# ÷ 3 Redux

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**Abstract**

Division by 3 is a classic problem of set theory, with a long and interesting history. Using the concept of "pointers", we present a proof that aims to be as simple, explicit and constructive as possible.

## 1 Introduction

The problem of "dividing by 3" can be stated stated as follows: given a bijection between the sets $A \times \{0,1,2\}$ and $B \times \{0,1,2\}$, prove, without using the axiom of choice, that there is a bijection between $A$ and $B$. Solving this problem is equivalent to solving division by $n$ for any $n$.

In his *Inaugural-Dissertation* from 1901, entitled "Untersuchungen aus der Mengenlehre", Felix Bernstein (of Cantor-Schröder-Bernstein fame) provided a detailed proof for "division by 2". In addition, he also provided a short proof outline for the general case of division by $n$. This paper was reprinted, with minor corrections, in the *Mathematische Annalen* in 1905, the world's leading mathematical publication at the time [1]. Bernstein's proof outline has stymied researchers for decades, indeed mathematicians of the caliber of Tarski and Conway more or less politely expressed their doubts as to whether Bernstein even had a proof.

Lindenbaum and Tarski claimed to have found a proof in 1926 [2], but the proof was lost. The proof was by Lindenbaum, who died in the Holocaust, and Tarski forgot. Sierpiński, Lindenbaum's PhD supervisor, worked on it but failed. Finally, Tarski published a proof in 1949 [3]. Later on, Doyle and Conway published another proof, which they claimed to be similar to Lindenbaum's lost proof. More details about the history and be fond in Doyle and Conway [4] and Doyle and Qiu [5], the latter also providing a new proof.

This paper provides an elementary and constructive proof, or at least as constructive as is possible given the problem at hand.

## 2 Preliminary Definitions and Theorems

**Notation 2.1** (Injection)**.** An injection from $B$ to $A$ is denoted $B \preccurlyeq A$.

**Notation 2.2** (Set Addition)**.** Given two sets $A$ and $B$, $A + B$ denotes a set that corresponds to the union of $A$ and $B$ in the case when $A$ and $B$ are disjoint. In general, we take this to be a set that can be put into bijection with the union of $A \times \{0\}$ and $B \times \{1\}$ which are disjoint by construction.

**Definition 2.3** (Swallowing)**.** We say that $B$ is swallowed by $A$, if $B + A \preccurlyeq A$. This is denoted $B \ll A$.

Swallowing is a generalisation of the Hilbert Hotel concept. There is a hotel with $A$ rooms, which are all occupied. $B$ new guests arrive. They are all able to check in by reassigning rooms. After rooms have been reassigned, the hotel may or may not be full.

Note that a set is said to be *Dedekind-infinite* if it swallows some non-empty set. This definition of infinity does not make any reference to numbers.

**Theorem 2.4** (Swallowing Criterion)**.** *$B \ll A$ is swallowed by $A$ if and only if we can associate to each element of $B$ a countably infinite sequence of distinct elements of $A$ such that the sequences of two different elements of $B$ do not overlap.*

*Proof.* When $B \ll A$, let $f$ be the injection from $B + A$ to $A$. We write $f^n$ for successive function composition, e.g. $f^2(x) = f \circ f(x)$. We want to prove that $B_n = (B, f(B), ..., f^n(B))$ is composed of pairwise disjoint sets. This is trivially true for $n = 0$, as $B_0$ has only one element. We proceed by induction and assume that the proposition is true for $n$. Because $f$ is injective, the respective images of the elements of $B_n$, i.e. $(f(B), f^2(B), ..., f^{n+1}(B))$, are also disjoint. As all these sets are subsets of $A$, which has no common element with $B$, we conclude that $B_{n+1} = (B, f(B), f^2(B), ..., f^{n+1}(B))$ is also composed of disjoint sets. This concludes the proof. As a result, for a given $b \in B$, the sequence $(b, f(b), f^2(b), ...)$ has the required properties.

Note that because f in an injection, we obviously have $B \approx f(B) \approx f^2(B) \approx ...$ as well as $A \approx f(A) \approx f^2(A) \approx ...$

Conversely, if for each $b \in B$ we have a sequence $c(b) = (c_b(1), c_b(2), c_b(3), ...)$ of elements of $A$, we denote by $C$ the set of all elements of all sequences $c(b)$. An injection $g$ from $A + B$ to $A$ can now be defined as follows:

$$g(x) = \begin{cases} x & \text{if } \ x \in A \backslash C \\ c_b(1) & \text{if } \ x = b \in B \\ c_b(i+1) & \text{if } \ x = c_b(i) \in C \end{cases} \tag{1}$$

It is easy to verify that this injection is actually a bijection. □

There is another, more complex way of proving that the sets $B, f(B), f^2(B), ...$ are pairwise disjoint. Its advantage is that it provides additional insight into the structure of the sets $f^i(A)$ and $f^i(B)$. First, note that for $n \geq 0$:

$$f^n(A + B) = f^n(A) + f^n(B) \quad \text{and} \quad f^n(A) \cap f^n(B) = \varnothing$$

This can be proved via an elementary induction argument. Next, we again use induction to show that:

$$f^n(A + B) \subseteq f^{n-1}(A)$$

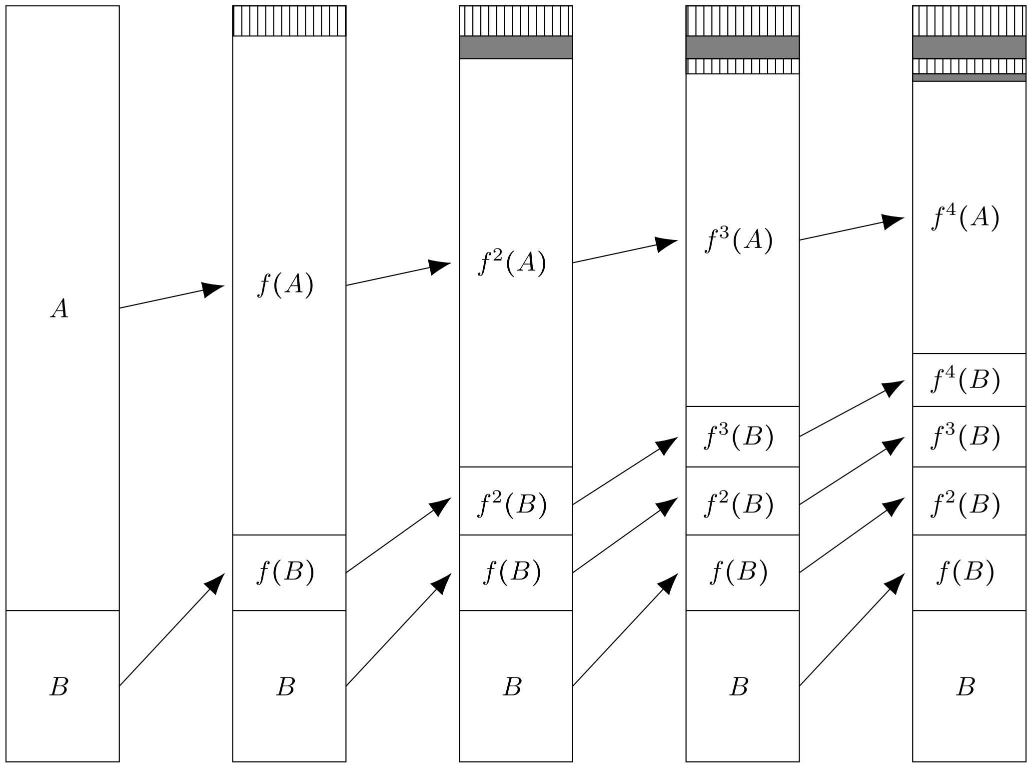


Figure 1: Visualising swallowing

The relation holds for $n = 1$. Assuming it holds for any $n \leq m$, we see that it also holds for $m + 1$ by writing:

$$f^{m+1}(A + B) = f \circ f^m(A + B)) \subseteq f \circ f^{m-1}(A) = f^m(A)$$

As a result, we obviously have:

$$f^n(A) \subseteq f^{n-1}(A) \quad \text{and} \quad f^n(B) \subseteq f^{n-1}(A)$$

We can finally write:

$$f^n(B) \subseteq f^{n-1}(A) \subseteq ... \subseteq f(A) \subseteq A$$

In other words, for $m < n$, we have $f^n(B) \subseteq f^m(A)$. Knowing that $f^m(A) \cap f^m(B) = \varnothing$, we conclude that $f^n(B) \cap f^m(B) = \varnothing$. We can apply induction one last time to conclude that the sets $f^n(B)$ are indeed pairwise disjoint.

**Observation 2.5** (Swallowing: canonical bijection)**.** In theorem 2.4 we have seen how, from an injection $f : A + B \to A$ we can obtain a set of countably infinite sequences, one for every element of B. The sequence $c(b) = (c_b(1), c_b(2), ...)$ is actually equal to $(f(b), f^2(b), ...)$. As before we can denote by $C$ the set of elements of all these sequences. We can then construct a bijection between $A + B$ and $A$:

$$g(x) = \begin{cases} x & \text{if} \quad x \in A \backslash C \\ f^{i+1}(b) & \text{if} \quad x = f^i(b) \end{cases} \tag{2}$$

**Observation 2.6** (Cantor's Theatre)**.** Cantor's theatre has two sections, A and B. It is a very successful theatre, and all seats are usually filled by season ticket holders. However section B is being closed for renovation, so Cantor has ordered new tickets for everyone from the Kronecker print shop. At the next representation, however, there is a problem: the theatre is almost empty even though everyone has gotten a seat. What will the press say? Unfazed, Cantor asks everyone to go back into the lobby and provides the following instructions: (1) All holders of original section A season passes go to your usual seats (2) holders of original B season passes, go claim your seats in section A using your new tickets (3) if you're an original season pass holder in section A, and someone comes to claim your seat using a new ticket, yield your seat and use your new ticket to claim a new seat in turn. As a result, everyone has a seat and no seats are empty. In addition, everyone had to change seat at most once.

**Examples 2.7.** If $A \approx B \approx \mathbb{N}^*$, consider the injection $f : A + B \to A$ defined by $f(a) = f(a,0) = 2a$ and $f(b) = f(b,1) = 3^b$. For a given element $b$ we have the sequence:

$$(b,1) \to (3^b,0) \to (2 \cdot 3^b,0) \to (2^2 \cdot 3^b,0) \to (2^3 \cdot 3^b,0) \to ...$$

We therefore have:

$$c_b(n) = 2^{n-1}3^b$$

The corresponding canonical bijection from $A + B$ to $A$ is:

$$g(x) = \begin{cases} 3^b & \text{if } x = (b,1) \in B \\ 2^{i+1}3^n & \text{if } x = (2^i 3^n, 0) \in A \text{ for some } i \in \mathbb{N}, n \in \mathbb{N}^* \\ a & \text{for any other } (a,0) \in A \end{cases}$$

**Examples 2.8.** If $A \approx B \approx \mathbb{N}^*$, consider the injection $f : A + B \to A$ defined by $f(a) = 2^a$ and $f(b) = 3^b$. This is obviously quite a "sparse" injection. For a given element $b$ we have the sequence:

$$(b,1) \to (3^b,0) \to (2^{3^b},0) \to (2^{2^{3^b}},0) \to ...$$

This corresponds to:

$$c_b(i) = (2 \uparrow\uparrow (i-1))^{3^b}$$

The canonical bijection is therefore:

$$g(x) = \begin{cases} 3^b & \text{if } x = (b,1) \in B \\ (2 \uparrow\uparrow (i+1))^{3^b} & \text{if } x = ((2 \uparrow\uparrow i)^{3^b}, 0) \in A \text{ for some } i \in \mathbb{N}, n \in \mathbb{N}^* \\ a & \text{for any other } (a,0) \in A \end{cases}$$

**Theorem 2.9** (Repeated Swallowing)**.** *When $B_1 \ll A$ and $B_2 \ll A$, then $B1 + B2 \ll A$. More generally, if every $B_1 .. B_k$ is swallowed by $A$, then $B_1 + ... + B_k$ is also swallowed by $A$.*

*Proof.* The intuition is simple: first $B_1$ check in, then $B_2$ check in etc.

Now let's prove this formally. $A + B_i \ll A$ means that there exists an injection $f_i : B_i + A \to A$. Let $C_i = B_i \cup ... \cup B_k$. We then define $g_i$ as a function from $A + C_i$ to $A$ such that:

$$g_i(x) = \begin{cases} f_i(x) \text{ if } x \in A + B_i \\ x \text{ otherwise} \end{cases}$$

The function $g_k \circ ... \circ g_1$ is then an injection from $A + C_k$ to $A$. When $k = 3$ this will look as in Figure 2.

□

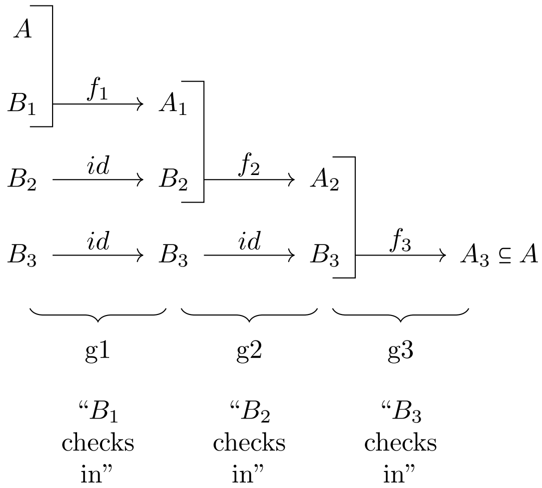


Figure 2: Successive swallowing

**Theorem 2.10.** *[Russian Doll theorem] Let $k$ be a a nonzero integer. For each integer $j \geq 1$, let $A(j), B(j,1), ..., B(j,k)$ be disjoint sets and define:*

$$A = \bigcup_{j\geq 1} A(j) \qquad B(j) = \bigcup_{i=1}^{k} B(j,i) \qquad C = \bigcap_{j\geq 1} B(j) \qquad M(j) = A(j) + B(j)$$

*We assume that for any $j \geq 1$, we have:*

$$A(j) \gtrcurlyeq B(j,1) \gtrcurlyeq B(j,2) \gtrcurlyeq ... \gtrcurlyeq B(j,k) \qquad \textit{and} \qquad M(j+1) \subseteq B(j)$$

*We can then conclude that $C \ll A$.*

Note that a number of injections are implicit in the preceding definitions. We introduce $B(j,0)$ as an alternative notation for $A(j)$ and define, for $i, j \geq 1$, the following injections:

$$\lambda_{j,i} : B(j,i) \mapsto B(j,i-1)$$

We can then define $\mu_{j,i} = \lambda_{j,i} \circ ... \circ \lambda_{j,1}$ which is an injection from $B(j,i)$ to $B(j,0) = A(j)$.

*Proof.* Figure 3 gives an idea of what is happening in the case $A(j+1) + B(j+1) = B(j)$.

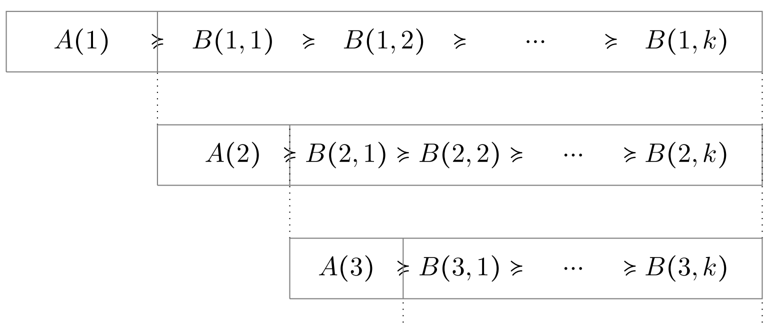


Figure 3: Russian dolls (first 3 levels)

At every step, the set we are dealing with is divided into $k+1$ parts, of which $A(j)$ is the "largest". And at every step we are removing (at least) a set of the size of this largest set. In the context of finite sets, we can thus write $|B(j+1)| \le \frac{k}{k+1} \cdot |M(j+1)| \le \frac{k}{k+1} \cdot |B(j)|$. This means that there is some integer $m$ such that $A(m) = B(m,1) = ... = B(m,k) = \varnothing$, which obviously implies $C = \varnothing$. But when $M$ is infinite, a different proof is necessary.

If $C = \varnothing$ the conclusion is trivial, so we will henceforth assume that $C \ne \varnothing$. This immediately implies that $B(j) \ne \varnothing$ and $A(j) \ne \varnothing$ for all $j$. Note that in this case the sets $A(j)$ are pairwise disjoint. Consider $c \in C$. By definition of $C$, there is a sequence $I_c = (i_{(c,j)})_{j \in \mathbb{N}^*}$ such that $\forall j \in \mathbb{N}^*, c \in B(j, i_{(c,j)})$. In other words,

$$
\begin{aligned}
c &\in B(1, i_{(c,1)}) \\
c &\in B(2, i_{(c,2)}) \\
&...
\end{aligned}
$$

As the elements of the sequence $I_c$ can only take $k$ different values, at least one of these values is bound to occur an infinite number of times. We define $m_c$ to be the smallest such value. And we define $C_m = \{c \in C \mid m_c = m\}$ for $m \in [k]$.

It is now quite easy to see that to each element of $C_m$ we can associate an infinite series of elements of $A = \bigcup_i A(i)$. Indeed, $c$ will be an element of an infinite collection of sets $B(j_{c1}, m)$, $B(j_{c2}, m)$, $B(j_{c3}, m)$, ... to which we can associate an infinite series of points in the disjoint sets $A(c1), A(c2), A(c3), ...$ via the injections $\mu$ which we defined above. In addition, the series of points in $A$ associated with two different elements of $C_m$ will not overlap. In other words, $C_m$ is swallowed by $A$. We can thus apply theorem 2.9 to conclude that $C = \bigcup C_m$ is swallowed by $A$.

□

**Theorem 2.11.** *[Exploding Doll theorem] Let $k$ be a a nonzero integer. For integers $j \ge 1$ and $i \in [k]$, let $A(j)$ and $B(j,i)$ be sets that are all pairwise disjoint and define:*

$$
B(j) = \bigcup_{i=1}^{k} B(j,i) \qquad M(j) = A(j) + B(j)
$$

*We assume that for any $j \ge 1$, we have:*

$$
A(j) \succcurlyeq B(j,1) \succcurlyeq B(j,2) \succcurlyeq ... \succcurlyeq B(j,k) \qquad \text{and} \qquad M(j+1) \preccurlyeq B(j)
$$

*For $j \ge 2$, we denote by $\psi_j$ the injection $M(j) \mapsto B(j-1)$ and by $\varphi_j = \psi_2 \circ ... \circ \psi_j$ the resulting injection $M(j) \mapsto B(1)$. We further define $\theta_j = \varphi_j^{-1}$, which is a function $B(1) \mapsto M(j)$ whose domain is necessarily a strict subset of $B(1)$ when $j \ge 3$.*

*We can then conclude that there exists a set $C \ll M(1)$ such that $\forall x \in M(1) \backslash C$, the sequence $x, \theta_1(x), \theta_2(x), ...$ has only a finite number of elements.*

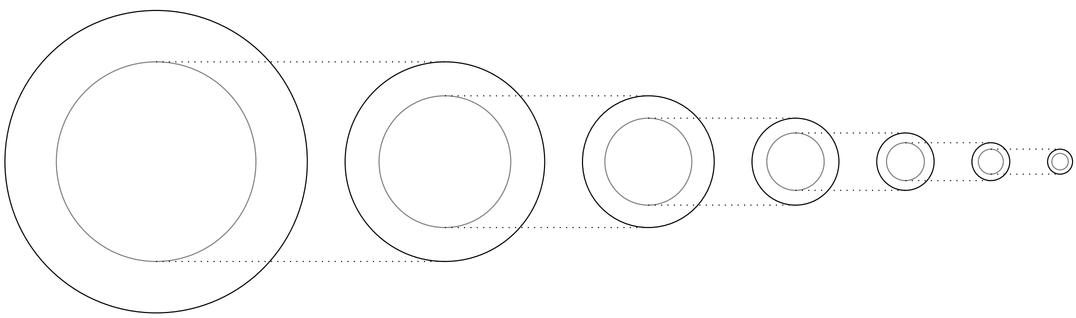

Figure 4: Exploding dolls

Informally, the preceding result means that, denoting $S = M(1) \backslash C$, the sequence $S, \theta_1(S), \theta_2(S), ...$ "converges" to $\varnothing$.

Note that this setup can be seen as follows. Starting with $B(1)$, we successively apply the functions $\psi_2^{-1}, \psi_3^{-1}, \ldots$ to obtain the sets $M(2) = \theta_2(B(1)), M(3) = \theta_3(B(1)), \ldots$. This is illustrated in figure 4. Each of these "ejected" sets has a well-defined structure, as it is split into subsets with clearly defined injections between them.

When considering sets with infinite cardinality, the sets $M(j)$ can remain "large" no matter how many steps are taken. For example consider a countable subset $\{a_j\}_{j\geq 1}$ of real numbers in $]0, 1[$. We can then obtain $M(j) = \{a_j + z \mid z \in \mathbb{Z}\}$ by defining $A(j) = a_j + (k+1)\mathbb{Z}$ and $B(j,i) = a_j + i + (k+1)\mathbb{Z}$ for $j \geq 1$ and $i \in [k]$. Then, taking any injection $\phi : \mathbb{Z} \mapsto \mathbb{Z} \backslash (k+1)\mathbb{Z}$, we can define the injection $\psi_j$ from $M(j)$ to $B(j-1)$ as $\psi_j(a_j + m) = a_{j-1} + \phi(m)$.

*Proof.* We define sets $A'(j)$ and $B'(j,i)$ as follows:

$$\begin{aligned} A'(1) &= A(1) & B'(1,i) &= B(1,i) \\ A'(2) &= \varphi_2(A(2)) & B'(2,i) &= \varphi_2(B(2,i)) \\ &\ldots & & \\ A'(j) &= \varphi_j(A(j)) & B'(j,i) &= \varphi_j(B(j,i)) \end{aligned}$$

These sets are all subsets of $M(1)$. We further define $M'(j) = A'(j) + B'(j)$ and note that Theorem 2.10 can then be applied to the sets $A'(j)$ and $B'(j,i)$. As a result, the set $C' = \bigcap_{j\geq 1} B'(j) = \bigcap_{j\geq 2} \varphi_j(B(j))$ is swallowed by $A' = \bigcup_{j\geq 1} A'(j) \subseteq M(1)$.

Consider $x \in M(1)$ such that the sequence $x, \theta_1(x), \theta_2(x), \ldots$ is infinite. If $y_j = \theta_j(x) = \varphi_j^{-1}(x)$ exists for $j \geq 3$, this means that $x = \varphi_j(y_i) = \varphi_{j-1} \circ \psi(y_j) \in \varphi_{j-1}(B(j-1))$. We can therefore conclude that $x$ must be an element of $C'$. Conversely, if $x \notin C'$, the above-mentioned sequence cannot be infinite. This completes the proof.

□

# 3 Dividing by 2

We briefly revisit division by 2, as a reminder of how much simpler it is compared to division by $n \geq 3$. We are given the bijection $2X \sim 2Y$, which we rewrite as $X_1 + X_2 \sim Y_1 + Y_2$. Denote $\beta$ the bijection between $2X$ and $2Y$, $\varphi$ the canonical bijection between $X_1$ and $X_2$ and $\psi$ the canonical bijection between $Y_1$ and $Y_2$. We further define $X_{ij} = Y_i \cap \beta(X_i)$, and similarly $Y_{ji}$, so that $X_{ij} \sim Y_{ji}$.

Following Bernstein [1], we can now take any element of $X_{12}$ (the "start set") and successively apply the bijections $\varphi$, $\beta$, $\psi$, $\beta$, $\varphi$, $\beta$, etc, stopping only if we end up in X21, which is the "stop set". This is shown in Figure 5.

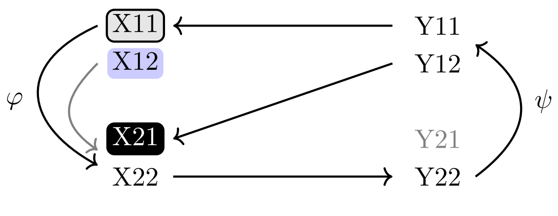


Figure 5: Setting up division by 2

Because we are applying bijections at each stage, the paths taken by two different elements of $X_{12}$ will have no common elements. Some elements of $X_{12}$ may end up having infinite paths and can therefore be swallowed by $X_{11}$. Those with finite paths are mapped injectively to $X_{21}$, and therefore also to $Y_{12}$. As a result, we obtain the injection $X_1 \preccurlyeq Y_1$. By symmetry we must also have $Y_1 \preccurlyeq X_1$, which yields the desired result via Cantor–Schröder–Bernstein.

# 4 Dividing by 3

We are given a bijection between 3X and 3Y.

Let's think of the set 3X as **families** composed of three **persons** each and labeled a, b, and c in order of seniority. They all sit comfortably in the lounge of Cantor's restaurant, each at a definite **spot**, waiting for dinner. Let's also think of 3Y as a set of tables, with each table having three seats labelled $\alpha$, $\beta$ and $\gamma$. Each person receives one **card** corresponding to a **seat** at a **table**, no two persons receive the same card, and a card for each seat of every table is given to some person (that is the bijection assumption). For example, guest $G3b$ (i.e. the person from group 3 with label b) could have card $T4\alpha$ (i.e. seat $\alpha$ at the table number 4).

Our aim is to ensure that, by exchanging cards, all families can sit together.

## 4.1 The alpha phase

During the alpha phase, the fact of holding a card labelled $\alpha$ gives a family the right to claim the entire corresponding table, with the members $(a, b, c)$ going to the respective seats $(\alpha, \beta, \gamma)$. We proceed in stages (1, 2, 3...). Each stage has 4 steps.

For each stage $i$, we denote by $W_i$ the people who are still in the lounge (or waiting room) at the beginning of that stage. At the beginning of stage 1, everyone will still be holding their original card. In subsequent steps, they may be holding either their original card, another card or a special hiding card (this will be explained below).

### 4.1.1 Step 1: Swap

This applies to families holding at least one $\alpha$ card. We want a definite rule to decide to which table to go, and who exactly goes to which seat. There is more than one way to do this. In fact this step could even be entirely skipped at the cost of some added complexity later.

The rule we will use is the following: the $\alpha$ card held by the most senior member determines the table they go to. This card is given to the $a$ family member, who is the lead for this phase. Then, if necessary, cards are swapped so that the other family members have the correct card for their seat.

In this first example, there are two $\alpha$ cards, the one held by the most senior family member (in this case $b$) is given to the family lead. As there are no more cards for table 2, there is nothing more to do.

| Member | original | reordered |
|---|---|---|
| $G1a$ | $T3\gamma$ | $T2\alpha$ |
| $G1b$ | $T2\alpha$ | $T3\gamma$ |
| $G1c$ | $T1\alpha$ | |

Table 1: Reordering with one direct card

In the second example, with family 3, we have two $\alpha$ cards, the first one determines the table (4) and is given to the family lead for this round. Then we make sure that $G3c$ gets the correct card.

| Member | original | reordered |
|---|---|---|
| $G3a$ | $T4\gamma$ | $T4\alpha$ |
| $G3b$ | $T4\alpha$ | $T2\alpha$ |
| $G3c$ | $T2\alpha$ | $T4\gamma$ |

Table 2: Reordering with two direct cards

Finally, in this third example all family members already have cards for the same table, they just need to reorder them.

| Member | original | reordered |
|---|---|---|
| $G3a$ | $T3\beta$ | $T3\alpha$ |
| $G3b$ | $T3\alpha$ | $T3\beta$ |
| $G3c$ | $T3\gamma$ | |

Table 3: Reordering with three direct cards

Once the card swapping is done, there are only four possibilities concerning the distribution of $\alpha$ cards: a family can have 3, 2, 1 or 0 of them. We denote the sets of people who are part of these families as $W_i(\alpha 3)$, $W_i(\alpha 2)$, $W_i(\alpha 1)$ and $W_i(\neg\alpha)$, respectively.

Let's define the following:

- $A(i)$ is the set of all $\alpha$ cards held by the most senior family members, i.e. family members $a$;
- $B(i,1)$ is the set of all $\alpha$ cards held by the second most senior $\alpha$ card holders in a family, when that family holds at least two $\alpha$ cards;
- $B(i,2)$ is the set of all $\alpha$ cards held by the third most senior $\alpha$ card holders in a family, when that family holds at least three $\alpha$ cards.
- $B(i) = B(i,1) + B(i,2)$

We note that:

$$A(i) \geqslant B(i,1) \geqslant B(i,2)$$

These sets are marked respectively in black, with vertical lines and with dots in Figure 6.

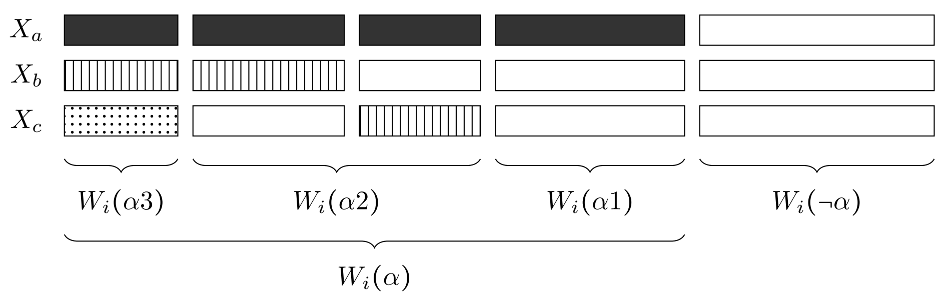


Figure 6: Types of families by $\alpha$ card holdings

#### 4.1.2 Step 2: Move

Again, this only applies to families holding at least one $\alpha$ card. They all know to which table they will go.

Those who already have the correct card take that card and go directly to their seat. This will always be the case of the family lead, as she will hold the $\alpha$ card for the destination table.

Others, if they do not yet have the correct card for their seat, do the following: they leave their original card on their spot in the lounge, go to the person holding the card they need, claim that card and replace it with a pointer to their spot. Then they take their claimed card and go to their seat. Note that they will always be able to find the card they need, as no-one else is able to claim the card.

A **pointer** represents the right to use a card that is located in a certain spot. Note that there is only ever a single pointer to a given spot, and that all the pointers created during stage $i$ are by members of $W_i(\alpha)$.

At the end of this step, all the families which initially held at least one $\alpha$ card, i.e. all members of the set $W_i(\alpha)$ will all have moved to their designated tables.

#### 4.1.3 Step 3: Pointers!

The people that now remain in the lounge are those in set $W_i(\neg\alpha)$. If their card was claimed in the preceding step, they now have a pointer that they can follow. Of course, their pointer could lead hem to a spot in which the card has also been replaced by a pointer, in which case they have to follow that pointer, and so on. So there are two cases:

- if, when following their pointer, they find a card in a finite number of steps, they take that card, together with all the pointers collected along the way, and return to their original spot;
- if the sequence of pointers is infinite, they collect these pointers and also return to their original spot. We consider this (countably) infinite series of to represent a special **hiding card**. Clearly, anyone holding such a card can be "swallowed" but we won't do that yet.

Note that any sequence of pointers, either attached to a card, or representing a hiding card is unique in the sense that two different sequences cannot have any pointer in common.

At the end of this step, some members of the set $W_i(\neg\alpha)$ may have received $\alpha$ cards by following pointers. These $\alpha$ cards must have been received from members of the set $B(i)$, as all members of $A(i)$ already had the correct card and did not need to claim it from someone else.

#### 4.1.4 Step 4: Conclude

If there are no more people in the waiting room, we are done! Everyone has found a seat at a table.

If there are still people in the waiting room, but they only have hiding cards, then they can be swallowed and we are also done.

If there are people in the waiting room holding $\alpha$ cards (which they must have received by following pointers), we move on to the next stage of the $\alpha$ phase.

If there are people in the waiting room holding cards, but none hold an $\alpha$ card, we end the $\alpha$ phase and enter the $\beta$ phase (if some hold $\beta$ cards) or go directly to the $\gamma$ phase.

#### 4.1.5 Example

An example of the first stage in a very simple finite setting is provided in Table 4.

### 4.2 Dealing with zombie cards

We still have one problem to solve: what if we cannot get rid of all the $\alpha$ cards, even if we take a (countably) infinite number of steps? We could sidestep the problem by attempting to take as many steps as necessary, but we do not think that this is a reasonable direction to take. Instead, we must acknowledge that our hope to complete the alpha phase at the first try appears to be

| Member | initial | swap | pointer | step1 | source | follow | step2 |
|---|---|---|---|---|---|---|---|
| $G1a$ | $T3\gamma$ | $T2\alpha$ | | $T2\alpha$ | | | |
| $G1b$ | $T2\alpha$ | $T3\gamma$ | $\mapsto G2c$ | $T2\beta$ | from $G3c$ | | |
| $G1c$ | $T1\alpha$ | | | $T2\gamma$ | from $G4a$ | | |
| $G2a$ | $T3\alpha$ | | | $T3\alpha$ | | | |
| $G2b$ | $T1\beta$ | | | $T3\beta$ | from $G4c$ | | |
| $G2c$ | $T1\gamma$ | | | $T3\gamma$ | from $G1b$ | | |
| $G3a$ | $T2\beta$ | $T4\alpha$ | | $T4\alpha$ | | | |
| $G3b$ | $T4\alpha$ | $T4\beta$ | | $T4\beta$ | | | |
| $G3c$ | $T4\beta$ | $T2\beta$ | $\mapsto G1b$ | $T4\gamma$ | from $G4b$ | | |
| $G4a$ | $T2\gamma$ | | $\mapsto G1c$ | | | | $T1\alpha$ |
| $G4b$ | $T4\gamma$ | | $\mapsto G3c$ | | | $\mapsto G1b \mapsto G2c$ | $T1\gamma$ |
| $G4c$ | $T3\beta$ | | $\mapsto G2b$ | | | | $T1\beta$ |

Table 4: Example of a swap/move/pointers! sequence

doomed to failure. All we can do is consider it a dry run and make note of the $\alpha$ cards that get passed on indefinitely. We might as well call these them *zombie cards*.

To each of these zombie cards we can associate a (countably) infinite series of pointers, and therefore also members of $3X$. And we do know the original holders of each of these cards. So we can simply have them removed from the game, and start anew. And after the second run, thanks to the *Exploding Dolls* theorem, there will be no loose ends. The proof, for now, is left to the reader.

### 4.3 Completing the process

Once we have gotten rid of the $\alpha$ cards, we deal with the $\beta$ cards, and then with the $\gamma$ cards. Anyone left after this can be "swallowed" and we're done.

## 5 Conclusion and Next Steps

Our interest in this topic started a few years ago, when we first encountered the "Division by three" paper by Doyle and Conway [4]. After reading and digesting it, we plunged ourselves into Bernstein's original. Where Tarski and others failed, we would succeed! Pure hybris of course, but it was fun. The project then lay dormant for well over a year, before we picked it up again, tried a new approach and very gradually came up with the one presented in this paper.

In the meantime, we also discovered a paper by Doyle and Qiu [5] and it seems that their earlier approach is quite close to what we came up with. So far we have not analysed it in detail, but note that the presentation is somewhat informal. Another paper by Schwartz [6], written specifically to explain Doyle and Qiu's proof, also has a mostly descriptive style.

From the Doyle/Qiu paper, we also became aware that Arie Hinkis has produced a proof (in [7]) which he claims follows Bernstein's argument. As of this writing, we have not yet had time to read this proof.

Therefore, much remains to be done. We might want to take a closer look at the more recent papers. Also, it could be useful to explain why Bernstein's original approach presents such seemingly insurmountable difficulties. Alas, *vita brevis* so for now we do not promise anything beyond what is in this document.

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