Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let B be a magnetic field on the domain Ω. We the have the following boundary value problem:

$$\operatorname{curl} B = 0, \tag{1}$$

$$\operatorname{div} B = 0 \text{ in } \Omega \tag{2}$$

$$B \cdot n = 0 \text{ on } \partial\Omega \text{ and} \tag{3}$$

$$\int_\gamma B \cdot dl = C_0 \tag{4}$$

where n is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology.

At first, let us introduce some basic notions about differential forms. We follow the brief introduction given by Arnold (cf. [1, Sec. 6.1]), but less details will be given. Let V be a real vector space with $\dim V = n$ and $\operatorname{Alt}^k V$ be the space of k-alternating maps from V^k to $\mathbb{R}$. For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\substack{i_1<...<i_k \\ i_{k+1}<...<i_{k+l}}} \operatorname{sgn}(i_1, ..., i_{k+l})\omega(v_{i_1}, ..., v_{i_k})\nu(v_{i_{k+1}}, ..., v_{i_{k+l}})$$

where $\operatorname{sgn}(i_1, ..., i_{k+l})$ is the sign of the permutation $(1, ..., k+l) \mapsto (i_1, ..., i_{k+l})$. This definition is not very intuitive. TBD: Examples in 3D.

Let $\{u_i\}_{i=1}^n$ be any basis of V and $\{u^i\}_{i=1}^n$ the correspoding dual basis. Then

$$\{u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k} | \, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\operatorname{Alt}^k V$. In particular, $\dim \operatorname{Alt}^k V = \binom{n}{k}$.

Given a inner product $\langle \cdot, \cdot \rangle_V$ on V we obtain an inner product on $\operatorname{Alt}^k V$ by defining

$$\langle u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}, u^{j_1} \wedge ... \wedge u^{j_k} \rangle_{\operatorname{Alt}^k V} := \det\left[(\langle u_{i_k}, u_{i_l} \rangle_V)_{1 \leq k,l \leq n}\right]$$

which is then extended to all of $\mathrm{Alt}^k V$ by linearity. We denote with $\|\cdot\|_{\mathrm{Alt}^k V}$ the induced norm. From this definition it follows directly that for a orthonormal basis b_1,..., b_n the corresponding basis $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\mathrm{Alt}^k V$.

$\mathrm{Alt}^n V$ is one-dimensional and so we have to choose a basis. We say that two orthonormal bases of V have the same orientation if the change of basis has positive determinant. That divides the orthonormal bases into two classes with different orientation. We choose one of these classes and call these orthonormal bases positively oriented. Take $\omega \in \mathrm{Alt}^n V$. Then $\omega(b_1, ..., b_n)$ is the same for any positively oriented orthornormal basis. We now define the *volume form* $\mathrm{vol} \in \mathrm{Alt}^n v$ by requiring it to be 1 on all positively oriented orthonormal bases. Using this volume form we can now define the *Hodge star operator* $* : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$ via the property

$$\omega \wedge \mu = \langle *\omega, \mu \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} \quad \forall \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^{n-k} V.$$

The Hodge star is an isometry, we have $** = (-1)^{k(n-k)}\mathrm{Id}$ and

$$\omega \wedge *\mu = \langle \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol} \quad \forall \omega, \mu \in \mathrm{Alt}^k V.$$

In particular in $\mathbb{R}^3$, we have $** = \mathrm{Id}$ i.e. $*$ is self-inverse.

Now we will move on to differential forms. Because we will mostly deal with the case $V = \mathbb{R}^n$ and denote $\mathrm{Alt}^k \mathbb{R}^n$ just as Alt^k. Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and denote the space of differential forms of degree k on Ω as $\Lambda^k(\Omega)$. We extend now the Hodge star operator to differential forms $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ by applying it pointwise.

Then we define the L_p-norm of a k-form ω for $1 \leq p < \infty$ as (cf. [4])

$$\|\omega\|_{L_p^k(\Omega)} := \left(\int_\Omega \|\omega\|_{\mathrm{Alt}^k}^p \right)^{1/p}$$

and for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x \in \Omega} \|\omega(x)\|_{\mathrm{Alt}^k}.$$

$L_p^k(\Omega)$ are the spaces of k-forms s.t. the corresponding L_p-norm is finite. For $p = 2$ we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the L_2 inner product

$$\langle \omega, \nu \rangle := \int_\Omega \langle \omega, \nu \rangle_{\mathrm{Alt}^k}. \tag{5}$$

Proposition 1. *The Hodge star operator* $* : L_2^k(\Omega) \to L_2^{n-k}(\Omega)$ *is a Hilbert space isometry.*

Proof. This follows directly from the definition of the inner product (5) and the fact that $*$ is an isometry when applied to alternating forms Alt^k. $\square$

Our next goal is to extend the exterior derivative d of smooth differential forms in the weak sense (cf. [4]). Let $\mathring{d} : L_2^k(\Omega) \to L_2^{k+1}(\Omega)$ be the exterior derivative as an unbounded operator with domain $D(\mathring{d}) = C_0^\infty\Lambda^k(\Omega)$ which are the smooth compactly supported differential forms of degree k. Analogous, let $\mathring{\delta} : L_2^k(\Omega) \to L_2^{k-1}(\Omega)$ be the codifferential operator $\mathring{\delta} := (-1)^{n(k-1)+1} * \mathring{d}*$ also with domain $C_0^\infty\Lambda^k(\Omega)$.

Then the exterior derivative $d\omega \in L_{k+1}^p(\Omega)$ is defined as the unique $(k+1)$-form in $L_{k+1}^p(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge *\phi = \int_\Omega \omega \wedge *\mathring{\delta}\phi \quad \forall \phi \in C_0^\infty\Lambda^k(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$\begin{aligned} W_p^k(\Omega) &= \left\{\omega \in L^p\Lambda^k(\Omega) \mid d\omega \in L_p^{k+1}(\Omega)\right\}, \\ W_{p,loc}^k(\Omega) &= \left\{\omega \; k\text{-form} \mid \omega|_A \in W_p^k(A) \text{ for every } A \subseteq \Omega \text{ compact}\right\}. \end{aligned}$$

For $\omega \in W_p^k(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W_p^k} := \left(\|\omega\|_{L_p^k}^p + \|d\omega\|_{L_p^k}^p\right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max\left\{\|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k}\right\}.$$

Definition 1 (L^p-cohomology)**.** We define the following subspaces of $W_p^k(\Omega)$, $1 \le p \le \infty$:

$$\begin{aligned} \mathfrak{B}_k &:= dW_p^{k-1}(\Omega) \text{ and} \\ \mathfrak{Z}_k &:= \{\omega \in W_p^k(\Omega) \mid d\omega = 0\}. \end{aligned}$$

We call the k-forms in $\mathfrak{B}_k$ exact and the forms in $\mathfrak{Z}_k$ closed. Because $d \circ d = 0$ we always have $\mathfrak{B}_k \subseteq \mathfrak{Z}_k$. Then we define the de Rham- or L^p-cohomology space $H_{p,dR}^k(\Omega)$ as the quotient space

$$H_{p,dR}^k(\Omega) := \mathfrak{Z}_k / \mathfrak{B}_k.$$

We want to examine the Hilbert space $L_2^k(\Omega)$ more closely (see [1, Sec. 6.2.6] for more details). We denote $H^k(d;\Omega) := W_2^k(\Omega)$. If the domain is clear we will leave it out. Note that the above definition of the exterior derivative is in the Hilbert space setting equivalent to defining d as the adjoint of $\mathring{\delta}$.

In order to extend $\mathring{\delta}$ as well, we will need the following Now we just define $\delta := (-1)^{n(k-1)+1} * d*$ as in the smooth setting. A simple computation shows that this is then the adjoint of $\mathring{d}$. Define

$$H^k(\delta;\Omega) := \{\omega \in L_2^k(\Omega)|\ *\omega \in H^{n-k}(d)\}.$$

Now take $\omega \in H^{k+1}(\delta)$ and $\phi \in C_0^\infty \Lambda^k(\Omega)$. Then

$$\begin{aligned}
\langle \delta\omega, \phi\rangle &= (-1)^{nk+1}\langle *d*\omega, \phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}\langle d*\omega, *\phi\rangle = (-1)^{nk+1}(-1)^{k(n-k)}\langle *\omega, \mathring{\delta}*\phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}(-1)^{n(n-k-1)+1}\langle *\omega, *\mathring{d}**\phi\rangle \\
&= (-1)^{n(n-1)+2}(-1)^{k(n-k)}\langle \omega, \mathring{d}**\phi\rangle \\
&= \langle \omega, \mathring{d}\phi\rangle
\end{aligned}$$

where we used repeatedly that $*$ is an isometry and $** = (-1)^{k(n-k)}\mathrm{Id}$. This shows that δ is the adjoint of $\mathring{d}$ and has the domain $H^k(\delta)$.

We then define additionally the space $\mathring{H}^k(d;\Omega)$ as the closure of $C_0^\infty\Lambda^k(\Omega) \subseteq H^k(d;\Omega)$ w.r.t. the $H^k(d)$-norm i.e. $\mathring{H}^k(d;\Omega)$ corresponds to k-forms in $H^k(d;\Omega)$ being zero on the boundary. TBD: There are several different ways to characterize zero boundary conditions in the L^2 setting. We have to choose the one that works best.

Before we can reformulate the boundary value problem in the language of differential forms we have to introduce some things from simplicial topology. A quick overview over the essential notions that we will use can for example be found in [1, Chapter 2]. A more thorough introduction can be found in [2, Chapter 4]. For the sake of brevity, we will follow the more intuitive approach taken by Arnold.

Definition 2 (Affine simplex)**.** Let x_0, x_1, ..., $x_k \in \mathbb{R}^n$ be affine independent. Then

$$[x_0, x_1, ..., x_k] := \operatorname{conv}\{x_0, ..., x_k\}$$

is called an affine k-simplex.

We will assume all simplices to be affine.

Definition 3 (Simplicial complex)**.** A *simplicial complex* K is a collection of affine simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of σ is in K,
2. $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is either empty or a face of both σ and τ.

We call $|K| := \bigcup\{\sigma | \sigma \in K\}$ the polyhedron of K.

For any topological space X a homeomorphism $\tau : |K| \to X$ is called *triangulation* of X. Let $\{x_1, x_2, ...\}$ be the vertices in the simplicial complex K. We fix an ordering of the vertices for every simplex. That means for any k every k simplex σ has a designated representation in the form of

$$\sigma = [x_{i_0}, x_{i_1}, ..., x_{i_k}].$$

Definition 4 (k-chain)**.** Let K be a simplicial complex. A formal linear combination of k simplices

$$c = \sum_{\sigma \in K\ k \text{ simplex}} c_\sigma \sigma$$

with $c_\sigma \in \mathbb{R}$ is called *k-chain*. The vector space of all k-chains is denoted by C_k^{c}.

These spaces of k-chains become now a chain complex by introducing the boundary operator ∂.

Definition 5 (Boundary)**.** For any simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$ we define the boundary

$$\partial[x_{i_0}, x_{i_1}, ..., x_{i_k}] := \sum_{j=0}^{k} (-1)^j [x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$$

where $[x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$ is the simplex without vertex x_{i_j}. We then extend the definition of the boundary operator linearly to k-chains $c = \sum_\sigma c_\sigma \sigma$ by

$$\partial c := \sum_\sigma c_\sigma \partial \sigma.$$

.

A crucial property of the chain operator is the following.

Proposition 2. $\partial \circ \partial = 0$.

Proof. This can be proven by direct computation, analogous to [2, Chap.4, Lemma 1.6] □

We call a k-chain c a *k-cycle* if $\partial c = 0$ and we call c a *k-boundary* if there exists a $(k+1)$-chain d s.t. $c = \partial d$. Let $Z_c^k \subseteq C_c^k$ be the subspace of k-cycles and $B_c^k \subseteq C_c^k$ the subspace of k-boundaries. We can now define the homology spaces of our simplicial complex.

Definition 6 (Chain homology)**.** The homology spaces H_c^k are the quotient spaces of cycles and boundaries i.e.

$$H_c^k := {Z_c^k}\big/{B_c^k}.$$

The homology spaces are independent of the chosen simplicial complex [**empty citation**].

Next, we define the space of *k-cochains* $C^k(K)$ which is nothing else than the dual space of the space of k-chains i.e.

$$C^k(K) := \mathrm{Hom}(C_c^k; \mathbb{R})$$

where $\mathrm{Hom}(X, Y)$ is the space of all vector space homomorphisms (i.e. linear mappings) from X to Y. We now introduce an operator between these spaces of cochains.

Definition 7. We define the operator $\delta : C^k K \to C^{k+1}(K)$ via

$$(\delta f)(c) := f(\partial c).$$

We call a cochain $f \in C^k(K)$ *closed* if $\delta f = 0$ and we call f *exact* if there is a $g \in C^{k+1}(K)$ s.t. $f = \delta g$.

We define the cohomology spaces analogous to homology spaces above.

Definition 8 (Cohomology of cochains)**.** Denote the space of closed k-cochains as $Z^k(K)$ and $B^k(K)$ the space of exact k-cochains. We then define the *cohomology spaces* $H^k(K)$ as

$$H^k(K) := {Z^k(K)}\big/{B^k(K)}.$$

We will later show that if we consider certain subspaces of cochains so called *p-summable* cochains that the L^p-cohomology defined above and the cohomology spaces of these p-summable cochains are isomorphic.

In order to show existence and uniqueness of solutions of the magnetostatic problem we rely on a result about the isomorphism of a simplicial cohomology space $H_p^k(K)$ and the L^p-cohomology space $H_{p,dR}^k(\overline{\Omega})$. This result was proven in [4]. In the diploma thesis of Nikolai Nowaczyk [5], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [4] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [4]). Therefore, we can apply the result to our case.

1 Isomorphism of Cohomology

1.1 Assumptions

Because $\overline{\Omega}$ from our problem is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [4, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ and take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^*g$ where g is the standard metric in ℓ^2. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on ℓ^2. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned} g_S|_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_k}{\partial x_j}(x) \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left(Df(x)^T Df(x) \right)_{ij} \\ &= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle , \end{aligned}$$

where D denotes the Jacobian. (TBD: This Jacobian as written here would technically be in $\mathbb{R}^{\infty \times n}$. Only finitely many lines are non-zero though, but this is not quite rigorous yet.)

We have two crucial assumptions on the triangulation for the result to hold (cf. [4, p.194]). We summarize them under *GKS-condition* named after the three authors of [4].

Assumption 1 (GKS-condition)**.** We will assume the following on the simplicial complex K and the triangulation τ:

1. The star of every vertex in K contains at most N simplices.
2. For the differential of τ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1,\ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

   where d denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most N simplices, which is fulfilled if we have a shape regular mesh.

Because τ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v\neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v\neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v\neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

1.2 Statement of the Isomorphism

From now on we will assume that the GKS condition is fulfilled. The isomorphism of the cohomology spaces from [4] uses several mappings between different cohomology spaces.

Definition 9 (Induced map)**.** Let V and W be real vector spaces, $X \subseteq V$, $Y \subseteq W$ be subspaces. For a linear map $L : V \to W$ with $L(X) \subseteq Y$ we define the induced map

$$[L] : {}^{V}\!/_{X} \to {}^{W}\!/_{Y},\ [v] \mapsto [Lv].$$

It is easy to check that the induced map is well-defined using the definition of quotient space.

The first isomorphism is induced from a linear mapping from the so called *S-forms* $S^k_p(K)$ to *p-summable k-cochains* $C^k_p(K)$ which will both be defined next.

Definition 10. We define the following norm of a k-cochain f

$$\|f\|_{C_p^k(K)} := \left(\sum_{c\, k\text{-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{f\ k\text{-cochain} |\ \|f\|_{C_p^k(K)} < \infty\}.$$

Take $\tau, \sigma \in K$ s.t. τ is a face of σ which we write as $\tau < \sigma$. It can be shown that the standard embedding $j : \tau \hookrightarrow \sigma$ induces an restriction operator $j_{\sigma,\tau}^* : W_\infty^*(\sigma) \to W_\infty^*(\tau)$ which is bounded (cf [4, p.191]).

Definition 11 (S-forms)**.** Let

$$\theta = \{\theta(\sigma) \in W_\infty^k(\sigma) | \sigma \in K\}$$

be a collection of differential k-forms. We call θ S-form of degree k if we have for all simplices $\mu < \sigma$

$$j_{\sigma,\mu}^* \theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{d\theta(\sigma) | \sigma \in K\} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

For $\theta \in S^k(K)$ we now define the norm

$$\|\theta\|_{S_p(K)} := \left(\sum_{\sigma \in K} \|\theta(\sigma)\|_{W_\infty^k(\sigma)}^p \right)^{1/p}.$$

$S_p^k(K)$ are the S-forms of degree k s.t. this norm is finite.

Using integration we can define define the homomorphism (see [4, p.191])

$$I : S_p^k(K) \to C_p^k(K),\ I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K.$$

With the exterior derivative d on S-forms as defined above we define

$$\mathcal{Z}_p^k := \{\theta \in S_p^k(K) |\ d\theta = 0\}$$
$$\mathcal{B}_p^k := dS_p^k(K)$$

and then the resulting cohomology space

$$\mathscr{H}_p^k(K) := {}^{\mathcal{Z}_p^k}\!\big/_{\mathcal{B}_p^k}.$$

We denote the standard cochain cohomology as $H_p^k(K)$. Then we have that the integration mapping $I : S_p^k(K) \to C_p^k(K)$ induces an isomorphism on the cohomologies i.e. $[I] : \mathscr{H}_p^k(K) \to H_p^k(K)$ is an isomorphism of vector spaces (see Theorem 1 in [4] and the proof thereof).

The next step is to obtain an isomorphism between the cohomology of S-forms $\mathscr{H}_p^k(K)$ and the L_p cohomology $H_{p,dR}^k(\overline{\Omega})$. At first, we define

$$\varphi : W_{\infty,loc}^k(M) \to S^k(K),\ \omega \mapsto \{\omega|_\sigma \mid \sigma \in K\}.$$

This is a well-defined vector space isomorphism ([4, p.191]). This way we can identify $W_{\infty,loc}^k(M)$ with $S^k(K)$. Using the isomorphism φ we now define $S_p^k(M) := \varphi^{-1} S_p^k(K)$. It can be shown that $S_p^k(M) \subseteq W_p^k(M)$. Let $\iota : S_p^k(M) \hookrightarrow W_p^k(M)$ be the inclusion operator. The inclusion induces an isomorphism on cohomology [4, Lemma 4, Corollary] i.e. $[\iota] : \mathscr{H}_p^k(K) \to H_{p,dR}^k(\overline{\Omega})$ is an isomorphism.

In conclusion, we get the following isomorphisms of cohomologies:

$$H_{p,dR}^k(\overline{\Omega}) \xrightarrow{[\iota]^{-1}} \mathscr{H}_p^k(K) \xrightarrow{[I]} H_p^k(K).$$

This result will be crucial in the next section to obtain a unique solution to our problem.

2 Existence and uniqueness of solutions

2.1 Well-definedness of the integral constraint

In the problem, we have the additional constraint

$$\int_\gamma B \cdot \mathrm{d}l = C_0$$

for some $C_0 \in \mathbb{R}$ and a closed bounded triangulated curve γ. This corresponds in the language of differential forms in the general case to the integration of a k-form ω over a k-chain γ

$$\int_\gamma \omega = C_0.$$

However, we only assume $\omega \in \mathring{H}^k(d;\Omega)$ so we have to check if and how this can be well defined.

Above, we introduced the integral operator I for $S^k_p(K)$ which can therefore be applied on $\omega \in S^k_p(\overline{\Omega})$ as

$$I(\omega) := I(\varphi(\omega)).$$

If we fix now the closed k-chain γ then $I(\cdot)(\gamma) = \int_\gamma$ becomes a functional on $S^k_p(\overline{\Omega})$, but it is a-priori not clear how to extend this to closed forms in $W^k_p(\overline{\Omega})$.

We know that $\int_\gamma d\eta = 0$ for $\eta \in S^{k-1}_p(\overline{\Omega})$ because otherwise I would not induce an isomorphism on cohomology. We extend this now by setting $\int_\gamma d\nu = 0$ for all $\nu \in W^{k-1}_p(\overline{\Omega})$. We have to check whether this is consistent with the definition above i.e. we have to show that if $d\nu \in S^k_p(M)$ for some $\nu \in W^{k-1}_p k(\overline{\Omega})$ then it must follow from the previous definition of the integral that indeed

$$\int_\gamma d\nu = 0$$

holds. Let $A \subseteq M$ be a bounded neighborhood of γ. We can then find $\tilde{\nu}$ s.t. $\tilde{\nu} \in W^{k-1}_q(A)$ for any $q > 1$ and $d\tilde{\nu} = d\nu$ [7, Thm 3.1.1]. Now it is possible to apply Stoke's theorem [3, Thm. 9] to get $\int_\gamma d\nu = 0$ and consequently we have shown consistency.

In order to extend the functional $\int_\gamma$ further we will use two operators $\mathscr{R}$ and $\mathscr{A}$ which are constructed in the second section of [4]. The precise definition and details of their construction are not relevant for our purposes because we will only use the following properties (cf. [4, Thm.2]).

Theorem 1. *Assume that the triangulation τ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L^k_{1,loc} \to L^k_{1,loc}$, $\mathscr{A} : L^k_{1,loc} \to L^{k-1}_{1,loc}$ such that*

1. *$\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ for $\omega \in W^k_{1,loc}(\overline{\Omega})$*
2. *for any $1 \le p \le \infty$, $\mathscr{R}(W^k_p(\overline{\Omega})) \subseteq S^k_p(\overline{\Omega})$.*

We can now use the operator $\mathscr{R}$ to define $\int_\gamma \omega$ for closed $\omega \in W^k_p(M)$ as

$$\int_\gamma \omega := \int_\gamma \mathscr{R}\omega.$$

This is consistent with the curve integral for S-forms because if $\omega \in S^k_p(M)$ closed then due to Thm. 1

$$\int_\gamma \mathscr{R}\omega = \int_\gamma \omega + d\mathscr{A}\omega + \mathscr{A}d\omega = \int_\gamma \omega.$$

2.2 Existence and uniqueness

We start with the following

Proposition 3. *Let γ be a closed k-chain s.t. the homology class $[\gamma]$ spans the homology space H_c^k. Then for any $C_0 \in \mathbb{R}\setminus\{0\}$ if we have closed cochains F, G s.t.*

$$F(\gamma) = G(\gamma) = C_0$$

then $[F] = [G]$ i.e. their cohomology classes are equal.

Proof. From [1, Sec. 2.5] we know that $\dim H^k(K) = \dim H_c^k$. Because F and G are closed we therefore have $\lambda_F, \lambda_G \in \mathbb{R}$ and a cohomology class $[b]$ s.t. $[F] = \lambda_F[b]$ and $[G] = \lambda_G[b]$. This is equivalent to the existence of $(k-1)$-cochains J_F, J_G s.t.

$$F = \lambda_F b + \delta J_F \text{ and } G = \lambda_G b + \delta J_G.$$

so

$$0 \neq \lambda_F b(\gamma) + \delta J_F(\gamma) = F(\gamma) = G(\gamma) = \lambda_G b(\gamma) + \delta J_F(\gamma).$$

Because γ is closed we have for any $(k-1)$-chain J, $\delta J(\gamma) = J(\partial\gamma) = 0$ and so we arrive at $\lambda_F = \lambda_G$ i.e. $[F] = [G]$. □

We will return now to the magnetostatic problem. In order to use the results above we will reformulate the problem in the notation of differential forms. There are two ways to identify a vector field with a differential form (cf. [1, Table 6.1 and p.70]) either as a 1-form or a 2-form. For a vector field B we define

$$\begin{aligned} &D^1\, B := B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \text{ and} \\ &D^2\, B := B_2\, dx_2 \wedge dx_3 - B_2\, dx_1 \wedge dx_3 + B_3\, dx_1 \wedge dx_2 \end{aligned}$$

as the corresponding 1-form and 2-form. Then the exterior derivative is $dD^2\,\omega$ corresponds to the divergence, the codifferential $\delta D^2\,\omega$ corresponds to the curl and the normal component being zero on the boundary corresponds to $\omega \in \mathring{H}^2(d)$.[**empty citation**].

If we then use the association of 3-forms with scalars we have the corresponding boundary value problem without the integral condition for 2-forms: Find $\omega \in \mathring{H}^2(d)$ s.t.

$$\delta\omega = 0, \tag{6}$$

$$d\omega = 0 \text{ in } \Omega. \tag{7}$$

Next, we have to add the integral condition. We use that we are in three dimensions so $** = (-1)^{k(n-k)}\tilde{\nu} = \tilde{\nu}$ [1, p.66] for any k-form $\tilde{\nu}$. and observe

$$\begin{aligned} *D^2\, B &= B_1\, **dx_1 + B_2\, **dx_2 + B_3\, **dx_3 = B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \\ &= D^1\, B. \end{aligned}$$

Then we have

$$\int_\gamma *D^2\, B = \int_\gamma D^1\, B = \int_\gamma B \cdot \mathrm{d}l.$$

In the last step we used the fact that the integration of a 1-form over a curve is equivalent to the curve integral of the associated vector field (cf. [1, Sec. 6.2.3]). Hence, we can add the integral condition

$$\int_\gamma *\omega = C_0 \tag{8}$$

and obtain the equivalent problem on differential forms.

We are now able to proof existence of a solution. Take a closed cochain $F \in C_2^1(K)$ s.t. $F(\gamma) = C_0$ and $F(\partial d) = 0$ for any 2-chain d as in Prop. 3. Then we know from Sec. 1.2 that there exists a unique $[\theta] \in \mathscr{H}_2^1(K)$ s.t. $[I]([\theta]) = [F]$. Let us take the 1-form $\eta := \varphi^{-1}\theta \in W_2^1(\omega)$. Then $\int_\gamma \eta = C_0$ holds. Because we need a 2-form we will take a closer look at $*\eta$. We have that the Hodge star operator $* : L_2^1(\Omega) \to L_2^2(\Omega)$ is a Hilbert space isometry.

First we compute

$$\delta * \eta = (-1)^{n(k-1)+1} * d * *\eta = (-1)^{n(k-1)+1} * d\eta = 0.$$

Using the Hodge decomposition for unbounded domains (cf. [6, Lemma 1]) we get therefore that there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^3(\delta, \Omega)$ and a harmonic $\omega \in \mathring{H}^2(d) \cap H^3(\delta)$ s.t.

$$*\eta = \lim_{i\to\infty} \delta\phi_i + \omega.$$

where the limit is in $L_2^1(\Omega)$.

We will show that ω is then a solution. Because ω is harmonic we already know that $d\omega = 0$, $\delta\omega = 0$ and $\omega \in \mathring{H}^2(d)$. It remains to show that the integral condition (8) is also satisfied.

At first we check that the integral is well-defined. We have $\delta\omega = 0$ which implies

$$d * \omega = * * d * \omega = (-1)^{n(k-1)+1} * \delta\omega = 0$$

so $*\omega$ is a closed 1-form. Therefore the integral is well-defined as shown in Sec. 2.1.

Take a ball of radius $R > 0$ around the origin with R large enough s.t. $\gamma \subseteq B_R$. Using the fact that the range of δ is closed on bounded domains (cf. [6, Lemma 7]) there is some $\phi_R \in H^3(\delta; B_R)$ s.t.

$$*\eta|_{B_R} = \omega|_{B_R} + \delta\phi_R.$$

We then get for the integral condition (8)

$$\begin{aligned}\int_\gamma *\omega &= \int_\gamma *\omega|_{B_R} = \int_\gamma *(*\eta|_{B_R} - \delta\phi_R) = \int_\gamma \eta|_{B_R} - (-1)^{3\cdot 2+1} d * \phi_R \\ &= \int_\gamma \eta|_{B_R} = C_0\end{aligned}$$

so the integral condition (8) is fulfilled and ω is indeed a solution.
TBD: Put all this in a theorem.

Theorem 2. *The solution of the problem is unique.*

Proof. Let $\omega, \tilde{\omega}$ both be solutions. Because $*\omega$ and $*\tilde{\omega}$ are closed the cochains $c \mapsto \int_c \mathscr{R} * \omega$ and $c \mapsto \int_c \mathscr{R} * \tilde{\omega}$ are closed.

Due to $\int_\gamma \mathscr{R}*\omega = \int_\gamma \mathscr{R}*\tilde{\omega}$ and the assumption that $[\gamma]$ spans the homology space we have with Prop. 3 $[I(\mathscr{R} * \omega)] = [I(\mathscr{R} * \tilde{\omega})]$ and because $[I]$ is an isomorphism $[\mathscr{R} * \omega] = [\mathscr{R} * \tilde{\omega}]$. Hence,

$$[*\tilde{\omega}] = [\mathscr{R} * \tilde{\omega}] = [\mathscr{R} * \omega] = [*\omega].$$

That is equivalent to the existence of some 0-form $\phi \in H^0(d)$ s.t. $*\omega = *\tilde{\omega} + d\phi$. We continue by applying the Hodge star operator to both sides and use the definition of the codifferential δ:

$$\omega = \tilde{\omega} + *d\phi = \tilde{\omega} + *d * *\phi = \tilde{\omega} + (-1)^{(n-k)(k-1)+1}\delta * \phi.$$

Then because ω and $\tilde{\omega}$ are harmonic we have $\omega, \tilde{\omega} \perp \delta H^3(\delta)$ and therefore

$$\omega = \tilde{\omega}.$$

□

If we now translate this back to standard vector calculus terms we have found the unique solution of the homogeneous magnetostatic on our domain Ω.

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