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Computing obstruction for compact Clifford-Klein form

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Abstract

In this package we develop functions for algorithms of finding homogeneous spaces of semisimple non-compact Lie groups which do not admit compact Clifford-Klein forms.

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Contents

Chapter 1

Notation

Notation for real Lie algebra is from CoReLG Package, [DFdG14]. Notice: we found some misspellings:

- "D",4,5 is $\mathfrak{so}(1,7)$
- "D",4,4 is $\mathfrak{so}(3,5)$
- "E",7,3 is $\mathfrak{e}_{7(-25)} = EVII$
- "E",7,4 is $\mathfrak{e}_{7(-5)} = EVI$

To be sure, check the rank or the dimension and check the result with the table in [OV90].

Example

```
gap> G:=RealFormById( "E", 7,3);
<Lie algebra of dimension 133 over SqrtField>
gap> rankG:=Dimension(CartanSubalgebra(G));
7
gap> rankRG:=Dimension(CartanSubspace(G));
3
gap> dimG:=Dimension(G);
133
gap> P:=CartanDecomposition( G ).P;
<vector space over SqrtField, with 54 generators>
gap> dimPforG:=Dimension(P);
54
gap> K:=CartanDecomposition( G ).K;
<Lie algebra of dimension 79 over SqrtField>
gap> rankK:= Dimension(CartanSubalgebra(K));
7
gap> dimK:= Dimension(K);
79
```

Chapter 2

Obstruction for compact Clifford-Klein form

In this chapter we describe functions for algorithms from [BJS+].

2.1 Technical functions

2.1.1 NonCompactDimension

▷ `NonCompactDimension(G)` (function)

For a real Lie algebra G constructed by the function *RealFormById* (from [DFdG14]), this function returns the non-compact dimension of G.

Example

```
gap> G:=RealFormById("E",6,2); # E6(6)
<Lie algebra of dimension 78 over SqrtField>
gap> dG:=NonCompactDimension(G);
42
```

2.1.2 PCoefficients

▷ `PCoefficients(type, rank)` (function)

Let G be the compact connected Lie group of the type *type* and the rank *rank*. Let $\Lambda P_G = \Lambda(y_1,...,y_l)$ be the exterior algebras over the spaces P_G of the primitive elements in $H^*(G)$. Denote the degrees as follows $|y_j| = 2p_j - 1, j = 1,...,l$. This function returns coefficients $p_1,\ldots,p_l, j = 1,...,l$.

Example

```
gap> PCoefficients("D",5);
[ 2, 4, 6, 8, 5 ]
```

2.1.3 PCalculate

▷ `PCalculate(pi, qi)` (function)

Here $pi = \{p_1, \ldots, p_l\}$ and $qi = \{q_1, \ldots, q_m\}$ are sets of coefficients ($l \geq m$). This function returns the polynomial: $P(t) = \prod_{j=m+1}^{l}(1+t^{2p_j-1})\prod_{i=1}^{m}(1-t^{2p_i})/(1-t^{2q_i})$.

Example

```
gap> PCalculate([4,2,3],[2,2]);
t^9+t^5+t^4+1
```

2.1.4 AllZeroDH

▷ `AllZeroDH(type, rank, id)` (function)

Let G^C be the complex Lie algebra of the type *type* and the rank *rank*. Let G be the real form of G^C with the index *id* (see *RealFormsInformation*,[DFdG14]). This function returns the set of degrees of $P(t)$ that have zero coefficients over all permutation (see Section 7 in [BJS+]).

Example

```
gap> AllZeroDH("F",4,2);
[ 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27 ]
```

References

[BJS$^+$] M. Bocheński, P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike. Semisimple subalgebras in simple lie algebras and a computational approach to the compact clifford-klein forms problem. https://arxiv.org/abs/1804.03484. 5, 6

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[OV90] A. Onishchik and E. Vinberg. *Lie Groups and Algebraic Groups*. Springer Series in Soviet Mathematics. Springer-Verlag Berlin Heidelberg, first edition, 1990. Original Russian edition published by Nauka, Moscow 1988. 4

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