1 Introduction

Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let B be a magnetic field on the domain Ω. We the have the following boundary value problem:

$$\operatorname{curl} B = 0, \tag{1.0.1}$$
$$\operatorname{div} B = 0 \text{ in } \Omega \tag{1.0.2}$$
$$B \cdot n = 0 \text{ on } \partial\Omega \text{ and} \tag{1.0.3}$$
$$\int_\gamma B \cdot dl = C_0 \tag{1.0.4}$$

where n is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology among other things...

2 Differential forms

2.1 Alternating maps

{sec:alternating

For the introduction of alternating maps we follow the short section in Arnold's book [2, Sec. 6.1.] combine it with material from [3, Sec. V.1]. However, more arguments and additional details are provided especially in Sec. 2.2 about scalar and vector proxies.

Definition 2.1.1 (Alternating k-linear form)**.** Let V be a real vector space with $\dim V = n$. We call map $\omega : V^k \to \mathbb{R}$ *k-linear form* if it is linear in every argument. We call a k-linear form *alternating* if the sign switches when two arguments are exchanged i.e.

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k), \text{ for } 1 \leq i < j \leq k, \quad v_1, ..., v_k \in V.$$

We denote the space k-linear forms on V as $\operatorname{Alt}^k V$. For the special case $k = 0$ we define $\operatorname{Alt}^0 V := \mathbb{R}$.

Let $\mathcal{S}_k$ be the set of all permutations $\{1, 2, ..., k\} \to \{1, 2, ..., k\}$. A permutation that only exchanges two numbers is called transposition and denoted by (i, j) for the transposition that exchanges (i, j) with $i \neq j$. Every permutation can be written as the composition of transpositions. This decomposition into transpositions is not unique. Take a permutation $\pi \in \mathcal{S}_k$ and decompose it into transpositions

$$\pi = \tau_p \circ \tau_{p-1} \circ ... \circ \tau_1.$$

The sign $\operatorname{sgn}(\pi)$ of a permutation $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is equal to $(-1)^p$. Even though the decomposition is not unique the parity of the number of transpositions is always the same and hence the sign is well-defined. For example, the permutation $(1, 2, 3, 4) \mapsto (2, 3, 1, 4)$ can be built by performing the transpositions $(1, 2)$ and $(1, 3)$ so $\operatorname{sgn}(\pi) = (-1)^2 = 1$.

Going back to alternating forms, this also means for any permutation $\pi : \{1, 2, ..., k\} \to \{1, 2, ..., k\}$ and $\omega \in \operatorname{Alt}^k V$

$$\omega(v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(k)}) = \operatorname{sgn}(\pi)\, \omega(v_1, v_2, ..., v_k).$$

Definition 2.1.2 (Wedge product)**.** For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\pi} \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, ..., v_{\pi(k)})\nu(v_{\pi(k+1)}, ..., v_{\pi(k+l)})$$

where we sum over all permutations $\pi : \{1, ..., k + l\} \to \{1, ..., k + l\}$ s.t. $\pi(1) < ... < \pi(k)$ and $\pi(k + 1) < ... < \pi(k + l)$.

Let us mention some important properties of the wedge product. It is associative, but not commutative. For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we have

$$\omega \wedge \mu = (-1)^{kl} \mu \wedge \omega. \qquad (2.1.1)$$ {eq:commutativit

Recalling the definition of the sign of a permutation $\pi \in \mathcal{S}_k$ we get for linear forms $\omega_1, \omega_2, ..., \omega_k \in V'$

$$\omega_{\pi(1)} \wedge \omega_{\pi(2)} \wedge ... \wedge \omega_{\pi(k)} = \operatorname{sgn}(\pi)\, \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k$$

and if a linear form appears twice then the expression is zero.

There is a useful formula for computing the wedge product of linear forms. We denote the dual space of V as V'. For $\omega_1, ..., \omega_k \in \operatorname{Alt}^1 V = V'$, $k \leq n$ we have the formula ([3, p.260])

$$\omega_1 \wedge ... \wedge \omega_k(v_1, ..., v_k) = \det(\omega_s(v_t))_{1 \leq s,t \leq n}. \qquad (2.1.2)$$ {eq:wedge_produc

This formula can be easily proven by induction using the definition of the wedge product and the determinant.

Let $\{b_i\}_{i=1}^n$ be any basis of V and $\{b^i\}_{i=1}^n$ the correspoding dual basis i.e. $b^i \in V'$, $b^i(b_j) = \delta_{ij}$ for $i, j = 1, 2, ..., n$. Then

$$\{b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \,|\, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\mathrm{Alt}^k V$. In particular, dim $\mathrm{Alt}^k V = \binom{n}{k}$.

Assume now that we are given an inner product on V, $\langle\cdot,\cdot\rangle_V$ where we will usually leave out the subscript if it is clear what space we mean. Recall the Riesz isomorphism $\Phi : V \to V'$ defined by

$$\Phi v(w) = \langle v, w\rangle.$$

Then we obtain an inner product on the dual space V' by using the Riesz isomorphism Φ

$$\langle \Phi v, \Phi w\rangle_{V'} := \langle v, w\rangle_V$$

which makes the Riesz isomorphism to an isometry.

Now we can define an inner product on $\mathrm{Alt}^k V$ by defining

$$\langle b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}, b^{j_1} \wedge ... \wedge b^{j_k}\rangle_{\mathrm{Alt}^k V} := \det(\langle b^{i_k}, b^{i_l}\rangle_V)_{1\leq k,l\leq n}$$

which is then extended to all of $\mathrm{Alt}^k V$ by linearity. We denote with $|\cdot|_{\mathrm{Alt}^k V}$ the induced norm. For an orthonormal basis u_1, ..., u_n the corresponding basis $u^{i_1} \wedge v^{i_2} \wedge ... \wedge u^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\mathrm{Alt}^k V$.

Next, we want to introduce the *pullback* as a natural linear mapping between spaces of alternating forms. Let V and W be finite-dimensional real vector spaces with ordered bases $(b_i)_{i=1}^n$ and $(c_j)_{j=1}^m$ respectively. We write a basis in standard brackets $(\cdot)$ if the basis is ordered. Let $L \in \mathcal{L}(V, W)$ where $\mathcal{L}(V, W)$ is the space of linear mappings from V to W. For $\omega \in \mathrm{Alt}^k W$ we define the pullback $L^*\omega \in \mathrm{Alt}^k V$ via

$$(L^*\omega)(v_1, ..., v_k) = \omega(L\, v_1, ..., L\, v_k).$$

It is then easy to see that L^* is a linear mapping from $\mathrm{Alt}^k W$ to $\mathrm{Alt}^k V$.

It is obvious from the definitions of the wedge product and the pullback that we have

$$L^*(\omega \wedge \nu) = L^*\omega \wedge L^*\nu \quad \forall \omega \in \mathrm{Alt}^k W,\ \nu \in \mathrm{Alt}^l W.$$

Proposition 2.1.3. *Let $A \in \mathbb{R}^{m\times n}$ be the matrix representation of L in the above bases i.e. $L\, b_i = \sum_{j=1}^m A_{ji} c_j$. Then we get the basis representation of the pullback*

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1\leq i_1<...<i_k\leq n} \det A_{(j_1,...,j_k),(i_1,...,i_k)}\, b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \tag{2.1.3}$$

{eq:basis_repres

where $A_{(j_1,...,j_k),(i_1,...,i_k)}$ is the matrix we get by choosing the rows $j_1, ..., j_k$ and the columns i_1, ... , i_k.

Proof. Because $\{b^{i_1}\wedge b^{i_2}\wedge...\wedge b^{i_k} \mid 1 \leq i_1 < ... < i_k \leq n\}$ and $\{c^{j_1}\wedge c^{j_2}\wedge...\wedge c^{j_k} \mid 1 \leq j_1 < ... < j_k \leq m\}$ are bases for $\operatorname{Alt}^k V$ and $\operatorname{Alt}^k W$ respectively, we can find $\lambda_{i_1...i_k}$ s.t.

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1\leq i_1<...<i_k\leq n} \lambda_{i_1...i_k} b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \tag{2.1.4}$$

{eq:basis_repres

Now recall the formula for the wedge product of 1-forms $\nu_i \in \operatorname{Alt}^1 V = V'$

$$\nu_1 \wedge ... \wedge \nu_k(v_1, ..., v_k) = \det \big(\nu_s(v_t)\big)_{1\leq s,t\leq n}.$$

Fix now $1 \leq l_1 < ... < l_k \leq n$. Then we get from this formula $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}(b_{l_1}, ..., b_{l_k}) = 1$ iif. $(i_1, ..., i_k) = (l_1, ..., l_k)$. Here it is important to remember that these indices are ordered. Plugging this in (2.1.4) gives us

$$\begin{aligned}
\lambda_{i_1...i_k} &= L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k})(b_{l_1}, ..., b_{l_k}) \\
&= c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}(L\, b_{l_1}, ..., L\, b_{l_k}) \\
&= \det \Big(c^{j_s}\big(\sum_{r_t=1}^{m} A_{r_t,l_t} c_{r_t}\big)\Big)_{1\leq s,t\leq k} \\
&= \det \Big(\sum_{r_t=1}^{m} A_{r_t,l_t} \delta_{j_s,r_t}\Big)_{1\leq s,t\leq k} \\
&= \det \big(A_{j_s,l_t}\big)_{1\leq s,t\leq k} \\
&= \det A_{(j_1,...,j_k),(i_1,...,i_k)}.
\end{aligned}$$

□

We want to emphasize the special case of the pullback of a n-linear map with $m = n$. So take $\omega \in \operatorname{Alt}^n W$. Then we know that $\dim \operatorname{Alt}^n W = \binom{n}{n} = 1$

and so $\omega = \lambda\, c^1 \wedge ... \wedge c^n$ for some $\lambda \in \mathbb{R}$. Then there only remains one summand in (2.1.3) and we obtain for this special case

$$L^*\omega = \lambda \det A\, b^1 \wedge ... \wedge b^n \tag{2.1.5}$$ {eq:pullback_alt

We want to examine $\mathrm{Alt}^n V$ a bit closer. $\mathrm{Alt}^n V$ is one-dimensional and so we can choose a basis by fixing a specific non-zero element. We want to choose one specific element called the *volume form* which will play a crucial role when we define integration on a manifold in Sec. **??**. We also need it to define the Hodge star operator below.

The choice of this volume form will depend on the orientation. We say that two bases of V have the same orientation if the change of basis has positive determinant. That divides the bases into two classes with different orientation. We choose one of these classes and call these bases positively oriented. In $\mathbb{R}^n$, the convention is to define the class as positively oriented which includes the standard orthonormal basis.

Definition 2.1.4 (Volume form)**.** Let $(b_i)_{i=1}^n$ be any positively oriented basis. Let G be the Gramian matrix i.e. $G_{ij} = \langle b_i, b_j\rangle$ which is always a symmetric positive definite matrix. Then we define the *volume form*

$$\mathrm{vol} := \sqrt{\det G}\, b^1 \wedge b^2 \wedge ... \wedge b^n.$$

We have the following defining property of the volume form.

Proposition 2.1.5. *For any ordered orthonormal basis $(u_i)_{i=1}^n$ we have*

$$\mathrm{vol}(u_1,\, u_2, ...,\, u_n) = (-1)^s.$$

with $s = 0$ if $(u_1, ..., u_n)$ has the same orientation as $(b_i)_{i=1}^n$ and $s = 1$ otherwise.

Proof. Let us define the matrix $B \in \mathbb{R}^{n\times n}$, $B_{k,i} = \langle b_i, u_k\rangle_V$ which is just the change of basis matrix from $(b_i)_{i=1}^n$ to $(u_i)_{i=1}^n$. Then using basic linear algebra we get $G = B^\top B$ and $\sqrt{\det G} = (-1)^s \det B$. Let now Ψ be the linear map with $\Psi b_i = u_i$. In the basis $(b_i)_{i=1}^n$ this has the matrix representation B^{-1} and so by using (2.1.5) we get

$$\begin{aligned}\mathrm{vol}(u_1,\, u_2, ...,\, u_n) &= \sqrt{\det G}\, b^1 \wedge ... \wedge b^n(\Psi b_1, ..., \Psi b_n)\\ &= (-1)^s \det B\, \Psi^*(b^1 \wedge ... \wedge b^n)(b_1, ..., b_n)\\ &= (-1)^s \det B\, \det B^{-1}\, (b^1 \wedge ... \wedge b^n)(b_1, ..., b_n) = (-1)^s.\end{aligned}$$

□

This property also defines the volume form uniquely so it is independent of the chosen basis. It only depends on the orientation. It also shows that vol is non-zero and thus

$$\mathrm{Alt}^n V = \mathrm{span}\{\mathrm{vol}\}.$$

Note that if we choose $\{b_i\}_i$ to be an orthonormal basis to begin with the Gramian matrix is just the identity and $\mathrm{vol} = b^1 \wedge ... \wedge b^n$. Especially in the case of $\mathbb{R}^n$ if we denote the standard dual basis by $\{dx^i\}_{i=1}^n$ then

$$\mathrm{vol} = dx^1 \wedge ... \wedge dx^n.$$

We will from now on assume that we fixed a orientation on V and with it the volume form vol. Using the resulting volume form on V we can now define the *Hodge star operator* with the following proposition.

Proposition 2.1.6. *There exists a isomorphism* $\star : Alt^k V \to Alt^{n-k} V$ *s.t.*

$$\omega \wedge \mu = \langle \star\omega, \mu \rangle_{Alt^{n-k} V} \mathrm{vol} \quad \forall \mu \in Alt^{n-k} V. \tag{2.1.6}$$ {eq:hodge_star_d

We call this isomorphism the Hodge star operator.

Proof. Let denote with vol' the dual basis element of vol i.e. $\mathrm{vol}'(\mathrm{vol}) = 1$. Let us fix $\omega \in \mathrm{Alt}^k V$. Then we can define the following linear form on $\mathrm{Alt}^{n-k} V$

$$\mu \mapsto \mathrm{vol}'(\omega \wedge \mu).$$

Then we define $\star\omega$ as the Riesz representative of this linear form that means we have $\mathrm{vol}'(\omega \wedge \mu) = \langle \star\omega, \mu \rangle_{\mathrm{Alt}^k V}$ for all $\mu \in \mathrm{Alt}^{n-k} V$ which implies (2.1.6). Checking linearity is trivial and will be omitted.

It is also clear from the uniqueness of the Riesz representative that the $\star\omega$ is uniquely determined by the above condition and thus $\star$ is injective. Since $\dim \mathrm{Alt}^k V = \binom{n}{k} = \binom{n}{n-k} = \dim \mathrm{Alt}^{n-k} V$ the injecivity implies surjectivity and $\star$ is an isomorphism. $\square$

Let us collect some important properties of the Hodge star operator without proof.

{prop:properties

Proposition 2.1.7. *Let* $(u_i)_{i=1}^n$ *be an ordered orthonormal basis of* V. *Let* $\{i_1, ..., i_n\} = \{1, ..., n\}$, $i_1 < i_2 < ... < i_k$ *and* $i_{k+1} < ... < i_n$.

$$\star(u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}) = \mathrm{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n} \tag{2.1.7}$$ {eq:hodge_star_o

where $\mathrm{sgn}(i_1, i_2, ..., i_n)$ *is the sign of the permutation* $j \mapsto i_j$. *In particular,* $\star$ *is an isometry since it maps orthonormal bases to orthonormal bases. Furthermore,* *(i)* $\star\star\omega = (-1)^{k(n-k)}\omega \quad \forall \omega \in Alt^k V$

(ii) $\omega \wedge \star\nu = \langle \omega, \nu \rangle_{Alt^k V} \text{ vol} \quad \forall \omega, \nu \in Alt^k V$.

Proof. Take $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ with $1 \leq j_{k+1} < ... < j_n \leq n$. Then

$$\begin{aligned}
&\langle \operatorname{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\mathrm{Alt}^{n-k} V} \\
&= \operatorname{sgn}(i_1, i_2, ..., i_n) \det \left(\langle u^{i_s}, u^{j_t} \rangle \right)_{k+1 \leq s,t \leq n} \\
&= \operatorname{sgn}(i_1, i_2, ..., i_n) \det \left(\delta_{i_s, j_t} \right)_{k+1 \leq s,t \leq n}.
\end{aligned}$$

In the last step we used the fact that u^i are orthonormal since u_i are. Now observe due to the ordering that $\det \left(\delta_{i_s, j_t} \right)_{k+1 \leq s,t \leq n}$ is one i.i.f. $i_s = j_s$ for $s = k+1, ..., n$ an is zero otherwise.

For the wedge product we get

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} = \begin{cases} \operatorname{sgn}(i_1, ..., i_n) \text{ vol}, & \text{if } (i_{k+1}, ..., i_n) = (j_{k+1}, ..., j_n) \\ 0, & \text{otherwise.} \end{cases}.$$

Comparing both expressions we just proved

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} = \langle \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\mathrm{Alt}^{n-k} V} \text{ vol}$$

and thus by linearity since the $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ form a basis of $\mathrm{Alt}^{n-k} V$ we get , $\star(u^{i_1} \wedge ... \wedge u^{i_k}) = \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}$ as claimed. We see from this that $\star$ maps the orthonormal basis to an orthonormal basis and is thus an isometry.

The other two claims follow from that easily. For any $\omega \in \mathrm{Alt}^k V$, $\mu \in \mathrm{Alt}^{n-k} V$

$$\begin{aligned}
\langle \star\star\omega, \mu \rangle_{\mathrm{Alt}^k V} \text{ vol} &= \star\omega \wedge \mu = (-1)^{k(n-k)} \mu \wedge \star\omega \\
&= (-1)^{k(n-k)} \langle \star\mu, \star\omega \rangle_{\mathrm{Alt}^{n-k} V} \text{ vol} \\
&= \langle (-1)^{k(n-k)} \omega, \mu \rangle_{\mathrm{Alt}^k V} \text{ vol}.
\end{aligned}$$

Then the first claim follows since $\mu \in \mathrm{Alt}^k V$ was arbitrary.

For the second claim, take $\omega, \nu \in \mathrm{Alt}^k V$, then

$$\omega \wedge \star\nu = \langle \star\nu, \star\omega \rangle_{\mathrm{Alt}^{n-k} V} \text{ vol} = \langle \omega, \nu \rangle_{\mathrm{Alt}^k V} \text{ vol}.$$

□

In particular in $\mathbb{R}^3$, we have $\star\star = \mathrm{Id}$ i.e. $\star$ is self-inverse. Notice also that we always have $\star 1 = \mathrm{vol}$.

Let us quickly derive the expression in any basis for the Hodge star applied to linear forms which we will need later. Let $\omega = \sum_i \omega_i b^i \in \mathrm{Alt}^1 V = V'$. Let us denote $g^{ij} = \langle b^i, b^j \rangle$ and $G_{ij} = \langle b_i, b_j \rangle$ again the Gramian matrix. Then we claim

$$\star\omega = \sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge \dots \wedge \widehat{b^j} \wedge \dots \wedge b^n \tag{2.1.8}$$ {eq:hodge_star_o

where $\widehat{b^j}$ means that this term is left. The proof is very simple in this case. Let ν denote the expression on the right hand side. For any $1 \le l \le n$ we get

$$\begin{aligned}
\nu \wedge b^l &= \left(\sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge \dots \wedge \widehat{b^j} \wedge \dots \wedge b^n \right) \wedge b^l \\
&= \sqrt{\det G} \sum_{j=1}^{n} \langle b^j, \sum_{i=1}^{n} \omega_i b^i \rangle (-1)^{j-1} b^1 \wedge b^2 \wedge \dots \wedge \widehat{b^j} \wedge \dots \wedge b^n \wedge b^l \\
&= \sqrt{\det G} \langle b^l, \omega \rangle (-1)^{(l-1)} (-1)^{(n-l)} b^1 \wedge b^2 \wedge \dots \wedge b^l \wedge \dots \wedge b^n \\
&= \langle (-1)^{n-1} \omega, b^l \rangle \,\mathrm{vol}\,.
\end{aligned}$$

where the inner product is always in the corresponding space of alternating forms. In the second step, we used that if $j \neq l$ then one basis element must appear twice in the wedge product which is then zero. If $j = l$ then $\mathrm{sgn}(1, 2, ..., \hat{l}, ..., n, l) = (-1)^{n-l}$ because we need $(n - l)$ transpostions to bring the indices into order. So we proved $(-1)^{n-1}\omega = \star\nu$ and thus $\star\omega = (-1)^{n-1} \star\star\nu = \nu$.

Then by linearity we obtain (2.1.6) and thus the given expression is indeed equal to $\star b^i$ as claimed.

2.2 Scalar and Vector proxies

{sec:scalar_and_

Now we want to relate alternating maps to elements of the vector space V itself or to scalars. Let us start with the easiest case. $\mathrm{Alt}^0 V$ are already scalars by definition. Now we can use the Hodge star operator which is an isometry $\star : \mathrm{Alt}^0 V \to \mathrm{Alt}^n V$ with $\star c = c\,\mathrm{vol}$. We call the real number that is associated with an element of $\mathrm{Alt}^n V$ *scalar proxy* i.e. the scalar proxy of $c \in \mathbb{R}$ is just $c\,\mathrm{vol} \in \mathrm{Alt}^n V$.

Next, we will move on to $\mathrm{Alt}^1 V$ and $\mathrm{Alt}^{n-1} V$. Let $\Phi : V \to V'$ denote the Riesz isomorphism which is an isometry. Because $V' = \mathrm{Alt}^1 V$ this gives us the correspondence of vectors and linear forms. Now we can once again use the Hodge star and obtain the isometry $\star\Phi : V \to \operatorname{Alt}^{n-1} V$. We call the vectors associated with an alternating 1- or $(n-1)$-linear map *vector proxy.*

These way to identfy alternating maps with scalars and vectors gives us the ability to look at the notions defined above in the context of scalars and vectors.

Let us look at the wedge product. We have for $v, w \in V$

$$\Phi v \wedge \star\Phi w = \langle \Phi v, \Phi w \rangle_{V'} \operatorname{vol} = \langle v, w \rangle_V \operatorname{vol}$$

which means that the wedge product of a linear from and an alternating $(n-1)$- linear map corresponds in proxies to the inner product.

Note that for $n = 2$ the situation is slightly ambiguous, see [2, p.67]. But this case will not be relevant in this thesis.

In the case of $V = \mathbb{R}^3$ with the standard basis vectors e_1, e_2 and e_3. Denote the resulting elements of the dual basis with e^1, e^2 and e^3 respectively. Take $v = v_1e_1 + v_2e_2 + v_3e_3 \in \mathbb{R}^3$ and recall that for a orthonormal basis the Riesz isomorphism maps basis elements to their dual basis elements i.e. $\Phi e_i = e^i$. Hence, we get $\Phi v = v_1e^1 + v_2e^2 + v_3e^3$. Take another $w \in \mathbb{R}^3$. Then using the $e^i \wedge e^j = -e^j \wedge e^i$ we get

$$\Phi v \wedge \Phi w = \star\Phi(v \times w). \tag{2.2.1}$$ {eq:cross_produc

That means in $3D$ in terms of vector proxies, the wedge product of two linear forms corresponds to the cross product. Note that (2.2.1) is formulated without using a specific basis and can therefore be computed using any basis i.e. if we have $v = \tilde{v}_1b_1 + \tilde{v}_2b_2 + \tilde{v}_3b_3$ and analogous for w we could still calculate the cross product directly as

$$v \times w = \Phi^{-1} \star \big(\Phi v \wedge \Phi w\big).$$

One has to take care though because if the basis is not orthonormal the Riesz isomorphism does not map basis elements b_i to their respective dual basis elements b^i. We have

$$\Phi b_i = \sum_{i=1}^{n} \langle b_j, b_i \rangle b^j$$

i.e. it has the gramian matrix G as basis representation. This is easy to see. Let $\Phi b_i = \sum_j \lambda_j b^j$. Then

$$\lambda_j = \Phi b_i(b_j) = \langle b_i, b_j \rangle.$$

As derived above, the Hodge star is not as trivial to compute either.

This illustrates the idea of a coordinate free description which is a important notion in differential geometry (see [**<empty citation>**]).

Similarly, we want to explore the pullback in terms of vector proxies as well. These will be important in the next section when we talk about the pullback of differential forms and apply these to the tranformation of integrals. In order to avoid complicated computations we will stick to orthonormal bases. Let $\{b_i\}_{i=1}^n$ be an ONB of V and $\{c_j\}_{j=1}^m$ be an ONB of W. Let $L : V \to W$ again be a linear map and A be the basis representation of it w.r.t. the two bases given i.e. $Lb_i = \sum_j A_{ji}c_j$. Then we get the pullback of linear forms in terms of vector proxies as $\Phi_V^{-1}L^*\Phi_W : W \to V$. Recall formula (2.1.5) which gives us for the pullback of one forms

$$L^*c^j = \sum_{i=1}^n A_{j,i}b^i \tag{2.2.2}$$ {eq:pullback_lin

so the matrix representation w.r.t. the given dual bases is $A^\top$. Then since we have $\Phi_V b_i = b^i$ and analogous for c^j the matrix representation of the Riesz isomorphism is just the identity. In total, the basis representation of $\Phi_V^{-1}L^*\Phi_W$ is then also $A^\top$.

For $m = n$ let us look at the pullback of alternating $(n-1)$-linear maps. In terms of vector proxies this can then be expressed as $\Phi_V^{-1} \star^{-1} L^* \star \Phi_W$. Note that we used the same symbol $\star$, but it is once applied in W and then the inverse in V. It can be shown with the same ideas and (2.1.7) that the matrix representation is the adjugate matrix $\mathrm{ad}(A)$ defined as

$$\mathrm{ad}(A)_{ij} = (-1)^{i+j} \det A_{-j,-i}$$

where $A_{-j,-i}$ is the matrix without the j-th row and i-th column. If A is invertible then $(\det A)\, A^{-1} = \mathrm{ad}(A)$.

The pullback of n-linear forms in terms of vector proxies is $\star^{-1}L^*\star$. Again n the case of $n = m$ and an orthonormal basis we get for $c \in \mathbb{R}$

$$\star^{-1}L^* \star c = \star^{-1}L^* c\,\mathrm{vol} = \star^{-1}c \,\det L \,\mathrm{vol} = c \,\det L.$$

2.3 Differential geometry

{sec:differentia

Before we define differential forms, let us start by revising some basics from differential geometry. We follow the approach from [3, Sec. II] for the most part.

2.3.1 Manifolds

In order to formulate the definition of a manifold, let us recall the definition of a topological space.

Definition 2.3.1 (Topological space)**.** A topological space is a set X together with collection of subsets of X denoted by $\mathcal{T}$ s.t.

- $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
- for $\{U_i \in \mathcal{T} \mid i \in \mathcal{I}\}$ for any index set $\mathcal{I}$, $\bigcup_{i \in \mathcal{I}} U_i \in \mathcal{T}$ and
- $\emptyset, X \in \mathcal{T}$.

The sets contained in $\mathcal{T}$ are called *open.*

For example, a metric space together with its usual open sets is a topological space. Another well known example of topologies which do not arise from a metric are the weak and weak-$\star$ topology on infinite dimensional spaces (see [4, Ch. 3]).

Definition 2.3.2 (Second countable topological space)**.** Let $(X, \mathcal{T})$ be a topological space. Then we call $\mathcal{B} \subseteq \mathcal{T}$ a basis for the topology of X if every open set (i.e. every set in $\mathcal{T}$) is a union of sets in $\mathcal{B}$. If a topological space has a countable basis it is called *second countable.*

$\mathbb{R}^n$ is an example of a second countable topological space. Consider the countable set of balls $\{B_r(x) \mid r \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ where $B_r(x)$ are the balls with center x and radius r. Then it is trivial to show that any open set in $\mathbb{R}^n$ is a union of of these balls. Hence, $\mathbb{R}^n$ is second countable.

Definition 2.3.3 (Hausdorff)**.** space Let $(X, \mathcal{T})$ be a topological space. We call $(X, \mathcal{T})$ *Hausdorff* if we can separate any two different points of X with disjoint neighborhoods. That means for any $x, y \in X$, $x \neq y$ there are $U_x, U_y \in \mathcal{T}$ s.t. $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

Example 2.3.4. Let (X, d) be a metric space and take $x, y \in X$, $x \neq y$. Then for the distance $\delta := d(x, y) > 0$ we can choose the open balls around x and y with radius $\delta/2$, denoted by $B_{\delta/2}(x)$ and $B_{\delta/2}(y)$. These are open and obviously disjoint. Therefore any metric space is Hausdorff.

Definition 2.3.5 (Manifold)**.** A n-dimensional C^α manifold with boundary, $\alpha \in \mathbb{N}$, is a second countable Hausdorff space M with an open cover $\{U_i\}_{i \in I}$ and a collection of maps called *charts* ϕ_i, $i \in I$ for some index set I s.t.

- $\phi_i : U_i \to V_i \subseteq \mathbb{R}_- \times \mathbb{R}^{n-1}$ are homeomorphisms

- for two charts ϕ_i, ϕ_j the *change of coordinates* or *chart transition* $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a C^α diffeomorphism. If $\alpha = 0$ it is a homeomorphism.

When we write (U_i, ϕ_i) we mean the chart ϕ_i has domain U_i.

The charts provide us with *local coordinates* $x_k : U_i \to \mathbb{R}^n$, $k = 1, ..., n$ with $\mathbf{x}(p) = (x_1(p), ..., x_n(p))^\top = \phi(p) \in \mathbb{R}^n$.

Let us take a look at some examples. First, take any open domain $\Omega \subseteq \mathbb{R}^n$. Then we can take the identity as a chart. Since $\mathbb{R}^n$ with the standard topology is Hausdorff and second coundable the same reasoning applies to open subdomains. Hence, Ω is a n-dimensional C^∞ manifold.

Proposition 2.3.6. *Every manifold has a countable atlas.*

Proof. Take any atlas $\{(U_i, \phi_i)\}_{i \in I}$. Because the manifold is second countable there exists a countable basis of the topology $\mathcal{B} = \{B_0, B_1, ...\}$. Every basis of a topological is a open cover which is easily checked. Define $\mathcal{A} = \{k \in \mathbb{N} \mid B_k \subseteq U_i \text{ for some } i \in I\}$. Then for every $k \in \mathcal{A}$ choose $i_k \in I$ s.t. $B_k \subseteq U_{i_k}$. Then $U_{i_k k \in \mathcal{A}}$ is an open cover and thus $\{(U_{i_k}, \phi_{i_k})\}_{k \in \mathcal{A}}$ is countable atlas. $\square$

Due to this proposition we will from now on assume that all the chosen atlas is countable. Note that here the second countability is essential. Some authors do not require this in the defintion of a manifold and then this proposition might fail. Another important property is the existence of a partition of unity on M assuming M is smooth which we will not prove.

Theorem 2.3.7 (Partition of unity)**.** *Let M be a smooth (i.e. C^∞) manifold with atlas $\{U_i, \phi_i)\}_{i=0}^N$, $N \leq \infty$. Then there exists a smooth partition of unity subordinate to the open cover $\{U_i\}_{i=0}^N$. That means there exists a family of non-negative smooth functions $\{\chi_i\}_{i=0}^N$ s.t. $\operatorname{supp} \chi_i \subseteq U_i$ and $\sum_{i=0}^N \chi_i(p) = 1$ for every $p \in M$.*

These partitions of unity are typically used to extend a construction that is done locally to the entire manifold as we will see later.

Let us denote $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$. Let us for the following equip $\mathbb{R}_- \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ with the subspace topology i.e. we call a set $V \subseteq \mathbb{R}_- \times \mathbb{R}^{n-1}$ open i.i.f. there exists an open set $V' \subseteq \mathbb{R}^n$ s.t. $V = V' \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$. This means e.g. that $B_1(0) \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$ is open which is not an open set in the standard topology of $\mathbb{R}^n$.

Definition 2.3.8 (Manifold with boundary)**.** We call M a *manifold with boundary* if the charts ϕ_i, $i \in I$ are homeomorphism from M into $\mathbb{R}_- \times \mathbb{R}^{n-1}$ endowed with the subspace topology. The *boundary* ∂M are the points that get mapped to $0 \times \mathbb{R}^{n-1}$ by the charts i.e. $\partial M = \bigcup_{i \in I} \phi_i^{-1}(0 \times \mathbb{R}^{n-1})$.

We should note the relationship between the boundary in the usual topological sense and the above definition of a boundary. Let $\Omega \subseteq \mathbb{R}^n$ be an open C^1 domain. Then $\partial\Omega$ in the usual topological sense is $\overline{\Omega} \setminus \partial\Omega$. However, if we see Ω as a manifold then the boundary $\partial\Omega$ is empty. If we look $\overline{\Omega}$ as a manifold with boundary, then the two notions agree.

Another important difference, is that in the case of manifold $\partial\partial M = \emptyset$, but for the general topological boundary that is not the case. If we have e.g. a single point $\{x\} \subseteq \mathbb{R}^n$ then $\partial\{x\} = \{x\}$.

From now on we want to investigate certain structures on a manifold that require some regularity to be defined. Therefore, we will assume $\alpha > 0$ i.e. our manifolds are differentiable.

Remark 2.3.9. There is also the concept of Lipschitzian manifolds. Then using Rademachers theorem some of the structures that are introduced below can be extended with some care. See [10] for a detailed discussion.

Definition 2.3.10 (Orientation of a manifold)**.** We call an atlas *oriented* if the Jacobian of the coordinate changes has positive determinant. A manifold that can be equipped with an oriented atlas is called *orientable*.

The next important concept we will recall are tangent spaces. It should be noted that there are different definitions of tangent space, but these lead to isomorphic notions (see e.g. [6, Sec. 1.B]).

Definition 2.3.11. Let M, N be an n- and m-dimensional manifold with or without boundary and a function $F : M \to N$. Take $p \in M$ and let (ϕ, U) and (ψ, V) be charts at p and $F(p)$ with local coordinates $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ respectively. Then we call the function

$$\bar{F}(x_1, ..., x_n) = \psi \circ F \circ \phi^{-1}(x_1, ..., x_n)$$

F expressed in local coordinates which depends on the chosen local coordinates.

For a point $p \in M$ and a neighorhood $U \subseteq M$ of p and a $V \subseteq N$ a neighborhood of $F(p)$ with local coordinate charts (ϕ, U) and (ψ, V) and resulting local coordinates $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ respectively and assume M and N to be at least C^1. We call a function $F : U \to N$ differentiable at p if the expression in local coordinates is differentiable i.e. if $\psi \circ f \circ \phi^{-1}$ is differentiable at $\phi(p)$. We define its Jacobian $DF(p) := D(\psi \circ F \circ \phi^{-1})(\phi(p))$ and we denote

$$\frac{\partial F_j}{\partial x_i}(p) = \frac{\partial (y_j \circ F \circ \phi^{-1})}{\partial x_i}(\phi(p)) \tag{2.3.1}$$

{eq:derivative_o

This definition of differentiabilty is independent of the chart. Let $(\tilde{U}, \tilde{\phi})$ with $p \in \tilde{U}$ be another chart and analogous $(\tilde{V}, \tilde{\psi})$ be local coordinate at $F(p)$. Then

$$\tilde{\psi} \circ F \circ \tilde{\phi}^{-1} = \tilde{\psi} \circ \psi^{-1} \circ \psi \circ F \circ \phi^{-1} \circ \phi \circ \tilde{\phi}^{-1}$$

and since the chart transitions are at least C^1-diffeomorphisms by definition $\tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$ is differentiable as well. It is important to notice though, that the values of the Jacobian and the derivatives depend on the chosen representation.

These type of definitions via local charts on a manifold are frequent in differential geometry. This is a proper definition if it is independent of the chosen chart. Because we do not want to bother with the technicalities of differential geometry too much we will very often leave out these types of proofs.

Definition 2.3.12 (Tangent space)**.** Let $I \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma : I \to M$ be a differentiable curve with $\gamma(0) = p \in M$. If 0 is on the boundary of I then we mean the one sided derivative. Let $f : M \to \mathbb{R}$ be a differentiable function. Let (ϕ, U) be a local chart. We define the the directional derivative $D_\gamma(f) := \frac{d}{dt} f(\gamma(t))|_{t=0}$. We call the functional $D_\gamma : C^1(U) \to \mathbb{R}$ a *tangent vector*. The real vector space of all tangent vectors at p is called the *tangent space* and denoted by T_pM

This begs the question why T_pM is actually a vector space. Let (U, ϕ) again be a local chart at p. We can express a tangent vector D_γ in local coordinates by

$$\begin{aligned} D_\gamma(f) &= \frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \frac{d}{dt} (f \circ \phi^{-1} \circ \phi)(\gamma(t))\Big|_{t=0} && (2.3.2) \\ &= \sum_{i=1}^{k} \frac{\partial (f \circ \phi^{-1})}{\partial x_i} (\phi(p))\, (x_i \circ \gamma)'(0) = (\sum_{i=1}^{k} v_i \frac{\partial}{\partial x_i}\Big|_p)(f). && (2.3.3) \end{aligned}$$

Thus we can express

$$D_\gamma = \sum_{i=1}^{k} (x_i \circ \gamma)'(0)\, \frac{\partial}{\partial x_i}\Big|_p .$$

We will now leave out the reference to p in the partial derivative. For the other direction take $\mathbf{v} = (v_1, v_2, ..., v_n)^\top \in \mathbb{R}^n$. We now want to find a differentiable curve γ s.t. $\gamma(0) = p$ and $D_\gamma = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$. For that define

$\gamma := \phi^{-1}(\phi(p) + \mathbf{v}t)$. Then

$$D_\gamma = \sum_{i=1}^{n} (x_i \circ \gamma)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} (x_i(p) + v_i t)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}.$$

So we have shown

$$T_pM = \operatorname{span} \left\{ \frac{\partial}{\partial x_i}\bigg|_p \right\}_{i=1}^{n}.$$

In order to show that this is a basis we have to prove linear independence. Assume we have $\sum_{i=1}^{n} \lambda_i \, \partial/\partial x_i|_p = 0$ Then because $x_j \circ \phi^{-1}(x) = x_j$ for $x \in \phi(U)$ and $1 \leq j \leq n$. Note x_j denote here the variable of $\mathbb{R}^n$ and the local coordinate $x_j : U \to \mathbb{R}$. Of course, this is a slight abuse of notation, but this a standard convention in differental geometry. Then we have

$$0 = \left(\sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i} \right) (x_j) = \sum_{i=1}^{n} \lambda_i \frac{\partial x_j}{\partial x_i} (\phi(p)) = \lambda_j$$

so $\frac{\partial}{\partial x_i}$ are linearly independent and thus a basis of T_pM.

Now that we introduced tangent spaces we will define the most important mapping between them.

Definition 2.3.13 (Differential)**.** Let M, N be an n and m dimensional manifold respectively. Take $p \in M$ and let $F : M \to N$ be differentiable at p. Then we define the differential of F at point p,

$$F_{*,p} : T_pM \to T_{F(p)}N, D_\gamma \mapsto D_{F \circ \gamma}.$$

Proposition 2.3.14. *Let $\{x_i\}_{i=1}^{m}$ and $\{y_j\}_{j=1}^{n}$ be local coordinates at p and $F(p)$ and let $\frac{\partial}{\partial x_i}|_p$ and $\frac{\partial}{\partial y_j}|_{F(p)}$ be the resulting bases for the tangent spaces at T_pM and $T_{F(p)}N$ respectively. Then the resulting matrix representation of the differential $F_{*,p} : T_pM \to T_{F(p)}N$ is the Jacobian as defined above so*

$$F_{*,p}\left(\frac{\partial}{\partial x_i}|_p \right) = \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i}(p) \, \frac{\partial}{\partial y_j}|_{F(p)}.$$

Proof. We will omitt the reference to the points of the partial derivatives. We choose $\gamma = \phi^{-1}(\phi(p) + e_i t)$. Then $D_\gamma = \frac{\partial}{\partial x_i}$ and by applying the chain rule and applying the above definitions, we compute

$$
\begin{aligned}
F_*(\frac{\partial}{\partial x_i})(f) = D_{F\circ\gamma}(f) &= \frac{d}{dt}(f(F\circ\gamma(t)))|_{t=0} \\
&= \frac{d}{dt}(f\circ\psi^{-1}\circ\psi\circ F\circ\phi^{-1}\circ\phi\circ\gamma(t))|_{t=0} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial f\circ\psi^{-1}}{\partial y_j}(\psi(F(p)))\frac{\partial(\psi\circ F\circ\phi^{-1})_j}{\partial x_i}(\phi(p))(x_i\circ\gamma)'(0) \\
&= \sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial f}{\partial y_j}(F(p)) = \left(\sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial y_j}\right)(f).
\end{aligned}
$$

□

For everything we did above we fixed one certain chart (U, ϕ) at p. Now the question arises what happens when we choose a different chart $(\tilde{U}, \tilde{\phi})$ instead. Going through the same steps we end up with the basis $\{\frac{\partial}{\partial y_j}\}_{j=1}^n$ which are the derivatives w.r.t. the chart $(\tilde{U}, \tilde{\phi})$. This is also a basis of the tangent space T_pM. Let us compute the change of basis. Using the chain rule we can easily compute that

$$
\frac{\partial(f\circ\phi^{-1})}{\partial x_i}(\phi(p)) = \sum_{j=1}^{n}\frac{\partial(f\circ\tilde{\phi}^{-1})}{\partial y_j}(\tilde{\phi}(p))\frac{\partial(\tilde{\phi}\circ\phi^{-1})_j}{\partial x_i}(\phi(p))
$$

and we recognize that the change of basis matrix is the Jacobian of the chart transition $D(\psi\circ\phi^{-1})(\phi(p))$.

A *vector field* X maps every point p to a tangent vector in the corresponding tangent space i.e. by using local coordinates

$$
X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i}
$$

with $X_i(p) \in \mathbb{R}$. The regularity of a vector field is defined via the regularity of its coefficent e.g. a vector field is differentiable if all its coefficents are. We must be careful though since we require higher regularity of the manifold for this to be well-defined. Assume X is differentiable i.e. X_i are differentiable. Let $(\tilde{U}, \tilde{\phi})$ be another local chart. From the change of basis of the tangent space we know that

$$
X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i} = \sum_{i,j=1}^{n} X_i(p)\frac{\partial(\tilde{\phi}\circ\phi)_j}{\partial x_i}(\phi(p))\frac{\partial}{\partial y_j}
$$

and we see the coefficents w.r.t. to the new basis are $\tilde{X}_j = \sum_{i=1}^n X_i(p)\frac{\partial(\tilde{\phi}\circ\phi)_j}{\partial x_i}(\phi(p))$. So we need $\frac{\partial(\tilde{\phi}\circ\phi)_j}{\partial x_i}$ to be differentiable as well which means that we want the manifold to be at least C^2.

We should briefly mention the question of orientation of the tangent spaces. Recall that we partitioned the bases of a finite-dimensional real vector in two orientations. We say that two bases have the same orientation if the change of basis matrix has positive determinant. We want to choose an orientation on the tangent spaces consistently which will be crucial when defining the volume form below.

Assume the manifold M is orientable and we have chosen an oriented atlas. For any tangent space T_p with local coordinates x_i near p, we define the resulting basis $\frac{\partial}{\partial x_i}$ as positively oriented which fixes the orientation of the vector space. We have to show that this is well-defined. But this is clear since if we take a different chart at p from the oriented atlas resulting in different local coordinates $\frac{\partial}{\partial y_j}$ we know that the change of basis is the Jacobian of the chart transition. But the Jacobian of the chart transition has positive determinant by definition and so the basis $\frac{\partial}{\partial y_j}$ is positively oriented as well.

2.4 Differential forms

{sec:differentia

Now that we introduced the necessary objects from differential geometry, we can finally define differential forms.

Definition 2.4.1 (Differential forms)**.** A differential k-form ω maps any point $p \in M$ to a alternating k-linear form $\omega_p \in \mathrm{Alt}^k\, T_pM$. We denote the space of differential k-forms on M as $\Lambda^k M$.

Let T_p^*M be the dual space of T_pM which is usually called *cotangent space*. As before let us choose a local chart $\phi : U \to \mathbb{R}^n$ with $p \in U$ and define $\frac{\partial}{\partial x_i}|_p$ as before. Denote the corresponding dual basis as dx^i, $i = 1, ..., n$ i.e. $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. From the consideration about alternating maps from section 2.1 we can now write any $\omega \in \Lambda^k M$ with

$$\omega_p = \sum_{1\leq i_1<...<i_k\leq n} a_{i_1,...,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

with $a_{i_1,...,i_k}(p) \in \mathbb{R}$. The regularity of differential forms is then defined via the regularity of these coefficents i.e. we call a differential form smooth if all the $a_{i_1,...,i_k}$ are smooth and we call a differential form differentiable if all the $a_{i_1,...,i_k}$ are differentiable and so on. These definition of regularity again require the manifold to be sufficiently regular as well.

We denote the space $C^\infty\Lambda^k M$ the space of smooth differential k-forms and analogous for other regularity. $C_0^\infty\Lambda^k(M)$ are the smooth differential forms with compact support contained in M i.e. $\operatorname{supp}\omega = \overline{\{p \in M \mid \omega_p \neq 0\}} \subseteq M \setminus \partial M$ where the closure is w.r.t. the topology on M.

In order to define the Hodge star and an inner product on differential forms we need that T_pM is an inner product space. A Riemannian metric gives us at every point $p \in M$ a symmetric, positive definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$.

Definition 2.4.2 (Riemannian metric)**.** A Riemannian metric g maps every point $p \in M$ to a symmetric, positive definite bilinear form $g_p : T_p \times T_p \to \mathbb{R}$. After choosing local coordinates x_i we frequently denote $g_{p,ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. We also require that for C^α vector fields X and Y the map $g(X,Y)$ is also C^α.

Manifolds on which a Riemannian metric can be defined are called *Riemannian manifolds.*

From now on, we will leave out the reference to the point p where appropriate. The Riemannian metric provides us with the inner product on every tangent space T_pM. We will from now on assume that M is an oriented Riemannian manifold of sufficient regularity and denote the Riemannian metric by g. Let $p \in M$ and T_pM be the tangent space at the point p. Due to our assumptions on M, this is an oriented inner product space of dimension n and we can apply all of the constructions from the previous chapter. We will define the volume form, vector and scalar proxies and finally pullbacks of differential forms.

Let us fix a point p and a chart ϕ at this point with local coordinates denoted by x_i, $i = 1, ..., n$. First, we have to check that we choose a orientation on T_pM as explained above.

The resulting gramian matrix is $(G_p)_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. So we have a volume form vol on M

$$\mathrm{vol}_p = \sqrt{\det G_p}\, dx^1 \wedge ... \wedge dx^n.$$

The volume form then depends on the chosen orientation, but not on the chosen local coordinates.

Now that we have a volume we can define the Hodge star operator $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$ simply by applying it pointwise i.e. $(\star\omega)_p = \star\omega_p$. We follow the analogous idea to define the wedge product $\wedge : \Lambda^k M \times \Lambda^l M \to \Lambda^{k+l} M$ via

$$(\omega \wedge \nu)_p = \omega_p \wedge \nu_p.$$

Recall from Prop. 2.1.7 that we then have for $\omega, \nu \in \Lambda^k M$ and $\mu \in \Lambda^{n-k} M$

$$
\begin{aligned}
g_p(\omega_p, \mu_p) &= \langle \star\omega_p, \mu_p \rangle_{\mathrm{Alt}^{n-k} T_pM} \operatorname{vol}_p \\
\star \star \omega &= (-1)^{k(n-k)} \omega \\
\omega_p \wedge \star\nu_p &= \langle \omega_p, \nu_p \rangle_{\mathrm{Alt}^k T_pM}.
\end{aligned}
$$

In order for the Hodge star to be well-defined the assumption of an orientation on our manifold is crucial.

We want to apply two important concepts from the previous section about alternating maps – vector proxies and pullbacks – to differential forms. The following is based on [2, Ch. 6], but many details have been added and additional examples are given. Recall, that for a real n-dimensional vector space V we had two ways to identify a vector $v \in V$ with an alternating map. Either as a linear form Φv where Φ is the Riesz isomorphism or as a $(n-1)$-linear alternating map $\star\Phi v$.

A vector field X maps every point $p \in M$ to a tangent vector $X(p) \in T_pM$. We can now identfy every vector field with a 1-form or a $(n-1)$-form. $p \mapsto \Phi_{T_pM} X(p)$ defines a 1-form and $p \mapsto \star\Phi_{T_pM} X(p)$ gives us a $(n-1)$-form. In differential geometry, the usual notation is $\Phi_{T_pM} X(p) = X^\flat(p)$. The inverse of $^\flat$ is $^\sharp$ i.e. $X = (X^\flat)^\sharp$. The isomorphisms $^\flat$ and $^\sharp$ are fittingly called *musical isomorphisms*. With these musical isomorphisms we can identify X with the 1-form $X^\flat$ or the $(n-1)$-form $\star X^\flat$. Vice versa, we find for $\omega \in \Lambda^1 M$ the *vector proxy* $\omega^\sharp$ and for and $(n-1)$-form $\nu \in \Lambda^{n-1} M$ we get $(\star^{-1}\nu)^\sharp$.

Denote with dx^i, $i = 1, ..., n$ the dual basis of $\frac{\partial}{\partial x_i}$. If the basis of the tangent space $\frac{\partial}{\partial x_i}$ are orthonormal then we have $(\frac{\partial}{\partial x_i})^\flat = dx^i$ because the Riesz isomorphism maps basis elements to their dual basis elements in this case.

Next, let us have a look at how we can extend pullbacks to differential forms. Recall again, that a linear map $L : V \to W$ with an n-dimensional real vector space V and an m-dimensional real vector space W we define its pullback $L^* : \mathrm{Alt}^k W \to \mathrm{Alt}^k V$ via

$$L^*\omega(v_1, ..., v_k) = \omega(Lv_1, ..., Lv_k).$$

We wish to do the analogous thing with differential forms.

Definition 2.4.3. For $\omega \in \Lambda^k N$ and a differentiable map $F : M \to N$ we define the pullback $F^*\omega$ as

$$(F^*\omega)_p = (F_{*,p})^* \omega_{F(p)}$$

or written differently for all $v_1, ..., v_k \in T_pM$

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(F_{*,p}v_1, ..., F_{*,p}v_k).$$

Now we can use the vector proxies and connect it to what we have done for alternating maps above.

Proposition 2.4.4. *So let X be a vector field on N. Take $p \in M$ and local coordinates x_i at p and y_j at $F(p)$. We assume that the bases $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ and $\{\frac{\partial}{\partial y_j}\}_{j=1}^m$ of T_pM and $T_{F(p)}N$ are orthonormal w.r.t. the Riemannian metrics on M and N respectively. Then we can write $X = \sum_{j=1}^m X_i \frac{\partial}{\partial x_i}$ and denote $\mathbf{X} = (X_1, ..., X_n)^\top$. Then we obtain for the pullback in terms of vector proxies of 1-forms*

$$(F^* X^\flat)^\sharp = \sum_{j=1}^n (DF(p)^\top \mathbf{X})_j \frac{\partial}{\partial y_j}.$$

If we assume $m = n$ additionally, then

$$(\star^{-1} F^* \star X^\flat)^\sharp = \sum_{j=1}^n (ad(DF(p))\mathbf{X})_j \frac{\partial}{\partial y_j}$$

where $ad(DF(p))$ is the adjugate matrix of $DF(p)$.

Proof. The resulting follows essentially immediately from applying the corresponding results for alternating maps. Using the definition of the $^\flat$ and $^\sharp$

$$(F^* X^\flat)^\sharp(p) = \Phi^{-1}_{T_pM} F^*_{*,p} \Phi_{T_{F(p)}N} X(p) = \sum_{j=1}^n (DF(p)^\top \mathbf{X})_j \frac{\partial}{\partial y_j}$$

where we used in the last step the matrix representation of vector proxies of 1-linear forms from (2.2.2). The analogous reasoning works for vector proxies of $(n-1)$-forms i.e. the second claim. □

Let $\hat{\Omega} = M$ and $\Omega = N$ be domains of $\mathbb{R}^n$ and assume F is a diffeomorphism. Then we can identify a vector field $\sum_{i=1}^m X_i \frac{\partial}{\partial x_i}$ with $\mathbf{X} = (X_1, ..., X_n)^\top$. We then can interpret $\mathbf{X}$ either as a vector proxy of a 1- or $(n-1)$-form and obtain the pullbacks

$$\mathcal{P}_F^1 \mathbf{X}(\hat{x}) := DF(\hat{x})\mathbf{X}(F(\hat{x})) \text{ and}$$
$$\mathcal{P}_F^{n-1} \mathbf{X}(\hat{x}) := (\det DF(\hat{x}))\, DF(\hat{x})^{-1}\mathbf{X}(F(\hat{x}))$$

where we used the fact that F is a diffeomorphism hence its Jacobian is invertible and then $\mathrm{ad}(DF(\hat{x})) = (\det DF(\hat{x}))\,(\hat{x})^{-1}\mathbf{X}(F(\hat{x}))$. This is widely known as the Piola transformation [5, Def. 9.8].

Now let us move on to scalar proxies. So let $\rho : N \to \mathbb{R}$ be just a scalar field i.e. a 0-form. In this case, we have the simple expression for the pullback $F^*\rho = \rho \circ F$.

But ρ could also be the scalar proxy of the n-form $\star\rho = \rho\,\mathrm{vol}_N$ (we assume once again as above $n = m$ and that our basis of the tangent spaces are orthonormal w.r.t. the Riemannian metric). Then in scalar proxies we obtain the pullback

$$\begin{aligned}
(\mathcal{P}^n_F\rho)(p) &= (\star^{-1}F^* \star \rho)_p = \star^{-1}(F^* \star \rho)_p = \star^{-1}(F_{*,p})^*(\star\rho)_{F(p)} \\
&= \star^{-1}(F_{*,p})^*\rho(F(p))dy^1 \wedge ... \wedge dy^n \\
&= \star^{-1}\rho(F(p)) \det DF(p)\, dx^1 \wedge ... \wedge dx^n \\
&= (\rho \circ F)(\det DF) \star^{-1} \mathrm{vol}_M = (\rho \circ F)(p) \det DF(p).
\end{aligned}$$

This is strikingly similar to the integrand in the standard transformation of integrals formula which will become crucial in the next section where we talk about the integration of differential forms. The next part of this section will be concerned with introducing a derivative for differential forms

Definition 2.4.5 (Exterior derivative)**.** Let $\omega \in C^1\Lambda^k(M)$ be given in local coordinates as

$$\omega_p = \sum_{1\leq i_1<...<i_k\leq n} a_{i_1,...,i_k}(p)dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

Then we define the exterior derivative $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$. By

$$(d\omega)_p = \sum_{1\leq i_1<...<i_k\leq n} \sum_{i=1}^{n} \frac{\partial a_{i_1,...,i_k}}{\partial x_i}(p)dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

We remind of that the derivative of $a_{i_1,...,i_k}$ is meant w.r.t. to local coordinates as defined at (2.3.1).

It can be shown that $d\omega$ is independent of the chosen coordinates. Let us mention some important properties of the exterior derivative. At first, we have $d \circ d = 0$ which will be important later on when we talk about cochain complexes in Section **??**.

The relation to the wedge product is described by a Leibniz-type formula. Let $\omega \in C^1\Lambda^kM$ and $\nu \in C^1\Lambda^lM$. Then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^l\omega \wedge d\nu. \qquad (2.4.1)$$ {eq:leibniz_form

The exterior derivative commutes with pullback i.e. for manifolds M and N a differentiable mapping $F : M \to N$ and $\omega \in \Lambda^kN$ we have $dF^*\omega = F^*d\omega$.

In terms of proxies this is related to very interesting results as we will see later.

Let us investigate the exterior derivative in the case when $M = \Omega \subseteq \mathbb{R}^n$ is an open subdomain. It turns out that by using scalar and vector proxies as introduced above we can identify the exterior derivative with well-known differential operators. We will use standard Euclidian coordinates meaning that our tangent basis $\frac{\partial}{\partial x_i}$ is orthonormal.

Let us start with a differentiable function $f : \Omega \to \mathbb{R}$ i.e. f is a 0-form. Then

$$(df)^\sharp = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \right)^\sharp = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

which we identify with the gradient ∇f. In other words, the exterior derivative is just the gradient in vector proxies.

Let $\mathbf{X}$ be a differentiable vector field on Ω with components X_i. This corresponds to the vector field $X = \sum_i X_i \frac{\partial}{\partial x_i}$. Let us view X as the vector proxy of an $(n-1)$-form. Then

$$\begin{aligned}
\star^{-1} d \star X^\flat = \star^{-1} d \star \sum_{i=1}^{n} X_i dx^i &= \star^{-1} d \sum_{i=1}^{n} X_i (-1)^{i-1} dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\
&= \star^{-1} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} (-1)^{i-1} dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\
&= \star^{-1} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} (-1)^{2(i-1)} \operatorname{vol} = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = \operatorname{div} \mathbf{X}.
\end{aligned}$$

The hat symbol used for $\widehat{dx^i}$ means that this term is left out.

In the case $n = 3$ if we identify $\mathbf{X}$ with a 1-form then we obtain using similiar computations that $(\star dX^\flat)^\sharp$ corresponds to $\operatorname{curl} \mathbf{X}$. So in 3D we can identify all the exterior derivatives with known differential operators and thereby putting them into a more general framework.

A very nice conclusion can be seen directly from the above computations. The expressions on the left hand side does not use any coordinates. Hence, we can use any coordinate system we want and can then compute e.g. the divergence in any coordinates we need. Note however that the computations are more cumbersome when the bases are not orthonormal.

Let us give an application of this fact. This example is not taken from the references. We will derive the divergence for arbitrary coordinates. So let $\mathbf{X} : \Omega \to \mathbb{R}^n$ be a differentiable vector field. By identifying this with the vector field $\sum_i X_i \frac{\partial}{\partial x_i}$ we computed above that

$$\operatorname{div} \mathbf{X} \operatorname{vol} = d \star X^\flat.$$

Now let us express $\mathbf{X}$ using different coordinates. Let $\phi : \Omega \to \hat{\Omega}$ be a diffeomorphism. This defines our local coordinates. Then the expression of $\mathbf{X}$ is these curvilinear coordinates is

$$\mathbf{X} = \sum_{j=1}^{n} \tilde{X}_j \frac{\partial \phi^{-1}}{\partial y_j}.$$

In terms of differential geometry, this corresponds to the representation of the vector field $X = \sum_j \tilde{X}_j \frac{\partial}{\partial y_j}$. Let $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)^\top$ and

$$G_{kl} = g(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}) = \frac{\partial \phi^{-1}}{\partial y_k} \cdot \frac{\partial \phi^{-1}}{\partial y_l}$$

Then we compute use the expression of the Hodge star operator for one forms (2.1.8) to compute

$$\begin{aligned}
(\operatorname{div} \mathbf{X}) \operatorname{vol} &= d \star X^\flat = d \star \sum_{j=1}^{n} (G\tilde{\mathbf{X}})_j dy^j \\
&= d \sum_{j=1}^{n} \sum_{k=1}^{n} (G\tilde{X})_j \sqrt{\det G} g^{jk} (-1)^{k-1} dy^1 \wedge dy^2 \wedge ... \wedge \widehat{dy^k} \wedge ... \wedge dy^n \\
&= \sum_{k=1}^{n} \frac{\partial(\sqrt{\det G} \sum_{j=1}^{n} g^{jk} (G\tilde{X})_j)}{\partial y_k} (-1)^{k-1} dy^k \wedge dy^1 \wedge dy^2 \wedge ... \wedge \widehat{dy^k} \wedge ... \wedge dy^n \\
&= \sum_{k=1}^{n} \frac{\partial(\sqrt{\det G} \tilde{\mathbf{X}})_k}{\partial y_k} (-1)^{2(k-1)} dy^1 \wedge dy^2 \wedge ... \wedge dy^k \wedge ... \wedge dy^n \\
&= \left[\frac{1}{\sqrt{\det G}} \sum_{k=1}^{n} \frac{\partial(\sqrt{\det G} \tilde{\mathbf{X}})_k}{\partial y_k} \right] \operatorname{vol}
\end{aligned}$$

and we find the well-known expression for the divergence in general coordinates

$$\operatorname{div} \mathbf{X} = \frac{1}{\sqrt{|\det G|}} \sum_{k=1}^{n} \frac{\partial(\sqrt{|\det G|}\, \tilde{X}_k)}{\partial y_k}.$$

As mentioned above let us investigate the consequences of the commutativity of the exterior derivative and the pullback in terms of vector and scalar proxies. Let $F : \widehat{\Omega} \to \Omega$ be a diffeomorphism with $\widehat{\Omega}, \Omega \in \mathbb{R}^n$. Let us again consider Euclidian coordinates on the domain and codomain. Then observe

$$
\begin{aligned}
&(\operatorname{div}\mathbf{X})(F(\hat{x}))\det DF(\hat{x})\widehat{\mathrm{vol}}_{\hat{x}} = (F^*(\operatorname{div}\mathbf{X}))_{\hat{x}} \wedge (F^*\,\mathrm{vol})_{\hat{x}} \\
&= \big(F^*\big(\operatorname{div}\mathbf{X} \wedge \mathrm{vol}\big)\big)_{\hat{x}} = \big(F^*\big(\operatorname{div}\mathbf{X}\,\mathrm{vol}\big)\big)_{\hat{x}} \\
&= \big(F^* d \star X^\flat\big)_{\hat{x}} = \big(dF^* \star X^\flat\big)_{\hat{x}}.
\end{aligned}
$$

and then

$$dF^* \star X^\flat = d \star \star^{-1} F^* \star X\flat = d \star \hat{X}^\flat = \widehat{\operatorname{div}}\widehat{\mathbf{X}}$$

by defining

$$(\star^{-1} F^* \star X^\flat)^\sharp =: \hat{X} = \sum_i \hat{\mathbf{X}}_i(\hat{x}) \frac{\partial}{\partial \hat{x}_i}$$

which we identify with the vector field

$$\widehat{\mathbf{X}}(\hat{x}) = (\widehat{X}_1(\hat{x}), \widehat{X}_2(\hat{x}), ..., \widehat{X}_n(\hat{x}))^\top$$

We then know from Sec. 2.1 and because we assume that we use orthonormal coordinates that

$$
\begin{aligned}
\hat{X}(\hat{x}) = (\star^{-1} F^* \star X\flat)^\sharp(\hat{x}) &= \sum_{i,j=1}^{n} \operatorname{ad}(DF(\hat{x}))_{ij} X_j(F(x)) \frac{\partial}{\partial \hat{x}_i} \\
&= \sum_{i,j=1}^{n} \det DF(\hat{x}) \big(DF(\hat{x})^{-1}\big)_{ij} X_j(F(x)) \frac{\partial}{\partial \hat{x}_i}
\end{aligned}
$$

and so we identify this with the vector field

$$\widehat{\mathbf{X}} = \det DF(\hat{x}) DF(\hat{x})^{-1}\mathbf{X}.$$

To summarize the situation in 3D,

Proposition 2.4.6. *Let $F : \hat{\Omega} \to \Omega$ be a diffeomorphism, $\rho \in C^1(\Omega)$ and $\mathbf{X} \in C^1(\Omega, \mathbb{R}^3)$. In scalar and vector proxies for 3D, we obtain the following expressions for pullbacks*

$$
\begin{aligned}
(\mathcal{P}_F^0 \rho)(\hat{x}) &= \rho(F(\hat{x})) \\
(\mathcal{P}_F^1 \mathbf{X})(\hat{x}) &= DF(\hat{x})^\top \mathbf{X}(F(\hat{x})) \\
(\mathcal{P}_F^2 \mathbf{X})(\hat{x}) &= \det DF(\hat{x}) DF(\hat{x})^{-1} \mathbf{X}(F(\hat{x})) \\
(\mathcal{P}_F^3 \rho)(\hat{x}) &= \det DF(\hat{x}) \rho(F(\hat{x}))
\end{aligned}
$$

and then the commuting properties

$$\widehat{\text{grad}}\mathcal{P}_F^0\rho = \mathcal{P}_F^1(\text{grad}\,\rho)$$
$$\widehat{\text{curl}}\mathcal{P}_F^1\mathbf{X} = \mathcal{P}_F^2(\text{curl}\,\mathbf{X})$$
$$\widehat{\text{div}}\mathcal{P}_F^2\mathbf{X} = \mathcal{P}_F^3(\text{div}\,\mathbf{X}).$$

We proved the last statement and the other two can be proven analogously. These commuting properties are useful for applications in finite elements (see e.g. [5, Sec. 14.3]).

2.5 Integration of differential forms

Throughout this chapter we assume that M is an orientable Riemannian manifold of dimension n. We assume that we have chosen a countable atlas $\{(U_i,\phi_i)\}_{i=1}^N$ with $N \leq \infty$. Before we can talk about integration of differential forms let us first investigate the integration of functions on a manifold.

2.5.1 Integration of functions on a manifold

We want to define integration in the framework of usual measure and integration theory which means defining it as a Lebesgue integral w.r.t. a measure on M which we have to define first along with a σ-algebra on M.

It is well known that the borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^n$ is generated by all open sets. This idea can be applied to any topological space X by defining the borel σ-algebra $\mathcal{B}(X)$ as the σ-algebra generated by all open sets as well. So we can simply use the topology on M to define our σ-algebra $\mathcal{B}(M)$. Because we know that all our charts $\phi_i : U_i \to \mathbb{R}^n$ are homeomorphisms it is very straightforward to show that a set $E \in \mathcal{B}(M)$ i.i.f. $\phi_i(E \cap U_i) \subseteq \mathbb{R}^n$ is Borel-measurable.

Now, we need to define a measure on the manifold. The following motivation is taken from [7, 3.H.2]. In Euclidian space $\mathbb{R}^n$ we use the standard Lebesgue measure that gives vvolume one to the unit cube. If we now take any vectors $v_1, v_2, ..., v_n \in \mathbb{R}^n$ then the parallelepiped spanned by these vectors has volume $\det(v_1 \mid v_2 \mid \cdots \mid v_n) = \sqrt{\det\big(\langle v_i, v_j\rangle\big)_{1\leq i,j\leq n}}$ where we used the standard inner product on $\mathbb{R}^n$.

If we now extend this idea to Riemannian manifolds then we are interested in the parallelepiped spanned by the tangent vectors $\frac{\partial}{\partial x_i}$ in the tangent space T_pM with $p \in M$. which would then be $\sqrt{\det\big(g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\big)_{1\leq i,j\leq n}} = \sqrt{\det G_p}$. Then we define for $E \subseteq U$ where U is the domain a chart

$\phi : U \to \mathbb{R}^n$ the measure

$$V(E) := \int_{\phi(E)} \sqrt{\det G_{\phi^{-1}(x)}} dx$$

which we will now extend globally to the whole manifold. Let $\{\chi_i\}_{i=1}^N$, $N \leq \infty$, be the partition of unity subordinate to the open cover given by the atlas $\{U_i\}_{i=0}^N$. Then we define the *Riemannian measure* for any $E \in \mathcal{B}(M)$

$$V(E) := \sum_{i=1}^{\infty} \int_{\phi_i(U_i \cap E)} \chi_i(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}(\phi_i^{-1}(x))} dx \in [0, \infty]$$

It can be shown that it is independent of the chosen oriented atlas using the transformation behaviour of $G^{(i)}$. But the orientation is crucial for it to be well-defined.

Proposition 2.5.1. *The Riemannian measure is independent of the chosen partition of unity and atlas if it is oriented the same.*

Proof. Let $\{(V_j, \psi_j)\}_{j=0}^{\infty}$ be a different atlas with the same orientation and $\{\rho_j\}_{j=0}^{\infty}$ be the partition of unity subordinate to it.

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{\psi_j(E \cap V_j)} \rho_j(\psi_j^{-1}(y)) \sqrt{\det G^{(j)}_{\psi^{-1}(y)}} \, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\psi_j(E \cap V_j \cap U_i)} \chi_i(\psi_j^{-1}(y)) \rho_j(\psi_j^{-1}(y)) \sqrt{\det G^{(j)}_{\psi_j^{-1}(y)}} \, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\phi_i(E \cap V_j \cap U_i)} \chi_i(\phi_i^{-1}(x)) \rho_j(\phi_i^{-1}(x)) \sqrt{\det G^{(j)}_{\phi_i^{-1}(x)}} \, \det D(\psi_j \circ \phi_i^{-1})(\phi_i(x)) \, dx
\end{aligned} \tag{2.5.1}$$

{eq:double_sum_r

where we used the fact that the χ_i sum up to one for the first equality and a simple tranformation of integral in the second equality using the chart transition $\phi_i^{-1} \circ \psi_j$. Since all summands are non-negative we can change the order of summation as we like. Then using the change of basis for the tangent space derived in Sec. 2.3 we get for $p \in U_i \cap V_j$

$$\begin{aligned}
(G^{(j)}_p)_{kl} &= g_p(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}) \\
&= \sum_{r,s=0}^{n} \frac{\partial(\phi_i \circ \psi_j^{-1})_r}{\partial y_k}(\psi(p)) \, \frac{\partial(\phi_i \circ \psi_j^{-1})_s}{\partial y_k}(\psi(p)) \, g_p(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_s}) \\
&= \Big(D(\phi_i \circ \psi_j^{-1})(\psi(p))^\top G^{(i)}_p D(\phi_i \circ \psi_j^{-1})(\psi(p)) \Big)_{kl}
\end{aligned}$$

and thus

$$\det G^{(j)}_{\phi^{-1}(x)} = \det G^{(i)}_{\phi_i^{-1}(x)} \left(\det D\big(\phi_i \circ \psi_j^{-1}\big)\big(\psi(\phi^{-1}(x))\big) \right)^2$$
$$= \det G^{(i)}_{\phi_i^{-1}(x)} \left(\det D(\psi_j \circ \phi_i^{-1})(x)^{-1}\right)^2.$$

Plugging this into (2.5.1) the determinant of the chart transition cancels out because $\det D(\psi_j \circ \phi_i^{-1})(x)^{-1} > 0$. We can integrate over $\phi_i(E \cap U_i)$ instead of $\phi_i(E \cap V_j \cap U_i)$ without changing the integral because $\operatorname{supp} \rho_j \subseteq V_j$.

$$\sum_{i=0}^{\infty} \int_{\phi_i(E\cap U_i)} \chi_i(\phi_i^{-1}(x)) \sum_{j=0}^{\infty} \rho_j(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}_{\phi_i^{-1}(x)}}\, dx$$
$$= \sum_{i=0}^{\infty} \int_{\phi_i(E\cap U_i)} \chi_i(\phi_i^{-1}(x)) \sum_{j=0}^{\infty} \rho_j(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}_{\phi_i^{-1}(x)}}\, dx$$
$$= V(E).$$

□

Now that we have the measure space $(M, \mathcal{B}(M), V)$ we can define integration in the usual Lebesgue way. It is easily shown that a function $f : M \to \mathbb{R}$ is measurable i.i.f. $f \circ \phi_i^{-1}$ is measurable for every chart ϕ_i. It is then simply an application of the definition of Lebesgue integration to show that for any measurable $f \geq 0$ we can express the integration as

$$\int_M f\, dV = \sum_{i=1}^{\infty} \int_{\phi_i(U_i)} \chi_i(\phi_i^{-1}(x)) f(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}(\phi_i^{-1}(x))}dx.$$

By introducing the integral as a Lebesgue integral w.r.t. the Riemannian measure we inherit the theoretical framework of Lebesgue integration. For example, we know that the the spaces $L^p(M, V)$ for $1 \leq p < \infty$, i.e. the p-integrable real-valued functions w.r.t. the Riemannian measure, are Banach spaces.

2.5.2 Integration of differential forms

{sec:integration

Once again, we should first ask ourselfs what a measurable differential form should be. We know that we can express our differential form locally for $p \in U_i$ using the local coordinates as

$$\omega_p = \sum_{1\leq i_1<\ldots<i_k\leq n} a_{i_1,\ldots,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

In the spirit of the the definition of differentiability in Sec. **??** we call a $\omega \in \Lambda^k(M)$ measurable if for every chart ϕ_i the coefficent functions of the differential form expressed in local coordinates are measurable. This notion is once again independent of the chosen coordinates.

Next, we will define integration of an n-form over an n dimensional manifold. At first, we do so for an open set $U \subseteq \mathbb{R}^n$. This is the simplest example of an n-dimensional manifold where we only have one chart which is the identify and the local coordinates are just our standard coordinates which we denote by z_i, $i = 1, ..., n$ and the resulting basis of the tangent space $\frac{\partial}{\partial z_i}$. Let ω be a measurable n-form on U so we can write

$$\omega_z = f(z)\, dz_1 \wedge dz_2 \wedge ... \wedge dz_n$$

for $z \in U$ with $f : U \to \mathbb{R}$ being Borel-measurable. We can now simply define

$$\int_U \omega = \int_U f(z)\, dz_1 dz_2 ... dz_n.$$

With this definition at hand we can now extend this definition to the manifold M. As it is often done in differential geometry we will work locally first and then extend this construction globally by using a partition of unity.

Let (U, ϕ) be a chart on M and assume $\operatorname{supp} \omega \subseteq U$. Then $(\phi^{-1})^* \omega$ is a n-form on $\phi(U) \subseteq \mathbb{R}^n$ and we can apply our prior definition. So now we just define

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^* \omega.$$

Now let us move on to the global definition. Let $\{(U_i, \phi_i)\}_{i=1}^{\infty}$ be an oriented atlas and let $\{\chi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to it. Then $\operatorname{supp} \chi_i \omega \subseteq U_i$ and we define

$$\int_M \omega := \sum_{i=1}^{\infty} \int_M \chi_i \omega.$$

We will now have a look how the integration of functions and of differential forms are related to each other. For the chart (ϕ_i, U_i) we denote the local coordinates as $x_k^{(i)}$, $k = 1, ..., n$ and the basis of 1-forms as $dx_{(i)}^k$. We know that for $p \in U_i$ we can write the volume form as

$$\mathrm{vol}_p = \sqrt{\det G^{(i)}(p)}\, dx_{(i)}^1 \wedge dx_{(i)}^2 \wedge ... \wedge dx_{(i)}^n.$$

where $G^(i)_{kl} = g(\frac{\partial}{\partial x^{(i)}_k}, \frac{\partial}{\partial x^{(i)}_l})$.

Because $(\phi_i^{-1})^* dx^k_{(i)} = dz^k$ – which follows directly from the definition – and the pullback commutes with the wedge product we have

$$\big((\phi_i^{-1})^* \operatorname{vol}\big)_z = \sqrt{\det G^{(i)}(\phi_i^{-1}(z))}\, dz^1 \wedge dz^2 \wedge \dots \wedge dz^n.$$

That means for a n-form $f \operatorname{vol}$ we can write the integral as

$$\int_M f \operatorname{vol} = \sum_{i=1}^{\infty} \int_{\phi_i(U_i)} \chi_i(\phi_i^{-1}(z)) f(\phi_i^{-1}(z)) \sqrt{\det G^{(i)}(\phi_i^{-1}(z))} dz^1 dz^2 \dots dz^n$$

and we see

$$\int_M f dV = \int_M f \operatorname{vol}$$

with the two different notions of integration. This also proves that the integral of differential n-forms is independent of the chosen coordinates as long as the orientation is respected. So we see that the the two definitions are essentially equivalent. The big advantage of considering these two approaches is that we know the integration of differential forms is within the framework of Lebesgue integration. It is then also clear how integrability for n-forms should be defined. We call an n-form $f \operatorname{vol}$ integrable if f is integrable w.r.t. the Riemannian measure.

2.5.3 Stokes' theorem and integration by parts

One of the most important results about the integration of differential forms is Stokes' theorem which we will state in this section. From it, we will obtain a integration by parts formula.

But before we do so we have to check how to define the restriction of a differential form to a submanifold $N \subseteq M$. We do that with the inclusion $\iota : N \hookrightarrow M$. Then for a smooth differential form $\omega \in C^\infty \Lambda^k(M)$ we define the restriction just via the pullback of the inclusion operator i.e. $\iota^* \omega \in C^\infty \Lambda^k(N)$.

For a k-dimensional submanifold N we then denote the integration of an k-form ω as

$$\int_N \iota^* \omega =: \int_N \omega.$$

Example 2.5.2. Let us investigate the integral of a 1-form over a curve. Let $\gamma : (0,1) \to \Omega$ be a smooth curve in the domain $\Omega \subseteq \mathbb{R}^n$. Denote $\Gamma = \gamma((0,1))$ and assume that $\gamma' \neq 0$ and $\gamma : (0,1) \to \Gamma$ is bijective. That means especially that Γ does not intersect itself. Then Γ is a manifold with boundary and the chart is γ^{-1}. Then let us calculate the Jacobian of the inclusion ι using the definition from (2.3.1). Since it is the most convenient, we take the identity as chart on Ω. Take $\gamma(t) \in \Gamma$. Then

$$D\iota(p)_{i,1} = D(\mathrm{Id} \circ \iota \circ \gamma)_{i,1}(t) = \frac{\partial \gamma_i}{\partial t}(t) = \gamma'(t).$$

This is then the matrix representation of the differential ι_* at $\gamma(t)$. Now take a at least continuous 1-form $\omega = \sum_{i=1}^n \omega_i dz^i$ on Ω. Then we know from the matrix representation of the pullback derived in Sec. 2.4 that

$$\iota^*\omega(t) = \sum_{i=1}^n \gamma_i'(t)\, \omega_i(\gamma(t))\, dt$$

and so

$$\int_\Gamma \omega = \int_0^1 \sum_{i=1}^n \gamma_i'(t)\, \omega_i(\gamma(t))\, dt.$$

Let us now look at this in terms of vector proxies. Take a vector field $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$ on Ω. Because we are using Euclidian coordinates we get

$$X^\flat = \sum_{i=1}^n X_i dz^i$$

and thus by denoting $\mathbf{X} = (X_1, X_2, ..., X_n)$ we get

$$\int_\Gamma X^\flat = \int_\Gamma \mathbf{X} \cdot dl$$

where the right hand side is the notation for the standard curve integral in $\mathbb{R}^n$. So we can conclude that, interpreted in vector proxies for a domain of $\mathbb{R}^n$, the integral of a 1-form corresponds to the usual curve integral.

Theorem 2.5.3 (Stokes). *Let M be a smooth oriented manifold with boundary ∂M. Let ω be a smooth compactly supported $(n-1)$-form. Then we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This theorem gives us a relation of the boundary and the exterior derivative which will be crucial in the topological context of differential forms which we will investigate in Sec. 3.3.

We can also derive a form of the integration by parts formula from it. Let $\omega \in C^1_0\Lambda^k(M)$ and $\mu \in C^1_0\Lambda^{n-k-1}(M)$. Then $\omega \wedge \mu \in C^1\Lambda^{n-1}(M)$. Recall the Leibniz rule for the exterior derivative

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu.$$

By integrating both sides over M and applying Stokes' theorem we obtain

$$\int_{\partial M} \omega \wedge \mu = \int_M d\omega \wedge \mu + (-1)^k \int_M \omega \wedge d\mu.$$

By using vector and scalar proxies as in Sec. 2.4 we this can be used to prove the well-known formula

$$\int_\Omega u \operatorname{div} \mathbf{F} \, dx = -\int_\Omega \operatorname{grad} u \cdot \mathbf{F} \, dx + \int_{\partial\Omega} u \, \mathbf{F} \cdot \mathbf{n} \, ds$$

where $\mathbf{n}$ is the unit normal, assuming Ω is a bounded Lipschitz domain and $\mathbf{F}$ and u are continuously differentiable or

$$\int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{F} \, dx = \int_\Omega \operatorname{curl} \mathbf{v} \cdot \ \mathbf{F} \, dx - \int_{\partial\Omega} (\mathbf{v} \times \mathbf{F}) \cdot \mathbf{n} \, ds$$

assuming $\mathbf{v}$ is a continuously differentiable vector field.

3 Singular homology

{sec:singular_ho

The curve integral constraint from the magnetostatic problem is very topological in nature and strongly related to the topology of the domain. In order to deal with this constraint and obtain the desired existence and uniqueness we require some tools from algebraic topology which we will introduce in this section. This material is taken from [3] where a lot more details and results can be found.

3.1 Homology groups

Denote with $\mathbb{R}^\infty$ the vector space of all real-valued sequences. Let $e_i \in \mathbb{R}^\infty$ for $i \in \mathbb{N}$ denote the sequences that that are zero for every index unequal to i and 1 for the index i. Note that in this thesis the natural numbers start at zero. Then we define the standard k-simplex Δ_k as

$$\Delta_k := \Big\{ \sum_{i=1}^k \lambda_i e_i \mid \sum_{i=0}^k \lambda_i = 1, \ 0 \le \lambda_i \le 1 \Big\} = \operatorname{conv}\{e_0, \ldots, e_k\}.$$

where conv is the usual convex combination.

Definition 3.1.1 (k-simplex)**.** Let X be a topological space. Then a *singular k-simplex* is a continuous map $\sigma_k : \Delta_k \to X$. We will frequently leave out the term 'singular' and refer to them just as k-simplices.

As the term 'singular' implies these simplices can be degenerated. For example, σ_k could just be constant for any k, so the object in the topological space corresponding to the k-simplex is just a point.

We can now introduce a algebraic structure by looking at finite formal sums of the form

$$\sum_{\sigma\ k\text{-simplex}} n_\sigma \sigma.$$

These formal sums form an abelian group which we refer to as the *singular k-chain group $C_k(X)$*.

We will now introduce an important homomorphism between these groups called the *boundary*.

Definition 3.1.2. Let $v_0, ..., v_k \in \mathbb{R}^n$. We define *affine singular k-simplex* as a special singular k-simplex denoted by

$$[v_0, ..., v_k] : \Delta_k \to \mathbb{R}^n, \sum_{i=0}^{k} \lambda_i e_i \mapsto \sum_{i=0}^{k} \lambda_i v_i.$$

As in the general case, the image can be a degenerated simplex in $\mathbb{R}^n$ since the v_i are not assumed to be affine independent.

We call the affine singular simplex $[e_0, ..., \hat{e}_i, ..., e_k] : \Delta_{k-1} \to \Delta_k$ the i-th face map. The $\hat{}$ means this vertex is left out. Here we tacitly used the natural inclusion $\mathbb{R}^{k+1} \subseteq \mathbb{R}^\infty$ so we have $\Delta_k \subseteq \mathbb{R}^{k+1}$. But this is just a way of representation.

With the face map we can now define the boundary operator.

Definition 3.1.3. For a singular k-simplex $\sigma : \Delta_k \to X$ we define its i-th face $\sigma^{(i)} := \sigma \circ F_i^k$ which is a $(k-1)$-simplex. We then define the *boundary* of σ as $\partial_k \sigma := \sum_{i=0}^{k} (-1)^i \sigma^{(i)}$. We extend this to a homomorphism between the chain groups

$$\partial_k : C_k(X) \to C_{k-1}(X), \sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma \partial_k \sigma.$$

In the case of $k = 0$, we set $\partial_0 = 0$.

We will frequently leave out the subscript and just write ∂ for the boundary if it is clear from the context.

A straightforward computation (cf. [3, Lemma 1.6]) shows the important property

$$\partial_k \circ \partial_{k+1} = 0.$$

This property implies that $\ker \partial_{k-1} \subseteq \operatorname{im} \partial_k$ is a subgroup. We call a chain $c \in C_k(X)$ *k-cycle* if $\partial_k c = 0$ and we call it *k-boundary* if $c \in \operatorname{im} \partial_{k+1}$. Denote the group of k-cycles as $Z_k(X)$ and the k-boundaries as $B_k(X)$. Since we are in the abelian setting this motivates us to define the resulting factor groups.

Definition 3.1.4 (Homology groups)**.** We define the *k-th homology group* of the topological space X as

$$H_k(X) := {Z_k(X)}\big/{B_k(X)}.$$

We denote the elements of the homology groups i.e. the equivalence classes of a k-cycle c as $[c] \in H_k(X)$.

If the k-th homology groups is finitely generated then we call the rank i.e. the number of generators the *k-th Betti number*. These Betti numbers are fundamental properties of the topological space. For example, the zeroth Betti number corresponds to the number of path-components of the space. In 3 dimensions, the first Betti number of a compact domain corresponds to the number of "holes", the second Betti number to number of enclosed "voids" in the domain. E.g. a filled torus has the zeroth Betti number one, the first Betti number also equal to one and the second equal to zero which can be proven using the Meyer-Vietoris sequence (see [3, Sec. IV.18]). We will not go into this in further since we do not want to focus too much on algebraic topology.

This construction can be put in an abstract algebraic framework in the following way. We call a collection of abelian groups C_i, $i \in \mathbb{Z}$ a graded group. Together with a collection of homomorphims $\partial_i : C_i \to C_{i-1}$ called *differentials* s.t. $\partial_{i-1} \circ \partial_i$ this is called a *chain complex* which we will denote by C_*.

Example 3.1.5. If we set $C_k(X) = 0$ for $k < 0$ this gives us a chain complex together with the boundary operator.

Completely analogous to above, we can define the homology groups

$$H_k(C_*) := {\ker \partial_k}\big/{\operatorname{im} \partial_{k+1}}.$$

Definition 3.1.6 (Chain map)**.** Let A_* and B_* be chain complexes. With a slight abuse of notation let us denote the differentials of both chain complexes just by ∂. Then a *chain map* $f : A^* \to B^*$ is a collection of homomorphisms $f_i : A_i \to B_i$ s.t. $f_{i-1} \circ \partial = \partial \circ f_i$.

Include commuting diagram. We will most times leave out the indices if it is clear what we mean. The crucial property of these chain maps is that they induce homomorphisms of the homology groups denoted as

$$[f_i] : H_i(A_*) \to H_i(B_*),\ [f_i]([a]) = [f_i([a])]$$

Let $\{C^i\}_{i\in\mathbb{N}}$ be a collection of abelian groups and homomorphisms $\partial^i : C^i \to C^{i+1}$ with $\partial^{i+1} \circ \partial^i = 0$ called *codifferentials*. Then we call this sequence a *cochain complex*. The only difference to chain complexes is that the index increases when applying the codifferential. Hence, they are basically the same from an algebraic point of view. By convention, we use superindices for anything that is related to cochain complexes.

We define *cochain maps* completely analogous to chain maps i.e. cochain maps commute with the codifferential.

The main motivation for cochain complexes comes from the *singular cochain complexes* which we will define in the next section.

Example 3.1.7 (De Rham complex)**.** Smooth differential forms provide us with another very important example. Let M be a smooth manifold. We will use the the notation introduced in Sec. 2.4. Then the smooth differential forms $C^\infty\Lambda^k(M)$ together with the exterior derivative give us a cochain complex which we call *de Rham complex*. Note that we have slightly more structure here since the $C^\infty\Lambda^k(M)$ are vector spaces and the exterior derivative a linear map i.e. a vector space homomorphism.

It turns out that the de Rham complex is closely related with the singular cochain complex. This relation will be investigated later in Sec. 3.3.

3.2 Cohomology groups

Let G be any abelian group and X be a topological space as before. Then we define the group of *k-cochains* $C^k(X; G)$ by

$$C^k(X; G) := \mathrm{Hom}(C_k(X), G)$$

i.e. the group of all homomorphisms from k-chains $C_k(X)$ to G. We generally use the superindex ^{k} if something is related to cochains and the subindex $_k$ if it is related to chains. Just as for chains we now introduce a homomorphism between the groups of cochains which transforms this into a cochain complex.

Definition 3.2.1 (Coboundary). We define the operator $\partial^k : C^k(X;G) \to C^{k+1}(X;G)$ via

$$(\partial^k f)(c) := f(\partial_{k+1} c).$$

for a $(k+1)$-chain c. We call a cochain $f \in C^k(K;G)$ *closed* if $\partial^k f = 0$ and we call f *exact* if there is a $g \in C^{k-1}(K;G)$ s.t. $f = \partial^{k-1} g$. As for the boundary map we will frequently leave away the superscript if the context is clear.

From the definition it is obvious that $\partial^{k+1} \circ \partial^k = 0$ and thus we have indeed a cochain complex which we call *singular cochain complex.* If there is no confusion with the general notion of cochain complex we will leave away the term 'singular'.

Definition 3.2.2 (Cochain cohomology). Denote the closed k-cochains as $Z^k(X;G)$ and the exact ones with $B^k(X;G)$. We then define the *cochain cohomology groups* $H^k(X;G)$ as

$$H^k(X;G) := {Z^k(X;G)}\big/{B^k(X;G)}.$$

Note that in the case of $G = \mathbb{R}$ this becomes a vector space.

Now of course there is the question how the homology and cohomology groups are related to each other. This question is answered by the *universal coefficent theorem.* But before we can formulate it we have to introduce exact sequences.

Definition 3.2.3 (Exact sequence). Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. Then this sequence of homomorphisms is called *exact* if $\operatorname{im} f_{i-1} = \ker f_i$.

The universal coefficent theorem in the case of simplicial homology states that the sequence

$$0 \to \operatorname{Ext}(H_{k-1}(K), G) \to H^k(K;G) \xrightarrow{\beta} \operatorname{Hom}(H_k(K), G) \to 0 \qquad (3.2.1)$$ {eq:univeral_coe

is exact. β is defined via

$$\beta([F])([c]) := F(c). \qquad (3.2.2)$$ {eq:isomorphism_

The definition of Ext can be found in [3], but it does not matter for our purpose because from now on we will assume $G = \mathbb{R}$ and $\operatorname{Ext}(H_{k-1}(X), \mathbb{R}) = 0$. This follows from the fact that $\mathbb{R}$ is a divisible and hence injective abelian group. The definition of these terms and the connections used can also be found in [3, Sec. V.6]. However, we will not dwelve into the algebraic background further. In the case of $G = \mathbb{R}$, we can conclude from the exactness of the above short sequence that $\ker \beta = 0$ and $\operatorname{im} \beta = \operatorname{Hom}(H_k(X), \mathbb{R})$. So β is an isomorphism.

3.3 De Rham's theorem

{sec:de_rhams_th

It turns out that the cochain cohomology is closely related to the cohomology of differential forms (**??**). Let us recall Stokes' theorem first which said that for a k-form $\omega \in C_0^\infty\Lambda^k(M)$ for a smooth k-dimensional oriented manifold M we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

The following details are taken from Section V.5 and and V.9 from [3]. We will only focus on the main ideas and avoid dwelving into the technical details. The interested reader can find more arguments in the given reference.

Let now σ be a smooth k-simplex i.e. $\sigma : \Delta_k \to M$ is smooth. We now define

$$\int_\sigma \omega = \int_{\Delta_k} \sigma^*\omega$$

and then we define the integral over a k-chain $c = \sum_\sigma n_\sigma \sigma$

$$\int_c \omega := \sum_\sigma n_\sigma \int_\sigma \omega.$$

This motivates us to introduce the homomorphism $I : C^\infty\Lambda^k(M) \to \mathrm{Hom}(C_k;\mathbb{R})$ defined by

$$I(\omega)(c) = \int_c \omega.$$

Remark 3.3.1. There are some technical details that we will not discuss in details here, but that should be mentioned. First, Δ_k is not a manifold. The $(k-2)$ skeleton are, simply speaking, the simplices of dimension $(k-2)$ i.e. the corners for $k = 2$ and the edges for $k = 3$. If we remove the $(k-2)$ skeleton then Δ_k is a manifold with boundary. But since this is a null-set w.r.t. the full simplex and the boundary as well, this does not matter for our arguments. Second, Because we are integrating over the Δ_k their orientation is important and has to be chosen consistently. We will not present here how this is done and will just assume it from now on.

Using Stokes's theorem and the fact that the exterior derivative commutes with the pullback we observe

$$\begin{aligned} I(d\omega)(c) &= \sum_\sigma n_\sigma \int_\sigma d\omega = \sum_\sigma n_\sigma \int_{\Delta_k} \sigma^* d\omega = \sum_\sigma n_\sigma \int_{\Delta_k} d\sigma^*\omega = \sum_\sigma n_\sigma \int_{\partial\Delta_k} \sigma^*\omega \\ &= \int_{\sum_\sigma n_\sigma \partial\sigma} \omega = I(\omega)(\partial c) = \partial\big(I(\omega)\big)(c) \end{aligned}$$

so we obtain

$$I(d\omega) = \partial\big(I(\omega)\big).$$

We see that I is a cochain map and thus induces a homomorphism on cohomology

$$[I] : H^k_{dR}(M) \to H^k(M;\mathbb{R}).$$

Using the notation and the definition of this map we can now formulate de Rham's theorem which will become very important later when proving existence and uniqueness in Sec. 5.

Theorem 3.3.2 (De Rham's theorem)**.** $[I]$ *is an isomorphism.*

4 Hilbert complexes

Another crucial tool for the proof will be the *Hodge decomposition* in 3D which relies on unbounded operators and Hilbert complexes. These will be introduces in this section. This section is essentially a recollection of the parts of chapter 3 and 4 of Arnold's book [2] which we will need. We will stay close to this source. At certain parts, some additional arguments and steps will be added, but the reference is already very detailed so there is not much to add. Throughout this section it will be assumed that the reader is familiar with the basic theory of Hilbert spaces and bounded linear operators. We will focus on real spaces exclusively.

4.1 Unbounded operators

{sec:unbounded_o

Definition 4.1.1 (Unbounded operators)**.** Let X and Y be Hilbert spaces. Then we call a linear mapping $T : D(T) \to Y$ with a subspace $D(T) \subseteq Y$ an *unbounded operator* from X to Y. We call $D(T)$ the *domain* of T.

We will write sometimes talk about an unbounded operator $T : X \to Y$ which means that T is not necessarily defined on all of X.

Note that this definition generalizes the standard operator. In particular, it includes the case when T is in fact bounded which can be slightly confusing, but we will stick to this common naming convention.

The domain is a crucial property of unbounded operators. We will sometimes denote the unbounded operator as the tuple $(T, D(T))$. If $D(T)$ is dense in X we call T *densely defined.* We say that two unbounded operators T and S from X to Y are equal if $D(T) = D(S)$ and $Tx = Sx$ for all $x \in D(T)$.

The standard example of an unbounded densely defined operator is the classical gradient with the domain $C_0^1(\Omega) \subseteq L^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ i.e. here we have $X = L^2(\Omega)$ and $Y = L^2(\Omega; \mathbb{R}^n)$. In short, grad is an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^1(\Omega)$. We could then denote it as $(\operatorname{grad}, C_0^1(\Omega))$. This also shows that the choice of domain is not unique. We could have instead chosen e.g. the different unbounded operator $(\operatorname{grad}, C_0^\infty(\Omega))$. Another example is the weak gradient with domain $H^1(\Omega)$ i.e. $(\operatorname{grad}, H^1(\Omega))$.

As for bounded operators we define the kernel or null space of an unbounded operator

$$\ker T = \{x \in D(T) \mid Tx = 0\}$$

and the image or range

$$\operatorname{im} T = \{Tx \mid x \in D(T)\}.$$

The only difference is to keep in mind that the unbounded operators are not defined on the whole X in general.

Recall that the graph of a function $f : X \to Y$ is defined as $\{(x, f(x)) \in X \times Y \mid x \in X\}$. Analogously, the graph of an unbounded operator T is

$$\Gamma(T) := \{(x, Tx) \mid x \in D(T)\}$$

which is obviously a subspace of $X \times Y$.

We define the *graph inner product* on $D(T)$ as

$$\langle x, z \rangle_{D(T)} := \langle x, z \rangle_X + \langle Tx, Tz \rangle_Y, \quad x, z \in D(T).$$

It is easy to show that this is indeed an inner product. We will call its induced norm the *graph norm*

$$\|x\|_{D(T)} = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}, \quad x \in D(T).$$

Even though this defines a norm, $D(T)$ might not be a Hilbert space because it is in general not complete w.r.t. this norm. Consider for example the unbounded operator $\operatorname{grad} : L^2(\Omega) \to L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^\infty(\Omega)$ and a $\Omega \subseteq \mathbb{R}^n$. The graph norm is then

$$\|\phi\|_{D(\operatorname{grad})} = \sqrt{\|\phi\|_{L^2(\Omega)}^2 + \|\operatorname{grad} \phi\|_{L^2(\Omega)}^2}, \quad \phi \in C_0^\infty(\Omega)$$

which is just the standard H^1-norm. But it is well-known that $C_0^\infty(\Omega)$ is in fact not closed w.r.t. this norm and thus not complete since the completion of it is the space $H^1_0(\Omega)$ i.e. the Sobolev space with zero trace on the boundary. Below in Prop. 4.1.3 we will provide a sufficient and necessary condition for the domain to be a Hilbert space when the graph norm is used.

The well-known closed graph theorem for bounded operators says that a linear operator from X to Y defined on all of X (in contrast to unbounded operators in general) is bounded i.i.f. its graph is closed in $X \times Y$ w.r.t. the norm $\|(x,y)\|_{X\times Y} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$. This motivates the following definition.

Definition 4.1.2 (Closed operator)**.** We call an unbounded operator $T : X \to Y$ *closed* if its graph $\Gamma(T)$ is closed w.r.t. the norm $\|\cdot\|_{X\times Y}$.

That means if we have a closed operator T and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \xrightarrow{X} x$ and $Tx_n \xrightarrow{Y} y$ for some $x \in X$ and $y \in Y$. Then $(x_n, Tx_n) \xrightarrow{X\times Y} (x,y)$ and since T is closed $(x,y) \in \Gamma(T)$ i.e. $x \in D(T)$ and $Tx = y$. This is just a rephrasing of the definition essentially so this characterizes closed operators equivalently.

{prop:closed_ope

Proposition 4.1.3. *An unbounded operator T is closed i.i.f. its domain $D(T)$, endowed with the graph inner product, is a Hilbert space.*

Proof. As mentioned above, the graph inner product is in fact an inner product on $D(T)$. So we have to show completeness. Assume that T is closed and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ that is Cauchy w.r.t. the graph norm. That implies that (x_n) must be Cauchy w.r.t. the X-norm and (Tx_n) must be Cauchy w.r.t. the Y-norm. Because X and Y are Hilbert spaces there exists $x \in X$ s.t. $x_n \to x$ and $y \in Y$ s.t. $Tx_n \to y$. Because T is closed we know $x \in D(T)$ so $D(T)$ is complete.

For the other direction, assume $D(T)$ is complete and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \to x \in X$ and $Tx_n \to y$ for some $y \in Y$. Because both sequences are convergent they are both Cauchy and thus (x_n) is Cauchy w.r.t. the graph norm. Due to the completeness of $D(T)$ that implies that $x \in D(T)$ and $x_n \xrightarrow{D(T)} x$ and

$$\|x_n - x\|^2_{D(T)} = \|x_n - x\|^2_X + \|Tx_n - Tx\|^2_Y \to 0$$

so $Tx_n \to Tx$ and thus $Tx = y$ which proves that T is closed. $\square$

As an example, take the unbounded operator $(\mathrm{grad}, H^1(\Omega))$ i.e. the weak gradient as an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega;\mathbb{R}^n)$ with domain $D(\mathrm{grad}) = H^1(\Omega)$. Then we described above that the graph norm here is just the H^1-norm. It is well-known that $H^1(\Omega)$ is a Hilbert space. Therefore the Prop. 4.1.3 tells us that $(\mathrm{grad}, H^1(\Omega))$ is a closed operator in contrast to $(\mathrm{grad}, C_0^\infty(\Omega))$ as described above.

The adjoint of bounded operators can be generalized to unbounded operators as well. Let us derive this step by step.

Assume $T : X \to Y$ is a densely defined unbounded operator. Let us fix a $y \in Y$ and look at the linear functional $l : D(T) \to \mathbb{R}$ given by

$$l(x) = \langle y, Tx \rangle_Y.$$

This functional is not necessarily bounded. But if it is i.e. if $l \in D(T)'$ then because $D(T)$ is dense in X we can extend it to a $\bar{l} \in X'$. Let $v \in X$ be its Riesz representative. That means we have

$$\langle v, x \rangle_X = l(x) = \langle y, Tx \rangle_Y \quad \forall x \in D(T).$$

Then when we define $v = T^* y$ and we recognize this as the defining property of the adjoint and define

$$D(T^*) := \{ y \in Y \mid \exists c_y \in \mathbb{R} : \langle y, Tx \rangle_Y \leq c_y \|x\|_X \forall x \in X \}$$

We would like to proof whether T^* is itself densely defined or closed. This can be done in an elegant way by invstigating the graphs of T and T^*. In order to do so, we have the following lemma.

{lem:rotated_gra

Lemma 4.1.4. *Let T be a densely defined unbounded operator from X to Y. Define the rotated graph of T^* as*

$$\tilde{\Gamma}(T^*) := \{(-x, y) \mid (y, x) \in \Gamma(T^*)\} = \{(-T^* y, y) \mid y \in D(T^*)\} \subseteq X \times Y.$$

Then we have

$$\Gamma(T)^\perp = \tilde{\Gamma}(T^*) \text{ and}$$
$$\overline{\Gamma(T)} = \tilde{\Gamma}(T^*)^\perp.$$

Proof. $(x, y) \in \Gamma(T)^\perp$ holds i.i.f.

$$0 = \langle (x, y), (v, Tv) \rangle_{X \times Y} = \langle x, v \rangle_X + \langle y, Tv \rangle_Y \quad \forall v \in D(T).$$

i.e.

$$\langle -x, v \rangle_X = \langle y, Tv \rangle_Y, \quad \forall v \in D(T).$$

This is just equivalent to saying that $-x = T^* y$ i.e.

$$(x, y) = (-T^* y, y) \in \tilde{\Gamma}(T^*)$$

which proves the first equality.

For the second equivalence recall the basic fact from Hilbert space theory that for any subspace of a Hilbert space V, $(V^\perp)^\perp = \overline{V}$. Hence, applying the orthogonal complement to both sides of the first equality gives us the second one. □

{cor:adjoint_of_

Corollary 4.1.5. *The adjoint T^* of a densely defined operator T is closed.*

Proof. Recall another basic fact from Hilbert space theory that the orthogonal complement of a space is always closed. So we know from the first equality that $\tilde{\Gamma}(T^*)$ is closed. It is then trivial to see from the definition of $\tilde{\Gamma}(T^*)$ that $\Gamma(T^*)$ is closed and thus T^* a closed unbounded operator. □

{prop:adjoint_of

Proposition 4.1.6. *Let T be a densely defined and closed unbounded operator. Then T^* is also densely defined and closed.*

Proof. We know from the previous corollary that T^* is closed. In order to prove density, once again recall a fact from Hilbert space theory that a subspace is dense i.i.f. its orthogonal complement is zero. So take $y \in D(T^*)^\perp$ arbitrary. We now have to show that $y = 0$ to complete the proof.

$$0 = \langle y, w\rangle_Y = \langle 0, -T^*w\rangle_X + \langle y, w\rangle_Y = \langle (0, y), (-T^*w, w)\rangle_{X\times Y} \quad \forall w \in D(T^*)$$

which just means

$$(0, y) \in \tilde{\Gamma}(T^*)^\perp \overset{\text{Lemma 4.1.4}}{=} \overline{\Gamma(T)} = \Gamma(T).$$

In the last line we used the fact, that T is closed. Thus $y = T0 = 0$ which concludes the proof. □

We will now take a closer look at the kernels and images of unbounded operators. Let us first notice a very clear result. If T is a closed unbounded operator then its kernel $\ker T$ is closed. This follows indeed from the definition. But this is not true for the image $\operatorname{im} T$. Let us take $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{im} T$ with $y_n \to y$. However if we now take the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $Tx_n = y_n$ we do not know if (x_n) converges or whether the limit is in $D(T)$ if it does converge. A very simple example is the inclusion operator $\iota : H^1(\Omega) \to L^2(\Omega)$. This is actually a bounded operator and hence closed since

$$\|\iota f\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)} \leq \|f\|_{H^1(\Omega)},$$

but its range $H^1(\Omega)$ is not closed in $L^2(\Omega)$.

Let us summarize the following relationships between the images and kernels of closed densely defined operators and their adjoints.

{prop:kernel_ima

Proposition 4.1.7. *Let $T : X \to Y$ be a closed densely defined operator. Then*

- $(\operatorname{im} T)^\perp = \ker T^*$

- $(\ker T)^\perp = \overline{\mathrm{im}(T^*)}$
- $(\mathrm{im}\, T^*)^\perp = \ker T$
- $(\ker T^*)^\perp = \overline{\mathrm{im}(T)}$

Proof. We will once again rely on Lemma 4.1.4 about the rotated graph. We will start with (iii)

$$x \in \ker T \Leftrightarrow (x,0) \in \Gamma(T) \overset{T \text{ closed}}{=} \overline{\Gamma(T)} = \tilde{\Gamma}(T^*)^\perp$$

The last statement is equivalent to saying that for any $y \in D(T^*)$ we have

$$0 = \langle (x,0), (-T^*y, y)\rangle_{X\times Y} = \langle x, -T^*y\rangle_X$$

which just means $x \in (\mathrm{im}\, T^*)^\perp$ and we proved (iii).

(ii) follows from that immediately by taking the orthogonal complement on both sides.

From Prop. 4.1.6 we know that T^* is closed and densely defined because T is. So the completely analogous reasoning with the roles of T and T^* exchanged gives us (i) and taking the orthogonal complement again proves (iv). $\square$

Let us investigate the situation in 3D with the common differential operators curl, grad and div on a domain Ω. Throughout the following, all function spaces will always be over an open domain $\Omega \subseteq \mathbb{R}^3$ unless specified otherwise. Take the unbounded operator $\mathrm{div} : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega)$ with domain C_0^∞ i.e. $(\mathrm{div}, C_0^\infty)$. When we take the adjoint of it then we know for $v \in D((\mathrm{div}, C_0^\infty)^*)$

$$\int_\Omega \mathrm{div}^*\, v \cdot \mathbf{u}\, dx = \int_\Omega v \;\mathrm{div}\, \mathbf{u}\, dx \quad \forall \mathbf{u} \in C_0^\infty(\Omega;\mathbb{R}^3)$$

Now if we take $\mathbf{u} = (\mathbf{u}_1, 0, 0)^\top$ then recognize

$$\int_\Omega (\mathrm{div}^*\, v)_1\, \mathbf{u}_1\, dx = \int_\Omega v\, \partial_1 \mathbf{u}_1\, dx \quad \forall \mathbf{u}_1 \in C_0^\infty \tag{4.1.1}$$ {eq:adjoint_grad

so we recognize that $-(\mathrm{div}^*\, v)_1$ is the weak derivative of w.r.t. the first coordinate i.e. $-\partial_1 v$ and analogous for the other coordinates so we recognize $\mathrm{div}^* = -\,\mathrm{grad}$. We further see that the domain s.t. (4.1.1) is fulfilled is $H^1(\Omega)$ by definition. That means we showed

$$(\mathrm{div}, C_0^\infty)^* = (-\,\mathrm{grad}, H^1).$$

Because C_0^∞ is dense in L^2 i.e. $(\operatorname{div}, C_0^\infty)$ is densely defined. We know from Cor. 4.1.5 that its adjoint $(-\operatorname{grad}, H^1)$ is closed and we can conclude from Prop. 4.1.3 that $H^1(\Omega)$ is in fact a Hilbert space when using the graph norm which is the H^1-norm here.

By interchanging div and grad in the above arguments we can conclude

$$(-\operatorname{grad}, C_0^\infty)^* = (\operatorname{div}, H(\operatorname{div}))$$

where $H(\operatorname{div})$ is the domain of the adjoint which is equivalent to saying that for $\mathbf{u} \in H(\operatorname{div})$ there exists a $\tilde{v} \in L^2$ s.t.

$$\int_\Omega \tilde{v}\phi \, dx = -\int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi \, dx \quad \forall \phi \in C_0^\infty$$

where we denote $\tilde{v} = \operatorname{div} \mathbf{v}$ i.e. the weak divergence.

We know that H^1 contains all smooth functions $C^\infty(\overline{\Omega})$ which are dense in L^2. Analogously, it can be shown that $H(\operatorname{div})$ includes all smooth vector valued functions $C^\infty(\overline{\Omega}; \mathbb{R}^3)$. Hence, both of $(\operatorname{grad}, H^1)$ and $(\operatorname{div}, H(\operatorname{div}))$ are densely defined.

That begs the question what in turn the adjoints of $(\operatorname{grad}, H^1)$ and $(\operatorname{div}, H(\operatorname{div}))$ are. In order to answer it, we take a look at the standard integration-by-parts formula assuming Ω is a Lipschitz domain

$$\int_\Omega u \operatorname{div} \mathbf{w} \, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{w} \, dx = \int_{\partial\Omega} u \, \mathbf{w} \cdot n \, ds \qquad (4.1.2)$$ {eq:integration_

where n is the outward unit normal of Ω. If we want $u \in D(\operatorname{div})$ then the boundary integral on the right hand side must vanish which would be the case if u is zero on the boundary.

We can now take two approaches. The first one follows [11] by saying a $u \in H^1$ has *zero boundary conditions* i.i.f. the right hand side vanishes for any $\mathbf{w} \in H(\operatorname{div})$. This has the advantage that this definition makes sense on any domain Ω independent of the regularity of the boundary.

The other approach followed by Arnold defines the trace operators. For H^1 this is very basic and leads to the classic trace operator $\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\Omega)$ assuming Ω has a Lipschitz boundary. Then we find $(\operatorname{div}, H(\operatorname{div}))^* = H_0^1(\Omega)$ where $H_0^1(\Omega)$ is the space of H^1-functions with zero trace. Then we obtain

$$(\operatorname{div}, H(\operatorname{div}))^* = (-\operatorname{grad}, H_0^1).$$

Vice versa, we see that if u is non-zero on the boundary then $\mathbf{w} \cdot n$ has to vanish. We can give mean to this for functions in $H(\operatorname{div})$ by defining a operator $\gamma_n : H(\div) \to H^{-1/2}$ s.t. if $\mathbf{w}$ is differentiable then $\gamma_n(\mathbf{w}) = \mathbf{w} \cdot n$. Then we would define

$$\mathring{H}(\div) = \{\mathbf{w} \in H(\mathrm{div}) \mid \gamma_n \, \mathbf{w} = 0\}$$

and we get

$$(\mathrm{grad}, H^1)^* = (\mathrm{div}, \mathring{H}(\mathrm{div})).$$

Now let us turn our attention to the remaining fundamenta differential operator curl. Recall that we have for $\mathbf{u}, \mathbf{v} \in C^1(\overline{\Omega})$ the integration-by-parts formula

$$\int_\Omega \mathbf{v} \cdot \mathrm{curl}\, \mathbf{u} \, dx = \int_\Omega \mathrm{curl}\, \mathbf{v} \cdot \mathbf{u} \, dx + \int_{\partial\Omega} \mathbf{v} \times n \cdot \mathbf{u} \, ds$$

if Ω is bounded. Based on this formula we obtain the weak curl as

$$(\mathrm{curl}, H(\mathrm{curl})) = (\mathrm{curl}, C_0^\infty)^*.$$

Then one is able to define an operator $\gamma_\tau : H(\mathrm{curl}) \to H^{-1/2}(\partial\Omega; \mathbb{R}^3)$ s.t. $\gamma_\tau \, \mathbf{w} = \mathbf{w} \times n$ for $\mathbf{w} \in C^1(\overline{\Omega}; \mathbb{R}^3)$ and then define $\mathring{H}(\mathrm{curl}) = \{\mathbf{w} \in H(\mathrm{curl}) \mid \gamma_\tau \, \mathbf{w} = 0\}$. It can be shown that $C^\infty(\overline{\Omega})$ is dense in $H(\mathrm{curl})$. Then we end up with

$$(\mathrm{curl}, H(\mathrm{curl}))^* = (\mathrm{curl}, \mathring{H}(\mathrm{curl})).$$

It can be shown that $C^\infty(\overline{\Omega})$ is dense in $H(\mathrm{curl})$. From Prop. **??** we then know that (grad, H^1), $(\mathrm{div}, H(\mathrm{div}))$ and $(\mathrm{curl}, H(\mathrm{curl}))$ as well as the corresponding spaces with homogeneous boundary condition are closed and densely defined. To summarize:

Theorem 4.1.8. *Let Ω be a bounded*

-

4.2 Hilbert complexes

Now we will combine the idea of cochain complexes from Section 3 with unbounded operators. Recall that a cochain complex is in full generality a sequence of groups $(G^i)_{i\in\mathbb{Z}}$ and group homomorphims $f^i : G^i \to G^{i+1}$ s.t. $f^{i+1} \circ f^i = 0$.

Definition 4.2.1 (Hilbert complex)**.** A Hilbert complex is a sequence of real Hilbert spaces $(W^k)_{k\in\mathbb{Z}}$ and a sequence of closed, densely defined unbounded operators $d^k : W^k \to W^{k+1}$ with domain $V^k \subseteq W^k$ s.t. $d^{k+1} \circ d^k = 0$.

We denote $\mathfrak{Z}^k := \ker d^k$ and $\mathfrak{B}^k := \operatorname{im} d^{k-1}$. Then it follows from the definition that $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$.

We know from **??** that unbounded operators are bounded w.r.t. the graph norm which means here that the restriction of the operators to their domain $d^k : V^k \to V^{k+1}$ are bounded operators when we use the graph norm on V^k. Note that $\mathfrak{B}^{k+1} \subseteq \mathfrak{Z}^{k+1} \subseteq V^{k+1}$ so this is well-defined. Because we assume d^k to be closed we know from Prop. 4.1.3 that V^k are Hilbert spaces w.r.t. the graph norm $\|\cdot\|_{V^k}$. So we see that d^k together with V^k is also a Hilbert complex which we call *domain complex*. In this Hilbert complex all operators are bounded. Notice because the operators are defined on the whole Hilbert space d^k this fits the definition of a cochain complex since vector spaces with summation are groups and the d^k are linear mappings and hence group homomorphisms.

Now let us investigate the adjoints of the operators in a Hilbert complex. Since we assume the operators to be closed and densely defined the adjoints exist and we denote with $d_k^* : W^k \to W^{k-1}$ the adjoint of d^k. Due to Prop. 4.1.6 we know that the adjoints are also closed and densely defined. We denote $V_k^* := D(d_k^*)$, $\mathfrak{Z}_k^* := \ker d_k^*$ and $\mathfrak{B}_k^* := \operatorname{im} d_k^*$. We will frequently leave out the indices from now on.

We can now apply Prop. 4.1.7 to this construction. Then we observe

$$\begin{aligned} \mathfrak{B}^\perp &= \mathfrak{Z}^*, \\ \mathfrak{Z}^\perp &= \overline{\mathfrak{B}^*}, \\ \mathfrak{B}^{*\perp} &= \mathfrak{Z} \text{ and} \\ \mathfrak{Z}^{*\perp} &= \overline{\mathfrak{B}^*}. \end{aligned}$$

Now recall the basic fact from Hilbert space theory that in any Hilbert space if we have any two subspaces $V \subseteq W$ then taking the orthogonal complements reverses the inclusion i.e. $V^\perp \supseteq W^\perp$. Then we get

$$\mathfrak{B}^* \subseteq \overline{\mathfrak{B}^*} = \mathfrak{Z}^\perp \subseteq \mathfrak{B}^\perp = \mathfrak{Z}^*. \tag{4.2.1}$$ {eq:image_kernel

So we recognize that $d_k^* : W^k \to W^{k-1}$ form a structure very similar to a Hilbert complex with the only difference being that the indices of the spaces decrease. We call this complex the *dual complex* of the Hilbert complex.

Definition 4.2.2. We call a $v \in V^k \cap V_k^*$ harmonic if $d^k v = 0$ and $d_k^* v = 0$. Denote the space of harmonic elements as $\mathfrak{H}^k$.

We can rewrite this as $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$. Using (4.2.1) we can write this as

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}_k^{*,\perp} = \mathfrak{B}^{k,\perp} \cap \mathfrak{Z}_k^*.$$

Now we can formulate the most important result of this chapter.

{thm:hodge_decom

Theorem 4.2.3 (Hodge decomposition)**.** *Let $d^k : W^k \to W^{k+1}$ form a Hilbert complex. Then we have*

$$\mathfrak{Z}^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \text{ and}$$
$$\mathfrak{Z}^*_k = \overline{\mathfrak{B}^*_k} \overset{\perp}{\oplus} \mathfrak{H}^k.$$

We obtain the Hodge decomposition of the space W^k

$$W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}^*_k}.$$

Proof. Let us first prove $\mathfrak{Z}^k \subseteq \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. Take $z \in \mathfrak{Z}^k$ arbitrary. From basic Hilbert theory we know that $W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{B}^{k,\perp}$. So we find $z = z_1 + z_2$ with $z_1 \in \overline{\mathfrak{B}^k}$ and $z_2 \in \mathfrak{B}^{k,\perp}$. Because $\mathfrak{Z}^k$ is closed $z_1 \in \overline{\mathfrak{B}^k} \subseteq \mathfrak{Z}^k$ and thus $z_2 = z - z_1 \in \mathfrak{Z}^k$ as well i.e. $z_2 \in \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{H}^k$ and so $z \in \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. The other inclusion is obvious since $\mathfrak{H} = \mathfrak{B}^{k,\perp} \cap \mathfrak{Z}^k$. The proof for the second equality is completely analogous.

For the Hodge decomposition, since $\mathfrak{Z}^k$ is closed

$$W^k = \mathfrak{Z}^k \overset{\perp}{\oplus} \mathfrak{Z}^{k,\perp} = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}^*_k}.$$

□

The Hodge decomposition is a very powerful tool whenever deal with Hilbert complexes.

4.2.1 L^2 de Rham complex in 3D

Let us investigate the situation for the differential operators grad, div and curl. All the necessary ingredients were already proven at the end of Sec. 4.1. Let Ω be a Lipschitz domain of $\mathbb{R}^3$.

We take $W^0 = W^3 = L^2(\Omega)$ and $W^1 = W^2 = L^2(\Omega;\mathbb{R}^3)$ and we set all other W^k to zero in order to obtain a sequence. Then we choose the operators $d^0 = \text{grad}$, $d^1 = \text{curl}$, $d^2 = \text{div}$ and for the domains we choose $V^0 = H^1(\Omega)$, $V^1 = H(\text{curl};\Omega)$, $V^2 = H(\text{div};\Omega)$ and $V^3 = L^2(\Omega) = W^3$. As before, we will leave out the refernence to the domain Ω now. All other d^k are just zero. The resulting domain complex is then

$$0 \to H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \to 0 \tag{4.2.2}$$ {eq:primal_de_rh

All these operators are closed and densely defined. It remains to show that $d^{k+1} \circ d^k = 0$. If $k < 0$ or $k > 2$ this is clear from the definition. Then we have from the definition of these operators for any $u \in H^1$ and $\mathbf{v} \in C_0^\infty(\Omega; \mathbb{R}^3)$

$$\int_\Omega \operatorname{curl} \operatorname{grad} u \cdot \mathbf{v} \, dx = \int_\Omega \operatorname{grad} u \cdot \operatorname{curl} \mathbf{v} \, dx = \int_\Omega u \, \operatorname{div} \operatorname{curl} \mathbf{v} \, dx = 0$$

which implies that $\operatorname{curl} \operatorname{grad} u = 0$. $\operatorname{div} \operatorname{curl} = 0$ is proven completely analogously. So (4.2.2) is indeed a Hilbert complex.

If Ω is bounded then the complex is closed i.e. all images are closed subspaces which should not be confused with being a closed operator as explained in the section about unbounded operators. The proof can be found in Sec. 4.3 of the main reference for this section [2].

The resulting dual domain complex is

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} \mathring{H}(\operatorname{div}) \xleftarrow{\operatorname{curl}} \mathring{H}(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H_0^1 \leftarrow 0.$$

For the harmonic elements – which are scalar and vector fields here – we obtain

$$\begin{aligned}
\mathfrak{H}^0 &= \{u \in H^1 \mid \operatorname{grad} u = 0\}, \\
\mathfrak{H}^1 &= \{\mathbf{u} \in H(\operatorname{curl}) \cap \mathring{H}(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\}, \\
\mathfrak{H}^2 &= \{\mathbf{u} \in \mathring{H}(\operatorname{curl}) \cap H(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\} \text{ and} \\
\mathfrak{H}^3 &= \{u \in H_0^1 \mid \operatorname{grad} u = 0\} = \{0\}.
\end{aligned}$$

For the last equality we used the fact that $\operatorname{grad} u = 0$ implies that u is constant almost everywhere with possibly different constants for different path components of Ω. But because we have homogeneous boundary conditions we get $u = 0$. Note that $\mathfrak{H}^1$ and $\mathfrak{H}^2$ are very similar. The only difference are the boundary conditions. $\mathbf{u} \in \mathfrak{H}^2 \subseteq \mathring{H}(\operatorname{curl})$ means that the generalized tangential trace $\gamma_\tau \mathbf{u}$ is zero. If $\mathbf{u} \in \mathfrak{H}^1 \subseteq \mathring{H}(\operatorname{div})$ then the generalized normal trace $\gamma_n \mathbf{u}$ vanishes.

This gives us the Hodge decomposition in the 3D case.

Theorem 4.2.4 (Hodge decomposition in 3D). *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain. Then we have the following decompositions of the kernels*

$$\begin{aligned}
\{\mathbf{u} \in H(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{grad} H^1} \overset{\perp}{\oplus} \mathfrak{H}^1 \\
\{\mathbf{u} \in H(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2 \\
\{\mathbf{u} \in \mathring{H}(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} \mathring{H}(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^1 \\
\{\mathbf{u} \in \mathring{H}(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{div} \mathring{H}(\operatorname{div})} \overset{\perp}{\oplus} \mathfrak{H}^2.
\end{aligned}$$

We can express L^2 as

$$\begin{aligned} L^2 &= \overline{\operatorname{div} \mathring{H}(\operatorname{div})} \overset{\perp}{\oplus} \{v \in H^1 \mid \operatorname{grad} v = 0\} \\ &= \overline{\operatorname{div} H(\operatorname{div})}. \end{aligned}$$

and for vector valued functions

$$\begin{aligned} L^2(\Omega; \mathbb{R}^3) &= \overline{\operatorname{grad} H(\operatorname{grad})} \overset{\perp}{\oplus} \mathfrak{H}^1 \overset{\perp}{\oplus} \overline{\operatorname{curl} \mathring{H}(\operatorname{curl})} \\ &= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2 \overset{\perp}{\oplus} \overline{\operatorname{grad} H_0^1}. \end{aligned}$$

Proof. This is just an application of the general Hodge decomposition Thm. 4.2.3 combined with what we derived above. □

Remark 4.2.5. Alternatively, we could have chosen the sequence with zero boundary conditions as the primal sequence i.e.

$$0 \to H_0^1 \xrightarrow{\operatorname{grad}} \mathring{H}(\operatorname{curl}) \xrightarrow{\operatorname{curl}} \mathring{H}(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2 \to 0$$

Then we can follow the exact same arguments to get the dual sequence

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} H(\operatorname{div}) \xleftarrow{\operatorname{curl}} H(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H^1 \leftarrow 0.$$

5 Existence and uniqueness of solutions

{sec:existence_a

In this section, we will apply the developed theory of the preceding chapters to prove the existence and uniqueness of the magnetostatic problem on exterior domains. With exterior domain we mean that our domain $\Omega \subseteq \mathbb{R}^3$ is the complement of a compact set. The main motivation for this problem is the special case of Ω being the complement of a torus. This is also the motivation behind the topological assumption that we will give. It might be useful to keep this example in mind.

Recall the magnetostatic problem we are studying.

{prob:magnetosta

Problem 5.0.1. Find $B \in H_0(\operatorname{div}; \Omega)$ s.t.

$$\operatorname{curl}\, B = 0, \tag{5.0.1}$$

$$\operatorname{div}\, B = 0 \text{ in } \Omega \text{ and} \tag{5.0.2}$$

$$\int_\gamma B \cdot dl = C_0. \tag{5.0.3}$$

Of course in order for the curve integral constraint to be well-defined we need to check the regularity of solutions. Then using the tools we developed in the previous sections we will proof existence and uniqueness.

5.1 Regularity of solutions

{sec:regularity_

We will rely on standard regularity results about elliptic systems of the following form. Take $A_{ij}^{\alpha\beta} \in \mathbb{R}$ for i, j, α, $\beta = 1, 2, 3$ and $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Then we have systems of the form

$$-\sum_{\alpha,\beta,j} \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta B_j) = f_i - \sum_\alpha \partial_\alpha F_i^\alpha \tag{5.1.1}$$ {eq:elliptic_sys

with data $f_i, F_i^\alpha \in L^2(\Omega)$. We call this system *elliptic* if A satisfies the Legendre condition i.e.

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^{3\times 3} \tag{5.1.2}$$ {eq:legendre_con

with $c > 0$. $|\xi|$ is here the Frobenius norm, but technically the chosen norm is irrelevant due to all norms on $\mathbb{R}^{3\times 3}$ being equivalent. We then call $B \in H^1_{loc}(\Omega)^3$ Recall definition of H^k_{loc} a weak solution of (5.1.1) if

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_\beta B_j \partial_\alpha \varphi_i dx = \int_\Omega \left\{ \sum_i f_i \varphi_i + \sum_{\alpha,i} F_i^\alpha \partial_\alpha \varphi_i \right\} dx \tag{5.1.3}$$ {eq:weak_ellipti

for all $\varphi \in C_0^1(\Omega; \mathbb{R}^3)$. This formulation is taken from [1, Sec. 1.3]. At first we will slightly modify the notion of weak solution.

{prop:weak_solut

Proposition 5.1.1. *Assume that we have an elliptic system with constant coefficients i.e. A is constant. Then (5.1.3) is fulfilled for all $\varphi \in C_0^1(\Omega; \mathbb{R}^3)$ if and only if it is fulfilled for $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$.*

Proof. This follows by a simple density argument. Assume that (5.1.3) is fulfilled for all test functions in $C_0^\infty(\Omega; \mathbb{R}^3)$. Now take $\varphi \in C_0^1(\Omega; \mathbb{R}^3)$ arbitrary. Because $\varphi \in H_0^1(\Omega)^3$ and $C_0^\infty(\Omega; \mathbb{R}^3)$ is dense in $H_0^1(\Omega)^3$ we can find a sequence $(\varphi^{(l)})_{l\in\mathbb{N}} \subseteq C_0^\infty(\Omega; \mathbb{R}^3)$ s.t. $\varphi^{(l)} \to \varphi$ in $H^1(\Omega)^3$. Thus the partial derivatives converge in $L^2(\Omega)$ and we get

$$\int_\Omega \sum_{i,j,\alpha,\beta} A_{ij}^{\alpha,\beta} \partial_\beta B_j \partial_\alpha \varphi_i dx = \sum_{i,j,\alpha,\beta} A_{ij}^{\alpha,\beta} \int_\Omega \partial_\beta B_j \lim_{l\to\infty} \partial_\alpha \varphi^{(l)} dx$$

$$\overset{L^2 \text{ limit}}{=} \lim_{l\to\infty} \int_\Omega \left\{ \sum_i f_i \varphi_i^{(l)} + \sum_{\alpha,i} F_i^\alpha \partial_\alpha \varphi_i^{(l)} \right\} dx = \int_\Omega \left\{ \sum_i f_i \varphi_i + \sum_{\alpha,i} F_i^\alpha \partial_\alpha \varphi_i \right\} dx.$$

Since $\varphi \in C_0^1(\Omega; \mathbb{R}^3)$ was arbitrary the first direction of the equivalence is proved. The other direction is trivial. $\square$

So we see that in the case of constant coefficents we can consider just smooth compactly supported functions as test functions.

Next, we will state the crucial result about the regularity of elliptic systems which will give us the desired regularity. This is Theorem 2.13 and Remark 2.16 in [1] in slightly less generality.

{thm:regularity_

Theorem 5.1.2. *Let Ω be an open domain in $\mathbb{R}^n$. Let A satisfy the Legendre condition (5.1.2). Then for every $B \in H^1_{loc}(\Omega)^3$ weak solution in the sense of (5.1.3) with $f \in H^k_{loc}(\Omega)^3$ and $F \in H^{k+1}_{loc}(\Omega;\mathbb{R}^{m\times n})$ we have $B \in H^{k+2}_{loc}(\Omega)^3$.*

{cor:smooth_solu

Corollary 5.1.3. *If under the assumptions of the previous theorem we consider the homogeneous problem, i.e.*

$$\int_\Omega \sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij}\partial_\beta B_j \partial_\alpha \varphi_i dx = 0$$

for all $\varphi \in C_0^\infty(\Omega;\mathbb{R}^3)$, then $B \in C^\infty(\Omega)$.

Proof. Here we have $F = 0$ and $f = 0$. Thus, $f \in H^k_{loc}(\Omega)^3$ for any $k \in \mathbb{N}$. Take any $Q \subseteq \Omega$ pre-compact. Then we know from Thm. 5.1.2. that $B \in H^{k+2}_{loc}(\Omega)^3$ and thus $B \in H^{k+2}(Q)^3$. Therefore, we can apply the standard Sobolev embedding theorem locally to get $B \in C^l(Q)$ for any $l \in \mathbb{N}$ and hence $B \in C^\infty(\Omega)$ since Q pre-compact was arbitrary. $\square$

It should be noted that this does not guarantee us any regularity on the boundary.

Before we can apply this result, we have to check whether a solution of our problem B is actually in $H^1_{loc}(\Omega)^3$.

{thm:solution_in

Theorem 5.1.4. *Assume $B \in H(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega)$. Then $B \in H^1_{loc}(\Omega)^3$.*

Note that we did not assume B to be a solution.

Proof. We know that for a function $u \in H_0(\mathrm{curl};U) \cap H(\mathrm{div};U)$ for some smooth domain U we have $u \in H^1(U)^3$ (cf. [9, Remark 3.48]). Our Ω is just assumed to be open so we can not apply this result directly because we have no information about the regularity of the boundary $\partial\Omega$.

Take $Q \subset\subset \Omega$ open and pre-compact. Then we can find an open cover of $\overline{Q}$ with a finite set of open balls $\{K_i\}_{i=1}^N$ s.t. $K_i \subseteq \Omega$ and

$$\overline{Q} \subseteq \bigcup_{i=1}^N K_i.$$

As a open cover of a compact set we can find a smooth partition of unity $\{\chi_i\}_{i=1}^N$ subordinate to $\{K_i\}_{i=1}^N$. $(B\chi_i)|_{K_i} \in H_0(\mathrm{curl}; K_i) \cap H(\mathrm{div}; K_i)$ and thus $(B\chi_i)|_{K_i} \in H^1(K_i)^3$ by the above mentioned result. Also because $B\chi_i$ has compact support in K_i we can extend it by zero to obtain $B\chi_i \in H^1(\mathbb{R}^3)^3$ where we abused the notation by denoting the extension the same. Whence,

$$B|_Q = (\sum_{i=1}^N \chi_i|_Q)B|_Q = \sum_{i=1}^N (\chi_i B)|_Q \in H^1(Q)$$

i.e. $B \in H^1_{loc}(\Omega)^3$. □

The following lemma is a reformulation of the differential operator $\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}$ which will be needed when we write our magnetostatic problem in the above standard elliptic form.

{lem:graddiv_cur

Lemma 5.1.5. *Let $F \in H^2_{loc}(\Omega)^3$. Then*

$$\operatorname{grad}\operatorname{div} F - \operatorname{curl}\operatorname{curl} F = \begin{pmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{pmatrix}.$$

Proof. By a simple calculation and changing the order of differentiation

$$\operatorname{grad}\operatorname{div} F = \begin{pmatrix} \partial_1^2 F_1 + \partial_1\partial_2 F_2 + \partial_1\partial_3 F_3 \\ \partial_1\partial_2 F_2 + \partial_2^2 F_2 + \partial_2\partial_3 F_3 \\ \partial_1\partial_3 F_1 + \partial_2\partial_3 F_2 + \partial_3^2 F_3 \end{pmatrix}$$

and

$$\operatorname{curl}\operatorname{curl} F = \operatorname{curl}\begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \begin{pmatrix} \partial_2(\partial_1 F_2 - \partial_2 F_1) - \partial_3(\partial_3 F_1 - \partial_1 F_3) \\ \partial_3(\partial_2 F_3 - \partial_3 F_2) - \partial_1(\partial_1 F_2 - \partial_2 F_1) \\ \partial_1(\partial_3 F_1 - \partial_1 F_3) - \partial_2(\partial_2 F_3 - \partial_3 F_2) \end{pmatrix}$$

$$= \begin{pmatrix} \partial_1\partial_2 F_2 - \partial_2^2 F_1 - \partial_3^2 F_1 + \partial_1\partial_3 F_3 \\ \partial_2\partial_3 F_3 - \partial_3^2 F_2 - \partial_1^2 F_2 + \partial_1\partial_2 F_3 \\ \partial_1\partial_3 F_3 - \partial_3^2 F_2 - \partial_2^2 F_3 + \partial_2\partial_3 F_2 \end{pmatrix}$$

and so by subtracting the two expressions

$$\operatorname{grad}\operatorname{div} F - \operatorname{curl}\operatorname{curl} F = \begin{pmatrix} \partial_1^2 F_1 + \partial_2^2 F_1 + \partial_3^2 F_1 \\ \partial_1^2 F_2 + \partial_2^2 F_2 + \partial_3^2 F_3 \\ \partial_1^2 F_3 + \partial_2^2 F_3 + \partial_3^2 F_3 \end{pmatrix} = \begin{pmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta F_3 \end{pmatrix}.$$

□

We want to rewrite this system in the expression of the elliptic system (5.1.1). We can rewrite the Laplacian

$$-\Delta F_i = -\sum_{\alpha=1}^{3} \partial_\alpha \partial_\alpha F_i = -\sum_{\alpha,\beta=1}^{3} \partial_\alpha \delta_{\alpha,\beta} \partial_\beta F_i = -\sum_{\alpha,\beta,j=1}^{3} \partial_\alpha \delta_{\alpha,\beta} \delta_{ij} \partial_\beta F_j$$

so we get $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. We have to check that the resulting differential operator is indeed elliptic, but this trivial because for any $(\xi_\alpha^i)_{1\leq i,\alpha\leq 3}$ we get

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j = \sum_{\alpha,\beta,i,j} \delta_{ij}\delta_{\alpha\beta} \xi_\alpha^i \xi_\beta^j = \sum_{\alpha,i} (\xi_\alpha^i)^2 = |\xi|^2$$

so the Legendre condition (5.1.2) is fulfilled and the resulting system is elliptic. The weak formulation is

$$\int_\Omega \sum_{\alpha,\beta,i,j} \delta_{ij}\delta_{\alpha\beta} \partial_\beta B_j \partial_\alpha \varphi_i dx = \sum_{i=1}^{3} \int_\Omega \nabla B_i \cdot \nabla \varphi_i dx.$$

Here we can assume $\varphi \in C_0^\infty(\Omega)^3$ due to Prop. 5.1.1.

{thm:smoothness_

Theorem 5.1.6 (Smoothness of solutions). *Let $\Omega \subseteq \mathbb{R}^3$ open and $B \in H(\operatorname{div};\Omega) \cup H(\operatorname{curl};\Omega)$ and*

$$\operatorname{curl} B = 0,$$
$$\operatorname{div} B = 0.$$

Then B is smooth.

Proof. Take $\varphi \in C_0^\infty(\Omega)^3$. Then

$$0 = \int_\Omega \operatorname{div} B \operatorname{div} \varphi + \operatorname{curl} B \cdot \operatorname{curl} \varphi dx = -\int_\Omega B \cdot (\operatorname{grad} \operatorname{div} \varphi - \operatorname{curl} \operatorname{curl} \varphi) dx$$

$$\overset{Lemma 5.1.5}{=} -\int_\Omega B \cdot \begin{pmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \Delta\varphi_3 \end{pmatrix} = \sum_{i=1}^{3} \int_\Omega \nabla B_i \cdot \nabla \varphi_i dx.$$

Note that the last integration by parts is well defined because $B \in H^1_{loc}(\Omega)$ according to Thm. 5.1.4. So B is a weak solution of the elliptic system given by $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. Because we look at the homogenous problem our right hand side is obviously smooth and thus B is smooth as well due to Cor. 5.1.3. □

Remark 5.1.7. Obviously, the above arguments can be generalized by using a non-zero right hand side of our problem. Then we will in general not obtain a smooth solution, but for a sufficiently regular right hand side the curve integral would still be well-defined. How much regularity? Source?

5.2 Existence and uniqueness

The curve integral condition is closely linked to the topology of our domain which we will have to use in our proof. This will rely on the tools of homology from Sec. 3. Because of this connection if we want the curve integral to give us uniqueness of the solution we need to assume certain topological properties. In our case, this will be the condition that our first homology group is generated by the curve that we are integrating over i.e.

$$H_1(\Omega) = \mathbb{Z}[\gamma].$$

Then we get the following existence and uniqueness result on the level of homology.

{prop:uniqueness

Proposition 5.2.1. *Assume that $H_1(\Omega) = \mathbb{Z}[\gamma]$ i.e. the homology class of the closed 1-chain γ is a generator of the first homology group. Then we have the following:*

(i) For any $C_0 \in \mathbb{R}$ there exists a closed 1-cochain $F \in Z^1(\Omega)$ with $F(\gamma) = C_0$,

(ii) any other $G \in Z^1(\Omega)$ with $G(\gamma) = C_0$ is in the same cohomology class i.e. $[F] = [G]$

i.e. the cochain is unique up to cohomology.

Proof. **Proof of (i)** Because $[\gamma]$ is a generator of the homology group we obtain a homomorphism $\hat{F} \in \mathrm{Hom}(H_1(\Omega), \mathbb{R})$ by fixing $\hat{F}([\gamma]) = C_0$. Note that $H^1(\Omega)$ is an abuse of notation because we mean here the first cohomology group of Ω and not the Sobolev space. In the context of singular homology this will always be the case.

This determines the other values. Recall the isomorphism $\beta : H^1(\Omega) \to \mathrm{Hom}(H_1(\Omega); \mathbb{R})$ from the universal coefficent theorem (3.2.1), $\beta([F]) = \tilde{F}$ where F and $\tilde{F}$ are in the same cohomology class. Then we know that there exists a $[F] \in H^1(\Omega)$ with $\beta([F]) = \hat{F}$ because β is a isomorphism. So we obtain

$$F(\gamma) = \beta([F])([\gamma]) = \hat{F}([\gamma]) = C_0.$$

Proof of (ii) Take $[c] \in H_1(\Omega)$ arbitrary. Then there exists $n \in \mathbb{Z}$ s.t. $[c] = n[\gamma]$. Using β from (3.2.1) We have

$$\beta([F])([c]) = \beta([F])(n[\gamma]) = n\beta([F])([\gamma]) = n\, F(\gamma) = n\, G(\gamma) = \beta([G])([c])$$

and thus $\beta([F]) = \beta([G])$. Because β is an isomorphism we arrive at $[F] = [G]$. $\square$

This abstract topological result can now be linked to the differential forms via the de Rham isomorphism from Sec. 3.3. We will formulate it in a way that demonstrates the connection of differential forms and cochains.

{cor:existence_u

Corollary 5.2.2. *Assume $H_1(\Omega) = \mathbb{Z}[\gamma]$ as above. Then*

(i) For any $C_0 \in \mathbb{R}$ there exists a closed smooth 1-form $\theta \in \mathfrak{Z}^1(\Omega)$ with

$$I(\theta)(\gamma) = \int_\gamma \theta = C_0$$

(ii) any other $\eta \in \mathfrak{Z}^1(\Omega)$ with

$$I(\eta)(\gamma) = \int_\gamma \eta = C_0$$

is in the same cohomology class of $H^1_{dR}(\Omega)$ i.e. $[\eta] = [\theta]$.

Proof. **Proof of (i)** Recall from Sec. 3.3 that the integration of differential forms over chains induces an isomorphism on cohomology $[I] : H^1_{dR}(\Omega) \to H^1(\Omega)$ which we call de Rham isomorphism. We know from Prop. 5.2.1 that there exists $F \in H^1(\Omega)$ s.t. $F(\gamma) = C_0$. The surjectivity of the de Rham isomorphism now gives us $[\theta] \in H^1(\Omega)$ s.t.

$$[I(\theta)] = [I]([\theta]) = [F]$$

i.e.

$$I(\theta) = F + \partial^0 J$$

with $J \in C^0$. Then,

$$I(\theta)(\gamma) = F(\gamma) + \partial^0 J(\gamma) = C_0 + J(\partial_1\gamma) \overset{\gamma \text{ closed}}{=} C_0.$$

Proof of (ii) We have $I(\eta)$ is a 1-cochain with $I(\eta)(\gamma) = C_0$. Thus, we can apply Prop. 5.2.1 to get

$$[I](\eta) = [I(\eta)] = [I(\theta)] = [I](\theta).$$

Because $[I]$ is an isomorphism we can conclude $[\eta] = [\theta]$. $\square$

{thm:existence}

Theorem 5.2.3 (Existence of solution)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be such that $\mathbb{R}^3 \setminus \Omega$ is compact. For the topology, we require that $H_1(\Omega) = \mathbb{Z}[\gamma]$ for a 1-chain γ. Assume further that there exists an ϵ-neighborhood*

$$\Omega_\epsilon := \{x \in \mathbb{R}^3 \mid d(x, \Omega) < \epsilon\}$$

s.t. $H_1(\Omega_\epsilon) = \mathbb{Z}[\gamma]$ as well. Then there exists a solution to Problem 5.0.1.

Let us say a view words about the topological assumption regarding Ω_ϵ. This just means that we can slightly increase the domain without changing the first homology group. As an example, think again of a torus in $\mathbb{R}^3$. Assuming the torus has non-empty interior we can slightly reduce the poloidal radius without changing the topology of its exterior domain.

Proof. At first we want to find a smooth differential 1-form $\theta \in C^\infty\Lambda^1(\Omega)$ with the desired curve integral. In order to do that we will increase the domain slightly. We start by referring to Cor. 5.2.2 to get a smooth differential form $\tilde{\theta} \in \Lambda^1(\Omega_\epsilon)$ with

$$\int_\gamma \tilde{\theta} = C_0. \tag{5.2.1}$$ {eq:integral_the

We now refer back to Sec. 2.4 and change back to vector proxies. Let $\tilde{\phi}$ be the vector proxy of $\tilde{\theta}$ i.e. by recalling the musical isomorphism **??** $\tilde{\theta}^\sharp = \tilde{\phi} \in C^\infty(\tilde{\Omega})^3$. Because $\tilde{\theta}$ is closed and (5.2.1) holds we obtain the corresponding properties of $\tilde{\phi}$ which are

$$\int_\gamma \tilde{\phi} \cdot dl = C_0$$
$$\operatorname{curl} \tilde{\phi} = 0.$$

We define ϕ by restricting $\tilde{\phi}$ to Ω. Let now K_R be the open ball around the origin with radius $R > 0$ large enough s.t. $\Omega^c \subseteq K_R$ and $\gamma \subseteq K_R$. We now restrict θ further to $\Omega_R := \Omega \cap K_R$. We denote the restriction with $\phi_R := \phi|_{\Omega_R}$.

We now need to construct a harmonic vector field with zero tangential trace s.t. we can extend it by zero. We do this by using the Hodge decomposition for 1-forms on Ω_R in the 3D case **??**. We project ϕ_R onto the harmonic fields to obtain B_R which has zero tangential trace. So there exists a sequence $(\psi_i)_{i\in\mathbb{N}} \in H^1(\Omega_R)$ s.t.

$$B_R = \phi_R - \lim_{i\to\infty} \nabla\psi_i$$

We also know that B_R is smooth because it is curl and divergence free from Thm.5.1.6. We want to check that the curve integral did not change. We know from **??** that the image of the exterior derivative is closed on bounded domains. Formulated in vector proxies, this means that $\nabla H^1(U)$ is closed in L^2 if U is a bounded domain. So we get that

$$\lim_{i\to\infty} \nabla\psi_i = \nabla\psi_R$$

with $\psi_R \in H^1(\Omega_R)$. Because B_R and ϕ_R are smooth ψ_R must be smooth as well and so we have

$$\int_\gamma B_R \cdot dl = \int_\gamma \phi_R \cdot dl.$$

Now extend B_R by zero onto $\mathbb{R}^3 \setminus K_R$. Denote this extension as $\overline{B}_R$. Because B_R has tangential trace zero and is curl-free its extension $\overline{B}_R \in H(\mathrm{curl};\Omega)$ and is also curl-free. Here it is important to remember that $\mathbb{R}^3 \setminus K_R \subseteq \Omega$. That means of course that $\overline{B}_R$ might not be smooth on ∂K_R as it might have a jump. Now we can once again use the Hodge decomposition of two forms. This time on the whole domain Ω to find a harmonic field B and a sequence $(\rho_i)_{i\in\mathbb{N}}$ s.t.

$$\overline{B_R} = B + \lim_{i\to\infty} \nabla\rho_i$$

since $\mathrm{curl}\,\overline{B_R} = 0$. Notice that because B is a harmonic vector field and because it is the vector proxy of a harmonic 2-form it already satisfies the Neumann boundary condition and it is divergence as well as curl free. That means B is a solution if the curve integral condition is satisfied. In order to see this, note that we have on K_R

$$B_R = B|_{K_R} + \lim_{i\to\infty} \nabla\rho_i|_{K_R}.$$

Because the image of the gradient is closed on bounded domains we have $\rho_R \in H^1(K_R)$ s.t.

$$B_R = B|_{K_R} + \nabla\rho_R.$$

With the same argument as above, ρ_R must be smooth and so

$$\int_\gamma B \cdot dl = \int_\gamma B|_{K_R} \cdot dl = \int_\gamma B_R \cdot dl = C_0$$

and thus B is indeed a solution. □

In the proof of uniqueness we will use the following lemma.

{lem:gradient_se

Lemma 5.2.4. *Let $\phi \in L^2_{loc}(\Omega_\epsilon)$ with $\nabla\phi \in L^2(\Omega)^3$. Then there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^1(\Omega)$ s.t. $\nabla\phi_i \to \nabla\phi$ in $L^2(\Omega)^3$.*

Proof. Take K_R the open ball around the origin with R large enough s.t. $(K_R)^c \subseteq \Omega$. Define $\Omega_R := K_R \cap \Omega$. Then $\overline{\Omega}_R \subseteq K_{R+1}$, where K_{R+1} is the open ball around the origin with radius $R+1$, Ω_R is a Lipschitz domain and

K_{R+1} is pre-compact and $\phi|_{\Omega_R} \in W^{1,2}(\Omega_R)$. Note that here we need the fact that $\phi \in L^2_{loc}(\Omega_\epsilon)$ because then Ω_R is pre-compact in Ω_ϵ. Therefore we can find an extension $E\phi \in W_0^{1,2}(\Omega_{R+1}) \hookrightarrow W^{1,2}(\mathbb{R}^3)$ (cf. [8, Sec. 1.5.1]). So we can now define

$$\bar{\phi} := \begin{cases} \phi & \text{in } \Omega \\ E\phi & \text{in } \Omega^c. \end{cases}$$

Then $\bar{\phi} \in L^2_{loc}(\mathbb{R}^3)$ and $\nabla\bar{\phi} \in L^2(\mathbb{R}^3)^3$. Then there exists a sequence $(\phi_l)_{l\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^3)$ s.t. $\nabla\phi_l \to \nabla\bar{\phi}$ in $L^2(\mathbb{R}^3)^3$ (cf. [12, Lemma 1.1]). By restricting ϕ_l to Ω we obtain the result. $\square$

Theorem 5.2.5. *Let the same assumptions hold as in Thm. 5.2.3. Then the solution of the problem is unique.*

Proof. Let B and $\tilde{B}$ both be solutions and denote with ω and $\tilde{\omega}$ the corresponding 1-forms i.e. $\omega = B^\flat$ and $\tilde{\omega} = \tilde{B}^\flat$. So we have $I(\omega)(\gamma) = I(\tilde{\omega})(\gamma) = C_0$ since

$$\int_\gamma \omega = \int_\gamma B^\flat = \int_\gamma B \cdot dl = \int_\gamma \tilde{B} \cdot dl = \int_\gamma \tilde{B}^\flat = \int_\gamma \tilde{\omega}$$

Then we know from Cor. 5.2.2 that ω and $\tilde{\omega}$ are in the same cohomology class in H^1_{dR}. So there exists a 0-form i.e. $\mu \in C^\infty(\Omega)$ s.t. $\omega - \tilde{\omega} = d\mu$. By applying $^\sharp$ on both sides

$$B - \tilde{B} = \omega^\sharp - \tilde{\omega}^\sharp = (d\mu)^\sharp = \operatorname{grad}\mu.$$

However, μ need not be in L^2 since Ω is unbounded. But we know that $\operatorname{grad}\mu \in L^2(\Omega)^3$ and $\mu \in L^2_{loc}(\Omega)$. Here we can now apply Lemma 5.2.4 and conclude

$$B - \tilde{B} \in \overline{\operatorname{grad} H^1(\Omega)}.$$

Now remembering the Hodge decomposition in the 3D case we know

$$B - \tilde{B} \in \overline{\operatorname{grad} H^1(\Omega)}^\perp$$

as well and thus $B = \tilde{B}$ which concludes the proof of uniqueness. $\square$

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