

The goal is to solve the following problem. Let  $\Omega \subseteq \mathbb{R}^n$  be an exterior polyhedral domain of a compact set i.e.  $\mathbb{R}^n \setminus \Omega$  is a compact polyhedron. We then have the following boundary value problem: For a fixed  $C_0 \in \mathbb{R}$  and closed bounded  $k$ -chain  $\gamma$  find  $\omega \in H\Lambda^k(\Omega)$  s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0, \\ \text{tr } \omega &= 0 \text{ and} \\ \int_{\gamma} \omega &= C_0 \end{aligned}$$

Because we consider polyhedral domains we assume that  $\gamma$  consists of finitely many  $k$ -simplices and that the cohomology class  $[\gamma]$  is a generator of the simplicial homology group. Our goal will be to show existence and uniqueness of solutions. In order to achieve this, we rely on a result about the isomorphism of a simplicial cohomology space  $H_p^k(K)$  which will be defined below and the  $L^p$ -cohomology space  $H_{p,dR}^k(\overline{\Omega})$  ( $dR$  short for de Rham). This result was proven in [Gol88]. In the diploma thesis by Nikolai Nowaczyk [Now11], which mostly is based on this paper, many additional details can be found. The first part will be to present this result. It should be noted that even though the results in the paper are proved explicitly for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [Gol88]). Therefore, we can apply the result to our case.

## 1 Isomorphism of Cohomology

### 1.1 Assumptions

We assume that  $\overline{\Omega}$  admits a smooth triangulation  $\tau : |K| \rightarrow \overline{\Omega}$  with  $|K|$  being the geometric realization of the simplicial complex  $K$ . Because  $\overline{\Omega}$  is itself a polyhedron we can assume that  $\overline{\Omega}$  and  $|K|$  are equal as subsets of  $\mathbb{R}^n$ . However, we will use different metrics. We use the Euclidian metric on  $\overline{\Omega}$  and we use the standard simplicial metric on  $|K|$  (cf. [Gol88, p.191]). This metric is defined as follows:

Choose some numbering of the vertices  $\{x_1, x_2, \dots\}$  and take  $f : |K| \rightarrow \ell^2$  where  $\ell^2$  is the Hilbert space of real-valued square-summable sequences s.t.  $f(x_i) = e_i$  with  $e_i \in \ell^2$  being the standard unit vectors and  $f$  is affine on every simplex. This mapping is unique.

Then we define the metric on  $|K|$  as the pullback  $g_S = f^*g$  where  $g$  is the standard metric in  $\ell^2$ . Let  $\langle \cdot, \cdot \rangle$  be the standard scalar product on  $\ell^2$ . Then

for  $x \in |K|$  and  $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x |K|$  we have

$$\begin{aligned}
g_S \left( \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^{\infty} \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^{\infty} \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k,l=1}^{\infty} v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k=1}^{\infty} v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \\
&= \sum_{i,j=1}^n \sum_{k=1}^{\infty} v_i w_j (Df(x)^T Df(x))_{ij} \\
&= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle,
\end{aligned}$$

where  $D$  denotes the Jacobian.

We have two crucial assumptions on the triangulation for the result to hold (cf. [Gol88, p.194]). We summarize them under *GKS-condition* named after the three authors of the [Gol88].

**Assumption 1** (GKS-condition). *We will assume the following on the simplicial complex  $K$  and the triangulation  $\tau$ :*

1. *The star of every vertex in  $K$  contains at most  $N$  simplices.*
2. *For the differential of  $\tau$  we have constants  $C_1, C_2 > 0$  s.t.*

$$\|d\tau|_x\| < C_1, \quad \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where  $d$  denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on  $|K|$  and  $\overline{\Omega}$ .

The first assumption is equivalent to every vertex being contained in at most  $N$  simplices, which is fulfilled if we have a shape regular mesh.

Since we have  $\overline{\Omega} = |K|$  as subsets of  $\mathbb{R}^n$   $\tau$  is just the identity in our case. So then the second assumption says that for every  $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v, v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

## 1.2 Statement of the Isomorphism

Take  $\tau, \sigma \in K$  s.t.  $\tau \leq \sigma$ . We define an extension operator  $j_{\sigma, \tau}^* : W_\infty^* \rightarrow W_\infty^*$  which is bounded (cf [Gol88, p.191]).

**Definition 1.** *Let*

$$\theta = \{\theta(\sigma) \in W_\infty^k(\sigma) | \sigma \in T\}$$

*be a collection of differential  $k$ -forms. We call  $\theta$  S-form of degree  $k$  if we have for all for simplices  $\mu \leq \sigma$*

$$j_{\sigma, \mu}^* \theta(\sigma) = \theta(\mu).$$

*We denote with  $S^k(K)$  the space of all S-forms of degree  $k$  over the chain complex  $K$ . For  $\theta \in S^k(K)$  we define  $d\theta := \{d\theta(\sigma) | \sigma \in K\} \in S^{k+1}(K)$ .  $S^*(K)$  is the resulting cochain complex.*

Using integration we can define the homomorphism (see [Gol88, p.191])

$$I : S_p^k(K) \rightarrow C_p^k(K), \quad I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K$$

which induces an isomorphism on cohomology (see Thm. 1 in [Gol88] and the proof thereof).

We say that  $\omega \in W_{\infty, loc}^k(M)$  if  $\omega|_A \in W_\infty^k(A)$  for every  $A \subseteq M$  compact. Then we define

$$\varphi_\tau : W_{\infty, loc}^k(M) \rightarrow S^k(K), \quad \omega \mapsto \{\tau|_\sigma^* \omega | \sigma \in K\}.$$

This is a well-defined vector space isomorphism ([Gol88, p.191]). This way we can identify  $W_{\infty, loc}^k(M)$  with  $S^k(K)$ . For S-forms of degree  $k$  we now define the norm

$$\|\theta\|_{S_p(K)} := \sum_{\sigma \in K} \|\theta(\sigma)\|_{W_\infty^k(\sigma)}.$$

$S_p^k(K)$  are the S-forms of degree  $k$  s.t. this norm is finite. Using the isomorphism  $\varphi_\tau$  we now define  $S_p^k(M) := (\varphi_\tau)^{-1} S_p^k(K)$ .

We then have  $S_p^k(M) \subseteq W_p^k(M)$  and the inclusion induces an isomorphism on cohomology [Gol88, Lemma 4, Corollary].

Above, we defined the integral operator  $I$  for  $S_p^k(K)$  which can be therefore be applied on  $S_p^k(M)$  as well. If we fix now a closed finite  $k$ -chain  $\gamma$ . Then  $I(\cdot)(\gamma) = \int_\gamma$  becomes a functional on  $S_p^k(M)$ , but is a-priori not

clear how to extend this to  $W_p^k(M)$ . We know that  $\int_\gamma d\eta = 0$  for  $\eta \in S_p^k(M)$  because otherwise  $I$  would not induce an isomorphism on cohomology. We extend this now by setting  $\int_\gamma d\nu = 0$  for all  $\nu \in W_p^{k-1}(M)$ . We have to check whether this is consistent with the definition above. Let  $\nu \in W_p^k(M)$  s.t.  $d\nu \in S_p^k(M)$ . Let  $A \subseteq M$  be a bounded neighborhood of  $\gamma$ . We can then find  $\tilde{\nu}$  s.t.  $\tilde{\nu} \in W_q(A)$  for any  $q > 1$  and  $d\tilde{\nu} = d\nu$  [Sch06, Thm 3.1.1]. We can then apply Stoke's theorem [GKS82, Thm. 9] to get  $\int_\gamma d\nu = 0$ . This shows consistency.

In the second part of [Gol88] they construct the operators  $\mathcal{R}$  and  $\mathcal{A}$ . The precise definition and construction of these operators is not relevant for our purposes because we will only use the following properties (cf. [Gol88, Thm.2]).

**Theorem 1.** *Assume that the triangulation  $\tau$  fulfills the GKS-condition. Then there exist linear mappings  $\mathcal{R} : L_{1,loc}^k \rightarrow L_{1,loc}^k$ ,  $\mathcal{A} : L_{1,loc}^k \rightarrow L_{1,loc}^{k-1}$  such that*

1.  $\mathcal{R}\omega - \omega = d\mathcal{A}\omega + \mathcal{A}d\omega$  for  $\omega \in W_{1,loc}^k(M)$
2. for any  $1 \leq p \leq \infty$ ,  $\mathcal{R}(W_p^k(M)) \subseteq S_p^k(M)$ .

We can now use this operator  $\mathcal{R}$  to define  $\int_\gamma \omega$  for closed  $\omega \in W_p^k(M)$  as

$$\int_\gamma \omega := \int_\gamma \mathcal{R}\omega.$$

This is consistent because if  $\omega \in S_p^k(M)$  closed then due to Thm. 1

$$\int_\gamma \mathcal{R}\omega = \int_\gamma \omega + d\mathcal{A}\omega + \mathcal{A}d\omega = \int_\gamma \omega.$$

## 2 Existence and uniqueness of solutions

### 2.1 Existence

Returning now back to the problem, we are now able to proof existence of a solution. Take a closed cochain  $F \in C_p^k(K)$  s.t.  $F(\gamma) = C_0$  and  $F(\partial d) = 0$  for  $(k-1)$ -chains  $d$ . Then we know from ??? that there exists a unique  $[\theta] \in \mathcal{H}_p^k(K)$  s.t.  $[I]([\theta]) = [F]$ . Let us take  $\eta := \varphi_\tau^{-1}\theta$ . Then  $\int_\gamma \eta = C_0$  holds. If we now take the Hodge decomposition  $L_2^k(M) = \mathfrak{B}^\sharp \oplus \mathcal{H}^k \oplus \mathfrak{B}_\sharp^*$  and define

$\omega$  as the projection of  $\eta$  onto the harmonic forms  $\mathcal{H}^k$ . Then we know that  $d\omega = 0$ ,  $\delta\omega = 0$  and  $\text{tr } \omega = 0$ . So we only have to show that

$$\int_{\gamma} \omega = C_0.$$

We know from the Hodge decomposition that there exists a sequence  $(\phi_i)_{i \in \mathbb{N}} \subseteq L_2^{k-1}(M)$  s.t.  $\omega = \eta - \lim_{i \rightarrow \infty} d\phi_i$ . Let now be  $R > 0$  large enough s.t.  $\gamma \subseteq B_R$ . Then we know that  $dW_2^{k-1}(B_R)$  is closed in  $L_2^k(B_R)$ . Therefore there exists  $\phi_R \in W_2^{k-1}(B_R)$  s.t.  $\lim_{i \rightarrow \infty} d\phi_i|_{B_R} = d\phi_R$ . So we have  $\omega|_{B_R} = \eta|_{B_R} - d\phi_R$  and

$$\int_{\gamma} \omega = \int_{\gamma} \omega|_{B_R} = \int_{\gamma} \eta|_{B_R} = C_0.$$

This proves existence.

## 2.2 Uniqueness

The first step is to show that the cochain chosen in the proof of existence is in fact unique if restricted to closed chains.

**Proposition 1.** *Let  $\gamma$  be a  $k$ -chain s.t. the homology class  $[\gamma]$  is a generator of the homology group. Assume for some  $C_0 \in \mathbb{R}$  there exist cochains  $F, G \in C_p^k(K)$  s.t.*

$$F(\gamma) = C_0 \text{ and } F(\partial d) = 0 \text{ for all } (k-1)\text{-chains } d$$

*and the same for  $G$ . Then the restriction of  $F$  and  $G$  to closed chains is the same.*

*Proof.* Take any closed  $k$ -chain  $c$ . Because  $\gamma$  is the generator of the homology group we have  $n \in \mathbb{Z}$  s.t.  $[c] = [n\gamma]$  where  $[\cdot]$  is the corresponding homology class. That means that we have some  $(k-1)$ -chain  $d$  s.t.  $c = n\gamma + \partial d$ . Using the properties of  $F$  and  $G$ ,

$$F(c) = F(n\gamma + \partial d) = nF(\gamma) = nC_0.$$

Because the same computation is valid for  $G$   $F(c) = G(c)$  follows.  $\square$

**Theorem 2.** *Assume that a co-chain as in Prop. 2.2 exists. Then the solution of the problem is unique.*

*Proof.* Let  $\omega, \tilde{\omega}$  both be solutions. Because  $\int_{\gamma} \omega = \int_{\gamma} \tilde{\omega}$  and  $\omega$  and  $\tilde{\omega}$  are closed we have due to Prop. 2.2 that  $\int_c \omega = \int_c \tilde{\omega}$  for any closed  $k$ -chain  $c$ . So we have for the induced homomorphism  $[I]([\mathcal{R}\omega]) = [I]([\mathcal{R}\tilde{\omega}])$  and therefore due to the isomorphism of cohomology  $[\mathcal{R}\omega] = [\mathcal{R}\tilde{\omega}]$ . Hence,

$$[\tilde{\omega}] = [\mathcal{R}\tilde{\omega}] = [\mathcal{R}\omega] = [\omega]$$

we get the equality of the cohomology classes.

That is equivalent to the existence of some  $(k-1)$ -form  $\phi \in W_2^{k-1}(\overline{\Omega})$  s.t.  $\omega = \tilde{\omega} + d\phi$ . Then because  $\omega$  and  $\tilde{\omega}$  are harmonic we have  $\omega, \tilde{\omega} \perp dW_2^{k-1}(\overline{\Omega})$  and therefore

$$\omega = \tilde{\omega}.$$

□