Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let B be a magnetic field on the domain. We the have the following boundary value problem:

$$\text{curl } B = 0, \tag{1}$$
$$\text{div } B = 0 \text{ in } \Omega \tag{2}$$
$$B \cdot n = 0 \text{ on } \partial\Omega \text{ and} \tag{3}$$
$$\int_\gamma B \cdot dl = C_0 \tag{4}$$

where n is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology.

Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and Alt^k the space of alternating k-linear maps on $\mathbb{R}^n$. Let $\langle\cdot,\cdot\rangle_{\text{Alt}^k}$ be the standard inner product of alterating k-linear maps and $\|\cdot\|_{\text{Alt}^k}$ be the induced norm. Then we define the L^p-norm of a k-form ω for $1 \leq p < \infty$ as (cf. [4])

$$\|\omega\|_{L^k_p(\Omega)} := \left(\int_\Omega \|\omega\|^p_{\text{Alt}^k}\right)^{1/p}$$

and for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x\in\Omega} \|\omega(x)\|_{\text{Alt}^k}.$$

For $p = 2$ we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the L^2 inner product

$$\langle\omega,\nu\rangle := \int_\Omega \omega \wedge *\nu.$$

$Lp^k(\Omega)$ are the spaces of k-forms s.t. the corresponding L^p-norm is finite.

Our next goal is to extend the exterior derivative d of smooth differential forms in the weak sense (cf. [4]). This is done analogous to the definition of the usual weak derivative. The exterior derivative $d\omega \in L^p\Lambda^{k+1}(\Omega)$ is defined as the unique $(k+1)$-form in $L^p\Lambda^{k+1}(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge \phi = (-1)^{k+1} \int_\Omega \omega \wedge d\phi \quad \forall \phi \in C_0^\infty \Lambda^{n-k-1}(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$\begin{aligned} W_p^k(\Omega) &= \left\{ \omega \in L^p\Lambda^k(\Omega) \,|\, d\omega \in L^p\Lambda^{k+1}(\Omega) \right\}, \\ W_{p,loc}^k(\Omega) &= \left\{ \omega \; k\text{-form} \,|\, \omega|_A \in W_\infty^k(A) \text{ for every } A \subseteq M \text{ compact} \right\}. \end{aligned}$$

For $\omega \in W_p^k(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W_p^k} := \left(\|\omega\|_{L_p^k}^p + \|d\omega\|_{L_p^k}^p \right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max \left\{ \|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k} \right\}.$$

Definition 1 (L^p-cohomology)**.** We define the following subspaces of $W_p^k(\Omega)$, $1 \leq p \leq \infty$:

$$\begin{aligned} \mathfrak{B}_k &:= dW_p^{k-1}(\Omega) \text{ and} \\ \mathfrak{Z}_k &:= \{\omega \in W_p^k(\Omega) |\, d\omega = 0\}. \end{aligned}$$

We call the k-forms in $\mathfrak{B}_k$ exact and the forms in $\mathfrak{Z}_k$ closed. Then we define the de Rham- or L^p-cohomology space $H_{p,dR}^k(\Omega)$ as the quotient space

$$H_{p,dR}^k(\Omega) := \mathfrak{Z}_k / \mathfrak{B}_k.$$

We want to examine the Hilbert space $L^2\Lambda^k(\Omega)$ more closely (see [1, Sec. 6.2.6] for more details). Because $d : L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$ is an unbounded, densely defined operator with domain $C_0^\infty\Lambda^k(\Omega)$ the adjoint δ is well-defined and is equal to the codifferential operator of differential forms. We then define additionally the space $\mathring{W}_2^k(\Omega)$ as the closure of $C_0^\infty(\Omega) \subseteq W_2^k(\Omega)$ w.r.t. the $W_2^k(\Omega)$-norm.

Before we formulate the boundary value problem that we will study we have to introduce some things from simplicial topology. A quick overview over the essential notions that we will use can for example be found in [1, Chapter 2]. A more thorough introduction can be found in [2, Chapter 4]. For the sake of brevity, we will follow the more intuitive approach taken by Arnold.

Definition 2 (Affine simplex)**.** Let x_1, x_2, ..., x_n be affine independent. Then

$$[x_1, x_2, ..., x_n] := \text{conv}\,\{x_1, ..., x_n\}$$

is called an affine simplex.

We will assume all simplices to be affine.

Definition 3 (Simplicial complex)**.** A *simplicial complex* K is a collection of affine simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of σ is in K,
2. σ, $\tau \in K \Rightarrow \sigma \cap \tau$ is in K.

We call $|K| := \bigcup\{\sigma | \sigma \in K\}$ the polyhedron of K.

For any topological space X a homeomorphism $\tau : |K| \to X$ is called *triangulation* of X. Let $\{x_1, x_2, ...\}$ be the vertices in the simplicial complex K. We fix an ordering of the vertices for every simplex. That means for any k every k simplex σ has a designated representation in the form of

$$\sigma = [x_{i_0}, x_{i_1}, ..., x_{i_k}].$$

Definition 4 (k-chain)**.** Let K be a simplicial complex. A formal linear combination of k simplices

$$c = \sum_{\sigma \in K \text{ } k \text{ simplex}} c_\sigma \sigma$$

with $c_\sigma \in \mathbb{R}$ is called *k-chain.* The vector space of all k-chains is denoted by C_k^{c}.

These spaces of k-chains become now a chain complex by introducing the boundary operator ∂.

Definition 5 (Boundary)**.** For any simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$ we define the boundary

$$\partial[x_{i_0}, x_{i_1}, ..., x_{i_k}] := \sum_{j=0}^{k} (-1)^j [x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$$

where $[x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$ is the simplex without vertex $\hat{x}_{i_j}$. We then extend the definition of the boundary operator linearly to k-chains $c = \sum_\sigma c_\sigma \sigma$ by

$$\partial c := \sum_\sigma c_\sigma \partial \sigma.$$

.

A crucial property of the chain operator is the following.

Proposition 1. $\partial \circ \partial = 0$.

Proof. This can be proven by direct computation, analogous to [2, Chap.4, Lemma 1.6] $\square$

We call a k-chain c a *k-cycle* if $\partial c = 0$ and we call c a *k-boundary* if there exists a $(k+1)$-chain d s.t. $c = \partial d$. Let $Z_c^k \subseteq C_c^k$ be the subspace of k-cycles and $B_c^k \subseteq C_c^k$ the subspace of k-boundaries. We can now define the homology spaces of our simplicial complex.

Definition 6 (Chain homology)**.** The homology spaces H_c^k are the quotient spaces of cycles and boundaries i.e.

$$H_c^k := {}^{Z_c^k}\!/\!_{B_c^k}.$$

The homology spaces are independent of the chosen simplicial complex [**empty citation**].

Next, we define the space of *k-cochains* $C^k(K)$ which is nothing else then the dual space of the space of k-chains i.e.

$$C^k(K) := \mathrm{Hom}(C_c^k; \mathbb{R})$$

where $\mathrm{Hom}(X, Y)$ is the space of all vector space homomorphisms (i.e. linear mappings) from X to Y. We now introduce an operator between these spaces of cochains.

Definition 7. We define the operator $\delta : C^k K \to C^{k+1}(K)$ via

$$(\delta f)(c) := f(\partial c).$$

We call a cochain $f \in C^k(K)$ *closed* if $\delta f = 0$ and we call f *exact* if there is a $g \in C^{k+1}(K)$ s.t. $f = \delta g$.

We define the cohomology spaces analogous to homology spaces above.

Definition 8 (Cohomology of cochains)**.** Denote the space of closed k-cochains as $Z^k(K)$ and $B^k(K)$ the space of exact k-cochains. We then define the *cohomology spaces* $H^k(K)$ as

$$H^k(K) := {}^{Z^k(K)}\!/\!_{B^k(K)}.$$

We will later show that if we consider certain subspaces of cochains so called *p-summable* cochains that the L^p-cohomology defined above and the cohomology spaces of these p-summable cochains are isomorphic.

In order to show existence and uniqueness of solutios of the magnetostatic problem we rely on a result about the isomorphism of a simplicial cohomology space $H^k_p(K)$ and the L^p-cohomology space $H^k_{p,dR}(\overline{\Omega})$. This result was proven in [4]. In the diploma thesis of Nikolai Nowaczyk [5], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [4] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [4]). Therefore, we can apply the result to our case.

1 Isomorphism of Cohomology

1.1 Assumptions

In order to formulate the assumptions necessary for the result from [4] to work we will define some basic things from simplicial topology theory. More details and references can be found in [2, Chapter 4.21].

Because $\overline{\Omega}$ is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [4, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ and take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^*g$ where g is the standard metric in ℓ^2. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on ℓ^2. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, $\sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned} g_S|_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j,l=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left(Df(x)^T Df(x) \right)_{ij} \\ &= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle , \end{aligned}$$

where D denotes the Jacobian.

We have two crucial assumptions on the triangulation for the result to hold (cf. [4, p.194]). We summarize them under *GKS-condition* named after the three authors of the [4].

Assumption 1 (GKS-condition)**.** We will assume the following on the simplicial complex K and the triangulation τ:

1. The star of every vertex in K contains at most N simplices.

2. For the differential of τ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1, \ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where d denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most N simplices, which is fulfilled if we have a shape regular mesh.

Because τ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

1.2 Statement of the Isomorphism

The isomorphism of the cohomology spaces from [4] uses several mappings between different cohomology spaces. The first isomorphism is induced from a linear mapping between from the so called *S-forms* $S_p^k(K)$ to *p-summable k-cochains* $C_p^k(K)$ which will both be defined next.

Definition 9. We define the following norm of a k-cochain f

$$\|f\|_{C_p^k(K)} := \left(\sum_{c\, k\text{-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{f\ k\text{-cochain}|\ \|f\|_{C_p^k(K)} < \infty\}.$$

Take $\tau, \sigma \in K$ s.t. τ is a face of σ which we write as $\tau < \sigma$. It can be shown that the standard embedding $j : \tau \hookrightarrow \sigma$ induces an restriction operator $j^*_{\sigma,\tau} : W^*_\infty(\sigma) \to W^*_\infty(\tau)$ which is bounded (cf [4, p.191]).

Definition 10 (S-forms)**.** Let

$$\theta = \{\theta(\sigma) \in W^k_\infty(\sigma) | \sigma \in K\}$$

be a collection of differential k-forms. We call θ S-form of degree k if we have for all for simplices $\mu < \sigma$

$$j^*_{\sigma,\mu}\theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{d\theta(\sigma)|\sigma \in K\} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

For $\theta \in S^k(K)$ we now define the norm

$$\|\theta\|_{S_p(K)} := \left(\sum_{\sigma \in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \right)^{1/p}.$$

$S_p^k(K)$ are the S-forms of degree k s.t. this norm is finite.

Using integration we can define define the homomorphism (see [4, p.191])

$$I : S_p^k(K) \to C_p^k(K),\ I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K.$$

With the exterior derivative d on S-forms as defined above we define

$$\begin{aligned}\mathscr{Z}_p^k &:= \{\theta \in S_p^k(K) |\, d\theta = 0\} \\ \mathscr{B}_p^k &:= dS_p^k(K)\end{aligned}$$

and then the resulting cohomology space

$$\mathscr{H}_p^k(K) := {}^{\mathscr{Z}_p^k}\!\big/_{\mathscr{B}_p^k}.$$

We denote the standard cochain cohomology as $H_p^k(K)$. Then we have that the integration mapping $I : S_p^k(K) \to C_p^k(K)$ induces an isomorphism on the cohomologies i.e. $[I] : \mathscr{H}_p^k(K) \to H_p^k(K)$ is an isomorphism of vector spaces (see Theorem 1 in [4] and the proof thereof).

The next step is to obtain an isomorphism between the cohomology of S-forms $\mathscr{H}_p^k(K)$ and the L^p cohomology $H_{p,dR}^k(\overline{\Omega})$. At first, we define

$$\varphi : W_{\infty,loc}^k(M) \to S^k(K),\ \omega \mapsto \{\omega|_\sigma \,|\, \sigma \in K\}.$$

This is a well-defined vector space isomorphism ([4, p.191]). This way we can identify $W_{\infty,loc}^k(M)$ with $S^k(K)$. Using the isomorphism φ we now define $S_p^k(M) := \varphi^{-1} S_p^k(K)$. It can be shown that $S_p^k(M) \subseteq W_p^k(M)$. Let $\iota : S_p^k(M) \hookrightarrow W_p^k(M)$ be the inclusion operator. The inclusion induces an isomorphism on cohomology [4, Lemma 4, Corollary] i.e. $[\iota] : \mathscr{H}_p^k(K) \to H_{p,dR}^k(\overline{\Omega})$ is an isomorphism.

In conclusion, we get the following isomorphisms of cohomologies:

$$H_{p,dR}^k(\overline{\Omega}) \xrightarrow{[\iota]^{-1}} \mathscr{H}_p^k(K) \xrightarrow{[I]} H_p^k(K).$$

2 Existence and uniqueness of solutions

2.1 Well-definedness of the integral constraint

In the problem, we have the additional constraint

$$\int_\gamma \omega = C_0.$$

for some $C_0 \in \mathbb{R}$ and a closed bounded k-chain γ. However, we only assume $\omega \in \mathring{W}_2^k(\Omega)$ so we have to check if and how this can be well defined.

Above, we introduced the integral operator I for $S_p^k(K)$ which can therefore be applied on $\omega \in S_p^k(M)$ as

$$I(\omega) := I(\varphi(\omega)).$$

If we fix now the closed k-chain γ then $I(\cdot)(\gamma) = \int_\gamma$ becomes a functional on $S_p^k(\overline{\Omega})$, but it is a-priori not clear how to extend this to closed forms in $W_p^k(\overline{\Omega})$.

We know that $\int_\gamma d\eta = 0$ for $\eta \in S_p^{k-1}(\overline{\Omega})$ because otherwise I would not induce an isomorphism on cohomology. We extend this now by setting $\int_\gamma d\nu = 0$ for all $\nu \in W_p^{k-1}(\overline{\Omega})$. We have to check whether this is consistent with the definition above i.e. we have to show that if $d\nu \in S_p^k(M)$ for some $\nu \in W_p^{k-1}k(\overline{\Omega})$ then it must follow from the previous definition of the integral that indeed

$$\int_\gamma d\nu = 0$$

holds. Let $A \subseteq M$ be a bounded neighborhood of γ. We can then find $\tilde{\nu}$ s.t. $\tilde{\nu} \in W_q^{k-1}(A)$ for any $q > 1$ and $d\tilde{\nu} = d\nu$ [7, Thm 3.1.1]. Now it is possible to apply Stoke's theorem [3, Thm. 9] to get $\int_\gamma d\nu = 0$ and consequently we have shown consistency.

In order to extend the functional $\int_\gamma$ further we will use two operators $\mathscr{R}$ and $\mathscr{A}$ which are constructed in the second section of [4]. The precise definition and details of their construction are not relevant for our purposes because we will only use the following properties (cf. [4, Thm.2]).

Theorem 1. *Assume that the triangulation τ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L_{1,loc}^k \to L_{1,loc}^k$, $\mathscr{A} : L_{1,loc}^k \to L_{1,loc}^{k-1}$ such that*

1. *$\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ for $\omega \in W_{1,loc}^k(\overline{\Omega})$*
2. *for any $1 \leq p \leq \infty$, $\mathscr{R}(W_p^k(\overline{\Omega})) \subseteq S_p^k(\overline{\Omega})$.*

We can now use the operator $\mathscr{R}$ to define $\int_\gamma \omega$ for closed $\omega \in W_p^k(M)$ as

$$\int_\gamma \omega := \int_\gamma \mathscr{R}\omega.$$

This is consistent with the curve integral for S-forms because if $\omega \in S_p^k(M)$ closed then due to Thm. 1

$$\int_\gamma \mathscr{R}\omega = \int_\gamma \omega + d\mathscr{A}\omega + \mathscr{A}d\omega = \int_\gamma \omega.$$

2.2 Existence and uniqueness

We start with the following

Proposition 2. *Let γ be a k-chain s.t. the homology class $[\gamma]$ is a generator of the homology group. For any $C_0 \in \mathbb{R}$ there exist a cochain $F \in C_p^k(K)$ s.t.*

$$F(\gamma) = C_0 \text{ and } F(\partial d) = 0 \text{ for all } (k-1)\text{-chains } d.$$

Furthermore, if we have another $G \in C_p^k(K)$ with these two properties then

$$F(c) = G(c) \quad \text{for all closed } k\text{-chains } c.$$

Proof. At first, we construct such a k-cochain F. Note that we assume $[\gamma]$ to be a generator of the homology space. Using this we define $\tilde{F} : H_p^k(K) \to \mathbb{R}$ by setting $\tilde{F}([\gamma]) = C_0$. Let $Z_p^k \subseteq C_p^k(K)$ denote the space of closed k-cochains. Then we define the linear map $\pi : Z_p^k \to H_p^k(K)$, $f \mapsto [f]$ and $F := \tilde{F} \circ \pi$. This F has then oviously the desired properties.

Next, we show uniqueness on closed chains. Take any closed k-chain c. Because $[\gamma]$ is the generator of the homology group we have $\lambda \in \mathbb{R}$ s.t. $[c] = [\lambda\, \gamma]$. That means that we have some $(k-1)$-chain d s.t. $c = \lambda\, \gamma + \partial d$. Using the properties of F and G,

$$F(c) = F(\lambda\, \gamma + \partial d) = \lambda F(\gamma) = \lambda\, C_0.$$

Because the same computation is valid for G, $F(c) = G(c)$ follows. $\square$

We will return now to the magnetostatic problem. In order to use the results above we will reformulate the problem in the notation of differential forms. There are two ways to identify a vector field with a differential form (cf. [1, Table 6.1 and p.70]) either as a 1-form or a 2-form. For a vector field B we define

$$\begin{aligned} &D^1\, B := B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \text{ and} \\ &D^2\, B := B_2\, dx_2 \wedge dx_3 - B_2\, dx_1 \wedge dx_3 + B_3\, dx_1 \wedge dx_2 \end{aligned}$$

as the corresponding 1-form and 2-form. Then the exterior derivative is $d\omega$ corresponds to the divergence, the codifferential δ corresponds to the curl and the normal component being zero corresponds to $\omega \in \mathring{W}_2^2(\Omega)$.[**empty citation**].

If we then use the association of 3-forms with scalars we have the corresponding boundary value problem without the integral condition for a 2-form $\omega \in \mathring{W}_2^2(\Omega)$:

$$\delta\omega = 0, \tag{5}$$
$$d\omega = 0 \text{ in } \Omega. \tag{6}$$

Next, we have to add the integral condition. We use $** = (-1)^{k(n-k)}\tilde{\nu} = \tilde{\nu}$ [1, p.66] for any differential form $\tilde{\nu}$. and observe

$$\begin{aligned} *D^2\, B &= B_1 \, **dx_1 + B_2 \, **dx_2 + B_3 \, **dx_3 = B_1 \, **dx_1 + B_2 \, **dx_2 + B_3 \, **dx_3 \\ &= D^1\, B. \end{aligned}$$

Then we have

$$\int_\gamma *D^2\, B = \int_\gamma D^1\, B = \int_\gamma B \cdot \mathrm{d}l.$$

In the last step we used the fact that the integration of a 1-form over a curve can is equivalent to the curve integral of the associated vector field (cf. [1, p. 6.2.3]). Hence, we can add the integral condition

$$\int_\gamma *\omega = C_0 \tag{7}$$

and obtain the equivalent problem on differential forms.

We are now able to proof existence of a solution. Take a closed cochain $F \in C_p^k(K)$ s.t. $F(\gamma) = C_0$ and $F(\partial d) = 0$ for $(k-1)$-chains d as in Prop. 2. Then we know from Sec. 1.2 that there exists a unique $[\theta] \in \mathscr{H}_2^1(K)$ s.t. $[I]([\theta]) = [F]$. Let us take the 1-form $\eta := \varphi^{-1}\theta \in W_2^1(\omega)$. Then $\int_\gamma \eta = C_0$ holds. Because we need a 2-form we will take a closer look at $*\eta$. We have that the Hodge star operator $* : L_2^1(\Omega) \to L_2^2(\Omega)$ is a Hilbert space isometry.

First we compute

$$\delta * \eta = (-1)^{n(k-1)+1} * d * *\eta = (-1)^{n(k-1)+1} * d\eta = 0.$$

Using the Hodge decomposition for unbounded domains (cf. [6, Lemma 1]) we get therefore that there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^3(\delta, \Omega)$ and a harmonic $\omega \in \mathcal{H}^2$ s.t.

$$*\eta = \lim_{i\to\infty} \delta\phi_i + \omega.$$

where the limit is in the L^2-sense.

We will show that ω is then a solution. Because ω is harmonic we already know that $d\omega = 0$, $\delta\omega = 0$ and $\omega \in \mathring{H}^2(d)$. It remains to show that the integral condition (7) is also satisfied.

At first we check that the integral is well-defined. We have $\delta\omega = 0$ which implies

$$d * \omega = * * d * \omega = (-1)^{n(k-1)+1} * \delta\omega = 0$$

so $*\omega$ is a closed 1-form. Therefore the integral is well-defined as shown in Sec. 2.1.

Take a ball of radius $R > 0$ around the origin with R large enough s.t. $\gamma \subseteq B_R$. Using the factthat the range of δ is closed on bounded domains (cf. [6, Lemma 7]) there is some $\phi_R \in H^3(\delta; B_R)$ s.t.

$$*\eta|_{B_R} = \omega|_{B_R} + \delta\phi_R.$$

We then get for the integral condition (7)

$$\begin{aligned}\int_\gamma *\omega &= \int_\gamma *\omega|_{B_R} = \int_\gamma *(*\eta|_{B_R} - \delta\phi_R) = \int_\gamma \eta|_{B_R} - (-1)^{3\cdot 2+1} d * \phi_R \\ &= \int_\gamma \eta|_{B_R} = C_0\end{aligned}$$

so the integral condition (7) is fulfilled and ω is a indeed solution.
TBD: Put all this in a theorem.

Theorem 2. *The solution of the problem is unique.*

Proof. Let $\omega, \tilde{\omega}$ both be solutions. Because $\int_\gamma *\omega = \int_\gamma *\tilde{\omega}$ and $*\omega$ and $*\tilde{\omega}$ are closed we have due to Prop. 2 that $\int_c *\omega = \int_c *\tilde{\omega}$ for any closed 1-chain c. So we have for the induced homomorphism $[I]([\mathscr{R} * \omega]) = [I]([\mathscr{R} * \tilde{\omega}])$ and because $[I]$ is an isomorphism $[\mathscr{R} * \omega] = [\mathscr{R} * \tilde{\omega}]$. Hence,

$$[*\tilde{\omega}] = [\mathscr{R} * \tilde{\omega}] = [\mathscr{R} * \omega] = [*\omega].$$

That is equivalent to the existence of some 0-form $\phi \in H^0(d)$ s.t. $*\omega = *\tilde{\omega} + d\phi$. We continue by applying the Hodge star operator to both sides and use the definition of the codifferential δ:

$$\omega = \tilde{\omega} + *d\phi = \tilde{\omega} + *d * *\phi = \tilde{\omega} + (-1)^{(n-k)(k-1)+1}\delta * \phi.$$

Then because ω and $\tilde{\omega}$ are harmonic we have $\omega, \tilde{\omega} \perp \delta H^3(\delta)$ and therefore

$$\omega = \tilde{\omega}.$$

$\square$

3 Application: Homogeneous magnetostatic problem on the exterior domain of a torus

As an application of this general boundary value problem we will have a look at the following magnetostatic problem. Let Ω be the exterior domain of a triangulated torus i.e. $\mathbb{R}^3 \setminus \Omega$ is a torus with triangulated surface. Let B be the magnetic field. We then have the following boundary value problem: It is natural to identify the magnetic field B with a 2-form ω. Then the exterior derivative is $d\omega$ corresponds to the divergence, the codifferential δ corresponds to the curl and the normal component being zero is corresponds to $\omega \in \mathring{W}_2^2(\Omega)$.[**empty citation**]. The curve integral is an integration over a one-dimensional manifold and corresponds therefore to the integration of a one-form. Therefore the condition **??** is can be expressed with the hodge star operator $*$ as

$$\int_\gamma *\omega = C_0.$$

Now we want to apply our previous results. Ω fulfills all required assumptions for the domain. Because γ goes around the torus once and the homology space H_c^1 is one-dimensional (TBD: This has to be referenced or proven). Therefore because γ is not a boundary $[\gamma]$ spans H_c^1. Now all assumptions are fulfilled and we can apply our result. We will do so on 1-forms and transfer the result to 2-forms using the Hodge star operator.

Existence

Let $\tilde{\omega} \in \mathring{W}_2^1(\Omega)$ be the unique solution of our general problem **??** and define $\omega := *\tilde{\omega}$. Then we use $** = (-1)^{k(n-k)}\tilde{\omega} = \tilde{\omega}$ [1, p.66] to get

$$d\omega = **\, d * \tilde{\omega} = *(-1)^{n(k-1)+1}\delta\tilde{\omega} = 0$$

and

$$\delta\omega = (-1)^{n(k-1)+1} * d * \omega = (-1)^{n(k-1)+1} * dom\tilde{e}ga = 0.$$

References

[1] Douglas N Arnold. *Finite Element Exterior Calculus*. SIAM, 2018.

[2] Glen E Bredon. *Topology and geometry*. Vol. 139. Graduate Texts in Mathematics. Springer, 2013.

[3] VM Gol'dshtein, VI Kuz'minov, and IA Shvedov. "Integration of differential forms of the classesW* p, q". In: *Siberian Mathematical Journal* 23.5 (1982), pp. 640–653.

[4] Gol'dshtein, V.M., Kuz'minov, V.I. & Shvedov, I.A. "De Rham isomorphism of the Lp-cohomology of noncompact Riemannian manifolds". In: *Sib Math J 29* 322.10 (1988), pp. 190–197.

[5] N Nowaczyk. "The de Rham Isomorphism and the L_p-Cohomology of non-compact Riemannian Manifolds". Friedrich-Wilhelms-Universität Bonn, 2011. URL: `https://nikno.de/index.php/publications/%20(last%20visited:%2016.11.2022)`.

[6] Rainer Picard. "Some decomposition theorems and their application to non-linear potential theory and Hodge theory". In: *Mathematical methods in the applied sciences* 12.1 (1990), pp. 35–52.

[7] Günter Schwarz. *Hodge Decomposition–A method for solving boundary value problems*. Springer, 2006.