# Cohomology Products in GAP, Explained in not Unbearable Detail, but Still Bad Enough to Require Being Seated while Reading

Marcus Bishop

July 24, 2006

The purpose of this document is to explain the implementation of cohomology products in the crime package for GAP including the Massey $n$-fold product. In this document, a composition of two functions $g \circ f$ is the function obtained by applying $f$ first and then $g$. The symbol $\circlearrowleft$ is used in diagrams to indicate that a polygon either commutes or anticommutes.

Let $G$ be a finite $p$-group for some prime $p$ and let $k = \mathbb{F}_p$. Also write $k$ for the trivial $kG$-module. We assume that we can calculate a $kG$-projective resolution $P_*$ of $k$, that is, for $n$ as large as we need, we can compute the integers $\{b_m : 0 \leq m \leq n\}$, the maps $\{\partial_m : 1 \leq m \leq n\}$ and the map $\epsilon$ such that

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \tag{1}$$

is exact, where $P_m = (kG)^{b_m}$. Later, we will assume moreover that $P_*$ is *minimal*, that is, that $\partial_m (P_m) \leq \operatorname{Rad}(P_{m-1})$ for all $m \geq 1$.

## 1 Cohomology Products

The following construction is taken from [?]. We begin with two cocycles $f : P_i \to k$ and $g : P_j \to k$, that is, that $f \circ \partial_{i+1} = g \circ \partial_{j+1} = 0$. We want to compute the cup product $fg : P_{i+j} \to k$.

We first convert $f$ into an chain map, resulting in the following commutative diagram.

$$\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "f_{m-1}"] & P_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "f_{i+2}"] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & P_i \arrow[d, "f_i"] \arrow[dr, "f"] & & \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k \arrow[r] & 0
\end{tikzcd} \tag{2}$$

1. Define $f_i$ such that $\epsilon \circ f_i = f$. This is possible by projectivity of $P_i$.
2. Define $f_{i+1}$ such that $\partial_1 \circ f_{i+1} = f_i \circ \partial_{i+1}$. This is possible by projectivity of $P_{i+1}$ since
$$\operatorname{im}(f_i \circ \partial_{i+1}) \leq \operatorname{im}(\partial_1) = \ker(\epsilon)$$
as $\epsilon \circ (f_i \circ \partial_{i+1}) = f \circ \partial_{i+1} = 0$.
3. Define $f_{i+2}$ such that $\partial_2 \circ f_{i+2} = f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of $P_{i+2}$ since
$$\operatorname{im}(f_{i+1} \circ \partial_{i+2}) \leq \operatorname{im}(\partial_2) = \ker(\partial_1)$$
as $\partial_1 \circ (f_{i+1} \circ \partial_{i+2}) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.
4. Define $f_m$ for $m > i + 2$ by recursion such that $\partial_{m-i} \circ f_m = f_{m-1} \circ \partial_m$. This is possible by projectivity of $P_m$ since
$$\operatorname{im}(f_{m-1} \circ \partial_m) \leq \operatorname{im}(\partial_{m-i}) = \ker(\partial_{m-i-1})$$
as $\partial_{m-i-1} \circ (f_{m-1} \circ \partial_m) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

Then the product $fg$ is calculated as $g \circ f_{i+j}$. The process above is used to compute the multiplication table used by the `CohomologyRing` command and is used to find generators by the `CohomologyGenerators` command.

## 2 The Yoneda Cocomplex

My understanding of the purpose of the Yoneda Cocomplex is the following. The definition of the Massey product below requires a cocomplex having an associative product. The product defined above, however, is defined only for $f$ and $g$ cocycles in $\operatorname{Hom}(P_*, k)$. The Yoneda cocomplex $Y$, on the other hand, has the same cohomology as $\operatorname{Hom}(P_*, k)$, but has an associative product defined for all cochains, namely composition. Moreover, we will show that via the isomorphism $\Phi : H^*(G, k) \to H^*(Y)$, composition in $Y$ agrees with the product defined in Section **??** up to the factor $(-1)^{\deg f \deg g}$, that is,
$$\Phi(fg) = (-1)^{\deg f \deg g} \Phi(g) \circ \Phi(f).$$

The following construction comes from [**?**].

**Definition 1.** *For* $i \geq 0$, *define*
$$Y^i = \prod_{m \geq i} \operatorname{Hom}_{kG}(P_m, P_{m-i}).$$

*Then an element* $f \in Y^i$ *is a collection of* $kG$*-homomorphisms* $\{f_m : P_m \to P_{m-i} : m \geq i\}$ *as in the following diagram.*

$$
\begin{array}{ccccccccccccccc}
P_n & \xrightarrow{\partial_n} & P_{n-1} & \longrightarrow \cdots \longrightarrow & P_m & \xrightarrow{\partial_m} & P_{m-1} & \xrightarrow{\partial_{m-1}} & P_{m-2} & \longrightarrow \cdots \longrightarrow & P_{i+1} & \xrightarrow{\partial_{i+1}} & P_i \\
\downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_m & & \downarrow f_{m-1} & & \downarrow f_{m-2} & & \downarrow f_{i+1} & & \downarrow f_i \\
P_{n-i} & \xrightarrow[\partial_{n-i}]{} & P_{n-i-1} & \longrightarrow \cdots \longrightarrow & P_{m-i} & \xrightarrow[\partial_{m-i}]{} & P_{m-i-1} & \xrightarrow[\partial_{m-i-1}]{} & P_{m-i-2} & \longrightarrow \cdots \longrightarrow & P_1 & \xrightarrow[\partial_1]{} & P_0
\end{array} \tag{3}
$$

Diagram (**??**) is not required to commute.

**Definition 2.** *Define* $Y = \bigoplus_{i\geq 0} Y^i$. $Y$ *is called the* Yoneda cocomplex *of* $P_*$. *We write* $\deg(f) = i$ *for* $f \in Y^i$. *Let* $f = \{f_m : m \geq i\} \in Y^i$ *and define*

$$
\begin{aligned}
\partial : Y^i &\to Y^{i+1} \\
f &\mapsto \left\{ f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m : m \geq 1 \right\}.
\end{aligned}
$$

We observe that cocycles in $Y$ are those elements $f$ for which (**??**) commutes if $\deg f$ is even and anticommutes if $\deg f$ is odd.

**Lemma 3.** $Y$ *with differentiation* $\partial$ *is a cocomplex, that is,* $\partial^2 = 0$.

*Proof.* Let $f \in Y^i$. We will show that $\partial^2 f = 0$ at the point $P_m$ in (**??**) for $m \geq i+2 = \deg(\partial^2 f)$. Follow along in the picture.

$$
\begin{aligned}
(\partial(\partial f))_m &= (\partial f)_{m-1} \circ \partial_m - (-1)^{i+1} \partial_{m-i-1} \circ (\partial f)_m \\
&= \left( f_{m-2} \circ \partial_{m-1} - (-1)^i \partial_{m-i-1} \circ f_{m-1} \right) \circ \partial_m \\
&\quad - (-1)^{i+1} \partial_{m-i-1} \circ \left( f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m \right) \\
&= f_{m-2} \circ \partial_{m-1} \circ \partial_m - \partial_{m-i-1} \circ \partial_{m-i} \circ f_m \\
&= 0
\end{aligned}
$$

■

**Theorem 4.** *The cohomology groups of* $Y$ *are* $H^*(G, k)$.

*Proof.* We will define a group isomorphism $\Phi : H^i(G, k) \to H^i(Y)$.

1. Let $f : P_i \to k$ be a cocycle in $\mathrm{Hom}^i_{kG}(P_*, k)$, that is, assume $f \circ \partial_{i+1} = 0$. Define $\Phi(f) = \{f_m : m \geq i\} \in Y^i$ as follows. The element $\Phi(f)$, together with $f$, is pictured in the following diagram.

$$
\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "f_{m-1}"] & P_{m-2} \arrow[r] \arrow[d, "f_{m-2}"] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "f_{i+2}"] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & P_i \arrow[d, "f_i"] \arrow[dr, "f"] & & \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k \arrow[r] & 0
\end{tikzcd} \tag{4}
$$

(a) Define $f_i$ such that $\epsilon \circ f_i = f$. This is possible by projectivity of $P_i$.

(b) Define $f_{i+1}$ such that $\partial_1 \circ f_{i+1} = (-1)^i f_i \circ \partial_{i+1}$. This is possible by projectivity of $P_{i+1}$ since
$$\mathrm{im}\left((-1)^i f_i \circ \partial_{i+1}\right) \leq \mathrm{im}(\partial_1) = \ker(\epsilon)$$
as $\epsilon \circ \left((-1)^i f_i \circ \partial_{i+1}\right) = (-1)^i f \circ \partial_{i+1} = 0$.

(c) Define $f_{i+2}$ such that $\partial_2 \circ f_{i+2} = (-1)^i f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of $P_{i+2}$ since
$$\mathrm{im}\left((-1)^i f_{i+1} \circ \partial_{i+2}\right) \leq \mathrm{im}(\partial_2) = \ker(\partial_1)$$
as $\partial_1 \circ \left((-1)^i f_{i+1} \circ \partial_{i+2}\right) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.

(d) Define $f_m$ for $m > i+2$ by recursion such that $\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \circ \partial_m$. This is possible by projectivity of $P_m$ since
$$\mathrm{im}\left((-1)^i f_{m-1} \circ \partial_m\right) \leq \mathrm{im}(\partial_{m-i}) = \ker(\partial_{m-i-1})$$
as $\partial_{m-i-1} \circ \left((-1)^i f_{m-1} \circ \partial_m\right) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

This completes the definition of $\Phi$. The maps $\{f_m : m \geq i\}$ defined in Steps **??-??** above satisfy
$$\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \partial_m.$$
In other words, $(\partial \Phi(f))_{m+1} = 0$ for all $m \geq i+1$ so that $\partial \Phi(f) = 0$. Thus, $\Phi(f)$ is a cocycle by construction.

2. We claim than any other choice of maps $\{f'_m : m \geq i\}$ satisfying the conditions in **??-??** above will be equivalent to $\{f_m : m \geq i\}$ in $H^i(Y)$. More precisely, if $f$ and $f'$ both satisfy conditions **??-??**, then will define a map $\theta \in Y^{i-1}$ such that $\partial\theta = f - f'$. Write $g_m = f_m - f'_m$ for $m \geq i$.

$$\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"] \arrow[dl, "\theta_{m-1}"'] & P_{m-2} \arrow[r, "\partial_{m-2}"] \arrow[d, "g_{m-2}"] \arrow[dl, "\theta_{m-2}"'] & P_{m-3} \arrow[r] \arrow[d, "g_{m-3}"] \arrow[dl, "\theta_{m-3}"'] & \cdots \arrow[r] & P_{i+2} \arrow[r, "\partial_{i+2}"] \arrow[d, "g_{i+2}"'] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "g_{i+1}"] \arrow[dl, "\theta_{i+1}"'] & P_i \arrow[r, "\partial_i"] \arrow[d, "g_i"] \arrow[dl, "\theta_i"'] & P_{i-1} \arrow[dl, "\theta_{i-1}"'] \\
P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r, "\partial_{m-i-2}"'] & P_{m-i-3} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k
\end{tikzcd} \tag{5}$$

(e) Take $\theta_{i-1} = 0$.

(f) Since $\epsilon \circ f_i = \epsilon \circ f'_i = f$, we have $\mathrm{im}(g_i) \leq \ker(\epsilon) = \mathrm{im}(\partial_1)$. Define $\theta_i$ such that $\partial_1 \circ \theta_i = (-1)^i g_i$. This is possible by projectivity of $P_i$. We rewrite the condition on $\theta_i$ for future reference as follows.
$$(\partial\theta)_i = 0 - (-1)^{i-1} \partial_1 \circ \theta_i = g_i \tag{6}$$

(g) By (**??**), we have

$$\partial_1 \circ \theta_i \circ \partial_{i+1} = (-1)^i g_i \circ \partial_{i+1} = \partial_1 \circ g_{i+1}$$

so that

$$\operatorname{im}(g_{i+1} - \theta_i \circ \partial_{i+1}) \leq \ker(\partial_1) = \operatorname{im}(\partial_2).$$

Define $\theta_{i+1}$ such that

$$\partial_2 \circ \theta_{i+1} = (-1)^i (g_{i+1} - \theta_i \circ \partial_{i+1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{i+1} = \theta_i \circ \partial_{i+1} - (-1)^{i-1} \partial_2 \circ \theta_{i+1} = g_{i+1} \tag{7}$$

(h) Assume by recursion that we have computed $\theta_{m-2}$ and $\theta_{m-3}$ such that

$$\partial_{m-i-1} \circ \theta_{m-2} = (-1)^i (g_{m-2} - \theta_{m-3} \circ \partial_{m-2}).$$

Then $\partial_{m-i-1} \circ \theta_{m-2} \circ \partial_{m-1} = (-1)^i g_{m-2} \circ \partial_{m-1} = \partial_{m-i-1} \circ g_{m-1}$ so that

$$\operatorname{im}(g_{m-1} - \theta_{m-2} \circ \partial_{m-1}) \leq \ker(\partial_{m-i-1}) = \operatorname{im}(\partial_{m-i}).$$

Define $\theta_{m-1}$ such that

$$\partial_{m-i} \circ \theta_{m-1} = (-1)^i (g_{m-1} - \theta_{m-2} \circ \partial_{m-1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{m-1} = \theta_{m-2} \circ \partial_{m-1} - (-1)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1} \tag{8}$$

This completes the definition of $\theta$. Then $\theta$ satisfies $\partial\theta = f - f'$ by (**??**), (**??**), and (**??**).

3. Suppose now that $f = \partial g$ for some cochain $g : P_{i-1} \to k$. Write $\Phi(g \circ \partial_i) = \{g_m : m \geq i\}$. We will construct $\theta$ such that $\Phi(\partial g) = \partial\theta$ for some $\theta \in Y^{i-1}$ as in the following diagram.

$$\begin{tikzcd}
P_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, "g_{m+1}"'] & P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] \arrow[dl, "\theta_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"'] \arrow[dl, "\theta_{m-1}"'] & P_{m-2} \arrow[r] \arrow[d, "g_{m-2}"'] \arrow[dl, "\theta_{m-2}"'] & \cdots \arrow[r] & P_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "g_{i+1}"'] & P_i \arrow[r, "\partial_i"] \arrow[d, "g_i"'] \arrow[dl, "\theta_i"'] & P_{i-1} \arrow[d, "g"] \arrow[dl, "\theta_{i-1}"'] \arrow[dl, phantom, shift right=3, "\circlearrowright"] \\
P_{m-i+1} \arrow[r, "\partial_{m-i+1}"'] & P_{m-i} \arrow[r, "\partial_{m-i}"'] & P_{m-i-1} \arrow[r, "\partial_{m-i-1}"'] & P_{m-i-2} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & k
\end{tikzcd}$$

(i) Define $\theta_{i-1}$ such that $\epsilon \circ \theta_{i-1} = g$. This is possible by projectivity of $P_{i-1}$.

(j) Since $\epsilon \circ \theta_{i-1} \circ \partial_i = g \circ \partial_i = \epsilon \circ g_i$, we have that

$$\operatorname{im}\left(g_i - \theta_{i-1} \circ \partial_i\right) \leq \ker\left(\epsilon\right) = \operatorname{im}\left(\partial_1\right).$$

Thus, by projectivity of $P_i$, we have $\theta_i$ such that

$$\partial_1 \circ \theta_i = \left(-1\right)^i \left(g_i - \theta_{i-1} \circ \partial_i\right).$$

Then

$$\left(\partial\theta\right)_i = \theta_{i-1} \circ \partial_i - \left(-1\right)^{i-1} \partial_1 \circ \theta_i = g_i.$$

(k) Assume by recursion that we have computed the maps $\theta_{m-1}$ and $\theta_{m-2}$ such that

$$\theta_{m-2} \circ \partial_{m-1} - \left(-1\right)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1}.$$

Then

$$\partial_{m-i} \circ g_m = \left(-1\right)^i g_{m-1} \circ \partial_m = \partial_{m-i} \circ \theta_{m-1} \circ \partial_m$$

so that

$$\operatorname{im}\left(g_m - \theta_{m-1} \circ \partial_m\right) \leq \ker\left(\partial_{m-i}\right) = \operatorname{im}\left(\partial_{m-i+1}\right).$$

Define $\theta_m$ such that

$$\partial_{m-i-1} \circ \theta_m = \left(-1\right)^i \left(g_m - \theta_{m-1} \circ \partial_m\right).$$

Then

$$\left(\partial\theta\right)_m = \theta_{m-1} \circ \partial_m - \left(-1\right)^{i-1} \partial_{m-i+1} \circ \theta_m = g_m.$$

This completes the definition of $\theta$. Then $g = \partial\theta$ by construction.

4. We will now show that $\Phi$ is a $k$-module homomorphism. Let $f, g : P_i \to k$ be cocycles and let $\alpha, \beta \in k$. Write $h = \alpha f + \beta g$. We want to show that $\Phi\left(h\right) = \alpha\Phi\left(f\right) + \beta\Phi\left(g\right)$. But $\epsilon \circ h_0 = \epsilon \circ \left(\alpha f_0 + \beta g_0\right) = \alpha f + \beta g$, so that we are in the situation of Step **??** above. Thus, $\Phi\left(h\right)$ and $\alpha\Phi\left(f\right) + \beta\Phi\left(g\right)$ are equivalent elements of $Y$.

5. By Steps **??** and **??**, we have that if $f$ and $f'$ are equivalent in $H^*\left(G, k\right)$, then $\Phi\left(f\right)$ and $\Phi\left(f'\right)$ are equivalent in $H^*\left(Y\right)$. This together with **??** shows that $\Phi$ is a well-defined $k$-module homomorphism.

6. Finally, $\Phi$ is a bijection, having inverse given by

$$\{f_m : m \geq i\} \mapsto \epsilon \circ f_i.$$

■

# 3 Products in $Y$

Consider the following product $Y^i \otimes Y^j \to Y^{i+j}$ on $Y$. Let $f \in Y^i$ and $g \in Y^j$ and consider the composition of the individual component maps of $f$ with those of $g$ such that legitimate compositions are obtained, as in the following diagram.

$$\begin{tikzcd}
P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & P_m \arrow[r, "\partial_m"] \arrow[d, "f_m"] & P_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & \cdots \arrow[r] & P_{i+j+1} \arrow[r, "\partial_{i+j+1}"] \arrow[d, "f_{i+j+1}"] & P_{i+j} \arrow[d, "f_{i+j}"] \\
P_{n-i} \arrow[r, "\partial_{n-i}"] \arrow[d, "g_{n-i}"] & P_{n-i-1} \arrow[r] \arrow[d, "g_{n-i-1}"] & \cdots \arrow[r] & P_{m-i} \arrow[r, "\partial_{m-i}"] \arrow[d, "g_{m-i}"] & P_{m-i-1} \arrow[r] \arrow[d, "g_{m-i-1}"] & \cdots \arrow[r] & P_{j+1} \arrow[r, "\partial_{j+1}"] \arrow[d, "g_{j+1}"] & P_j \arrow[d, "g_j"] \\
P_{n-i-j} \arrow[r, "\partial_{n-i-j}"] & P_{n-i-j-1} \arrow[r] & \cdots \arrow[r] & P_{m-i-j} \arrow[r, "\partial_{m-i-j}"] & P_{m-i-j-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0
\end{tikzcd} \tag{9}$$

Observe that we have thrown away the maps $\{f_m : i \leq m \leq i+j-1\}$. I suppose that the natural symbol for the object in (**??**) would be $g \circ f$, to emphasize the fact that we're talking about the component-wise composition of two elements of $Y$ and *not* a cohomology product.

**Observation 5.** $\partial (g \circ f) = g \circ \partial f + (-1)^{\deg f} \partial g \circ f$.

*Proof.* Write $i = \deg (f)$ and $j = \deg (g)$ as in (**??**). We will show the claim at the point $P_m$ in (**??**) for $m \geq i+j+1 = \deg (\partial (g \circ f))$. Follow along in the picture.

$$\begin{aligned}
\left(g \circ \partial f + (-1)^i \partial g \circ f\right)_m &= g_{m-i-1} \circ \left(f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m\right) \\
&\quad + (-1)^i \left(g_{m-i-1} \circ \partial_{m-i} - (-1)^j \partial_{m-i-j} \circ g_{m-i}\right) \circ f_m \\
&= g_{m-i-1} \circ f_{m-1} \circ \partial_m - (-1)^{i+j} \partial_{m-i-j} \circ g_{m-i} \circ f_m \\
&= (\partial (g \circ f))_m
\end{aligned}$$

∎

**Claim 6.** *Composition in* $Y$ *induces via* $\Phi$ *an associative binary operation*

$$H^i (G, k) \otimes H^j (G, k) \to H^{i+j} (G, k)$$

*making* $H^* (G, k)$ *into a ring with 1.*

# 4 Relationships among products on $H^* (G, k)$

Let $f \in H^i (G, k)$ and $g \in H^j (G, k)$. Consider the following products on $H^* (G, k)$.

1. The *cup product* $fg$ defined in Section **??**

2. The product induced from composition in $Y$

$$(f,g) \overset{\Phi}{\mapsto} \big(\Phi(f), \Phi(g)\big) \overset{\circ}{\mapsto} \Phi(g) \circ \Phi(f) \overset{\Phi^{-1}}{\mapsto} \epsilon \circ \big(\Phi(g) \circ \Phi(f)\big)_{i+j}$$

3. The *Massey 2-fold product* $\langle f, g \rangle$, defined more generally in Section **??** below,

$$(f,g) \overset{\Phi}{\mapsto} \big(\Phi(f), \Phi(g)\big) \overset{\langle\cdot\rangle}{\mapsto} (-1)^{i}\, \Phi(g) \circ \Phi(f) \overset{\Phi^{-1}}{\mapsto} (-1)^{i}\, \epsilon \circ \big(\Phi(g) \circ \Phi(f)\big)_{i+j}$$

The cup product is calculated as $g \circ f_{i+j}$, where $f_{i+j}$ is as in (**??**), whereas product **??** is calculated as $g \circ f_{i+j}$, where $f_{i+j}$ is as in (**??**). Comparing (**??**) and (**??**), we see that the two $f_i$'s are the same, the $f_{i+1}$'s differ by $(-1)^{i}$, the $f_{i+2}$'s differ by $(-1)^{2i}$, and in general, the $f_{i+m}$'s differ by $(-1)^{im}$. Thus, products **??** and **??** differ by $(-1)^{ij}$, that is,

$$\Phi^{-1}\big(\Phi(g) \circ \Phi(f)\big) = (-1)^{ij} fg$$

so that

$$\Phi(fg) = (-1)^{ij}\, \Phi(g) \circ \Phi(f)$$

and therefore

$$\Phi(fg) = (-1)^{i(j+1)} \langle f, g \rangle .$$

We observe that product **??** is associative (see [**?**]), and that product **??** is also associative, consisting of composition of functions. The Massey product, however, is not associative in general.

# 5 Massey Products

The idea of the Massey product is to extend the cohomology product to an $n$-fold product for $n \geq 2$. The following definition is adapted from [**?**].

**Definition 7.** *For $k \geq 2$, let $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ be cocycles in $Y$. The* Massey $k$-fold product $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ *is defined provided that for each pair $(i,j)$ with $1 \leq i < j \leq k$ other than $(1,k)$, the lower-degree product $\left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$ is defined and vanishes as an element of $H^*(Y)$, that is, if for each qualifying $(i,j)$, there exists $u^{i,j} \in Y$ such that $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$. In this situation, the value of $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ is defined to be*

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}}$$

*where the symbols $u^{1,1}$ and $u^{k,k}$ are taken to be $f^{(1)}$ and $f^{(k)}$ respectively and $\overline{u} = (-1)^{\deg(u)}\, u$.*

Observe that in the case $k = 2$, the condition on $(i, j)$ is vacuously satisfied, so that $\langle f, g \rangle = g \circ \overline{f}$.

Traditionally, one organizes the information in Definition **??** in an array, such as the following,

$$\begin{array}{cccc} f^{(1)} & u^{1,2} & u^{1,3} & \\ & f^{(2)} & u^{2,3} & u^{2,4} \\ & & f^{(3)} & u^{3,4} \\ & & & f^{(4)} \end{array}$$

and traces the top row with one hand while tracing the rightmost column with the other hand as $t$ runs from 1 to 3. In this case, we have

$$\left\langle f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \right\rangle = u^{2,4} \circ \overline{f^{(1)}} + u^{3,4} \circ \overline{u^{1,2}} + f^{(4)} \circ \overline{u^{1,3}}.$$

**Lemma 8.** *$\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ is a cocycle in $Y$.*

The reason for the sign appearing in Definition **??** becomes apparent is the following proof.

*Proof.* We begin by making a general observation about $Y$. Suppose $f \in Y^i$ and that $g = \partial\theta$ for some $\theta \in Y^{j-1}$ as in the following diagram.

$$\begin{tikzcd} & P_{i+j+m+1} \arrow[r, "\partial_{i+j+m+1}"] \arrow[d, "f_{i+j+m+1}"] & P_{i+j+m} \arrow[d, "f_{i+j+m}"] \\ & P_{j+m+1} \arrow[r, "\partial_{j+m+1}"] \arrow[dl, "\theta_{j+m+1}"'] \arrow[d, "g_{j+m+1}"] & P_{j+m} \arrow[dl, "\theta_{j+m}"'] \arrow[d, "g_{j+m}"] \\ P_{m+2} \arrow[r, "\partial_{m+2}"'] & P_{m+1} \arrow[r, "\partial_{m+1}"'] & P_m \end{tikzcd}$$

Then by Observation **??**, we have

$$\begin{aligned} (g \circ f)_{i+j+m+1} &= g_{j+m+1} \circ f_{i+j+m+1} \\ &= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\ &= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\ &\quad - (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} + (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} \\ &= -(-1)^i (\theta \circ (\partial f))_{i+j+m+1} + (-1)^i \partial (\theta \circ f)_{i+j+m+1} \end{aligned}$$

so that as elements of $H^*(Y)$, we have

$$\partial\theta \circ f = -(-1)^i \theta \circ \partial f. \tag{10}$$

Now we compute the derivative of $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$.

$$\begin{aligned}
\partial \left( \sum_{t=1}^{k-1} (-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ u^{1,t} \right) &= \sum_{t=1}^{k-1} \left( (-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ \partial u^{1,t} + \partial u^{t+1,k} \circ u^{1,t} \right) \\
&= \sum_{t=1}^{k-1} \left( -\partial u^{t+1,k} \circ u^{1,t} + \partial u^{t+1,k} \circ u^{1,t} \right) \\
&= 0
\end{aligned}$$

■

**Observation 9.** *The condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$ *forces*

$$\deg\left(u^{i,j}\right) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j$$

$$\textit{and} \qquad \deg\left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

**Troubling Observation 10.** $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ *is not uniquely defined, unless for each* $(i,j)$ *the condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$ *is satisfied by exactly one cochain* $u^{i,j}$*.*

Suppose that we are given cocycles $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ and we want to compute the map $u^{i,j}$ for some $(i,j)$ with $1 \le i < j \le k$ other than $(1,k)$. Assume that recursively, we have computed all of the maps in the following array.

$$\begin{array}{ccccc}
f^{(i)} & u^{i,i+1} & \cdots & u^{i,j-1} & \\
& f^{(i+1)} & & u^{i+1,j-1} & u^{i+1,j} \\
& & & & \vdots \\
& & & f^{(j-1)} & u^{j-1,j} \\
& & & & f^{(j)}
\end{array}$$

The map $u^{i,j}$ will be such that

$$\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle = \sum_{t=i}^{j-1} u^{t+1,j} \circ \overline{u^{i,t}} \tag{11}$$

where $u^{i,i} = f^{(i)}$ and $u^{j,j} = f^{(j)}$. Write $g$ for the map on the right-hand side of (**??**). Write

$$d = \deg(g) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

The relevant maps are all pictured below.

$$
\begin{tikzcd}
P_m \arrow[r, "\partial_m"] \arrow[d, "g_m"'] & P_{m-1} \arrow[r, "\partial_{m-1}"] \arrow[d, "g_{m-1}"'] \arrow[dl, "u^{i,j}_{m-1}"'] & P_{m-2} \arrow[r, "\partial_{m-2}"] \arrow[d, "g_{m-2}"'] \arrow[dl, "u^{i,j}_{m-2}"'] & P_{m-3} \arrow[r] \arrow[d] \arrow[dl, "u^{i,j}_{m-3}"'] & \cdots \arrow[r] & P_{d+2} \arrow[r, "\partial_{d+2}"] \arrow[d, "g_{d+2}"'] & P_{d+1} \arrow[r, "\partial_{d+1}"] \arrow[d, "g_{d+1}"'] \arrow[dl, "u^{i,j}_{d+1}"'] & P_d \arrow[r, "\partial_d"] \arrow[d, "g_d"'] \arrow[dl, "u^{i,j}_{d}"'] \arrow[dl, phantom, "\circlearrowright", near end] & {} \arrow[dl, "u^{i,j}_{d-1}"'] \\
P_{m-d} \arrow[r, "\partial_{m-d}"'] & P_{m-d-1} \arrow[r, "\partial_{m-d-1}"'] & P_{m-d-2} \arrow[r, "\partial_{m-d-2}"'] & P_{m-d-3} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r, "\partial_2"'] & P_1 \arrow[r, "\partial_1"'] & P_0 \arrow[r, "\epsilon"'] & {}
\end{tikzcd}
\tag{12}
$$

We assume now that $P_*$ is minimal, that is, that $\partial_m(P_m) \leq \operatorname{Rad}(P_{m-1})$ for all $m \geq 1$. This implies that $\partial f = 0$ for *any* cochain $f$, that is, we have $\partial_{i+1} \circ f = 0$ for any $kG$-homomorphism $f : P_i \to k$.

The map $u^{i,j} \in Y^{d-1}$ is constructed as follows.

1. We take $u^{i,j}_{d-1} = 0$.

2. The assumption that $\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \rangle = g$ vanishes as an element of $H^d(Y)$ tells us that $\epsilon \circ g_d$ vanishes as an element of $H^d(G,k)$. But since $P_*$ is *minimal*, this means that $\epsilon \circ g_d$ is actually the zero map. Then by projectivity of $P_d$, there exists $u^{i,j}_d$ such that $\partial_1 \circ u^{i,j}_d = (-1)^d g_d$. Observe that this means
$$\left(\partial u^{i,j}\right)_d = 0 - (-1)^{d-1} \partial_1 \circ u^{i,j}_d = g_d.$$

3. The map $g$ is a cocycle by Lemma **??**. This means that the *rectangles* in (**??**) either commute or anticommute, depending on whether $d$ is even or odd. Thus,
$$\partial_1 \circ \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) = \partial_1 \circ g_{d+1} - (-1)^d g_d \circ \partial_{d+1} = 0$$
so that
$$\operatorname{im}\left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) \leq \ker(\partial_1) = \operatorname{im}(\partial_2).$$
Thus, there exists $u^{i,j}_{d+1}$ such that
$$\partial_2 \circ u^{i,j}_{d+1} = (-1)^d \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right).$$
Observe that this means
$$\left(\partial u^{i,j}\right)_{d+1} = u^{i,j}_d \circ \partial_{d+1} - (-1)^{d-1} \partial_2 \circ u^{i,j}_{d+1} = g_{d+1}.$$

4. Assume by recursion that we have constructed that maps $u^{i,j}_{m-2}$ and $u^{i,j}_{m-3}$ such that
$$\partial_{m-d-1} \circ u^{i,j}_{m-2} = (-1)^d \left(g_{m-2} - u^{i,j}_{m-3} \circ \partial_{m-2}\right).$$
Thus
$$\partial_{m-d-1} \circ \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right) = \partial_{m-d-1} \circ g_{m-1} - (-1)^d g_{m-2} \circ \partial_{m-1} = 0$$

so that

$$\operatorname{im}\left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right) \leq \ker\left(\partial_{m-d-1}\right) = \operatorname{im}\left(\partial_{m-d}\right).$$

Thus, there exists $u^{i,j}_{m-1}$ such that

$$\partial_{m-d} \circ u^{i,j}_{m-1} = (-1)^d \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1}\right).$$

Observe that this means

$$\left(\partial u^{i,j}\right)_{m-1} = u^{i,j}_{m-2} \circ \partial_{m-1} - (-1)^{d-1} \partial_{m-d} \circ u^{i,j}_{m-1} = g_{m-1}.$$

This completes the construction of $u^{i,j}$. By construction, we have $\partial\left(u^{i,j}\right) = g$.

Finally, observe that in the last step in the calculation of $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)}\right\rangle$, which is actually the *first* step, as this is a recursive process, it is only necessary to calculate $u^{1,k-1}$, but none of the maps $u^{1,m}$ for $2 \leq m \leq k-2$, and none of the maps $u^{m,k}$ for $2 \leq m \leq k-1$. In effect, the sum

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}} = \sum_{t=1}^{k-2} u^{t+1,k} \circ \overline{u^{1,t}} + f^{(k)} \circ \overline{u^{1,k-1}}$$

appearing in Definition **??** is calculated as

$$\sum_{t=1}^{k-2} \overset{!}{\boxed{u^{t+1,k}_{\deg u^{t+1,k}}}} \circ \overline{u^{1,t}_{\deg u^{t+1,k} + \deg u^{1,t}}} + f^{(k)}_{\deg f^{(k)}} \circ \overline{u^{1,k-1}_{\deg f^{(k)} + \deg u^{1,k-1}}},$$

But $u^{t+1,k}_{\deg u^{t+1,k}} = 0$ by construction (see Step **??** above), so the sum reduces to a single term. This is not the case with the intermediate maps $u^{i,j}$ with $j - i \leq k-2$.

# References


[1] Inger Christin Borge. *A cohomological approach to the classification of p-groups*. PhD thesis, Oxford, `http://www.maths.abdn.ac.uk/~bensondj/html/archive/borge.html`, 2001.

[2] Jon F. Carlson, Lisa Townsley, Luis Valeri-Elizondo, and Mucheng Zhang. *Cohomology rings of finite groups*, volume 3 of *Algebras and Applications*. Kluwer Academic Publishers, Dordrecht, 2003.

[3] David Kraines. Massey higher products. *Trans. Amer. Math. Soc.*, 124:431–449, 1966.