1 with the following vertices of three types; and edges given in Table 1.

(i) points of $\mathcal{E}$ on ℓ,
(ii) points of $\mathcal{E}$ that are on a plane of $\mathcal{E}$ that contains ℓ,
(iii) the remaining points of $\mathcal{E}$.

Note that it can be shown using geometric techniques that Γ_1 is regular if and only if $q=2$, and that in this case Γ_1 is strongly regular with the same parameters as the point-graph $\Gamma_{\mathcal{E}}$ of $\mathcal{E}$. This can also be proved using the Godsil-McKay construction as follows. The graph Γ_1 is constructed from $\Gamma_{\mathcal{E}}$ by altering the edges through points Q of type (ii), and R of type (iii). Consider the partition $\{\mathcal{X},\mathcal{Y}\}$ of $\Gamma_{\mathcal{E}}$ where $\mathcal{X}$ contains the vertices of type (ii), and $\mathcal{Y}$ contains the vertices of type (i) and (iii). Then geometric techniques can be used to show that this partition satisfies the conditions of Result 1.1 if and only if $q=2$, and so Γ_1 is strongly regular when $q=2$. □

We now give our general construction of a series of infinite families of strongly regular

Table 1: Edges in Γ_1

Vertex pair	Vertex types	Vertex pair is an edge of Γ_1:
P, P'	P, P' are type (i)	always (note PP' is a line of $\mathcal{E}$)
P, Q	P is type (i), Q is type (ii)	always (note PQ is a line of $\mathcal{E}$)
Q, Q'	Q, Q' are type (ii)	when QQ' is a line of $\mathcal{E}$
P, R	P is type (i), R is type (iii)	when PR is a line of $\mathcal{E}$
R, R'	R, R' are type (iii)	when RR' is a line of $\mathcal{E}$
Q, R	Q is type (ii), R is type (iii)	when QR is a 2-secant of $\mathcal{E}$

graphs. First we define a partition of the vertices of the point-graph of $\mathcal{Q}_n$.

Definition 2.2 *In* $\mathrm{PG}(n, q)$*, let* $\mathcal{Q}_n$ *be a non-singular quadric with projective index* g*, and let* Γ *be the point-graph of* $\mathcal{Q}_n$*. For each integer* s *with* $0 \le s < g$ *(where* g *is the projective index), let* α_s *be an* s*-dimensional subspace that is contained in* $\mathcal{Q}_n$*. Let*

- $\mathcal{X}_s$ *be the vertices of* Γ *which correspond to points of* $\mathcal{Q}_n$ *that: do not lie in* α_s*; and lie in an* $(s+1)$*-dimensional subspace that contains* α_s *and is contained in* $\mathcal{Q}_n$*;*
- $\mathcal{Y}_s$ *be the remaining vertices of* Γ *(note that this includes the points of* α_s*).*

Note that if $s = g$, then $\mathcal{X}_s$ is empty, so we need $s < g$. We show in Section 3 that $\{\mathcal{X}_s, \mathcal{Y}_s\}$ is a Godsil-McKay partition if and only if $q = 2$. Hence using Result 1.1, we can construct another strongly regular graph. The main result of this article is to prove the following result (Section 3) and to determine when we obtain new strongly regular graphs (Section 5).

Theorem 2.3 *In* $\mathrm{PG}(n, 2)$*, let* $\mathcal{Q}_n$ *be a non-singular quadric of projective index* $g \ge 1$ *with point-graph* Γ*. For each integer* s*,* $0 \le s < g$*, the partition* $\{\mathcal{X}_s, \mathcal{Y}_s\}$ *defined in Definition 2.2 is a Godsil-McKay partition. Hence the graph* Γ_s *obtained using the Godsil-McKay construction with the partition* $\{\mathcal{X}_s, \mathcal{Y}_s\}$ *is a strongly regular graph with the same parameters as* Γ*.*

When working with Γ_s, it is useful to partition the vertices of the graphs Γ, Γ_s (and so the corresponding points of $\mathcal{Q}_n$) into three sets with respect to the s-dimensional subspace α_s (as we did in Example 2.1).

- Vertices of type (i) correspond to points in α_s.
- Vertices of type (ii) correspond to points that lie in an $(s+1)$-dimensional subspace that is contained in $\mathcal{E}$ and contains α_s (and are not of type (i)).
- Vertices of type (iii) correspond to the remaining points of $\mathcal{E}$.

Note that $\mathcal{X}_s$ contains all the vertices of type (ii), and $\mathcal{Y}_s$ contains all the vertices of type (i) and (iii).

3 Proof of Theorem 2.3

In this section, let $\mathcal{Q}_n$ non-singular quadric in $\mathrm{PG}(n,q)$ with projective index $g = r-1$. Let α_s be a subspace of dimension s, $0 \le s < g$, contained in $\mathcal{Q}_n$. Let $\{\mathcal{X}_s, \mathcal{Y}_s\}$ be the partition defined on the point-graph Γ of $\mathcal{Q}_n$, as in Definition 2.2. We show that $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Conditions I and II of Result 1.1. First we count the points in $\mathcal{X}_s$.

Lemma 3.1 *Let x be the number of points in $\mathcal{X}_s$, then*

1. *if* $\mathcal{Q}_n = Q^-(2r+1,q)$, $x = \dfrac{q^{s+1}(q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}$;
2. *if* $\mathcal{Q}_n = Q^+(2r+1,q)$, $x = \dfrac{q^{s+1}(q^{r-s-1}+1)(q^{r-s}-1)}{(q-1)}$;
3. *if* $\mathcal{Q}_n = Q(2r,q)$, $x = \dfrac{q^{s+1}(q^{r-s-1}+1)(q^{r-s-1}-1)}{(q-1)}$.

Proof We prove this in the case $\mathcal{Q}_n$ is $\mathcal{E} = Q^-(2r+1,q)$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H} = Q^-(2r+1,q)$ and $\mathcal{P} = Q(2r,q)$ are proved in a very similar manner.

By [6, Theorem 22.5.1], the number of subspaces of dimension s contained in $\mathcal{E}$ is

$$\frac{\Big((q^{r-s+1}+1)(q^{r-s+2}+1)\cdots(q^{r+1}+1)\Big)\times\Big((q^{r-s}-1)(q^{r-s+1}-1)\cdots(q^{r}-1)\Big)}{(q-1)(q^2-1)\cdots(q^{s+1}-1)}$$

(note this can also be used to count the number of subspaces of dimension $s+1$ contained in $\mathcal{E}$, which we will also need). Further, [5, Theorem 3.1] shows that the number of subspaces of dimension s in a subspace of dimension $s+1$ is

$$\frac{q^{s+2}-1}{q-1}.$$

By [6], the number of subspaces of dimension $s+1$ that contain α_s and are contained in $\mathcal{E}$ is a constant. To calculate it, we count pairs (Π, Σ) where Π is an s-dimensional subspace contained in $\mathcal{E}$, Σ is an $(s+1)$-dimensional subspace contained in $\mathcal{E}$, and $\Pi \subset \Sigma$. This count gives the number of subspaces of dimension $s+1$ that contain α_s and are contained in $\mathcal{E}$ is

$$\frac{(q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}. \tag{1}$$

Each of these subspace contains q^{s+1} points that are not in α_s. Hence $|\mathcal{X}_s|$ is (1) times q^{s+1} as required. □

Next we show that $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Condition I of Result 1.1.

Lemma 3.2 *Let G be the subgraph of Γ on the vertices in $\mathcal{X}_s$. Then G is a regular graph with degree k where*

1. *if $\mathfrak{Q}_n = Q^-(2r+1,q)$, then $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-1}+1)(q^{r-s-2}-1)}{(q-1)}$;*
2. *if $\mathfrak{Q}_n = Q^+(2r+1,q)$, then $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-2}+1)(q^{r-s-1}-1)}{(q-1)}$;*
3. *if $\mathfrak{Q}_n = Q(2r,q)$, then $k = (q^{s+1}-1) + \dfrac{q^{s+2}(q^{r-s-2}+1)(q^{r-s-2}-1)}{(q-1)}$.*

Proof We prove this in the case $\mathfrak{Q}_n$ is $\mathcal{E} = Q^-(2r+1,q)$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathfrak{Q}_n$ is $\mathcal{H} = Q^-(2r+1,q)$ and $\mathcal{P} = Q(2r,q)$ are proved in a very similar manner. Let Q be a vertex in $\mathcal{X}_s$, so in $\mathrm{PG}(2r+1,q)$, Q is a point of $\mathcal{E}$ such that the $(s+1)$-dimensional space $\Sigma = \langle Q, \alpha_s\rangle$ is contained in $\mathcal{E}$. A vertex Q' in $\mathcal{X}_s$ is adjacent to Q if the line QQ' is contained in $\mathcal{E}$. We can partition the lines of $\mathcal{E}$ through Q into three groups: A contains the lines of $\mathcal{E}$ through Q that lie in Σ; B contains the lines of $\mathcal{E}$ through Q (not in A) that lie in an $(s+2)$-dimensional subspace that contains Σ and is contained in $\mathcal{E}$; and C contains the remaining lines of $\mathcal{E}$ through Q.

The number of lines in A is the number of lines through a point in an s-dimensional subspace should the s be $s+1$?, which by [5, Theorem 3.1] is

$$\frac{(q^{s+1}-1)}{(q-1)}. \tag{2}$$

Each of the lines in A contains Q and meets α_s in one point. So each line in A gives rise to $q-1$ vertices in $\mathcal{X}_s$ which are adjacent to Q in G. In total, A contributes $(q-1)\times|A| = (q^{s+1}-1)$ neighbours of Q in G.

The count in (1) can be used to show that the number of subspace of dimension $s+2$ that contain the $(s+1)$-space $\Sigma = \langle Q, \alpha_s\rangle$ and are contained in $\mathcal{E}$ is $(q^{r-s-1}+1)(q^{r-s-2}-1)/(q-1)$. Shall we remove from here Note that this number is 0 if $s = g-1$. to here, as discussed later? Similarly, (2) can be generalised to show that the number of lines through Q that lie in a subspace of dimension $s+2$, and do not lie in the $(s+1)$-space Σ is $\Big((q^{s+2}-1)/(q-1)\Big) - \Big((q^{s+1}-1)/(q-1)\Big) = q^{s+1}$. Hence

$$|B| = q^{s+1} \times \frac{(q^{r-s-1}+1)(q^{r-s-2}-1)}{(q-1)}.$$

Each line in B contains one point of Σ and the remaining q points correspond to q vertices that lie in $\mathcal{X}_s$ (and are not considered in A). That is, each line in B contributes q neighbours to Q in the graph G. So in total, B contributes $q \times |B| = q^{s+2}(q^{r-s-1}+1)(q^{r-s-2}-1)/(q-1)$ neighbours to Q in the graph G.

Let ℓ be a line in C, so ℓ contains Q, but the $(s+2)$-space $\Pi = \langle \alpha_s, \ell \rangle$ is not contained in $\mathcal{E}$. Suppose that ℓ contains another point Q' that corresponds to a vertex in $\mathcal{X}_s$. Then $\Pi \cap \mathcal{E}$ contains the two distinct $(s+1)$-dimensional subspaces $\Sigma = \langle \alpha_s, Q \rangle$ and $\Sigma' = \langle \alpha_s, Q' \rangle$. As Π is not contained in $\mathcal{E}$, Π meets $\mathcal{E}$ in exactly Σ, Σ'. Thus ℓ is not a line of $\mathcal{E}$. So ℓ contains exactly two points Q, Q' that correspond to vertices of $\mathcal{X}_s$, and they are not adjacent in G (as $QQ' = \ell$ is not a line of $\mathcal{E}$). Thus C contributes 0 neighbours to Q in the graph G.

Finally, summing the neighbours to Q obtained from cases A, B, C gives the required result. Note that if $s = g-1$, and so $s = r-2$, the second term is zero, and so the degree of G is $q^{r-1}-1$. $\square$

Now we look at Condition II of Result 1.1.

Lemma 3.3 *The partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$ satisfies Condition II of Result 1.1 if and only if $q = 2$.*

Proof We prove this in the case $\mathcal{Q}_n$ is $\mathcal{E} = Q^-(2r+1, q)$, which has projective index $g = r-1$ and point-graph denoted $\Gamma_{\mathcal{E}}$. The cases when $\mathcal{Q}_n$ is $\mathcal{H} = Q^-(2r+1, q)$ and $\mathcal{P} = Q(2r, q)$ are proved in a very similar manner.

We show that each vertex in $\mathcal{Y}_s$ is adjacent to 0, $\frac{1}{2}|\mathcal{X}_s|$ or $\frac{1}{2}|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$.

First consider a vertex P in $\mathcal{Y}_s$ of type (i). For every point Q of type (ii), PQ is a line of $\mathcal{E}$, so P, Q are adjacent vertices in $\Gamma_{\mathcal{E}}$. That is, each vertex of type (i) in $\mathcal{Y}_s$ is adjacent to the $|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$.

Now consider a vertex R in $\mathcal{Y}_s$ of type (iii). We count the number of vertices Q of type (ii) for which RQ is a line of $\mathcal{E}$. We will show that this is $\frac{1}{2}|\mathcal{X}_s|$ if and only if $q = 2$. Let Σ be a subspace of $\mathcal{E}$ of dimension $s+1$ that contains α_s. So $\Sigma \backslash \alpha_s$ consists of points of type (ii), hence $R \notin \Sigma$. Consider the $(s+2)$-space $\Pi = \langle \Sigma, R \rangle$. As R is of type (iii), $\langle \alpha_s, R \rangle$ is not contained in $\mathcal{E}$, so Π is not contained in $\mathcal{E}$. Further $\Pi \cap \mathcal{E}$ contains the $(s+1)$-space Σ and the point $R \notin \Sigma$. Hence $\Pi \cap \mathcal{E}$ is two distinct $(s+1)$-spaces (I think we might need a reference for this - just noticed that Property * in Section 4 might be what we need - or something equivalent?), one is Σ, the other we denote by Σ'. As Σ' contains R of type (iii), Σ' does not contain α_s. Hence $\Sigma' \cap \Sigma = \Omega$ is an s-space distinct from α_s. Consider a line m joining R to a point Q in $\Omega \backslash (\Omega \cap \alpha_s)$, and note that Q' (should this be Q?)is type (ii). As $m \subseteq \Omega \subset \mathcal{E}$ - should this be $m \subseteq \Sigma' \subset \mathcal{E}$, m is a line of $\mathcal{E}$. As Π is not in $\mathcal{E}$, m contains a unique point of type (ii), namely Q. So the remaining points of m are of type (iii). That is, in the graph $\Gamma_{\mathcal{E}}$, m gives rise to one neighbour of R in $\mathcal{X}_s$, namely Q. Thus each point in $\Omega\backslash(\Omega\cap\alpha_s)$ gives a unique neighbour of type (ii) to R in $\Gamma_{\mathcal{E}}$. This is true for every $(s+1)$-space that contains α_s and is contained in $\mathcal{E}$. Moreover, each neighbour of R in $\mathcal{X}_s$ corresponds to a point that lies in exactly one such $(s+1)$-space, so arises exactly once in this way. Further, there are no other lines of $\mathcal{E}$ through R that contain a point of type (ii). By (1), there are

$$\frac{(q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}$$

$(s+1)$-dimensional spaces that contain α_s and are contained in $\mathcal{E}$. Further, $|\Omega\backslash(\Omega\cap\alpha_s)| = q^s$. Hence in $\Gamma_{\mathcal{E}}$, there are

$$\frac{q^s\ (q^{r-s}+1)(q^{r-s-1}-1)}{(q-1)}$$

neighbours of R in $\mathcal{X}_s$. To satisfy Condition II of Result 1.1, we want this to be 0, $\frac{1}{2}|\mathcal{X}_s|$ or $|\mathcal{X}_s|$. Note that since it is $< |\mathcal{X}_s|$ (calculated in Lemma 3.1), we see that this can occur if and only if $q = 2$ (when it is equal to $\frac{1}{2}|\mathcal{X}_s|$), or 0 (when $r-s-1=0$). However by Definition 2.2, $s < g = r-1$, so $r-s-1$ is never zero. Thus $q = 2$ and R is adjacent to $\frac{1}{2}|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$.

Thus vertices in $\mathcal{Y}_s$ are adjacent to either $|\mathcal{X}_s|$ (if of type (i)) or $\frac{1}{2}|\mathcal{X}_s|$ vertices in $\mathcal{X}_s$ (if of type (iii)), if and only if $q = 2$. That is, Condition II of Result 1.1 is satisfied in Γ if and only if $q = 2$. $\square$

We use these lemmas to provide a proof of Theorem 2.3.

Proof of Theorem 2.3 Let s be an integer with $0 \leq s < g$, and so we need $g \geq 1$. Lemmas 3.2 and 3.3, show that for a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n, 2)$, the partition defined in Definition 2.2 satisfies Conditions I and II of Result 1.1. Hence by Result 1.1, for any s, $0 \leq s < g$, the graph Γ_s is a strongly regular graph with the same parameters as Γ. $\square$

It is useful to note that the proof of Lemma 3.3 gives a description of the edges in the graph Γ_s. That is, let P, P' be vertices of type (i), Q, Q' vertices of type (ii), and R, R' vertices of type (iii). Then $\{P, P'\}$, $\{P, Q\}$, $\{P, R\}$, $\{Q, Q'\}$, $\{R, R'\}$ are edges of Γ_s if PP', PQ, PR, QQ', RR' are lines of $\mathcal{Q}_n$ respectively; and $\{Q, R\}$ is an edge of Γ_s if QR is a 2-secant of $\mathcal{Q}_n$. In summary, we have:

Corollary 3.4 *Let Γ_s, $0 \leq s < g$ be the graph constructed in Theorem 2.3. The adjacencies in Γ_s are the same as those given in Table 1.*

Remark 3.5 Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$. As we need $g \geq 1$ for our construction to work: when $\mathcal{Q}_n$ is a hyperbolic quadric, we need $n \geq 3$; when $\mathcal{Q}_n$ is a parabolic quadric, we need $n \geq 4$; and when $\mathcal{Q}_n$ is an elliptic quadric, we need $n \geq 5$.

Remark 3.6 We note that if $q \neq 2$, then geometric techniques similar to those used in Lemmas 3.1, 3.2 and 3.3 show that the graph Γ_s with $s > 0$ is *not* regular.

4 Maximal cliques of Γ_s

In this section, we classify and count the maximal cliques in each graph Γ_s. We will make repeated use of the following property of polar spaces, see [6, Section 26.1].

Property (*) Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$, Σ a generator of $\mathcal{Q}_n$, and X a point of $\mathcal{Q}_n$ not in Σ. Then there is a unique generator Π of $\mathcal{Q}_n$ that contains X and meets Σ in a $(g-1)$-space. Further, the points in Σ which lie on a line of $\mathcal{Q}_n$ through X are exactly the points in $\Sigma \cap \Pi$.

4.1 Description of Maximal Cliques of Γ_s

In this section, let $\mathcal{Q}_n$ be a non-singular quadric of $\mathrm{PG}(n, 2)$ of projective index g with point-graph Γ. Let α_s be an s-dimensional space of $\mathcal{Q}_n$ giving rise to the Godsil-McKay partition $\{\mathcal{X}_s, \mathcal{Y}_s\}$ and the new graph Γ_s as in Theorem 2.3.

We first describe the maximal cliques in the point-graph Γ of $\mathcal{Q}_n$. The projective index g of $\mathcal{Q}_n$ is the dimension of the largest subspaces contained in $\mathcal{Q}_n$, these g-spaces are called *generators* of $\mathcal{Q}_n$ (see [6, Chapter 22] for more details). A generator contains $2^{g+1} - 1$ points, and any subspace of $\mathcal{Q}_n$ is contained in a generator of $\mathcal{Q}_n$. Hence the maximal cliques of Γ correspond to generators of $\mathcal{Q}_n$, and so contain $2^{g+1} - 1$ vertices. We want to study maximal cliques in Γ_s, we begin by studying cliques of Γ_s of size $2^{g+1} - 1$. We define a *g-clique* of Γ_s to be a clique of size $2^{g+1} - 1$. The next lemma describes two types of g-cliques of Γ_s. Note that the first type corresponds to generators of $\mathcal{Q}_n$ containing α_s, and so corresponds to maximal cliques of the original graph Γ. Figure 1 illustrates both types of g-cliques of Lemma 4.1.

Figure 1: g-cliques of Γ_s

Lemma 4.1 *Let Γ_s, $0 \leq s < g$, be the graph constructed as in Theorem 2.3.*

A. Let Σ be a generator of $\mathcal{Q}_n$ that contains α_s. Then the vertices of Γ_s corresponding to the points of Σ form a g-clique of Γ_s.

B. Let Π, Σ be two generators of $\mathcal{Q}_n$ such that: Σ contains α_s; Π does not contain α_s; and Π, Σ meet in a $(g-1)$-dimensional space. Let $\mathcal{C}_a$ be the $2^s - 1$ points of $\alpha_s \cap \Pi$; $\mathcal{C}_b$ be the $2^g - 2^s$ points of Σ that are not in α_s or Π; and $\mathcal{C}_c$ be the 2^g points of $\Pi \backslash \Sigma$. Then $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ corresponds to a g-clique of the graph Γ_s.

Proof For part A, let Σ be a generator of $\mathcal{Q}_n$ that contains α_s. Let $\mathcal{C}$ be the set of vertices of Γ_s that correspond to the points of Σ. As $\mathcal{C}$ consists of vertices of type (i) and (ii) only, two vertices of $\mathcal{C}$ are adjacent if the corresponding two points lie on a line of $\mathcal{Q}_n$. As Σ is contained in $\mathcal{Q}_n$, every pair of distinct points in Σ lie in a line of $\mathcal{Q}_n$. Hence every pair of distinct vertices in $\mathcal{C}$ are adjacent, so $\mathcal{C}$ is a clique. Further, Σ contains $2^{g+1}-1$ points, so $|\mathcal{C}| = 2^{g+1}-1$. Thus $\mathcal{C}$ is a g-clique of Γ_s.

We now consider the set described in part B. First note that the three sets $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ are pairwise disjoint, and $\mathcal{C}_a$ consists of points of type (i); $\mathcal{C}_b$ consists of points of type (ii); and $\mathcal{C}_c$ consists of points of type (iii). Further, the number of points in $\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c$ can be calculated by straightforward counting points in projective spaces.

Suppose $|\mathcal{C}_a|, |\mathcal{C}_b|, |\mathcal{C}_c| > 1$, and let $P, P' \in \mathcal{C}_a$, $Q, Q' \in \mathcal{C}_b$, $R, R' \in \mathcal{C}_c$. We note the following pairs lie in a subspace of $\mathcal{Q}_n$, and so lie on a line of $\mathcal{Q}_n$: $P, P' \in \alpha_s \subset \mathcal{Q}_n$, $Q, Q' \in \Sigma \subset \mathcal{Q}_n$, $P, Q \in \Sigma \subset \mathcal{Q}_n$, $P, R \in \Pi \subset \mathcal{Q}_n$, $R, R' \in \Pi \subset \mathcal{Q}_n$. Hence the corresponding pairs of vertices are all adjacent in Γ_s. So to show that $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ is a clique, we need to show that QR is a 2-secant of $\mathcal{Q}_n$. Similarly, if any of $|\mathcal{C}_a|, |\mathcal{C}_b|, |\mathcal{C}_c|$ is ≤ 1, we only need to show that QR is a 2-secant of $\mathcal{Q}_n$.

Consider the line QR. It lies in the $(g+1)$-space $\langle \Pi, \Sigma \rangle$, which meets $\mathcal{Q}_n$ in exactly Π and Σ. As $Q \in \Sigma \backslash \Pi$ and $R \in \Pi \backslash \Sigma$, the line QR is not contained in $\mathcal{Q}_n$, so it is a 2-secant of $\mathcal{Q}_n$. Hence QR is an edge of Γ_s. That is, $\mathcal{C}_a \cup \mathcal{C}_b \cup \mathcal{C}_c$ is a set of $2^{g+1}-1$ vertices of Γ_s such that any two vertices are adjacent, and so it is a g-clique of Γ_s. $\square$

We will show that the only maximal cliques in Γ_s are the g-cliques of Type A and B. We begin with some preliminary lemmas. First note that the g-cliques of Type A contain no points of type (iii), and we will now show that the converse also holds.

Lemma 4.2 *Let $\mathcal{C}$ be a g-clique of Γ_s, $0 \leq s < g$, that contains no vertices of type* (iii)*, then $\mathcal{C}$ is of Type A.*

Proof Let $\mathcal{C}$ be a g-clique of Γ_s, $0 \leq s < g$, that contains no vertices of type (iii). Suppose $\mathcal{C}$ is not contained in a generator of $\mathcal{Q}_n$. We can enumerate the number (shouldn't this be "enumerate the points" or "count the number of points"?) of points of $\mathcal{Q}$ in each generator of $\mathcal{Q}_n$. Let Σ be a generator of $\mathcal{Q}_n$ that contains the maximum number of points of $\mathcal{C}$. As $\mathcal{C}$ is not contained in Σ, there is a point A of $\mathcal{C}$ that is not in Σ. By property (*), there is a unique generator Π of $\mathcal{Q}_n$ that contains A and meets Σ in a $(g-1)$-space. Further, the points of Σ that lie on a line of $\mathcal{Q}_n$ through A are exactly the points of $\Sigma \cap \Pi$. As $\mathcal{C}$ contains no points of type (iii), edges in $\mathcal{C}$ correspond to lines of $\mathcal{Q}_n$, so the points of $\mathcal{C}$ (should $\mathcal{C}$ be changed to $\mathcal{C} \cap \Sigma$) lie in $\Sigma \cap \Pi$. That is, $|\Pi \cap \mathcal{C}| \geq |\Sigma \cap \mathcal{C}| + 1$, which contradicts the choice of Σ being the generator with the largest intersection with $\mathcal{C}$. Hence $\mathcal{C}$ is contained in a generator of $\mathcal{Q}_n$. As $|\mathcal{C}| = 2^{g+1}-1$, the vertices of $\mathcal{C}$ correspond exactly to the points of this generator, and so $\mathcal{C}$ is a Type A g-clique. $\square$

Lemma 4.3 *Every generator of $\mathcal{Q}_n$ contains a point of type (ii).*

Proof DELETE THIS: Let Π be a generator of $\mathcal{Q}_n$. If Π contains α_s, then Π contains only points of type (i) and (ii), and as $s < g$, Π contains at least one point of type (ii). Suppose Π meets α_s in a subspace α_t of dimension t, with $-1 \leq t \leq s-1$. Let P_1 be a point of $\alpha_s \backslash \alpha_t$, and let $\alpha_{t+1} = \langle P_1, \alpha_t \rangle$, so α_{t+1} has dimension $t+1$. As $P_1 \notin \Pi$, by property (*) there exists a unique generator Σ_1 of $\mathcal{Q}_n$ containing P_1 and meeting Π in a $(g-1)$-space H_1. Now each point $X \in H_1$ either lies in α_t or is on a line of $\mathcal{Q}_n$ with each point in α_{t+1} (But α_s could be contained in Π?). If $\alpha_{t+1} \neq \alpha_s$, that is, if $t+1 < s$, we can repeat this process with $P_2 \in \alpha_s \backslash \alpha_{t+1}$. We use property (*) to get a generator Σ_2 of $\mathcal{Q}_n$ that contains P_2 and meets Π in a $(g-1)$-space which contains α_t. Further, Σ_2 meets H_1 in a $(g-2)$-space denoted H_2. (Is it clear that $H_2 \neq H_1$? - we could argue this if we had a count of how many generators through a $g-1$ space) Repeating this process a total of $s-t$ times, we eventually obtain H_{s-t} of dimension $(g-(s-t)) = g-s+t$ in Π. Note that as $s < g$ we have $g-s+t > t$ and so $H_{s-t} \backslash \alpha_t$ is not empty. Let $X \in H_{s-t} \backslash \alpha_t$, then X is on a line of $\mathcal{Q}_n$ with all the points of α_s. Thus X is type (ii). As $H_{s-t} \subset \Pi$, $X \in \Pi$, so Π contains at least one point of type (ii) as required.

ADD THE FOLLOWING:

Let Π be a generator of $\mathcal{Q}_n$. If Π contains α_s, then Π contains only points of type (i) and (ii), and as $s < g$, Π contains at least one point of type (ii). Suppose Π meets α_s in a subspace α_t of dimension t, with $-1 \leq t \leq s-1$. Let P_1 be a point of $\alpha_s \backslash \alpha_t$, and as $P_1 \notin \Pi$, by property (*) there exists a unique generator Σ_1 of $\mathcal{Q}_n$ containing P_1 and meeting Π in a $(g-1)$-space H_1, which necessarily contains α_t. Let $\Sigma_1^s = \alpha_s \cap \Sigma_1 \supseteq \langle \alpha_t, P_1 \rangle$. The points of H_1 not of type (i) is on a line of $\mathcal{Q}_n$ with each point in Σ_1^s.

If $\Sigma_1^s \neq \alpha_s$, we can repeat this process with $P_2 \in \alpha_s \backslash \Sigma_1^s$. We use property (*) to get a generator Σ_2 of $\mathcal{Q}_n$ that contains P_2 and meets Π in a $(g-1)$-space which contains α_t. Further, Σ_2 meets H_1 in at least a $(g-2)$-dimensional space denoted H_2, which necessarily contains α_t. The points of H_2 which are not of type (i) are on a line of Q_n with each point in $\Sigma_2^2 = \alpha_s \cap \Sigma_2 \supseteq \langle \alpha_t, P_1, P_2 \rangle$.

Repeating this process at most $s-t$ times, we eventually obtain $H_{s-t} \subseteq \Pi$ of dimension at least $d = g-(s-t)$, with the property that all the points of H_{s-t} which are not of type (i), are on a line of Q_n with all the points in α_s, and so are of type (ii). We need to show that H_{s-t} *does* contain points other than that of α_t.

We consider the dimension of H_{s-t} compared to that of its subspace α_t. We have that the dimension of H_{s-t} less that of α_t is at least $d-t = g-(s-t)-t = g-s > 0$, by definition of s. Hence the set $H_{s-t} \backslash \alpha_t$ is non-empty and contains points of type (ii). Thus $\Pi \supset H_{s-t}$ contains at least one point of type (ii) as required. □

We now show that there are only two types of g-cliques in Γ_s, namely those of Type A and B described in Lemma 4.1.

Lemma 4.4 *In the quadric graph, any clique lies on a generator.*

Proof Suppose a clique $\mathcal{C}$ is not contained in any generator. Let Π be the generator which has maximal intersection with $\mathcal{C}$, so there is a point $P \in \mathcal{C}$ which is not in Π. Consider the generator Σ which contains P and meets Π in a $(g-1)$–space H. Then the points of H is exactly the points on Π which are on a line with P. Thus $\mathcal{C} \cap \Pi \subseteq H$. However Σ contains P and $\mathcal{C} \cap \Pi$, thus Σ has at least one more point of $\mathcal{C}$ than Π, contradicting the definition of Π, proving that $\mathcal{C}$ lies on a generator. $\square$

Lemma 4.5 *Let $\mathcal{C}$ be a g-clique in Γ_s, $0 \leq s < g$, then $\mathcal{C}$ is a g-clique of Type A or B.*

Proof Let $\mathcal{C}$ be a g-clique of Γ_s and denote the subsets of vertices of $\mathcal{C}$ of type (i), (ii), (iii) by $\mathcal{C}_{\rm i}$, $\mathcal{C}_{\rm ii}$, $\mathcal{C}_{\rm iii}$ respectively. If $\mathcal{C}_{\rm iii} = \emptyset$, then by Lemma 4.2, $\mathcal{C}$ corresponds to a generator of $\mathcal{Q}_n$ containing α_s, and so is of Type A. So suppose $\mathcal{C}_{\rm iii} \neq \emptyset$.

REMOVE THE FOLLOWING:

We begin by constructing two generators of $\mathcal{Q}_n$ whose union contains the g-clique $\mathcal{C}$. Let $P \in \alpha_s$ and $R \in \mathcal{C}_{\rm iii}$, so P is type (i) and R is type (iii), and hence PR is a line of $\mathcal{Q}_n$. Moreover, as $\mathcal{C}$ is a clique, for any two points $R, R' \in \mathcal{C}_{\rm iii}$, RR' is a line of $\mathcal{Q}_n$. Further, note that α_s is a subspace. Thus any two points in $\alpha_s \cup \mathcal{C}_{\rm iii}$ lie on a line of $\mathcal{Q}_n$, and so $\langle \alpha_s, \mathcal{C}_{\rm iii} \rangle$ is a subspace contained in $\mathcal{Q}_n$. I think the required result is that every clique lies on a generator Hence $\langle \alpha_s, \mathcal{C}_{\rm iii} \rangle$ is contained in a generator of $\mathcal{Q}_n$, denoted Π. Thus $\langle \mathcal{C}_{\rm i}, \mathcal{C}_{\rm iii} \rangle$ is contained in Π. A similar argument shows that $\langle \mathcal{C}_{\rm i}, \mathcal{C}_{\rm ii} \rangle$ is contained in a generator Σ of $\mathcal{Q}_n$. We now show that $\mathcal{C}_{\rm ii}$ is not empty. Suppose $\mathcal{C}_{\rm ii} = \emptyset$, then $\mathcal{C}$ is contained in Π, and as $|\mathcal{C}| = 2^{g+1} - 1$, we have $\mathcal{C} = \mathcal{C}_{\rm i} \cup \mathcal{C}_{\rm iii} = \Pi$. However, by Lemma 4.3, Π contains at least one point of type (ii), a contradiction. Thus $\mathcal{C}_{\rm ii} \neq \emptyset$. So let $Q \in \mathcal{C}_{\rm ii}$ and $R \in \mathcal{C}_{\rm iii}$, then $\{Q, R\}$ is an edge of Γ_s, hence QR is a 2-secant of $\mathcal{Q}_n$. As $Q \in \Sigma \subset \mathcal{Q}_n$, we have $R \notin \Sigma$. Similarly $R \in \Pi$ and QR a 2-secant implies $Q \notin \Pi$. In summary, we have

$$\mathcal{C}_{\rm i} \subset \alpha_s \cap \Pi \cap \Sigma; \quad \mathcal{C}_{\rm ii} \subset \Sigma \backslash \Pi; \quad \mathcal{C}_{\rm iii} \subset \Pi \backslash \Sigma; \quad \mathcal{C} \subset \Sigma \cup \Pi.$$

Next we determine the size of $\mathcal{C}_{\rm i}$, $\mathcal{C}_{\rm ii}$ and $\mathcal{C}_{\rm iii}$. As $\mathcal{C}_{\rm iii} \neq \emptyset$, there is a point $R \in \mathcal{C}_{\rm iii}$, so $R \notin \Sigma$. By property (*), there is a unique generator Π' of $\mathcal{Q}_n$ that contains R and meets Σ in a $(g-1)$-space denoted $H = \Sigma \cap \Pi'$. If H contained α_s, then $\langle R, \alpha_s \rangle \subset \Pi'$ would be a subspace of $\mathcal{Q}_n$, which implies that R is type (ii), a contradiction. Thus $H \cap \alpha_s$ is an $(s-1)$-space. If $P \in \mathcal{C}_{\rm i}$, then $P, R \in \mathcal{C}$, so P, R are adjacent in Γ_s and so PR is a line of $\mathcal{Q}_n$. Thus $P \in H$, and so $P \in H \cap \alpha_s$. Hence $|\mathcal{C}_{\rm i}| \leq |H \cap \alpha_s| = 2^s - 1$. Now by the construction of H, each point in $H \backslash \alpha_s$ lies on a line of $\mathcal{Q}_n$ with R, and each point of $\Sigma \backslash (H \cup \alpha_s)$ lies on a 2-secant of $\mathcal{Q}_n$ with R. So the type (ii) points of $\mathcal{C}$ are contained in $\Sigma \backslash (H \cup \alpha_s)$. That is, $|\mathcal{C}_{\rm ii}| \leq |\Sigma \backslash (H \cup \alpha_s)| = (2^{g+1} - 1) - \big((2^g - 1) + 2^s\big) = 2^g - 2^s$.

As $\mathcal{C}_{\text{ii}} \neq \emptyset$, there is a point $Q \in \mathcal{C}_{\text{ii}}$, so $Q \in \Sigma\backslash\Pi$. By property (*), there is a unique generator Σ' of $\mathcal{Q}_n$ that contains Q and meets Π in a $(g-1)$-space. Hence Q is on a line of $\mathcal{Q}_n$ with the $2^g - 1$ points of $\Pi \cap \Sigma'$; and Q is on a 2-secant of $\mathcal{Q}_n$ with the $(2^{g+1}-1)-(2^g-1) = 2^g$ points of $\Pi\backslash\Sigma'$. If R is a point of $\mathcal{C}_{\text{iii}}$, then as $Q, R \in \mathcal{C}$, they are adjacent in Γ_s and so QR is a 2-secant of $\mathcal{Q}_n$. Hence the points of $\mathcal{C}_{\text{iii}}$ lie in $\Pi\backslash\Sigma'$, and so $|\mathcal{C}_{\text{iii}}| \leq 2^g$.

Further, as $|\mathcal{C}| = 2^{g+1}-1$, we need equality in all three of these bounds, that is, $|\mathcal{C}_{\text{i}}| = 2^s-1$, $|\mathcal{C}_{\text{ii}}| = 2^g - 2^s$, and $|\mathcal{C}_{\text{iii}}| = 2^g$. Moreover,

$$\mathcal{C}_{\text{i}} = \alpha_s \cap \Pi', \quad \mathcal{C}_{\text{ii}} = \Sigma\backslash(\alpha_s \cup \Pi'), \quad \mathcal{C}_{\text{iii}} = \Pi\backslash\Sigma'. \tag{3}$$

To show that $\mathcal{C}$ is a g-clique of Type B, we need to show that $\Pi = \Pi'$ and $\Sigma = \Sigma'$. Suppose that $\Pi \neq \Pi'$, so $\Pi \cap \Pi'$ has dimension at most $g-1$, that is $|\Pi \cap \Pi'| \leq 2^g - 1$. As Π contains $\mathcal{C}_{\text{iii}}$, and $|\mathcal{C}_{\text{iii}}| = 2^g > |\Pi \cap \Pi'|$, there exists a point $R^* \in \mathcal{C}_{\text{iii}}$ with $R^* \in \Pi\backslash\Pi'$. By Property (*), there exists a unique generator Π^* of $\mathcal{Q}_n$ which contains R^* and meets Σ in a $(g-1)$-space. Further, for each point $X \in \Sigma\backslash\Pi^*$, XR^* is a 2-secant of $\mathcal{Q}_n$. Thus $\mathcal{C}_{\text{ii}} \subset \Sigma\backslash\Pi^*$. By (4), $\mathcal{C}_{\text{ii}} = \Sigma\backslash(\alpha_s \cup \Pi')$, moreover we have $|\Sigma\backslash(\alpha_s \cup \Pi')| = |\Sigma\backslash(\alpha_s \cup \Pi^*)|$. Hence $\Sigma \cap \Pi' = \Sigma \cap \Pi^*$, and so $\Pi' \cap \Pi^*$ is a $(g-1)$-space in Σ. Recall that $R \in \Pi'$, and by assumption $R^* \in \Pi^*\backslash\Pi'$, so $\Pi' \neq \Pi^*$. Thus $\langle \Pi', \Pi^* \rangle$ is a $(g+1)$-space, and so meets $\mathcal{Q}_n$ in exactly the two generators Π', Π^*. Now $R, R^* \in \mathcal{C}_{\text{iii}}$, so $\{R, R^*\}$ is an edge of Γ_s, and so RR^* is a line of $\mathcal{Q}_n$. As $R^* \in \Pi^*\backslash\Pi$, and RR^* is a line of $\mathcal{Q}_n$ in $\langle \Pi', \Pi^* \rangle$, we have $R \in \Pi^*$. So $R \in \Pi^* \cap \Pi' \subset \Sigma$, contradicting the choice of $R \notin \Sigma$. Hence $\Pi = \Pi'$. Thus Σ meets Π in a $(g-1)$-space, so by the construction of Σ', we have $\Sigma = \Sigma'$.

We begin by constructing two generators of $\mathcal{Q}_n$ whose union contains the g-clique $\mathcal{C}$. As $\mathcal{C}$ is a clique, then $\mathcal{C}_{\text{i}} \cup \mathcal{C}_{\text{iii}}$ is a clique, and since $\mathcal{C}_{\text{i}} \cup \mathcal{C}_{\text{iii}}$ contains only points of type (i) and (iii), it follows that every two such points is on a line of Q_n, so $\mathcal{C}_{\text{i}} \cup \mathcal{C}_{\text{iii}}$ is a clique of $\mathcal{Q}_n$ and so lie on a generator Π of $\mathcal{Q}_n$. (have lemma above to support this)

Now consider the points of α_s and $\mathcal{C}_{\text{ii}}$. By definition of points of type (ii), every point of $\mathcal{C}_{\text{ii}}$ is on a line with every point of α_s. So $\alpha_s \cup \mathcal{C}_{\text{ii}}$ is a clique of $\mathcal{Q}_n$ and so lies on a generator Σ of $\mathcal{Q}_n$.

For $\mathcal{C}$ to be of type B, we need to show that $\Pi \cap \Sigma$ has dimension $g-1$.

We now show that $\mathcal{C}_{\text{ii}}$ is not empty. Suppose $\mathcal{C}_{\text{ii}} = \emptyset$, then $\mathcal{C}$ is contained in Π, and as $|\mathcal{C}| = 2^{g+1}-1$, we have $\mathcal{C} = \mathcal{C}_{\text{i}} \cup \mathcal{C}_{\text{iii}} = \Pi$. However, by Lemma 4.3, Π contains at least one point of type (ii), a contradiction. Thus $\mathcal{C}_{\text{ii}} \neq \emptyset$.

So let $Q \in \mathcal{C}_{\text{ii}}$ and $R \in \mathcal{C}_{\text{iii}}$, then $\{Q, R\}$ is an edge of Γ_s, hence QR is a 2-secant of $\mathcal{Q}_n$. As $Q \in \Sigma \subset \mathcal{Q}_n$, we have $R \notin \Sigma$. Similarly $R \in \Pi$ and QR a 2-secant implies $Q \notin \Pi$. In summary, we have

$$\mathcal{C}_{\text{i}} \subset \alpha_s \cap \Pi \cap \Sigma; \quad \mathcal{C}_{\text{ii}} \subset \Sigma\backslash\Pi; \quad \mathcal{C}_{\text{iii}} \subset \Pi\backslash\Sigma; \quad \mathcal{C} \subset \Sigma \cup \Pi.$$

Next we determine the size of $\mathcal{C}_{\rm i}$, $\mathcal{C}_{\rm ii}$ and $\mathcal{C}_{\rm iii}$. As $\mathcal{C}_{\rm iii} \neq \emptyset$, there is a point $R \in \mathcal{C}_{\rm iii}$, so $R \notin \Sigma$. By property (*), there is a unique generator Π' of $\mathcal{Q}_n$ that contains R and meets Σ in a $(g-1)$-space denoted $H = \Sigma \cap \Pi'$. If H contained α_s, then $\langle R, \alpha_s \rangle \subset \Pi'$ would be a subspace of $\mathcal{Q}_n$, which implies that R is type (ii), a contradiction. Thus $H \cap \alpha_s$ is an $(s-1)$-space. If $P \in \mathcal{C}_{\rm i}$, then $P, R \in \mathcal{C}$, so P, R are adjacent in Γ_s and so PR is a line of $\mathcal{Q}_n$. Thus $P \in H$, and so $P \in H \cap \alpha_s$. Hence $|\mathcal{C}_{\rm i}| \leq |H \cap \alpha_s| = 2^s - 1$. Now by the construction of H, each point in $H \backslash \alpha_s$ lies on a line of $\mathcal{Q}_n$ with R, and each point of $\Sigma \backslash (H \cup \alpha_s)$ lies on a 2-secant of $\mathcal{Q}_n$ with R. So the type (ii) points of $\mathcal{C}$ are contained in $\Sigma \backslash (H \cup \alpha_s)$. That is, $|\mathcal{C}_{\rm ii}| \leq |\Sigma \backslash (H \cup \alpha_s)| = (2^{g+1} - 1) - \big((2^g - 1) + 2^s\big) = 2^g - 2^s$.

As $\mathcal{C}_{\rm ii} \neq \emptyset$, there is a point $Q \in \mathcal{C}_{\rm ii}$, so $Q \in \Sigma \backslash \Pi$. By property (*), there is a unique generator Σ' of $\mathcal{Q}_n$ that contains Q and meets Π in a $(g-1)$-space. Hence Q is on a line of $\mathcal{Q}_n$ with the $2^g - 1$ points of $\Pi \cap \Sigma'$; and Q is on a 2-secant of $\mathcal{Q}_n$ with the $(2^{g+1} - 1) - (2^g - 1) = 2^g$ points of $\Pi \backslash \Sigma'$. If R is a point of $\mathcal{C}_{\rm iii}$, then as $Q, R \in \mathcal{C}$, they are adjacent in Γ_s and so QR is a 2-secant of $\mathcal{Q}_n$. Hence the points of $\mathcal{C}_{\rm iii}$ lie in $\Pi \backslash \Sigma'$, and so $|\mathcal{C}_{\rm iii}| \leq 2^g$.

Further, as $|\mathcal{C}| = 2^{g+1} - 1$, we need equality in all three of these bounds, that is, $|\mathcal{C}_{\rm i}| = 2^s - 1$, $|\mathcal{C}_{\rm ii}| = 2^g - 2^s$, and $|\mathcal{C}_{\rm iii}| = 2^g$. Moreover,

$$\mathcal{C}_{\rm i} = \alpha_s \cap \Pi', \quad \mathcal{C}_{\rm ii} = \Sigma \backslash (\alpha_s \cup \Pi'), \quad \mathcal{C}_{\rm iii} = \Pi \backslash \Sigma'. \tag{4}$$

To show that $\mathcal{C}$ is a g-clique of Type B, we need to show that $\Pi = \Pi'$ and $\Sigma = \Sigma'$. Suppose that $\Pi \neq \Pi'$, so $\Pi \cap \Pi'$ has dimension at most $g-1$, that is $|\Pi \cap \Pi'| \leq 2^g - 1$. As Π contains $\mathcal{C}_{\rm iii}$, and $|\mathcal{C}_{\rm iii}| = 2^g > |\Pi \cap \Pi'|$, there exists a point $R^* \in \mathcal{C}_{\rm iii}$ with $R^* \in \Pi \backslash \Pi'$. By Property (*), there exists a unique generator Π^* of $\mathcal{Q}_n$ which contains R^* and meets Σ in a $(g-1)$-space. Further, for each point $X \in \Sigma \backslash \Pi^*$, XR^* is a 2-secant of $\mathcal{Q}_n$. Thus $\mathcal{C}_{\rm ii} \subset \Sigma \backslash \Pi^*$. By (4), $\mathcal{C}_{\rm ii} = \Sigma \backslash (\alpha_s \cup \Pi')$, moreover we have $|\Sigma \backslash (\alpha_s \cup \Pi')| = |\Sigma \backslash (\alpha_s \cup \Pi^*)|$. Hence $\Sigma \cap \Pi' = \Sigma \cap \Pi^*$, and so $\Pi' \cap \Pi^*$ is a $(g-1)$-space in Σ. Recall that $R \in \Pi'$, and by assumption $R^* \in \Pi^* \backslash \Pi'$, so $\Pi' \neq \Pi^*$. Thus $\langle \Pi', \Pi^* \rangle$ is a $(g+1)$-space, and so meets $\mathcal{Q}_n$ in exactly the two generators Π', Π^*.is this right? - some justification? Now $R, R^* \in \mathcal{C}_{\rm iii}$, so $\{R, R^*\}$ is an edge of Γ_s, and so RR^* is a line of $\mathcal{Q}_n$. As $R^* \in \Pi^* \backslash \Pi$, and RR^* is a line of $\mathcal{Q}_n$ in $\langle \Pi', \Pi^* \rangle$, we have $R \in \Pi^*$. So $R \in \Pi^* \cap \Pi' \subset \Sigma$, contradicting the choice of $R \notin \Sigma$. Hence $\Pi = \Pi'$. Thus Σ meets Π in a $(g-1)$-space, so by the construction of Σ', we have $\Sigma = \Sigma'$. $\square$

Lemma 4.6 *The maximum size of a clique in Γ_s is $2^{g+1} - 1$.*

Proof REMOVE:

Suppose Γ_s contains a clique $\mathcal{K}$ of size 2^{g+1}, and let $X, Y \in \mathcal{K}$. Then by Theorem 4.5, $\mathcal{K} \backslash X$ is a g-clique of Type A or B, and so the number of vertices of each type in $\mathcal{K} \backslash X$ satisfies Table 2.

INSERT THIS:

Suppose Γ_s contains a clique $\mathcal{K}$ of size 2^{g+1}. Then for each vertex X in $\mathcal{K}$, by Theorem 4.5, $\mathcal{K}\backslash X$ is a g-clique of Type A or B, whose vertices are given in Table 2. As $g > 0$ it follows that if one $\mathcal{K}\backslash X$ is of type B, then every $\mathcal{K}\backslash X$ is of type B. However, if we then remove a vertex Y of type different to X, then $\mathcal{K}\backslash Y$ does not satisfy either column. Similarly, if one of $\mathcal{K}\backslash X$ is of type A, then if we remove vertex Y of type different to X, then $\mathcal{K}\backslash Y$ does not satisfy either column. Hence $\mathcal{K}$ does not exist.

Table 2: Number of vertices of each type in each g-clique

	g-clique A	g-clique B
vertex type (i)	$2^{s+1}-1$	2^s-1
vertex type (ii)	$2^{g+1}-2^{s+1}$	2^g-2^s
vertex type (iii)	0	2^g

REMOVE It is straightforward to check that if $\mathcal{K}\backslash X$ satisfies one of the columns in Table 2, then $\mathcal{K}\backslash Y$ does not satisfy either column, contradicting Theorem 4.5. Hence $\mathcal{K}$ does not exist.

$\square$

In summary, we have classified the maximal cliques of Γ_s as follows.

Theorem 4.7 *Let $\mathcal{Q}_n$ be a non-singular quadric of $\mathrm{PG}(n,2)$ of projective index $g \geq 1$, and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 2.3. If $\mathcal{C}$ is a maximal clique of Γ_s, then $\mathcal{C}$ is a g-clique of Type A or B.*

4.2 Counting maximal cliques

In the previous section, we classified the maximal cliques in the graph Γ_s. In this section we count them.

Theorem 4.8 *Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n,2)$ of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 2.3.*

1. *If $\mathcal{Q}_n = Q^-(2r+1,2)$, then*
   (a) Γ has $(2^2+1)(2^3+1)\cdots(2^{r+1}+1)$ maximal cliques.
   (b) Γ_s has $(2^2+1)(2^3+1)\cdots(2^{r-s}+1)\big(2^{r+2}-2^{r-s+1}+1\big)$ maximal cliques.

2. *If* $\mathcal{Q}_n = Q^+(2r+1,2)$, *then*
   (a) Γ *has* $(2^0+1)(2^1+1)\cdots(2^r+1)$ *maximal cliques.*
   (b) Γ_s *has* $(2^0+1)(2^1+1))\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s}+1\big)$ *maximal cliques.*
3. *If* $\mathcal{Q}_n = Q(2r,2)$, *then*
   (a) Γ *has* $(2^1+1)(2^2+1)\cdots(2^r+1)$ *maximal cliques.*
   (b) Γ_s *has* $(2^1+1)(2^2+1)\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s+1}+1\big)$ *should be* $(2^1+1)(2^2+1)\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s}+1\big)$ *maximal cliques.*

Proof For part 1, let $\mathcal{Q}_n = \mathcal{E} = Q^-(2r+1,2)$ have point graph Γ. The maximal cliques of Γ correspond exactly to the generators of $\mathcal{E}$; and the number of generators of $\mathcal{E}$ is $(2^2+1)(2^3+1)\cdots(2^{r+1}+1)$ by [6, Theorem 22.5.1]. This proves 1(a).

Now consider the graph Γ_s, $0 \le s < g = r-1$. Let $N_{s,\mathrm{A}}$, $N_{s,\mathrm{B}}$ be the number of maximal cliques of Γ_s of Type A and B respectively. By Lemma 4.1, $N_{s,\mathrm{A}}$ is equal to the number of generators of $\mathcal{E}$ that contain α_s, and so by [6, Theorem 22.4.7],

$$N_{s,\mathrm{A}} = (2^2+1)(2^3+1)\cdots(2^{r-s}+1). \tag{5}$$

To count the maximal cliques of Type B, by Lemma 4.1 we need to count the number of pairs of generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s, and Π meets Σ in a $(g-1)$-space not containing α_s. The number of choices for Σ is the number of generators of $\mathcal{E}$ that contain α_s which by (5) is $N_{s,\mathrm{A}}$. Once Σ is chosen, we count the number of choices for Π. The number of $(g-1)$-spaces contained in Σ but not containing α_s equals the number of $(g-1)$-spaces contained in Σ minus the number of $(g-1)$-spaces contained in Σ which contain α_s. This is $(2^{g+1}-1)-(2^{g-s}-1) = 2^{g+1}-2^{g-s}$. By [6, Lemma 22.4.8], the number of generators of $\mathcal{E}$ that meet Σ in a fixed $(g-1)$-space is four. Hence the number of choices for Π is $(2^{g+1}-2^{g-s}) \times 4 = 2^{g+3}-2^{g-s+2}$. Thus $N_{s,\mathrm{B}} = N_{s,\mathrm{A}}\big(2^{g+3}-2^{g-s+2}\big) = N_{s,\mathrm{A}}\big(2^{r+2}-2^{r-s+1}\big)$ as the projective index of $\mathcal{E}$ is $g = r-1$. Hence the total number of maximal cliques of Γ_s is $N_{s,\mathrm{A}} + N_{s,\mathrm{B}} = N_{s,\mathrm{A}}\big(2^{r+2}-2^{r-s+1}+1\big)$ as required. This completes the proof of part 1. The proofs of parts 2 and 3 are similar. $\square$

Theorem 4.9 *Let* $\mathcal{Q}_n$ *be a non-singular quadric in* $\mathrm{PG}(n,2)$ *of projective index* $g \ge 1$. *Let* Γ_s, $0 \le s < g$, *be the graph constructed in Theorem 2.3.*

1. *If* $\mathcal{Q}_n = Q^-(2r+1,2)$, *then the number of maximal cliques of* Γ_s, $0 \le s < g-1$, *containing a vertex of Type*
   (i) *is* $(2^2+1)(2^3+1)\cdots(2^{r-s}+1)\big(2^{r+1}-2^{r-s+1}+1\big)$,
   (ii) *is* $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s}+1\big)$,
   (iii) *is* $(2^2+1)(2^3+1)\cdots(2^{r-s}+1)$.
   If $s = g-1 = r-2$, *then the number of maximal cliques of* Γ_s *containing a vertex of Type* (i), (ii), *and* (iii) *is* $5(2^{r+1}-3)$, $2^{r+1}-3$ *and* 5 *respectively.*
2. *If* $\mathcal{Q}_n = Q^+(2r+1,2)$, *then the number of maximal cliques of* Γ_s, $0 \le s < g-1$, *containing a vertex of Type*

(i) *is* $(2^0+1)(2^1+1)\cdots(2^{r-s-1}+1)\big(2^r-2^{r-s}+1\big)$,

(ii) *is* $(2^0+1)(2^1+1)\cdots(2^{r-s-2}+1)\big(2^r-2^{r-s-1}+1\big)$,

(iii) *is* $(2^0+1)(2^1+1)\cdots(2^{r-s-1}+1)$.

If $s=g-1=r-1$*, then the number of maximal cliques of* Γ_s *containing a vertex of Type* (i), (ii)*, and* (iii) *is* $2(2^r-1)$, 2^r *and 1 respectively.*

3. *If* $\mathcal{Q}_n=Q^-(2r+1,2)$ *should be* $\mathcal{Q}_n=Q(2r,2)$*?, then the number of maximal cliques of* Γ_s, $0\le s<g-1$*, containing a vertex of Type*

(i) *is* $(2^1+1)(2^2+1)\cdots(2^{r-s-1}+1)\big(2^r-2^{r-s}+1\big)$,

(ii) *is* $(2^1+1)(2^2+1)\cdots(2^{r-s-2}+1)\big(2^r-2^{r-s-1}+1\big)$,

(iii) *is* $(2^1+1)(2^2+1)\cdots(2^{r-s-1}+1)$.

If $s=g-1=r-2$*, then the number of maximal cliques of* Γ_s *containing a vertex of Type* (i), (ii)*, and* (iii) *is* $3(2^r-3)$, 2^r-1 *and 3 respectively.*

Proof For part 1, let $\mathcal{Q}_n=\mathcal{E}=Q^-(2r+1,2)$ and let P be a vertex of Γ_s of type (i), so in $\mathrm{PG}(2r+1,2)$, $P\in\alpha_s$. All the maximal cliques of Γ_s of Type A contain α_s. So by (5), P lies in $N_{s,\mathrm{A}}=(2^2+1)(2^3+1)\cdots(2^{r-s}+1)$ maximal cliques of Type A. To form a maximal clique of Γ_s of Type B that contains P, we need two generators Σ,Π of $\mathcal{E}$ such that Σ contains α_s, Π meets Σ in a $(g-1)$-space not containing α_s, and $P\in\Pi$. We count the number of pairs Σ, Π satisfying this. First, the number of choices for Σ equals the number of generators of $\mathcal{E}$ containing α_s which is $N_{s,\mathrm{A}}$. The number of $(g-1)$-spaces of Σ that contain P is 2^g-1, and the number of $(g-1)$-spaces of Σ that contain α_s and P is $2^{g-s}-1$. Hence the number of $(g-1)$-spaces of Σ that contain P, but do not contain α_s is $(2^g-1)-(2^{g-s}-1)=2^g-2^{g-s}$. By [6, Lemma 22.4.8], the number of generators of $\mathcal{E}$ that meet Σ in a fixed $(g-1)$-space is four. In total, the number of maximal cliques of Type B containing P is $N_{s,\mathrm{A}}\times(2^g-2^{g-s})\times 4=N_{s,\mathrm{A}}\big(2^{r+1}-2^{r-s+1}\big)$ as $\mathcal{E}$ has projective index $g=r-1$. Hence the total number of maximal cliques of Γ_s containing P is $N_{s,\mathrm{A}}\big(2^{r+1}-2^{r-s+1}+1\big)$ as required.

Now let Q be a vertex of Γ_s of type (ii). The number of maximal cliques of Type A containing Q equals the number of generators of $\mathcal{E}$ containing α_s and Q which by [6, Theorem 22.4.7] is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)$. To count the maximal cliques of Γ_s that contain Q, we need to count pairs of generators Σ,Π of $\mathcal{E}$ such that Σ contains α_s and Q, and Π meets Σ in a $(g-1)$-space not containing α_s or Q. The number of choices for Σ is calculated above to be $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)$. Further, the number of $(g-1)$-spaces of Σ containing α_s is $2^{g-s}-1$; the number of $(g-1)$-spaces of Σ containing α_s and Q is $2^{g-s-1}-1$; and the number of $(g-1)$-spaces of Σ containing Q is 2^g-1. Hence the number of $(g-1)$-spaces of Σ that do not contain α_s and do not contain Q is $(2^{g+1}-1)-(2^{g-s}-1)-(2^g-1)+(2^{g-s-1}-1)=2^g-2^{g-s-1}$. As before, each of these $(g-1)$-spaces lies in four suitable choices for the generator Π of $\mathcal{E}$. Hence the number of maximal cliques of Type B containing Q is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\times(2^g-2^{g-s-1})\times 4=(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s}\big)$ as $\mathcal{E}$ has projective index $g=r-1$. Hence the total number of maximal cliques containing Q is $(2^2+1)(2^3+1)\cdots(2^{r-s-1}+1)\big(2^{r+1}-2^{r-s}+1\big)$ as required.

Let R be a vertex of Γ_s of type (iii), then R is in no maximal cliques of Type A. To count the maximal cliques of Γ_s of Type B containing R, we need to count pairs of generators Σ, Π of $\mathcal{E}$ such that Σ contains α_s, Π meets Σ in a $(g-1)$-space not containing α_s, and Π contains R. The number of choices for Σ equals the number of generators of $\mathcal{E}$ containing α_s which is $N_{s,\mathrm{A}}$ by (5). As Σ contains α_s, it contains no points of type (iii), so $R \notin \Sigma$. So by property (*), there is a unique generator of $\mathcal{E}$ that contains R and meets Σ in a $(g-1)$-space H. Further, if H contained α_s, then $\langle R, \alpha_s \rangle$ would be contained in $\mathcal{E}$, and so R would be type (ii), a contradiction, so H does not contain α_s. So for each Σ, there is a unique choice for Π that can be used to form a Type B maximal clique containing R. Hence the number of maximal cliques of Γ_s containing R is $N_{s,\mathrm{A}} = (2^2+1)(2^3+1)\cdots(2^{r-s}+1)$ as required. This completes the proof of part 1. The proofs of parts 2 and 3 are similar.

Note that if $s = 0$, then as $\Gamma_s \cong \Gamma$ then the number of cliques through points of type (i), (ii) and (iii) are all the same. $\square$

5 The graphs Γ_s are all non-isomorphic

Theorem 5.1 *Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n,2)$ of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 2.3. Then Γ_s is isomorphic to Γ if and only if $s = 0$.*

Proof We want to define an isomorphism between Γ and Γ_0. Consider $s = 0$, so in $\mathrm{PG}(n,2)$, the subspace α_0 is a point denoted P (that is, there is a unique point of type (i)). We define a mapping ϕ of Γ as follows. The map ϕ fixes the vertex P and fixes each vertex of type (iii). Let Q be a vertex of type (ii) of Γ. Then Q corresponds to a point of $\mathrm{PG}(n,2)$. Further, PQ is a line of $\mathcal{Q}_n$, and so meets $\mathcal{Q}_n$ in a third point that corresponds to a vertex of Γ of type (ii), we let $\phi(Q)$ be this third point, so $\phi(\phi(Q)) = Q$. So ϕ is an isomorphism that maps Γ to a graph denoted Γ', with incidence inherited from Γ, that is, vertices X and Y lie on a line of $\mathcal{Q}_n$ if and only if $\phi(X)$ and $\phi(Y)$ are adjacent in Γ'.

We now show that Γ' is Γ_0. In Γ', let Q_1, Q_2 be vertices of type (ii), and R, R' vertices of type (iii). We consider in turn the different types of edges of Γ', and show that they satisfy Table 1. To simplify notation, let $Q_1^* = \phi^{-1}(Q_1)$ and $\phi^{-1}(Q_2) = Q_2^*$. Firstly, $\{P, Q_1\}$ is an edge of Γ' if and only if $\{P, Q_1^*\}$ is an edge of Γ if and only if $PQ_1^* = PQ_1$ is a line of $\mathcal{Q}_n$. Similarly, $\{P, R\}$ (respectively $\{R, R'\}$) is an edge of Γ if and only if PR (respectively RR') is a line of $\mathcal{Q}_n$. Now $\{Q_1, R\}$ is an edge of Γ should this be Γ'?, if $\{Q_1^*, R\}$ is an edge of Γ, that is, Q_1^*R is a line of $\mathcal{Q}_n$.The plane $\langle P, Q_1^*, R \rangle$ is not contained in $\mathcal{Q}_n$ (as R is type (iii)), so it meets $\mathcal{Q}_n$ in exactly the lines PQ_1^*, Q_1^*R.need a reference here As Q_1 is the third point on the line PQ_1^*, we have that Q_1R is a 2-secant of $\mathcal{Q}_n$.

Finally suppose $\{Q_1, Q_2\}$ is an edge of Γ', so $\{Q_1^*, Q_2^*\}$ is an edge of Γ. If the line Q_1Q_2 contains P, then $Q_1^* = Q_2$ and $Q_2^* = Q_1$, so $\{Q_1, Q_2\}$ is an edge of Γ and so Q_1Q_2 is a line of $\mathcal{Q}_n$. Now suppose Q_1Q_2 does not contain P. Then $\{Q_1^*, Q_2^*\}$ an edge of Γ implies $Q_1^*Q_2^*$ is a line of $\mathcal{Q}_n$. Hence the plane $\langle P, Q_1^*, Q_2^*\rangle$ contains at least three lines, namely PQ_1^*, PQ_2^* and $Q_1^*Q_2^*$, and so is contained in $\mathcal{Q}_n$. need a reference here Further, it contains Q_1 and Q_2, so Q_1Q_2 is a line of $\mathcal{Q}_n$. In summary, we have shown that the edges of Γ' satisfy Table 1. So by Corollary 3.4, Γ' is Γ_s with α_s a point, that is, Γ' is Γ_0.

We now show that Γ_s with $s > 1$ is not isomorphic to the graph Γ. The maximal cliques of Γ correspond exactly to the generators of $\mathcal{Q}_n$. Let $\mathcal{Q}_n = \mathcal{E} = Q^-(2r+1, 2)$, then the number of maximal cliques of Γ through a vertex X of Γ equals the number of generators of $\mathcal{E}$ containing a point of $\mathcal{E}$. By [6, Theorem 22.4.7], this is $(2^2+1)(2^3+1)\cdots(2^r+1)$.

REMOVE: Let P be a vertex of Γ of type (i), so P is also a vertex of Γ_s of type (i). If Γ is isomorphic to Γ_s, then the number of maximal cliques containing P is the same for both graphs. So by Theorem 4.9, we need

$$(2^2+1)(2^3+1)\cdots(2^{r-s}+1)\big(2^{r+1}-2^{r-s+1}+1\big) = (2^2+1)(2^3+1)\cdots(2^r+1).$$

This holds if and only if $2^{r+1}-2^{r-s+1}+1 = (2^{r-s+1}+1)\cdots(2^r+1)$ which holds if and only if $s = 0$.

ADD:

Thus in Γ', points of type (i) and (iii) have the same number of maximal cliques. Consider Theorem 4.9. If $s < g-1$ then we need $2^{r+1}-2^{r-s+1}+1 = 1$, and so $s = 0$. If $s = g-1$ then $r = 1$ and $s = -1$, so this case does not occur.

Hence Γ_s with $s > 1$ is not isomorphic to Γ. The proof of this in the cases when $\mathcal{Q}_n$ is $Q^+(2r+1, 2)$ or $Q(2r, 2)$ are similar. □

Theorem 5.2 *Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$ of projective index $g \geq 1$. Let Γ be the point-graph of $\mathcal{Q}_n$ and let Γ_s, $0 \leq s < g$, be the graph constructed in Theorem 2.3. Then the graphs $\Gamma_0, \Gamma_1, \ldots, \Gamma_{g-1}$ are distinct up to isomorphism.*

Proof First suppose that $\mathcal{Q}_n = \mathcal{E} = Q^-(2r+1, 2)$, and let s_1, s_2 be two integers with $0 \leq s_1 < s_2 < g$. Then the number of maximal cliques in Γ_{s_1} and Γ_{s_2} are given in Theorem 4.8(1). These two numbers are equal if and only if

$$2^{r+2} - 2^{r-s_2+1} + 1 \quad = \quad (2^{r-s_2+1}+1)\cdots(2^{r-s_1}+1)\big(2^{r+2}-2^{r-s_1+1}+1\big). \tag{6}$$

Now $2^{r+2} >$ LHS (6) = RHS (6) $> 2^{r-s_1} \times (2^{r+2}-2^{r-s_1})$ and so $2^{s_1+2} > 2^{r+2}-2^{r-s_1} > 2^{r+1}$, hence $s_1+2 > r+1$. However, this contradicts $s_1 < s_2 < g = r-1$. [Can just put this as a comment if you think the argument is ok so don't have to work it out to check it] Thus Γ_{s_1} and Γ_{s_2} are not isomorphic if s_1 and s_2 are distinct. The proof in the cases when $\mathcal{Q}_n$ is $Q^+(2r+1, 2)$ or $Q(2r, 2)$ are similar.

□

5.1 Kantor's graphs

In [7], Kantor constructs a strongly regular graph Γ_K from a non-singular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n, q)$ with the same parameters as the point-graph Γ of $\mathcal{Q}_n$. Kantor conjects that the graph Γ_K is not the same as Γ except in the case when $\mathcal{Q}_n = Q^+(7, q)$. We show that Γ_K is not isomorphic to the graphs Γ_s when $s > 0$. Kantor's construction works when the quadric $\mathcal{Q}_n$ contains a spread, however, we do not need to describe the details of Kantor's graphs to prove non-isomorphism. We use [7, Lemma 3.3] which shows that Γ_K contains a partition of the vertices into maximal cliques (which contain $2^{g+1} - 1$ vertices). We show that Γ_s, $s > 0$ cannot contain such a partition.

Theorem 5.3 *Let $\mathcal{Q}_n$ be a non-singular quadric in $\mathrm{PG}(n, 2)$ of projective index $g \geq 1$. Let Γ_s, $0 < s < g$ be the graph constructed in Theorem 2.3. Let Γ_K be the graph constructed from $\mathcal{Q}_n$ in [7]. Then Γ_K is not isomorphic to Γ_s, $0 < s < g$.*

Proof We show that the vertices of Γ_s, $s > 0$ cannot be partitioned into maximal cliques. Suppose $s > 0$, and let $\mathcal{C}, \mathcal{C}'$ be maximal cliques of Γ_s of type A, and $\mathcal{K}, \mathcal{K}'$ be maximal cliques of Γ_s of type B. Now $\mathcal{C}$, $\mathcal{C}'$ both contain α_s so they are not disjoint. Further, $\mathcal{K}$ contains at least one point in α_s as $s > 0$. So $\mathcal{C}, \mathcal{K}$ are not disjoint. Now consider $\mathcal{K}, \mathcal{K}'$. They both meet α_s in a subspace of dimension $s - 1$. If $s \geq 2$, then two subspaces of dimension $s - 1$ contained in an s-space meet in at least a point, and so $\mathcal{K}, \mathcal{K}'$ share at least a point. Thus if $s \geq 2$, any two cliques of Γ_s share at least one vertex. Now suppose $s = 1$, so α_1 is a line. To partition the three points of the line α_1 using maximal cliques, we need three maximal cliques of type B, one through each point. Moreover, any partition of Γ_1 into maximal cliques cannot contain any further maximal clique. We show that Γ_1 cannot be partitioned into three maximal cliques. First, a maximal clique has $2^{g+1} - 1$ points, so three pairwise disjoint maximal cliques contain $x = 3(2^{g+1} - 1)$ points, with either $g = r - 1$ or r. As $0 < s < g$, it follows that $g \geq 2$. Thus for the elliptic and parabolic case we have $r \geq 3$ and for the hyperbolic case we have $r \geq 2$. However, $Q^-(2r + 1, 2)$ contains $2^{2r+1} - 2^r - 1$ points, $Q^+(2r + 1, 2)$ contains $2^{2r+1} + 2^r - 1$ points and $Q(2r, 2)$ contains $2^{2r} - 1$ points. None of these numbers is equal to x when $r \geq 2$. Hence we cannot partition the vertices of Γ_s, $s > 0$ into maximal cliques. Thus by [7, Lemma 3.3], Γ_s is not isomorphic to Γ_K. □

6 Conclusion

In summary, Table 3 lists the parameters of the strongly regular graphs arising from the point graph of each type of non-singular quadric. Further, we list the number of non-isomorphic graphs with these parameters arising from our construction (note that one of these is the point graph of the quadric).

Table 3: Parameters of the strongly regular graphs Γ_s

quadric	$Q^-(2r+1,2)$, $r \geq 2$	$Q^+(2r+1,2)$, $r \geq 1$	$Q(2r,2)$, $r \geq 2$
v	$2^{2r+1} - 2^r - 1$	$2^{2r+1} + 2^r - 1$	$2^{2r} - 1$
k	$2^{2r} - 2^r - 2$	$2^{2r} + 2^r - 2$	$2^{2r-1} - 2$
λ	$2^{2r-1} - 2^r - 3$	$2^{2r-1} + 2^r - 3$	$2^{2r-2} - 3$
μ	$2^{2r-1} - 2^{r-1} - 1$	$2^{2r-1} + 2^{r-1} - 1$	$2^{2r-2} - 1$
# non-isomorphic graphs	$r-1$	r	$r-1$

7 The automorphism group of Γ_s

The aim of this section is to determine the automorphism group of Γ_s, and we will show the following:

Theorem 7.1 *Consider the graph Γ_s defined above with $s > 0$. Then $\mathrm{Aut}(\Gamma_s) = \mathrm{Aut}(\Gamma)_{\alpha_s}$.*

Note that in the case $s = 0$, we have $\Gamma_0 \cong \Gamma$, so in this case $\mathrm{Aut}(\Gamma_0) = \mathrm{Aut}(\Gamma)$.

The graph Γ defined from the quadric $\mathcal{Q}_n$, is in some sense independent of the projective space $\mathrm{PG}(n, q)$ in which $\mathcal{Q}_n$ is embedded. However, the result is stronger:

Result 7.2 *[XREF] Suppose $n \geq 3$. Consider the graph Γ whose vertices are the points lying on a non-singular quadric $\mathcal{Q}_n$, whose vertices are adjacent if the corresponding points of $\mathcal{Q}_n$ lie on a line contained in $\mathcal{Q}_n$. Then the automorphism group $\mathrm{Aut}(\Gamma)$ is ismorphic to $\mathrm{PGO}(n+1, q)$, the subgroup of the automorphism group of $\mathrm{PG}(n, 2)$ fixing $\mathcal{Q}_n$.*

Lemma 7.3 *Let γ be any 1-1 mapping on the points of $\mathcal{Q}_n$ that preserves the generators of $\mathcal{Q}_n$. Then the action of γ on the vertices of Γ induces an automorphism of Γ.*

Proof Suppose $\mathcal{Q}_n$ is an elliptic quadric. We first show that every subspace X of dimension d $(0 \leq d \leq g)$ lying on $\mathcal{Q}$ is the exact intersection of the generators containing it. By [6, Theorem 22.4.7], the number of generators containing X is $(q^2+1)(q^3+1)\cdots(q^{\frac{1}{2}(n-1)-d}+1)$. Note that this value is different for each dimension d $(0 \leq d \leq g)$. Now suppose the intersection X' of the generators containing X is bigger than X. So there exists $P \in X' \backslash X$ with the generators containing X are the generators containing $\langle X, P\rangle$, and the dimension of $\langle X, P\rangle$ is one more than that of X. This contradicts the count of the generators containing a subspace, given above.

Consider a subspace X lying on $\mathcal{Q}_n$, where X is the exact intersection of generators $G_1, \ldots, G_a$ say. Thus under γ, X maps to a set X^γ, and as γ preserves the generators, $G_1^\gamma, \ldots, G_a^\gamma$ are also generators, and we have $X^\gamma = \cap_{i=1}^a G_i^\gamma$. As the intersection of the generators define the subspaces in $\mathcal{Q}_n$, it follows that X^γ is a subspace of $\mathcal{Q}_n$, that is γ preserves the subspaces of $\mathcal{Q}_n$.

We now show that show such a map is an automorphism of Γ. Two points A, B are adjacent in Γ, if and only if they lie on a line $\ell = \{A, B, \ldots\}$ of Q_n. As γ preserves subspaces of $\mathcal{Q}_n$, in particular lines, then $\ell^\gamma = \{A^\gamma, B^\gamma, \ldots, \}$ is a line of Q_n and so A^γ, B^γ are adjacent in Γ. If A and B are non-adjacent, but A^γ, B^γ are adjacent, then applying γ^{-1} (which preserves subspaces, as γ does) shows that A, B are adjacent in Γ, a contradiction. Hence γ is an automorphism of Γ.

The proof for the parabolic and hyperbolic quadrics are similar. □

We now introduce Witt's theorem and its implications over finite fields of characteristic 2.

Recall that a quadratic form $\mathcal{Q}_n(x_0, \ldots, x_n)$ (not necessarily non-singular) over V, the vector space of dimension $n+1$ over $\mathrm{GF}(q)$, is a homogeneous polynomial of degree 2 over $\mathrm{GF}(q)$, given by

$$\mathcal{Q}_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} x_i x_j.$$

The $(n+1) \times (n+1)$ matrix $A = [a_{ij}]$ defines a bilinear form $B(x, y)$ over the vectors x, y of V by: $B(x, y) = \mathcal{Q}_n(x + y) - \mathcal{Q}_n(x) - \mathcal{Q}_n(y)$.

A *quadratic space* is a pair $(V, \mathcal{Q}_n)$ where $V, \mathcal{Q}_n$ are defined above.

Definition 7.4 *Two n-dimensional quadratic spaces $(V, \mathcal{Q}_n)$ and $(V', \mathcal{Q}'_n)$ are* isometric *if there exists an invertible linear transformation $T \colon V \to V'$ called an* isometry *such that $\mathcal{Q}_n(v) = \mathcal{Q}'_n(Tv)$ for all $v \in V$.*

Result 7.5 (Witt's theorem XREF) *Let (V, b) be a finite-dimensional vector space over an arbitrary field K together with a non-degenerate symmetric or skew-symmetric bilinear form. If $f : U \to U'$ is an isometry between two subspaces of V then f extends to an isometry of (V, b) if and only if $f(U \cap rad(V)) = f(U) \cap rad(V)$.*

We interpret the above result for projective spaces. In this case the radical is the empty set when n is odd, or q is odd. In the case when both n and q are even, $\mathcal{Q}_n$ is a parabolic quadric and the radical is a the point $N \notin \mathcal{Q}_n$ which is called the *nucleus*, being the intersection of the all the tangent spaces. Now consider any two subspaces X, Y of the same dimension of $\mathrm{PG}(n, 2)$ and a quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n, 2)$.

Consider the corresponding quadratic spaces $(X, X \cap \mathcal{Q}_n)$, $(Y, Y \cap \mathcal{Q}_n)$. So an isometry γ is an automorphism of $\mathrm{PG}(n, q)$ with the property that $\gamma\colon X \mapsto Y$, $X \cap \mathcal{Q}_n \mapsto Y \cap \mathcal{Q}_n$. Witt's Theorem then says there is an automorphism β of $\mathrm{PG}(n, q)$ fixing $\mathcal{Q}_n$ with $\beta|_X = \gamma|_X$ if and only if either

1. n or q is odd; or
2. n, q are both even, and either both X, Y contain the nucleus, or both do not contain the nucleus.

In the case when X (and hence Y) lie on $\mathcal{Q}_n$ (that is, $X, Y \subseteq \mathcal{Q}_n$), neither contain the nucleus (if it exists). So Witt's Theorem says that for any automorphism $\gamma\colon X \mapsto Y$, we can find β with $\beta|_X = \gamma|_X$ and β fixing $\mathcal{Q}_n$. So for example we may take X, Y to be generators of $\mathcal{Q}_n$, so there is an automorphism α of $\mathrm{PG}(n, q)$ mapping X to Y fixing $\mathcal{Q}_n$, so the group fixing $\mathcal{Q}_n$ is transitive on the generators. If we take X, Y to be points of $\mathcal{Q}_n$, then there is an automorphism α of $\mathrm{PG}(n, q)$ mapping X to Y fixing $\mathcal{Q}_n$, so the group fixing $\mathcal{Q}_n$ is transitive on the points of $\mathcal{Q}_n$. In general, the group fixing $\mathcal{Q}_n$ is transitive on the r-dimensional subspaces contained in $\mathcal{Q}_n$.

If we consider $X = Y \subseteq \mathcal{Q}_n$ and an automorphism group transitive on the points of X, then there is a subgroup of the automorphism subgroup of $\mathrm{PG}(n, q)$ fixing $\mathcal{Q}_n$ and transitive on the points of X.

Lemma 7.6 *The stabilizer A_s $(0 \le s < g)$ of α_s in the group fixing $\mathcal{Q}_n$ is transitive on the each of the following three sets: the points of type (i), the points of type (ii) and the points of type (iii).*

Proof First note by [8, Theorem 2.10], for any $r \ge 1$, the group $G = \mathrm{PGL}(r+1, q)$ acting on $X = \mathrm{PG}(r, q)$ is transitive on on sets of $r+2$ points, with every $r+1$ independent.

Consider two points P, P' of type (i), that is, $P, P' \in \alpha_s \subseteq \mathcal{Q}_n$ and as there is an automorphism of $\mathrm{PG}(n, q)$ mapping P to P' fixing α_s, it follows by Witt's Theorem that there is an automorphism mapping P to P', fixing α_s and fixing $\mathcal{Q}_n$. Thus A_s is transitive on the points of type (i).

Now consider two points Q, Q' of type (ii). Let $X = \langle Q, \alpha_s \rangle$ and $Y = \langle Q', \alpha_s \rangle$, so $X, Y \subseteq \mathcal{Q}_n$. Since there is an automorphism $\gamma \colon X \mapsto Y$ fixing α_s and mapping Q to Q', by Witt's Theorem there is one which does this and fixes $\mathcal{Q}_n$. Thus A_s is transitive on the points of type (ii).

Finally, consider two points R, R' of type (iii). Let $X = \langle R, \alpha_s \rangle$ and $Y = \langle R', \alpha_s \rangle$. As R, R' is of type (iii), X and Y are not contained entirely within $\mathcal{Q}_n$. However, as $X \cap \mathcal{Q}_n$ contains the space α_s, it follows (XREF) that it contains another s dimensional space β_X of $\mathcal{Q}_n$. Note that by property (*), there is a generator containing R which meets in a hyperplane some generator Π containing α_s, hence β_X contains X. Similarly define $\beta_Y \subseteq Y$ with β_Y containing Y. X (and similarly Y) contain no other points of Q, since if it contained another point A, then a line ℓ through A would contain three points of Q, namely A, $\ell \cap \alpha_s$ and $\ell \cap \beta_X$, and as ℓ only contains three points, it follows that $\ell \subseteq \mathcal{Q}_n$ and so $X \subseteq \mathcal{Q}_n$,(XREF) a contradiction. So X and Y both meet $\mathcal{Q}_n$ in a pair of hyperplanes (α_s, β_X) and (α_s, β_Y) with $X \in \beta_X$ and $Y \in \beta_Y$. Thus we can find an automorphism of $\mathrm{PG}(n, q)$ which fixes α_s and maps Q to Q', so by Witt's theorem there is one which does that and also fixes $\mathcal{Q}_n$, hence A_s is transitive on the points of type (iii). $\square$

We show that the different types of vertices in Γ_s have different number of maximal cliques, hence $\mathrm{Aut}(\Gamma_s)$ has at least three orbits on the vertices.

Lemma 7.7 *In Γ_s $(0 < s < g)$, when counting the maximal cliques, the number through points of type (i) is greater than the number through points of than type (ii), which is greater than the number through points of type (iii).*

or

In Γ_s $(0 < s < g)$, the number of maximal cliques through points of type (i), (ii) and (iii) are all different.

Proof We only prove the result for the elliptic quadric $\mathcal{Q}_n^-(2r+1, 2)$, the other cases are similar.

We will show that for $s > 0$ the number of maximal cliques through points of type (i) is greater than type (ii), greater than type (iii). Comparing the number of cliques through points of type (i), (ii) and (ii), from Theorem 4.9, it is sufficient to show that

$$A = (2^{r-s}+1)(2^{r+1}-2^{r-s+1}+1) > B = 2^{r+1}-2^{r-s}+1 > C = 2^{r-s}+1.$$

We calculate $A - B = 2^{r-s}(2^{r+1}-2^{r-s+1}) > 0$ as $s > 0$, and $B - C = 2^{r+1}-2^{r-s+1} > 0$ as $s > 0$, and so $A > B > C$ as required. $\square$

In the above proof, in the case $s = 0$ we see that $A-B = B-C = 0$. That is, there are the same number of g-cliques through each point of Γ_0. This is because $\Gamma_0 \cong \Gamma$ (Theorem 5.1) and the automorphism group of Γ is transitive on the vertices of Γ.

Theorem 7.8 *The group $A_s = Aut(\Gamma)_{\alpha_s}$ induces an automorphism group on Γ_s, and so $Aut(\Gamma)_{\alpha_s} \subseteq Aut(\Gamma_s)$. Further, if $s > 0$, then $\mathrm{Aut}(\Gamma_s)$ has exactly three orbits on the vertices of Γ_s, namely the points of each type.*

Proof We now show that subgroup of the group fixing $\mathcal{Q}_n$ which fixing α_s induces an automorphism group on Γ_s. Let γ be an automorphism of $\mathrm{PG}(n,2)$ fixing $\mathcal{Q}_n$ and α_s. As γ is an automorphism fo Γ, it preserves adjacency and non-adjacency in Γ. Note that the only difference between Γ and Γ_s is between the points Q of type (ii) and R of type (iii), more specifically, Q is adjacent to R in Γ if and only if Q is non-adjacent to R in Γ_s. It follws that γ induces an automorphism of Γ_s. Thus, by Lemma 7.6, $\mathrm{Aut}(\Gamma_s)$ has at most three orbits on the vertices of Γ_s, being the points of each type. The result now follows since by Lemma 7.7, $\mathrm{Aut}(\Gamma_s)$ has at least three orbits. $\square$

We will now show that every automorphism of Γ_s is an automorphism of Γ, and this will completely determine the automorphism group of Γ_s.

The following is well known, https://cameroncounts.files.wordpress.com/2015/04/pps1.pdf,
in the proof of Theorem 7.7 (or here is another reference: http://www.e-booksdirectory.com/details.php?eboo
Is it available on xarchive? Perhaps we should ask him to put it on there for posterity)

Result 7.9 *Let Γ be the strongly regular graph associated with the quadric $\mathcal{Q}_n$. Then the g-cliques of Γ (of size $2^{g+1}-1$) are in one to one correspondence with the generators of $\mathcal{Q}_n$.*

Lemma 7.10 *We can recover Γ (and hence $\mathcal{Q}_n$, as a set of points and subspaces) from Γ_s $(0 \le s < g)$.*

Proof We wish to recover Γ and hence $\mathcal{Q}_n$. If $s = 0$ then $\Gamma = \Gamma_0$ by Theorem 5.1. So suppose $s > 0$. The vertices of Γ are the vertices of Γ_s. By Lemma 7.7 we can distinguish the vertices of each type in Γ_s by the number of maximal cliques through them. Thus we can reconstruct the original graph Γ by keeping the adjacencies the same, except reversing the adjacencies between points of type (ii) and type (iii).

By Result 7.9, from Γ we can obtain all the generators of $\mathcal{Q}_n$, and by intersecting the generators pairwise, we can firstly recover the hyperplanes of each generator, and by continuing this process, recover the lattice of subspaces of the generators. At the end of this process, we have: the points of $\mathcal{Q}_n$, all the lines contained in $\mathcal{Q}_n$; the planes contained in $\mathcal{Q}_n$; ...; the g-spaces contained in $\mathcal{Q}_n$. $\square$

For each generator of $\mathcal{Q}_n$ define a *generator set* G as a set of points of Γ_s which corresponds to a generator of $\mathcal{Q}_n$, which by Result 7.9 corresponds to a g-clique of Γ.

Lemma 7.11 *Let γ be an automorphism of Γ_s $(0 < s < g)$. Then γ preserves the generators sets of Γ_s.*

Proof The generators sets and the g-cliques of Γ_s which contain only points of type (i) and type (ii) are the same, so there is nothing to show in this case. Now consider a generator set G of Γ_s consisting of points of type (i), (ii) and (iii) and sets A, B, C representing points of these types, so $G = A \cup B \cup C$ (A, B, C possibly empty). As $s > 0$, by Lemma 7.7, γ preserves the points of each type, that is A^γ, B^γ and C^γ are points of type (i), (ii) and (iii) respectively. Now the elements of $A \cup B$ are a clique, so the elements of $A^\gamma \cup B^\gamma$ are a clique, similarly the elements of $A^\gamma \cup C^\gamma$ form a clique. As the elements of B are not adjacent in Γ_s to any element of C, and γ is an automorphism, it follows that the elements of B^γ are not adjacent in Γ_s to any element of C^γ. Thus in Γ, $(A \cup B \cup C)^\gamma$ is a g-clique, and by Result 7.9 $(A \cup B \cup C)^\gamma$ is a generator set of Γ_s. Thus γ permutes the generators sets of Γ_s. $\square$

Proof *of Theorem 7.1* Consider any automorphism γ of Γ_s. By Lemma 7.11 it preserves the generator sets of Γ_s, and by Lemma 7.3 it is an automorphism of Γ. Thus $\mathrm{Aut}(\Gamma_s) \subseteq \mathrm{Aut}(\Gamma)$. By Theorem 7.8 $\mathrm{Aut}(\Gamma_s)$ fixes α_s, and so $\mathrm{Aut}(\Gamma_s) = \mathrm{Aut}(\Gamma_s)_{\alpha_s} \subseteq \mathrm{Aut}(\Gamma)_{\alpha_s}$. By Theorem 7.8 $\mathrm{Aut}(\Gamma)_{\alpha_s} \subseteq \mathrm{Aut}(\Gamma_s)$ which completes the proof. $\square$

References

[1] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language, *J. Symbolic Comput.*, **24** (1997), 235–265.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier. *Distance-regular graphs.* Ergebnisse der Mathematik 3.18, Springer, Heidelberg, 1989.

[3] A. E. Brouwer and W. H. Haemers. *Spectra of graphs.* Springer, 2012.

[4] C. D. Godsil and B. D. McKay. Constructing cospectral graphs. *Aequationes Math.*, **25** (1982) 257–268.

[5] *Projective Geometry over Finite Fields, Second Edition.* Oxford University Press, 1998.

[6] J. W. P. Hirschfeld. and J. A. Thas. *General Galois Geometries.* Oxford University Press, 1991.

[7] W. M. Kantor. Strongly regular graphs defined by spreads. *Israel J. Math.*, **41** (1982) 298–312.

[8] Z.-x. Wan. *Geometry of classical groups over finite fields*. Studentlitteratur, 1993.

[9] D. E. Taylor. *The geometry of the classical groups*, Sigma series in pure mathematics, Heldermann Verlag, 1992.