Volatile Kalman Filter Scaling Law Notes

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Introduction

A central goal for learning theories in psychology and neuroscience is to model how animals and humans learn associations between cues and outcomes. Typically, this process is often modeled by updating beliefs based on prediction errors, modulated by a learning rate. *Behrens et al.* (2007) proposed that learning rates are dynamically modulated by the *volatility* of the environment: when environmental changes are more frequent, the learning rate should increase, and when changes are less frequent, it should decrease. However, the computational model introduced by *Behrens et al.* to describe this relationship is computationally complex and likely not biologically feasible.

To address this, *Piray and Daw* (2020) introduced the *Volatile Kalman Filter* (VKF) as tractable model of learning in volatile environments. The model has three parameters: the observation noise σ^2, initial volatility estimate v_0, and volatility update parameter λ. At each time step, the posterior mean m_t and variance w_t of the latent state s_t is estimated according to the following algorithm.

1. Compute the Kalman Gain: $k_t = (w_{t-1} + v_{t-1})/(w_{t-1} + v_{t-1} + \sigma^2)$
2. Update the Posterior Mean: $m_t = m_{t-1} + k_t(o_t - m_{t-1})$
3. Update the Posterior Variance: $w_t = (1 - k_t)(w_{t-1} + v_{t-1})$
4. Update Volatility Estimate: $v_t = v_{t-1} + \lambda((m_t - m_{t-1})^2 + w_{t-1} + w_t - 2w_{t-1,t} - v_{t-1})$

Where o_t is the observation at time t and $w_{t-1,t}$ is the autocovariance $w_{t-1,t} = cov[x_t, x_{t-1}]$ and can be shown to be given by $w_{t-1,t} = (1 - k_t)w_{t-1}$.

In this project, we investigate how the magnitude of the prediction error $|o_t - m_{t-1}|$ impacts the volatility estimate v_t for both current at future update steps. To make this precise, we first let $\mathbf{v} \in \mathbb{R}^T$ denote the volatility estimates given by the VKF for a simulation of T time steps, so that $\mathbf{v}(t) = v_t$. Likewise, let $\delta \in \mathbb{R}^T$ be the same for the magnitude of the prediction error, so $\delta(t) = |m_{t-1} - o_t|$. Now, let the rolling average of δ at time t be denoted by $\delta_{\text{avg}}(t)$, which can be computed as

$$\delta_{\text{avg}}(t) = \frac{1}{n}\sum_{k=1}^{n_p} \delta(t - k)$$

where $n_p \in \mathbb{N}$ is the number of preceding elements included in the average and $\delta(t-k)$ represents the prediction error at previous time steps. We further let $\delta_{\text{avg}}^n \in \mathbb{R}^{T-n}$ to be the vector containing the rolling averages for time steps $n+1 \le t \le T$.

We define $n_r \le T/2$ to be the integer such that maximizes the pearson correlation between $\delta_{\text{avg}}^{n_r}$ and $\mathbf{v}_{n+1:}$. Note that we take $n_r \le T/2$ in all experiments to avoid finding spurious correlations between small vectors. This condition can likely be considerably relaxed, however. Further, we define n_ρ to be the integer that maximizes Spearman's rank correlation coefficient for the same vectors.

The computational experiments outline below investigate the connection between n_r, n_ρ, and λ. In particular, our main finding will be to demonstrate that

$$n_r = \frac{c_r}{\lambda},\ n_\rho = \frac{c_\rho}{\lambda},$$

where c_r and c_ρ depend only on the other parameters in the simulation.

Experiment One

In the first experiment, we assume that the latent state dynamics match are those assumed by the derivation of VKF. That is, that the latent state at time t is given by

$$s_t = s_{t-1} + e_t,$$

where e_t is a noise term drawn from $\mathcal{N}(0, z_t^{-1})$. The precision z_t is updated according to

$$z_t = R z_{t-1} \varepsilon_t,$$

where $R \ge 1$ is a constant and ε_t is a random variable drawn from a beta distribution governed by

$$p(\varepsilon_t) = \mathcal{B}(\varepsilon_t | \eta\nu, (1-\eta)\nu),$$

where $\eta = R^{-1}$ and $\nu = \frac{1}{2(1-\eta)}$. Observation were generated by the Gaussian distribution

$$p(o_t|s_t) = \mathcal{N}(o_t|s_t, \sigma_o^2),$$

where σ_o^2 is the observation variance, which is fixed across time.

First, we ran the simulation of the latent state for $T = 500$ time steps with initial latent state and precision $s_0 = 10$ and $z_0 = 10$, respectively. The observation variance was set at $\sigma_o^2 = 0.1$, and $(1-\eta) = 0.001$, which controls how the precision z_t is updated over time. VKF was then run on the observations generated by the simulation with parameters $\lambda = 0.2$, $v_0 = 1.0$, and $\sigma^2 = 0.3$. Figure 1 depicts the true latent state s_t, the predictions of VKF m_t, and the observations for each time step. Figures 2 and 3 illustrate the volatility estimates and the learning rates (Kalman gain) derived from the simulation, respectively.

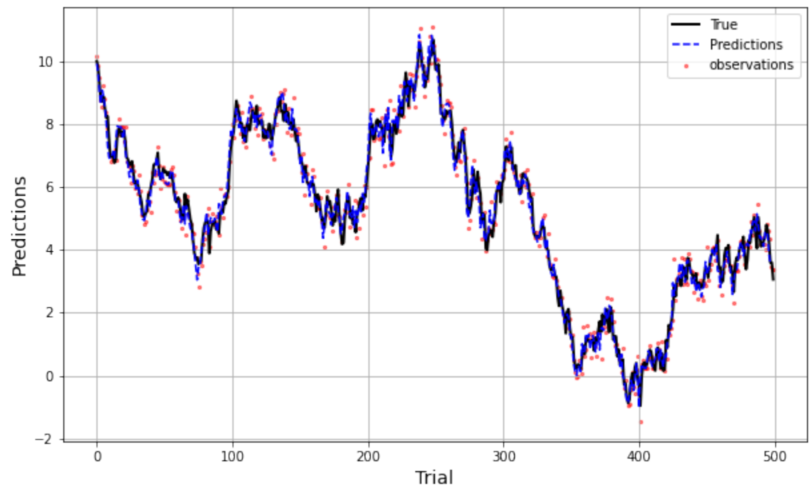


Figure 1: Simulation results for $T = 500$ time steps showing the true latent state (s_t), the predictions of the Volatile Kalman Filter (VKF; m_t), and the observations. The initial latent state and precision were set to $s_0 = 10$ and $z_0 = 10$, respectively, with observation variance $\sigma_o^2 = 0.1$ and precision update parameter $(1-\eta) = 0.001$. VKF was applied to the simulated observations with parameters $\lambda = 0.2$, $v_0 = 1.0$, and $\sigma^2 = 0.3$.

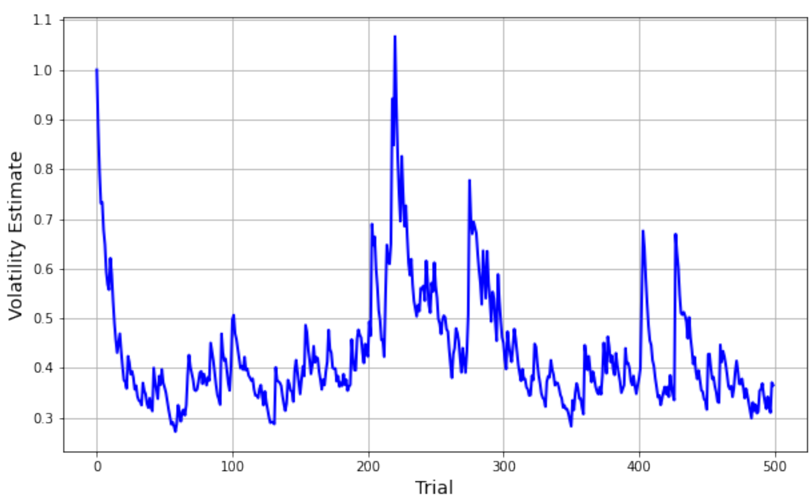


Figure 2: Volatility estimates (v_t) produced by the Volatile Kalman Filter (VKF) during the simulation of $T = 500$ time steps. The volatility estimate reflects the model's inferred uncertainty about the environment's rate of change, with initial parameters set to $\lambda = 0.2$, $v_0 = 1.0$, and $\sigma^2 = 0.3$.

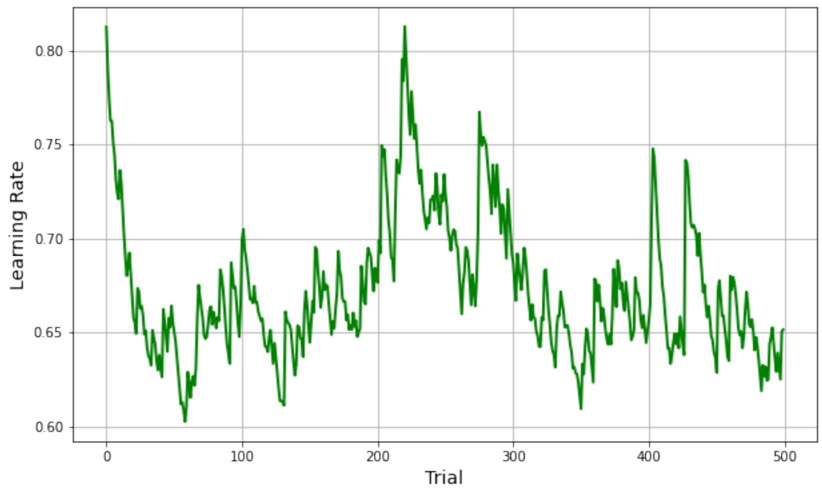


Figure 3: Learning rates (Kalman gain, k_t) computed by the Volatile Kalman Filter (VKF) for the simulation of $T = 500$ time steps. The Kalman gain adapts dynamically to changes in the environment, influenced by the volatility estimates and observation variance ($\sigma^2 = 0.3$).

To further explore the relationship between the magnitude of the prediction error and the volatility estimates, we calculated Pearson and Spearman correlation coefficients between the rolling average of the prediction error $\delta_{\text{avg}}(t)$ and the volatility estimates $\mathbf{v}(t)$ across varying window sizes n_p. The results are shown in Figure 4, which depicts the correlation values as a function of the rolling average window size n_p.

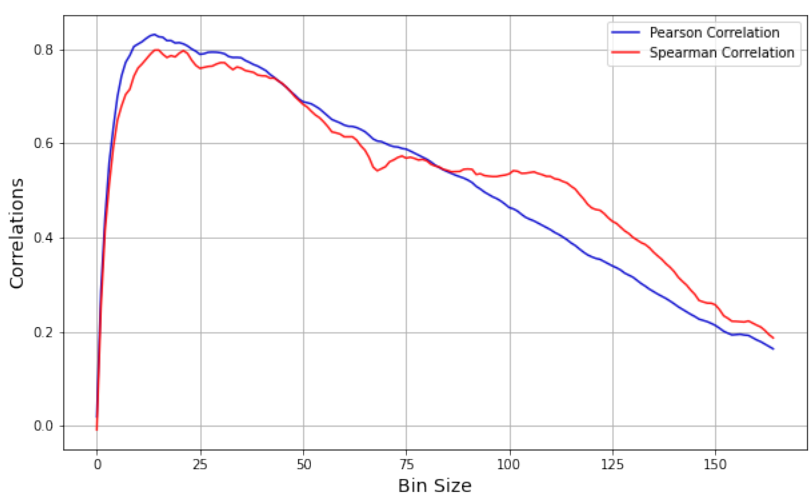


Figure 4

The plots reveal that the Pearson correlation increases initially with larger window sizes, peaking at $n_r = 14$, indicating the strongest linear relationship for this specific scale. Similarly, the Spearman correlation, which captures monotonic relationships, reaches its maximum at $n_\rho = 15$. The trends in both correlation measures decline beyond their respective peaks, indicating that overly large windows dilute the relationship between prediction error and volatility estimates.

To investigate, the relationship between n_r, n_ρ and λ, we repeat the simulation 100 times, varying λ between 10^-2 and 1 on a logarithmic scale. Figure 5 shows how illustrates how the optimal rolling average sizes n_r and n_ρ change as a function of λ

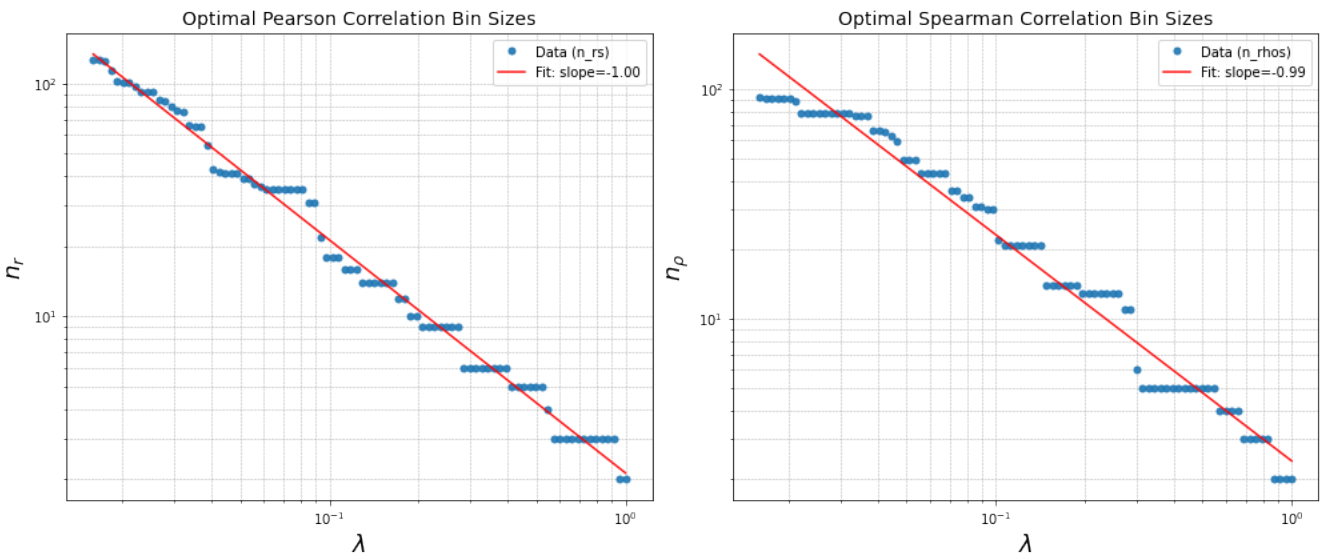


Figure 5: Log-log plots of n_r and n_ρ against λ.

The results show a clear linear relationship on a log-log scale, consistent with the hypothesized inverse proportionality between n_r, n_ρ and λ. Specifically:

- For the Pearson correlation, the slope of the regression line is approximately -1.00, with an intercept of 0.33 and $R^2 = 0.99$. This suggests that the relationship between n_r and λ is nearly perfectly described by the equation:

$$n_r \cdot \lambda \approx 2.13.$$

- For the Spearman correlation, the slope is -0.99, with an intercept of 0.38 and $R^2 = 0.98$. This indicates a similar relationship for n_ρ:

$$n_\rho \cdot \lambda \approx 2.41.$$

[1]

[1] Due to some numerical issues, I calculated these values using the values from only the last 90 simulations. This can be easily fixed to produce the same result, however (which I may try to do when I have more time. . .).

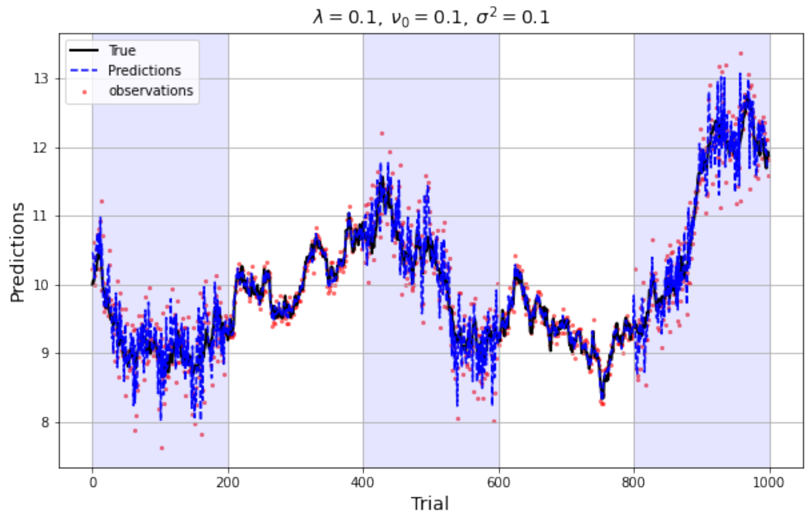
Experiment Two

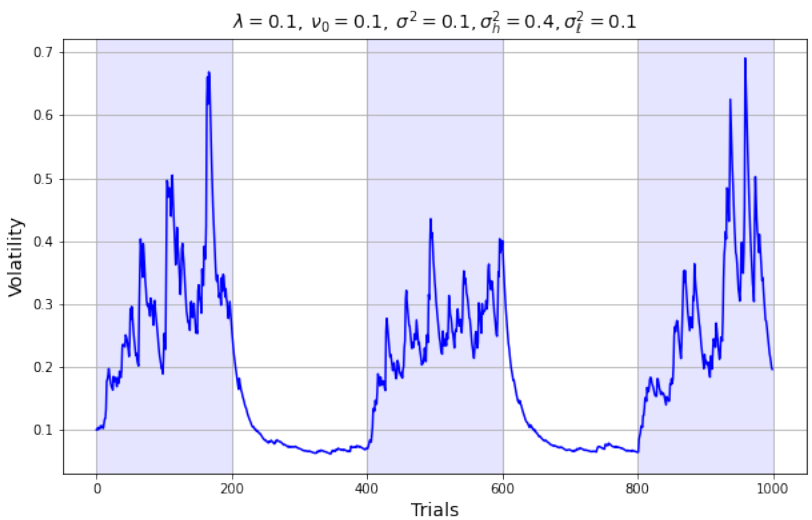
Assume that $s_t \sim \mathcal{N}(s_{t-1}, \nu)$ for constant process variance ν. Further, assume that

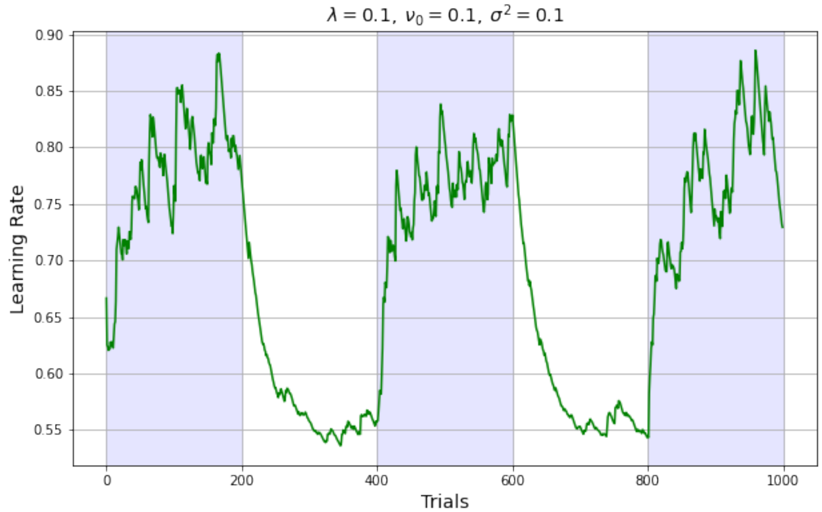
$$o_t \sim \mathcal{N}(s_t, \sigma_t^2).$$

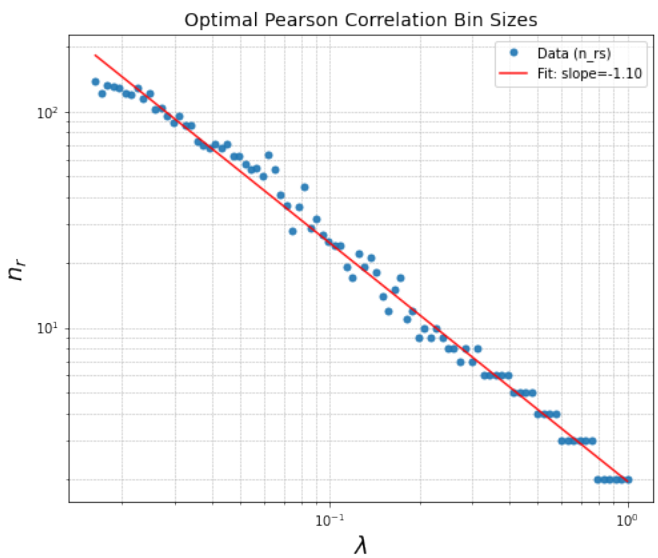
Throught the trial, observation noise σ_t^2 switches between two values $\sigma_\ell^2, \sigma_h^2$, where $\sigma_\ell^2 < \sigma_h^2$.

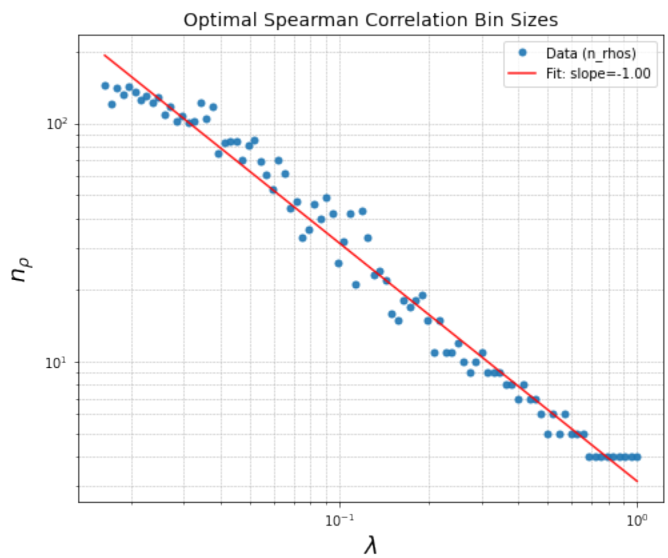
Forthcoming.






$\lambda = 0.1,\ \nu_0 = 0.1,\ \sigma^2 = 0.1$
Learning Rate
Trials
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0
200
400
600
800
1000


Optimal Pearson Correlation Bin Sizes
Data (n_rs)
Fit: slope=-1.10
n_r
λ
10^2
10^1
10^{-1}
10^0


Optimal Spearman Correlation Bin Sizes
Data (n_rhos)
Fit: slope=-1.00
n_ρ
λ
10^2
10^1
10^{-1}
10^0