TheoryPrep **(Due: 6/21/20)**

Assignment #3: Functions and Cardinality

Name: Student name(s)

We've taken some problems from the course texts.

Problem 1: Do Our Definitions Make Sense

Learning goal: As discussed in the videos, whenever you see a definition that's supposed to capture an intuitive idea like "size", you should make sure the definitions have the properties you would expect. In this section you will prove some of these properties. Hint: you'll mostly be applying the definitions of surjectivity and injectivity.

(a) If $|A| \leq |B|$ and $|B| \leq |C|$, then we'd expect $|A| \leq |C|$ (this is called transitivity). According to our definitions, this means we need to prove that if there exist surjections $f : B \to A$ and $g : C \to B$ then there must be a surjection from $C \to A$.

(b) If $|A| = |B|$ and $|B| = |C|$, then we'd expect $|A| = |C|$ (this is again the transitivity property). According to our definitions, this means we should prove, if there exists a bijection $f : A \to B$ and a bijection $g : B \to C$, then there exists a bijection from A to C.

(c) If two sets have the same cardinality, we'd expect be able to write either $|A| = |B|$ or $|B| = |A|$ (this is called the symmetric property–it's not true for $<$ for example). Thus, we want to prove that there exists a bijection $f : A \to B$ if and only if there exists a bijection from $g : B \to A$.

(d) In the videos we discussed how an surjection from $A \to B$ defines $|A| \geq |B|$ and an injection from $A \to B$ intuitively corresponds to $|A| \leq |B|$. In explaining why these definitions were consistent, we used the fact that a surjection $f : A \to B$ exists if and only if an injection $g : B \to A$ exists (you can assume the axiom of choice if you want–you don't need to worry about it though if you feel all the assumptions in your proof are fine). Prove this fact.

Problem 2: Problems about Functions

Learning goal: This section has a number of problems about functions that require a diversity of techniques. Working on them will develop your proof skills and understanding of the mathematical concepts around functions.

(a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not bijective. Prove that it is injective but not surjective.

(b) Let $f : A \to B$ and $g : B \to C$ for some sets A, B, C. Prove that if $g \circ f$ is a bijection, then f is an injection and g is a surjection.

(c) Let $f : A \to B$ and $g : B \to C$ for some sets A, B, C. Prove or disprove: If $g \circ f$ is a bijection, then g is an injection or f is a surjection.

(d) Let $f : D \to D$ be a function where D is nonempty. Indicate which of the below conditions are logically equivalent to f being injective (g is a function mapping $D \to D$). You do not need to prove the equivalence, but you should justify it to yourself.

1. $\forall x, y \in D. x = y$ or $f(x) \neq f(y)$
2. $\forall x, y \in D. x = y \Rightarrow f(x) = f(y)$
3. $\forall x, y \in D. x \neq y \Rightarrow f(x) \neq f(y)$

4. $\forall x, y \in D. f(x) = f(y) \Rightarrow x = y$
5. $\neg(\exists x, y \in D. x \neq y$ and $f(x) = f(y))$
6. $\neg(\exists x \in D \forall y \in D. f(x) = f(y))$
7. $\exists g \forall x \in D. f(g(x)) = x$
8. $\exists g \forall x \in D. g(f(x)) = x$

Problem 3: Longer Proofs

Learning Goal: These proofs require multiple different techniques and steps which will help you understand how to do and clearly write up multi-step proofs.

(a) We first introduce a couple of definitions. Let $\{0,1\}^*$ be the set of all finite length binary strings (so things like $0, 110, 110101011$). Let $F = \{f : \mathbb{N} \to \{0,1\}^*\}$ be the set of all functions mapping the natural numbers to finite binary strings ($\{0,1\}^*$ is the set of all finite length binary strings).

We might be interested in knowing which of these sets (if either) is larger. Intuitively, it seems like F should be much larger, but as we know, comparing infinite cardinalities is not as easy as that. Prove that F is indeed larger, or in other words that there exists no surjection from $\{0,1\}^* \to F$.

(b) Let $E = \{ae^{bx} + c | a, b, c \in \mathbb{R}, a \neq 0, b \neq 0\}$ be the set of non-constant exponential functions and $L = \{ax + b | a, b \in \mathbb{R}, a \neq 0\}$ be the set of non-constant linear functions.

We inductively define S to contain functions f which satisfy any of the following 3 conditions:

1. $f \in E$
2. $f \in L$
3. $f = g \circ h$, where $g \in S$ and $h \in S$

That is, we start with all the linear and exponential functions in S and then add any function we can get by composing them together.

Prove that if $f \in S$, then f is injective.

(c) Let $\{1,2,3\}^\omega$ be the set of infinite strings of $1, 2$, and 3 and $\{4,5\}^\omega$ be the set of infinite strings of 4 and 5. Prove that a bijection exists between $\{4,5\}^\omega$ and $\{1,2,3\}^\omega \times \{1,2,3\}^\omega$

(d) (Challenge) Prove Schroder-Bernstein.

Problem 4: Review

Learning goal: It's important to keep skills fresh. We review material from the previous two weeks.

(a) Consider a modified version of football, where the teams can score either 4 or 7 points at a time. Find and prove the minimum point value k such that a team can achieve any score any number of points greater than or equal to k.

(b) There are some interesting closure relationships between rationals and irrationals (here closure means that if you do an operation to elements of a set–like adding two elements or taking the square root of an element–you stay in the set). For example, we proved in a previous video if n^2 is irrational, then n is also irrational. We might also note that if a, b are rational, then so is $a + b$ or ab. In this problem we consider a more complex operation of exponentiation. Prove that both the rationals and the irrationals are not closed under exponentiation. This means, prove that there exists $a, b \in \mathbb{Q}$ such that $a^b \notin \mathbb{Q}$, and prove there exists $a, b \in \mathbb{R} - \mathbb{Q}$ ($\mathbb{R} - \mathbb{Q}$ is the irrationals) such that $a^b \notin \mathbb{R} - \mathbb{Q}$. As a hint, try using $\sqrt{2}^{\sqrt{2}}$. You may assume $\sqrt{2}$ is irrational.

Problem 5: Logistics

Purpose: This helps us make sure the course is going at the right speed!

(a) How long did you spend on the videos and readings this week?

(b) How long (including time in problem sessions) did you spend on this problem set?

(c) Do you have any feedback about the course in general (did the videos and readings sufficiently prepare you for the problem set)?