Finite Incidence Geometry in GAP

GAP in Algebraic Research

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Contents

Some history on (finite) incidence geometry

- Beniamino Segre, classical algebraic geometry.

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- Jacques Tits, geometry of finite simple groups.

Some history on (finite) incidence geometry

- Beniamino Segre, classical algebraic geometry.
- Jacques Tits, geometry of finite simple groups.
- many others ....

FinInG

- Finite Incidence Geometry in GAP.
- Development started in 2005.
- Authors: John Bamberg, Anton Betten, Philippe Cara, Jan De Beule, Michel Lavrauw, and Max Neunhöffer
- `http://www.fining.org`
- `https://bitbucket.org/jdebeule/fining`
- `https://www.gap-system.org/Packages/fining.html`
- support 'at' fining.org

Projective spaces

- Let K be a (skew)field.
- Let $V(n+1, K)$ be a (left) vector space over K of dimension $n+1$.

Definition

A *projective space* of dimension n over the skewfield K is the geometry consisting of the sub vector spaces of $V(n+1, K)$, with incidence being symmetrized set theoretic containment.

We denote this projective space as $\mathrm{PG}(n, K)$. The points, lines, planes, $\ldots$, hyperplanes, respectively, are the 1-, 2-, 3-, $\ldots$, n-dimensional subspaces of $V(n+1, K)$.

My first projective space in GAP

```
ext-104-247:~ jdebeule$ gap4r10
 ┌───────┐   GAP 4.10.0 of 01-Nov-2018
 │  GAP  │   https://www.gap-system.org
 └───────┘   Architecture: x86_64-apple-darwin18.0.0-default64
 Configuration:  gmp 6.1.2
 Loading the library and packages ...
 Packages:   AClib 1.3.1, Alnuth 3.1.0, AtlasRep 1.5.1, AutoDoc 2018.09.20,
             AutPGrp 1.10, Browse 1.8.8, CRISP 1.4.4, Cryst 4.1.18,
             CrystCat 1.1.8, CTblLib 1.2.2, FactInt 1.6.2, FGA 1.4.0,
             GAPDoc 1.6.2, IO 4.5.4, IRREDSOL 1.4, LAGUNA 3.9.0,
             Polenta 1.3.8, Polycyclic 2.14, PrimGrp 3.3.2, RadiRoot 2.8,
             ResClasses 4.7.1, SmallGrp 1.3, Sophus 1.24, SpinSym 1.5,
             TomLib 1.2.7, TransGrp 2.0.4, utils 0.59
 Try '??help' for help. See also '?copyright', '?cite' and '?authors'
gap> ColorPrompt(true);
gap> LoadPackage("fining");
```

Figure: Load FinFinG

My first projective space in GAP

- `GF`
- `ProjectiveSpace`, alternatively `PG`

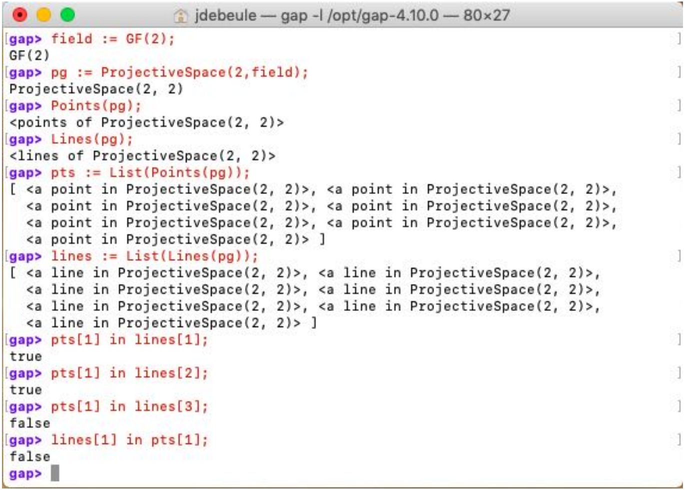
My first projective space in GAP

- `GF`
- `ProjectiveSpace`, alternatively `PG`
- `Points`
- `Lines`
- `List`

My first projective space in GAP

- `GF`
- `ProjectiveSpace`, alternatively `PG`
- `Points`
- `Lines`
- `List`
- `in`

My first projective space in GAP



jdebeule — gap -l /opt/gap-4.10.0 — 80×27

```
gap> field := GF(2);
GF(2)
gap> pg := ProjectiveSpace(2,field);
ProjectiveSpace(2, 2)
gap> Points(pg);
<points of ProjectiveSpace(2, 2)>
gap> Lines(pg);
<lines of ProjectiveSpace(2, 2)>
gap> pts := List(Points(pg));
[ <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)> ]
gap> lines := List(Lines(pg));
[ <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)> ]
gap> pts[1] in lines[1];
true
gap> pts[1] in lines[2];
true
gap> pts[1] in lines[3];
false
gap> lines[1] in pts[1];
false
gap>
```

Figure: Fano plane in FinInG

Incidence geometry

Definition (based on e.g. [2])

An *incidence structure* is a tuple $(\mathcal{S}, T, t, I)$, such that

- $\mathcal{S}$ a non empty set called the set of *elements*,
- the set T is a finite, non empty set, called the *set of types*,
- $t : \mathcal{S} \to T$, is a function and for $x \in \mathcal{S}$, $t(x)$ is called its *type*,
- the relation I is a symmetric relation on $\mathcal{S}$, called the *incidence relation*,
- two elements of the same type are never incident.

The rank of an incidence structure $(\mathcal{S}, T, t, I)$ is $|T|$, the cardinality of T.

Incidence geometry

Definition

A *flag* of an incidence structure $(\mathcal{S}, T, I)$ is a set $F \subset \mathcal{S}$ such that every two elements of F are incident.

From the definition it follows immediately that a flag contains at most one element of a given type.

Definition

A flag F is *maximal* if it is not contained in a larger flag.

Incidence geometry

Definition

An incidence structure is called an *incidence geometry* if every maximal flag contains an element of each type.

A projective space as incidence geometry

- `Rank`
- `TypesOfElementsOfIncidenceStructure`
- `ElementsOfIncidenceStructure`
- `Points`, `Lines`, `Planes`, `Solids`, `Hyperplanes`
- `IsIncident`

A projective space as incidence geometry

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> pg := PG(2,2);
ProjectiveSpace(2, 2)
gap> pts := List(Points(pg));
[ <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)>, <a point in ProjectiveSpace(2, 2)>,
  <a point in ProjectiveSpace(2, 2)> ]
gap> lines := List(Lines(pg));
[ <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)>, <a line in ProjectiveSpace(2, 2)>,
  <a line in ProjectiveSpace(2, 2)> ]
gap> p := pts[1];
<a point in ProjectiveSpace(2, 2)>
gap> l := lines[1];
<a line in ProjectiveSpace(2, 2)>
gap> IsIncident(p,l);
true
gap> IsIncident(l,p);
true
gap> p * l;
true
```

A projective space as incidence geometry

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> pg := PG(7,8);
ProjectiveSpace(7, 8)
gap> ElementsOfIncidenceStructure(pg);
<All elements of ProjectiveSpace(7, 8)>
gap> ElementsOfIncidenceStructure(pg,1);
<points of ProjectiveSpace(7, 8)>
gap> ElementsOfIncidenceStructure(pg,2);
<lines of ProjectiveSpace(7, 8)>
gap> TypesOfElementsOfIncidenceStructure(pg);
[ "point", "line", "plane", "solid", "proj. 4-space", "proj. 5-space",
  "proj. 6-space" ]
gap> Points(pg);
<points of ProjectiveSpace(7, 8)>
gap> Lines(pg);
<lines of ProjectiveSpace(7, 8)>
gap> Planes(pg);
<planes of ProjectiveSpace(7, 8)>
gap> Solids(pg);
<solids of ProjectiveSpace(7, 8)>
gap> Hyperplanes(pg);
<proj. 6-subspaces of ProjectiveSpace(7, 8)>
gap>
```

Some examples

- The projective space $\mathrm{PG}(n, K)$ is an incidence **geometry** of rank n.
- Define
  - $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\} \cup \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}$,
  - $T = \{$“integer”,“set of integers”$\}$,
  - t is the obvious function from $\mathcal{S}$ to T,
  - the incidence relation I is symmetrized containment.

  Then $(\mathcal{S}, T, t, I)$ is an incidence geometry of rank 2. We could have called elements of type “integer” points as well, and elements of type “set of integers” lines as well, and vice versa.

Graph theoretical point of view

- An finite incidence structure of rank *r* is a multipartite graph (*r* partitions), together with a type function on the set of vertices and where the incidence relation is simply the adjacency relation of the graph.
- Given a finite incidence structure, it is easy to create this *incidence graph*.
- The flags are the cliques of the graph.

An incidence structure in FinInG

```
jdebeule — gap -l /opt/gap-4.10.0 — 80×24
gap> S := IncidenceStructure(els,incidence,type,typeset);
Incidence structure of rank 2
gap>
gap> graph := IncidenceGraph(S);
rec(
  adjacencies := [ [ 8, 9, 10 ], [ 8, 11, 12 ], [ 8, 13, 14 ], [ 9, 11, 13 ],
      [ 9, 12, 14 ], [ 10, 11, 14 ], [ 10, 12, 13 ], [ 1, 2, 3 ],
      [ 1, 4, 5 ], [ 1, 6, 7 ], [ 2, 4, 6 ], [ 2, 5, 7 ], [ 3, 4, 7 ],
      [ 3, 5, 6 ] ], group := Group(()), isGraph := true,
  names := [ 1 .. 14 ], order := 14,
  representatives := [ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 ],
  schreierVector := [ -1, -2, -3, -4, -5, -6, -7, -8, -9, -10, -11, -12, -13,
      -14 ] )
gap> group := AutomorphismGroup(graph);
Group([ (4,5)(6,7)(11,12)(13,14), (2,3)(6,7)(11,13)(12,14), (4,6)(5,7)(9,10)
(13,14), (2,4)(3,5)(8,9)(12,13), (1,2)(5,6)(9,11)(10,12), (1,8)(2,9)(3,10)
(4,11)(5,12)(6,13)(7,14) ])
gap> StructureDescription(group);
"PSL(3,2) : C2"
gap>
```

Elements and objects

- An element of an incidence geometry is represented by an object.
- `VectorSpaceToElement` and `UnderlyingObject`.

Elements and objects

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> pg := PG(2,2);
ProjectiveSpace(2, 2)
gap> p := VectorSpaceToElement(pg,[1,0,0]*Z(2));
<a point in ProjectiveSpace(2, 2)>
gap> l := VectorSpaceToElement(pg,[[1,0,0],[0,1,0]]*Z(2));
<a line in ProjectiveSpace(2, 2)>
gap> UnderlyingObject(p);
<cvec over GF(2,1) of length 3>
gap> Unpack(last);
[ Z(2)^0, 0*Z(2), 0*Z(2) ]
gap> UnderlyingObject(l);
<cmat 2x3 over GF(2,1)>
gap> Unpack(last);
[ [ Z(2)^0, 0*Z(2), 0*Z(2) ], [ 0*Z(2), Z(2)^0, 0*Z(2) ] ]
gap>
```

Elements and shadows

Definition

The *shadow of type i* of an element, respectively flag, is the set of elements of type i incident with the element, respectively with all elements of the flag.

Elements and shadows

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> pg := PG(3,9);
ProjectiveSpace(3, 9)
gap> p := VectorSpaceToElement(pg,[1,0,0,0]*Z(9)^0);
<a point in ProjectiveSpace(3, 9)>
gap> Lines(p);
<shadow lines in ProjectiveSpace(3, 9)>
gap> Planes(p);
<shadow planes in ProjectiveSpace(3, 9)>
gap> plane := HyperplaneByDualCoordinates(pg,[0,1,0,0]*Z(9)^0);
<a plane in ProjectiveSpace(3, 9)>
gap> Points(plane);
<shadow points in ProjectiveSpace(3, 9)>
gap> Lines(plane);
<shadow lines in ProjectiveSpace(3, 9)>
gap> p * plane;
true
gap> flag := FlagOfIncidenceStructure(pg,[p,plane]);
<a flag of ProjectiveSpace(3, 9)>
gap> shadow := ShadowOfFlag(pg,flag,2);
<shadow lines in ProjectiveSpace(3, 9)>
gap>
```

More examples of incidence geometries ...

...available in FinInG.

- Coset geometries
- Affine spaces.
- Finite classical polar spaces.

Desarguesian projective spaces

Theorem

Let K be a skewfield and $n \geq 2$. The projective geometry $\mathrm{PG}(n, K)$ is Desarguesian.

Desarguesian projective spaces

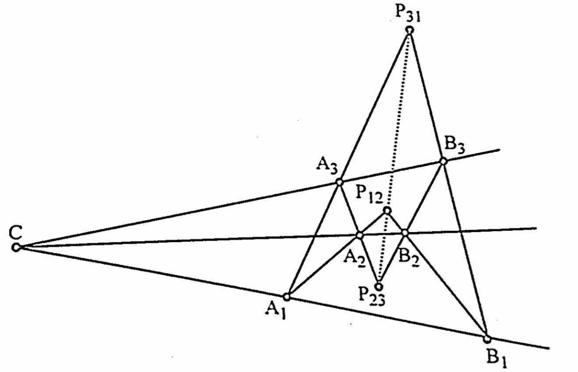


Figure: The theorem of Desargues, figure taken from [1]

.

Pappian projective spaces

Theorem

Let K be a field and $n \geq 2$. The projective geometry $\mathrm{PG}(n, K)$ is Pappian.

Pappian projective spaces

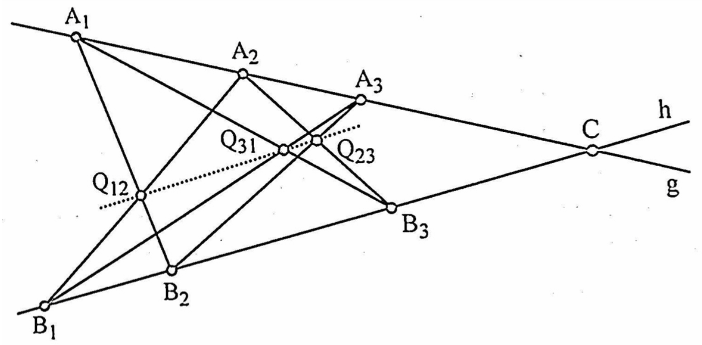


Figure: The theorem of Pappus, figure taken from [1]

.

Axiomatic projective planes

Definition

An axiomatic projective plane $\mathcal{P}$ is a point line geometry satisfying the following axioms

- Any two points determine a unique line.
- Any two lines meet in exactly one point.
- There are four points of which no three are collinear.

Axiomatic projective spaces

Definition

An axiomatic projective space $\mathcal{P}$ is a point line geometry satisfying the following axioms

- Any two points determine a unique line.
- (Veblen-Young[a]) Let A, B, C, and D be four points such that the line AB intersects the line CD. Then the line AC also intersects the line BD.
- Any line is incident with at least three points.
- There are at least two lines.

[a] sometimes also called the axiom of Pasch

Axiomatic projective spaces

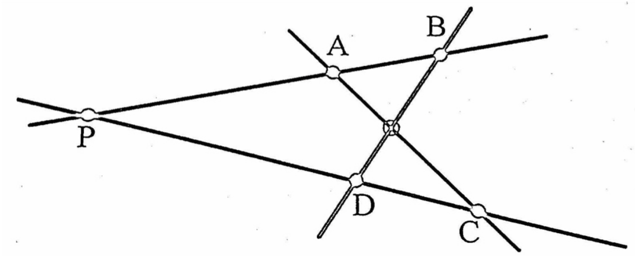


Figure: The axiom of Veblen-Young, figure taken from [1]

.

Classical theorems

Theorem

A projective space $\mathcal{P}$ is Desarguesian if and only if $\mathcal{P} = \mathrm{PG}(n, K)$ for some $n \geq 2$ and K a skewfield. A projective space $\mathcal{P}$ of dimension at least 3 is Desarguesian.

Theorem

A projective space $\mathcal{P}$ is Pappian if and only if $\mathcal{P} = \mathrm{PG}(n, K)$ for some $n \geq 2$ and K a field.

One more classical theorem

Theorem (Hessenberg)

If the theorem of Pappus hold in the projective space $\mathcal{P}$, then also the theorem of Desargues.

Classical theorems

Corollary (by Wedderburn)

A finite projective space $\mathcal{P}$ is Pappian if and only if it is Desarguesian.

Theorem (Fundamental theorem of projective geometry)

All the collineations of $\mathrm{PG}(n, K)$ are induced by the group of semilinear bijective maps of the underlying vector space.

A finite projective space $\mathrm{PG}(n, K)$, $K = \mathbb{F}_q$, will be denoted $\mathrm{PG}(n, q)$.

Groups and projective groups

G, acting on $V(n+1,K)$	$G/\mathrm{Sc}(n+1,K)$, acting on $\mathrm{PG}(n,K)$	name
$\Gamma\mathrm{L}(n+1,K)$	$\mathrm{P}\Gamma\mathrm{L}(n+1,K)$	collineations
$\mathrm{GL}(n+1,K)$	$\mathrm{PGL}(n+1,K)$	homographies
$(n+1,K)$	$\mathrm{PSL}(n+1,K)$	special homographies

Projective groups in FinInG

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> pgl43 := CollineationGroup(PG(3,3));
The FinInG collineation group PGL(4,3)
gap> Order(pgl43);
12130560
gap> PGL(4,3);
Group([ (15,16)(17,20)(18,22)(19,21)(23,32)(24,34)(25,33)(26,38)(27,40)(28,39)
(29,35)(30,37)(31,36), (1,2,5,14,16,22,40,9,26,33,4,11,32)(3,8,23,15,19,31,27,
39,12,35,7,20,34)(6,17,25,18,28,36,10,29,24,21,37,13,38) ])
gap> Order(last);
12130560
gap> psl32 := SpecialHomographyGroup(PG(2,2));
The FinInG PSL group PSL(3,2)
gap> Order(psl32);
168
gap> Action(psl32,Points(PG(2,2)),OnProjSubspaces);
Group([ (1,3)(5,7), (1,4,2)(3,5,6) ])
gap> PSL(3,2);
Group([ (4,6)(5,7), (1,2,4)(3,6,5) ])
gap> Action(psl32,Lines(PG(2,2)),OnProjSubspaces);
Group([ (2,5)(3,7), (1,2,4)(3,5,6) ])
gap>
```

Groups and actions

- `OnProjSubspaces`: action function for elements of a FinInG projective group and a subspace of a projective space.
- Compatible with generic gap functions for groups, actions, orbits and stabilizers.
- Particular implementations: `FiningOrbits`, `FiningStabiliser`, `FiningSetwiseStabiliser`, and variants.

Non Desarguesian projective planes

- There exist many non Desarguesian finite projective planes.
- From the algebraic point of view, the *coordinatizing structure* is intereseting. This leads to the study of e.g. semi fields and other algebraic structures.
- The so called *finite translation planes* can be constructed from a spread of a finite three dimensional projective space.

Definition

A *collineation* of an incidence geometry is an incidence and type preserving bijection of the set of elements.

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> blocks := [[1,2,3],[1,4,5],[1,6,7],[2,4,6],[3,4,7],[2,5,7],[3,5,6]];
[ [ 1, 2, 3 ], [ 1, 4, 5 ], [ 1, 6, 7 ], [ 2, 4, 6 ], [ 3, 4, 7 ],
  [ 2, 5, 7 ], [ 3, 5, 6 ] ]
gap> plane := GeneralisedPolygonByBlocks(blocks);
<projective plane order 2>
gap> group := CollineationGroup(plane);
Group([ (4,5)(6,7)(11,12)(13,14), (4,6)(5,7)(9,10)(13,14), (2,3)(6,7)(11,13)
(12,14), (2,4)(3,5)(8,9)(12,13), (1,2)(5,6)(9,11)(10,12) ])
gap> StructureDescription(group);
"PSL(3,2)"
gap> 
```

Quadratic and sesquilinear forms

Definition

Let K be a field. A *sesquilinear form* on the vector space $V(n, K)$ is a map $f : V \to K$ such that

- $f(x + y, z) = f(x, z) + f(y, z)$, and $f(x, y + z) = f(x, y) + f(x, z), \forall x, y, z \in V$,
- $f(\alpha x, y) = \alpha f(x, y), \forall x, y \in V$ and $\forall \alpha \in K$, and,
- $f(x, \alpha y) = \alpha^\theta f(x, y), \forall x, y \in V$ and $\forall \alpha \in K$, for a fixed automorphism of K.

When $\theta = 1_K$, then f is bilinear.

Quadratic and sesquilinear forms

Definition

Let K be a field. A *quadratic form* on the vector space $V(n, K)$ is a map $Q : V \rightarrow K$ such that

$$Q(\alpha v) = \alpha^2 v, \forall \alpha \in K, \forall v \in V .$$

Definition

Let Q be a quadratic form on the vector space V. Then the *associated bilinear form* is the bilinear form

$$f_Q : V \rightarrow K : f_Q(v, w) := Q(v + w) - Q(v) - Q(w) .$$

isotropic subspaces

Definition

Let f be a sesquilinear form on a vector space $V = V(n, K)$. Then

- a vector $v \in V$ is isotropic if $f(v, v) = 0$,
- $\mathrm{Rad}(f) = \{v \in V | f(v, w) = 0\,, \forall w \in V\}$

Definition

Let f be a sesquilinear form on a vector space $V = V(n, K)$. Then a subspace $S \subset V$ is *totally isotropic* if

$$f(v, w) = 0\,, \forall v, w \in S\,.$$

singular subspaces

Definition

Let Q be a quadratic form on a vector space V. Then

- a vector $v \in V$ is singular if $Q(v) = 0$,
- $\mathrm{Rad}(Q) = \{v \in V | Q(v) = 0\} \cap \mathrm{Rad}(f_Q)$.

Definition

Let Q be a quadratic form on a vector space V. Then a subspace $S \subset V$ is *totally singular* if S and totally isotropic with relation to f_Q and all vectors $v \in S$ are singular with relation to Q.

Fields of odd characteristic

Lemma

Let f be a bilinear form on the vector space $V(n, K)$, K a field of odd characteristic. Then f determines a unique quadratic form Q such that $f = f_Q$.

Proof.

Define $Q : V \to K : v \mapsto \frac{1}{2}f(v, v)$ $\square$

Fields of even characteristic

Lemma

Let Q be a quadratic form on the vector space $V = V(n, K)$, K a field of even characteristic. Then the associated bilinear form f_Q is alternating.

Proof.

$f_Q(v, v) := Q(2v) - 2Q(v) = 0, \forall v \in V$ $\square$

Finite classical polar spaces

Definition

A *finite classical polar space* is the geometry consisting of the totally isotropic or, respectively, totally singular subspaces with relation to a sesquilinear or, respectively quadratic, form on a vector space $V(n, K)$, K a finite field.

- From the definition it follows that a finite classical polar space is naturally embedded in a finite projective space.
- The rank of the polar space as incidence geometry is the Witt index of the underlying form.

Maps on formed spaces

Definition

Let f be a sesquilinear form on the vector space $V = V(n, K)$. Let $\phi : V \to V$ be a map on V. We call ϕ a

- semi-similarity if $f(v, w) = \lambda f(\phi(v), \phi(w))^\theta \; \forall v, w \in V$, for a fixed $\lambda \in K$ and θ a fixed automorphism of K.
- a similarity $f(v, w) = \lambda f(\phi(v), \phi(w))$ for all $v, w \in V$, for a fixed $\lambda \in K$,
- an isometry $f(v, w) = f(\phi(v), \phi(w))$.

Classical theorems

Theorem

Let $\mathcal{S}$ be a finite classical polar space with underlying sesquilinear or quadratic form f. All collineations of $\mathcal{S}$ are induced by the bijective semi-similarities on the formed vector space (V, f).

Axiomatic polar spaces

- Foundations by Veldkamp, Freudenthal, Tits, Buekenhout, Shult, and others, see e.g. [2].
- We restrict to finite generalized quadrangles.

Definition

An *finite generalized quadrangle* is a point-line geometry $\mathcal{S}$ satisfying the following axioms.

- Each point is incident with $1 + t$ lines, $t \geq 1$, and two points are incident with at most one line.
- Each line is incident with $1 + s$ lines, $s \geq 1$, and two lines are incident with at most one line.
- If the point P is not on the line l, then P is collinear with exactly one point of l.

Classical theorems

Definition

The rank of a polar space is one more than the dimension of the subspaces of maximal dimension.

Theorem

A polar space of rank at least three is classical, i.e. it is determined by a form on a vector space over a skewfield.

Groups of collineations of fcps

	$\mathrm{W}(d,q)$	$\mathrm{Q}^+(d,q)$	$\mathrm{Q}^-(d,q)$	$\mathrm{Q}(d,q)$	$\mathrm{H}(d,q^2)$
SI	$\mathrm{PSp}(d,q)$	$\mathrm{PSO}^+(d,q)$	$\mathrm{PSO}^-(d,q)$	$\mathrm{PSO}(d,q)$	$\mathrm{PSU}(d,q^2)$
I	$\mathrm{PSp}(d,q)$	$\mathrm{PGO}^+(d,q)$	$\mathrm{PGO}^-(d,q)$	$\mathrm{PGO}(d,q)$	$\mathrm{PGU}(d,q^2)$
Sim	$\mathrm{PGSp}(d,q)$	$\mathrm{P\Delta O}^+(d,q)$	$\mathrm{P\Delta O}^-(d,q)$	$\mathrm{PGO}(d,q)$	$\mathrm{PGU}(d,q^2)$
Col	$\mathrm{P\Gamma Sp}(d,q)$	$\mathrm{P\Gamma O}^+(d,q)$	$\mathrm{P\Gamma O}^-(d,q)$	$\mathrm{P\Gamma O}(d,q)$	$\mathrm{P\Gamma U}(d,q^2)$

- SI: special isometry group, I: isometry group, Sim: similarity group, Col: collineation group.
- $\mathrm{W}(d,q)$, $\mathrm{Q}^+(d,q)$, $\mathrm{Q}^-(d,q)$: d must be odd.
- $\mathrm{Q}(d,q)$: d must be even.

Projective finite classical groups

jdebeule — gap -l /opt/gap-4.10.0 — 80×24

```
gap> CollineationGroup(HermitianPolarSpace(3,4));
PGammaU(4,2^2)
gap> SpecialIsometryGroup(ParabolicQuadric(2,9));
PSO(0,3,9)
gap> SimilarityGroup(HermitianPolarSpace(4,16));
PGU(5,4^2)
gap>
```

Geometry morphisms

Definition

Let $\mathcal{S} = (S, T, t, I)$ and $\mathcal{S}' = (S', T', t', I')$ be two incidence geometries. A *geometry morphism* is a map $\phi : S \to S'$, such that

$$\forall A, B \in S : t(A) = t(B) \implies t'(\phi(A)) = t'(\phi(B)) \, ,$$

and

$$\forall A, B \in S, \phi(A) \; I' \; \phi(B) \implies A \; I \; B,$$

$$\forall A, B \in S : t'(\phi(A)) \neq t'(\phi(B)) \implies (A \; I \; B \implies \phi(A) \; I' \; \phi(B)),$$

Examples of geometry morphisms

- IsomorphismPolarSpaces
- NaturalEmbeddingBySubspace
- KleinCorrespondence
- NaturalEmbeddingByFieldReduction

Generalized *d*-gons

Definition

A finite generalized *d*-gon is a point-line geometry of which the incidence graph has diameter *d* and girth 2*d*.

Theorem

If Γ *is a finite* thick *generalized d-gon, then* $d \in \{3, 4, 6, 8\}$.

Separate class of geometries

- Finite classical polar spaces of rank 2 are the finite classical generalized quadrangles.
- Split Cayley hexagon and Triality hexagon are available.
- Generic function to construct GPs from elements (of e.g. a projective space).

Finite Geometry and friends

http://summerschool.fining.org

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