1 Variational formulation of the magnetostatic in 2D

In 2D we have the Hilbert complex

?????????

where we have the curl which is the scalar curl and curl which is the vector valued curl defined as.

It is easily derived that we have the integration by parts formula if $\mathbf{v} \in C_b^1(\overline{\Omega})$. Here Ω is from now on assumed bounded and Lipschitz.

The 2D magnetostatic problem is then Let $J \in B_0^*$ be given. Then

Problem 1.0.1 (2D magnetostatic problem)**.** Find $\mathbf{B} \in H_0(\mathrm{div}) \cap H(\mathrm{curl})$ s.t.

$$\mathrm{curl}\,\mathbf{B} = J,$$
$$\mathrm{div}\,\mathbf{B} = 0, + \text{ additional constraints}$$

The additional constraints are necessary to give a unique solution. We will focus, just as in the first part, on a curve integral as additional constraint. Another option would be an orthogonality constraint as in **??**.

1.1 Mixed formulation

In order to solve this problem numerically using finite elements we have to choose a suitable variational formulation of the problem. We will use the following For any $J \in \mathrm{curl}\,H(\mathrm{curl})$ find $\sigma \in H_0^1$, $B \in H_0(\mathrm{div})$ and $p \in \mathfrak{H}^1$ s.t.

$$\langle \sigma, \tau \rangle - \langle u, \mathrm{curl}\tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \tag{1.1.1}$$ {eq:first_eq_mix

$$\langle \mathrm{curl}\sigma, \mathbf{v} \rangle + \langle \mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathbf{v} \rangle + \langle \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in V^k \text{ and+ additional constraints} \tag{1.1.2}$$ {eq:second_eq_mi

This formulation is of course much more complicated than the first one **??**, but this formulation will turns out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will answer it in two propositions. We will treat the harmonic constraint separately.

Proposition 1.1.1. *For any $J \in L^2$, (1.1.1) and (1.1.2) hold i.i.f. $\sigma = 0$, $\mathbf{p} = 0$ and $\mathrm{curl}\,\mathbf{B} = J$ and $\mathrm{div}\,\mathbf{B} = 0$ i.e. the magnetostatic* **??** *is fulfilled without the additional constraint.*

Proof. Assume $(\sigma, \mathbf{B}, \mathbf{p})$ is a solution of (1.1.1) and (1.1.2). Then the first equation is

$$\langle \sigma + J, \tau \rangle = \langle \mathbf{B}, \mathrm{curl}\tau \rangle \quad \forall \tau \in H_0^1$$

which is equivalent to $\mathbf{B} \in H(\mathrm{curl})$ and $J + \sigma = \mathrm{curl}\,\mathbf{B}$.

Now assume additionally, that (1.1.2) holds. Then by choosing $\mathbf{v}\mathfrak{p} \in \mathfrak{H}^1$ we get from the definition of the harmonic forms and $\mathfrak{H}^1 \perp \mathrm{curl}H_0^1$ from the Hodge decomposition and thus

$$\langle \mathrm{curl}\sigma, \mathfrak{p} \rangle + \langle du, d\mathfrak{p} \rangle + \langle p, \mathfrak{p} \rangle = \langle p, \mathfrak{p} \rangle = 0$$

and so $\mathfrak{p} = 0$. Then we can choose $\mathbf{v} = \mathrm{curl}\sigma$ to get

$$\langle \mathrm{curl}\sigma, \mathrm{curl}\sigma \rangle + \langle \mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathrm{curl}\sigma \rangle + \langle \mathbf{p}, \mathrm{curl}\sigma \rangle = \|\mathrm{curl}\sigma\|^2.$$

Because $\sigma \in H_0^1$ this gives us $\sigma = 0$. Also we have then $J = \mathrm{curl}\,\mathbf{B}$. At last we choose $\mathbf{v} = \mathbf{B}$ which gives us $\mathrm{div}\,\mathbf{B} = 0$ and thus we proved the first direction.

The other implication is clear i.e. if $\mathbf{B} \in H(\mathrm{curl}) \cap H_0(\mathrm{div})$ with $\mathrm{curl}\,\mathbf{B} = J$ and $\mathrm{div}\,\mathbf{B} = 0$ then the variational formulation clearly holds. □

If we now add the same additional constraints to both formulations of the problem then they will remain equivalent.

1.2 Curve integral constraint

We want to add now a curve integral constraint. We first want to rewrite the curve integral in variational form.

Let Γ be a closed curve with parametrization $\gamma : [0, |\Gamma|]$ s.t. $|\gamma'(t)| = 1$ and assume that γ is bijective i.e. the curve does not intersect itself. Since Γ is a closed curve it encompasses a set K Add drawing Let $\mathbf{n}$ be the unit normal of K. Then we know that $\mathbf{n} \perp \gamma'$.

If we now take $\mathbf{B}$ that

$$n \times \mathbf{B} = -\mathbf{B} \times n = -\mathbf{B} \cdot R_{\pi/2}\mathbf{n}$$

then $R_{\pi/2}\mathbf{n}$ is either γ' or $-\gamma'$. Assume w.l.o.g. that $R_{\pi/2}\mathbf{n} = \gamma'$ and thus

$$-\mathbf{B} \cdot R_{\pi/2}\mathbf{n} = -\mathbf{B} \cdot \gamma'.$$

So we see that we can write

$$n \times \mathbf{B} = -\mathbf{B} \cdot \gamma'$$

and so the curve integral becomes

$$\int_\Gamma \mathbf{B} \cdot dl = \int_0^{|\Gamma|} \mathbf{B}(\gamma(t)) \cdot \gamma'(t)\, dt = -\int_0^{|\Gamma|} n(\gamma(t)) \times \mathbf{B}(\gamma(t))\, dt = -\int_\Gamma n \times \mathbf{B}.$$

Defie ψ s.t. $\psi = 0$ on $\partial\Omega_{in}$ and $\psi = 1$ on Γ and constant one outside Then we observe

$$\int_\Omega \mathrm{curl}\psi \cdot \mathbf{B}\, dx = \int_\Omega \psi\, J\, dx - \int_\partial \Omega n \times \mathbf{B}\, dl = \int_\Omega \psi\, J\, dx - \int_\Gamma \mathbf{B} \cdot dl$$

Note that even though right hand side requires some regularity for $\mathbf{B}$ the left hand side makes sense even if $\mathbf{B}$ is only in L^2! So if we are in a situation where the we have curve integral given then we can add this constraint like this.

Let us assume we are given that the curve integral

$$\int_\Gamma \mathbf{B} \cdot dl = C_0$$

assuming this makes sense. Then we choose ψ and then get the constraint

$$\langle \mathrm{curl}\psi, \mathbf{B} \rangle = \langle J, \psi \rangle + -?C_0.$$

Note that there are not test functions involved since ψ is fixed. We define $C_1 := \langle J, \psi \rangle + -?C_0$. Then to get a variational formulation we multiply **??** with an arbitrary $\mu \in \mathbb{R}$. Then we reformulate the mixed variational form slightly.

Let $J \in L^2$, $\mathbf{p} \in \mathfrak{H}^1$. Find $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\mathrm{div})$, $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle u, \mathrm{curl}\tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \tag{1.2.1}$$ {eq:first_eq_mix

$$\langle \mathrm{curl}\sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in V^k, \tag{1.2.2}$$ {eq:second_eq_mi

$$\mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = \mu C_1 \quad \forall \mu \in \mathbb{R}. \tag{1.2.3}$$

which gives us the variational formulation of the magnetostatic problem with curve integral constraint. We will study the well-posedness of this formulation next. Using the analogous reasoning as in **??** we see that the first two equations are still equivalent to the magnetostatic problem without additional constraint even though we use now Need assumption that $\dim \mathfrak{H}^1 = 1$.

Defining $X := H_0^1 \times H_0(\mathrm{div}) \times \mathbb{R}$ and the bilinear form $a : X \times X \to \mathbb{R}$

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \langle \sigma, \tau \rangle - \langle u, \mathrm{curl}\tau \rangle + \langle \mathrm{curl}\sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle - \mu \langle \mathrm{curl}\psi, \mathbf{B} \rangle.$$

This allows us to rewrite **??** in the standard form

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = -\langle J, \tau\rangle - \mu C_1 \quad \forall(\tau, \mathbf{v}, \mu) \in X.$$

Note that the bilinear form a is not symmetric.

Lemma 1.2.1. *Define $T : X \to X$ as*

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\mathbf{p}, -\alpha\langle \mathbf{p}, \alpha\langle \mathfrak{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\rangle).$$

Assume c_ψ is positive. Then T is surjective.

Proof. Take $(\tau, \mathbf{v}, \mu) \in X$ arbitrary. Then Now we choose $\sigma = (1+1/c_P)^{-1}(\tau + (1/c_P^2)\,\mathbf{curl}^{-1}\,\mathbf{v})$ and $\mathbf{B}_{\mathfrak{B}} = \mathbf{v}_{\mathfrak{B}} - \mathbf{curl}\,\sigma$. So

$$\sigma - 1/c_P^2\,\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}} = \sigma - 1/c_P^2(\mathbf{curl}^{-1}\,\mathbf{v} - \sigma) = (1 + 1/c_P^2)\sigma - 1/c_P^2\,\mathbf{curl}^{-1}\,\mathbf{v} = \tau.$$

We simply choose $\mathbf{B}_{\mathfrak{B}^*} = \mathbf{v}_{\mathfrak{B}^*}$. For the harmonic part we observe for $\mathbf{v}_{\mathfrak{H}} = c_v p$
Let us look at the system

$$\begin{pmatrix} 1 & \beta \\ \alpha & 1/c_\psi \end{pmatrix} \begin{pmatrix} \kappa_u \\ \lambda \end{pmatrix} = \begin{pmatrix} c_v \\ \mu \end{pmatrix}$$

Now since $c_\psi > 0$ and $\alpha < 0$, $\beta > 0$ we get $1/c_\psi - \alpha\beta \neq 0$ and the system has a solution. Then we see

$$\mathbf{v}_{\mathfrak{H}} = c_v p = p(\kappa_u + \beta\lambda) = \mathbf{B}_{\mathfrak{H}} + \beta\lambda p$$

and

$$\mu = \alpha\kappa_u + 1/c_\psi\lambda = \alpha\kappa_u\|p\|^2 + 1/c_\psi\lambda = \alpha\langle \mathbf{B}, \mathbf{p}\rangle + 1/c_\psi\lambda.$$

And so in comining all that we arrive at $T(\sigma, \mathbf{B}, \mathbf{p}) = (\tau, \mathbf{v}, \mathbf{p})$. $\square$

Theorem 1.2.2. *a satisfies a inf-sup condition with γ depending on the Poincaré constant as well as ψ.*

Proof. We will use T-coercivity to prove it.

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\mathbf{p}, -\alpha\langle \mathbf{p}, \alpha\langle \mathfrak{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\rangle)$$

with $\beta = \frac{3c_1^2c_P^2}{c_\psi^2}$ and $\alpha = -\frac{c_\psi}{4c_1^2c_P^2}$. Then T is bijective **??**. We split up $d\psi = d\psi_0 + c_\psi \mathfrak{p}$ to get

$$
\begin{aligned}
&a(\sigma, \mathbf{B}, \lambda; T(\sigma, \mathbf{B}, \lambda)) \\
&= \langle \sigma, \sigma - \frac{1}{c_P^2}\rho\rangle - \langle \mathbf{B}, \mathbf{curl}\,\sigma - \frac{1}{c_P^2}\mathbf{curl}\,\rho\rangle + \langle \operatorname{div}\mathbf{B}, \operatorname{div}\mathbf{curl} \\
&\quad + \sigma \operatorname{div}\mathbf{B} + \beta\lambda\mathfrak{p}\rangle + \langle \lambda\mathbf{p}, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathfrak{p}\rangle - (\alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi})\langle \mathbf{B}, \mathbf{curl}\,\psi\rangle \\
&= \|\sigma\|^2 - \frac{1}{c_P^2}\langle \sigma, \rho\rangle + \frac{1}{c_P^2}\|B_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\beta - \alpha c_\psi \|\mathbf{B}_{\mathfrak{H}}\|^2 \\
&\quad - \alpha\langle \mathfrak{p}, \mathbf{B}\rangle\langle \mathbf{B}, \mathbf{curl}\,\psi_0\rangle - \frac{\lambda}{c_\psi}\langle B_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle \\
&\ldots \geq \|\sigma\|^2 - \left(\frac{1}{2}\|\sigma\|^2 + \frac{\|B_{\mathfrak{B}}\|^2}{2c_P^2}\right) + \frac{1}{c_P^2}\|B_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 \\
&\quad + \lambda^2\beta - \alpha c_\psi\|\mathbf{B}_{\mathfrak{H}}\|^2 - \left(\frac{\epsilon_1\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2}{2} + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2\epsilon_1}\right) - \left(\frac{\lambda^2}{2\epsilon_2 c_\psi^2} + \frac{\epsilon_2\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2}\right)
\end{aligned}
$$

Choose $\epsilon_1 = 4c_1^2c_P^2$ to get

$$
\begin{aligned}
&\frac{1}{2}\|\sigma\|^2 + \frac{1}{2c_P^2}\|B_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{1}{2\epsilon_2 c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(-\alpha c_\psi - \frac{4c_1^2c_P^2\alpha^2}{2}\right) - \|B_{\mathfrak{B}}\|^2\frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2} - \|B_{\mathfrak{B}}\|^2\frac{\epsilon_2\|\mathbf{curl}\,\psi_0\|^2}{2}
\end{aligned}
$$

Now choose $\epsilon_2 = \frac{1}{4c_1^2c_P^2}$ and plug in the definition of α to get bound it from below with

$$
\begin{aligned}
&\frac{1}{2}\|\sigma\|^2 + \|B_{\mathfrak{B}}\|^2\left(\frac{1}{2c_P^2} - \frac{1}{8c_P^2} - \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2}\right) + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{4c_1^2c_P^2}{2c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(\frac{c_\psi^2}{4c_1^2c_P^2} - \frac{c_1^2c_P^2c_\psi^2}{8c_1^4c_P^4}\right)
\end{aligned}
$$

and finally by using .... and $\beta = \frac{3c_1^2c_P^2}{c_\psi^2}$

$$
\frac{1}{2}\|\sigma\|^2 + \frac{1}{4c_P^2}\|B_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \frac{1}{2c_P^2B_{\mathfrak{B}^*}} + \frac{c_\psi}{8c_1^2c_P^2} + \frac{1}{2}\|\operatorname{div}\mathbf{B}\|^2 + \frac{c_1^2c_P^2}{c_\psi^2}\lambda^2 + \frac{c_\psi^2}{8c_1^2c_P^2}\|\mathbf{B}_{\mathfrak{H}}\|^2
$$

$\square$

Theorem 1.2.3 (Stability)**.** *The system is stable. For solution* $(\sigma, \mathbf{B}, \mathbf{p}) \in X$ *we get*

$$\|\sigma\|_V + \|\mathbf{B}\|_V + |\lambda| \leq \frac{\|J\| + |C_1|}{\gamma}.$$

Proof. The statement follows immediately from **??** and the fact that

$$|l(\tau, \mathbf{v}, \mu)| = | - \langle J, \tau \rangle - C_1 \mu| \leq (\|J\| + C_1)\|\tau, \mathbf{v}, \mu\|_X$$

and thus $\|l\|_{X'} \leq \|J\| + |C_1|$. $\square$

2 Discrete Hilbert complex

In order to approximate the Hodge Laplacian problem we want to use finite elements. We want to use them in a way that we can rebuild the structure of the Hilbert complex in our discretization. This section follows Sec. 5.2 in Arnold's book [1].

Let us assume the we have finite dimensional subspaces $V_h^k \subseteq V^k$. Then we define completely analogous to the continuous case,

$$\begin{aligned} \mathfrak{Z}_h^k &:= \{v \in V_h^k \mid dv = 0\} = \ker d \cap V_h^k \\ \mathfrak{B}_h^k &:= \{dv \mid v \in V_h^{k-1}\}. \end{aligned}$$

We can now also define the discrete harmonic forms. Now the situation is slightly different however. We will not use the continuous adjoint d_k^* to define it. Instead,

$$\mathfrak{H}_h^k := \{v \in \mathfrak{Z}_h^k \mid v \perp \mathfrak{B}_h^k\} = \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k,\perp}.$$

Notice that we have $\mathfrak{Z}_h^k \subseteq \mathfrak{Z}^k$ and $\mathfrak{B}_h^k \subseteq \mathfrak{b}^k$, but due to $\mathfrak{B}_h^{k,\perp} \supseteq \mathfrak{B}^{k,\perp}$ we have in general

$$\mathfrak{H}^k = \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k,\perp} \not\subseteq \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{H}^k.$$

There are three crucial properties that are necessary for stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want that the discrete spaces V_h^k approximate the continuous ones V^j. This can be generally summarized that

$$\lim_{h \to 0} \inf_{v_h \in V_h^j} \|w - v_h\| = 0, \quad \forall w \in V^j.$$

This is usually satisfied if we use established finite elements for a given space e.g. if we take Lagragian FE if $V = H^1$ or Raviart-Thomas if $V = H(\text{div})$ [**<empty citation>**].

The next property is more restrictive. We require that $dV_h^{k-1} \subseteq V_h^k$ and $dV_h^j \subseteq V_h^{j+1}$. This shows that the we cannot simply use arbitrary discrete subspaces independent from one another. We say the spaces have to be compatible [**<empty citation>**]. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. **??** directly to it. Let us do that.

Denote the restriction of d^j to V_h^j as d_h^j. Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \to V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to d_h the adjoint $d_j^* h$ is not the restriction of the adjoint the continuous adjoint d_j^*. In general, $V_h \nsubseteq V^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

and its dual complex

$$V_h^{k-1} \xleftarrow{d_k^*} V_h^k \xleftarrow{d_{k+1}^*} V_h^{k+1}$$

From the general Hilbert complex theory (Thm. **??**) we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \overset{\perp}{\oplus} \mathfrak{H}_h^j \overset{\perp}{\oplus} \mathfrak{B}_{jh}^*.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. Especially the question how well the discrete harmonic forms approximate the contiinous one will be looked at more closely.

The third crucial assumption is the existence of *bounded cochain projections* π_h. This is a projection that is a cochain map in the sense of cochain complexes **??** i.e. the following diagram commutes: π_h are either bounded in the V or in the W-norm where W-boundedness implies V boundedness. The cochain projection will play an important role in the stablity of the discrete system.

Let us now answer the question about the difference between discrete and continous harmonic forms. In order to do that we need some way to measure the "difference" between two subspaces.

Definition 2.0.1 (Gap between subspaces)**.** For a Banach space W with subspaces Z_1 and Z_2. Let S_1 and S_2 be the unit spheres in Z_1 and Z_2 respectively i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$. Then we define the gap between these subspaces as

$$\operatorname{gap}(Z_1, Z_2) = \max\{\sup_{z_1 \in S_1} \operatorname{dist} z_1, Z_2, \sup_{z_2 \in S_2} \operatorname{dist} z_2, Z_1\}$$

This definition is from [**kato perturbation theory**] and defines a metric on the set of closed subspaces of W (see **??**Remark p.198]) If W is a Hilbert space – as it is throughout this section – and Z_1 and Z_2 are closed then the $\operatorname{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|$ i.e. the difference in operator norm of the orthogonal projections onto Z_1 and Z_2. This gives us a measure of distance between spaces which we can now apply to the question of the difference of the difference between discrete and continous harmonic forms.

Proposition 2.0.2 (Gap between harmonic forms)**.** *Assume that the discrete complex* **??** *admits a V-bounded cochain projection π_h. Then*

$$\|(I - P_{\mathfrak{H}_h^k})q\|_V \leq \|(I - \pi_h^k)q\|_V, \forall q \in \mathfrak{H}^k$$
$$\|(I - P_{\mathfrak{H}^k})q_h\|_V \leq \|(I - \pi_h^k)P_{\mathfrak{H}^k} q\|_V, \forall q \in \mathfrak{H}^k, \forall q_h \in \mathfrak{H}_h^k$$

and then

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q \in \mathfrak{H}, \|q\|=1} \|(I - \pi_h^k)q\|_V$$

Proof. See [1, Thm. 5.2]. □

Do not forget the continuous poincare inequality

Proposition 2.0.3 (Discrete Poincare inequality)**.** *Assume that we have a V-bounded cochain projection π_h for the discrete Hilbert complex* **??**. *Then*

$$\|v\|_V \leq c_P \|\pi_h\|_V \|dv\|, \quad \forall v \in \mathfrak{Z}_h^{k\perp} \cap V_h$$

with c_P being the Poincare constant from **??**.

Proof. This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}^{k,\perp} \cap V^k) = \mathfrak{B} \supseteq \mathfrak{B}_h$ we find $z \in \mathfrak{Z}^{k\perp} \cap V_h$ s.t. $dz = dv$. We can apply now the continuous Poincare inequality **??** to get $\|z\|_V \leq c_P \|dz\|_V = c_P \|dv_h\|_V$. Now we can combine the different assumptions about the discrete Hilbert complex teo get $v_h - \pi_h z \in V_h^k$. Now we can use the fact that π_h is a cochain map and the fact that π_h is a projection:

$$d\pi_h^k z = \pi_h^{k+1} dz = \pi_h^{k+1} dv_h = dv_h$$

For the last equality we used also the fact that we have a discrete complex i.e. $d^k V_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \pi_h z) = 0$ i.e. $(v_h - \pi_h z) \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v, v_h - \pi_h z \rangle = \langle v, v_h - \pi_h z \rangle + \langle dv, d(v_h - \pi_h z) \rangle = \langle v, v_h - \pi_h z \rangle_V$$

so $v_h - \pi_h z$ is V orthogonal to v_h. So

$$\|v_h\|_V^2 = \langle v_h, \pi_h^k z \rangle_V + \langle v_h, v_h - \pi_h^k z \rangle_V = \langle v_h, \pi_h^k z \rangle_V \leq \|\pi_h\|_V \|dv\| \overset{Poincare ineq.}{\leq} c_P \|\pi_h\|_V \|dv\|_V$$

□

So we get the inf sup condition with $c_{P,h} = c_P \|\pi_h\|_V$ instead of c_P and obtain well-posedness.

3 Discretized magnetostatic problem

Let us apply this discretized Hilbert complex to the 2D Hilbert complex **??** to get $V_h^0 \subseteq H_0^1$, $V_h^1 \subseteq H_0(\mathrm{div})$ and $V_h^2 \subseteq L^2$ with

$$V_h^0 \xrightarrow{\mathbf{curl}} V_h^1 \xrightarrow{\mathrm{div}} V_h^2$$

and the dual complex

$$V_h^0 \xleftarrow{\widetilde{\mathrm{curl}}_h} V_h^1 \xleftarrow{-\widetilde{\mathrm{grad}}_h} V_h^2$$

where $\widetilde{\mathrm{curl}}_h$ is the adjoint of **curl** and corresponds thus to weak form of curl and the same for $\widetilde{\mathrm{grad}}_h$. The discretized version of the magnetostatic problem then states: Find $\mathbf{B}_h \in V_h^1$ s.t.

$$\widetilde{\mathrm{curl}}_h \mathbf{B} = J \text{ and } \mathrm{div}\, \mathbf{B} = 0.$$

Note that the divergence is enforced strongly while the curl is only enforced weakly. As explained in **??** we will add the curve integral constraint as in **??**. This gives us the following discrete formulation. Find $\sigma_h \in V_h^0$, $\mathbf{B} \in V_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathrm{curl}\tau_h \rangle = -\langle J, \tau_h \rangle \quad \forall \tau_h \in V_h^0,$$
$$\langle \mathrm{curl}\sigma_h, \mathbf{v}_h \rangle + \langle \mathrm{div}\, \mathbf{B}_h, \mathrm{div}\, \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle = 0 \quad \forall \mathbf{v}_h \in V_h^1,$$
$$\mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle = \mu C_1 \quad \forall \mu \in \mathbb{R}.$$

Here we assume for simplicity that $\mathbf{curl}\, \psi \in V_h^1$. Since we can choose ψ this is not unreasonable.

Note that this trial and test space is indeed conforming, but we choose a discrete harmonic form $\mathbf{p}_h \in \mathfrak{H}_h^1$ so the resulting bilinear forms are different. We assume that $\dim \mathfrak{H}_h^1 = \dim \mathfrak{H}^1 = 1$. The stability now follows from the exact same arguments as in **??** so we obtain a inf sup condition with a different constant γ_h that involves $c_{P,h}$ from **??**.

4 Implementation in 2D

4.1 Splines

For the discretization we will use the push forward of tensor product splines on a rectangular reference domain $\hat{\Omega}$. We use geometric degrees of freedom which we will introduce below **??**. This section is a recollection of [**broken FEEC framework on mapped multipatch**] since we use the same method as presented in this paper also to fix notation.

We will use two different types knot sequences, non-periodic and periodic ones. For the non-periodic ones we take an open knot sequence

$$0 = \xi_0 = \xi_1 = ... = \xi_p < \xi_{p+1} < \xi_{p+2} < ... < \xi_{n-1} < \xi_n = \xi_{n+1} = ... = \xi_{n+p}$$

with $n = N + p$ where N is the number of cells. Note that all the knot multiplicities in the interior are one and thus our spline space has maximal regularity. We then define $\mathcal{N}_i^q$ be the normalized B-spline [**multipatch paper**]. We then define the spline spce $\mathbb{S}_q = \mathbb{S}_q(\boldsymbol{\xi}) = span \mathcal{N}_i^q \mid i = 0, ..., n-1$. Since we have maximal regularity we get that

$$\{v \in C^{q-1} \mid v|_{\xi_{q+j,q+j+1}} \in \mathbb{P}_q\}.$$

$\mathcal{N}_0^{p-1}$ vanishes.

For the periodic case, the B-splines starting at knots ... are periodically extended to the start of the integral.

We now can take tensor product of spline spaces. We use the notation with $\mathbf{q} \in \{p-1, p\}^2$ and we define with $\mathbf{i} \in [N_0] \times [N_1]$

$$\mathcal{N}_{\mathbf{i}}^{\mathbf{q}} \mathcal{N}_{i_1}^{q_1} \mathcal{N}_{i_2}^{q_2}.$$

We write this as $\mathbb{S}_{\mathbf{q}} = \mathbb{S}_{q_1} \otimes \mathbb{S}_{q_2}$ The spline spaces used in the tensor product can also be periodic or only one of them can be periodic. On $\hat{\Omega}$ we obtain the following discrete Hilbert complex

$$\mathbb{S}_{p,p} \xrightarrow{\mathbf{curl}} \mathbb{S}_{p-1,p} \xrightarrow{\text{div}} \mathbb{S}_{p-1,p-1}$$

and we denote $\hat{V}^0 = \mathbb{S}_{p,p}$, $\hat{V}^1 = \mathbb{S}_{p-1,p}$ and $\hat{V}^2 = \mathbb{S}_{p-1,p-1}$. These spaces are conforming for $p > 1$?

Let us now investigate the degrees of freedom. We will use geometric degrees of freedom i.e. each degree of freedom can be associated with some geometrical element of our domain. We define Greville points by

$$\zeta_i := \frac{\xi_{i+1} + ... + \xi_{i+p}}{p}$$

i.e. the knot averages for $i = 0, ..., n-1$. Then the spline interpolation at these points is well-defined (see [**I think that was in the isogeometric analysis book**]).

This gives us the following geometric elements nodes, edges and cells

$$\begin{aligned} \hat{\mathsf{n}}_{\mathbf{i}} &:= (\zeta_{i_1}, \zeta_{i_2}) for \mathbf{i} \in \hat{\mathcal{M}}^0 \\ \hat{\mathsf{e}}_{d,\mathbf{i}} &:= [\hat{\mathsf{n}}_{\mathbf{i}}, \hat{\mathsf{n}}_{\mathbf{i}+\mathbf{e}_d}] for (d, \mathbf{i}) \in \hat{\mathcal{M}}^1 \\ \hat{\mathsf{n}}_{\mathbf{i}} &:= [\hat{\mathsf{e}}_{1,\mathbf{i}}, \hat{\mathsf{e}}_{1,\mathbf{i}}] = [\zeta_{i_1}, \zeta_{i_1+1}] \times [\zeta_{i_2}, \zeta_{i_2+1}] for \mathbf{i} \in \hat{\mathcal{M}}^2 \end{aligned}$$

with $[\cdot]$ being the convex hull. As before, $\mathbf{e}_d$ for $d = 1, 2$ is the standard basis vector of $\mathbb{R}^2$. The set of multiindices are defined as

$$\begin{aligned} \hat{\mathcal{M}}^0 &:= [n-1]^2 \\ \hat{\mathcal{M}}^1 &:= \{(d, \mathbf{i} =) \mid \mathbf{i} \in \hat{\mathcal{M}}^0, d \in \{1, 2\}\} \\ \hat{\mathcal{M}}^2 &:= [n-2]^2 \end{aligned}$$

Does this stuff go through for periodic case?

Now that we have defined the geometric elements we define the corresponding degrees of freedom as

$$\begin{aligned} \sigma_{\mathbf{i}}^{\hat{0}}(v) &:= v(\hat{\mathsf{n}}_{\mathbf{i}}) for \mathbf{i} \in \hat{\mathcal{M}}^0 \\ \hat{\sigma}_{\mathbf{i}}^2(v) &:= \int_{\hat{c}_{\mathbf{i}}} v for (\mathbf{i}) \in \hat{\mathcal{M}}^2 \end{aligned}$$

We $\mathbf{e}_d^{\perp}$ as the rotation of $\mathbf{e}_d$ by π_2 in counter clockwise direction i.e. $\mathbf{e}_1^{\perp} = \mathbf{e}_2$ and $\mathbf{e}_2^{\perp} = -\mathbf{e}_1$. Something about orientation

These degrees of freedome are unisolvent (explanation) and we can thus define our basis functions $\hat{\Lambda}^l_\mu$, $\mu \in \mathcal{M}^l$ as the basis which is dual to the degrees of freedom in the sense

$$\hat{\sigma}^l_\mu(\hat{\Lambda}^l_\nu) = \delta_{\mu,\nu} \quad for \mu, \nu \in \mathcal{M}^l.$$

The question is now on what function spaces these degrees of freedom are defined. We note first that the standard choice as described above with $\hat{V}^0 = H^1_0(\hat{\Omega})$ $\hat{V}^1 = H_0(\text{div})$ and $\hat{V}^2 = L^2(\hat{\Omega})$ can not work because the evaluation at point values is not stable for H^1 in 2D. Thus, we need to choose function spaces with higher regularity or integrability.

Let us define the spaces

$$W^1_{1,2}(\hat{\Omega}) := \{v \in L^1(\hat{\Omega}) \mid \partial_1 \partial_2 v \in L^1(\hat{\Omega})\}$$
$$W^1_d(\hat{\Omega}) := \{v \in L^1(\hat{\Omega}) \mid \partial_d v \in L^1(\hat{\Omega})\}$$

Why do we set the sequence to the spaces with L^1 sub instead of the intersection?

This now gives us the discrete setting on the reference domain $\hat{\Omega}$ the idea is now to define the basis functions on the physical domain Ω by a push forward of the basis functions on the reference domain. This is now the inverse of the analogous operations to **??** in 2D.

We will stick to the single patch case meaning that there is a diffeomorphism $F : \hat{\Omega} \to \Omega$. Then we define the pullbacks

$$\mathcal{P}^0_F : v \mapsto \hat{v} := v \circ F$$
$$\mathcal{P}^1_F : \mathbf{v} \mapsto \hat{\mathbf{v}} := \det DF \, DF^{-1}\mathbf{v}$$
$$\mathcal{P}^2_F : v \mapsto \hat{v} := (\det DF) v$$

which map functions on the physical domain Ω to functions on the reference domain $\hat{\Omega}$. Then we have the commuting properties

$$\widehat{\mathbf{curl}} \mathcal{P}^0_F v = \mathcal{P}^1_F \, \mathbf{curl} \, v$$
$$\widehat{\text{div}} \mathcal{P}^1_F \mathbf{v} = \mathcal{P}^1_F \, \text{div} \, v$$

Using the pullbacks we define the pushforwards as $\mathcal{F}^l := (\mathcal{P}^l_F)^{-1}$ and then we get the basis functions on the physical domain

$$\Lambda^l_\mu := \mathcal{F}^l \hat{\Lambda}^l_\mu$$

and then

$$V^l_h := \text{span}\{\Lambda^l_\mu \mid \mu \in \mathcal{M}^l\}$$

are our discrete spaces. Approximation properties???

Using the geometric degrees from **??** we can now construct the corresponding by

$$\sigma_\mu^l := \hat{\sigma}_\mu^l \circ \mathcal{P}_F^l$$

Then we have by construction that $\sigma_\mu^l(\Lambda_\nu^l) = \delta_{\mu,\nu}$.

Then these degrees of freedom commute with the corresponding differential operators as desired from **??**. They also correspond to geometric elements. σ^0 corresponds to point values in the physical domain, σ^1 to the fluxes through the image of edges and σ^2 to the integral over the mapped cells.

This very simple geometric interpretation gives us the ability to enforce the boundary condition directly by setting the corresponding degrees of freedom to zero.

For $V_h^0 \subseteq H_0^1$ we want the trace on the boundary to vanish which means that we set the values at the boundary nodes to zero. Thus, for $\mathtt{n}_{\mathbf{i}}$ on the boundary we want $\sigma_{\mathbf{i}}^0(v) = 0$.

For $V_h^1 \subseteq H_0(\mathrm{div})$ we want to have the normal trace zero. So when $\mathtt{e}_{d,\mathbf{i}}(\mathbf{v})$ is boundary edge we require This is then achieved when $\sigma_{d,\mathbf{i}}^1(\mathbf{v}) = 0$.

We now define the spaces $\bar{V}_h^l$ which are the correspoding spaces without any boundary conditions. Then we define the projections $P_h^l : \bar{V}_h^l \to \bar{V}_h^l$ which set the boundary degrees of freedom to zero. They have a very simple matrix representation $\mathbb{P}_h^l$ $(\mathbb{P}_h^l)_{\mu,\nu} = 1$ i.i.f. $\mu = \nu$ and μ does not correspond to a geometric element on the boundary. They are easily constructed by taking the identity matrix and setting the diagonal entries to zero that belong to boundary degrees of freedom.

On the lower level these basis functions are not used explicitely. Instead we use B-splines to compute the corresponding mass matrices etc. We will not go to deep into the details of implementation however. More details about the use of B-splines and the connection with the basis Λ_μ^l can be found in [**multipatch˙paper**]