Week 1 TheoryPrep Readings

May 2020

Selected from Mathematics for Computer Science (Lehman, Leighton, Meyer 2017) and Building Blocks for Computer Science (Fleck 2013)

Introduction

This text explains how to use mathematical models and methods to analyze problems that arise in computer science. Proofs play a central role in this work because the authors share a belief with most mathematicians that proofs are essential for genuine understanding. Proofs also play a growing role in computer science; they are used to certify that software and hardware will *always* behave correctly, something that no amount of testing can do.

Simply put, a proof is a method of establishing truth. Like beauty, "truth" sometimes depends on the eye of the beholder, and it should not be surprising that what constitutes a proof differs among fields. For example, in the judicial system, *legal* truth is decided by a jury based on the allowable evidence presented at trial. In the business world, *authoritative* truth is specified by a trusted person or organization, or maybe just your boss. In fields such as physics or biology, *scientific* truth is confirmed by experiment.[1] In statistics, *probable* truth is established by statistical analysis of sample data.

Philosophical proof involves careful exposition and persuasion typically based on a series of small, plausible arguments. The best example begins with "Cogito ergo sum," a Latin sentence that translates as "I think, therefore I am." This phrase comes from the beginning of a 17th century essay by the mathematician/philosopher, René Descartes, and it is one of the most famous quotes in the world: do a web search for it, and you will be flooded with hits.

Deducing your existence from the fact that you're thinking about your existence is a pretty cool and persuasive-sounding idea. However, with just a few more lines

[1] Actually, only scientific *falsehood* can be demonstrated by an experiment—when the experiment fails to behave as predicted. But no amount of experiment can confirm that the *next* experiment won't fail. For this reason, scientists rarely speak of truth, but rather of *theories* that accurately predict past, and anticipated future, experiments.

of argument in this vein, Descartes goes on to conclude that there is an infinitely beneficent God. Whether or not you believe in an infinitely beneficent God, you'll probably agree that any very short "proof" of God's infinite beneficence is bound to be far-fetched. So even in masterful hands, this approach is not reliable.

Mathematics has its own specific notion of "proof."

Definition. A *mathematical proof* of a *proposition* is a chain of *logical deductions* leading to the proposition from a base set of *axioms*.

The three key ideas in this definition are highlighted: *proposition*, *logical deduction*, and *axiom*. Chapter 1 examines these three ideas along with some basic ways of organizing proofs. Chapter 2 introduces the Well Ordering Principle, a basic method of proof; later, Chapter 5 introduces the closely related proof method of induction.

If you're going to prove a proposition, you'd better have a precise understanding of what the proposition means. To avoid ambiguity and uncertain definitions in ordinary language, mathematicians use language very precisely, and they often express propositions using logical formulas; these are the subject of Chapter 3.

The first three Chapters assume the reader is familiar with a few mathematical concepts like sets and functions. Chapters 4 and 8 offer a more careful look at such mathematical data types, examining in particular properties and methods for proving things about infinite sets. Chapter 7 goes on to examine recursively defined data types.

0.1 References

[14], [49], [1]

1 What is a Proof?

Start

1.1 Propositions

Definition. A *proposition* is a statement (communication) that is either true or false.

For example, both of the following statements are propositions. The first is true, and the second is false.

Proposition 1.1.1. *2 + 3 = 5.*

Proposition 1.1.2. *1 + 1 = 3.*

Being true or false doesn't sound like much of a limitation, but it does exclude statements such as "Wherefore art thou Romeo?" and "Give me an A!" It also excludes statements whose truth varies with circumstance such as, "It's five o'clock," or "the stock market will rise tomorrow."

Unfortunately it is not always easy to decide if a claimed proposition is true or false:

Claim 1.1.3. *For every nonnegative integer n the value of $n^2 + n + 41$ is prime.*

(A *prime* is an integer greater than 1 that is not divisible by any other integer greater than 1. For example, 2, 3, 5, 7, 11, are the first five primes.) Let's try some numerical experimentation to check this proposition. Let

$$p(n) ::= n^2 + n + 41.^{1} \tag{1.1}$$

We begin with $p(0) = 41$, which is prime; then

$$p(1) = 43,\ p(2) = 47,\ p(3) = 53, \ldots,\ p(20) = 461$$

are each prime. Hmmm, starts to look like a plausible claim. In fact we can keep checking through $n = 39$ and confirm that $p(39) = 1601$ is prime.

But $p(40) = 40^2 + 40 + 41 = 41 \cdot 41$, which is not prime. So Claim 1.1.3 is false since it's not true that $p(n)$ is prime *for all* nonnegative integers n. In fact, it's not hard to show that *no* polynomial with integer coefficients can map all

[1] The symbol ::= means "equal by definition." It's always ok simply to write "=" instead of ::=, but reminding the reader that an equality holds by definition can be helpful.

nonnegative numbers into prime numbers, unless it's a constant (see Problem 1.26). But this example highlights the point that, in general, you can't check a claim about an infinite set by checking a finite sample of its elements, no matter how large the sample.

By the way, propositions like this about *all* numbers or all items of some kind are so common that there is a special notation for them. With this notation, Claim 1.1.3 would be

$$\forall n \in \mathbb{N}.\ p(n) \text{ is prime.} \tag{1.2}$$

Here the symbol $\forall$ is read "for all." The symbol $\mathbb{N}$ stands for the set of *nonnegative integers*: 0, 1, 2, 3, ... (ask your instructor for the complete list). The symbol "$\in$" is read as "is a member of," or "belongs to," or simply as "is in." The period after the $\mathbb{N}$ is just a separator between phrases.

Here are two even more extreme examples:

Conjecture. *[Euler] The equation*

$$a^4 + b^4 + c^4 = d^4$$

has no solution when a, b, c, d are positive integers.

Euler (pronounced "oiler") conjectured this in 1769. But the conjecture was proved false 218 years later by Noam Elkies at a liberal arts school up Mass Ave. The solution he found was $a = 95800, b = 217519, c = 414560, d = 422481$.

In logical notation, Euler's Conjecture could be written,

$$\forall a \in \mathbb{Z}^+ \forall b \in \mathbb{Z}^+ \forall c \in \mathbb{Z}^+ \forall d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4.$$

Here, $\mathbb{Z}^+$ is a symbol for the positive integers. Strings of $\forall$'s like this are usually abbreviated for easier reading:

$$\forall a, b, c, d \in \mathbb{Z}^+.\ a^4 + b^4 + c^4 \neq d^4.$$

Here's another claim which would be hard to falsify by sampling: the smallest possible x, y, z that satisfy the equality each have more than 1000 digits!

False Claim. $313(x^3 + y^3) = z^3$ *has no solution when* $x, y, z \in \mathbb{Z}^+$.

It's worth mentioning a couple of further famous propositions whose proofs were sought for centuries before finally being discovered:

Proposition 1.1.4 (Four Color Theorem)**.** *Every map can be colored with 4 colors so that adjacent*[2] *regions have different colors.*

[2]Two regions are adjacent only when they share a boundary segment of positive length. They are not considered to be adjacent if their boundaries meet only at a few points.

Several incorrect proofs of this theorem have been published, including one that stood for 10 years in the late 19th century before its mistake was found. A laborious proof was finally found in 1976 by mathematicians Appel and Haken, who used a complex computer program to categorize the four-colorable maps. The program left a few thousand maps uncategorized, which were checked by hand by Haken and his assistants—among them his 15-year-old daughter.

There was reason to doubt whether this was a legitimate proof—the proof was too big to be checked without a computer. No one could guarantee that the computer calculated correctly, nor was anyone enthusiastic about exerting the effort to recheck the four-colorings of thousands of maps that were done by hand. Two decades later a mostly intelligible proof of the Four Color Theorem was found, though a computer is still needed to check four-colorability of several hundred special maps.[3]

Proposition 1.1.5 (Fermat's Last Theorem)**.** *There are no positive integers x, y and z such that*

$$x^n + y^n = z^n$$

for some integer $n > 2$.

In a book he was reading around 1630, Fermat claimed to have a proof for this proposition, but not enough space in the margin to write it down. Over the years, the Theorem was proved to hold for all n up to 4,000,000, but we've seen that this shouldn't necessarily inspire confidence that it holds for *all* n. There is, after all, a clear resemblance between Fermat's Last Theorem and Euler's false Conjecture. Finally, in 1994, British mathematician Andrew Wiles gave a proof, after seven years of working in secrecy and isolation in his attic. His proof did not fit in any margin.[4]

Finally, let's mention another simply stated proposition whose truth remains unknown.

Conjecture 1.1.6 (*Goldbach*)**.** *Every even integer greater than 2 is the sum of two primes.*

Goldbach's Conjecture dates back to 1742. It is known to hold for all numbers up to 10^{18}, but to this day, no one knows whether it's true or false.

[3]The story of the proof of the Four Color Theorem is told in a well-reviewed popular (non-technical) book: "Four Colors Suffice. How the Map Problem was Solved." *Robin Wilson*. Princeton Univ. Press, 2003, 276pp. ISBN 0-691-11533-8.

[4]In fact, Wiles' original proof was wrong, but he and several collaborators used his ideas to arrive at a correct proof a year later. This story is the subject of the popular book, *Fermat's Enigma* by Simon Singh, Walker & Company, November, 1997.

For a computer scientist, some of the most important things to prove are the correctness of programs and systems—whether a program or system does what it's supposed to. Programs are notoriously buggy, and there's a growing community of researchers and practitioners trying to find ways to prove program correctness. These efforts have been successful enough in the case of CPU chips that they are now routinely used by leading chip manufacturers to prove chip correctness and avoid some notorious past mistakes.

Developing mathematical methods to verify programs and systems remains an active research area. We'll illustrate some of these methods in Chapter 5.

1.2 Predicates

A *predicate* can be understood as a proposition whose truth depends on the value of one or more variables. So "n is a perfect square" describes a predicate, since you can't say if it's true or false until you know what the value of the variable n happens to be. Once you know, for example, that n equals 4, the predicate becomes the true proposition "4 is a perfect square". Remember, nothing says that the proposition has to be true: if the value of n were 5, you would get the false proposition "5 is a perfect square."

Like other propositions, predicates are often named with a letter. Furthermore, a function-like notation is used to denote a predicate supplied with specific variable values. For example, we might use the name "P" for predicate above:

$$P(n) ::= \text{“}n \text{ is a perfect square”},$$

and repeat the remarks above by asserting that $P(4)$ is true, and $P(5)$ is false.

This notation for predicates is confusingly similar to ordinary function notation. If P is a predicate, then $P(n)$ is either *true* or *false*, depending on the value of n. On the other hand, if p is an ordinary function, like n^2+1, then $p(n)$ is a *numerical quantity*. **Don't confuse these two!**

1.3 The Axiomatic Method

The standard procedure for establishing truth in mathematics was invented by Euclid, a mathematician working in Alexandria, Egypt around 300 BC. His idea was to begin with five *assumptions* about geometry, which seemed undeniable based on direct experience. (For example, "There is a straight line segment between every

pair of points".) Propositions like these that are simply accepted as true are called *axioms*.

Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs." A *proof* is a sequence of logical deductions from axioms and previously proved statements that concludes with the proposition in question. You probably wrote many proofs in high school geometry class, and you'll see a lot more in this text.

There are several common terms for a proposition that has been proved. The different terms hint at the role of the proposition within a larger body of work.

- Important true propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a theorem.

These definitions are not precise. In fact, sometimes a good lemma turns out to be far more important than the theorem it was originally used to prove.

Euclid's axiom-and-proof approach, now called the *axiomatic method*, remains the foundation for mathematics today. In fact, just a handful of axioms, called the Zermelo-Fraenkel with Choice axioms (ZFC), together with a few logical deduction rules, appear to be sufficient to derive essentially all of mathematics. We'll examine these in Chapter 8.

1.5 Proving an Implication

Start

Propositions of the form "If P, then Q" are called *implications*. This implication is often rephrased as "P IMPLIES Q."

Here are some examples:

- (Quadratic Formula) If $ax^2 + bx + c = 0$ and $a \neq 0$, then
$$x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a.$$
- (Goldbach's Conjecture 1.1.6 rephrased) If n is an even integer greater than 2, then n is a sum of two primes.
- If $0 \leq x \leq 2$, then $-x^3 + 4x + 1 > 0$.

There are a couple of standard methods for proving an implication.

1.5.1 Method #1

In order to prove that P IMPLIES Q:

1. Write, "Assume P."
2. Show that Q logically follows.

Example

Theorem 1.5.1. *If* $0 \le x \le 2$*, then* $-x^3 + 4x + 1 > 0$*.*

Before we write a proof of this theorem, we have to do some scratchwork to figure out why it is true.

The inequality certainly holds for $x = 0$; then the left side is equal to 1 and $1 > 0$. As x grows, the $4x$ term (which is positive) initially seems to have greater magnitude than $-x^3$ (which is negative). For example, when $x = 1$, we have $4x = 4$, but $-x^3 = -1$ only. In fact, it looks like $-x^3$ doesn't begin to dominate until $x > 2$. So it seems the $-x^3 + 4x$ part should be nonnegative for all x between 0 and 2, which would imply that $-x^3 + 4x + 1$ is positive.

So far, so good. But we still have to replace all those "seems like" phrases with solid, logical arguments. We can get a better handle on the critical $-x^3 + 4x$ part by factoring it, which is not too hard:

$$-x^3 + 4x = x(2 - x)(2 + x)$$

Aha! For x between 0 and 2, all of the terms on the right side are nonnegative. And a product of nonnegative terms is also nonnegative. Let's organize this blizzard of observations into a clean proof.

Proof. Assume $0 \le x \le 2$. Then x, $2-x$ and $2+x$ are all nonnegative. Therefore, the product of these terms is also nonnegative. Adding 1 to this product gives a positive number, so:

$$x(2 - x)(2 + x) + 1 > 0$$

Multiplying out on the left side proves that

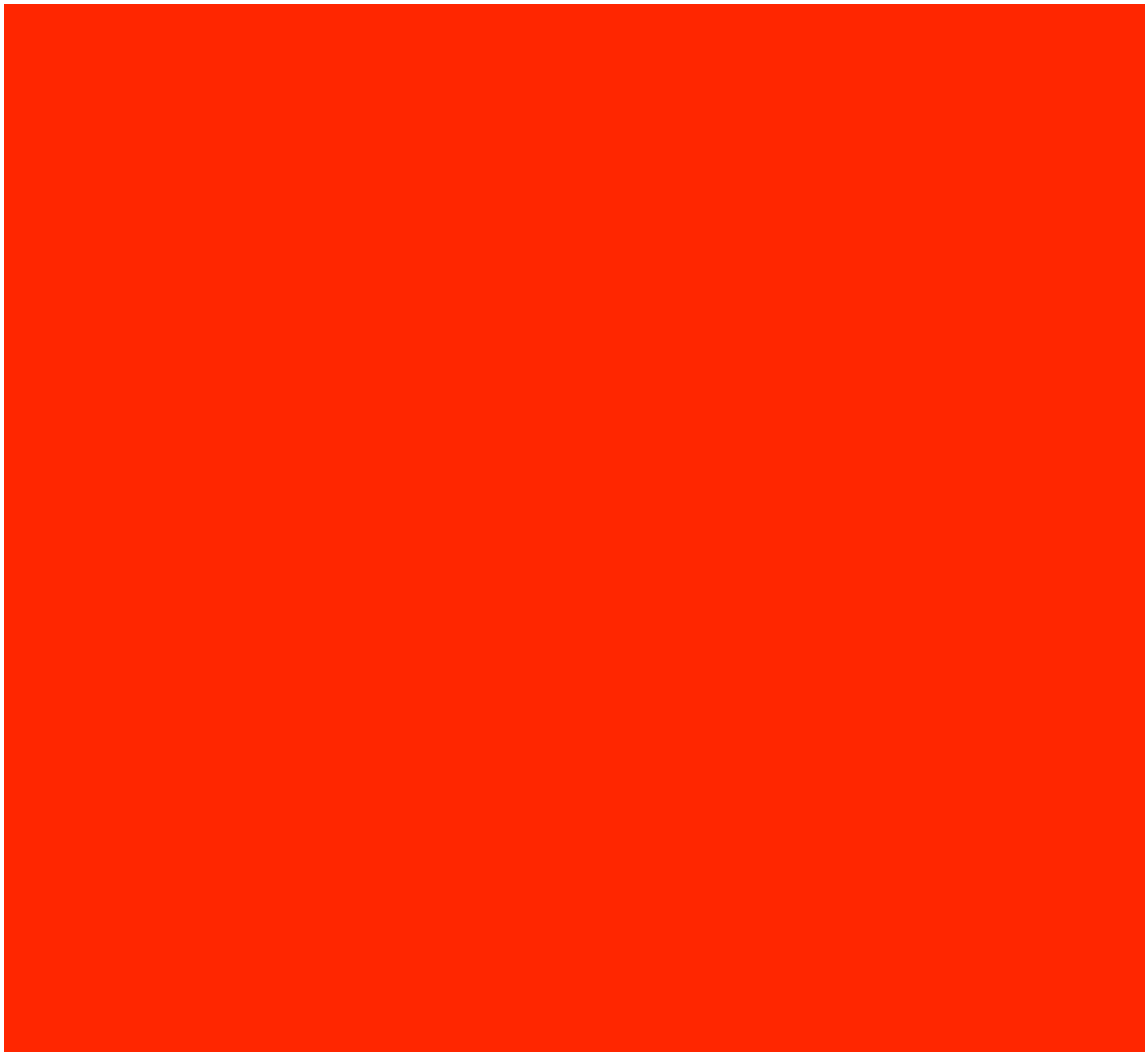
$$-x^3 + 4x + 1 > 0$$

as claimed. ■

There are a couple points here that apply to all proofs:

- You'll often need to do some scratchwork while you're trying to figure out the logical steps of a proof. Your scratchwork can be as disorganized as you like—full of dead-ends, strange diagrams, obscene words, whatever. But keep your scratchwork separate from your final proof, which should be clear and concise.

- Proofs typically begin with the word "Proof" and end with some sort of delimiter like □ or "QED." The only purpose for these conventions is to clarify where proofs begin and end.



1.6 Proving an "If and Only If"

Many mathematical theorems assert that two statements are logically equivalent; that is, one holds if and only if the other does. Here is an example that has been known for several thousand years:

> Two triangles have the same side lengths if and only if two side lengths and the angle between those sides are the same.

The phrase "if and only if" comes up so often that it is often abbreviated "iff."

1.6.1 Method #1: Prove Each Statement Implies the Other

The statement "P IFF Q" is equivalent to the two statements "P IMPLIES Q" and "Q IMPLIES P." So you can prove an "iff" by proving *two* implications:

1. Write, "We prove P implies Q and vice-versa."
2. Write, "First, we show P implies Q." Do this by one of the methods in Section 1.5.
3. Write, "Now, we show Q implies P." Again, do this by one of the methods in Section 1.5.

1.6.2 Method #2: Construct a Chain of Iffs

In order to prove that P is true iff Q is true:

1. Write, "We construct a chain of if-and-only-if implications."
2. Prove P is equivalent to a second statement which is equivalent to a third statement and so forth until you reach Q.

This method sometimes requires more ingenuity than the first, but the result can be a short, elegant proof.

Example

The *standard deviation* of a sequence of values $x_1, x_2, \ldots, x_n$ is defined to be:

$$\sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2}{n}} \tag{1.3}$$

where μ is the average or *mean* of the values:

$$\mu ::= \frac{x_1+x_2+\cdots+x_n}{n}$$

Theorem 1.6.1. *The standard deviation of a sequence of values $x_1, \ldots, x_n$ is zero iff all the values are equal to the mean.*

For example, the standard deviation of test scores is zero if and only if everyone scored exactly the class average.

Proof. We construct a chain of "iff" implications, starting with the statement that the standard deviation (1.3) is zero:

$$\sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\cdots+(x_n-\mu)^2}{n}} = 0. \tag{1.4}$$

Now since zero is the only number whose square root is zero, equation (1.4) holds iff

$$(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2 = 0. \tag{1.5}$$

Squares of real numbers are always nonnegative, so every term on the left-hand side of equation (1.5) is nonnegative. This means that (1.5) holds iff

$$\text{Every term on the left-hand side of (1.5) is zero.} \tag{1.6}$$

But a term $(x_i - \mu)^2$ is zero iff $x_i = \mu$, so (1.6) is true iff

$$\text{Every } x_i \text{ equals the mean.}$$

■

1.8 Proof by Contradiction

In a *proof by contradiction*, or *indirect proof*, you show that if a proposition were false, then some false fact would be true. Since a false fact by definition can't be true, the proposition must be true.

Proof by contradiction is *always* a viable approach. However, as the name suggests, indirect proofs can be a little convoluted, so direct proofs are generally preferable when they are available.

Method: In order to prove a proposition P by contradiction:

1. Write, "We use proof by contradiction."
2. Write, "Suppose P is false."
3. Deduce something known to be false (a logical contradiction).
4. Write, "This is a contradiction. Therefore, P must be true."

Example

We'll prove by contradiction that $\sqrt{2}$ is irrational. Remember that a number is *rational* if it is equal to a ratio of integers—for example, $3.5 = 7/2$ and $0.1111\cdots = 1/9$ are rational numbers.

Theorem 1.8.1. *$\sqrt{2}$ is irrational.*

Proof. We use proof by contradiction. Suppose the claim is false, and $\sqrt{2}$ is rational. Then we can write $\sqrt{2}$ as a fraction n/d in *lowest terms*.

Squaring both sides gives $2 = n^2/d^2$ and so $2d^2 = n^2$. This implies that n is a multiple of 2 (see Problems 1.15 and 1.16). Therefore n^2 must be a multiple of 4. But since $2d^2 = n^2$, we know $2d^2$ is a multiple of 4 and so d^2 is a multiple of 2. This implies that d is a multiple of 2.

So, the numerator and denominator have 2 as a common factor, which contradicts the fact that n/d is in lowest terms. Thus, $\sqrt{2}$ must be irrational. ■

1.9 *Good* Proofs in Practice

One purpose of a proof is to establish the truth of an assertion with absolute certainty, and mechanically checkable proofs of enormous length or complexity can accomplish this. But humanly intelligible proofs are the only ones that help someone understand the subject. Mathematicians generally agree that important mathematical results can't be fully understood until their proofs are understood. That is why proofs are an important part of the curriculum.

To be understandable and helpful, more is required of a proof than just logical correctness: a good proof must also be clear. Correctness and clarity usually go together; a well-written proof is more likely to be a correct proof, since mistakes are harder to hide.

In practice, the notion of proof is a moving target. Proofs in a professional research journal are generally unintelligible to all but a few experts who know all the terminology and prior results used in the proof. Conversely, proofs in the first weeks of a beginning course like 6.042 would be regarded as tediously long-winded by a professional mathematician. In fact, what we accept as a good proof later in the term will be different from what we consider good proofs in the first couple of weeks of 6.042. But even so, we can offer some general tips on writing good proofs:

State your game plan. A good proof begins by explaining the general line of reasoning, for example, "We use case analysis" or "We argue by contradiction."

Keep a linear flow. Sometimes proofs are written like mathematical mosaics, with juicy tidbits of independent reasoning sprinkled throughout. This is not good. The steps of an argument should follow one another in an intelligible order.

A proof is an essay, not a calculation. Many students initially write proofs the way they compute integrals. The result is a long sequence of expressions without explanation, making it very hard to follow. This is bad. A good proof usually looks like an essay with some equations thrown in. Use complete sentences.

Avoid excessive symbolism. Your reader is probably good at understanding words, but much less skilled at reading arcane mathematical symbols. Use words where you reasonably can.

Revise and simplify. Your readers will be grateful.

Introduce notation thoughtfully. Sometimes an argument can be greatly simplified by introducing a variable, devising a special notation, or defining a new term. But do this sparingly, since you're requiring the reader to remember all that new stuff. And remember to actually *define* the meanings of new variables, terms, or notations; don't just start using them!

Structure long proofs. Long programs are usually broken into a hierarchy of smaller procedures. Long proofs are much the same. When your proof needed facts that are easily stated, but not readily proved, those fact are best pulled out as preliminary lemmas. Also, if you are repeating essentially the same argument over and over, try to capture that argument in a general lemma, which you can cite repeatedly instead.

Be wary of the "obvious." When familiar or truly obvious facts are needed in a proof, it's OK to label them as such and to not prove them. But remember that what's obvious to you may not be—and typically is not—obvious to your reader.

Most especially, don't use phrases like "clearly" or "obviously" in an attempt to bully the reader into accepting something you're having trouble proving. Also, go on the alert whenever you see one of these phrases in someone else's proof.

Finish. At some point in a proof, you'll have established all the essential facts you need. Resist the temptation to quit and leave the reader to draw the "obvious" conclusion. Instead, tie everything together yourself and explain why the original claim follows.

Creating a good proof is a lot like creating a beautiful work of art. In fact, mathematicians often refer to really good proofs as being "elegant" or "beautiful." It takes a practice and experience to write proofs that merit such praises, but to get you started in the right direction, we will provide templates for the most useful proof techniques.

Throughout the text there are also examples of *bogus proofs*—arguments that look like proofs but aren't. Sometimes a bogus proof can reach false conclusions because of missteps or mistaken assumptions. More subtle bogus proofs reach correct conclusions, but do so in improper ways such as circular reasoning, leaping to unjustified conclusions, or saying that the hard part of the proof is "left to the reader." Learning to spot the flaws in improper proofs will hone your skills at seeing how each proof step follows logically from prior steps. It will also enable you to spot flaws in your own proofs.

The analogy between good proofs and good programs extends beyond structure. The same rigorous thinking needed for proofs is essential in the design of critical computer systems. When algorithms and protocols only "mostly work" due to reliance on hand-waving arguments, the results can range from problematic to catastrophic. An early example was the Therac 25, a machine that provided radiation therapy to cancer victims, but occasionally killed them with massive overdoses due to a software race condition. A further example of a dozen years ago (August 2004) involved a single faulty command to a computer system used by United and American Airlines that grounded the entire fleet of both companies—and all their passengers!

It is a certainty that we'll all one day be at the mercy of critical computer systems designed by you and your classmates. So we really hope that you'll develop the ability to formulate rock-solid logical arguments that a system actually does what you think it should do!

1.10 References

[14], [1], [49], [18], [22]

3.6 Predicate Formulas

Start

3.6.1 Quantifiers

The "for all" notation $\forall$ has already made an early appearance in Section 1.1. For example, the predicate

$$\text{“}x^2 \geq 0\text{”}$$

is always true when x is a real number. That is,

$$\forall x \in \mathbb{R}.\, x^2 \geq 0$$

is a true statement. On the other hand, the predicate

$$\text{“}5x^2 - 7 = 0\text{”}$$

is only sometimes true; specifically, when $x = \pm\sqrt{7/5}$. There is a "there exists" notation $\exists$ to indicate that a predicate is true for at least one, but not necessarily all objects. So

$$\exists x \in \mathbb{R}.\, 5x^2 - 7 = 0$$

is true, while

$$\forall x \in \mathbb{R}.\, 5x^2 - 7 = 0$$

is not true.

[2] **P** stands for problems whose instances can be solved in time that grows polynomially with the size of the instance. **NP** stands for ***n****ondeterministtic* ***p****olynomial time*, but we'll leave an explanation of what that is to texts on the theory of computational complexity.

There are several ways to express the notions of "always true" and "sometimes true" in English. The table below gives some general formats on the left and specific examples using those formats on the right. You can expect to see such phrases hundreds of times in mathematical writing!

Always True

For all $x \in D$, $P(x)$ is true.	For all $x \in \mathbb{R}$, $x^2 \geq 0$.
$P(x)$ is true for every x in the set D.	$x^2 \geq 0$ for every $x \in \mathbb{R}$.

Sometimes True

There is an $x \in D$ such that $P(x)$ is true.	There is an $x \in \mathbb{R}$ such that $5x^2 - 7 = 0$.
$P(x)$ is true for some x in the set D.	$5x^2 - 7 = 0$ for some $x \in \mathbb{R}$.
$P(x)$ is true for at least one $x \in D$.	$5x^2 - 7 = 0$ for at least one $x \in \mathbb{R}$.

All these sentences "quantify" how often the predicate is true. Specifically, an assertion that a predicate is always true is called a *universal quantification*, and an assertion that a predicate is sometimes true is an *existential quantification*. Sometimes the English sentences are unclear with respect to quantification:

If you can solve any problem we come up with,
then you get an A for the course. (3.19)

The phrase "you can solve any problem we can come up with" could reasonably be interpreted as either a universal or existential quantification:

you can solve *every* problem we come up with, (3.20)

or maybe

you can solve *at least one* problem we come up with. (3.21)

To be precise, let Probs be the set of problems we come up with, Solves(x) be the predicate "You can solve problem x," and G be the proposition, "You get an A for the course." Then the two different interpretations of (3.19) can be written as follows:

$(\forall x \in \text{Probs. Solves}(x))$ IMPLIES G,	for (3.20),
$(\exists x \in \text{Probs. Solves}(x))$ IMPLIES G.	for (3.21).

3.6.2 Mixing Quantifiers

Many mathematical statements involve several quantifiers. For example, we already described

> Goldbach's Conjecture 1.1.6: Every even integer greater than 2 is the sum of two primes.

Let's write this out in more detail to be precise about the quantification:

> For every even integer n greater than 2, there exist primes p and q such that $n = p + q$.

Let Evens be the set of even integers greater than 2, and let Primes be the set of primes. Then we can write Goldbach's Conjecture in logic notation as follows:

$$\underbrace{\forall n \in \text{Evens}}_{\substack{\text{for every even}\\ \text{integer } n > 2}} \underbrace{\exists p \in \text{Primes}\, \exists q \in \text{Primes}.}_{\substack{\text{there exist primes}\\ p \text{ and } q \text{ such that}}} n = p + q.$$

3.6.3 Order of Quantifiers

Swapping the order of different kinds of quantifiers (existential or universal) usually changes the meaning of a proposition. For example, let's return to one of our initial, confusing statements:

"Every American has a dream."

This sentence is ambiguous because the order of quantifiers is unclear. Let A be the set of Americans, let D be the set of dreams, and define the predicate $H(a, d)$ to be "American a has dream d." Now the sentence could mean there is a single dream that every American shares—such as the dream of owning their own home:

$$\exists d \in D\, \forall a \in A.\, H(a, d)$$

Or it could mean that every American has a personal dream:

$$\forall a \in A\, \exists d \in D.\, H(a, d)$$

For example, some Americans may dream of a peaceful retirement, while others dream of continuing practicing their profession as long as they live, and still others may dream of being so rich they needn't think about work at all.

Swapping quantifiers in Goldbach's Conjecture creates a patently false statement that every even number ≥ 2 is the sum of *the same* two primes:

$$\underbrace{\exists p \in \text{Primes}\, \exists q \in \text{Primes}.}_{\substack{\text{there exist primes}\\ p \text{ and } q \text{ such that}}} \underbrace{\forall n \in \text{Evens}}_{\substack{\text{for every even}\\ \text{integer } n > 2}} n = p + q.$$

3.6.4 Variables Over One Domain

When all the variables in a formula are understood to take values from the same nonempty set D it's conventional to omit mention of D. For example, instead of $\forall x \in D\, \exists y \in D.\, Q(x, y)$ we'd write $\forall x \exists y.\, Q(x, y)$. The unnamed nonempty set that x and y range over is called the *domain of discourse*, or just plain *domain*, of the formula.

It's easy to arrange for all the variables to range over one domain. For example, Goldbach's Conjecture could be expressed with all variables ranging over the domain $\mathbb{N}$ as

$$\forall n.\, n \in \text{Evens IMPLIES } (\exists\, p\, \exists q.\, p \in \text{Primes AND } q \in \text{Primes AND } n = p + q).$$

3.6.5 Negating Quantifiers

There is a simple relationship between the two kinds of quantifiers. The following two sentences mean the same thing:

> Not everyone likes ice cream.
>
> There is someone who does not like ice cream.

The equivalence of these sentences is an instance of a general equivalence that holds between predicate formulas:

$$\text{NOT}(\forall x.\, P(x)) \quad \text{is equivalent to} \quad \exists x.\, \text{NOT}(P(x)). \tag{3.22}$$

Similarly, these sentences mean the same thing:

> There is no one who likes being mocked.
>
> Everyone dislikes being mocked.

The corresponding predicate formula equivalence is

$$\text{NOT}(\exists x.\, P(x)) \quad \text{is equivalent to} \quad \forall x.\, \text{NOT}(P(x)). \tag{3.23}$$

Note that the equivalence (3.23) follows directly by negating both sides the equivalence (3.22).

The general principle is that *moving a* NOT *to the other side of an "∃" changes it into "∀," and* vice versa.

Start

4 Mathematical Data Types

We have assumed that you've already been introduced to the concepts of sets, sequences, and functions, and we've used them informally several times in previous sections. In this chapter, we'll now take a more careful look at these mathematical data types. We'll quickly review the basic definitions, add a few more such as "images" and "inverse images" that may not be familiar, and end the chapter with some methods for comparing the sizes of sets.

4.1 Sets

Informally, a *set* is a bunch of objects, which are called the *elements* of the set. The elements of a set can be just about anything: numbers, points in space, or even other sets. The conventional way to write down a set is to list the elements inside curly-braces. For example, here are some sets:

$$A = \{\text{Alex, Tippy, Shells, Shadow}\} \qquad \text{dead pets}$$
$$B = \{\text{red, blue, yellow}\} \qquad \text{primary colors}$$
$$C = \{\{a, b\}, \{a, c\}, \{b, c\}\} \qquad \text{a set of sets}$$

This works fine for small finite sets. Other sets might be defined by indicating how to generate a list of them:

$$D ::= \{1, 2, 4, 8, 16, \ldots\} \qquad \text{the powers of 2}$$

The order of elements is not significant, so $\{x, y\}$ and $\{y, x\}$ are the same set written two different ways. Also, any object is, or is not, an element of a given set—there is no notion of an element appearing more than once in a set.[1] So, writing $\{x, x\}$ is just indicating the same thing twice: that x is in the set. In particular, $\{x, x\} = \{x\}$.

The expression "$e \in S$" asserts that e is an element of set S. For example, $32 \in D$ and blue $\in B$, but Tailspin $\notin A$—yet.

Sets are simple, flexible, and everywhere. You'll find some set mentioned in nearly every section of this text.

[1] It's not hard to develop a notion of *multisets* in which elements can occur more than once, but multisets are not ordinary sets and are not covered in this text.

4.1.1 Some Popular Sets

Mathematicians have devised special symbols to represent some common sets.

symbol	set	elements
$\emptyset$	the empty set	none
$\mathbb{N}$	nonnegative integers	$\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	rational numbers	$\frac{1}{2},\ -\frac{5}{3},\ 16,$ etc.
$\mathbb{R}$	real numbers	$\pi,\ e,\ -9,\ \sqrt{2},$ etc.
$\mathbb{C}$	complex numbers	$i,\ \frac{19}{2},\ \sqrt{2} - 2i,$ etc.

A superscript "$+$" restricts a set to its positive elements; for example, $\mathbb{R}^+$ denotes the set of positive real numbers. Similarly, $\mathbb{Z}^-$ denotes the set of negative integers.

4.1.2 Comparing and Combining Sets

The expression $S \subseteq T$ indicates that set S is a *subset* of set T, which means that every element of S is also an element of T. For example, $\mathbb{N} \subseteq \mathbb{Z}$ because every nonnegative integer is an integer; $\mathbb{Q} \subseteq \mathbb{R}$ because every rational number is a real number, but $\mathbb{C} \not\subseteq \mathbb{R}$ because not every complex number is a real number.

As a memory trick, think of the "$\subseteq$" symbol as like the "$\leq$" sign with the smaller set or number on the left-hand side. Notice that just as $n \leq n$ for any number n, also $S \subseteq S$ for any set S.

There is also a relation $\subset$ on sets like the "less than" relation $<$ on numbers. $S \subset T$ means that S is a subset of T, but the two are *not* equal. So just as $n \not< n$ for every number n, also $A \not\subset A$, for every set A. "$S \subset T$" is read as "S is a *strict subset* of T."

There are several basic ways to combine sets. For example, suppose

$$X ::= \{1, 2, 3\},$$
$$Y ::= \{2, 3, 4\}.$$

Definition 4.1.1.

- The *union* of sets A and B, denoted $A \cup B$, includes exactly the elements appearing in A or B or both. That is,

$$x \in A \cup B \quad \text{IFF} \quad x \in A \text{ OR } x \in B.$$

So $X \cup Y = \{1, 2, 3, 4\}$.

- The *intersection* of A and B, denoted $A \cap B$, consists of all elements that appear in *both* A and B. That is,

$$x \in A \cap B \quad \text{IFF} \quad x \in A \text{ AND } x \in B.$$

  So, $X \cap Y = \{2, 3\}$.

- The *set difference* of A and B, denoted $A - B$, consists of all elements that are in A, but not in B. That is,

$$x \in A - B \quad \text{IFF} \quad x \in A \text{ AND } x \notin B.$$

  So, $X - Y = \{1\}$ and $Y - X = \{4\}$.

Often all the sets being considered are subsets of a known domain of discourse D. Then for any subset A of D, we define $\overline{A}$ to be the set of all elements of D *not* in A. That is,

$$\overline{A} ::= D - A.$$

The set $\overline{A}$ is called the *complement* of A. So

$$\overline{A} = \emptyset \text{ IFF } A = D.$$

For example, if the domain we're working with is the integers, the complement of the nonnegative integers is the set of negative integers:

$$\overline{\mathbb{N}} = \mathbb{Z}^{-}.$$

We can use complement to rephrase subset in terms of equality

$$A \subseteq B \text{ is equivalent to } A \cap \overline{B} = \emptyset.$$

4.1.3 Power Set

The set of all the subsets of a set A is called the *power set* $\text{pow}(A)$ of A. So

$$B \in \text{pow}(A) \quad \text{IFF} \quad B \subseteq A.$$

For example, the elements of $\text{pow}(\{1, 2\})$ are $\emptyset$, $\{1\}$, $\{2\}$ and $\{1, 2\}$.

More generally, if A has n elements, then there are 2^n sets in $\text{pow}(A)$—see Theorem 4.5.5. For this reason, some authors use the notation 2^A instead of $\text{pow}(A)$.

4.1.4 Set Builder Notation

An important use of predicates is in *set builder notation*. We'll often want to talk about sets that cannot be described very well by listing the elements explicitly or by taking unions, intersections, etc., of easily described sets. Set builder notation often comes to the rescue. The idea is to define a *set* using a *predicate*; in particular, the set consists of all values that make the predicate true. Here are some examples of set builder notation:

$$\begin{aligned} A &::= \{n \in \mathbb{N} \mid n \text{ is a prime and } n = 4k + 1 \text{ for some integer } k\},\\ B &::= \{x \in \mathbb{R} \mid x^3 - 3x + 1 > 0\},\\ C &::= \{a + bi \in \mathbb{C} \mid a^2 + 2b^2 \leq 1\},\\ D &::= \{L \in \text{books} \mid L \text{ is cited in this text}\}. \end{aligned}$$

The set A consists of all nonnegative integers n for which the predicate

"n is a prime and $n = 4k + 1$ for some integer k"

is true. Thus, the smallest elements of A are:

$$5, 13, 17, 29, 37, 41, 53, 61, 73, \ldots.$$

Trying to indicate the set A by listing these first few elements wouldn't work very well; even after ten terms, the pattern is not obvious. Similarly, the set B consists of all real numbers x for which the predicate

$$x^3 - 3x + 1 > 0$$

is true. In this case, an explicit description of the set B in terms of intervals would require solving a cubic equation. Set C consists of all complex numbers $a + bi$ such that:

$$a^2 + 2b^2 \leq 1$$

This is an oval-shaped region around the origin in the complex plane. Finally, the members of set D can be determined by filtering out journal articles in from the list of references in the Bibliography 22.5.

[2]This is actually the first of the ZFC axioms for set theory mentioned at the end of Section 1.3 and discussed further in Section 8.3.2.