1 Differential forms

Before we define differential forms, let us start by revising some basics from differential geometry. We follow the approach from [2, Sec. II.5]. It should be noted that there are different definitions of tangent space, but these lead to isomorphic notions (see e.g. [3, Sec. 1.B]). Let M be a smooth manifold without boundary. For a point $p \in M$ and a neighborhood U we call a function $f : U \to \mathbb{R}$ differentiable if for a local chart $\phi : U \to \mathbb{R}^k$ we have that $f \circ \phi^{-1}$ is differentiable at $\phi(p)$.

Let $I \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma : I \to M$ be a differentiable curve with $\gamma(0) = p \in M$. For a differentiable $f : U \to \mathbb{R}$ we define the the directional derivative $D_\gamma(f) := \frac{d}{dt} f(\gamma(t))|_{t=0}$. We call the functional $D_\gamma : C^1(U) \to \mathbb{R}$ tangent vector. With the help of a local chart ϕ we can now express these derivations by

$$\begin{aligned} D_\gamma(f) &= \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt}(f \circ \phi^{-1} \circ \phi)(\gamma(t))|_{t=0} && (1.0.1) \\ &= \sum_{i=1}^{k} \frac{\partial (f \circ \phi^{-1})}{\partial x_i}(\phi(p)) \, \frac{d}{dt}[\phi^{-1}(\gamma(t))]_i|_{t=0} = (\sum_{i=1}^{k} v_i \frac{\partial}{\partial x_i})(f). && (1.0.2) \end{aligned}$$

In the last step, we abused the location by denoting by f also the corresponding expression in local coordinates i.e. $f \circ \phi^{-1}$. This way we identified D_γ with $v \in \mathbb{R}^k$ by $v_i := \frac{d}{dt}[\phi^{-1}(\gamma(t))]_i|_{t=0}$. If we use a curve $\tilde{\gamma}$ s.t. $D_{\tilde{\gamma}} = D_\gamma$ and the same local chart ϕ then we obtain the same vector so this identification is unique.

Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let B be a magnetic field on the domain Ω. We the have the following boundary value problem:

$$\begin{aligned} \operatorname{curl} B &= 0, && (1.0.3) \\ \operatorname{div} B &= 0 \text{ in } \Omega && (1.0.4) \\ B \cdot n &= 0 \text{ on } \partial\Omega \text{ and} && (1.0.5) \\ \int_\gamma B \cdot dl &= C_0 && (1.0.6) \end{aligned}$$

where n is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology among other things...

At first, let us introduce some basic notions about differential forms. We follow the brief introduction given by Arnold (cf. [1, Sec. 6.1]), but less details will be given.

1.1 Alternating maps

Let V be a real vector space with $\dim V = n$. Then k-linear maps are of the form

$$\omega : \underbrace{V \times V \times ... \times V}_{k \text{ times}} \to \mathbb{R}$$

that are linear in every component. We call a k-linear form *alternating* if the sign switches when two arguments are exchanged i.e.

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k), \text{ for } 1 \leq i < j \leq k, \quad v_1, ..., v_k \in V.$$

Denote the space of alternating maps by $\operatorname{Alt}^k V$.

For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\pi} \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, ..., v_{\pi(k)})\nu(v_{\pi(k+1)}, ..., v_{\pi(k+l)})$$

where we sum over all permutations $\pi : \{1, ..., k+l\} \to \{1, ..., k+l\}$ s.t. $\pi(1) < ... < \pi(k)$ and $\pi(k+1) < ... < \pi(k+l)$. This definition is not very intuitive. TBD: Examples in 3D.

Let $\{u_i\}_{i=1}^n$ be any basis of V and $\{u^i\}_{i=1}^n$ the correspoding dual basis i.e. $u^i : V \to \mathbb{R}$, $u^i(u_j) = \delta_{ij}$. Then

$$\{u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k} \,|\, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\operatorname{Alt}^k V$. In particular, $\dim \operatorname{Alt}^k V = \binom{n}{k}$.

Given a inner product $\langle \cdot, \cdot \rangle_V$ on V we obtain an inner product on $\operatorname{Alt}^k V$ by defining

$$\langle u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}, u^{j_1} \wedge ... \wedge u^{j_k} \rangle_{\operatorname{Alt}^k V} := \det\left[(\langle u_{i_k}, u_{i_l} \rangle_V)_{1 \leq k, l \leq n}\right]$$

which is then extended to all of $\operatorname{Alt}^k V$ by linearity. We denote with $\|\cdot\|_{\operatorname{Alt}^k V}$ the induced norm. From this definition it follows directly that for a orthonormal basis b_1, ..., b_n the corresponding basis $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\operatorname{Alt}^k V$.

$\mathrm{Alt}^n V$ is one-dimensional and so can choose a basis by fixing a specific non-zero element. We say that two orthonormal bases of V have the same orientation if the change of basis has positive determinant. That divides the orthonormal bases into two classes with different orientation. We choose one of these classes and call these orthonormal bases positively oriented. In $\mathbb{R}^n$, the convention is to define the class as positively oriented which includes the standard orthonormal basis. Take $\omega \in \mathrm{Alt}^n V$. Then $\omega(b_1, ..., b_n)$ is the same for any positively oriented orthornormal basis. We now define the *volume form* $\mathrm{vol} \in \mathrm{Alt}^n V$ by requiring it to be 1 on all positively oriented orthonormal bases. Using this volume form we can now define the *Hodge star operator* $\star : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$ via the property

$$\omega \wedge \mu = \langle \star\omega, \mu \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} \quad \forall \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^{n-k} V.$$

The Hodge star is an isometry, we have $\star\star = (-1)^{k(n-k)}\mathrm{Id}$ and

$$\omega \wedge \star\mu = \langle \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol} \quad \forall \omega, \mu \in \mathrm{Alt}^k V.$$

In particular in $\mathbb{R}^3$, we have $\star\star = \mathrm{Id}$ i.e. $\star$ is self-inverse.

$\mathrm{Alt}^0 V$ is just $\mathbb{R}$ by definition. So the Hodge star now gives us a natural isomorphism from $\mathrm{Alt}^n V \to \mathbb{R}$. We call the real number that is associated with an element of $\mathrm{Alt}^n V$ *scalar proxy*. Similarly we have that $\mathrm{Alt}^1 V = V'$, the usual dual space of V. Because V is finite-dimensional we know that $V \cong V'$ using the Riesz representation. Then with the help of the Hodge star we obtain a natural isomorphism from $\mathrm{Alt}^{n-1} V \to V$. We call the vector corresponding to a 1- or $(n-1)$-alternating map *vector proxy*.

Note that for $n = 2$ the situation is slightly ambiguous, see [1, p.67]. But this case will not be relevant in this thesis.

Let us take a closer look at the case of $V = \mathbb{R}^3$ with the standard basis vectors e_1, e_2 and e_3. Denote the resulting elements of the dual basis with dx^i. Then for some $v \in V$ we get the corresponding 1-alternating map $v_1\, dx^1 + v_2\, dx^2 + v_3\, dx^3 \in \mathrm{Alt}^1$ and the 2-alternating map $v_1\, dx^2 \wedge dx^3 - v_2\, dx^1 \wedge dx^3 + v_3\, dx^1 \wedge dx^2 \in \mathrm{Alt}^1$. For the scalar proxies, take $c \in \mathbb{R}$. Then $c \in \mathrm{Alt}^0$ of course and because we have $\mathrm{vol} = dx^1 \wedge dx^2 \wedge dx^3$ we get the corresponding $c\, dx^1 \wedge dx^2 \wedge dx^3 \in \mathrm{Alt}^3$.

We can now use this relation to investigate the meaning of the wedge product for vector proxies. Let us denote Φ^k as the isomorphism to the k-alternating map with this scalar or vector as proxy. Let $v, w \in \mathbb{R}^3$. Then

$$\begin{aligned} &\Phi^1 v \wedge \Phi^1 w = \Phi^2(v \times w) \text{ and} \\ &\Phi^1 v \wedge \Phi^2 w = \Phi^3(v \cdot w). \end{aligned}$$

Note that for a 0-alternating map (i.e. a scalar) the wedge product is nothing but the vector-scalar multiplication.

So in 3 dimensions, we can always relate the operations of differential forms to the operations of the corresponding proxies.

1.2 Differential forms

Before we define differential forms, let us start by revising some basics from differential geometry. We follow the approach from [2, Sec. II.5]. It should be noted that there are different definitions of tangent space, but these lead to isomorphic notions (see e.g. [3, Sec. 1.B]). Let M be a smooth manifold without boundary. For a point $p \in M$ and a neighorhood U we call a function $f : U \to \mathbb{R}$ differentiable if for a local chart $\phi : U \to \mathbb{R}^k$ we have that $f \circ \phi^{-1}$ is differentiable at $\phi(p)$. Note that we assume that our manifold is differentiable so all chart transitions are differentiable and thus this definition is independent of the chosen chart.

Let $I \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma : I \to M$ be a differentiable curve with $\gamma(0) = p \in M$. For a differentiable $f : U \to \mathbb{R}$ we define the the directional derivative $D_\gamma(f) := \frac{d}{dt} f(\gamma(t))|_{t=0}$. We call the functional $D_\gamma : C^1(U) \to \mathbb{R}$ tangent vector. Let us fix a chart $\phi : U \to \mathbb{R}^n$ and let us define

$$\frac{\partial}{\partial x_i}|_p : C^1(U) \to \mathbb{R} f \mapsto \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)).$$

The value depend in general on the chosen chart. With the help of a local chart ϕ we can now express these derivations by

$$\begin{aligned} D_\gamma(f) &= \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt}(f \circ \phi^{-1} \circ \phi)(\gamma(t))|_{t=0} && (1.2.1) \\ &= \sum_{i=1}^{k} \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p)) \phi^{-1} \circ (\gamma)_i'(0) && (1.2.2) \\ &= \left(\sum_{i=1}^{k} \frac{\partial}{\partial x_i}|_p \phi^{-1} \circ (\gamma)_i'(0) \right)(f) && (1.2.3) \end{aligned}$$

Thus we can express

$$D_\gamma = \sum_{i=1}^{k} \frac{\partial}{\partial x_i}|_p \phi^{-1} \circ (\gamma)_i'(0).$$

So we have that $T_pM = \text{span}\left\{\frac{\partial}{\partial x_i}|_p\right\}$. We will show that this indeed a basis. Assume we have $\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}|_p = 0$ Then because $\phi_j \circ \phi^{-1}(x) = x_j$ for $x \in \phi(U)$ and $1 \leq j \leq n$. Then we have

$$0 = \left(\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i}|_p\right)(\phi_j) = \sum_{i=1}^n \lambda_i \frac{\partial x_j}{\partial x_i}(\phi(p)) = \lambda_j$$

so $\left\{\frac{\partial}{\partial x_i}|_p\right\}$ is linearly independent and thus a basis of T_pM.

Definition 1.2.1 (Differential forms)**.** A differential k-form ω is a maps any point $p \in M$ to a alternating k-linear mapping $\omega_p \in \text{Alt}^k T_pM$. We denote the space of differential k-forms on M as $\Lambda^k M$.

Let T_p^*M be the dual space of T_pM which is usually called *cotangent space*. As before let us choose a local chart $\phi : U \to \mathbb{R}^n$ with $p \in U$ and define $\frac{\partial}{\partial x_i}|_p$ as before. Because we know that this is a basis of T_pM we denote the corresponding dual basis as dx^i, $i = 1, ..., n$. From the consideration about alternating maps from section **??** we can now write any $\omega \in \Lambda^k M$ with

$$\omega_p = \sum_{1 \leq i_1 < ... < i_k \leq n} a_{i_1,...,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

with $a_{i_1,...,i_k}(p) \in \mathbb{R}$. The regularity of differential forms is then defined via the regularity of these coefficents i.e. we call a differential form smooth if all the $a_{i_1,...,i_k}$ are smooth and we call a differential form differentiable if all the $a_{i_1,...,i_k}$ are differentiable and so on. We denote the space $C^\infty \Lambda^k M$ the space of smooth differential k-forms and analogous for other regularity.

$C_0^\infty \Lambda^k(M)$ are the smooth compactly supported differential forms which will become very crucial later when we discuss Sobolev spaces of differential forms (see Sec. **??**). Note that the support is not defined via the coefficent functions as above, but directly on the manifold i.e. $\text{supp}\, \omega = \overline{\{p \in M \mid \omega_p \neq 0\}} \subseteq M$ where the closure is w.r.t. the topology on M.

In order to define the Hodge star and an inner product on differential forms we need that T_pM is an inner product space.

A Riemannian metric gives us at every point $p \in M$ a symmetric, positive definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$. Additionally, a Riemannian metric is assumed to be smooth in the sense that for smooth vector fields X and Y we have $p \mapsto g_p(X, Y)$ is a smooth function. The degree of smoothness depends on the context. For example, for a differentiable manifold it is only required to be differentiable as well. More details... Manifolds on which a Riemannian metric is defined are called *Riemannian manifolds*. The

Riemannian metric provides us with the inner product on every tangent space T_pM.

Now we will move on to differential forms on a smooth, oriented, Riemannian manifold M of dimension n with or without boundary. We denote the Riemannian metric by g. Let $p \in M$ and T_pM be the tangent space at the point p. Due to our assumptions on M, this is an inner product space of dimension n and we can apply all of the constructions from the previous chapter Are you sure?. We have a volume form vol on M so the previous constructions using the Hodge star operator are well-defined. [3].

We define the Hodge star operator to differential forms $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ simply by applying it pointwise. In order for the Hodge star to be well-defined the assumption of an orientation on our manifold is crucial. We do the same for the exterior product to get $\wedge : \Lambda^k \times \Lambda^l \to \Lambda^{k+l}$.

Next, we will define integration of an n-form over an n dimensional manifold. At first, we do so for an open set $U \subseteq \mathbb{R}^n$. This is the simplest example of an n-dimensional manifold where we only have one chart which is the identify and the local coordinates are just our standard coordinates. Let ω be an n-form on U so we can write

$$\omega_x = f(x)dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

for $x \in U$. We can now simply define

$$\int_U \omega = \int_U f(x)dx.$$

With this definition at hand we can now extend this definition to any smooth oriented n-dimensional manifold M. As it is often done in differential geometry we will work locally first and then extend this construction globally by using a partition of unity.

Let (U, ϕ) be a chart on M and assume $\operatorname{supp} \omega \subseteq U$. Then $(\phi^{-1})^*\omega$ is a n-form on $\phi(U) \subseteq \mathbb{R}^n$ and we can apply our prior definition. So now we just define

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^*\omega.$$

It can be shown that this definition does not depend on the chart if we choose the atlas corresponding to the orientation of the manifold.

Now let us move on to the global definition. Let $\{(U_i, \phi_i)\}_{i=1}^{\infty}$ be a oriented atlas and let $\{\xi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to it. Then $\operatorname{supp} \xi_i\omega \subseteq U_i$ and we define

$$\int_M \omega := \sum_{i=1}^{\infty} \int_M \xi_i\omega.$$

This definition is also independent of the chosen chart and partition of unity. We will omit the proof. In Bredon this is defined only for differentiable functions but Arnold uses this in a L^2 setting. So there must be a way to generalize the ideas of integrability etc.

Now that we defined integration we will also introduce a natural way to differentiate differential forms. Let $\omega \in \Lambda^k(M)$ be given in local coordinates with some chart (U, ϕ) as above i.e.

$$\omega_p = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1,\ldots,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

Then we define the exterior derivative $d : Lambda^k(M) \to Lambda^{k+1}(M)$. By

$$(d\omega)_p = \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{i=1}^{n} \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_i}(\phi(p)) dx^k \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

As the other definitions above this is also independent of the chosen chart.

If the manifold is oriented and we have thus a Hodge star operator. Then we define the *codifferential operator* $\delta := (-1)^{n(k-1)+1} \star d \star$ which is then an operator $\Lambda^k(M) \to Lambda^{k-1}(M)$.

The exterior derivative and the codifferential both require the differential form to be differentiable. Later we will extend this in weak sense so classical differentiabilty is no longer required (see **??**).

Then we define the L_p-norm of a k-form ω for $1 \le p < \infty$ as (cf. [4])

$$\|\omega\|_{L_p^k(\Omega)} := \left(\int_\Omega \|\omega(x)\|_{\mathrm{Alt}^k}^p \, dx \right)^{1/p}$$

and for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x \in \Omega} \|\omega(x)\|_{\mathrm{Alt}^k}.$$

$L_p^k(\Omega)$ are the spaces of k-forms s.t. the corresponding L_p-norm is finite. For $p = 2$ we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the L_2 inner product

$$\langle \omega, \nu \rangle := \int_\Omega \langle \omega(x), \nu(x) \rangle_{\mathrm{Alt}^k} \, dx = \int_\Omega \omega \wedge \star \nu \qquad (1.2.4)$$ {eq:def_inner_pr

Proposition 1.2.2. *The Hodge star operator $\star : L_2^k(\Omega) \to L_2^{n-k}(\Omega)$ is a Hilbert space isometry.*

Proof. This follows directly from the definition of the inner product (1.2.4) and the fact that $\star$ is an isometry when applied to alternating forms Alt^k. $\square$

Our next goal is to extend the exterior derivative d of smooth differential forms in the weak sense (cf. [4]). Let $\mathring{d} : L_2^k(\Omega) \to L_2^{k+1}(\Omega)$ be the exterior derivative as an unbounded operator with domain $D(\mathring{d}) = C_0^\infty\Lambda^k(\Omega)$ which are the smooth compactly supported differential forms φ of degree k with $\operatorname{supp}\varphi \subseteq \operatorname{int} M$.

Note that when we talk about smoothness or regularity of differential forms we always mean the regularity of the coefficents when the form is expressed via local charts.

Analogous, let $\mathring{\delta} : L_2^k(\Omega) \to L_2^{k-1}(\Omega)$ be the codifferential operator $\mathring{\delta} := (-1)^{n(k-1)+1} \star \mathring{d} \star$ also with domain $C_0^\infty\Lambda^k(\Omega)$.

Then the exterior derivative $d\omega \in L_p^{k+1}(\Omega)$ is defined as the unique $(k+1)$-form in $L_p^{k+1}(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge \star\phi = \int_\Omega \omega \wedge \star\mathring{\delta}\phi \quad \forall \phi \in C_0^\infty\Lambda^k(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$W_p^k(\Omega) = \left\{\omega \in L_p^k(\Omega) \mid d\omega \in L_p^{k+1}(\Omega)\right\},$$
$$W_{p,loc}^k(\Omega) = \left\{\omega \text{ } k\text{-form} \mid \omega|_A \in W_p^k(A) \text{ for every open } A \subseteq \Omega \text{ s.t. } \overline{A} \subseteq \Omega \text{ is compact}\right\}.$$

For $\omega \in W_p^k(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W_p^k(\Omega)} := \left(\|\omega\|_{L_p^k(\Omega)}^p + \|d\omega\|_{L_p^k(\Omega)}^p\right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max\left\{\|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k}\right\}.$$

{rem:identificat

Remark 1.2.3. Throughout this thesis, we will mostly deal with open subdomains $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary. Then Ω is a smooth submanifold of $\mathbb{R}^n$ and $\overline{\Omega}$ is a Lipschitz manifold with boundary. If we now assume that $\Omega = \operatorname{int}\overline{\Omega}$ then $W_p^k(\Omega)$ and $W_p^k(\overline{\Omega})$ are essentially the same. Take $\omega \in W_p^k(\Omega)$ and extend it arbitrarily to $\overline{\omega} \in W_p^k(\overline{\Omega})$. Then because $\partial\Omega$ is a null set $\overline{\omega} \in L_2^k(\overline{\Omega})$ and because the definition of the exterior derivative uses only smooth funtions with compact support contained in $\operatorname{int}\overline{\Omega} = \Omega$ we get that $d\overline{\omega} = d\omega \in L_2^k(\overline{\Omega})$ (again by choosing arbitrary values on the boundary). From now on we will in the assumed setting treat the spaces $W_p^k(\overline{\Omega})$ and $W_p^k(\Omega)$ as the same.

Definition 1.2.4 (L^p-cohomology)**.** We define the following subspaces of $W_p^k(\Omega)$, $1 \leq p \leq \infty$:

$$\begin{aligned}\mathfrak{B}_k &:= dW_p^{k-1}(\Omega) \text{ and}\\ \mathfrak{Z}_k &:= \{\omega \in W_p^k(\Omega) | \, d\omega = 0\}.\end{aligned}$$

We call the k-forms in $\mathfrak{B}_k$ exact and the forms in $\mathfrak{Z}_k$ closed. Because $d \circ d = 0$ we always have $\mathfrak{B}_k \subseteq \mathfrak{Z}_k$. Then we define the de Rham- or L^p-cohomology space $H^k_{p,dR}(\Omega)$ as the quotient space

$$H^k_{p,dR}(\Omega) := \mathfrak{Z}_k \big/ \mathfrak{B}_k.$$

We want to examine the Hilbert space $L_2^k(\Omega)$ more closely (see [1, Sec. 6.2.6] for more details). We denote $H^k(d;\Omega) := W_2^k(\Omega)$. If the domain is clear we will leave it out. Note that the above definition of the exterior derivative is in the Hilbert space setting equivalent to defining d as the adjoint of $\mathring{\delta}$.

In order to extend $\mathring{\delta}$ as well, we will need the following

Definition 1.2.5 (Codifferential operator)**.** Analogous to the smooth case, we define the *codifferential operator* for any k as an unbounded operator $\delta : L_2^k(\Omega) \to L_2^{k-1}$ as

$$\delta := (-1)^{n(k-1)+1} \star d \star$$

with domain

$$D(\delta) = \{\omega \in L_2^k(\Omega) | \star \omega \in H^{n-k}(d)\} =: H^k(\delta; \Omega).$$

Proposition 1.2.6. *$\delta = \mathring{d}^*$ i.e. δ is the adjoint of $\mathring{d}$.*

Proof. Denote with $D(\mathring{d}^*) \subseteq L_2^{k-1}(\Omega)$ the domain of the adjoint. Now take $\omega \in H^k(\delta)$ and $\phi \in C_0^\infty \Lambda^k(\Omega)$. Then

$$\begin{aligned}&\langle \delta\omega, \phi\rangle = (-1)^{nk+1} \langle \star d \star \omega, \phi\rangle\\ &= (-1)^{nk+1}(-1)^{k(n-k)}\langle d \star \omega, \star\phi\rangle = (-1)^{nk+1}(-1)^{k(n-k)}\langle \star\omega, \mathring{\delta} \star \phi\rangle\\ &= (-1)^{nk+1}(-1)^{k(n-k)}(-1)^{n(n-k-1)+1}\langle \star\omega, \star\mathring{d} \star \star\phi\rangle\\ &= (-1)^{n(n-1)+2}(-1)^{k(n-k)}\langle \omega, \mathring{d} \star \star\phi\rangle\\ &= \langle \omega, \mathring{d}\phi\rangle\end{aligned}$$

where we used repeatedly that $\star$ is an isometry and $\star\star = (-1)^{k(n-k)}\mathrm{Id}$. This shows that $H^{k+1}(\delta) \subseteq D(\mathring{d}^*)$ and that $\mathring{d}^*\omega = \delta\omega$. Now for the other inclusion assume that $\omega \in D(\mathring{d}^*)$ and take $\phi \in C_0^\infty\Lambda^{n-k}(\Omega)$ arbitrary.

$$\langle \star\omega, \mathring{\delta}\phi\rangle = \pm\langle \omega, \mathring{d}\star\phi\rangle = \pm\langle \mathring{d}^*\omega, \star\phi\rangle = \pm\langle \star\mathring{d}^*\omega, \phi\rangle.$$

Here we use $\pm$ to mean that we choose the sign correctly, s.t. all the operations are correct. Then by choosing the sign appropriately we find that $\pm\star\mathring{d}^*\omega = d\star\omega$ and therefore $\star\omega \in H^{n-k-1}(d)$ so we proved $D(\mathring{d}^*) \subseteq H^{k+1}(\delta)$ and we are done. □

In order to deal with the boundary of our domain we introduce Homogeneous boundary conditions for these Sobolev spaces of differential forms.

{def:zero_bounda

Definition 1.2.7 (Zero boundary condition)**.** We say that $\omega \in H^k(d;\Omega)$ has zero boundary condition if

$$\langle d\omega, \chi\rangle_{L_2^{k+1}(\Omega)} = \langle \omega, \delta\chi\rangle_{L_2^k(\Omega)} \quad \forall \chi \in H^{k+1}(\delta;\Omega).$$

Denote $\mathring{H}^k(d;\Omega) := \{\omega \in H^k(d;\Omega) |\, \omega \text{ has zero boundary condition}\}$.

Of course we should justify why this is a reasonable definition. If Ω is lipschitz and bounded we have the integration by parts formula (cf. [1, Thm. 6.3])

$$\int_\Omega d\omega \wedge \mu = (-1)^k \int_\Omega \omega \wedge d\mu + \int_{\partial\Omega} \operatorname{tr}\mu \wedge \operatorname{tr}\omega \quad \text{for } \omega \in H^1\Lambda^k(\Omega),\ \mu \in H^{n-k-1}(d;\Omega)$$

where $H^1\Lambda^k(\Omega)$ are the differential forms with all coefficients being in $H^1(\Omega)$ (here we mean just the standard Sobolev space). Let now $\omega \in \mathring{H}^k(d)$. Then if we use the integration by parts formula and $\langle d\omega, \mu\rangle_{L^{k+1}(\Omega)} = \langle \omega, \delta\mu\rangle_{L^k(\Omega)}$ we get after some computation using the Hodge star

$$\langle \operatorname{tr}\omega, \star\operatorname{tr}\star\mu\rangle_{L^k(\Omega)} = 0 \quad \forall \mu \in H^1\Lambda^k(\Omega).$$

The trace operator $\operatorname{tr} : H^1\Lambda^k(\Omega) \to H^{1/2}\Lambda^k(\Omega)$ is surjective [1, Thm. 6.1].

$$\star\mathrm{tr}\star H^1\Lambda^{k+1}(\Omega) = \star\mathrm{tr} H^1\Lambda^{n-k-1}(\Omega) = \star H^{1/2}\Lambda^{n-k-1}(\Omega) = H^{1/2}\Lambda^k(\Omega)$$

is dense in $L_2^k(\Omega)$. Thus $\operatorname{tr}\omega = 0$. So in the case of bounded Lipschitz domains this definition is reasonable. The reason why we chose to define it as in Def. 1.2.7 is that is easily extendible to unbounded domains and the regularity of the boundary is not an issue.

Then we define the spaces

$$H_0^k(d;\Omega) := \{\omega \in H^k(d;\Omega) | d\omega = 0\}$$
$$\mathring{H}_0^k(d;\Omega) := \{\omega \in \mathring{H}^k(d;\Omega) | d\omega = 0\}$$

i.e. the spaces of closed forms. We will use the analogous definition for $H_0^k(\delta;\Omega)$ and $\mathring{H}_0^k(\delta;\Omega)$ which we call coclosed forms. We then define the spaces of harmonic forms

$$\mathring{H}_0^k(d,\delta;\Omega) := \{\omega \in \mathring{H}^k(d;\Omega)|\, d\omega = 0, \delta\omega = 0\}.$$

With this one can prove the Hodge decomposition ([1, Lemma 1])

$$L_2^k(\Omega) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \overset{\perp}{\oplus} \overline{\delta H^{k+1}(\delta)} \tag{1.2.5}$$ {hodge_decomposi

and furthermore for the closed and coclosed forms respectively,

$$\mathring{H}_0^k(d) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \tag{1.2.6}$$ {decomposition_c

$$H_0^k(\delta) = \overline{\delta H^{k+1}(\delta)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta). \tag{1.2.7}$$ {decomposition_c

2 Simplicial topology

Before we can reformulate the boundary value problem in the language of differential forms we have to introduce some things from simplicial topology. This material is taken from [2] where a lot more details and results can be found.

2.1 Simplicial complex

Definition 2.1.1 (Affine simplex)**.** Let x_0, x_1, ..., $x_k \in \mathbb{R}^n$ be affine independent. Then

$$[x_0, x_1, ..., x_k] := \mathrm{conv}\{x_0, ..., x_k\}$$

is called an affine k-simplex. With conv we mean the convex combination.

We will assume all simplices to be affine.

Definition 2.1.2 (Simplicial complex)**.** A *simplicial complex* K is a collection of affine simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of σ is in K,

2. $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is either empty or a face of both σ and τ.

We call $|K| := \bigcup\{\sigma|\sigma \in K\}$ the polyhedron of K and we denote the set of all k-simplices as $K^{(k)}$.

For any topological space X a homeomorphism $\tau : |K| \to X$ is called *triangulation* of X. Let $\{x_1, x_2, ...\}$ be the vertices in the simplicial complex K. We fix an ordering of the vertices for every simplex. That means for any k every k simplex σ has a designated representation in the form of

$$\sigma = [x_{i_0}, x_{i_1}, ..., x_{i_k}].$$

Definition 2.1.3 (Star)**.** The *star* of a vertex x_i is the collection of all simplices $\sigma \in K$ s.t. x_i is contained in σ.

We call a simplicial complex *star-bounded* for some number N if the star of any vertex contains at most N simplices.

Does this imply quasi-regularity? Star-boundedness will be a crucial assumption later (see Assumption **??**). Now we will introduce algebraic structures on the simplicial complex.

Definition 2.1.4 (k-chain)**.** Let K be a simplicial complex. By $C_k(K)$ we will denote the free abelian group on the k-simplices i.e. the abelian group of all formal finite sums

$$\sum_\sigma n_\sigma \sigma$$

with σ being k-simplices. The elements of $C_k(K)$ (i.e. sums of the above form) are called *k-chains*.

Definition 2.1.5 (Boundary)**.** We have the *boundary operator* $\partial_k : C_k(K) \to C_{k-1}(K)$. We first define it for any simplex $[x_{i_0}, x_{i_1}, ..., x_{i_k}]$

$$\partial[x_{i_0}, x_{i_1}, ..., x_{i_k}] := \sum_{j=0}^{k} (-1)^j [x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$$

where $[x_{i_0}, x_{i_1}, ..., \hat{x}_{i_j}, ..., x_{i_k}]$ is the simplex without vertex x_{i_j}. We then extend the definition of the boundary operator linearly to k-chains $c = \sum_\sigma c_\sigma \sigma$ by

$$\partial c := \sum_\sigma c_\sigma \partial \sigma.$$

By convention we define $\partial_0 = 0$. We will leave out the index k if it is clear from the context or irrelevant.

A crucial property of the chain operator is the following.

Proposition 2.1.6. $\partial \circ \partial = 0$.

Proof. This can be proven by direct computation, analogous to [2, Chap. 4, Lemma 1.6] $\square$

We can now generalize the notions of the chain groups by using factor groups. Let $L \subseteq K$ be a simplicial complex which we will call simplicial subcomplex. Then we define the factor group

$$C_k(K, L) := {}^{C_k(K)}\!\big/_{C_k(L)}.$$

We denote the elements of $C_k(K, L)$ i.e. the equivalence class by $\underline{c}$ for some $c \in C_k(K)$. Because L is a simplicial complex we have $\partial_k c \in C_{k-1}(L)$ for any $c \in C_k(L)$. Thus it induces a homomorphism on the factor groups denoted by

$$\underline{\partial}_k : C_k(K, L) \to C_{k-1}(K, L),\ \underline{\partial}_k \underline{c} := \underline{\partial_k c}.$$

Note that $C(K, L)$ generalizes the definition of $C_k(K, L)$ because we can choose $L = \emptyset$. Then the abelian group on $\emptyset$ is just $\{0\}$ and thus $C_k(K, L) = C_k(K)$.

Remark 2.1.7. The last equality is technically not precise of course. However, for any group G with identity ι the resulting factor group ${}^{G}\!\big/_{\{\iota\}}$ every equivalence class $\underline{g}$ would only include g itself so we can trivially identify G with ${}^{G}\!\big/_{\{\iota\}}$ and we will do so from now on.

We want to investigate how we can understand $\underline{c}$.

Proposition 2.1.8. *Let $c \in C_k(K)$ be arbitrary. Then*

$$\underline{c} = \underline{0} \leftrightarrow c = \sum_{\tau \in L} n_\tau \tau.$$

As usual the right hand side is a finite sum. So for two k-chains $c = \sum_\sigma n_\sigma \sigma \in C_k(K)$ and $c' = \sum_\sigma n'_\sigma \sigma$ we have

$$\underline{c} = \underline{c'} \leftrightarrow n_\sigma = n'_\sigma \quad \forall \sigma \in K \setminus L. \tag{2.1.1}$$ {eq:equivalence_

Proof. $c \in \underline{0}$ is just equivalent to saying that $c \in C_k(L)$ by definition of the factor group and thus we can write $c = \sum\limits_{\tau \in L} n_\tau \tau$. (2.1.1) follows from that because $\underline{c} = \underline{c'}$ is equivalent to $\underline{c - c'} = \underline{0}$. $\square$

In this sense we can identify $C_k(K, L)$ with formal sums of simplices in $K^{(k)} \setminus L^{(k)}$.

2.2 Simplicial (Co-)Homology

We call a k-chain $\underline{c} \in C_k(K,L)$ a *k-cycle* if $\underline{\partial c} = 0$ and we call $\underline{c}$ a *k-boundary* if there exists $\underline{d}$ s.t. $\underline{c} = \underline{\partial d}$. Let $Z_k(K,L) = \ker \underline{\partial_k}$ and $B_k(K,L) = \operatorname{im} \underline{\partial_{k-1}}$. We can now define the simplicial homology groups of our simplicial complex.

Definition 2.2.1 (Simplicial homology)**.** The relative homology groups $H_k(K,L)$ are defined as

$$H_k(K) := {}^{Z_k(K,L)}\!\big/_{B_k(K,L)}.$$

Remark 2.2.2. We call them relative homology group because define them on the for $C_k(K,L)$. The standard homology groups $H_k(K)$ (i.e. taking $L = \emptyset$) are independent of the chosen triangulation of $|K|$ [2, p.248] because they are isomorphic to the singular homology groups which only depend on $|K|$

Let G be any abelian group. Then we define the group of *k-cochains* $C^k(K,L;G)$ by

$$C^k(K,L;G) := \operatorname{Hom}(C_k(K,L),\, G)$$

i.e. the group of all homomorphisms from $C_k(K,L)$ to G. We generally use the superindex ^{k} if something is related to cochains and the subindex $_k$ if it is related to chains. We now introduce an operator between the groups of cochains.

$C^k(K,L;G)$ can be identified with the cochains that vanish on L.

{prop:relative_c

Proposition 2.2.3. *Define*

$$\tilde{C}^k(K,L;G) := \{F \in C^k(K;G) \mid F(c) = 0 \quad \forall c \in C_k(L)\}.$$

Then any $F \in \tilde{C}^k(K,L;G)$ induces and homomorphism $\underline{F} \in C^k(K,L;G)$ and $j : \tilde{C}^k(K,L;G) \to C^k(K,L;G)$, $F \mapsto \underline{F}$ is an isomorphism.

Proof. Because $C_k(L) \subseteq \ker F$ is a subgroup we get the induced homomorphism

$$\underline{F} : C_k(K,L) = {}^{C_k(K)}\!\big/_{C_k(L)} \to {}^{G}\!\big/_{0} = G.$$

j **is injective.** Let $\underline{F}(\underline{c}) = 0$ for every $c \in C_k(K)$. Then $F(c) = \underline{F}(\underline{c}) = 0$ so $F = 0$. j **is surjective.** Let $\varphi \in C^k(K,L;G)$ be arbitrary. Then define $F \in C^k(K)$, $F(c) := \varphi(\underline{c})$. We first have to show that $F \in \tilde{C}^k(K,L)$. For any $d \in C_k(L)$ we get $\underline{d} = 0 \in C_k(K,L)$ and $F(d) = 0$. Then we have for any $\underline{c} \in C_k(K,L)$, $\underline{F}(\underline{c}) = F(c) = \varphi(\underline{c})$ i.e. $\underline{F} = \varphi$ which proves surjectivity. □

Proposition 2.2.4. *j as defined in Prop. 2.2.3 is a chain map.*

Proof. Let $F \in \tilde{C}^k(K, L)$ be arbitrary. Then

$$\underline{\partial^k F}(\underline{c}) = \partial^k F(c) = F(\partial_k F) = \underline{F}(\underline{\partial_k c}) = \underline{F}(\underline{\partial}_k \underline{c}) = \underline{\partial}^k \underline{F}(\underline{c})$$

i.e. $j \circ \partial^k = \underline{\partial}^k \circ j$ and j is a chain map. $\square$

Corollary 2.2.5. *The induced map $[j] : H^k(K, L; G) \to \tilde{H}^k(K, L; G)$ is an isomorphism.*

Definition 2.2.6. We define the operator $\delta : C^k(K; G) \to C^{k+1}(K; G)$ via

$$(\delta f)(c) := f(\partial c).$$

for a $(k+1)$-chain c. We call a cochain $f \in C^k(K; G)$ *closed* if $\delta f = 0$ and we call f *exact* if there is a $g \in C^{k+1}(K; G)$ s.t. $f = \delta g$.

We define the cohomology spaces analogous to homology spaces above.

Definition 2.2.7 (Simplicial cohomology)**.** Denote the closed k-cochains as $Z^k(K; G)$ and the exact ones with $B^k(K; G)$. We then define the *simplicial cohomology groups* $H^k(K)$ as

$$H^k(K; G) := {}^{Z^k(K;G)}\!\big/_{B^k(K;G)}.$$

Note that in the case of $G = \mathbb{R}$ this becomes a vector space. We will later show that if we consider certain subspaces of cochains so called *p-summable* cochains (see Def. **??**) that the L_p-cohomology defined above and the cohomology spaces of these p-summable cochains are isomorphic.

Now of course there is the question how the homology and cohomology groups are related to each other. This question is answered by the *universal coefficent theorem.* But before we can formulate it we have to introduce exact sequences.

Definition 2.2.8 (Exact sequence)**.** Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. Then this sequence of homomorphisms is called *exact* if $\operatorname{im} f_{i-1} = \ker f_i$.

The universal coefficent theorem in the case of simplicial homology states that the sequence

$$0 \to \operatorname{Ext}(H_{k-1}(K), G) \to H^k(K; G) \xrightarrow{\beta} \operatorname{Hom}(H_k(K), G) \to 0 \qquad (2.2.1)$$ {eq:univeral_coe

is exact. β is defined via $\beta([F])([c]) := F(c)$. The definition of Ext can be found in [2], but it does not matter for our purpose because from now on we will assume $G = \mathbb{R}$ and $\mathrm{Ext}(H_{k-1}(K), \mathbb{R}) = 0$. This follows from the fact that $\mathbb{R}$ is a divisible and hence injective abelian group. The definition these terms and the connections used can also be found in [2, Sec. V.6]. However, we will not dwelve into the algebraic background further. We can conclude from the exactness of the above short sequence that $\ker \beta = 0$ and $\mathrm{im}\, \beta = \mathrm{Hom}(H_k(K), \mathbb{R})$. So β is a isomorphism.

2.3 Existence of uniqueness of cochain

As an application, we will show the following proposition which will be used later to show existence and uniqueness of a solution of the magnetostatic problem.

{prop:uniqueness

Proposition 2.3.1. *Assume that $H_1(K, L) = \mathbb{Z}[\gamma]$ i.e. the homology class of the closed 1-chain γ is a generator of the first homology group. Then we have the following:*

(i) For any $C_0 \in \mathbb{R}$ there exists a closed 1-chain $F \in Z^1(K, L)$ with $F(\gamma) = C_0$,

(ii) any other $G \in Z^1(K, L)$ with $G(\gamma) = C_0$ is in the same cohomology class i.e. $[F] = [G]$.

Proof. **Proof of (i)** Because $[\gamma]$ is a generator of the homology group we obtain a homomorphism $\hat{F} \in \mathrm{Hom}(H_1(K, L), \mathbb{R})$ by fixing $\hat{F}([\gamma]) = C_0$. This determines the other values. Then we know from (2.2.1) that there exists a $[F] \in H^1(K, L)$ with $\beta([F]) = \hat{F}$ because β is a isomorphism. So we obtain

$$F(\gamma) = \beta([F])([\gamma]) = \hat{F}([\gamma]) = C_0.$$

Proof of (ii) Take $[c] \in H_1(K, L)$ arbitrary. Then there exists $n \in \mathbb{Z}$ s.t. $[c] = n[\gamma]$. Using β from (2.2.1) We have

$$\beta([F])([c]) = \beta([F])(n[\gamma]) = n\beta([F])([\gamma]) = n\, F(\gamma) = n\, G(\gamma) = \beta([G])([c])$$

and thus $\beta([F]) = \beta([G])$. Because β is an isomorphism we arrive at $[F] = [G]$. □

The following lemma will be needed later in Sec. **??** to prove existence and uniqueness of a solution of the magnetostatic problem. It is now useful to think of K as a triangulation of an exterior domain and L be the triangulation of the exterior of some ball $\mathbb{R}^3 \setminus B_R$ with $R > 0$ large enough. Then we assume for the simplicial subcomplex L that its geometric realization $|L|$ is homeomorphic to $\mathbb{R}^3 \setminus B_R$.

First we need to understand how the zero-th homology group looks like. When we say path-connected we mean in the sense of simplicial topology i.e. we can connect all vertices with a 1-chain.

{lem:zeroth_homo

Lemma 2.3.2. *Let K be a path-connected simplicial complex. Then the homology class of every vertex $[x_i] \in H_0(K)$ is a generator of the homology group.*

Proof. The proof works exactly as the proof of Theorem IV.2.1 in [2] where we use the notion of path-connected on simplicial complexes. □

{lem:inclusion_z

Lemma 2.3.3. *Let K, L be simplicial complexes with $L \subseteq K$ and assume they are path-connected Then the inclusion $\iota : C_0(L) \hookrightarrow C_0(K)$ induces an isomorphism on homology.*

Proof. This follows immediately from Lemma 2.3.2. □

{lem:isom_chains

Lemma 2.3.4. *Let K and L be as in the previous lemma. Assume additionally that $|L|$ is homeomorphic to $\mathbb{R}^3 \setminus B_R$. Then the natural surjection $j : C_1(K) \to C_1(K, L)$ induces an isomorphism on homology.*

Proof. We have the exact sequence

$$H_1(L) \xrightarrow{[\iota_1]} H_1(K) \xrightarrow{[j_1]} H_1(K, L)$$
$$\xrightarrow{[\partial_1]} H_0(L) \xrightarrow{[\iota_0]} H_0(K) \xrightarrow{[j_0]} H_0(K, L).$$

We know that $H_1(L) \cong H_1(|L|) \cong H_1(\mathbb{S}^2) = 0$. Because $[\iota_0]$ is an isomorphism due to Lemma 2.3.3 $\mathrm{im}[\partial_1] = 0$. Thus, we get the exact sequence

$$0 \xrightarrow{[\iota_1]} H_1(K) \xrightarrow{[j_1]} H_1(K, L) \xrightarrow{[\partial_1]} 0$$

and so $[j_1]$ is a isomorphism. □

Now using all these lemmas and results we can now prove the following important lemma that will be needed later in Sec. **??**

{lem:existence_v

Lemma 2.3.5. *Let K and L be as in the previous lemma and assume that $H_1(K) = \mathbb{Z}[\gamma]$. Then there exists a cochain $F \in \tilde{C}^1(K, L)$ s.t. $F(\gamma) = C_0$.*

Proof. We know from Lemma 2.3.4 that $[j] : H_1(K) \to H_1(K, L)$ is a isomorphism. Thus, we have $[j]([\gamma]) = [\underline{\gamma}] \in H_1(K, L)$ is a generator. We can now apply Prop. 2.3.1 to get a $\tilde{F} \in C^1(K, L)$ s.t. $\tilde{F}(\underline{\gamma}) = C_0$. Now we continue with Prop. 2.2.3 to get that there exists a unique $F \in \tilde{C}^1(K, L)$ s.t. $jF = \underline{F} = \tilde{F}$ and we arrive at

$$F(\gamma) = \underline{F}(\underline{\gamma}) = \tilde{F}(\underline{\gamma}) = C_0.$$

□

3 Isomorphism of Cohomology

In order to show existence and uniqueness of solutions of the magnetostatic problem we use a lot of results and tools from [4]. In the diploma thesis of Nikolai Nowaczyk [6], which mostly is based on this paper, many additional details can be found. The results which are important for our progress will be presented in the next section. It should be noted that even though the results in [4] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [4]). Therefore, we can apply the result to our case.

In the first section we will introduce S-forms which can be seen as a intermediate object between differential forms and we will go over the basic results from [4] how these S-forms are connected with cochains on our simplicial complex K. Then we will look at how S-forms and differential forms are connected. At last, we will introduce two operators and their properties which are introduced in [4] and will prove to be very useful later on.

Because $\overline{\Omega}$ from our problem is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [4, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ and take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^*g$ where g is the standard metric in ℓ^2. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on ℓ^2. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned} g_S|_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_k}{\partial x_j}(x) \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left(Df(x)^T Df(x) \right)_{ij} \\ &= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle, \end{aligned}$$

where D denotes the Jacobian. (TBD: This Jacobian as written here would technically be in $\mathbb{R}^{\infty \times n}$. Only finitely many lines are non-zero though, but this is not quite rigorous yet.)

We have two crucial assumptions on the triangulation for the result to hold (cf. [4, p.194]). We summarize them under *GKS-condition* named after the three authors of [4].

{ass:gks_conditi

Assumption 3.0.1 (GKS-condition)**.** We will assume the following on the simplicial complex K and the triangulation τ:

(i) The star of every vertex in K contains at most N simplices.

(ii) For the differential of τ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1, \ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where d denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most N simplices, which is fulfilled if we have a shape regular mesh.

This has to be shown.

Because τ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

3.1 S-forms

Definition 3.1.1 (Induced map)**.** Let V and W be real vector spaces, $X \subseteq V$, $Y \subseteq W$ be subspaces. For a linear map $L : V \to W$ with $L(X) \subseteq Y$ we define the induced map

$$[L] : {}^{V}\!/_{X} \to {}^{W}\!/_{Y},\ [v] \mapsto [Lv].$$

It is easy to check that the induced map is well-defined using the definition of quotient space.

The first isomorphism is induced from a linear mapping from the so called *S-forms* $S_p^k(K)$ to *p-summable k-cochains* $C_p^k(K)$ which will both be defined next.

Definition 3.1.2. We define the following norm of a k-cochain f

$$\|f\|_{C_p^k(K)} := \left(\sum_{\sigma \in K^{(k)}} |f(\sigma)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{ f \in C^k(K) |\ \|f\|_{C_p^k(K)} < \infty \}.$$

Take $\tau, \sigma \in K$ s.t. τ is a face of σ which we write as $\tau < \sigma$. We need a restriction operator $j^*_{\tau,\sigma} : W^k_{\infty,loc}(\sigma) \to W^k_{\infty,loc}(\tau)$. This is done by extending $\omega \in W^k_{\infty,loc}(\sigma)$ first to some $\tilde{\omega} \in W^k_{\infty,loc}(U)$ with U an open neighborhood of σ in the affine hull of σ. Then $\tau \subseteq U$ and we apply a restriction operator $j^*_{\tau,U}$ and define $j^*_{\tau,\sigma}\omega := j^*_{\tau,U}\tilde{\omega}$. This restriction operator is then well defined, bounded and independent of the chosen extension $\tilde{\omega}$ (cf. [4, p.191]). It should be emphasized that this restriction only works for W^k_∞ and fails for W^k_p, $p < \infty$.

Definition 3.1.3 (S-forms)**.** Let

$$\theta = \{ \theta(\sigma) \in W^k_\infty(\sigma) | \sigma \in K \}$$

be a collection of differential k-forms. We call θ *S-form of degree k* if we have for all simplices $\mu < \sigma$

$$j^*_{\sigma,\mu}\theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{ d\theta(\sigma) |\ \sigma \in K \} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

For $\theta \in S^k(K)$ we now define the norm

$$\|\theta\|_{S_p(K)} := \left(\sum_{\sigma\in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)}\right)^{1/p}.$$

$S^k_p(K)$ are the S-forms of degree k s.t. this norm is finite.

Let $L \subseteq K$ be a simplicial subcomplex. Then we define the restriction operator

$$j^*_{K,L} : S^k(K) \to S^k(L), \{\theta(\sigma) \mid \sigma \in K\} \mapsto \{\theta(\sigma) \mid \sigma \in L\}.$$

Then we define $S^k(K, L) := \ker j^*_{K,L}$ and analogously $S^k_p(K, L)$. So these are the S-forms that vanish on a simplicial subcomplex. As in the case of cochains (compare Sec. **??**) this generalizes the first definition because we get $S^k(K, \emptyset) = S^k(K)$.

The integration of an S-form $\theta \in S^k(K)$ over a chain $c = \sum n_\sigma \sigma \in C_k(K)$ is then simply defined as

$$\int_c \theta := \sum_\sigma n_\sigma \int_\sigma \theta(\sigma).$$

We also write $I(\theta)(c) := \int_c \theta$. Then the integration mapping $I : S^k(K, L) \to C^k(K, L)$ is a homomorphism (see [4, p.191]). The reason why S-forms are a useful concept is because they provide an intermediate step between differential forms and cochains.

With the exterior derivative d on S-forms as defined above we define

$$\mathscr{Z}^k(K, L) := \{\theta \in S^k(K, L)|\, d\theta = 0\}$$
$$\mathscr{B}^k(K, L) := dS^{k-1}(K, L)$$

and then the resulting cohomology space

$$\mathscr{H}^k(K, L) := {}^{\mathscr{Z}^k(K, L)}\!\big/_{\mathscr{B}^k(K, L)}.$$

We also define the in the case of p-summable S-forms

$$\mathscr{Z}^k_p(K, L) := \{\theta \in S^k_p(K, L)|\, d\theta = 0\}$$

as well $\mathscr{B}^k_p(K, L)$ and $\mathscr{H}^k_p(K, L)$ analogously.

Then we have that the integration mapping $I : S^k_p(K, L) \to C^k_p(K, L)$ induces an isomorphism on the cohomologies i.e. $[I] : \mathscr{H}^k_p(K, L) \to \tilde{H}^k_p(K, L)$ is an isomorphism of vector spaces (see [4, Thm. 1] and the proof thereof).

3.2 Isomorphism between cohomologies of S-forms and L^p-cohomology

The next step is to obtain an isomorphism between the cohomology of S-forms $\mathscr{H}_p^k(K)$ and the L_p cohomology $H_{p,dR}^k(\overline{\Omega})$. Do I really do that?

In order to achieve that we need a some connection between differential forms and S-forms. Take $\omega \in W_{\infty,loc}^k(\overline{\Omega})$. Using the analogous reasoning as above, we use the restriction operators $j_{\sigma,\tau}^*$ to restrict ω to the simplices. Denote the resulting forms as $\omega(\sigma) \in W_{\infty,loc}^k(\sigma)$. So we obtain an S-form $\{\omega(\sigma) | \, \sigma \in K\}$. This way we constructed an operator

$$\varphi : W_{\infty,loc}^k(\overline{\Omega}) \to S^k(K), \; \omega \mapsto \{\omega(\sigma) | \, \sigma \in K\} \tag{3.2.1}$$ {eq:forms_Sforms

This operator φ is an isomorphism [4, Lemma 1]. To also obtain corresponding forms to the p-summable S-forms we simply define $S_p^k(\overline{\Omega}) := \varphi^{-1} S_p^k(K)$. It can be shown that $S_p^k(\overline{\Omega}) \subseteq W_p^k(\overline{\Omega})$. Now if we have some $\omega \in S_p^k(\overline{\Omega})$ then we can look at two possible norms. From the inclusion we know that $\|\omega\|_{W_p^k(\overline{\Omega})} < \infty$. But we can also use the norm induced by φ denoted by

$$\|\omega\|_{S_p^k(\overline{\Omega})} := \|\varphi\omega\|_{S_p^k(K)}. \tag{3.2.2}$$ {eq:induced_norm

Let us briefly show that this is indeed a norm.

Proposition 3.2.1. *Let K be a simplicial complex and $\overline{\Omega} = |K|$. Then $\|\cdot\|_{S_p^k(\overline{\Omega})}$ is a norm and φ as defined at (3.2.1) is an isometry.*

Proof. Subadditivity and absolute homogeneity follow directly from the linearity of φ and the corresponding property of $\|\cdot\|_{S_p^k(K)}$. For the positive definiteness

$$\|\omega\|_{S_p^k(\overline{\Omega})} = 0 \Leftrightarrow \|\varphi\omega\|_{S_p^k(K)} \Leftrightarrow \varphi\omega = 0 \overset{\varphi \text{ isom.}}{\Leftrightarrow} \omega = 0.$$

φ is an isometry by construction of the norm. □

We then also have that the inclusion operator $\iota : S_p^k(\overline{\Omega}) \hookrightarrow W_p^k(\overline{\Omega})$ is bounded ([4, Lemma 4]).

{rem:local_const

Remark 3.2.2. In the paper [4], no difference is made between the spaces $S_p^k(\overline{\Omega})$ and $S_p^k(K)$ because they can be identified with each other via φ. However, we will follow the approach from Nowaczyk's thesis [6] to treat them separately because it is more precise and less confusing.

We can now extend our definition of integration on a simplex to any differential form $\omega \in W^k_{\infty,loc}(\overline{\Omega})$ thanks to the well-definedness of the restriction operators. We can do this by first applying φ which means nothing but applying these restrictions to obtain an S-form $\varphi\omega \in S^k(K)$ for which we have defined the integration above. Note that this is exactly how integration is defined for continuous differential forms. The only difference is that we had to give an additional argument because the restriction can not be applied directly as in the continuous case. We will sometimes just write $I(\omega) := I(\varphi\omega)$ or $\int_\sigma \omega = \int_\sigma (\varphi\omega)(\sigma)$ for any simplex $\sigma \in K$.

Now we also want to introduce a regularization operator in order to obtain a form in $S^k_p(\overline{\Omega})$ from form $W^k_p(\overline{\Omega})$. This is done using two operators $\mathscr{R} : L^k_{1,loc}(\overline{\Omega}) \to L^k_{1,loc}(\overline{\Omega})$ and $\mathscr{A} : L^k_{1,loc}(\overline{\Omega}) \to L^{k-1}_{1,loc}(\overline{\Omega})$. We will not go over their construction but instead we will only use some properties that we collect in the following theorem (cf. [4, Thm.2]).

{thm:operators}

Theorem 3.2.3. *Assume that the triangulation τ fulfills the GKS-condition defined at Def.* **??**. *Then there exist linear mappings $\mathscr{R} : L^k_{1,loc}(\overline{\Omega}) \to L^k_{1,loc}(\overline{\Omega})$, $\mathscr{A} : L^k_{1,loc}(\overline{\Omega}) \to L^{k-1}_{1,loc}(\overline{\Omega})$ such that*

(i) $\mathscr{R}(W^k_{1,loc}(\overline{\Omega})) \subseteq W^k_{1,loc}(\overline{\Omega})$, $\mathscr{A}(W^k_{1,loc}(\overline{\Omega})) \subseteq W^{k-1}_{1,loc}(\overline{\Omega})$ *and* $\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ *and* $d\mathscr{R}\omega = \mathscr{R}d\omega$ *for* $\omega \in W^k_{1,loc}(\overline{\Omega})$

(ii) for any $1 \le p \le \infty$, $\mathscr{R}(W^k_p(\overline{\Omega})) \subseteq S^k_p(\overline{\Omega})$ *and* $\mathscr{A}(S^k_p(\overline{\Omega})) \subseteq S^{k-1}_p(\overline{\Omega})$

(iii) $\mathscr{R} : W^k_p(\overline{\Omega}) \to (S^k_p(\overline{\Omega}), \|\cdot\|_{S^k_p(\overline{\Omega})})$ *is bounded (here we mean with* $(S^k_p(\overline{\Omega}), \|\cdot\|_{S^k_p(\overline{\Omega})})$ *the space* $S^k_p(\overline{\Omega})$ *endowed with the norm* $\|\cdot\|_{S^k_p(\overline{\Omega})}$*))*

Let $\iota : S^k_p(\overline{\Omega}) \hookrightarrow W^k_p(\overline{\Omega})$ be the inclusion operator. The inclusion induces an isomorphism on cohomology [4, Lemma 4, Corollary] i.e. $[\iota \circ \varphi^{-1}] : \mathscr{H}^k_p(K) \to H^k_{p,dR}(\overline{\Omega})$ is an isomorphism.

Remark 3.2.4. Let $L \subseteq K$ be a full-dimensional simplicial subcomplex. Let $\mathscr{R}_L$ be the regularization operator constructed for this simplicial subcomplex. The construction of the regularization operator is done locally on the stars of the vertices. Therefore if we now look at a vertex $x_i \in L$ s.t. its star Σ_i does not include any boundary vertices of L then we have $\mathscr{R}\omega = \mathscr{R}_L(\omega|_{|L|})$ for any $\omega \in W^k_{1,loc}(|K|)$ if we choose the ordering of the vertices appropriately. We will not prove this because it would require to go over the construction of the operator which we want to avoid as it is quite technical. Thus, we refer to [4, Sec. 2] for further details.

4 Existence and uniqueness of solutions

4.1 Reformulation of the problem

We will return now to the magnetostatic problem. In order to use the results above we will reformulate the problem in the notation of differential forms. From now on we assume $n = 3$ i.e. we are in three dimensional space. There are two ways to identify a vector field with a differential form (cf. [1, Table 6.1 and p.70]) either as a 1-form or a 2-form. For a vector field B we define

$$
\begin{aligned}
&F^1\, B := B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \text{ and} \\
&F^2\, B := B_2\, dx_2 \wedge dx_3 - B_2\, dx_1 \wedge dx_3 + B_3\, dx_1 \wedge dx_2
\end{aligned}
$$

as the corresponding 1-form and 2-form. Then the exterior derivative is $dF^2\,\omega$ corresponds to the divergence, the codifferential $\delta F^2\,\omega$ corresponds to the curl and the normal component being zero on the boundary corresponds to $\omega \in \mathring{H}^2(d)$.[**empty citation**].

If we then use the association of 3-forms with scalars we have the corresponding boundary value problem without the integral condition for 2-forms: Find $\omega \in \mathring{H}^2(d)$ s.t.

$$
\delta\omega = 0, \tag{4.1.1}
$$

$$
d\omega = 0 \text{ in } \Omega. \tag{4.1.2}
$$

Next, we have to add the integral condition. We remind the reader that we are in three dimensions so $** = \mathrm{Id}$ and observe

$$
\begin{aligned}
*F^2\, B &= B_1\, **dx_1 + B_2\, **dx_2 + B_3\, **dx_3 = B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \\
&= F^1\, B.
\end{aligned}
$$

: Actually, we already use the Hodge star to define the vector proxies. So of course the vector proxy of the Hodge star will be the same. Then we have

$$
\int_\gamma *F^2\, B = \int_\gamma F^1\, B = \int_\gamma B \cdot \mathrm{d}l.
$$

In the last step we used the fact that the integration of a 1-form over a curve is equivalent to the curve integral of the associated vector field (cf. [1, Sec. 6.2.3]). Hence, we can add the integral condition

$$
\int_\gamma *\omega = C_0. \tag{4.1.3}
$$

{integral_condit

However, we have only $\omega \in \mathring{H}^2_0(d,\delta)$ so $*\omega \in H^1(d)$ so this integral might not be well defined. In order to deal with this, we will again use the operator $\mathscr{R}$ from Sec. **??**.

We know from Thm. 3.2.3 that $\mathscr{R}*\omega \in S^1_2(\overline{\Omega})$. Using the operator φ from **??** we obtain $\varphi\mathscr{R}*\omega \in S^1_2(K)$. Now we know from Sec. **??** that the integration mapping $I : S^1_2(K) \to C^1_2(K)$ is well-defined. Denote $\bar{I} := I \circ \varphi \circ \mathscr{R}$ and let us replace the integral condition (**??**) with

$$\bar{I}(*\omega)(\gamma) = C_0.$$

Of course, we have to justify why this is reasonable. So let us take $\eta \in S^1_2(\overline{\Omega})$. That means we can integrate it directly using the definition from **??**. Let us also assume that η is closed and $\int_\gamma \eta = C_0$. Then we know from Thm. 3.2.3

$$\mathscr{R}\eta = \eta + d\mathscr{A}\eta + \mathscr{A}d\eta \overset{\eta \text{ closed}}{=} \eta + d\mathscr{A}\eta.$$

We know further that $\mathscr{A}\eta \in S^0_2(\overline{\Omega})$. Apply φ on both sides and use that fact that it commutes **??** with the exterior derivative to get

$$\varphi\mathscr{R}\eta = \varphi\eta + d\varphi\mathscr{A}\eta.$$

$[I]$ is an isomorphism of cohomology and thus sends exact S-forms to exact cochains. γ is closed so $I(d\varphi\mathscr{A}\eta)(\gamma) = 0$ and we conclude

$$\bar{I}(\eta)(\gamma) = I(\varphi\mathscr{R}\eta)(\gamma) = I(\varphi\eta) \overset{\text{by def.}}{=} I(\eta),$$

i.e. the integral remains unchanged if we integrate closed forms over closed chains. Because $*\omega$ is closed we thus do not change the integral if the curve integral $*\omega$ would already have been well defined before.

To summarize we obtain the following problem.

{prob:magnetosta

Problem 4.1.1. Find $\omega \in \mathring{H}^2(d;\Omega)$ s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0 \text{ in } \Omega, \\ \bar{I}(*\omega)(\gamma) &= C_0. \end{aligned}$$

We will examine existence and uniqueness of this problem in the next section.

4.2 Existence and uniqueness

We start with the following

{prop:integral_e

Proposition 4.2.1. *Let $(\phi_i)_{i\in\mathbb{N}} \subseteq H^0(d)$ s.t. $(d\phi_i)_{i\in\mathbb{N}}$ is convergent and let $\gamma \in C_1(K)$ be a bounded closed 1-chain. Then*

$$\bar{I}(\lim_{i\to\infty} d\phi_i)(\gamma) = 0.$$

Proof. Because $[\bar{I}]$ is an isomorphism of cohomology $\bar{I}$ has to send exact forms to exact cochains. Because γ is a closed 1-chain we obtain $\bar{I}(d\psi)(\gamma) = 0$ for every $\psi \in H^0(d)$. Because $\mathscr{R}$, ϕ and I are all bounded $\bar{I}$ is a continuous operator **??** and so we have $\bar{I}(\lim_{i\to\infty} d\phi_i) = \lim_{i\to\infty} \bar{I}(d\phi_i)$ with the limit of cochains taken w.r.t. the norm in $C_2^1(K)$. Now let J_γ be the set of indices of simplices contained in γ i.e.

$$\gamma = \sum_{i\in J_\gamma} \sigma_i.$$

Here we remind that we fixed the ordering and indices for our simplicial complex **??**. Let $N_\gamma := |J_\gamma|$ which is finite. Then

$$\begin{aligned}
&\left|\left(\lim_{i\to\infty} \bar{I}(d\phi_j)\right)(\gamma)\right| \leq \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\right)(\gamma)\right| \\
&\leq \sum_{k\in J_\gamma} \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_i)\right)(\sigma_k)\right| \\
&\leq N_\gamma^{1/2} \left(\sum_{k\in J_\gamma} \left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\right)(\sigma_k)\right|^2\right)^{1/2} \\
&\leq N_\gamma^{1/2} \|\lim_{i\to\infty} \bar{I}(d\phi_i) - \bar{I}(d\phi_j)\|_{C_2^1(K)} \leq N_\gamma^{1/2}\epsilon
\end{aligned}$$

and thus

$$\left|\left(\lim_{i\to\infty} \bar{I}(d\phi_i)\right)(\gamma)\right| = 0$$

because ϵ was arbitrarily small. This can be done more elegantly by identifying $C_2^1(K)$ with ℓ^2. See notes from 28.12 □

{lem:extension_b

Lemma 4.2.2. *Let Ω, U be open and $U \subseteq \Omega$. Let $\nu \in \mathring{H}^k(d;U)$. Define $\bar{\nu}$ as the zero extension of ν in $\Omega \setminus U$. Then $\bar{\nu} \in \mathring{H}^k(d;\Omega)$ and $d\bar{\nu} = d\nu$ in U and $d\bar{\nu} = 0$ in $\Omega \setminus U$.*

Proof. The proof works exactly as in the standard Sobolev case. Take $\varphi \in C_0^\infty\Lambda^k(\Omega)$ arbitrary. Define $\bar{\nu}' = d\nu$ in U and $\bar{\nu}' = 0$ in $\Omega \setminus U$. We want to show that

$$\int_\Omega \bar{\nu} \wedge *\delta\varphi = \int_\Omega \bar{\nu}' \wedge *\varphi$$

which implies that $\bar{\nu}' = d\bar{\nu}$. If $\operatorname{supp}\varphi \subseteq U$ this follows by applying the definition of $d\nu$. If $\operatorname{supp}\varphi \subseteq \operatorname{int}(\Omega \setminus U)$ then it is also trivial since both $\bar{\nu}$ and $\bar{\nu}'$ are zero in $\Omega \setminus U$.

Otherwise, we recognize $\varphi|_U \in C_c^\infty\Lambda^{k+1}(U)$. Thus we get from the definition of the zero boundary condition (Def. 1.2.7)

$$\langle \bar{\nu}, \delta\varphi \rangle_{L_2^k(\Omega)} = \langle \nu, \delta\varphi \rangle_{L_2^k(U)} = \langle d\nu, \varphi \rangle_{L_2^{k+1}(U)} = \langle \bar{\nu}', \varphi \rangle_{L_2^{k+1}(\Omega)}.$$

Thus, $\bar{\nu}' = d\bar{\nu}$. Because $d\nu \in L_2^{k+1}(U)$, $d\bar{\nu} \in L_2^{k+1}(\Omega)$ and the lemma is proven. □

{thm:existence}

Theorem 4.2.3 (Existence of solution)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be such that $\mathbb{R}^3 \setminus \Omega$ is compact and $\operatorname{int}\overline{\Omega} = \operatorname{int}\Omega$. Assume there exists a simplicial complex K s.t. $\overline{\Omega} = |K|$ and a simplicial subcomplex $L \subseteq K$ s.t. $|L|$ is homeomorphic to the exterior of a ball with radius $R > 0$ and $K \setminus L$ is finite. Both K and L are assumed path-connected in the sense of simplicial complexes (cf.* **??***). For K we require that $H_1(K) = \mathbb{Z}[\gamma]$ for a 1-chain γ. Assume that the GKS-condition 3.0.1 holds for K. Then there exists a solution to Problem 4.1.1.*

Proof. Under the given assumptions we know that we can find a $F \in \tilde{C}^1(K, L)$ (see Lemma 2.3.5) with $F(\gamma) = C_0$. Because $K \setminus L$ is finite F is obviously 2-summable i.e. $F \in \tilde{C}_2^1(K, L)$ Then we use the fact that the integration mapping $I : S_2^1(K, L) \to \tilde{C}_2^1(K, L)$ induces an isomorphism on cohomology $[I] : \mathscr{H}_2^1(K, L) \to \tilde{H}_2^1(K, L)$. So there exists a S-form $\tilde{\theta} \in S_2^1(K, L)$ s.t. $[I(\tilde{\theta})] = [F] \in \tilde{H}_2^1(K, L)$. Thus we have some $J \in \tilde{C}_2^0(K, L)$ s.t. $I(\tilde{\theta}) = F + \partial J$ and thus

$$I(\tilde{\theta})(\gamma) = F(\gamma) + J(\partial\gamma) \overset{\gamma \text{ closed}}{=} F(\gamma) = C_0.$$

Now define $\theta := \varphi^{-1}\tilde{\theta}$. So we found already a 1-form that has the desired curve integral.

$\tilde{\theta} \in S_2^1(K, L) \subseteq S_2^1(K)$ so $\theta \in S_2^1(\overline{\Omega}) \subseteq W_2^1(\overline{\Omega})$. We now use the fact that $\operatorname{int}\overline{\Omega} = \operatorname{int}\Omega$ and remember the resulting identification of the Sobolev spaces $W_2^1(\overline{\Omega})$ and $W_2^1(\Omega)$ (cf. Remark 1.2.3) to obtain $\theta \in H^1(d; \Omega)$. Next, we will use the Hodge decomposition (1.2.5). Now we project $\star\theta$ onto harmonic forms to get $\omega \in \mathring{H}^2_0(d,\delta;\Omega)$. So because $\star\theta$ is co-closed we obtain a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^3(\delta)$ s.t.

$$\star\theta = \omega + \lim_{i\to\infty} \delta\phi_i$$

where the limit is in the L^2-sense. Apply the Hodge star operator on both sides, use the fact that it is an isometry and thus continuous and then remember the definition of δ to get a sequence $(\psi_i)_{i\in\mathbb{N}} \subseteq H^0(d)$ s.t.

$$\theta = \star\omega + \lim_{i\to\infty} d\psi_i.$$

Then we get

$$\int_\gamma \mathscr{R} \star \omega = \int_\gamma \mathscr{R}(\theta - \lim_{i\to\infty} d\psi_i) \overset{\text{Prop.4.2.1}}{=} \int_\gamma \mathscr{R}\theta \overset{\text{Prop.??}}{=} \int_\gamma \theta = C_0.$$

In the penultimate step equality we used the fact that $\mathscr{R}$ does not change the integral because $\theta \in S^1(K)$ is closed. Thus ω fulfills the integral condition. Because $\omega \in \mathring{H}^2_0(d,\delta;\Omega)$ all other conditions are satisfied as well and ω is a solution. □

In the proof of uniqueness we will use the following lemma. We are now back in the realm of standard vector analysis so all the notation is to be seen in this light (e.g. H^1 is here the standard Sobolov space and not a Sobolev space of differential forms).

{lem:gradient_se

Lemma 4.2.4. *Let $\phi \in L^2_{loc}(\Omega)$ with $\nabla\phi \in L^2(\Omega)$. Then there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^1(\Omega)$ s.t. $\nabla\phi_i \to \nabla\phi$ in $L^2(\Omega)^3$.*

Proof. Take B_R with R large enough s.t. $B_R^c \subseteq \Omega$. Define $\Omega_R := B_R \cap \Omega$. Then $\overline{\Omega}_R \subseteq B_{R+1}$ and Ω_R is a Lipschitz domain and B_{R+1} is pre-compact and $\phi|_{\Omega_R} \in W^{1,2}(\Omega_R)$. Therefore we can find an extension $E\phi \in W_0^{1,2}(\Omega_{R+1}) \hookrightarrow W^{1,2}(\mathbb{R}^3)$ (cf. [5, Sec. 1.5.1]). So we can now define

$$\bar{\phi} := \begin{cases} \phi & \text{in } \Omega \\ E\phi & \text{in } \Omega^c. \end{cases}$$

Then $\bar{\phi} \in L^2_{loc}(\mathbb{R}^3)$ and $\nabla\bar{\phi} \in L^2(\mathbb{R}^3)^3$. Then there exists a sequence $(\phi_l)_{l\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^3)$ s.t. $\nabla\phi_l \to \nabla\bar{\phi}$ in $L^2(\mathbb{R}^3)^3$ (cf. [7, Lemma 1.1]). By restricting ϕ_l to Ω we obtain the result. □

Theorem 4.2.5. *Let the same assumptions hold as in Thm. 4.2.3. Then solution of the problem is unique.*

Proof. Let ω and $\tilde{\omega}$ both be solutions. Then $\star\omega$ is closed thus $\mathscr{R}\star\omega \in S^1_2(\overline{\Omega})$ is also closed because

$$d\mathscr{R}\star\omega = \mathscr{R}d\star\omega = 0.$$

The same holds for $\tilde{\omega}$. $I(\varphi\mathscr{R}\star\omega), I(\varphi\mathscr{R}\star\tilde{\omega}) \in C^1_2(K)$ are closed 1-cochains with $I(\varphi\mathscr{R}\star\omega)(\gamma) = I(\varphi\mathscr{R}\star\tilde{\omega})(\gamma)$. Thus Prop. 2.3.1 implies $[I(\varphi\mathscr{R}\star\omega)] = [I(\varphi\mathscr{R}\star\tilde{\omega})] \in H^1(K)$. Because $[I] : \mathscr{H}^1(K) \to H^1(K)$ is an isomorphism (see **??**) we have

$$[\varphi\mathscr{R}\star\omega] = [\varphi\mathscr{R}\star\tilde{\omega}]$$

Here the cohomology classes are in the cohomology spaces of S-forms $\mathscr{H}^1(K)$. That means there is a S-form $\theta \in S^0(K)$ s.t.

$$\begin{aligned}
&\varphi\mathscr{R}\star\omega - \varphi\mathscr{R}\star\omega = d\theta \\
&\Rightarrow \mathscr{R}\star\omega - \mathscr{R}\star\omega = \varphi^{-1}d\theta = d\varphi^{-1}\theta =: d\tilde{\theta}.
\end{aligned}$$

We have $\tilde{\theta} \in S^0(\overline{\Omega}) \subseteq W^0_{\infty,loc}(\overline{\Omega})$ and $d\tilde{\theta} \in S^1_2(\overline{\Omega}) \subseteq W^1_2(\overline{\Omega})$.

Using the properties of $\mathscr{R}$ there exists a $\eta \in L^0_{2,loc}(\Omega)$ with $d\eta \in L^1_2(\Omega)$ s.t.

$$\star\omega - \star\tilde{\omega} = -d\mathscr{A}(\star\omega - \star\tilde{\omega}) + d\theta = d\eta.$$

By applying the Hodge star operator on both sides we find $\mu \in L^3_{2,loc}(\Omega)$ with $\delta\eta \in L^2_2(\Omega)$ s.t.

$$\omega - \tilde{\omega} = \delta\mu. \qquad (4.2.1) \quad \{\texttt{difference\_solu}$$

But because μ is only in $L^3_{2,loc}(\Omega)$ we can not immediately conclude that $\delta\mu = 0$.

Let us briefly return to vector proxies. Let B and $\tilde{B}$ be vector proxies of ω and $\tilde{\omega}$ respectively. (4.2.1) then translates to $B, \tilde{B} \in \mathring{H}(\mathrm{div})$ and $B - \tilde{B} = \nabla\phi$ with $\phi \in L^2_{loc}(\Omega)$ and $\nabla\phi \in L^2(\Omega)^3$. Because B and $\tilde{B}$ are both harmonic we have

$$0 = \int_\Omega (B - \tilde{B}) \cdot \nabla f \, dx = \int_\Omega \nabla\phi \cdot \nabla f \, dx \quad \forall f \in H^1(\Omega).$$

Thank to Lemma 4.2.4 we know that $\nabla\phi \in \overline{\nabla H^1(\Omega)}$. Because B and $\tilde{B}$ are harmonic we have $B - \tilde{B} \in \overline{\nabla H^1(\Omega)}^\perp$ and hence $\nabla\phi = 0$ and $B = \tilde{B}$. Because the corresponding vector proxies are equal we we obtain $\omega = \tilde{\omega}$. $\square$

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