# An Example CRIME calculation: The cohomology ring of $Q_8$

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Let $G = Q_8 = \left\langle x, y \mid x^2 = y^2 = (xy)^2, x^4 = 1 \right\rangle = \left\langle x, y, z \mid x^2 = y^2 = z = (xy)^2, x^4 = 1 \right\rangle$. Observe that $z$ in the second presentation is redundant, but simplifies the notation later. In GAP, we execute the following commands.

```
gap> G:=SmallGroup(8,4);
<pc group of size 8 with 3 generators>
gap> Pcgs(G);
Pcgs([ f1, f2, f3 ])
```

Then a little manipulation in GAP reveals that `f1`, `f2`, and `f3`, correspond with $x$, $y$, and $z$ from the presentation above, and with $i$, $j$, and $-1$ from the standard presentation of $Q_8$.

Let $k = \mathbb{F}_2$. It's well known that $k$ has a periodic minimal $kG$-projective resolution. To see this, we start with the following commands.

```
gap> C:=CohomologyObject(G);
<object>
gap> ProjectiveResolution(C,10);
[ 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2 ]
```

`ProjectiveResolution` returns the $kG$-ranks of the terms of the minimal projective resolution. These numbers are called the *Betti numbers* of the resolution. Therefore, this tells us that $k$ has a minimal $kG$-projective resolution

$$P_* : \quad \cdots \longrightarrow kG \xrightarrow{\partial_4} kG \xrightarrow{\partial_3} (kG)^{\oplus 2} \xrightarrow{\partial_2} (kG)^{\oplus 2} \xrightarrow{\partial_1} kG \xrightarrow{\epsilon} k \longrightarrow 0 \quad (1)$$

We can see from (1) that $P_*$ appears to be periodic, but we confirm this below by looking at the boundary maps. The map $\epsilon$ is the usual augmentation $\epsilon \left( \sum_g \alpha_g g \right) = \sum_g \alpha_g$.

Since $P_*$ is minimal, the cohomology groups $H^i(G) = \mathrm{Ext}^i(k, k)$ are simply

$$\mathrm{Hom}_{kG}(P_i, k) = k^{b_i}.$$

Here, $b_i$ is the $(i+1)$st element in the list returned by `ProjectiveResolution`, so the first element in this list is the dimension of $P_0$. Thus, the Betti numbers give the ranks of the cohomology groups as well.

To look at the boundary maps, we need some notation. Recall that for $G$ a $p$-group of size $p^n$ and $k$ a field of characteristic $p$, which is exactly the situation that we're in in this example, the group algebra $kG$ has a basis

$$\mathcal{B}' = \left\{ x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \,\middle|\, 0 \le a_1, a_2, \ldots a_n \le p-1 \right\} \tag{2}$$

where $x_1, x_2, \ldots x_n$ is a polycyclic generating set for $G$. In fact, the fact that $\mathcal{B}'$ is a basis merely expresses the fact the $x_1, x_2, \ldots x_n$ is a polycyclic generating set. In the example $G = Q_8$, arranging the $(a_1, a_2, \ldots, a_n)$'s in reverse lexicographic order, we have

$$\begin{aligned}
\mathcal{B}' &= \left(\; 1, \quad x, \quad y, \quad xy, \quad z, \quad xz, \quad yz, \quad xyz \;\right) \\
&= \left(\; 1, \quad i, \quad j, \quad k, \quad -1, \quad -i, \quad -j, \quad -k \;\right).
\end{aligned}$$

However, a more computationally efficient basis of $kG$ is the following.

$$\mathcal{B} = \left\{ (x_1 - 1)^{a_1} (x_2 - 1)^{a_2} \ldots (x_n - 1)^{a_n} \,\middle|\, 0 \le a_1, a_2, \ldots a_n \le p-1 \right\} \tag{3}$$

Let $I = x + 1$, $J = y + 1$, and $K = xy + 1$. Observe that $I^2 = J^2 = z + 1$. Observe also that $K = I + J + IJ$. The element $K$ was included to make the boundary maps below look more symmetric. Then in the example $G = Q_8$ we have

$$\mathcal{B} = \left(\; 1, \quad I, \quad J, \quad IJ, \quad I^2, \quad I^3, \quad I^2J, \quad I^3J \;\right)$$

The boundary maps returned by `BoundaryMaps` are with respect to the basis $\mathcal{B}$.

```
gap> Display(BoundaryMap(C,1));
. 1 . . . . . .
. . 1 . . . . .
gap> Display(BoundaryMap(C,2));
. 1 . . . . . . . . 1 . . . . .
. . 1 . . . . . . 1 1 1 . . . .
gap> Display(BoundaryMap(C,3));
. . 1 . . . . . 1 1 1 . . . .
gap> Display(BoundaryMap(C,4));
. . . . . . . 1
gap> Display(BoundaryMap(C,5));
. 1 . . . . . .
. . 1 . . . . .
```

Observe first that $\partial_5 = \partial_1$, so we see that $P_*$ is in fact periodic as mentioned above. The matrices for $\partial_n$ give only the image of $1_G$ from each direct factor of $P_n$, since the images of the the other elements of $P_n$ are determined by linearity. [1] For example, since

$$\partial_1 : P_1 = kG \oplus kG \to P_0 = kG,$$

the matrix returned above tells us that $\partial_1(1_G, 0) = I$ and $\partial_1(0, 1_G) = J$. Summarizing the information above, we have the following.

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[1] Note to users: if the matrices giving the action of $kG$ on itself with respect to $\mathcal{B}$, or the full matrices for the $\partial_n$'s would be useful to users, please let me know. I could include functions to return them, but I hesitate to overload the user with superfluous information.

$$\partial_n = \begin{cases} \begin{pmatrix} I \\ J \end{pmatrix} & \text{if } n \equiv 1 \pmod 4 \\[2mm] \begin{pmatrix} I & J \\ J & K \end{pmatrix} & \text{if } n \equiv 2 \pmod 4 \\[2mm] \begin{pmatrix} J & K \end{pmatrix} & \text{if } n \equiv 3 \pmod 4 \\[2mm] \begin{pmatrix} I^3 J \end{pmatrix} & \text{if } n \equiv 0 \pmod 4 \end{cases} \qquad (n \geq 1) \tag{4}$$

The matrices in (4) are meant to be multiplied on the right as usual in GAP.

Now since $H^1(G) = \mathrm{Hom}_{kG}(P_1, k)$, we have a natural basis $\{\eta_1, \eta_2\}$ of $H^1(G)$ where $\eta_1$ is the map sending $(1_G, 0) \mapsto 1_k$ and $(0, 1_G) \mapsto 0$ and $\eta_2$ is the other way around.

Then the following are chain maps representing $\eta_1$ and $\eta_2$.

$$\tag{5}$$



In the rows of the diagrams in (5) we have copies of $P_*$, while in the columns, we have maps making the diagrams commute. These maps were produced by inspection and by . . . well, let's just say that I used GAP a tiny bit. Fortunately, this is exactly what the CRIME package does for us, as we will see below.

For the purpose of multiplication, the pictures in (5) represent $\eta_1$ and $\eta_2$, so the composition of the two pictures represents the product, as in the following picture.

$$\tag{6}$$



From (6), we can see that $\eta_1 \eta_2 = \zeta_2$ where $\{\zeta_1, \zeta_2\}$ is the natural basis of $H^2(G)$. This is the map going from $P_2$ in the top row to $k$ in the bottom, as in the diagrams in (5).

By composing the first diagram with itself, we find that $\eta_1^2 = \zeta_1$. Similarly, by more chain map production and composition, we find that $\eta_2 \zeta_2$ is a nonzero element of degree 3, but that no product of elements of degree $< 4$ produces a nonzero element of degree 4.

Let $\{\xi\}$ be the natural basis of $H^4(G)$. We lift $\xi$ to a chain map.

$$\tag{7}$$



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This time, the production of the chain map is easy because of the periodicity of $P_*$. From (7), we see that all the elements of degree 4–7 arise as products of $\xi$ with elements of degree 0–3, which in turn are products of $\eta_1$ and $\eta_2$.

Thus, by recursion, we find that $\eta_1$, $\eta_2$, and $\xi$ generate the entire ring $H^*(G)$. This is precisely what **GAP** tells us from the following commands.

```
gap> CohomologyGenerators(C,10);
[ 1, 1, 4 ]
gap> A:=CohomologyRing(C,10);
<algebra of dimension 17 over GF(2)>
gap> LocateGeneratorsInCohomologyRing(C);
[ v.2, v.3, v.7 ]
```

`CohomologyGenerators` merely tells us the degrees of the generators, and they agree with those which we computed above.

The ring returned by `CohomologyRing` has basis `[A.1, A.2, .. A.17]` corresponding with the concatenation of the natural bases of the $H^i(G)$'s. Thus, `A.1` is the identity element, `A.2` and `A.3` correspond with $\eta_1$ and $\eta_2$, `A.4` and `A.5` correspond with $\zeta_1$ and $\zeta_2$, etc. Observe that $17 = \sum_{i=0}^{10} b_i$ which explains the dimension of `A`. The true cohomology ring is infinite-dimensional, so that `A` can be seen as a degree-10-truncation, that is, $A \cong H^*(G)/J_{>10}$ where $J_{>10}$ is the subring of all elements of degree $> 10$.

The following commands verify the calculations mentioned above.

```
gap> A.2^2;
v.4
gap> A.2*A.3;
v.5
gap> A.3*A.5;
v.6
```

The command `LocateGeneratorsInCohomologyRing` tells us that $\eta_1$, $\eta_2$, and $\xi$ correspond with `A.2`, `A.3`, and `A.7`, which we had already deduced by degree considerations, but if $\dim H^4(G)$ had been greater than 1, we wouldn't have known which element corresponded with $\xi$.

Finally, **GAP** gives us a presentation of $H^*(G)$ with the following command.

```
gap> CohomologyRelators(C,10);
[ [ z, y, x ], [ z^2+z*y+y^2, y^3 ] ]
```

This tells us that
$$H^*(G) \cong k[z, y, x]/(z^2 + yz + y^2, y^3)$$
is a polynomial ring in the variables $z$, $y$ and $x$, modulo the ideal generated by $z^2 + yz + y^2$ and $y^3$.