The goal is to solve the following problem. Let $\Omega \subseteq \mathbb{R}^n$ be a an exterior polyhedral domain of a compact set i.e. $\mathbb{R}^n \setminus \Omega$ is a compact polyhedron. We then have the following boundary value problem: For a fixed $C_0 \in \mathbb{R}$ and closed bounded k-chain γ find $\omega \in H\Lambda^k(\Omega)$ s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0, \\ \mathrm{tr}\ \omega &= 0 \text{ and} \\ \int_\gamma \omega &= C_0 \end{aligned}$$

Because we consider polyhedral domains we assume that γ consists of finitely many k-simplices and that the cohomology class $[\gamma]$ is a generator of the simplicial homology group. Our goal will be to show existence and uniqueness of solutions. In order to show this, we rely on a result about the isomorphism of a simplicial cohomology space $H_p^k(K)$ which will be defined below and the L^p-cohomology space $H_{p,dR}^k(\bar{\Omega})$ (dR short for de Rham). This result was proven in [Gol88]. In the diploma thesis by Nikolai Nowaczyk [Now11], which mostly is based on this paper, many additional details can be found. The first part will be to present this result.

1 Isomorphism of Cohomology

We assume that the $\bar{\Omega}$ admits a smooth triangulation $\tau : |K| \to \bar{\Omega}$ with $|K|$ being the geometric realization of the simplicial complex K. Because $\bar{\Omega}$ is itself a polyhedron we can assume that $\bar{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$. However, we will use different metrics. We use the Euclidian metric on $\bar{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [Gol88, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ Take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as $g_S = f^* g$ where g is the standard metric in ℓ^2. That means for a differential

We have two crucial assumptions.

Assumption 1. *Star-boundedness The star of every vertex in the mesh contains at most N simplices.*

This assumption is for example fulfilled if we consider a shape-regular triangulation of our domain [**???**].

In order to formulate the second assumption we have to note that for our triangulation τ we have that $M = |K|$. Therefore τ is just the identity in our case. However, we use a different metric on M and $|K|$. On our domain M we use the Euclidian metric. On $|K|$ however, we have to use the standard simplicial metric which is defined as follows (cf. [Gol88, p.191]). We enumerate the vertices of the triangulation as x_1, x_2, ... Let $f : |K| \to \ell^2$ be an embedding of the triangulation $|K|$ into $\mathbb{R}^\infty$ s.t. $f(x_i) = e_i$ where e_i are the standard unit vectors of $\mathbb{R}^\infty$ and f is affine on every simplex. Keeping this in mind, we have the following assumption

Assumption 2. *We have constants $C_1, C_2 > 0$ s.t. for every $x \in M$ we have*

$$\|d\tau|_x\| < C_1,\ \|d\tau^{-1}|_\tau(x)\| < C_2$$

where the operator norm is to be understood w.r.t. the metrics on $|K|$ and M.

TO BE DONE

Take $\tau, \sigma \in K$ s.t. $\tau \leq \sigma$. We define an extension operator $j^*_{\sigma,\tau} : W^*_\infty \to W^*_\infty$ which is bounded (cf [Gol88, p.191]).

Definition 1. *Let*

$$\theta = \{\theta(\sigma) \in W^k_\infty(\sigma) | \sigma \in T\}$$

be a collection of differential k-forms. We call θ S-form of degree k if we have for all for simplices $\mu \leq \sigma$

$$j^*_{\sigma,\mu}\theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{d\theta(\sigma)|\sigma \in K\} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

Using integration we can define define the homomorphism (see [Gol88, p.191])

$$I : S^k_p(K) \to C^k_p(K),\ I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K$$

which induces an isomorphism on cohomology (see Thm. 1 in [Gol88] and the proof thereof).

We say that $\omega \in W^k_{\infty,loc}(M)$ if $\omega|_A \in W^k_\infty(A)$for every$A \subseteq M$compact. Then we define

$$\varphi_\tau : W^k_{\infty,loc}(M) \to S^k(K),\ \omega \mapsto \{\tau|^*_\sigma\omega|\sigma \in K\}.$$

This is a well-defined vector space isomorphism ([Gol88, p.191]). This way we can identify $W^k_{\infty,loc}(M)$ with $S^k(K)$. For S-forms of degree k we now define the norm

$$\|\theta\|_{S_p(K)} := \sum_{\sigma\in K}\|\theta(\sigma)\|_{W^k_\infty(\sigma)}.$$

$S^k_p(K)$ are the S-forms of degree k s.t. this norm is finite. Using the isomorphism φ_τ we now define $S^k_p(M) := (\varphi_\tau)^{-1}S^k_p(K)$.

We then have $S^k_p(M) \subseteq W^k_p(M)$ and the inclusion induces an isomorphism on cohomology [Gol88, Lemma 4, Corollary].

Above, we defined the integral operator I for $S^k_p(K)$ which can be therefore be applied on $S^k_p(M)$ as well. If we fix now a closed finite k-chain γ. Then $I(\cdot)(\gamma) = \int_\gamma$ becomes a functional on $S^k_p(M)$, but is is a-priori not clear how to extend this to $W^k_p(M)$. We know that $\int_\gamma d\eta = 0$ for $\eta \in S^k_p(M)$ because otherwise I would not induce a isomorphism on cohomology. We extend this now by setting $\int_\gamma d\nu = 0$ for all $\nu \in W^{k-1}_p(M)$. We have to check whether this is consistent with the definition above. Let $\nu \in W^k_p(M)$ s.t. $d\nu \in S^k_p(M)$. Let $A \subseteq M$ be a bounded neighborhood of γ. We can then find $\tilde\nu$ s.t. $\tilde\nu \in W_q(A)$ for any $q > 1$ and $d\tilde\nu = d\nu$ [Sch06, Thm 3.1.1]. We can then apply Stoke's theorem [GKS82, Thm. 9] to get $\int_\gamma d\nu = 0$. This shows consistency.

In the second part of [Gol88] they construct the operators $\mathscr{R}$ and $\mathscr{A}$. The precise definition and construction of these operators is not relevant for our purposes because we will only use the following properties (cf. [Gol88, Thm.2]).

Theorem 1. *Assume that the triangulation τ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L^k_{1,loc} \to L^k_{1,loc}$, $\mathscr{A} : L^k_{1,loc} \to L^{k-1}_{1,loc}$ such that*

1. $\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ *for* $\omega \in W^k_{1,loc}(M)$
2. *for any* $1 \le p \le \infty$, $\mathscr{R}(W^k_p(M)) \subseteq S^k_p(M)$.

We can now use this operator $\mathscr{R}$ to define $\int_\gamma \omega$ for closed $\omega \in W^k_p(M)$ as

$$\int_\gamma \omega := \int_\gamma \mathscr{R}\omega.$$

This is consistent because if $\omega \in S^k_p(M)$ closed then due to Thm. 1

$$\int_\gamma \mathscr{R}\omega = \int_\gamma \omega + d\mathscr{A}\omega + \mathscr{A}d\omega = \int_\gamma \omega.$$

1.1 Existence of a solution

Returning now back to the problem, we are now able to proof existence of a solution. Take a closed cochain $F \in C_p^k(K)$ s.t. $F(\gamma) = C_0$ and $F(\partial d) = 0$ for (k-1)-chains d. Then we know from ??? that there exists a unique $[\theta] \in \mathscr{H}_p^k(K)$ s.t. $[I]([\theta]) = [F]$. Let us take $\eta := \varphi_\tau^{-1}\theta$. Then $\int_\gamma \eta = C_0$ holds. If we now take the Hodge decomposition $L_2^k(M) = \bar{\mathfrak{B}}^{\mathfrak{k}} \bigoplus \mathcal{H}^k \bigoplus \bar{\mathfrak{B}}_{\mathfrak{k}}^*$ and define ω as the projection of η onto the harmonic forms $\mathcal{H}^k$. Then we know that $d\omega = 0$, $\delta\omega = 0$ and $\operatorname{tr}\omega = 0$. So we only have to show that

$$\int_\gamma \omega = C_0.$$

We know from the Hodge decomposition that there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq L_2^{k-1}(M)$ s.t. $\omega = \eta - \lim_{i\to\infty} d\phi_i$. Let now be $R > 0$ large enough s.t. $\gamma \subseteq B_R$. Then we know that $dW_2^{k-1}(B_R)$ is closed in $L_2^k(B_R)$. Therefore there exists $\phi_R \in W_2^{k-1}(B_R)$ s.t. $\lim_{i\to\infty} d\phi_i|_{B_R} = d\phi_R$. So we have $\omega|_{B_R} = \eta|_{B_R} - d\phi_R$ and

$$\int_\gamma \omega = \int_\gamma \omega|_{B_R} = \int_\gamma \eta|_{B_R} = C_0.$$

This proves existence.

1.2 Existence and uniqueness of solutions

The first step is to show that the cochain chosen in the proof of existence is in fact unique if restricted to closed chains.

Proposition 1. *Let γ be a k-chain s.t. the homology class $[\gamma]$ is a generator of the homology group. Assume for some $C_0 \in \mathbb{R}$ there exist cochains $F, G \in C_p^k(K)$ s.t.*

$$F(\gamma) = C_0 \text{ and } F(\partial d) = 0 \text{ for all } (k-1)\text{-chains } d$$

and the same for G. Then the restriction of F and G to closed chains is the same.

Proof. Take any closed k-chain c. Because γ is the generator of the homology group we have $n \in \mathbb{Z}$ s.t. $[c] = [n\,\gamma]$ where $[\cdot]$ is the corresponding homology class. That means that we have some $(k-1)$-chain d s.t. $c = n\,\gamma + \partial d$. Using the properties of F and G,

$$F(c) = F(n\,\gamma + \partial d) = nF(\gamma) = n\,C_0.$$

Because the same computation is valid for G $F(c) = G(c)$ follows. □

Theorem 2. *Assume that a co-chain as in Prop. 1.2 exists. Then the solution of the problem is unique.*

Proof. Let $\omega, \tilde{\omega}$ both be solutions. Because $\int_\gamma \omega = \int_\gamma \tilde{\omega}$ and ω and $\tilde{\omega}$ are closed we have due to Prop. 1.2 that $\int_c \omega = \int_c \tilde{\omega}$ for any closed k-chain c. So we have for the induced homomorphism $[I]([\mathscr{R}\omega]) = [I]([\mathscr{R}\tilde{\omega}])$ and therefore due to the isomorphism of cohomology $[\mathscr{R}\omega] = [\mathscr{R}\tilde{\omega}]$. Hence,

$$[\tilde{\omega}] = [\mathscr{R}\tilde{\omega}] = [\mathscr{R}\omega] = [\omega]$$

we get the equality of the cohomology classes.

That is equivalent to the existence of some $(k-1)$-form $\phi \in W_2^{k-1}(\bar{\Omega})$ s.t. $\omega = \tilde{\omega} + d\phi$. Then because ω and $\tilde{\omega}$ are harmonic we have $\omega, \tilde{\omega} \perp dW_2^{k-1}(\bar{\Omega})$ and therefore

$$\omega = \tilde{\omega}.$$

□