

# NoCK

## **Computing obstruction for compact Clifford-Klein form**

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## Abstract

In this package we develop functions for algorithms of finding homogeneous spaces of semisimple non-compact Lie groups which do not admit compact Clifford-Klein forms.

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# Chapter 1

## Notation

Notation for real Lie algebra is from [DFdG14] Package. Notice: We found some misspelling:

- "D",4,5 is  $\mathfrak{so}(1,7)$
- "D",4,4 is  $\mathfrak{so}(3,5)$
- "E",7,3 is  $\mathfrak{e}_{7(-25)} = EVII$
- "E",7,4 is  $\mathfrak{e}_{7(-5)} = EVI$

To be sure, check rank or dimension and check result with table in [OV90].

Example

```
gap> G:=RealFormById( "E", 7,3);
<Lie algebra of dimension 133 over SqrtField>
gap> rankG:=Dimension(CartanSubalgebra(G));
7
gap> rankRG:=Dimension(CartanSubspace(G));
3
gap> dimG:=Dimension(G);
133
gap> P:=CartanDecomposition( G ).P;
<vector space over SqrtField, with 54 generators>
gap> dimPforG:=Dimension(P);
54
gap> K:=CartanDecomposition( G ).K;
<Lie algebra of dimension 79 over SqrtField>
gap> rankK:= Dimension(CartanSubalgebra(K));
7
gap> dimK:= Dimension(K);
79
```

## Chapter 2

# Obstruction for compact Clifford-Klein form

In this chapter we describe functions for algorithms from [BJS<sup>+</sup>].

### 2.1 Technical functions

#### 2.1.1 NonCompactDimension

▷ `NonCompactDimension(G)` (function)

For a real Lie algebra  $G$  constructed by the function *RealFormById* (from [DFdG14]), this function returns non-compact dimension of  $G$ .

Example

```
gap> G:=RealFormById("E",6,2); # E6(6)
<Lie algebra of dimension 78 over SqrtField>
gap> dG:=NonCompactDimension(G);
42
```

#### 2.1.2 PCoefficients

▷ `PCoefficients(type, rank)` (function)

Let  $G$  be the compact connecte Lie group of type *type* and rank *rank*. Let  $\Lambda P_G = \Lambda(y_1, \dots, y_l)$  be the exterior algebras over the spaces  $P_G$  of the primitive elements in  $H^*(G)$ . Denote the degrees as follows  $|y_j| = 2p_j - 1, j = 1, \dots, l$ . This function returns coefficients  $p_1, \dots, p_l, j = 1, \dots, l$ .

Example

```
gap> PCoefficients("D",5);
[ 2, 4, 6, 8, 5 ]
```

#### 2.1.3 PCalculate

▷ `PCalculate(pi, qi)` (function)

Here  $pi = \{p_1, \dots, p_l\}$  and  $qi = \{q_1, \dots, q_m\}$  are sets of coefficients ( $l \geq m$ ). This function returns polynomial:  $P(t) = \prod_{j=m+1}^l (1 + t^{2p_j-1}) \prod_{i=1}^m (1 - t^{2p_i}) / (1 - t^{2q_i})$ .

Example

```
gap> PCalculate([4,2,3],[2,2]);
t^9+t^5+t^4+1
```

### 2.1.4 AllZeroDH

▷ AllZeroDH(*type*, *rank*, *id*)

(function)

Let  $G^C$  be the complex Lie algebra of type *type* and rank *rank*. Let  $G$  be the real form of  $G^C$  with index *id* (see *RealFormsInformation*, [DFdG14]). This function returns set of degrees of  $P(t)$  that have zero coefficients over all permutation (see Section 7 in [BJS<sup>+</sup>]).

Example

```
gap> AllZeroDH("F",4,2);
[ 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27 ]
```

# References

- [BJS<sup>+</sup>] M. Bocheński, P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike. Semisimple subalgebras in simple lie algebras and a computational approach to the compact clifford-klein forms problem. in preparation. 5, 6
- [DFdG14] H. Dietrich, P. Faccin, and W. A. de Graaf. Corelg, computation with real lie groups, version 1.20. <http://users.monash.edu/~heikod/corelg/>, Dec 2014. Refereed GAP package. 4, 5, 6
- [OV90] A. Onishchik and E. Vinberg. *Lie Groups and Algebraic Groups*. Springer Series in Soviet Mathematics. Springer-Verlag Berlin Heidelberg, first edition, 1990. Original Russian edition published by Nauka, Moscow 1988. 4



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