Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let B be a magnetic field on the domain Ω. We the have the following boundary value problem:

$$\text{curl } B = 0, \tag{1}$$
$$\text{div } B = 0 \text{ in } \Omega \tag{2}$$
$$B \cdot n = 0 \text{ on } \partial\Omega \text{ and} \tag{3}$$
$$\int_\gamma B \cdot dl = C_0 \tag{4}$$

where n is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology.

At first, let us introduce some basic notions about differential forms. We follow the brief introduction given by Arnold (cf. [1, Sec. 6.1]), but less details will be given. Let V be a real vector space with $\dim V = n$ and $\operatorname{Alt}^k V$ be the space of k-alternating maps from V^k to $\mathbb{R}$. For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\substack{i_1 < ... < i_k \\ i_{k+1} < ... < i_{k+l}}} \operatorname{sgn}(i_1, ..., i_{k+l}) \omega(v_{i_1}, ..., v_{i_k}) \nu(v_{i_{k+1}}, ..., v_{i_{k+l}})$$

where $\operatorname{sgn}(i_1, ..., i_{k+l})$ is the sign of the permutation $(1, ..., k+l) \mapsto (i_1, ..., i_{k+l})$. This definition is not very intuitive. TBD: Examples in 3D.

Let $\{u_i\}_{i=1}^n$ be any basis of V and $\{u^i\}_{i=1}^n$ the correspoding dual basis. Then

$$\{u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k} \,|\, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\operatorname{Alt}^k V$. In particular, $\dim \operatorname{Alt}^k V = \binom{n}{k}$.

Given a inner product $\langle \cdot, \cdot \rangle_V$ on V we obtain an inner product on $\operatorname{Alt}^k V$ by defining

$$\langle u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}, u^{j_1} \wedge ... \wedge u^{j_k} \rangle_{\operatorname{Alt}^k V} := \det [(\langle u_{i_k}, u_{i_l} \rangle_V)_{1 \leq k,l \leq n}]$$

which is then extended to all of $\mathrm{Alt}^k V$ by linearity. We denote with $\|\cdot\|_{\mathrm{Alt}^k V}$ the induced norm. From this definition it follows directly that for a orthonormal basis b_1,..., b_n the corresponding basis $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\mathrm{Alt}^k V$.

$\mathrm{Alt}^n V$ is one-dimensional and so we have to choose a basis. We say that two orthonormal bases of V have the same orientation if the change of basis has positive determinant. That divides the orthonormal bases into two classes with different orientation. We choose one of these classes and call these orthonormal bases positively oriented. Take $\omega \in \mathrm{Alt}^n V$. Then $\omega(b_1, ..., b_n)$ is the same for any positively oriented orthornormal basis. We now define the *volume form* $\mathrm{vol} \in \mathrm{Alt}^n v$ by requiring it to be 1 on all positively oriented orthonormal bases. Using this volume form we can now define the *Hodge star operator* $* : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$ via the property

$$\omega \wedge \mu = \langle *\omega, \mu \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} \quad \forall \omega \in \mathrm{Alt}^k V,\, \mu \in \mathrm{Alt}^{n-k} V.$$

The Hodge star is an isometry, we have $** = (-1)^{k(n-k)}\mathrm{Id}$ and

$$\omega \wedge *\mu = \langle \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol} \quad \forall \omega, \mu \in \mathrm{Alt}^k V.$$

In particular in $\mathbb{R}^3$, we have $** = \mathrm{Id}$ i.e. $*$ is self-inverse.

Now we will move on to differential forms. We will mostly deal with the case $V = \mathbb{R}^n$ and denote $\mathrm{Alt}^k \mathbb{R}^n$ just as Alt^k. Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and denote the space of differential forms of degree k on Ω as $\Lambda^k(\Omega)$. We extend the Hodge star operator to differential forms $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ simply by applying it pointwise.

Then we define the L_p-norm of a k-form ω for $1 \leq p < \infty$ as (cf. [3])

$$\|\omega\|_{L_p^k(\Omega)} := \left(\int_\Omega \|\omega\|_{\mathrm{Alt}^k}^p \right)^{1/p}$$

and for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x \in \Omega} \|\omega(x)\|_{\mathrm{Alt}^k}.$$

$L_p^k(\Omega)$ are the spaces of k-forms s.t. the corresponding L_p-norm is finite. For $p = 2$ we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the L_2 inner product

$$\langle \omega, \nu \rangle := \int_\Omega \langle \omega, \nu \rangle_{\mathrm{Alt}^k}. \tag{5}$$

Proposition 1. *The Hodge star operator* $* : L_2^k(\Omega) \to L_2^{n-k}(\Omega)$ *is a Hilbert space isometry.*

Proof. This follows directly from the definition of the inner product (5) and the fact that $*$ is an isometry when applied to alternating forms Alt^k. $\square$

Our next goal is to extend the exterior derivative d of smooth differential forms in the weak sense (cf. [3]). Let $\mathring{d} : L_2^k(\Omega) \to L_2^{k+1}(\Omega)$ be the exterior derivative as an unbounded operator with domain $D(\mathring{d}) = C_0^\infty\Lambda^k(\Omega)$ which are the smooth compactly supported differential forms of degree k. Analogous, let $\mathring{\delta} : L_2^k(\Omega) \to L_2^{k-1}(\Omega)$ be the codifferential operator $\mathring{\delta} := (-1)^{n(k-1)+1} * \mathring{d} *$ also with domain $C_0^\infty\Lambda^k(\Omega)$.

Then the exterior derivative $d\omega \in L_{k+1}^p(\Omega)$ is defined as the unique $(k+1)$-form in $L_{k+1}^p(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge *\phi = \int_\Omega \omega \wedge *\mathring{\delta}\phi \quad \forall \phi \in C_0^\infty\Lambda^k(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$W_p^k(\Omega) = \left\{\omega \in L_p^k(\Omega) \,|\, d\omega \in L_p^{k+1}(\Omega)\right\},$$
$$W_{p,loc}^k(\Omega) = \left\{\omega \; k\text{-form} \,|\, \omega|_A \in W_p^k(A) \text{ for every open } A \subseteq \Omega \text{ s.t. } \overline{A} \subseteq \Omega \text{ is compact}\right\}.$$

For $\omega \in W_p^k(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W_p^k} := \left(\|\omega\|_{L_p^k}^p + \|d\omega\|_{L_p^k}^p\right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max\left\{\|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k}\right\}.$$

Definition 1 (L^p-cohomology)**.** We define the following subspaces of $W_p^k(\Omega)$, $1 \le p \le \infty$:

$$\mathfrak{B}_k := dW_p^{k-1}(\Omega) \text{ and}$$
$$\mathfrak{Z}_k := \{\omega \in W_p^k(\Omega) |\, d\omega = 0\}.$$

We call the k-forms in $\mathfrak{B}_k$ exact and the forms in $\mathfrak{Z}_k$ closed. Because $d \circ d = 0$ we always have $\mathfrak{B}_k \subseteq \mathfrak{Z}_k$. Then we define the de Rham- or L^p-cohomology space $H_{p,dR}^k(\Omega)$ as the quotient space

$$H_{p,dR}^k(\Omega) := \mathfrak{Z}_k \big/ \mathfrak{B}_k.$$

We want to examine the Hilbert space $L_2^k(\Omega)$ more closely (see [1, Sec. 6.2.6] for more details). We denote $H^k(d;\Omega) := W_2^k(\Omega)$. If the domain is clear we will leave it out. Note that the above definition of the exterior derivative is in the Hilbert space setting equivalent to defining d as the adjoint of $\mathring{\delta}$.

Now we just define $\delta := (-1)^{n(k-1)+1} * d*$ as in the smooth setting. We will show that this is the adjoint of $\mathring{d}$. Denote with $D(\mathring{d}^*) \subseteq L_2^{k+1}(\Omega)$ the domain of the adjoint. Define

$$H^k(\delta;\Omega) := \{\omega \in L_2^k(\Omega) | *\omega \in H^{n-k}(d)\}.$$

Now take $\omega \in H^{k+1}(\delta)$ and $\phi \in C_0^\infty \Lambda^k(\Omega)$. Then

$$\begin{aligned}
&\langle \delta\omega, \phi\rangle = (-1)^{nk+1}\langle *d*\omega, \phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}\langle d*\omega, *\phi\rangle = (-1)^{nk+1}(-1)^{k(n-k)}\langle *\omega, \mathring{\delta}*\phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}(-1)^{n(n-k-1)+1}\langle *\omega, *\mathring{d}**\phi\rangle \\
&= (-1)^{n(n-1)+2}(-1)^{k(n-k)}\langle \omega, \mathring{d}**\phi\rangle \\
&= \langle \omega, \mathring{d}\phi\rangle
\end{aligned}$$

where we used repeatedly that $*$ is an isometry and $** = (-1)^{k(n-k)}\mathrm{Id}$. This shows that $H^{k+1}(\delta) \subseteq D(\mathring{d}^*)$ and that $\mathring{d}^*\omega = \delta\omega$. Now for the other inclusion assume that $\omega \in D(\mathring{d}^*)$ and take $\phi \in C_0^\infty\Lambda^{n-k}(\Omega)$ arbitrary.

$$\langle *\omega, \mathring{\delta}\phi\rangle = \pm\langle \omega, \mathring{d}*\phi\rangle = \pm\langle \mathring{d}^*\omega, *\phi\rangle = \pm\langle *\mathring{d}^*\omega, \phi\rangle.$$

Here we use $\pm$ to mean that we choose the sign correctly, s.t. all the operations are correct. Then by choosing the sign appropriately we find that $\pm*\mathring{d}^*\omega = d*\omega$ and therefore $*\omega \in H^{n-k-1}(d)$ so we proved $D(\mathring{d}^*) \subseteq H^{k+1}(\delta)$ and we are done.

We then define additionally the space $\mathring{H}^k(d;\Omega)$ as the closure of $C_0^\infty\Lambda^k(\Omega) \subseteq H^k(d;\Omega)$ w.r.t. the $H^k(d)$-norm i.e. $\mathring{H}^k(d;\Omega)$ corresponds to k-forms in $H^k(d;\Omega)$ being zero on the boundary. TBD: There are several different ways to characterize zero boundary conditions in the L^2 setting. We have to choose the one that works best. Then we define the spaces

$$\begin{aligned}
H_0^k(d;\Omega) &:= \{\omega \in H^k(d;\Omega) | d\omega = 0\} \\
\mathring{H}_0^k(d;\Omega) &:= \{\omega \in \mathring{H}^k(d;\Omega) | d\omega = 0\}
\end{aligned}$$

i.e. the spaces of closed forms. We will use the analogous definition for $H_0^k(\delta;\Omega)$ and $\mathring{H}_0^k(\delta;\Omega)$ which we call coclosed forms. We then define the spaces of harmonic forms

$$\mathring{H}_0^k(d,\delta;\Omega) := \{\omega \in \mathring{H}^k(d;\Omega) | \, d\omega = 0, \delta\omega = 0\}.$$

With this one can prove the Hodge decomposition ([1, Lemma 1])

$$L_2^k(\Omega) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \overset{\perp}{\oplus} \overline{\delta H^{k+1}(\delta)} \tag{6}$$

and furthermore for the closed and coclosed forms respectively,

$$\mathring{H}_0^k(d) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \tag{7}$$

$$H_0^k(\delta) = \overline{\delta H^{k+1}(\delta)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta). \tag{8}$$

Before we can reformulate the boundary value problem in the language of differential forms we have to introduce some things from simplicial topology. This material is taken from [2] where a lot more details and results can be found.

Definition 2 (Affine simplex)**.** Let x_0, x_1, ..., $x_k \in \mathbb{R}^n$ be affine independent. Then

$$[x_0, x_1, ..., x_k] := \mathrm{conv}\{x_0, ..., x_k\}$$

is called an affine k-simplex.

We will assume all simplices to be affine.

Definition 3 (Simplicial complex)**.** A *simplicial complex* K is a collection of affine simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of σ is in K,
2. σ, $\tau \in K \Rightarrow \sigma \cap \tau$ is either empty or a face of both σ and τ.

We call $|K| := \bigcup\{\sigma | \sigma \in K\}$ the polyhedron of K.

For any topological space X a homeomorphism $\tau : |K| \to X$ is called *triangulation* of X. Let $\{x_1, x_2, ...\}$ be the vertices in the simplicial complex K. We fix an ordering of the vertices for every simplex. That means for any k every k simplex σ has a designated representation in the form of

$$\sigma = [x_{i_0}, x_{i_1}, ..., x_{i_k}].$$

Definition 4 (k-chain)**.** Let K be a simplicial complex. By $C_k(K)$ we will denote the free abelian group on the k-simplices i.e. the abelian group of all formal finite sums

$$\sum_\sigma n_\sigma \sigma$$

with σ being k-simplices. The elements of $C_k(K)$ (i.e. sums of the above form) are called *k-chains*.

These groups of k-chains become now a chain complex by introducing the boundary operator ∂.

Definition 5 (Boundary)**.** For any simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$ we define the boundary

$$\partial[x_{i_0}, x_{i_1}, ..., x_{i_k}] := \sum_{j=0}^{k} (-1)^j [x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$$

where $[x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$ is the simplex without vertex x_{i_j}. We then extend the definition of the boundary operator linearly to k-chains $c = \sum_\sigma c_\sigma \sigma$ by

$$\partial c := \sum_\sigma c_\sigma \partial \sigma.$$

.

A crucial property of the chain operator is the following.

Proposition 2. $\partial \circ \partial = 0$.

Proof. This can be proven by direct computation, analogous to [2, Chap.4, Lemma 1.6] □

We call a k-chain c a *k-cycle* if $\partial c = 0$ and we call c a *k-boundary* if there exists a $(k+1)$-chain d s.t. $c = \partial d$. Let $Z_k(K) \subseteq C_k(K)$ be the subgroup of k-cycles and $B_k(K) \subseteq C_k(K)$ the subgroup of k-boundaries. We can now define the simplicial homology groups of our simplicial complex.

Definition 6 (Simplicial homology)**.** The homology groups $H_k(K)$ are the quotient groups of cycles and boundaries i.e.

$$H_k(K) := {}^{Z_k(K)}\!/_{B_k(K)}.$$

The homology groups are independent of the chosen simplicial complex. [**empty citation**].

Let G be any group. Then we define the group of *k-cochains* $C^k(K; G)$ by

$$C^k(K; G) := \mathrm{Hom}(C_k(K), G)$$

i.e. the group of all homomorphisms from k-chains to G. We generally use the superindex if something is related to cochains and the subindex if it is related to chains. We now introduce an operator between these spaces of cochains.

Definition 7. We define the operator $\delta : C^k(K;G) \to C^{k+1}(K;G)$ via

$$(\delta f)(c) := f(\partial c).$$

for a $(k+1)$-chain c. We call a cochain $f \in C^k(K;G)$ *closed* if $\delta f = 0$ and we call f *exact* if there is a $g \in C^{k+1}(K;G)$ s.t. $f = \delta g$.

We define the cohomology spaces analogous to homology spaces above.

Definition 8 (Simplicial cohomology)**.** Denote the closed k-cochains as $Z^k(K;G)$ and the exact ones with $B^k(K;G)$. We then define the *simplicial cohomology groups* $H^k(K)$ as

$$H^k(K;G) := {Z^k(K;G)}\big/{B^k(K;G)}.$$

Note that in the case of $G = \mathbb{R}$ this becomes a vector space. We will later show that if we consider certain subspaces of cochains so called *p-summable* cochains that the L^p-cohomology defined above and the cohomology spaces of these p-summable cochains are isomorphic.

Now of course there is the question how the homology and cohomology groups are related to each other. This question is answered by the *universal coefficent theorem.* But before we can formulate it we have to introduce exact sequences.

Definition 9 (Exact sequence)**.** Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. Then this sequence of homomorphisms is called *exact* if $\operatorname{im} f_{i-1} = \ker f_i$.

The universal coefficent theorem in the case of simplicial homology states that the sequence

$$0 \to \operatorname{Ext}(H_{k-1}(K), G) \to H^k(K;G) \xrightarrow{\beta} \operatorname{Hom}(H_k(K), G) \to 0 \qquad (9)$$

is exact. β is defined via $\beta([F])([c]) := F(c)$. The definition of Ext can be found in [2], but it does not matter for our purpose because from now on we will assume $G = \mathbb{R}$ and $\operatorname{Ext}(H_{k-1}(K), \mathbb{R}) = 0$. This follows from the fact that $\mathbb{R}$ is a divisible and hence injective abelian group. The definition these terms and the connections used can also be found in [2, Ch. V.6]. However, we will not dwelve into the algebraic background further. We can conclude from the exactness of the above short sequence that $\ker \beta = 0$ and $\operatorname{im} \beta = \operatorname{Hom}(H_k(K), \mathbb{R})$. So β is a isomorphism.

As an application, we will show the following proposition which will be used later to show uniqueness of a solution of the magnetostatic problem.

Proposition 3. *Assume that* $H_1(K) = \mathbb{Z}[\gamma]$ *i.e. the homology class of the closed* 1*-chain* γ *is a generator of the first homology group. Then we have the following:*

(i) For any $C_0 \in \mathbb{R}$ *there exists a closed* 1*-chain* $F \in Z^1(K)$ *with* $F(\gamma) = C_0$*,*

(ii) any other $G \in Z^1(K)$ *with* $G(\gamma) = C_0$ *is in the same cohomology class i.e.* $[F] = [G]$.

Proof. **Proof of (i)** Because $[\gamma]$ is a generator of the homology group we obtain a homomorphism $\hat{F} \in \mathrm{Hom}(H_1(K), \mathbb{R})$ by fixing $\hat{F}([\gamma]) = C_0$. This determines the other values. Then we know from (9) that there exists a $[F] \in H^1(K)$ with $\beta([F]) = \hat{F}$ because β is a isomorphism. So we obtain

$$F(\gamma) = \beta([F])([\gamma]) = \hat{F}([\gamma]) = C_0.$$

Proof of (ii) Take $[c] \in H_1(K)$ arbitrary. Then there exists $n \in \mathbb{Z}$ s.t. $[c] = n[\gamma]$. Using β from 9 We have

$$\beta([F])([c]) = \beta([F])(n[\gamma]) = n\beta([F])([\gamma]) = n\,F(\gamma) = n\,F(\gamma) = \beta([G])([c])$$

and thus $\beta([F]) = \beta([G])$. Because β is an isomorphism we arrive at $[F] = [G]$. $\square$

In order to show existence and uniqueness of solutions of the magnetostatic problem we rely on a result about the isomorphism of a simplicial cohomology space $H_p^k(K)$ and the L^p-cohomology space $H_{p,dR}^k(\overline{\Omega})$. This result was proven in [3]. In the diploma thesis of Nikolai Nowaczyk [4], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [3] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [3]). Therefore, we can apply the result to our case.

1 Isomorphism of Cohomology

Before we state the theorem about the isomorphism of we will first formulate a crucial assumption for this result to hold.

Because $\overline{\Omega}$ from our problem is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [3, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ and take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^* g$ where g is the standard metric in ℓ^2. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on ℓ^2. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, $\sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned}
g_S|_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_k}{\partial x_j}(x) \\
&= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left(Df(x)^T Df(x) \right)_{ij} \\
&= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle,
\end{aligned}$$

where D denotes the Jacobian. (TBD: This Jacobian as written here would technically be in $\mathbb{R}^{\infty \times n}$. Only finitely many lines are non-zero though, but this is not quite rigorous yet.)

We have two crucial assumptions on the triangulation for the result to hold (cf. [3, p.194]). We summarize them under *GKS-condition* named after the three authors of [3].

Assumption 1 (GKS-condition)**.** We will assume the following on the simplicial complex K and the triangulation τ:

1. The star of every vertex in K contains at most N simplices.
2. For the differential of τ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1, \ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where d denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most N simplices, which is fulfilled if we have a shape regular mesh.

Because τ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

1.1 S-forms

From now on we will assume that the GKS condition is fulfilled.

Definition 10 (Induced map)**.** Let V and W be real vector spaces, $X \subseteq V$, $Y \subseteq W$ be subspaces. For a linear map $L : V \to W$ with $L(X) \subseteq Y$ we define the induced map

$$[L] : {}^{V}\!\diagup_{X} \to {}^{W}\!\diagup_{Y},\ [v] \mapsto [Lv].$$

It is easy to check that the induced map is well-defined using the definition of quotient space.

The first isomorphism is induced from a linear mapping from the so called *S-forms* $S^k_p(K)$ to *p-summable k-cochains* $C^k_p(K)$ which will both be defined next.

Definition 11. We define the following norm of a k-cochain f

$$\|f\|_{C^k_p(K)} := \left(\sum_{c\, k\text{-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C^k_p(K) := \{f\ k\text{-cochain} |\ \|f\|_{C^k_p(K)} < \infty\}.$$

Take $\tau, \sigma \in K$ s.t. τ is a face of σ which we write as $\tau < \sigma$. We need a restriction operator $j^*_{\tau,\sigma} : W^k_{\infty,loc}(\sigma) \to W^k_{\infty,loc}(\tau)$. This is done by extending a $\omega \in W^k_{\infty,loc}(\sigma)$ first to some $\tilde{\omega} \in W^k_{\infty,loc}(U)$ with U an open neighborhood of σ in the affine hull of σ. Then $\tau \subseteq U$ and we apply a restriction operator $j^*_{\tau,U}$ and define $j^*_{\tau,\sigma}\omega := j^*_{\tau,U}\tilde{\omega}$. This restriction is then well defined, bounded and independent of the chosen extension $\tilde{\omega}$ (cf. [3, p.191]). It should be emphasized that this restriction only works for W^k_∞ and fails for W^k_p, $p < \infty$.

Definition 12 (S-forms)**.** Let

$$\theta = \{\theta(\sigma) \in W^k_\infty(\sigma) | \sigma \in K\}$$

be a collection of differential k-forms. We call θ S-form of degree k if we have for all simplices $\mu < \sigma$

$$j^*_{\sigma,\mu}\theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{d\theta(\sigma) | \sigma \in K\} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

For $\theta \in S^k(K)$ we now define the norm

$$\|\theta\|_{S_p(K)} := \left(\sum_{\sigma \in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \right)^{1/p}.$$

$S^k_p(K)$ are the S-forms of degree k s.t. this norm is finite.

The integration of an S-form $\theta \in S^k(K)$ over a chain $c = \sum n_\sigma \sigma \in C_k(K)$ is then simply defined as

$$\int_c \theta := \sum_\sigma n_\sigma \int_\sigma \theta(\sigma).$$

We also write $I(\theta)(c) := \int_c \theta$. Then the integration mapping $I : S^k(K) \to C^k(K)$ is a homomorphism (see [3, p.191]). The reason why S-forms are a useful concept is because they provide an intermediate step between differential forms and cochains.

With the exterior derivative d on S-forms as defined above we define

$$\mathscr{Z}^k := \{\theta \in S^k(K) |\, d\theta = 0\}$$
$$\mathscr{B}^k := dS^{k-1}(K)$$

and then the resulting cohomology space

$$\mathscr{H}^k(K) := {\mathscr{Z}^k}/{\mathscr{B}^k}.$$

We also define the in the case of p-summable S-forms

$$\mathscr{Z}^k_p := \{\theta \in S^k_p(K) |\, d\theta = 0\}$$

as well $\mathscr{B}^k_p$ and $\mathscr{H}^k_p(K)$ analogously.

Then we have that the integration mapping $I : S^k(K) \to C^k(K)$ induces an isomorphism on the cohomologies i.e. $[I] : \mathscr{H}^k(K) \to H^k(K)$ is an isomorphism of vector spaces (see [3, p.191]). Also $I : S^k_p(K) \to C^k_p(K)$ is well defined and induces an isomorphism on the respective cohomology spaces as well (see [3, Thm. 1] and the proof thereof).

1.2 Isomorphism between cohomologies of S-forms and L^p-cohomology

The next step is to obtain an isomorphism between the cohomology of S-forms $\mathscr{H}_p^k(K)$ and the L_p cohomology $H^k_{p,dR}(\overline{\Omega})$. In order to achieve that we need a some connection between differential forms and S-forms. Take $\omega \in W^k_{\infty,loc}(\overline{\Omega})$. Using the analogous reasoning as above, we use the restriction operators $j^*_{\sigma,\tau}$ to restrict ω to the simplices. Denote the resulting forms as $\omega(\sigma) \in W^k_{\infty,loc}(\sigma)$. So we obtain an S-form $\{\omega(\sigma)|\, \sigma \in K\}$. This way we constructed an operator $\varphi : W^k_{\infty,loc}(\overline{\Omega}) \to S^k(K)$. This operator φ is an isomorphism [3, Lemma 1]. To also obtain corresponding forms to the p-summable S-forms we simply define $S^k_p(\overline{\Omega}) := \phi^{-1}S^k_p(K)$. It can be shown that $S^k_p(\overline{\Omega}) \subseteq W^k_p(\overline{\Omega})$ and that the inclusion operator is bounded ([3, Lemma 4]).

Now we also want to go in the other direction i.e. we want to obtain a $\theta \in S^k_p(\overline{\Omega}) \subseteq W^k_p(\overline{\Omega})$. This is done using two operators $\mathscr{R} : L^k_{1,loc}(\overline{\Omega}) \to L^k_{1,loc}(\overline{\Omega})$ and $\mathscr{A} : L^k_{1,loc}(\overline{\Omega}) \to L^{k-1}_{1,loc}(\overline{\Omega})$. The precise definition and details of their construction are not relevant for our purposes because we will only use the some properties that we collect in the following theorem (cf. [3, Thm.2]).

Theorem 1. *Assume that the triangulation τ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L^k_{1,loc}(\overline{\Omega}) \to L^k_{1,loc}(\overline{\Omega})$, $\mathscr{A} : L^k_{1,loc}(\overline{\Omega}) \to L^{k-1}_{1,loc}(\overline{\Omega})$ such that*

(i) $\mathscr{R}(W^k_{1,loc}(\overline{\Omega})) \subseteq W^k_{1,loc}(\overline{\Omega})$, $\mathscr{A}(W^k_{1,loc}(\overline{\Omega})) \subseteq W^{k-1}_{1,loc}(\overline{\Omega})$ and $\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ and $d\mathscr{R}\omega = \mathscr{R}d\omega$ for $\omega \in W^k_{1,loc}(\overline{\Omega})$

(ii) for any $1 \le p \le \infty$, $\mathscr{R}(W^k_p(\overline{\Omega})) \subseteq S^k_p(\overline{\Omega})$ and $\mathscr{A}(S^k_p(\overline{\Omega})) \subseteq S^{k-1}_p(\overline{\Omega})$

(iii) $\mathscr{R} : W^k_p(\overline{\Omega}) \to (S^k_p(\overline{\Omega}), \|\cdot\|_{S^k_p(\overline{\Omega})})$ is bounded.

Let $\iota : S^k_p(M) \hookrightarrow W^k_p(M)$ be the inclusion operator. The inclusion induces an isomorphism on cohomology [3, Lemma 4, Corollary] i.e. $[\iota] : \mathscr{H}^k_p(K) \to H^k_{p,dR}(\overline{\Omega})$ is an isomorphism.

2 Existence and uniqueness of solutions

2.1 Reformulation of the problem

We will return now to the magnetostatic problem. In order to use the results above we will reformulate the problem in the notation of differential forms. There are two ways to identify a vector field with a differential form (cf. [1,

Table 6.1 and p.70]) either as a 1-form or a 2-form. For a vector field B we define

$$\begin{aligned} F^1 B &:= B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \text{ and} \\ F^2 B &:= B_2\, dx_2 \wedge dx_3 - B_2\, dx_1 \wedge dx_3 + B_3\, dx_1 \wedge dx_2 \end{aligned}$$

as the corresponding 1-form and 2-form. Then the exterior derivative is $dF^2\,\omega$ corresponds to the divergence, the codifferential $\delta F^2\,\omega$ corresponds to the curl and the normal component being zero on the boundary corresponds to $\omega \in \mathring{H}^2(d)$.[**empty citation**].

If we then use the association of 3-forms with scalars we have the corresponding boundary value problem without the integral condition for 2-forms: Find $\omega \in \mathring{H}^2(d)$ s.t.

$$\delta\omega = 0, \tag{10}$$

$$d\omega = 0 \text{ in } \Omega. \tag{11}$$

Next, we have to add the integral condition. We use that we are in three dimensions so $** = (-1)^{k(n-k)}\tilde{\nu} = \tilde{\nu}$ [1, p.66] for any k-form $\tilde{\nu}$. and observe

$$\begin{aligned} *F^2\, B = B_1\, **dx_1 + B_2\, **dx_2 + B_3\, **dx_3 = B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \\ = F^1\, B. \end{aligned}$$

Then we have

$$\int_\gamma *F^2\, B = \int_\gamma F^1\, B = \int_\gamma B \cdot \mathrm{d}l.$$

In the last step we used the fact that the integration of a 1-form over a curve is equivalent to the curve integral of the associated vector field (cf. [1, Sec. 6.2.3]). Hence, we can add the integral condition

$$\int_\gamma *\omega = C_0. \tag{12}$$

However, we have only $\omega \in \mathring{H}^2_0(d, \delta)$ so $*\omega \in H^1(\delta)$ so this integral might not be well defined. In order to deal with this, we will again use the operator $\mathscr{R}$ and $\bar{I}$ from Sec. **??**. Instead of using (12) directly we use the condition

$$\int_\gamma \mathscr{R} * \omega = C_0.$$

This is equivalent to $(\bar{I} * \omega)(\gamma) = C_0$. We know that this is well-defined.

We want to give some justification about why this is a reasonable extension. For any closed $\eta \in W_p^1(\overline{\Omega})$ we have $\mathscr{R}\eta = \eta - d\mathscr{A}\eta$ from Thm. 1. For smooth $\phi \in \Lambda^0(\overline{\Omega})$ we immediately get from the standard Stoke's theorem that $\int_c d\phi = 0$ for all closed 1-chains c. If we assume sufficient regularity on $\mathscr{R}\eta$, $d\mathscr{A}\eta$ and η then we would have indeed

$$\int_\gamma \mathscr{R}\eta = \int_\gamma \eta - d\mathscr{A}\eta = \int_\gamma \eta.$$

Remark 1. This justification can be done more rigorously with the help of S-forms (cf. [3]) which correspond to $W_{\infty,loc}(\overline{\Omega})$ with an additional decaying condition. Then we obtain that the integral is consistent with the integral on these S-forms.

To summarize we obtain the following problem.

Problem 1. Find $\omega \in \mathring{H}^2(d;\Omega)$ s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0 \text{ in } \Omega, \\ \bar{I}(*\omega)(\gamma) &= C_0. \end{aligned}$$

We will examine existence and uniqueness of this problem in the next section.

2.2 Existence and uniqueness

From now on we assume $n = 3$ i.e. we are in three dimensional space. We start with the following

Proposition 4. *Let $(\phi_i)_{i\in\mathbb{N}} \subseteq H^0(d)$ s.t. $(d\phi_i)_{i\in\mathbb{N}}$ is convergent and let $\gamma \in C_1(K)$ be a bounded closed 1-chain. Then*

$$\bar{I}(\lim_{i\to\infty} d\phi_i)(\gamma) = 0.$$

Proof. Because $[\bar{I}]$ is an isomorphism of cohomology $\bar{I}$ has to send exact forms to exact cochains. Because γ is a closed 1-chain we obtain $\bar{I}(d\psi)(\gamma) = 0$ for every $\psi \in H^0(d)$. $\bar{I}$ is a continuous operator **??** and so we have $\bar{I}(\lim_{i\to\infty} d\phi_i) = \lim_{i\to\infty} \bar{I}(d\phi_i)$ with the limit of cochains taken w.r.t. the norm in $C_2^1(K)$. Now let J_γ be the set of indices of simplices contained in γ i.e.

$$\gamma = \sum_{i\in J_\gamma} \sigma_i.$$

Here we remind that we fixed the ordering and indices for our simplicial complex **??**. Let $N_\gamma := |J_\gamma|$ which is finite. Then

$$\begin{aligned}
&\left|\left(\lim_{i\to\infty} \bar{I}(d\phi_j)\right)(\gamma)\right| \leq \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\right)(\gamma)\right| \\
&\leq \sum_{k\in J_\gamma} \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_i)\right)(\sigma_k)\right| \\
&\leq N_\gamma^{1/2} \left(\sum_{k\in J_\gamma} \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\right)(\sigma_k)\right|^2\right)^{1/2} \\
&\leq N_\gamma^{1/2} \|\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\|_{C_2^1(K)} \leq N_\gamma^{1/2}\epsilon
\end{aligned}$$

and thus

$$\left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i)\right)(\gamma)\right| = 0$$

because ϵ was arbitrarily small. This can be done more elegantly by identifying $C_2^1(K)$ with ℓ^2. See notes from 28.12 □

Lemma 1. *Let Ω, U be open and $U \subseteq \Omega$. Let $\nu \in \mathring{H}^k(d;U)$. Define $\bar{\nu}$ as the zero extension of ν in $\Omega \setminus U$. Then $\bar{\nu} \in \mathring{H}^k(d;\Omega)$ and $d\bar{\nu} = d\nu$ in U and zero in $\Omega \setminus U$.*

Proof. The proof works exactly as in the standard Sobolev case. Take $\varphi \in C_0^\infty\Lambda^k(\Omega)$ arbitrary. Define $\bar{\nu}' = d\nu$ in U and $\bar{\nu}' = 0$ in $\Omega \setminus U$. We want to show that

$$\int_\Omega \bar{\nu} \wedge *\delta\varphi = \int_\Omega \bar{\nu}' \wedge *\varphi$$

which implies that $\bar{\nu}' = d\bar{\nu}$. If $\operatorname{supp}\varphi \subseteq U$ this follows by applying the definition of $d\nu$. If $\operatorname{supp}\varphi \subseteq \operatorname{int}(\Omega \setminus U)$ then it is also trivial since both $\bar{\nu}$ and $\bar{\nu}'$ are zero in $\Omega \setminus U$.

Otherwise, we recognize $\varphi|_U \in C_c^\infty\Lambda^{k+1}(U)$. Thus we get from the definition of the zero boundary condition **??**

$$\langle \bar{\nu}, \delta\varphi\rangle_{L_2^k(\Omega)} = \langle \nu, \delta\varphi\rangle_{L_2^k(U)} = \langle d\nu, \varphi\rangle_{L_2^k(U)} = \langle \bar{\nu}', \varphi\rangle_{L_2^k(\Omega)}.$$

Thus, $\bar{\nu}' = d\bar{\nu}$. Because $d\nu \in L_2^{k+1}(U)$, $d\bar{\nu} \in L_2^{k+1}(\Omega)$ and the lemma is proven. □

Now we construct a solution. Let $L \subseteq K$ be a bounded simplicial subcomplex s.t. $\gamma \in C_1(L)$ and the star of every vertex from γ does not touch the boundary of L. We also require that $H_1(L) = \mathbb{Z}[\gamma]$. Let $\mathscr{R}_L$ be the regularization operator from Thm. 1, but now constructed for L instead of K.

We know from 3 that a closed cochain $F \in C^1(L)$ exists s.t. $F(\gamma) = C_0$. We know from **??** that $[I] : \mathscr{H}^1(L) \to H^1(L)$ is a isomorphism. Thus we find a $\theta \in S^1(L)$ s.t. $[I(\theta)] = [F]$ i.e. $I(\theta) = F + \delta J$ for some $J \in C^0(L)$. Thus $I(\theta) = F(\gamma) + (\delta J)(\gamma) \overset{\gamma \text{ closed}}{=} F(\gamma) = C_0$.

By our definition of integration **??** we have then $I(\theta) = C_0$ and due to Lemma **??**

$$\int_\gamma \mathscr{R}_L \theta = \int_\gamma \theta = C_0.$$

We now use the Hodge decomposition (6) and project θ onto the harmonic functions to obtain $\omega_L \in \mathring{H}^1_0(d, \delta; |L|)$ with

$$\omega_L = \theta_L - d\psi$$

with $\psi \in H^0(d; |L|)$.

$$\int_\gamma \mathscr{R}_L \omega_L = \int_\gamma \mathscr{R}_L(\theta - d\psi) = \int_\gamma \mathscr{R}_L \theta = \int_\gamma \theta = C_0.$$

Because $\omega_L \in \mathring{H}^1(d; |L|)$ we can use Lemma 1 extend it by zero to obtain $\bar{\omega}_L \in H^1(d; \Omega)$ which is closed. Because the construction of the operator $\mathscr{R}$ is local on the stars of the vertices (cf. [3]) we have $\mathscr{R}_L \omega_L = \mathscr{R} \bar{\omega}_L$ on every simplex attached to γ because $\mathscr{R}_L$ and $\mathscr{R}$ are constructed in the same manner. Therefore $\int_\gamma \mathscr{R}_L \omega_L = \int_\gamma \mathscr{R} \bar{\omega}_L = C_0$. Now we project $*\bar{\omega}_L$ onto harmonic forms to get $\omega \in \mathring{H}^2_0(d, \delta; \Omega)$. Because the domain Ω is unbounded the image of d and δ are not closed anymore. So we obtain a sequence $(\phi_i)_{i \in \mathbb{N}} \subseteq H^3(\delta)$ s.t.

$$*\bar{\omega}_L = \omega + \lim_{i \to \infty} \delta \phi_i.$$

where the limit is in the L^2-sense. Apply the Hodge star operator on both sides, use the fact that it is an isometry and thus continuous and then remember the definition of δ to get a sequence $(\psi_i)_{i \in \mathbb{N}} \subseteq H^0(d)$ s.t.

$$\bar{\omega}_L = *\omega + \lim_{i \to \infty} d\psi_i.$$

Then we get

$$\int_\gamma \mathscr{R} * \omega = \int_\gamma \mathscr{R}(\bar{\omega}_L - \lim_{i\to\infty} d\psi_i) = \int_\gamma \mathscr{R}\bar{\omega}_L = C_0.$$

Thus ω fulfills the integral condition. Because $\omega \in \mathring{H}_0^2(d,\delta;\Omega)$ all other conditions are satisfied as well and ω is a solution.

In the proof of uniqueness we will use the following lemma. We are now back in the realm of standard vector analysis so all the notation is to be seen in this light (e.g. H^1 is here the standard Sobolov space and not related to differential forms).

Lemma 2. *Let $\phi \in L^2_{loc}(\Omega)$ with $\nabla\phi \in L^2(\Omega)$. Then there exists a sequence $(\phi_l)_{i\in\mathbb{N}} \subseteq H^1(\Omega)$ s.t. $\nabla\phi_i \to \nabla\phi$ in $L^2(\Omega)^3$.*

Proof. Take B_R with R large enough s.t. $B_R^c \subseteq \Omega$. Define $\Omega_R := B_R \cap \Omega$. Then $\overline{\Omega}_R \subseteq B_{R+1}$ and Ω_R is a Lipschitz domain and B_{R+1} is pre-compact and $\phi|_{\Omega_R} \in W^{1,2}(\Omega_R)$. Therefore we can find an extension $E\phi \in W_0^{1,2}(\Omega_{R+1}) \hookrightarrow W^{1,2}(\mathbb{R}^3)$ (cf. [**sobolev**]). So we can now define

$$\bar{\phi} := \begin{cases} \phi & \text{in } \Omega \\ E\phi & \text{in } \Omega^c. \end{cases}$$

Then $\phi \in L^2_{loc}(\mathbb{R}^3)$ and $\nabla\bar{\phi} \in L^2(\mathbb{R}^3)^3$. Then there exists a sequence $(\phi_l)_{l\in\mathbb{N}}$ s.t. $\nabla\phi_l \to \nabla\bar{\phi}$ in $L^2(\mathbb{R}^3)^3$ (cf. [**simader**]). By restricting ϕ_l to Ω we obtain the result. □

Theorem 2. *The solution of the problem is unique.*

Proof. Let ω and $\tilde{\omega}$ both be solutions. Then $*\omega$ is closed thus $\mathscr{R} * \omega \in S_2^1(\overline{\Omega})$ is also closed because

$$d\mathscr{R} * \omega = \mathscr{R} d * \omega = 0.$$

The same holds for $\tilde{\omega}$. $I(\mathscr{R}*\omega), I(\mathscr{R}*\tilde{\omega}) \in C_2^1(K)$ are closed 1-cochains with $I(\mathscr{R}*\omega)(\gamma) = I(\mathscr{R}*\tilde{\omega})(\gamma)$. Thus Prop. 3 implies $[I(\mathscr{R}*\omega)] = [I(\mathscr{R}*\tilde{\omega})] \in H^1(K;\mathbb{R})$. Now we use the Whitney transformation. We know that $[I]$ and $[w]$ are inverses on cohomology from **??**. That means we get

$$[\varphi\mathscr{R} * \omega] = [wI\varphi\mathscr{R} * \omega] = [wI\varphi\mathscr{R} * \tilde{\omega}] = [\varphi\mathscr{R} * \tilde{\omega}].$$

Here the cohomology classes are in the cohomology spaces of S-forms $\mathscr{H}_2^1(K)$. That means there is a S-form $\theta \in S^0(K)$ s.t.

$$\begin{aligned} &\varphi\mathscr{R} * \omega - \varphi\mathscr{R} * \omega = d\theta \\ &\Rightarrow \mathscr{R} * \omega - \mathscr{R} * \omega = \varphi^{-1} d\theta = d\varphi^{-1}\theta =: d\tilde{\theta}. \end{aligned}$$

We have $\tilde{\theta} \in S^0(\overline{\Omega}) \subseteq W^0_{\infty,loc}(\overline{\Omega})$ and $d\tilde{\theta} \in S^1_2(\overline{\Omega}) \subseteq W^1_2(\overline{\Omega})$.

Using the properties of $\mathscr{R}$ there exists a $\eta \in L^0_{2,loc}(\Omega)$ with $d\eta \in L^1_2(\Omega)$ s.t.

$$*\omega - *\tilde{\omega} = -d\mathscr{A}(*\omega - *\tilde{\omega}) + d\theta = d\eta.$$

By applying the Hodge star operator on both sides we find $\mu \in L^3_{2,loc}(\Omega)$ with $\delta\eta \in L^2_2(\Omega)$ s.t.

$$\omega - \tilde{\omega} = \delta\mu. \tag{13}$$

But because μ is only in $L^3_{2,loc}(\Omega)$ we can not immediately conclude that $\delta\mu = 0$.

Let us briefly return to vector proxies. Let B and $\tilde{B}$ be vector proxies of ω and $\tilde{\omega}$ respectively. (13) then translates to $B, \tilde{B} \in \mathring{H}(\mathrm{div})$ and $B - \tilde{B} = \nabla\phi$ with $\phi \in L^2_{loc}(\Omega)$ and $\nabla\phi \in L^2(\Omega)^3$. Because B and $\tilde{B}$ are both harmonic we have

$$0 = \int_\Omega (B - \tilde{B}) \cdot \nabla f \, dx = \int_\Omega \nabla\phi \cdot \nabla f \, dx \quad \forall f \in H^1(\Omega).$$

Thank to Lemma **??** we know that $\nabla\phi \in \overline{\nabla H^1(\Omega)}$. Because B and $\tilde{B}$ are harmonic we have $B - \tilde{B} \in \overline{\nabla H^1(\Omega)}^{\perp}$ and hence $\nabla\phi = 0$ and $B = \tilde{B}$. Because the corresponding vector proxies are equal we we obtain $\omega = \tilde{\omega}$. □

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