1 Introduction

Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. We want to use the curve integral along this curve as an additional constraint. Let $\mathbf{B}$ be the magnetic field. Then we obtain the following problem: Find a vector field $\mathbf{B}$ on Ω s.t.

$$\operatorname{curl} \mathbf{B} = 0, \tag{1.0.1}$$

$$\operatorname{div} \mathbf{B} = 0 \text{ in } \Omega \tag{1.0.2}$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and} \tag{1.0.3}$$

$$\int_\gamma \mathbf{B} \cdot dl = C_0 \tag{1.0.4}$$

where $\mathbf{n}$ is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to find the proper formulation of this problem. This requires appropriate assumptions on the domain Ω and restating the problem in terms of the correct Sobolev spaces which will be explained later. We want to prove existence and uniqueness of solutions.

We will first spend a lot of work in properly introducing the tools needed for the proof. In Section 2 will introduce differential forms with all necessary operations on them. Section 3 is then a very short introduction to singular homology from algebraic topology. This will be needed to properly formulate and treat the curve integral condition described above. It will culminate in de Rham's theorem which can be seen as the connection between differential forms and singular homology. In Section 4, the focus lies on introducing unbounded operators and the concept of a Hilbert complex which will be fundamental tool in our proof of existence and uniqueness. At last, we will use the developed tools in Section 5 to formulate the problem rigorously and proof existence and uniqueness under reasonable assumptions. Throughout the first chapters, we will provide the definitions, basic results and we will show the proofs of many of them if they are not too long or trivial. Since all the presented topics are vast these sections should be seen as a very basic introduction.

2 Differential forms

{sec:differentia

We will introduce differential forms on manifolds. We start with the abstract formulation of alternating forms on finite dimensional real vector spaces in Section 2.1. After a short introduction of the essential notions from differential geometry in Section 2.2 we will finally define differential forms and operations on them in Section 2.3. The integration of differential forms will be developed in its own section 2.4 at the end.

2.1 Alternating forms

{sec:alternating

Understanding alternating forms is essential in understanding differential forms since these provide us with an alternating form at any point of a manifold. That is the reason why we will properly introduce the concept in this secion first. For the introduction of alternating forms we follow the short section in Arnold's book [2, Sec. 6.1.] combine it with material from [3, Sec. V.1]. However, more arguments and additional details are provided especially in Sec. 2.1.2 about scalar and vector proxies.

2.1.1 Basic definitions

Definition 2.1.1 (Alternating k-linear form)**.** Let V be a real vector space with $\dim V = n$. We call map $\omega : V^k \to \mathbb{R}$ *k-linear form* if it is linear in every argument. We call a k-linear form *alternating* if the sign switches when two arguments are exchanged i.e.

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k), \text{ for } 1 \leq i < j \leq k, \quad v_1, ..., v_k \in V.$$

We denote the space of alternating k-linear forms on V as $\operatorname{Alt}^k V$. For the special case $k = 0$ we define $\operatorname{Alt}^0 V := \mathbb{R}$.

Let $\mathcal{S}_k$ be the set of all permutations $\{1, 2, ..., k\} \to \{1, 2, ..., k\}$. A permutation that only exchanges two numbers is called transposition and denoted by (i, j) for the transposition that exchanges (i, j) with $i \neq j$. Every permutation can be written as the composition of transpositions. This decomposition into transpositions is not unique. Take a permutation $\pi \in \mathcal{S}_k$ and decompose it into transpositions

$$\pi = \tau_p \circ \tau_{p-1} \circ ... \circ \tau_1.$$

The sign $\operatorname{sgn}(\pi)$ of a permutation $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is equal to $(-1)^p$. Even though the decomposition is not unique the parity of the number of transpositions is always the same and hence the sign is well-defined. For example, the permutation $(1,2,3,4) \mapsto (2,3,1,4)$ can be built by performing the transpositions $(1,2)$ and $(1,3)$ so $\operatorname{sgn}(\pi) = (-1)^2 = 1$.

Going back to alternating forms, this also means for any permutation $\pi : \{1,2,...,k\} \to \{1,2,...,k\}$ and $\omega \in \operatorname{Alt}^k V$

$$\omega(v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(k)}) = \operatorname{sgn}(\pi)\, \omega(v_1, v_2, ..., v_k).$$

Definition 2.1.2 (Wedge product)**.** For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\pi} \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, ..., v_{\pi(k)})\nu(v_{\pi(k+1)}, ..., v_{\pi(k+l)})$$

where we sum over all permutations $\pi \in \mathcal{S}_{k+l}$ s.t. $\pi(1) < ... < \pi(k)$ and $\pi(k+1) < ... < \pi(k+l)$.

Let us mention some important properties of the wedge product. It is associative, but not commutative. For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we have

$$\omega \wedge \mu = (-1)^{kl} \mu \wedge \omega. \tag{2.1.1}$$

{eq:commutativit

Recalling the definition of the sign of a permutation $\pi \in \mathcal{S}_k$ we get for linear forms $\omega_1, \omega_2, ..., \omega_k \in V'$

$$\omega_{\pi(1)} \wedge \omega_{\pi(2)} \wedge ... \wedge \omega_{\pi(k)} = \operatorname{sgn}(\pi)\, \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k$$

and if a linear form appears twice then the expression is zero.

There is a useful formula for computing the wedge product of linear forms. We denote the dual space of V as V'. For $\omega_1, ..., \omega_k \in \operatorname{Alt}^1 V = V'$, $k \leq n$ we have the formula ([3, p.260])

$$\omega_1 \wedge ... \wedge \omega_k(v_1, ..., v_k) = \det(\omega_s(v_t))_{1 \leq s,t \leq n}. \tag{2.1.2}$$

{eq:wedge_produc

This formula can be easily proven by induction using the definition of the wedge product and the determinant.

Let $\{b_i\}_{i=1}^n$ be any basis of V and $\{b^i\}_{i=1}^n$ the correspoding dual basis i.e. $b^i \in V'$, $b^i(b_j) = \delta_{ij}$ for $i, j = 1, 2, ..., n$. Then

$$\{b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \,|\, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\operatorname{Alt}^k V$. In particular, $\dim \operatorname{Alt}^k V = \binom{n}{k}$.

Assume now that we are given an inner product $\langle \cdot, \cdot \rangle_V$ on V, where we will usually leave out the subscript if it is clear what space we mean. Recall the Riesz isomorphism $\Phi : V \to V'$ defined by

$$\Phi v(w) = \langle v, w \rangle.$$

Then we obtain an inner product on the dual space V' by using the Riesz isomorphism Φ

$$\langle \Phi v, \Phi w \rangle_{V'} := \langle v, w \rangle_V$$

which makes the Riesz isomorphism to an isometry.

Now we can define an inner product on $\mathrm{Alt}^k V$ by defining

$$\langle b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}, b^{j_1} \wedge ... \wedge b^{j_k} \rangle_{\mathrm{Alt}^k V} := \det(\langle b^{i_k}, b^{i_l} \rangle_V)_{1 \leq k, l \leq n} \qquad (2.1.3)$$ {eq:inner_produc

which is then extended to all of $\mathrm{Alt}^k V$ by linearity. We denote with $|\cdot|_{\mathrm{Alt}^k V}$ the induced norm. For an orthonormal basis u_1, ..., u_n the corresponding basis $u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\mathrm{Alt}^k V$. Next, we want to introduce the *pullback* as a natural linear mapping between spaces of alternating forms.

Definition 2.1.3. Let V and W be finite-dimensional real vector spaces with ordered bases $(b_i)_{i=1}^n$ and $(c_j)_{j=1}^m$ respectively. We write a basis in round brackets $(\cdot)$ if it is ordered. Let $L \in \mathcal{L}(V, W)$ where $\mathcal{L}(V, W)$ is the space of linear mappings from V to W. For $\omega \in \mathrm{Alt}^k W$ we define the pullback $L^*\omega \in \mathrm{Alt}^k V$ via

$$(L^*\omega)(v_1, ..., v_k) = \omega(L\, v_1, ..., L\, v_k).$$

It is then easy to see that L^* is a linear mapping from $\mathrm{Alt}^k W$ to $\mathrm{Alt}^k V$. It is obvious from the definitions of the wedge product and the pullback that we have

$$L^*(\omega \wedge \nu) = L^*\omega \wedge L^*\nu \quad \forall \omega \in \mathrm{Alt}^k W,\ \nu \in \mathrm{Alt}^l W.$$

Proposition 2.1.4. *Let $A \in \mathbb{R}^{m \times n}$ be the matrix representation of L in the above bases i.e. $L\, b_i = \sum_{j=1}^m A_{ji} c_j$. Then we get the basis representation of the pullback*

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1 \leq i_1 < ... < i_k \leq n} \det A_{(j_1,...,j_k),(i_1,...,i_k)}\, b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \qquad (2.1.4)$$ {eq:basis_repres

where $A_{(j_1,...,j_k),(i_1,...,i_k)}$ is the matrix we get by choosing the rows $j_1, ..., j_k$ and the columns i_1, ... , i_k.

Proof. Because $\{b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \mid 1 \leq i_1 < ... < i_k \leq n\}$ and $\{c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k} \mid 1 \leq j_1 < ... < j_k \leq m\}$ are bases for $\mathrm{Alt}^k V$ and $\mathrm{Alt}^k W$ respectively, we can find $\lambda_{i_1 ... i_k}$ s.t.

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1 ... i_k} b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \qquad (2.1.5)$$ {eq:basis_repres

Now recall the formula for the wedge product of 1-forms $\nu_i \in \mathrm{Alt}^1 V = V'$

$$\nu_1 \wedge ... \wedge \nu_k(v_1, ..., v_k) = \det\big(\nu_s(v_t)\big)_{1\le s,t\le n}.$$

Fix now $1 \le l_1 < ... < l_k \le n$. Then we get from this formula $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}(b_{l_1}, ..., b_{l_k}) = 1$ i.i.f. $(i_1, ..., i_k) = (l_1, ..., l_k)$. Here it is important to remember that these indices are ordered. Plugging this in (2.1.5) gives us

$$\begin{aligned}
\lambda_{l_1 \dots l_k} &= L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k})(b_{l_1}, ..., b_{l_k}) \\
&= c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}(L\, b_{l_1}, ..., L\, b_{l_k}) \\
&= \det\big(c^{j_s}(\sum_{r_t=1}^{m} A_{r_t,l_t} c_{r_t})\big)_{1\le s,t\le k} \\
&= \det\big(\sum_{r_t=1}^{m} A_{r_t,l_t} \delta_{j_s,r_t}\big)_{1\le s,t\le k} \\
&= \det\big(A_{j_s,l_t}\big)_{1\le s,t\le k} \\
&= \det A_{(j_1,\dots,j_k),(l_1,\dots,l_k)}.
\end{aligned}$$

□

We want to emphasize the special case of the pullback of an alternating n-linear form with $m = n$. So take $\omega \in \mathrm{Alt}^n W$. Then we know that $\dim \mathrm{Alt}^n W = \binom{n}{n} = 1$ and so $\omega = \lambda\, c^1 \wedge ... \wedge c^n$ for some $\lambda \in \mathbb{R}$. Then there only remains one summand in (2.1.4) and we obtain for this special case

$$L^*\omega = \lambda \det A\, b^1 \wedge ... \wedge b^n \qquad (2.1.6)$$ {eq:pullback_alt

We want to examine $\mathrm{Alt}^n V$ a bit closer. $\mathrm{Alt}^n V$ is one-dimensional and so we can choose a basis by fixing a specific non-zero element. We want to choose one specific element called the *volume form* which will play a crucial role when we define integration on a manifold in Sec. 2.4. We also need it to define the Hodge star operator below.

The choice of this volume form will depend on the orientation. We say that two ordered bases of V have the same orientation if the change of basis has positive determinant. That divides the ordered bases into two classes with different orientation. We choose one of these classes and call these bases positively oriented. In $\mathbb{R}^n$, the convention is to define the class as positively oriented which includes the standard orthonormal basis.

Definition 2.1.5 (Volume form)**.** Let $(b_i)_{i=1}^n$ be any positively oriented basis. Let G be the Gramian matrix i.e. $G_{ij} = \langle b_i, b_j\rangle$ which is always a symmetric positive definite matrix. Then we define the *volume form*

$$\mathrm{vol} := \sqrt{\det G}\, b^1 \wedge b^2 \wedge ... \wedge b^n.$$

We have the following defining property of the volume form.

Proposition 2.1.6. *For any ordered orthonormal basis* $(u_i)_{i=1}^n$ *we have*

$$\mathrm{vol}(u_1,\, u_2, ..., \, u_n) = (-1)^s.$$

with $s = 0$ *if* $(u_1, ..., u_n)$ *has the same orientation as* $(b_i)_{i=1}^n$ *and* $s = 1$ *otherwise.*

Proof. Let us define the matrix $B \in \mathbb{R}^{n\times n}$, $B_{k,i} = \langle b_i, u_k\rangle_V$ which is just the change of basis matrix from $(b_i)_{i=1}^n$ to $(u_i)_{i=1}^n$. Then using basic linear algebra we get $G = B^\top B$ and $\sqrt{\det G} = (-1)^s \det B$. Let now Ψ be the linear map with $\Psi b_i = u_i$. In the basis $(b_i)_{i=1}^n$ this has the matrix representation B^{-1} and so by using (2.1.6) we get

$$\begin{aligned} \mathrm{vol}(u_1,\, u_2, ..., \, u_n) &= \sqrt{\det G}\, b^1 \wedge ... \wedge b^n (\Psi b_1, ..., \Psi b_n) \\ &= (-1)^s \det B\, \Psi^*(b^1 \wedge ... \wedge b^n)(b_1, ..., b_n) \\ &= (-1)^s \det B\, \det B^{-1}\, (b^1 \wedge ... \wedge b^n)(b_1, ..., b_n) = (-1)^s. \end{aligned}$$

□

This property also defines the volume form uniquely so it is independent of the chosen basis. It only depends on the orientation. It also shows that vol is non-zero and thus

$$\mathrm{Alt}^n V = \mathrm{span}\{\mathrm{vol}\}.$$

Note that if we choose $\{b_i\}_i$ to be an orthonormal basis to begin with the Gramian matrix is just the identity and $\mathrm{vol} = b^1 \wedge ... \wedge b^n$. Especially in the case of $\mathbb{R}^n$ if we denote the standard basis by $\{e_i\}_{i=1}^n$ and the resulting dual basis as $\{e^i\}_{i=1}^n$ then

$$\mathrm{vol} = e^1 \wedge ... \wedge e^n.$$

We will from now on assume that we fixed an orientation on V and with it the volume form vol. Using the resulting volume form on V we can now define the *Hodge star operator* with the following proposition.

Proposition 2.1.7. *There exists a isomorphism* $\star : Alt^k V \to Alt^{n-k} V$ *s.t.*

$$\omega \wedge \mu = \langle \star\omega, \mu\rangle_{Alt^{n-k} V}\, \mathrm{vol} \quad \forall \mu \in Alt^{n-k} V. \tag{2.1.7}$$ {eq:hodge_star_d

We call this isomorphism the Hodge star operator.

Proof. Let us denote with vol' the dual basis of V' corresponding of vol i.e. $\mathrm{vol}'(\mathrm{vol}) = 1$. Let us fix $\omega \in \mathrm{Alt}^k V$. Then we can define the following linear form on $\mathrm{Alt}^{n-k} V$

$$\mu \mapsto \mathrm{vol}'(\omega \wedge \mu).$$

Then we define $\star\omega$ as the Riesz representative of this linear form that means we have $\mathrm{vol}'(\omega \wedge \mu) = \langle \star\omega, \mu \rangle_{\mathrm{Alt}^{n-k} V}$ for all $\mu \in \mathrm{Alt}^{n-k} V$ which is equivalent (2.1.7). Checking linearity is trivial and will be omitted.

It is also clear from the uniqueness of the Riesz representative that the $\star\omega$ is uniquely determined by the above condition and thus $\star$ is injective. Since $\dim \mathrm{Alt}^k V = \binom{n}{k} = \binom{n}{n-k} = \dim \mathrm{Alt}^{n-k} V$ the injecivity implies surjectivity and $\star$ is an isomorphism. $\square$

Let us collect some important properties of the Hodge star operator.

{prop:properties

Proposition 2.1.8. *Let $(u_i)_{i=1}^n$ be an ordered orthonormal basis of V. Let $\{i_1, ..., i_n\} = \{1, ..., n\}$, $i_1 < i_2 < ... < i_k$ and $i_{k+1} < ... < i_n$. Then*

$$\star(u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}) = \mathrm{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n} \tag{2.1.8}$$ {eq:hodge_star_o

where $\mathrm{sgn}(i_1, i_2, ..., i_n)$ is the sign of the permutation $j \mapsto i_j$. In particular, $\star$ is an isometry since it maps orthonormal bases to orthonormal bases. Furthermore,

(i) $\star \star \omega = (-1)^{k(n-k)}\omega \quad \forall \omega \in \mathit{Alt}^k V$

(ii) $\omega \wedge \star\nu = \langle \omega, \nu \rangle_{\mathit{Alt}^k V} \mathrm{vol} \quad \forall \omega, \nu \in \mathit{Alt}^k V.$

Proof. Take $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ with $1 \le j_{k+1} < ... < j_n \le n$. Then

$$\begin{aligned} &\langle \mathrm{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\mathrm{Alt}^{n-k} V} \\ &= \mathrm{sgn}(i_1, i_2, ..., i_n) \det \left(\langle u^{i_s}, u^{j_t} \rangle \right)_{k+1 \le s,t \le n} \\ &= \mathrm{sgn}(i_1, i_2, ..., i_n) \det \left(\delta_{i_s, j_t} \right)_{k+1 \le s,t \le n}. \end{aligned}$$

In the last step we used the fact that u^i are orthonormal since u_i are. Now observe due to the ordering that $\det \left(\delta_{i_s,j_t} \right)_{k+1 \le s,t \le n} = 1$ i.i.f. $i_s = j_s$ for $s = k+1, ..., n$ and is zero otherwise.

For the wedge product we get

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} = \begin{cases} \mathrm{sgn}(i_1, ..., i_n)\ \mathrm{vol}, & \text{if } (i_{k+1}, ..., i_n) = (j_{k+1}, ..., j_n) \\ 0, & \text{otherwise.} \end{cases}.$$

Comparing both expressions we just proved

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} = \langle \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\operatorname{Alt}^{n-k} V} \operatorname{vol}.$$

Because the $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ for $j_{k+1} < ... < j_n$ are a basis of $\operatorname{Alt}^{n-k} V$ we can deduce that

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge \nu = \langle \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}, \nu \rangle_{\operatorname{Alt}^{n-k} V} \operatorname{vol}.$$

and thus $\star(u^{i_1} \wedge ... \wedge u^{i_k}) = \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}$ as claimed. We see from this that $\star$ maps the orthonormal basis of $\operatorname{Alt}^k V$ to an orthonormal basis of $\operatorname{Alt}^{n-k} V$ and is thus an isometry.

The other two claims follow from that easily. For any $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^{n-k} V$

$$\begin{aligned}
\langle \star \star \omega, \mu \rangle_{\operatorname{Alt}^k V} \operatorname{vol} &= \star\omega \wedge \mu = (-1)^{k(n-k)} \mu \wedge \star\omega \\
&= (-1)^{k(n-k)} \langle \star\mu, \star\omega \rangle_{\operatorname{Alt}^{n-k} V} \operatorname{vol} \\
&= \langle (-1)^{k(n-k)} \omega, \mu \rangle_{\operatorname{Alt}^k V} \operatorname{vol}.
\end{aligned}$$

Then the first claim follows since $\mu \in \operatorname{Alt}^k V$ was arbitrary.

For the second claim, take $\omega, \nu \in \operatorname{Alt}^k V$, then

$$\omega \wedge \star\nu = \langle \star\nu, \star\omega \rangle_{\operatorname{Alt}^{n-k} V} \operatorname{vol} = \langle \omega, \nu \rangle_{\operatorname{Alt}^k V} \operatorname{vol}.$$

□

In particular in $\mathbb{R}^3$, we have $\star\star = \operatorname{Id}$. Notice also that we always have $\star 1 = \operatorname{vol}$.

Let us quickly derive the expression in any basis for the Hodge star applied to linear forms which we will need later. Let $\omega = \sum_i \omega_i b^i \in \operatorname{Alt}^1 V = V'$. Let us denote $g^{ij} = \langle b^i, b^j \rangle$ and $G_{ij} = \langle b_i, b_j \rangle$ again the Gramian matrix. Then we claim

$$\star\omega = \sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge ... \wedge \widehat{b^j} \wedge ... \wedge b^n \tag{2.1.9}$$ {eq:hodge_star_o

where $\widehat{b^j}$ means that this term is left. The proof is very simple in this case. Let ν denote the expression on the right hand side of (2.1.9). For any $1 \leq l \leq n$

we get

$$
\begin{aligned}
\nu \wedge b^l &= \left(\sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge ... \wedge \widehat{b^j} \wedge ... \wedge b^n \right) \wedge b^l \\
&= \sqrt{\det G} \sum_{j=1}^{n} \langle b^j, \sum_{i=1}^{n} \omega_i b^i \rangle (-1)^{j-1} b^1 \wedge b^2 \wedge ... \wedge \widehat{b^j} \wedge ... \wedge b^n \wedge b^l \\
&= \sqrt{\det G} \langle b^l, \omega \rangle (-1)^{(l-1)} (-1)^{(n-l)} b^1 \wedge b^2 \wedge ... \wedge b^l \wedge ... \wedge b^n \\
&= \langle (-1)^{n-1} \omega, b^l \rangle \operatorname{vol}.
\end{aligned}
$$

where the inner product is always in the corresponding space of alternating forms. In the second step, we used that if $j \neq l$ then one basis element must appear twice in the wedge product which is then zero. If $j = l$ then $\operatorname{sgn}(1, 2, ..., \hat{l}, ..., n, l) = (-1)^{n-l}$ because we need $(n - l)$ transpostions to bring the indices into order. So we proved $(-1)^{n-1}\omega = \star\nu$ and thus $\star\omega = (-1)^{n-1} \star \star\nu = \nu$. This shows also that the explicit expression of the Hodge star becomes much more complicated if the basis is not orthonormal.

2.1.2 Scalar and Vector proxies

{sec:scalar_and_

Now we want to relate alternating maps to elements of the vector space V itself or to scalars. Let us start with the easiest case. $\operatorname{Alt}^0 V$ are already scalars by definition. Now we can use the Hodge star operator which is an isometry $\star : \operatorname{Alt}^0 V \to \operatorname{Alt}^n V$ with $\star c = c \operatorname{vol}$. We call the real number that is associated with an element of $\operatorname{Alt}^n V$ *scalar proxy* i.e. the scalar proxy of $c \in \mathbb{R}$ is just $c \operatorname{vol} \in \operatorname{Alt}^n V$.

Next, we will move on to $\operatorname{Alt}^1 V$ and $\operatorname{Alt}^{n-1} V$. Let $\Phi : V \to V'$ denote the Riesz isomorphism which is an isometry. Because $V' = \operatorname{Alt}^1 V$ this gives us the correspondence of vectors and linear forms. Now we can once again use the Hodge star and obtain the isometry $\star\Phi : V \to \operatorname{Alt}^{n-1} V$. We call the vectors associated with an alternating 1- or $(n-1)$-linear form *vector proxy*.

These ways to identify alternating forms with scalars and vectors gives us the ability to look at the notions defined above in the context of scalars and vectors. Let us look at the wedge product. We have for $v, w \in V$

$$
\Phi v \wedge \star\Phi w = \langle \Phi v, \Phi w \rangle_{V'} \operatorname{vol} = \langle v, w \rangle_V \operatorname{vol}
$$

which means that the wedge product of a linear form and an alternating $(n-1)$- linear form corresponds in proxies to the inner product.

In the case of $V = \mathbb{R}^3$ with the standard basis vectors e_1, e_2 and e_3. Denote the resulting elements of the dual basis with e^1, e^2 and e^3 respectively.

Take $v = v_1e_1 + v_2e_2 + v_3e_3 \in \mathbb{R}^3$ and recall that for a orthonormal basis the Riesz isomorphism maps basis elements to their dual basis elements i.e. $\Phi e_i = e^i$. Hence, we get $\Phi v = v_1e^1 + v_2e^2 + v_3e^3$. Take another $w \in \mathbb{R}^3$. Then using the $e^i \wedge e^j = -e^j \wedge e^i$ we get

$$\Phi v \wedge \Phi w = \star\Phi(v \times w). \tag{2.1.10}$$ {eq:cross_produc

That means in $3D$ in terms of vector proxies, the wedge product of two linear forms corresponds to the cross product. Note that (2.1.10) is formulated without using a specific basis and can therefore be computed using any basis i.e. if we have $v = \tilde{v}_1b_1 + \tilde{v}_2b_2 + \tilde{v}_3b_3$ and analogous for w we could still calculate the cross product directly as

$$v \times w = \Phi^{-1} \star \big(\Phi v \wedge \Phi w\big).$$

One has to take care though because if the basis is not orthonormal the Riesz isomorphism does not map basis elements b_i to their respective dual basis elements b^i. Instead we have

$$\Phi b_i = \sum_{i=1}^{n} \langle b_j, b_i\rangle b^j$$

i.e. it has the gramian matrix G as basis representation. This is easy to see. Let $\Phi b_i = \sum_j \lambda_j b^j$. Then

$$\lambda_j = \Phi b_i(b_j) = \langle b_i, b_j\rangle.$$

As derived above, the Hodge star is not as trivial to compute either.

Similarly, we want to explore the pullback in terms of vector proxies as well. These will be important in the next section when we talk about the pullback of differential forms and apply these to the tranformation of integrals. In order to avoid complicated computations we will stick to orthonormal bases. Let $\{b_i\}_{i=1}^n$ be an ONB of V and $\{c_j\}_{j=1}^m$ be an ONB of W. Let $L : V \to W$ again be a linear map and A be the basis representation of it w.r.t. the two bases given i.e. $Lb_i = \sum_j A_{ji}c_j$. Then we get the pullback of linear forms in terms of vector proxies as $\Phi_V^{-1}L^*\Phi_W : W \to V$. Recall formula (2.1.6) which gives us for the pullback of one forms

$$L^*c^j = \sum_{i=1}^{n} A_{j,i}b^i \tag{2.1.11}$$ {eq:pullback_lin

so the matrix representation w.r.t. the given dual bases is $A^\top$. Then since we have $\Phi_V b_i = b^i$ and analogous for c^j the matrix representation of the

Riesz isomorphism is just the identity. In total, the basis representation of $\Phi_V^{-1}L^*\Phi_W$ is then also $A^\top$.

For $m = n$ let us look at the pullback of alternating $(n-1)$-linear maps. In terms of vector proxies this can then be expressed as $\Phi_V^{-1} \star^{-1} L^* \star \Phi_W$. Note that we used the same symbol $\star$, but it is once applied in W and then the inverse in V. It can be shown with the same ideas and (2.1.8) that the matrix representation is the adjugate matrix $\mathrm{ad}(A)$ defined as

$$\mathrm{ad}(A)_{ij} = (-1)^{i+j} \det A_{-j,-i}$$

where $A_{-j,-i}$ is the matrix without the j-th row and i-th column. If A is invertible then $(\det A)\, A^{-1} = \mathrm{ad}(A)$.

The pullback of n-linear alternating forms in terms of scalar proxies is $\star^{-1}L^*\star$. Again in the case of $n = m$ and an orthonormal basis we get for $c \in \mathbb{R}$

$$\star^{-1}L^* \star c = \star^{-1}L^*(c\, c^1 \wedge c^2 \wedge ... \wedge c^n) = \star^{-1}c \det L\, b^1 \wedge b^2 \wedge ... \wedge b^n = c \det L.$$

2.2 Basics from differential geometry

{sec:differentia

Before we define differential forms, let us start by revising some basics from differential geometry. We follow the approach from [3, Sec. II] for the most part.

2.2.1 Manifolds

In order to formulate the definition of a manifold, let us recall the definition of a topological space.

Definition 2.2.1 (Topological space)**.** A topological space is a set X together with a collection of subsets of X denoted by $\mathcal{T}$ s.t.

- $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
- For $\{U_i \in \mathcal{T} \mid i \in \mathcal{I}\}$ with any index set $\mathcal{I}$, $\bigcup_{i\in\mathcal{I}} U_i \in \mathcal{T}$ and
- $\emptyset, X \in \mathcal{T}$.

The sets contained in $\mathcal{T}$ are called *open*.

For example, a metric space together with its usual open sets is a topological space. A well known example of topologies which do not arise from a metric are the weak and weak-$\star$ topology on infinite dimensional spaces (see [4, Ch. 3]).

Definition 2.2.2 (Second countable topological space)**.** Let $(X, \mathcal{T})$ be a topological space. Then we call $\mathcal{B} \subseteq \mathcal{T}$ a basis for the topology of X if every open set (i.e. every set in $\mathcal{T}$) is a union of sets in $\mathcal{B}$. If a topological space has a countable basis it is called *second countable.*

$\mathbb{R}^n$ with the standard norm is an example of a second countable topological space. Consider the countable set of balls $\{B_r(x) \mid 0 < r \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ where $B_r(x)$ are the balls with center x and radius r. Then it is trivial to show that any open set in $\mathbb{R}^n$ is a union of of these balls. Hence, $\mathbb{R}^n$ is second countable.

Definition 2.2.3 (Hausdorff space)**.** Let $(X, \mathcal{T})$ be a topological space. We call $(X, \mathcal{T})$ *Hausdorff* if we can separate any two different points of X with disjoint neighborhoods. That means for any $x, y \in X$, $x \neq y$ there are $U_x, U_y \in \mathcal{T}$ s.t. $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

Example 2.2.4. Let (X, d) be a metric space and take $x, y \in X$, $x \neq y$. Then for the distance $\delta := d(x, y) > 0$ we can choose the open balls around x and y with radius $\delta/2$, denoted by $B_{\delta/2}(x)$ and $B_{\delta/2}(y)$. These are open and obviously disjoint. Therefore any metric space is Hausdorff.

Definition 2.2.5 (Manifold)**.** A n-dimensional C^α manifold M, $\alpha \in \mathbb{N}$ (in this thesis the natural numbers start with zero), is a second countable Hausdorff space M with an open cover $\{U_i\}_{i \in I}$ and a collection of maps called *charts* $\boldsymbol{\phi}_i$, $i \in I$ for some index set I s.t.

- $\boldsymbol{\phi}_i : U_i \to V_i \subseteq \mathbb{R}^n$ are homeomorphisms
- for two charts $\boldsymbol{\phi}_i$, $\boldsymbol{\phi}_j$ the *change of coordinates* or *chart transition* $\boldsymbol{\phi}_j \circ \boldsymbol{\phi}_i^{-1} : \boldsymbol{\phi}_i(U_i \cap U_j) \to \boldsymbol{\phi}_j(U_i \cap U_j)$ is a C^α diffeomorphism. If $\alpha = 0$ it is a homeomorphism.

When we write $(U_i, \boldsymbol{\phi}_i)$ we mean the chart $\boldsymbol{\phi}_i$ with domain U_i.

Vector valued quantities like charts will be denoted in bold symbols throughout this thesis. The charts provide us with *local coordinates* $x_k : U_i \to \mathbb{R}^n$, $k = 1, ..., n$ with $\mathbf{x}(p) = (x_1(p), ..., x_n(p))^\top = \boldsymbol{\phi}(p) \in \mathbb{R}^n$. Let us take a look at an example. Take any open domain $\Omega \subseteq \mathbb{R}^n$. Then we can take the identity as a chart. Since $\mathbb{R}^n$ with the standard topology is Hausdorff and second coundable the same reasoning applies to open subdomains. Hence, Ω is a n-dimensional C^∞ manifold.

Proposition 2.2.6. *Every manifold has a countable atlas.*

Proof. Take any atlas $\{(U_i, \boldsymbol{\phi}_i)\}_{i\in I}$. Because the manifold is second countable there exists a countable basis $\mathcal{B} = \{B_0, B_1, ...\}$ of the topology. Every basis of a topological is a open cover which is easily checked. Define $\mathcal{A} = \{k \in \mathbb{N} \mid B_k \subseteq U_i \text{ for some } i \in I\}$. Then for every $k \in \mathcal{A}$ choose $i_k \in I$ s.t. $B_k \subseteq U_{i_k}$. Then $\{U_{i_k}\}_{k\in\mathcal{A}}$ is an open cover and thus $\{(U_{i_k}, \boldsymbol{\phi}_{i_k})\}_{k\in\mathcal{A}}$ is a countable atlas. $\square$

Due to this proposition we will from now on assume that all the chosen atlases are countable. Note that here the second countability is essential. Some authors do not require this in the defintion of a manifold and then this proposition might not hold (cf. [6, 1.A.2]). Another important property is the existence of a partition of unity on M assuming M is smooth which we will not prove.

Theorem 2.2.7 (Partition of unity)**.** *Let M be a smooth (i.e. C^∞) manifold with atlas $\{U_i, \boldsymbol{\phi}_i)\}_{i=0}^N$, $N \leq \infty$. Then there exists a smooth partition of unity subordinate to the open cover $\{U_i\}_{i=0}^N$. That means there exists a family of non-negative smooth functions $\{\chi_i\}_{i=0}^N$ s.t.* $\operatorname{supp}\chi_i \subseteq U_i$ *and* $\sum_{i=0}^N \chi_i(p) = 1$ *for every $p \in M$.*

These partitions of unity are typically used to extend a construction that is done locally to the entire manifold as we will see later.

Let us denote $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$. Let us for the following equip $\mathbb{R}_- \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ with the subspace topology i.e. a set $V \subseteq \mathbb{R}_- \times \mathbb{R}^{n-1}$ is open i.i.f. there exists an open set $V' \subseteq \mathbb{R}^n$ s.t. $V = V' \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$. This means e.g. that $B_1(0) \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$ is open which is not an open set in the standard topology of $\mathbb{R}^n$.

{def:manifold_wi

Definition 2.2.8 (Manifold with boundary)**.** We call M a *manifold with boundary* if the charts $\boldsymbol{\phi}_i$, $i \in I$ are homeomorphism from M into $\mathbb{R}_- \times \mathbb{R}^{n-1}$ endowed with the subspace topology. The *boundary* ∂M are the points that get mapped to $\{0\} \times \mathbb{R}^{n-1}$ by the charts i.e. $\partial M = \bigcup_{i\in I} \boldsymbol{\phi}_i^{-1}(\{0\} \times \mathbb{R}^{n-1})$.

Remark 2.2.9. We should note the relationship between the boundary in the usual topological sense and the above definition of a boundary of a manifold. Let $\Omega \subseteq \mathbb{R}^n$ be an open C^1 domain. Then $\partial\Omega$ in the usual topological sense is $\overline{\Omega} \setminus \Omega$. However, if we see Ω as a manifold with boundary then the boundary $\partial\Omega$ in the sense of Def. 2.2.8 is empty.

Another important difference, is that in the case of manifold $\partial\partial M = \emptyset$, but for the general topological boundary that is not the case. If we have e.g. a single point $\{x\} \subseteq \mathbb{R}^n$ then $\partial\{x\} = \{x\}$.

2.2.2 Tangent spaces

{sec:tangent_spa

From now on we want to investigate certain structures on a manifold that require some regularity to be defined. Therefore, we will assume $\alpha > 0$ i.e. our manifolds are differentiable.

Remark 2.2.10. There is also the concept of Lipschitzian manifolds. Then using Rademachers theorem some of the structures that are introduced below can be extended with some care. See [11] for a detailed discussion.

Definition 2.2.11 (Orientation of a manifold)**.** We call an atlas *oriented* if the Jacobian of the coordinate changes has positive determinant. A manifold that can be equipped with an oriented atlas is called *orientable*.

The next important concept we will recall are tangent spaces. It should be noted that there are different definitions of tangent space, but these lead to isomorphic notions (see e.g. [7, Sec. 1.B]).

Definition 2.2.12. Let M, N be an n- and m-dimensional manifold with or without boundary and a function $F : M \to N$. Take $p \in M$ and let $(\boldsymbol{\phi}, U)$ and $(\boldsymbol{\psi}, V)$ be charts at p and $F(p)$ with local coordinates $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ respectively. Then we call the function

$$\bar{F}(x_1, ..., x_n) = \boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1}(x_1, ..., x_n)$$

F expressed in local coordinates which depends on the chosen local coordinates.

For a point $p \in M$ and neighorhoods $U \subseteq M$ of p and $V \subseteq N$ of $F(p)$ with local coordinate charts $(\boldsymbol{\phi}, U)$ and $(\boldsymbol{\psi}, V)$ and resulting local coordinates $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ respectively and assume M and N to be at least C^1. We call a function $F : U \to N$ differentiable at p if the expression in local coordinates is differentiable i.e. if $\boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1}$ is differentiable at $\boldsymbol{\phi}(p)$. We define its Jacobian $DF(p) := D(\boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1})(\boldsymbol{\phi}(p))$ and we denote

$$\frac{\partial F_j}{\partial x_i}(p) = \frac{\partial(y_j \circ F \circ \boldsymbol{\phi}^{-1})}{\partial x_i}(\boldsymbol{\phi}(p)) \qquad (2.2.1)$$ {eq:derivative_o

It is important to notice, that the values of the Jacobian and the derivatives depend on the chosen representation. However, the definition of differentiability is independent of the chart. Let $(\tilde{U}, \tilde{\boldsymbol{\phi}})$ with $p \in \tilde{U}$ be another chart and analogous $(\tilde{V}, \tilde{\boldsymbol{\psi}})$ be local coordinate at $F(p)$. Then

$$\tilde{\boldsymbol{\psi}} \circ F \circ \tilde{\boldsymbol{\phi}}^{-1} = \tilde{\boldsymbol{\psi}} \circ \boldsymbol{\psi}^{-1} \circ \boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1} \circ \boldsymbol{\phi} \circ \tilde{\boldsymbol{\phi}}^{-1}$$

and since the chart transitions are at least C^1-diffeomorphisms by definition $\tilde{\boldsymbol{\psi}} \circ F \circ \tilde{\boldsymbol{\phi}}^{-1}$ is differentiable as well. These types of definitions via local charts on a manifold are frequent in differential geometry. This is a proper definition if it is independent of the chosen chart. Because we do not want to bother with the technicalities of differential geometry too much we will very often leave out these types of proofs.

Definition 2.2.13 (Tangent space)**.** Let $I \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma : I \to M$ be a differentiable curve with $\gamma(0) = p \in M$. If 0 is on the boundary of I then we mean the one sided derivative. Let $f : M \to \mathbb{R}$ be a differentiable function. Let $(\boldsymbol{\phi}, U)$ be a local chart. We define the the directional derivative $D_\gamma(f) := \frac{d}{dt} f(\gamma(t))|_{t=0}$. We call the functional $D_\gamma : C^1(U) \to \mathbb{R}$ a *tangent vector*. The real vector space of all tangent vectors at p is called the *tangent space* and denoted by T_pM

This begs the question why T_pM is actually a vector space. Let $(U, \boldsymbol{\phi})$ again be a local chart at p. We can express a tangent vector D_γ in local coordinates by

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \frac{d}{dt}(f \circ \boldsymbol{\phi}^{-1} \circ \boldsymbol{\phi})(\gamma(t))\Big|_{t=0} \tag{2.2.2}$$

$$= \sum_{i=1}^{k} \frac{\partial (f \circ \boldsymbol{\phi}^{-1})}{\partial x_i}(\boldsymbol{\phi}(p))\,(x_i \circ \gamma)'(0) = (\sum_{i=1}^{k} v_i \frac{\partial}{\partial x_i}\Big|_p)(f) \tag{2.2.3}$$

by taking $v_i = (x_i \circ \gamma)'(0)$. Thus we can express

$$D_\gamma = \sum_{i=1}^{k} (x_i \circ \gamma)'(0)\,\frac{\partial}{\partial x_i}\Big|_p .$$

We will now leave out the reference to p in the partial derivative. For the other direction take $\mathbf{v} = (v_1, v_2, ..., v_n)^\top \in \mathbb{R}^n$. We now want to find a differentiable curve γ s.t. $\gamma(0) = p$ and $D_\gamma = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$. For that define $\gamma(t) := \boldsymbol{\phi}^{-1}(\boldsymbol{\phi}(p) + \mathbf{v}t)$. Then

$$D_\gamma = \sum_{i=1}^{n} (x_i \circ \gamma)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} (x_i(p) + v_i t)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}.$$

This shows that T_pM is a linear space and

$$T_pM = \operatorname{span} \left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^{n} .$$

In order to show that this is a basis we have to prove linear independence. Assume we have $\sum_{i=1}^{n} \lambda_i \, \partial/\partial x_i|_p = 0$ Then because $x_j \circ \boldsymbol{\phi}^{-1}(x) = x_j$ for $\mathbf{x} \in \boldsymbol{\phi}(U)$ and $1 \leq j \leq n$. Note x_j denote here the variable of $\mathbb{R}^n$ and the local coordinate $x_j : U \to \mathbb{R}$. Of course, this is a slight abuse of notation, but this a standard convention in differental geometry. Then we have

$$0 = \left(\sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i} \right) (x_j) = \sum_{i=1}^{n} \lambda_i \frac{\partial (x_j \circ \boldsymbol{\phi}^{-1})}{\partial x_i} (\boldsymbol{\phi}(p)) = \lambda_j$$

so $\frac{\partial}{\partial x_i}$ are linearly independent and thus a basis of T_pM. Thus we have shown

Proposition 2.2.14. *The set of tangent vectors at a point $p \in M$ is a vector space. The derivatives w.r.t. the local coordinates $\frac{\partial}{\partial x_i}$ are tangent vectors and*

$$T_pM = \operatorname{span} \left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^{n} .$$

Now that we introduced tangent spaces we will define the most important mapping between them.

Definition 2.2.15 (Differential)**.** Let M, N be an n and m dimensional manifold respectively. Take $p \in M$ and let $F : M \to N$ be differentiable at p. Then we define the differential of F at point p,

$$F_{*,p} : T_pM \to T_{F(p)}N, D_\gamma \mapsto D_{F \circ \gamma}.$$

Proposition 2.2.16. *Let $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ be local coordinates at p and $F(p)$ of the charts $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and let $\frac{\partial}{\partial x_i}|_p$ and $\frac{\partial}{\partial y_j}|_{F(p)}$ be the resulting bases for the tangent spaces at T_pM and $T_{F(p)}N$ respectively. Then the resulting matrix representation of the differential $F_{*,p} : T_pM \to T_{F(p)}N$ is the Jacobian as defined above so*

$$F_{*,p} \left(\frac{\partial}{\partial x_i}|_p \right) = \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_i}(p) \, \frac{\partial}{\partial y_j}|_{F(p)}.$$

Proof. We will omitt the reference to the points of the partial derivatives. We choose $\gamma = \boldsymbol{\phi}^{-1}(\boldsymbol{\phi}(p) + e_i t)$. Then $D_\gamma = \frac{\partial}{\partial x_i}$ and by applying the chain rule and applying the above definitions, we compute

$$\begin{aligned}
F_*(\frac{\partial}{\partial x_i})(f) = D_{F\circ\gamma}(f) &= \frac{d}{dt}(f(F\circ\gamma(t)))|_{t=0} \\
&= \frac{d}{dt}(f\circ\boldsymbol{\psi}^{-1}\circ\boldsymbol{\psi}\circ F\circ\boldsymbol{\phi}^{-1}\circ\boldsymbol{\phi}\circ\gamma(t))|_{t=0} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial f\circ\boldsymbol{\psi}^{-1}}{\partial y_j}(\boldsymbol{\psi}(F(p)))\frac{\partial(\boldsymbol{\psi}\circ F\circ\boldsymbol{\phi}^{-1})_j}{\partial x_i}(\boldsymbol{\phi}(p))(x_i\circ\gamma)'(0) \\
&= \sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial f}{\partial y_j}(F(p)) = \left(\sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial y_j}\right)(f).
\end{aligned}$$

□

For everything we did above we fixed one certain chart $(U,\boldsymbol{\phi})$ at p. Now the question arises what happens when we choose a different chart $(\tilde{U},\tilde{\boldsymbol{\phi}})$ instead. Going through the same steps we end up with another basis $\{\frac{\partial}{\partial y_j}\}_{j=1}^{n}$ of the tangent space T_pM which are the derivatives w.r.t. the chart $(\tilde{U},\tilde{\boldsymbol{\phi}})$. Let us compute the change of basis. Using the chain rule we can easily compute that

$$\frac{\partial(f\circ\boldsymbol{\phi}^{-1})}{\partial x_i}(\boldsymbol{\phi}(p)) = \sum_{j=1}^{n}\frac{\partial(f\circ\tilde{\boldsymbol{\phi}}^{-1})}{\partial y_j}(\tilde{\boldsymbol{\phi}}(p))\frac{\partial(\tilde{\boldsymbol{\phi}}\circ\boldsymbol{\phi}^{-1})_j}{\partial x_i}(\boldsymbol{\phi}(p))$$

and we recognize that the change of basis matrix is the Jacobian of the chart transition $D(\boldsymbol{\psi}\circ\boldsymbol{\phi}^{-1})(\boldsymbol{\phi}(p))$.

A *vector field* X maps every point p to a tangent vector in the corresponding tangent space i.e. by using local coordinates

$$X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i}$$

with $X_i(p)\in\mathbb{R}$. The regularity of a vector field is defined via the regularity of its coefficent e.g. a vector field is differentiable if all its coefficents are. We must be careful though since we require higher regularity of the manifold for this to be well-defined. Assume X is differentiable i.e. X_i are differentiable. Let $(\tilde{U},\tilde{\boldsymbol{\phi}})$ be another local chart. From the change of basis of the tangent space we know that

$$X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i} = \sum_{i,j=1}^{n} X_i(p)\frac{\partial(\tilde{\boldsymbol{\phi}}\circ\boldsymbol{\phi})_j}{\partial x_i}(\boldsymbol{\phi}(p))\frac{\partial}{\partial y_j}$$

and we see the coefficents w.r.t. to the new basis are $\tilde{X}_j = \sum_{i=1}^n X_i(p) \frac{\partial (\tilde{\boldsymbol{\phi}} \circ \boldsymbol{\phi})_j}{\partial x_i} (\boldsymbol{\phi}(p))$. So we need $\frac{\partial (\tilde{\boldsymbol{\phi}} \circ \boldsymbol{\phi})_j}{\partial x_i}$ to be differentiable as well which means that we want the manifold to be at least C^2.

We should briefly mention the question of orientation of the tangent spaces. Recall that we partitioned the bases of a finite-dimensional real vector in two orientations. We say that two bases have the same orientation if the change of basis matrix has positive determinant. We want to choose an orientation on the tangent spaces consistently which will be crucial when defining the volume form below.

Assume the manifold M is orientable and we have chosen an oriented atlas. For any tangent space T_p with local coordinates x_i near p, we define the resulting basis $\frac{\partial}{\partial x_i}$ as positively oriented which fixes the orientation of the vector space. We have to show that this is well-defined. But this is clear since if we take a different chart at p from the oriented atlas resulting in different local coordinates $\frac{\partial}{\partial y_j}$ we know that the change of basis is the Jacobian of the chart transition. But the Jacobian of the chart transition has positive determinant by definition and so the basis $\frac{\partial}{\partial y_j}$ is positively oriented as well.

2.3 Differential forms

{sec:differentia

Now that we introduced the necessary objects from differential geometry, we can finally define differential forms on manifolds.

Definition 2.3.1 (Differential forms)**.** A differential k-form ω maps any point $p \in M$ to a alternating k-linear form $\omega_p \in \operatorname{Alt}^k T_p M$. We denote the space of differential k-forms on M as $\Lambda^k M$.

Let $T_p^* M$ be the dual space of $T_p M$ which is usually called *cotangent space*. As before let us choose a local chart $\boldsymbol{\phi} : U \to \mathbb{R}^n$ with $p \in U$ and define $\frac{\partial}{\partial x_i}|_p$ as before. Denote the corresponding dual basis as dx^i, $i = 1, ..., n$ i.e. $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. From the consideration about alternating maps from section 2.1 we can now write any $\omega \in \Lambda^k M$ with

$$\omega_p = \sum_{1 \leq i_1 < ... < i_k \leq n} a_{i_1,...,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

with $a_{i_1,...,i_k}(p) \in \mathbb{R}$. The regularity of differential forms is then defined via the regularity of these coefficents i.e. we call a differential form smooth if all the $a_{i_1,...,i_k}$ are smooth and we call a differential form differentiable if all the $a_{i_1,...,i_k}$ are differentiable and so on. These definitions of regularity again require the manifold to be sufficiently regular as well.

We denote the space $C^\infty\Lambda^k M$ the space of smooth differential k-forms and analogous for other regularity. $C_c^\infty\Lambda^k(M)$ are the smooth differential forms with compact support where the closure is w.r.t. the topology on M. Note that for a manifold with boundary $\omega \in C_c^\infty\Lambda^k(M)$ are not necessarily zero on the boundary.

In order to define the Hodge star and an inner product on differential forms we need that T_pM is an inner product space. A Riemannian metric gives us at every point $p \in M$ a symmetric, positive definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$.

Definition 2.3.2 (Riemannian metric)**.** A Riemannian metric g maps every point $p \in M$ to a symmetric, positive definite bilinear form $g_p : T_p \times T_p \to \mathbb{R}$. After choosing local coordinates $\{x_i\}_{i=1}^n$ we frequently denote $g_{p,ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. We also require that for C^α vector fields X and Y the map $g(X,Y)$ is also C^α . A manifold with a Riemannian metric is called *Riemannian manifold.*

From now on, we will leave out the reference to the point p where appropriate. The Riemannian metric provides us with the inner product on every tangent space T_pM.

As explained in the section about alternating multilinear forms, T_p^*M is also a inner product space where the inner product is defined via the Riesz isomorphism we denote this inner product as $\langle\cdot,\cdot\rangle_{T_p^*M}$. Just as defined in (2.1.3) we also obtain an inner product on $\mathrm{Alt}^k T_pM$.

We will from now on assume that M is an oriented Riemannian manifold of sufficient regularity and denote the Riemannian metric by g. Let $p \in M$ and T_pM be the tangent space at the point p. Due to our assumptions on M, this is an oriented inner product space of dimension n and we can apply all of the constructions from the previous chapter. We will define the volume form, vector and scalar proxies and finally pullbacks of differential forms.

Let us fix a point p and a chart $\boldsymbol{\phi}$ at this point with local coordinates denoted by x_i, $i = 1, ..., n$. The resulting gramian matrix is $(G_p)_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. So we have a volume form vol on M

$$\mathrm{vol}_p = \sqrt{\det G_p} dx^1 \wedge ... \wedge dx^n.$$

The volume form then depends on the chosen orientation, but not on the chosen local coordinates.

Now that we have a volume form we can define the Hodge star operator $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$ simply by applying it pointwise i.e. for a differential form $\omega \in \Lambda^k M$, $(\star\omega)_p = \star\omega_p$. We follow the analogous idea to define the wedge product $\wedge : \Lambda^k M \times \Lambda^l M \to \Lambda^{k+l} M$ via

$$(\omega \wedge \nu)_p = \omega_p \wedge \nu_p.$$

Recall from Prop. 2.1.8 that we then have for $\omega, \nu \in \Lambda^k M$ and $\mu \in \Lambda^{n-k} M$

$$\begin{aligned} \omega_p \wedge \mu_p &= \langle \star\omega_p, \mu_p \rangle_{\mathrm{Alt}^{n-k} T_pM} \operatorname{vol}_p \\ \star\star\,\omega &= (-1)^{k(n-k)}\omega \\ \omega_p \wedge \star\nu_p &= \langle \omega_p, \nu_p \rangle_{\mathrm{Alt}^k T_pM} \operatorname{vol}. \end{aligned}$$

In order for the Hodge star to be well-defined the assumption of an orientation on our manifold is crucial.

We want to apply two important concepts from the previous section about alternating maps – vector proxies and pullbacks – to differential forms. The following is based on [2, Ch. 6], but many details have been added and additional examples are given. Recall, that for a real n-dimensional vector space V we had two ways to identify a vector $v \in V$ with an alternating map. Either as a linear form Φv where Φ is the Riesz isomorphism or as a $(n-1)$-linear alternating map $\star\Phi v$.

We can now identify every vector field with a 1-form or a $(n-1)$-form. $p \mapsto \Phi_{T_pM} X(p)$ defines a 1-form and $p \mapsto \star\Phi_{T_pM} X(p)$ gives us a $(n-1)$-form. In differential geometry, the usual notation is $\Phi_{T_pM} X(p) = X^\flat(p)$. The inverse of $^\flat$ is the $^\sharp$ operator. The isomorphisms $^\flat$ and $^\sharp$ are fittingly called *musical isomorphisms*. With these musical isomorphisms we can identify X with the 1-form $X^\flat$ or the $(n-1)$-form $\star X^\flat$. Vice versa, we find for $\omega \in \Lambda^1 M$ the *vector proxy* $\omega^\sharp$ and for an $(n-1)$-form $\nu \in \Lambda^{n-1} M$ we get $(\star^{-1}\nu)^\sharp$.

Recall that the matrix representation of the Riesz isomorphism is the Gramian matrix $G = (g_{ij})_{1 \le i,j \le n}$. So if we have a vector field $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and define $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$ then we can compute the associated 1-form

$$X^\flat = \sum_{j=1}^{n} (G\mathbf{X})_j dy^j = \sum_{i,j=1}^{n} g_{ij} X_i dy^j.$$

Denote with dx^i, $i = 1, ..., n$ the dual basis of $\frac{\partial}{\partial x_i}$. If the basis of the tangent space $\frac{\partial}{\partial x_i}$ are orthonormal then we have $(\frac{\partial}{\partial x_i})^\flat = dx^i$ because the Riesz isomorphism maps basis elements to their dual basis elements in this case.

Next, let us have a look at how we can extend pullbacks to differential forms. Recall again that for a linear map $L : V \to W$ with an n-dimensional real vector space V and an m-dimensional real vector space W we define its pullback $L^* : \mathrm{Alt}^k W \to \mathrm{Alt}^k V$ via

$$L^*\omega(v_1, ..., v_k) = \omega(Lv_1, ..., Lv_k).$$

We wish to do the analogous thing with differential forms.

Definition 2.3.3. For $\omega \in \Lambda^k N$ and a differentiable map $F : M \to N$ we define the pullback $F^*\omega$ as

$$(F^*\omega)_p = (F_{*,p})^* \omega_{F(p)}$$

or written differently for all $v_1, ..., v_k \in T_pM$

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(F_{*,p}v_1, ..., F_{*,p}v_k).$$

As an immediate consequence of this definition, we inherit from the corresponding fact about alternating forms the fact that the pullback commutes with the wedge product i.e. for $\omega \in \Lambda^k M$, $\nu \in \Lambda^l M$

$$F^*(\omega \wedge \nu) = F^*\omega \wedge F^*\nu$$

Now we can use the vector proxies and connect it to what we have done for alternating maps above.

{prop:pullback_v

Proposition 2.3.4. *So let X be a vector field on N. Take $p \in M$ and local coordinates $\{x_i\}_{i=1}^n$ at p and $\{y_j\}_{j=1}^m$ at $F(p)$. We assume that the bases $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ and $\{\frac{\partial}{\partial y_j}\}_{j=1}^m$ of T_pM and $T_{F(p)}N$ are orthonormal w.r.t. the Riemannian metrics on M and N respectively. Then we can write $X = \sum_{j=1}^m X_i \frac{\partial}{\partial x_i}$ and define $\mathbf{X} = (X_1, ..., X_n)^\top$. Then we obtain for the pullback in terms of vector proxies of* 1*-forms*

$$(F^*X^\flat)^\sharp(p) = \sum_{j=1}^n (DF(p)^\top \mathbf{X}(F(p)))_j \frac{\partial}{\partial y_j}.$$

If we assume $m = n$ additionally, then

$$(\star^{-1} F^* \star X^\flat)^\sharp = \sum_{j=1}^n (ad(DF(p))\mathbf{X})_j \frac{\partial}{\partial y_j}$$

where $ad(DF(p))$ is the adjugate matrix of $DF(p)$.

Proof. The resulting follows essentially immediately from applying the corresponding results for alternating maps. Using the definition of the $^\flat$ and $^\sharp$

$$(F^*X^\flat)^\sharp(p) = \Phi_{T_pM}^{-1} F_{*,p}^* \Phi_{T_{F(p)}N} X(p) = \sum_{j=1}^n (DF(p)^\top \mathbf{X}(F(p)))_j \frac{\partial}{\partial y_j}$$

where we used in the last step the matrix representation of the pullback of vector proxies of of 1-linear forms from (2.1.11). The analogous reasoning works for vector proxies of $(n-1)$-forms i.e. the second claim. □

Let $M = \hat{\Omega} \subseteq \mathbb{R}^n$ and $N = \Omega \subseteq \mathbb{R}^m$ and assume F is a diffeomorphism. Then we can choose the identity as a chart and obtain the bases of the tangent spaces $\frac{\partial}{\partial \hat{x}_i}$ Then we can identify a vector field $\sum_{i=1}^{m} X_i \frac{\partial}{\partial x_i}$ with $\mathbf{X} = (X_1, ..., X_n)^\top$. We then can interpret $\mathbf{X}$ either as a vector proxy of a 1- or $(n-1)$-form and obtain the pullbacks

$$\begin{aligned} \mathcal{P}_F^1 \mathbf{X}(\hat{x}) &:= DF(\hat{x})^\top \mathbf{X}(F(\hat{x})) \text{ and} \\ \mathcal{P}_F^{n-1} \mathbf{X}(\hat{x}) &:= (\det DF(\hat{x}))\, DF(\hat{x})^{-1} \mathbf{X}(F(\hat{x})) \end{aligned} \tag{2.3.1}$$ {eq:piola_transf

where we used the fact that F is a diffeomorphism hence its Jacobian is invertible and then $\operatorname{ad}(DF(\hat{x})) = (\det DF(\hat{x}))\, DF(\hat{x})^{-1}$. (2.3.1) is widely known as the Piola transformation [5, Def. 9.8].

{rem:vector_prox

Remark 2.3.5. Notice that for $n = 2$, this gives us two ways to identify a vector field with a 1-form. For an vector field $X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2}$, $X^\flat$ and $\star X^\flat$ are both 1-forms. Depending on the choice, this results in one of the pullbacks of Prop. 2.3.4.

Now let us move on to scalar proxies. So let $\rho : \Omega \to \mathbb{R}$ be just a scalar field i.e. a 0-form. In this case, we have the simple expression for the pullback $F^*\rho = \rho \circ F$.

But ρ could also be the scalar proxy of the n-form $\star\rho = \rho \operatorname{vol}$ with $\operatorname{vol} = dx^1 \wedge dx^2 \wedge ... \wedge dx^n$ being the volume form on Ω. On $\hat{\Omega}$ we denote the volume form $\widehat{\operatorname{vol}} = d\hat{x}^1 \wedge d\hat{x}^2 \wedge ... \wedge d\hat{x}^n$. Then in scalar proxies we obtain the pullback

$$\begin{aligned} (\mathcal{P}_F^n \rho)(\hat{x}) &= (\star^{-1} F^* \star \rho)_{\hat{x}} = \star^{-1}(F^* \star \rho)_{\hat{x}} = \star^{-1}(F_{*,\hat{x}})^*(\star\rho)_{F(\hat{x})} \\ &= \star^{-1}(F_{*,\hat{x}})^* \rho(F(\hat{x})) dx^1 \wedge ... \wedge dx^n \\ &= \star^{-1} \rho(F(\hat{x})) \det DF(p)\, d\hat{x}^1 \wedge ... \wedge d\hat{x}^n \\ &= (\rho \circ F)(\det DF) \star^{-1} \widehat{\operatorname{vol}} = (\rho \circ F)(\hat{x}) \det DF(\hat{x}). \end{aligned}$$

This is strikingly similar to the integrand in the standard transformation of integrals formula which will become crucial in the next section where we talk about the integration of differential forms. The next part of this section will be concerned with introducing a derivative for differential forms

Definition 2.3.6 (Exterior derivative)**.** Let $\omega \in C^1\Lambda^k(M)$ be given in local coordinates as

$$\omega_p = \sum_{1 \le i_1 < ... < i_k \le n} a_{i_1,...,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

Then we define the exterior derivative $d : C^1\Lambda^k(M) \to C^0\Lambda^{k+1}(M)$. By

$$(d\omega)_p = \sum_{1\le i_1<\ldots<i_k\le n} \sum_{i=1}^{n} \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_i}(p) dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

We remind that the derivative of $a_{i_1,\ldots,i_k}$ is meant w.r.t. to local coordinates as defined at (2.2.1).

It can be shown that $d\omega$ is independent of the chosen coordinates. Let us mention some important properties of the exterior derivative.

We call a differential form $\omega \in \Lambda^k M$ *closed* if $d\omega = 0$ and *exact* if there exists $\nu \in \Lambda^{k-1}$ s.t. $d\nu = \omega$. We denote the closed k-forms as $\mathfrak{Z}^k$ and the exact k-forms as $\mathfrak{B}^k$. We have $d \circ d = 0$ and thus $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$.

The relation to the wedge product is described by a Leibniz-type formula. Let $\omega \in C^1\Lambda^k M$ and $\nu \in C^1\Lambda^l M$. Then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^k \omega \wedge d\nu. \tag{2.3.2}$$

{eq:leibniz_form

The exterior derivative commutes with pullback i.e. for manifolds M and N a differentiable mapping $F : M \to N$ and $\omega \in \Lambda^k N$ we have $dF^*\omega = F^*d\omega$. In terms of proxies this is related to very interesting results as we will see later.

Let us investigate the exterior derivative in the case when $M = \Omega \subseteq \mathbb{R}^n$ is an open subdomain. It turns out that by using scalar and vector proxies as introduced above we can identify the exterior derivative with well-known differential operators. We will use standard Euclidian coordinates meaning that our tangent basis $\frac{\partial}{\partial x_i}$ is orthonormal.

Let us start with a differentiable function $f : \Omega \to \mathbb{R}$ i.e. f is a 0-form. Then

$$(df)^\sharp = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \right)^\sharp = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

which we identify with grad f. In other words, the vector proxy of the exterior derivative of a 0-form corresponds to the gradient.

Let $\mathbf{X}$ be a differentiable vector field on Ω with components X_i. This corresponds to the vector field $X = \sum_i X_i \frac{\partial}{\partial x_i}$. Let us view X as the vector proxy of the $(n-1)$-form $\star X^\flat$. Then

$$\begin{aligned}\star^{-1} d \star X^\flat &= \star^{-1} d \star \sum_{i=1}^n X_i dx^i = \star^{-1} d \sum_{i=1}^n X_i (-1)^{i-1} dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ &= \star^{-1} \sum_{i=1}^n \frac{\partial X_i}{\partial x_i} (-1)^{i-1} dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ &= \star^{-1} \sum_{i=1}^n \frac{\partial X_i}{\partial x_i} (-1)^{2(i-1)} \operatorname{vol} = \sum_{i=1}^n \frac{\partial X_i}{\partial x_i} = \operatorname{div} \mathbf{X}.\end{aligned}$$

The hat symbol used for $\widehat{dx^i}$ means that this term is left out. So in the case of a $(n-1)$-form it corresponds to the divergence.

In the case $n = 3$ if we identify $\mathbf{X}$ with a 1-form then we obtain using similiar computations that $(\star dX^\flat)^\sharp$ corresponds to $\operatorname{curl} \mathbf{X}$. So in 3D we can identify all the exterior derivatives with known differential operators and thereby putting them into a more general framework.

A very nice conclusion can be seen directly from the above computations. The expressions on the left hand side does not use any coordinates. Hence, we can use any coordinate system we want and can then compute e.g. the divergence in any coordinates we need. Note however that the computations are more cumbersome when the bases are not orthonormal.

Let us give an application of this fact. This example is not taken from the references. We will derive the divergence for arbitrary coordinates. So let $\mathbf{X} : \Omega \to \mathbb{R}^n$ be a differentiable vector field. By identifying this with the vector field $\sum_i X_i \frac{\partial}{\partial x_i}$ we computed above that

$$\operatorname{div} \mathbf{X} \operatorname{vol} = d \star X^\flat.$$

Now let us express $\mathbf{X}$ using different coordinates. Let $\boldsymbol{\phi} : \Omega \to \hat{\Omega} \subseteq \mathbb{R}^n$ be a diffeomorphism. This defines our local coordinates. Then the expression of $\mathbf{X}$ is these curvilinear coordinates is

$$\mathbf{X} = \sum_{j=1}^n \tilde{X}_j \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_j}.$$

In terms of differential geometry, this corresponds to the representation of the vector field $X = \sum_j \tilde{X}_j \frac{\partial}{\partial \hat{x}_j}$. Let $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)^\top$ and

$$G_{kl} = g(\frac{\partial}{\partial \hat{x}_k}, \frac{\partial}{\partial \hat{x}_l}) = \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_k} \cdot \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_l}$$

Then we compute use the expression of the Hodge star operator for one forms (2.1.9) to compute

$$
\begin{aligned}
(\operatorname{div}\mathbf{X})\,\mathrm{vol} &= d\star X^\flat = d\star\sum_{j=1}^n (G\tilde{\mathbf{X}})_j d\hat{x}^j \\
&= d\sum_{j=1}^n\sum_{k=1}^n (G\tilde{\mathbf{X}})_j\sqrt{\det G}g^{jk}(-1)^{k-1}d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge\widehat{d\hat{x}^k}\wedge\ldots\wedge d\hat{x}^n \\
&= \sum_{k=1}^n \frac{\partial(\sqrt{\det G}\sum_{j=1}^n g^{jk}(G\tilde{\mathbf{X}})_j)}{\partial\hat{x}_k}(-1)^{k-1}d\hat{x}^k\wedge d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge\widehat{d\hat{x}^k}\wedge\ldots\wedge d\hat{x}^n \\
&= \sum_{k=1}^n \frac{\partial(\sqrt{\det G}\tilde{\mathbf{X}})_k}{\partial\hat{x}_k}(-1)^{2(k-1)}d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge d\hat{x}^k\wedge\ldots\wedge d\hat{x}^n \\
&= \left[\frac{1}{\sqrt{\det G}}\sum_{k=1}^n\frac{\partial(\sqrt{\det G}\tilde{\mathbf{X}})_k}{\partial\hat{x}_k}\right]\mathrm{vol}
\end{aligned}
$$

where we used that $(G^{-1})_{ij} = \langle dx^i, dx^j\rangle_{T_p^*M} = g^{ij}$ and we find the well-known expression for the divergence in general coordinates

$$\operatorname{div}\mathbf{X} = \frac{1}{\sqrt{\det G}}\sum_{k=1}^n\frac{\partial(\sqrt{\det G}\,\tilde{X}_k)}{\partial y_k}.$$

As mentioned above let us investigate the consequences of the commutativity of the exterior derivative and the pullback in terms of vector and scalar proxies. Let $F:\widehat{\Omega}\to\Omega$ be a diffeomorphism with $\widehat{\Omega},\Omega\in\mathbb{R}^n$ with volume forms $\widehat{\mathrm{vol}}$ and vol. Let us again consider Euclidian coordinates on the domain and codomain. Then observe

$$
\begin{aligned}
\mathcal{P}_F^n(\operatorname{div}\mathbf{X})(\hat{x})\widehat{\mathrm{vol}}_{\hat{x}} &= (\operatorname{div}\mathbf{X})(F(\hat{x}))\det DF(\hat{x})\widehat{\mathrm{vol}}_{\hat{x}} = (F^*(\operatorname{div}\mathbf{X}))_{\hat{x}}\wedge(F^*\,\mathrm{vol})_{\hat{x}} \\
&= \big(F^*\big(\operatorname{div}\mathbf{X}\wedge\mathrm{vol}\big)\big)_{\hat{x}} = \big(F^*\big(\operatorname{div}\mathbf{X}\,\mathrm{vol}\big)\big)_{\hat{x}} \\
&= \big(F^* d\star X^\flat\big)_{\hat{x}} = \big(dF^*\star X^\flat\big)_{\hat{x}}\,.
\end{aligned}
$$

where we used that F^* commputes with the pullback and the wedge product. Note that the wedge product is just the scalar multiplication if one of the factors is a 0-form. Take $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$. Then we know from Prop. 2.3.4 and (2.3.1) by defining $(\hat{X}_1, \hat{X}_2, ..., \hat{X}_n)^\top = \hat{\mathbf{X}} = \mathcal{P}_F^{n-1}\mathbf{X}$ and $\hat{X} = \sum_{i=1}^n \hat{X}_i\frac{\partial}{\partial\hat{x}_i}$

$$dF^*\star X^\flat = d\star\star^{-1}F^*\star X^\flat = d\star\hat{X}^\flat = \widehat{\operatorname{div}}\hat{\mathbf{X}}\,\widehat{\mathrm{vol}}$$

and we recognize

$$\mathcal{P}_F^n(\operatorname{div}\mathbf{X}) = \widehat{\operatorname{div}}\mathcal{P}_F^{n-1}\mathbf{X}.$$

To summarize the situation in 3D,

{prop:pullback_a

Proposition 2.3.7. *Let $F : \hat{\Omega} \to \Omega$ be a diffeomorphism, $\rho \in C^1(\Omega)$ and $\mathbf{X} \in C^1(\Omega, \mathbb{R}^3)$. In scalar and vector proxies for 3D, we obtain the following expressions for pullbacks*

$$\begin{aligned}
(\mathcal{P}_F^0 \rho)(\hat{x}) &= \rho(F(\hat{x})) \\
(\mathcal{P}_F^1 \mathbf{X})(\hat{x}) &= DF(\hat{x})^\top \mathbf{X}(F(\hat{x})) \\
(\mathcal{P}_F^2 \mathbf{X})(\hat{x}) &= \det DF(\hat{x}) DF(\hat{x})^{-1} \mathbf{X}(F(\hat{x})) \\
(\mathcal{P}_F^3 \rho)(\hat{x}) &= \det DF(\hat{x}) \rho(F(\hat{x}))
\end{aligned}$$

and then the commuting properties

$$\begin{aligned}
\widehat{\mathrm{grad}} \mathcal{P}_F^0 \rho &= \mathcal{P}_F^1 (\mathrm{grad}\, \rho) \\
\widehat{\mathrm{curl}} \mathcal{P}_F^1 \mathbf{X} &= \mathcal{P}_F^2 (\mathrm{curl}\, \mathbf{X}) \\
\widehat{\mathrm{div}} \mathcal{P}_F^2 \mathbf{X} &= \mathcal{P}_F^3 (\mathrm{div}\, \mathbf{X}).
\end{aligned}$$

We proved the last statement and the other two can be proven analogously. These commuting properties are useful for applications in finite elements (see e.g. [5, Sec. 14.3]).

2.4 Integration of differential forms

{sec:integration

Differential k-forms can be integrated over k-dimensional manifolds and the properties of this operation, in particular Stokes' theorem, are one main motivation for working with differential forms. In many books, integration is only defined for smooth and compactly supported forms (see e.g. [3, Sec. V.3]). This approach is easier, but slightly unsatisfying since then we can not put the integration of forms into the framework of standard Lebesgue integration theory and it is not possible to define L^p and Sobolev spaces of differential forms properly. Thus, even though we will actually only integrate smooth differential forms over compact manifolds in the proofs of existence and uniqueness, we want to introduce the integration more generally. For details about the more general theory of functional spaces on manifolds, see [8, Sec. 10.2.4].

Throughout this chapter we assume that M is a smooth orientable Riemannian manifold of dimension n. We assume that we have chosen a countable oriented atlas $\{(U_i, \boldsymbol{\phi}_i)\}_{i=1}^N$ with $N \leq \infty$. Before we can talk about integration of differential forms let us first investigate the integration of functions on a manifold.

2.4.1 Integration of functions on a manifold

We want to define integration in the framework of usual measure and integration theory which means defining it as a Lebesgue integral w.r.t. a measure on M which we have to define first along with a σ-algebra on M.

It is well known that the borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^n$ is generated by all open sets. This idea can be applied to any topological space X by defining the borel σ-algebra $\mathcal{B}(X)$ as the σ-algebra generated by all open sets as well. So we can simply use the topology on M to define our σ-algebra $\mathcal{B}(M)$. Because we know that all our charts $\boldsymbol{\phi}_i : U_i \to \mathbb{R}^n$ are homeomorphisms it is very straightforward to show that a set $E \in \mathcal{B}(M)$ i.i.f. $\boldsymbol{\phi}_i(E \cap U_i) \subseteq \mathbb{R}^n$ is Borel-measurable.

Now, we need to define a measure on the manifold. The following motivation is taken from [6, 3.H.2]. In Euclidian space $\mathbb{R}^n$ we use the standard Lebesgue measure that gives volume one to the unit cube. If we now take any vectors $v_1, v_2, ..., v_n \in \mathbb{R}^n$ then the parallelepiped spanned by these vectors has volume $\det(v_1 \mid v_2 \mid \cdots \mid v_n) = \sqrt{\det\left(\langle v_i, v_j\rangle\right)_{1\leq i,j\leq n}}$ where we used the standard inner product on $\mathbb{R}^n$.

If we now extend this idea to Riemannian manifolds then we are interested in the parallelepiped spanned by the tangent vectors $\frac{\partial}{\partial x_i}$ in the tangent space T_pM with $p \in M$ which would then have volume $\sqrt{\det\left(g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)_{1\leq i,j\leq n}} = \sqrt{\det G_p}$. Then we define for a Borel-measurable $E \subseteq U$ where U is the domain a chart $\boldsymbol{\phi} : U \to \mathbb{R}^n$ the measure

$$V(E) := \int_{\boldsymbol{\phi}(E)} \sqrt{\det G_{\boldsymbol{\phi}^{-1}(x)}}dx$$

which we will now extend globally to the whole manifold. We do this by "gluing" the local measures together using a partition of unity subordinate to our open cover. Let $\{\chi_i\}_{i=1}^N$, $N \leq \infty$, be the partition of unity subordinate to the open cover given by the atlas $\{U_i\}_{i=0}^N$. Then we define the *Riemannian measure* for any $E \in \mathcal{B}(M)$

$$V(E) := \sum_{i=1}^{\infty} \int_{\boldsymbol{\phi}_i(U_i\cap E)} \chi_i(\boldsymbol{\phi}_i^{-1}(x))\sqrt{\det G^{(i)}(\boldsymbol{\phi}_i^{-1}(x))}dx \in [0, \infty] \qquad (2.4.1)$$

{eq:riemannian_m

It can be shown that it is independent of the chosen oriented atlas using the transformation behaviour of $G^{(i)}$. But the orientation is crucial for it to be well-defined.

Proposition 2.4.1. *The Riemannian measure is independent of the chosen partition of unity and atlas if it is oriented the same.*

Proof. Let $\{(V_j, \boldsymbol{\psi}_j)\}_{j=0}^{\infty}$ be a different atlas with the same orientation and $\{\rho_j\}_{j=0}^{\infty}$ be a partition of unity subordinate to it. If we now define the Riemannian measure (2.4.1) using this atlas, then

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{\boldsymbol{\psi}_j(E\cap V_j)} \rho_j(\boldsymbol{\psi}_j^{-1}(y)) \sqrt{\det G^{(j)}_{\boldsymbol{\psi}^{-1}(y)}}\, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\boldsymbol{\psi}_j(E\cap V_j \cap U_i)} \chi_i(\boldsymbol{\psi}_j^{-1}(y))\rho_j(\boldsymbol{\psi}_j^{-1}(y)) \sqrt{\det G^{(j)}_{\boldsymbol{\psi}_j^{-1}(y)}}\, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\boldsymbol{\phi}_i(E\cap V_j \cap U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(x))\rho_j(\boldsymbol{\phi}_i^{-1}(x)) \sqrt{\det G^{(j)}_{\boldsymbol{\phi}_i^{-1}(x)}}\, \det D(\boldsymbol{\psi}_j \circ \boldsymbol{\phi}_i^{-1})(\boldsymbol{\phi}_i(x))\, dx
\end{aligned} \tag{2.4.2}$$

{eq:double_sum_r

where we used the fact that the χ_i sum up to one for the first equality and a simple tranformation of integral in the second equality using the chart transition $\boldsymbol{\phi}_i^{-1} \circ \boldsymbol{\psi}_j$. Since all summands are non-negative we can change the order of summation as we like. Then using the change of basis for the tangent space derived in Sec. 2.2 we get for $p \in U_i \cap V_j$

$$\begin{aligned}
(G^{(j)}_p)_{kl} &= g_p\Big(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\Big) \\
&= \sum_{r,s=0}^{n} \frac{\partial(\boldsymbol{\phi}_i \circ \boldsymbol{\psi}_j^{-1})_r}{\partial y_k}(\boldsymbol{\psi}(p))\, \frac{\partial(\boldsymbol{\phi}_i \circ \boldsymbol{\psi}_j^{-1})_s}{\partial y_l}(\boldsymbol{\psi}(p))\, g_p\Big(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_s}\Big) \\
&= \Big(D(\boldsymbol{\phi}_i \circ \boldsymbol{\psi}_j^{-1})(\boldsymbol{\psi}(p))^\top G^{(i)}_p D(\boldsymbol{\phi}_i \circ \boldsymbol{\psi}_j^{-1})(\boldsymbol{\psi}(p))\Big)_{kl}
\end{aligned}$$

and thus

$$\begin{aligned}
\det G^{(j)}_{\boldsymbol{\phi}^{-1}(x)} &= \det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)} \Big(\det D\big(\boldsymbol{\phi}_i \circ \boldsymbol{\psi}_j^{-1}\big)\big(\boldsymbol{\psi}(\boldsymbol{\phi}^{-1}(x))\big)\Big)^2 \\
&= \det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)} \left(\det D(\boldsymbol{\psi}_j \circ \boldsymbol{\phi}_i^{-1})(x)^{-1}\right)^2 .
\end{aligned}$$

Plugging this into (2.4.2) the determinant of the chart transition cancels out because $\det D(\boldsymbol{\psi}_j \circ \boldsymbol{\phi}_i^{-1})(x)^{-1} > 0$. We can integrate over $\boldsymbol{\phi}_i(E \cap U_i)$ instead of $\boldsymbol{\phi}_i(E \cap V_j \cap U_i)$ without changing the integral because $\operatorname{supp} \rho_j \subseteq V_j$.

$$\begin{aligned}
&\sum_{i=0}^{\infty} \int_{\boldsymbol{\phi}_i(E\cap U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(x)) \sum_{j=0}^{\infty} \rho_j(\boldsymbol{\phi}_i^{-1}(x)) \sqrt{\det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)}}\, dx \\
&= \sum_{i=0}^{\infty} \int_{\boldsymbol{\phi}_i(E\cap U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(x)) \sqrt{\det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)}}\, dx \\
&= V(E).
\end{aligned}$$

□

Now that we have the measure space $(M, \mathcal{B}(M), V)$ we can define integration in the usual Lebesgue way. It is easily shown that a function $f : M \to \mathbb{R}$ is measurable i.i.f. $f \circ \boldsymbol{\phi}_i^{-1}$ is measurable for every chart $\boldsymbol{\phi}_i$. It is then simply an application of the definition of Lebesgue integration to show that for any measurable $f \geq 0$ we can express the integration as

$$\int_M f\, dV = \sum_{i=1}^{\infty} \int_{\boldsymbol{\phi}_i(U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(x)) f(\boldsymbol{\phi}_i^{-1}(x)) \sqrt{\det G^{(i)}(\boldsymbol{\phi}_i^{-1}(x))} dx.$$

By introducing the integral as a Lebesgue integral w.r.t. the Riemannian measure we inherit the theoretical framework of Lebesgue integration. For example, we know that the the spaces $L^p(M, V)$ for $1 \leq p < \infty$, i.e. the p-integrable real-valued functions w.r.t. the Riemannian measure, are Banach spaces.

2.4.2 Integration of differential forms

{sec:integration

Once again, we should first ask ourselfs what a measurable differential form should be. We know that we can express our differential form locally for $p \in U_i$ using the local coordinates as

$$\omega_p = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1,\ldots,i_k}(p) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

In the spirit of the definition of differentiability in Sec. 2.2.2 we call a $\omega \in \Lambda^k(M)$ measurable if for every chart $\boldsymbol{\phi}_i$ the coefficent functions of the differential form expressed in local coordinates are measurable. This notion is once again independent of the chosen coordinates.

Next, we will define integration of an n-form over an n dimensional manifold. At first, we do so for an open set $U \subseteq \mathbb{R}^n$. This is the simplest example of an n-dimensional manifold where we only have one chart which is the identity and the local coordinates are just our standard coordinates which we denote by z_i, $i = 1, \ldots, n$ and the resulting basis of the tangent space $\frac{\partial}{\partial z_i}$. Let ω be a measurable n-form on U so we can write

$$\omega_z = f(z)\, dz^1 \wedge dz^2 \wedge \ldots \wedge dz^n$$

for $z \in U$ with $f : U \to \mathbb{R}$ being Borel-measurable. We can now simply define

$$\int_U \omega = \int_U f(z)\, dz^1 dz^2 \ldots dz^n.$$

With this definition at hand we can now extend this definition to the manifold M. As it is often done in differential geometry we will work locally first and then extend this construction globally by using a partition of unity.

Let $(U, \boldsymbol{\phi})$ be a chart on M and assume $\operatorname{supp} \omega \subseteq U$. Then $(\boldsymbol{\phi}^{-1})^* \omega$ is a n-form on $\boldsymbol{\phi}(U) \subseteq \mathbb{R}^n$ and we can apply our prior definition. So now we just define

$$\int_M \omega := \int_{\boldsymbol{\phi}(U)} (\boldsymbol{\phi}^{-1})^* \omega.$$

Now let us move on to the global definition. Let $\{(U_i, \boldsymbol{\phi}_i)\}_{i=1}^{\infty}$ be an oriented atlas and let $\{\chi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to it. Then $\operatorname{supp} \chi_i \omega \subseteq U_i$ and we define

$$\int_M \omega := \sum_{i=1}^{\infty} \int_M \chi_i \omega.$$

The well-definedness will be proven below.

We will now have a look how the integration of functions and of differential forms are related to each other. For the chart $(\boldsymbol{\phi}_i, U_i)$ we denote the local coordinates as $x_k^{(i)}$, $k = 1, ..., n$ and the basis of 1-forms as $dx_{(i)}^k$. We know that for $p \in U_i$ we can write the volume form as

$$\mathrm{vol}_p = \sqrt{\det G^{(i)}(p)}\, dx_{(i)}^1 \wedge dx_{(i)}^2 \wedge ... \wedge dx_{(i)}^n.$$

where $G_{kl}^{(i)} = g(\frac{\partial}{\partial x_k^{(i)}}, \frac{\partial}{\partial x_l^{(i)}})$.

Because $(\boldsymbol{\phi}_i^{-1})^* dx_{(i)}^k = dz^k$ – which follows directly from the definition – and the pullback commutes with the wedge product we have

$$\big((\boldsymbol{\phi}_i^{-1})^* \,\mathrm{vol}\big)_z = \sqrt{\det G^{(i)}(\boldsymbol{\phi}_i^{-1}(z))}\, dz^1 \wedge dz^2 \wedge ... \wedge dz^n.$$

That means for a n-form $f \,\mathrm{vol}$ we can write the integral as

$$\int_M f \,\mathrm{vol} = \sum_{i=1}^{\infty} \int_{\boldsymbol{\phi}_i(U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(z)) f(\boldsymbol{\phi}_i^{-1}(z)) \sqrt{\det G^{(i)}(\boldsymbol{\phi}_i^{-1}(z))} dz^1 dz^2 ... dz^n$$

and we see

$$\int_M f dV = \int_M f \,\mathrm{vol}$$

with the two different notions of integration. This also proves that the integral of differential n-forms is independent of the chosen coordinates as long as the orientation is respected. So we see that the the two definitions are essentially equivalent. The big advantage of considering these two approaches is that we know the integration of differential forms is within the framework of Lebesgue integration. It is then also clear how integrability for n-forms should be defined. We call an n-form $f\,\mathrm{vol}$ integrable if f is integrable w.r.t. the Riemannian measure.

2.4.3 Stokes' theorem and integration by parts

One of the most important results about the integration of differential forms is Stokes' theorem which we will state in this section. From it, we will obtain a integration by parts formula.

But before we do so we have to check how to define the restriction of a differential form to a submanifold $N \subseteq M$. A submanifold N is just a manifold contained in M. As a further restriction, we always assume that we use the subspace topology on a submanifold i.e. a set $U \subseteq N$ is open i.i.f. there exists an open $U' \subseteq M$ s.t. $U = U' \cap N$.

We now define the restriction with the inclusion $\iota : N \hookrightarrow M$. Then for a smooth differential form $\omega \in C^\infty\Lambda^k(M)$ we define the restriction just via the pullback of the inclusion operator i.e. $\iota^*\omega \in C^\infty\Lambda^k(N)$. For a k-dimensional submanifold N we then denote the integration over N of a k-form ω as

$$\int_N \iota^*\omega =: \int_N \omega.$$

{ex:integration_

Example 2.4.2. Let us investigate the integral of a 1-form over a curve. Let $\gamma : (0,1) \to \Omega$ be a smooth curve in the domain $\Omega \subseteq \mathbb{R}^n$. Denote $\Gamma = \gamma((0,1))$ and assume that $\gamma' \neq 0$ and $\gamma : (0,1) \to \Gamma$ is bijective. That means especially that Γ does not intersect itself. Then Γ is a manifold and the chart is γ^{-1}. Then let us calculate the Jacobian of the inclusion ι using the definition from (2.2.1). Since it is the most convenient, we take the identity as chart on Ω. Take $\gamma(t) \in \Gamma$. Then

$$D\iota(\gamma(t))_{i,1} = D(\mathrm{Id} \circ \iota \circ \gamma)_{i,1}(t) = \frac{\partial \gamma_i}{\partial t}(t)$$

which means

$$D\iota(\gamma(t)) = \gamma'(t)^\top$$

This is then the matrix representation of the differential ι_* at $\gamma(t)$. Now take an at least continuous 1-form $\omega = \sum_{i=1}^n \omega_i dz^i$ on Ω. Then we know from the matrix representation of the pullback derived in Sec. 2.3 that

$$\iota^*\omega(t) = \sum_{i=1}^{n} \gamma_i'(t)\, \omega_i(\gamma(t))\, dt$$

and so

$$\int_\Gamma \omega = \int_0^1 \sum_{i=1}^{n} \gamma_i'(t)\, \omega_i(\gamma(t))\, dt.$$

Let us now look at this in terms of vector proxies. Take a vector field $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$ on Ω. Because we are using Euclidian coordinates we get

$$X^\flat = \sum_{i=1}^{n} X_i dz^i$$

and thus by denoting $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$ we get

$$\int_\Gamma X^\flat = \int_\Gamma \mathbf{X} \cdot dl$$

where the right hand side is the notation for the standard curve integral in $\mathbb{R}^n$. So we can conclude that, interpreted in vector proxies for a domain of $\mathbb{R}^n$, the integral of a 1-form corresponds to the usual curve integral.

Theorem 2.4.3 (Stokes)**.** *Let M be a smooth oriented manifold with boundary ∂M. Let ω be a smooth compactly supported $(n-1)$-form. Then we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Remember that compactly supported is meant in the topology on a manifold with boundary which means that ω can be non-zero on ∂M. This theorem gives us a relation of the boundary and the exterior derivative which will be crucial in the topological context of differential forms which we will investigate in Sec. 3.3.

We can also derive a form of the integration by parts formula from it. Let $\omega \in C_c^\infty\Lambda^k(M)$ and $\mu \in C_c^\infty\Lambda^{n-k-1}(M)$. Then $\omega \wedge \mu \in C_c^\infty\Lambda^{n-1}(M)$. Recall the Leibniz rule for the exterior derivative

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu.$$

By integrating both sides over M and applying Stokes' theorem we obtain

$$\int_{\partial M} \omega \wedge \mu = \int_M d\omega \wedge \mu + (-1)^k \int_M \omega \wedge d\mu.$$

By using vector and scalar proxies as in Sec. 2.3 this can be used along with the expression of the restriction of a $(n-1)$-form to prove the well-known formula

$$\int_\Omega u \operatorname{div} \mathbf{F} \, dx = -\int_\Omega \operatorname{grad} u \cdot \mathbf{F} \, dx + \int_{\partial\Omega} u \, \mathbf{F} \cdot \mathbf{n} \, ds$$

where $\mathbf{n}$ is the unit normal, assuming $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain and $\mathbf{F}$ and u are continuously differentiable. In 3D, using the restriction of a 1-form, one can show

$$\int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{F} \, dx = \int_\Omega \operatorname{curl} \mathbf{v} \cdot \ \mathbf{F} \, dx - \int_{\partial\Omega} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{F} \, ds$$

assuming $\mathbf{v}$ is a continuously differentiable vector field. But because these integration-by-parts formulas are well known we will not prove them here again.

3 Singular homology

{sec:singular_ho

The curve integral constraint from the magnetostatic problem is very topological in nature and strongly related to the topology of the domain. In order to deal with this constraint and obtain the desired existence and uniqueness we require some tools from algebraic topology which we will introduce in this section. This material is taken from [3] where a lot more details and results can be found.

3.1 Homology groups

Denote with $\mathbb{R}^\infty$ the vector space of all real-valued sequences. Let $e_i \in \mathbb{R}^\infty$ for $i \in \mathbb{N}$ denote the sequences that are zero for every index unequal to i and 1 for the index i. Note that the natural numbers start at zero in this thesis. Then we define the standard k-simplex Δ_k as

$$\Delta_k := \left\{ \sum_{i=0}^{k} \lambda_i e_i \mid \sum_{i=0}^{k} \lambda_i = 1, \ 0 \leq \lambda_i \leq 1 \right\} = \operatorname{conv}\{e_0, ..., e_k\}.$$

where conv is the usual convex combination.

Definition 3.1.1 (k-simplex)**.** Let X be a topological space. Then a *singular k-simplex* is a continuous map $\sigma_k : \Delta_k \to X$. We will frequently leave out the term 'singular' and refer to them just as k-simplices. We will very often also call the geometric object in our topological space i.e. $\sigma_k(\Delta_k)$ the singular k-simplex. It should be clear from the context whether we mean the function or the subset.

As the term 'singular' implies these simplices can be degenerated. For example, σ_k could just be constant, so the object in the topological space corresponding to the k-simplex is just a point.

We can now introduce a algebraic structure by looking at finite formal sums of the form

$$\sum_{\sigma\ k\text{-simplex}} n_\sigma \sigma.$$

These formal sums form an abelian group which we refer to as the *singular k-chain group $C_k(X)$*.

We will now introduce an important homomorphism between these groups called the *boundary.*

Definition 3.1.2. Let $\mathbf{v}_0, ..., \mathbf{v}_k \in \mathbb{R}^n$. We define *affine singular k-simplex* as a special singular k-simplex denoted by

$$[\mathbf{v}_0, ..., \mathbf{v}_k] : \Delta_k \to \mathbb{R}^n, \sum_{i=0}^{k} \lambda_i e_i \mapsto \sum_{i=0}^{k} \lambda_i \mathbf{v}_i.$$

As in the general case, the image can be a degenerated simplex in $\mathbb{R}^n$ since the $\mathbf{v}_i$ are not assumed to be affine independent.

We call the affine singular simplex

$$[e_0, ..., \hat{e}_i, ..., e_k] : \Delta_{k-1} \to \Delta_k \tag{3.1.1}$$ {eq:face_map}

the *i-th face map* which we denote by F_i^k and sometimes the ^{k} will be left out. The ˆ means this vertex is left out. Here we tacitly used the natural inclusion $\mathbb{R}^{k+1} \subseteq \mathbb{R}^\infty$ so we have $\Delta_k \subseteq \mathbb{R}^{k+1}$. But this is just a way of representation.

With the face map we can now define the boundary operator.

Definition 3.1.3 (Boundary)**.** For a singular k-simplex $\sigma : \Delta_k \to X$ we define its i-th face $\sigma^{(i)} := \sigma \circ F_i^k$ which is a $(k-1)$-simplex. We then define the *boundary* of σ as $\partial_k \sigma := \sum_{i=0}^{k} (-1)^i \sigma^{(i)}$. We extend this to a homomorphism between the chain groups

$$\partial_k : C_k(X) \to C_{k-1}(X), \sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma \partial_k \sigma.$$

In the case of $k = 0$, we set $\partial_0 = 0$.

We will frequently leave out the subscript and just write ∂ for the boundary if it is clear from the context. A straightforward computation (cf. [3, Lemma 1.6]) shows the important property

$$\partial_k \circ \partial_{k+1} = 0.$$

This property implies that $\ker \partial_{k-1} \subseteq \operatorname{im} \partial_k$ is a subgroup. We call a chain $c \in C_k(X)$ *k-cycle* or *closed* if $\partial_k c = 0$ and we call it *k-boundary* or *exact* if $c \in \operatorname{im} \partial_{k+1}$. Denote the group of k-cycles as $Z_k(X)$ and the k-boundaries as $B_k(X)$. Since we are in the abelian setting this motivates us to define the resulting factor groups.

Definition 3.1.4 (Homology groups)**.** We define the *k-th homology group* of the topological space X as the factor group

$$H_k(X) := {}^{Z_k(X)}\!/\!_{B_k(X)}.$$

We denote the elements of the homology groups i.e. the equivalence classes of a k-cycle c as $[c] \in H_k(X)$.

If the k-th homology groups is finitely generated then we call the rank i.e. the number of generators the *k-th Betti number*. These Betti numbers are fundamental properties of the topological space. For example, the zeroth Betti number corresponds to the number of path-components of the space. In 3 dimensions, the first Betti number of a compact domain corresponds to the number of "holes", the second Betti number to number of enclosed "voids" in the domain [2, p.14]. E.g. a filled torus has the zeroth Betti number one, the first Betti number also equal to one and the second equal to zero which can be proven using the Meyer-Vietoris sequence (see [3, Sec. IV.18]). We will not go into this in further since we do not want to focus too much on algebraic topology.

This construction can be put in an abstract algebraic framework in the following way. We call a collection of abelian groups C_i, $i \in \mathbb{Z}$ a graded group. Together with a collection of homomorphims $\partial_i : C_i \to C_{i-1}$ called *differentials* s.t. $\partial_{i-1} \circ \partial_i = 0$ this is called a *chain complex* which we will denote by C_*.

Example 3.1.5. If we set $C_k(X) = 0$ for $k < 0$ then the groups of k-chains with the boundary operator form a chain complex .

Completely analogous to above, we can define the homology groups of a abstract chain complex

$$H_k(C_*) := {}^{\ker \partial_k}\!/\!_{\operatorname{im} \partial_{k+1}}.$$

Definition 3.1.6 (Chain map)**.** Let A_* and B_* be chain complexes. With a slight abuse of notation let us denote the differentials of both chain complexes just by ∂. Then a *chain map* $f : A^* \to B^*$ is a collection of homomorphisms $f_i : A_i \to B_i$ s.t. $\partial \circ f_{i-1} = f_i \circ \partial$ i.e. the following diagram is commutative

$$\begin{tikzcd} \dots \arrow[r, "\partial"] & A_{i-1} \arrow[r, "\partial"] \arrow[d, "f_{i-1}"] & A_i \arrow[r, "\partial"] \arrow[d, "f_i"] & \dots \\ \dots \arrow[r, "\partial"] & B_{i-1} \arrow[r, "\partial"] & B_i \arrow[r, "\partial"] & \dots \end{tikzcd}$$

We will most times leave out the indices if it is clear what we mean. The crucial property of these chain maps is that they induce homomorphisms of the homology groups denoted as

$$[f_i] : H_i(A_*) \to H_i(B_*),\ [f_i]([a]) = [f_i(a)]$$

3.2 Cohomology groups

{sec:cohomology_

Let us start with the abstract definition of a cochain complex. Let $\{C^i\}_{i\in\mathbb{N}}$ be a collection of abelian groups and homomorphisms $\partial^i : C^i \to C^{i+1}$ with $\partial^{i+1} \circ \partial^i = 0$ called *codifferentials*. Then we call this sequence a *cochain complex*. The only difference to chain complexes is that the index increases when applying the codifferential. Hence, they are basically the same from an algebraic point of view. By convention, we use superindices for anything that is related to cochain complexes.

We define *cochain maps* completely analogous to chain maps i.e. cochain maps commute with the codifferential.

The main motivation for cochain complexes comes from the *singular cochain complexes*. Let G be any abelian group and X be a topological space as before. Then we define the group of *k-cochains* $C^k(X; G)$ by

$$C^k(X; G) := \mathrm{Hom}(C_k(X),\, G)$$

i.e. the group of all homomorphisms from k-chains $C_k(X)$ to G. Just as for chains we now introduce a homomorphism between the groups of cochains which transforms this into a cochain complex.

Definition 3.2.1 (Coboundary)**.** We define the operator $\partial^k : C^k(X; G) \to C^{k+1}(X; G)$ via

$$(\partial^k f)(c) := f(\partial_{k+1} c).$$

for a $(k+1)$-chain c. We call a cochain $f \in C^k(X; G)$ *closed* if $\partial^k f = 0$ and we call f *exact* if there is a $g \in C^{k-1}(X; G)$ s.t. $f = \partial^{k-1} g$. As for the boundary map we will frequently leave away the superscript if the context is clear.

Notice that this naming is analogous to the naming for closed and exact forms. This is no coincidence as we will see in the Sec. 3.3.

From the definition it is obvious that $\partial^{k+1} \circ \partial^k = 0$ and thus we have indeed a cochain complex which we call *singular cochain complex*. If there is no confusion with the general notion of cochain complex we will leave away the term 'singular'.

Definition 3.2.2 (Singular cochain cohomology)**.** Denote the closed k-cochains as $Z^k(X;G)$ and the exact ones with $B^k(X;G)$. We then define the *cochain cohomology groups* $H^k(X;G)$ as

$$H^k(X;G) := {}^{Z^k(X;G)}\!\big/_{B^k(X;G)}.$$

Note that in the case of $G = \mathbb{R}$ this becomes a vector space. Now of course there is the question how the homology and cohomology groups are related to each other. This question is answered by the *universal coefficent theorem*. But before we can formulate it we have to introduce exact sequences.

{def:exact_seque

Definition 3.2.3 (Exact sequence)**.** Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. Then this sequence of homomorphisms is called *exact* if $\operatorname{im} f_{i-1} = \ker f_i$.

The universal coefficent theorem in the case of simplicial homology states that the sequence

$$0 \to \operatorname{Ext}(H_{k-1}(K), G) \to H^k(K;G) \xrightarrow{\beta} \operatorname{Hom}(H_k(K), G) \to 0 \qquad (3.2.1)$$ {eq:univeral_coe

is exact. β is defined via

$$\beta([F])([c]) := F(c). \qquad (3.2.2)$$ {eq:isomorphism_

The definition of Ext can be found in [3], but it does not matter for our purpose because from now on we will assume $G = \mathbb{R}$ and in this case $\operatorname{Ext}(H_{k-1}(X), \mathbb{R}) = 0$. This follows from the fact that $\mathbb{R}$ is a divisible and hence injective abelian group. The definition of these terms and the connections used can also be found in [3, Sec. V.6]. However, we will not dwelve into the algebraic background further. In the case of $G = \mathbb{R}$, we can conclude from the exactness of the above short sequence that $\ker \beta = 0$ and $\operatorname{im} \beta = \operatorname{Hom}(H_k(X), \mathbb{R})$. So β is an isomorphism.

3.3 De Rham's theorem

{sec:de_rhams_th

It turns out that the singular cochain cohomology is closely related to the cohomology of differential forms, the *de Rham cohomology* which is introduced next.

Let M be a smooth n-dimensional manifold. We will use the the notation introduced in Sec. 2.3. Then the smooth differential forms $C^\infty\Lambda^k(M)$ together with the exterior derivative give us a cochain complex which we call *de Rham complex* due to the property $d \circ d = 0$. Note that we have slightly more structure here since the $C^\infty\Lambda^k(M)$ are vector spaces and the exterior derivative a linear map i.e. a vector space homomorphism. Let us denote the exterior derivative on the space of k-forms as d^k. Then we define

$$\mathfrak{B}^k(M) := \operatorname{im} d^{k-1}$$
$$\mathfrak{Z}^k(M) := \ker d^k$$

which are the exact and closed forms respectively. We define $d^k = 0$ for $k < 0$ and $k > n$. We then define the *de Rham cohomology group*

$$H^k_{dR}(M) := {}^{\mathfrak{Z}^k(M)}\!\big/_{\mathfrak{B}^k(M)}$$

It turns out that the de Rham complex is closely related with the singular cochain complex which is the topic of this section.

Let us recall Stokes' theorem first which said that for a k-form $\omega \in C^\infty_c\Lambda^k(M)$ for a smooth k-dimensional oriented manifold M we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

The following details are taken from Section V.5 and and V.9 from [3]. We will only focus on the main ideas and avoid dwelving into the technical details. The interested reader can find more arguments in the given reference.

Let now σ be a smooth k-simplex i.e. $\sigma : \Delta_k \to M$ is smooth. We will solely focus on smooth simplices from now on. Let $C_k^{\text{smooth}}(M)$ be the abelian group generated by smooth k-simplices and then $H^k_{\text{smooth}}(M;\mathbb{R})$ be the cochain groups constructed analogous to the standard case. Then in fact, $H^k_{\text{smooth}}(M;\mathbb{R})$ and $H^k(M;\mathbb{R})$ are isomorphic. This is the reason why it is sufficient for this section to deal with smooth simplices only and we will, by abuse of notation, refer to them as $C_k(M)$ and the homology groups as $H_k(M)$ etc.

We now define

$$\int_\sigma \omega = \int_{\Delta_k} \sigma^*\omega$$

and then we define the integral over a k-chain $c = \sum_\sigma n_\sigma \sigma$

$$\int_c \omega := \sum_\sigma n_\sigma \int_\sigma \omega.$$

This motivates us to introduce the homomorphism $I : C^\infty\Lambda^k(M) \to \mathrm{Hom}(C_k; \mathbb{R})$ defined by

$$I(\omega)(c) = \int_c \omega.$$

Remark 3.3.1. There are some technical details that we will not discuss in details here, but that should be mentioned. First, Δ_k is not a manifold. The $(k-2)$-skeleton are the boundaries of the faces i.e. the corners for $k = 2$ and the edges for $k = 3$. If we remove the $(k-2)$-skeleton then Δ_k is a manifold with boundary. But since this is a null-set w.r.t. the full simplex and the boundary as well, this does not matter for our arguments. Second, because we are integrating over the Δ_k their orientation is important and has to be chosen consistently. We will not present the details here. The only important fact is that the simplices are oriented in a way s.t.

$$\int_{[e_0,\dots,\widehat{e_i},\dots,e_k]} \nu = (-1)^i \int_{\Delta_{k-1}} F_i^* \nu \tag{3.3.1}$$ {eq:integral_bou

for any integrable $(k-1)$-form on the face $[e_0, ..., \widehat{e_i}, ..., e_k]$ where the F_i are the face maps defined at (3.1.1). Include a picture Please consult [2, Sec. V.5] for more details.

Now, we have to remember the definition of the boundary of a singular simplex and the face map F_i^k which serves as a chart of the $k-1$ dimensional manifold $[e_0, ..., \widehat{e_i}, ..., e_k]$. Then using Stokes' theorem

$$\begin{aligned} I(d\omega)(\sigma) &= \int_\sigma d\omega = \int_{\Delta_k} \sigma^* d\omega = \int_{\Delta_k} d\sigma^*\omega = \int_{\partial\Delta_k} \sigma^*\omega \\ &= \sum_{i=0}^k \int_{[e_0,\dots,\widehat{e_i},\dots,e_k]} \sigma^*\omega \overset{(3.3.1)}{=} \sum_{i=0}^k (-1)^i \int_{\Delta_{k-1}} F_i^*\sigma^*\omega = \sum_{i=0}^k (-1)^i \int_{\Delta_{k-1}} (\sigma\circ F_i)^*\omega \\ &= \sum_{i=0}^k (-1)^i \int_{\sigma\circ F_i} \omega = I(\omega)(\partial\sigma) = \partial\big(I(\omega)\big)(\sigma). \end{aligned}$$

so we obtain

$$I(d\omega) = \partial\big(I(\omega)\big).$$

We see that I is a cochain map and thus induces a homomorphism on cohomology

$$[I] : H^k_{dR}(M) \to H^k(M; \mathbb{R}).$$

Using the notation and the definition of this map we can now formulate de Rham's theorem which will become very important later when proving existence and uniqueness in Sec. 5.

{thm:de_rhams_th

Theorem 3.3.2 (De Rham's theorem)**.** *Let M be a smooth orientable manifold. Then $[I] : H^k_{dR}(M) \to H^k(M; \mathbb{R})$ is an isomorphism.*

This is quite a deep result that requires more tools than we introduced and is difficult to prove so we will just state it here. Even though it is a very short statement it has very deep implications. It implies that the de Rham isomorphism reflects fundamental topological properties of the manifold.

4 Hilbert complexes

{sec:hilbert_com

In this section, we move away from geometry and topology to functional analysis. A crucial tool for the proof will be the *Hodge decomposition* in 3D which relies on unbounded operators and Hilbert complexes. These will be introduced in this section. This section is essentially a recollection of the parts of chapter 3 and 4 of Arnold's book [2] which we will need.

We start by introducing the concept of unbounded operators on real Hilbert spaces and introduce the adjoint for these. Then we will apply this theory to the differential operators grad, curl and div in 3D. In the second part we will introduce Hilbert complexes which combines the idea of cochain complexes and unbounded operators on Hilbert spaces. This will lead to the Hodge decomposition which is an important tool that we will need in the proof of existence and uniqueness.

Throughout this section it will be assumed that the reader is familiar with basics of functional analysis, especially Hilbert spaces, and basic knowledge about Sobolev spaces. We will focus on real spaces exclusively.

4.1 Unbounded operators

{sec:unbounded_o

We will provide the basic definitions about unbounded operators and propositions about those. After defining unbounded operators we will talk about closed and densely defined operators mainly and the adjoint. Most of the proofs are very short and will be provided in detail.

Definition 4.1.1 (Unbounded operators)**.** Let X and Y be Hilbert spaces. Then we call a linear mapping $T : D(T) \to Y$ with a subspace $D(T) \subseteq Y$ an *unbounded operator* from X to Y. We call $D(T)$ the *domain* of T.

We will talk about an unbounded operator $T : X \to Y$ which means that T is not necessarily defined on all of X.

Note that this definition generalizes the standard operator. In particular, it includes the case when T is in fact bounded which can be slightly confusing, but we will stick to this common naming convention.

The domain is a crucial property of unbounded operators. We will sometimes denote the unbounded operator as the tuple $(T, D(T))$. If $D(T)$ is dense in X we call T *densely defined*. We say that two unbounded operators T and S from X to Y are equal if $D(T) = D(S)$ and $Tx = Sx$ for all $x \in D(T)$.

An easy example of an unbounded densely defined operator is the classical gradient with the domain $C_0^1(\Omega) \subseteq L^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open i.e. here we have $X = L^2(\Omega)$ and $Y = L^2(\Omega; \mathbb{R}^n)$. In short, grad is an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^1(\Omega)$. We could then denote it as $(\mathrm{grad}, C_0^1(\Omega))$. This also shows that the choice of domain is not unique. We could have instead chosen e.g. the different unbounded operator $(\mathrm{grad}, C_0^\infty(\Omega))$. Another example is the weak gradient with domain $H^1(\Omega)$ i.e. $(\mathrm{grad}, H^1(\Omega))$ all of which are densely defined.

As for bounded operators we define the kernel or null space of an unbounded operator

$$\ker T = \{x \in D(T) \mid Tx = 0\}$$

and the image or range

$$\operatorname{im} T = \{Tx \mid x \in D(T)\}.$$

The only difference to keep in mind is that the unbounded operators are not defined on the whole X in general.

Recall that the graph of a function $f : X \to Y$ is defined as $\{(x, f(x)) \in X \times Y \mid x \in X\}$. Analogously, the graph of an unbounded operator T is

$$\Gamma(T) := \{(x, Tx) \mid x \in D(T)\}$$

which is obviously a subspace of $X \times Y$.

We define the *graph inner product* on $D(T)$ as

$$\langle x, z\rangle_{D(T)} := \langle x, z\rangle_X + \langle Tx, Tz\rangle_Y, \quad x, z \in D(T).$$

It is easy to show that this is indeed an inner product. We will call its induced norm the *graph norm*

$$\|x\|_{D(T)} = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}, \quad x \in D(T).$$

Even though this defines a norm, $D(T)$ might not be a Hilbert space because it is in general not complete w.r.t. this norm. Consider for example the unbounded operator $\operatorname{grad} : L^2(\Omega) \to L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^\infty(\Omega)$ and a $\Omega \subseteq \mathbb{R}^n$ open. The graph norm is then

$$\|\phi\|_{D(\mathrm{grad})} = \sqrt{\|\phi\|_{L^2(\Omega)}^2 + \|\operatorname{grad}\phi\|_{L^2(\Omega)}^2}, \quad \phi \in C_0^\infty(\Omega)$$

which is just the standard H^1-norm. But it is well-known that $C_0^\infty(\Omega)$ is in fact not closed w.r.t. this norm and thus not complete since the completion of it is the space $H_0^1(\Omega)$ i.e. the Sobolev space with zero trace on the boundary. Below in Prop. 4.1.3 we will provide a sufficient and necessary condition for the domain to be a Hilbert space when the graph norm is used.

The well-known closed graph theorem for bounded operators says that a linear operator from X to Y defined on all of X (in contrast to unbounded operators in general) is bounded i.i.f. its graph is closed in $X \times Y$ w.r.t. the norm $\|(x,y)\|_{X\times Y} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$. This motivates the following definition.

Definition 4.1.2 (Closed operator)**.** We call an unbounded operator $T : X \to Y$ *closed* if its graph $\Gamma(T)$ is closed w.r.t. the norm $\|\cdot\|_{X\times Y}$.

That means if we have a closed operator T and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \xrightarrow{X} x$ and $Tx_n \xrightarrow{Y} y$ for some $x \in X$ and $y \in Y$. Then $(x_n, Tx_n) \xrightarrow{X\times Y} (x,y)$ and since T is closed $(x,y) \in \Gamma(T)$ i.e. $x \in D(T)$ and $Tx = y$. This is just a rephrasing of the definition essentially so this characterizes closed operators equivalently.

{prop:closed_ope

Proposition 4.1.3. *An unbounded operator T is closed i.i.f. its domain $D(T)$, endowed with the graph inner product, is a Hilbert space.*

Proof. As mentioned above, the graph inner product is in fact an inner product on $D(T)$. So we have to show completeness. Assume that T is closed and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ that is Cauchy w.r.t. the graph norm. That implies that (x_n) must be Cauchy w.r.t. the X-norm and (Tx_n) must be Cauchy w.r.t. the Y-norm so both sequences are convergent. Because X and Y are Hilbert spaces there exists $x \in X$ s.t. $x_n \to x$ and $y \in Y$ s.t. $Tx_n \to y$. Because T is closed we know $x \in D(T)$ so $D(T)$ is complete.

For the other direction, assume $D(T)$ is complete and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \to x \in X$ and $Tx_n \to y$ for some $y \in Y$. Because both sequences are convergent they are both Cauchy and thus (x_n) is Cauchy w.r.t. the graph norm. Due to the completeness of $D(T)$ that implies that $x \in D(T)$ and $x_n \xrightarrow{D(T)} x$ and

$$\|x_n - x\|^2_{D(T)} = \|x_n - x\|^2_X + \|Tx_n - Tx\|^2_Y \to 0$$

so $Tx_n \to Tx$ and thus $Tx = y$ which proves that T is closed. $\square$

As an example, take the unbounded operator $(\mathrm{grad}, H^1(\Omega))$ i.e. the weak gradient as an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega;\mathbb{R}^n)$ with domain $D(\mathrm{grad}) = H^1(\Omega)$. Then we described above that the graph norm here is just the H^1-norm. It is well-known that $H^1(\Omega)$ is a Hilbert space. Therefore Prop. 4.1.3 tells us that $(\mathrm{grad}, H^1(\Omega))$ is a closed operator in contrast to $(\mathrm{grad}, C_0^\infty(\Omega))$ as described above.

The adjoint of bounded operators can be generalized to unbounded operators as well. Let us derive this step by step.

Assume $T : X \to Y$ is a densely defined unbounded operator. Let us fix a $y \in Y$ and look at the linear functional $l : D(T) \to \mathbb{R}$ given by

$$l(x) = \langle y, Tx\rangle_Y.$$

This functional is not necessarily bounded w.r.t. the norm on X. But if it is i.e. if $l \in D(T)'$ then because $D(T)$ is dense in X we can extend it to a $\bar{l} \in X'$. Let $v \in X$ be its Riesz representative. That means we have

$$\langle v, x\rangle_X = \langle l, x\rangle_{X'\times X} = \langle y, Tx\rangle_Y \quad \forall x \in D(T).$$

$\langle l, x\rangle_{X'\times X}$ is the usual duality pairing and we will frequenty just write $\langle l, x\rangle$. Then we define $v = T^*y$ and recognize this as the defining property of the adjoint and define

$$D(T^*) := \{y \in Y \mid \exists c_y \in \mathbb{R} : \langle y, Tx\rangle_Y \leq c_y \|x\|_X \forall x \in X\}$$

It is easy to check that this is a linear subspace of Y.

Proposition 4.1.4. *$T^* : Y \to X$ is a linear unbounded operator with domain $D(T^*)$.*

Proof. Note first that T^*y is well-defined for $y \in D(T^*)$ since the T^*y is the Riesz representative of $x \mapsto \langle y, Tx\rangle_Y$ and the Riesz representative is well-known to be unique.

We only have to show that T^* is linear. Take $y_1, y_2 \in D(T^*)$ and $\lambda \in \mathbb{R}$. Then

$$\langle T^*(y_1 + \lambda y_2), x\rangle_X = \langle y_1 + \lambda y_2, Tx\rangle_Y = \langle T^* y_1, x\rangle + \lambda\langle T^* y_2, x\rangle = \langle T^* y_1 + \lambda T^* y_2, x\rangle_X.$$

for all $x \in D(T)$. Because $D(T)$ is dense in X this implies $T^*(y_1 + \lambda y_2) = T^* y_1 + \lambda T^* y_2$. □

We would like to proof whether T^* is itself densely defined or closed. This can be done in an elegant way by investigating the graphs of T and T^*. But $\Gamma(T)$ is a subspace of $X \times Y$ and $\Gamma(T^*)$ of $Y \times X$. To compare the two, we introduce a rotation operator. For any real vector spaces V and W we define the rotation operator

$$R_{V\times W} : V \times W \to W \times V, (v, w) \mapsto (-w, v).$$

It is obvious that when V and W are normed spaces $R_{V\times W}$ is an isometry and we have $R_{W,V} R_{V,W} Z = Z$ for any subspace $Z \subseteq V \times W$. Using this rotation operator we can formulate the following lemma.

{lem:rotated_gra

Lemma 4.1.5. *Let T be a densely defined unbounded operator from X to Y. Then we have*

$$\Gamma(T)^\perp = R_{Y,X}\Gamma(T^*) \text{ and}$$
$$\overline{\Gamma(T)} = \big(R_{Y,X}\Gamma(T^*)\big)^\perp.$$

Proof. $(x, y) \in \Gamma(T)^\perp$ holds i.i.f.

$$0 = \langle (x, y), (v, Tv)\rangle_{X\times Y} = \langle x, v\rangle_X + \langle y, Tv\rangle_Y \quad \forall v \in D(T).$$

i.e.

$$\langle -x, v\rangle_X = \langle y, Tv\rangle_Y, \quad \forall v \in D(T).$$

This is just equivalent to saying that $-x = T^* y$ i.e.

$$(x, y) = (-T^* y, y) \in R_{Y,X}\Gamma(T^*)$$

which proves the first equality.

For the second equivalence recall the basic fact from Hilbert space theory that for any subspace of a Hilbert space V, $(V^\perp)^\perp = \overline{V}$. Hence, applying the orthogonal complement to both sides of the first equality gives us the second one. □

{cor:adjoint_of_

Corollary 4.1.6. *The adjoint T^* of a densely defined operator T is closed.*

Proof. Recall another basic fact from Hilbert space theory that the orthogonal complement of a space is always closed. So we know from the first equality that $R_{Y,X}\Gamma(T^*)$ is closed. Since $R_{Y,X}$ is an isometry we conclude that $\Gamma(T^*)$ is closed and thus T^* a closed unbounded operator. $\square$

{prop:adjoint_of

Proposition 4.1.7. *Let T be a densely defined and closed unbounded operator. Then T^* is also densely defined and closed.*

Proof. We know from the previous corollary that T^* is closed. In order to prove density, once again recall a fact from Hilbert space theory that a subspace is dense i.i.f. its orthogonal complement is zero. So take $y \in D(T^*)^\perp$ arbitrary. We now have to show that $y = 0$ to complete the proof.

$$0 = \langle y, w\rangle_Y = \langle 0, -T^*w\rangle_X + \langle y, w\rangle_Y = \langle (0, y), (-T^*w, w)\rangle_{X\times Y} \quad \forall w \in D(T^*)$$

which just means

$$(0, y) \in \big(R_{Y,X}\Gamma(T^*)\big)^\perp \overset{\text{Lemma 4.1.5}}{=} \overline{\Gamma(T)} = \Gamma(T).$$

In the last line we used the fact, that T is closed. Thus $y = T0 = 0$ which concludes the proof. $\square$

{prop:T_starstar

Proposition 4.1.8. *If T is a closed and densely defined operator, then $T^{**} = T$*

Proof. This is another application of Lemma 4.1.5. Because T is closed, $\Gamma(T) = \big(R_{Y,X}\Gamma(T^*)\big)^\perp$. From the previous proposition we know that T^* is closed and densely defined operator as well and so $\Gamma(T^*) = \big(R_{X,Y}\Gamma(T^{**})\big)^\perp$. T^{**} is closed and hence its graph as well, so from the properties of the rotation we know that

$$\begin{aligned}\Gamma(T^{**}) = \overline{\Gamma(T^{**})} = \Gamma(T^{**})^{\perp\perp} &= \big(R_{Y,X}R_{X,Y}\Gamma(T^{**})\big)^{\perp\perp}\\ &= \Big(R_{Y,X}\big(R_{X,Y}\Gamma(T^{**})\big)^\perp\Big)^\perp = \Big(R_{Y,X}\Gamma(T^*)\Big)^\perp = \Gamma(T)\end{aligned}$$

Thus T and T^{**} have the same graph, which means that they are equal. $\square$

We will now take a closer look at the kernels and images of unbounded operators. Let us first notice a very clear result. If T is a closed unbounded operator then its kernel $\ker T$ is closed. This follows indeed from the definition. But this is not true for the image $\operatorname{im} T$. Let us take $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{im} T$

with $y_n \to y$. If we now take the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $Tx_n = y_n$ we do not know if (x_n) converges or whether the limit is in $D(T)$ if it does converge. A very simple example is the inclusion operator $\iota : H^1(\Omega) \to L^2(\Omega)$. This is actually a bounded operator and hence closed since

$$\|\iota f\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)} \leq \|f\|_{H^1(\Omega)},$$

but its range $H^1(\Omega)$ is not closed in $L^2(\Omega)$.

Let us summarize the following relationships between the images and kernels of closed densely defined operators and their adjoints.

{prop:kernel_ima

Proposition 4.1.9. *Let $T : X \to Y$ be a closed densely defined operator. Then*

(i) $(\operatorname{im} T)^\perp = \ker T^*$

(ii) $(\ker T)^\perp = \overline{\operatorname{im}(T^*)}$

(iii) $(\operatorname{im} T^*)^\perp = \ker T$

(iv) $(\ker T^*)^\perp = \overline{\operatorname{im}(T)}$

Proof. We will once again rely on Lemma 4.1.5 about the rotated graph. We will start with (iii)

$$x \in \ker T \Leftrightarrow (x,0) \in \Gamma(T) \overset{T \text{ closed}}{=} \overline{\Gamma(T)} = \big(R_{Y,X}\Gamma(T^*)\big)^\perp$$

The last statement is equivalent to saying that for any $y \in D(T^*)$ we have

$$0 = \langle (x,0), (-T^*y, y)\rangle_{X\times Y} = \langle x, -T^*y\rangle_X$$

which just means $x \in (\operatorname{im} T^*)^\perp$ and we proved (iii).

(ii) follows from that immediately by taking the orthogonal complement on both sides.

From Prop. 4.1.7 we know that T^* is closed and densely defined because T is. So the completely analogous reasoning with the roles of T and T^* exchanged gives us (i) and taking the orthogonal complement again proves (iv). ☐

4.1.1 Differential operators in 3D

{sec:adjoints_di

Let us investigate the situation in 3D with the common differential operators curl, grad and div on a domain $\Omega \subseteq \mathbb{R}^3$. At first, we only assume Ω to be open, but we will later introduce some assumptions on the boundary of it.

We will follow [2, Sec. 3.4] for the most part. However, all the results in Arnold's book are provided for bounded domains only which is insufficient for our situation since we want to cover the case where Ω is unbounded with compact boundary. Thus, we will generalize the proofs provided in the reference to this case which always involves applying some additional cut-off type argument.

Take the unbounded operator $\operatorname{div} : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega)$ with the smooth compactly supported vector valued functions $C_0^\infty(\Omega;\mathbb{R}^3)$ as domain i.e. $(\operatorname{div}, C_0^\infty(\Omega;\mathbb{R}^3))$ which we will from now on denote as $(\operatorname{div}, C_0^\infty)$. At first we recognize that the operator $(\operatorname{div}, C_0^\infty(\Omega;\mathbb{R}^3))$ is densely defined and thus the the adjoint exists. When we take the adjoint of it then we know for $v \in D((\operatorname{div}, C_0^\infty)^*)$

$$\int_\Omega \operatorname{div}^* v \cdot \mathbf{u}\, dx = \int_\Omega v \operatorname{div} \mathbf{u}\, dx \quad \forall \mathbf{u} \in C_0^\infty(\Omega;\mathbb{R}^3)$$

As before, vector valued quantites are written in bold. Now if we take $\mathbf{u} = (u_1, 0, 0)^\top$ then

$$\int_\Omega (\operatorname{div}^* v)_1\, u_1\, dx = \int_\Omega v\, \partial_1 u_1\, dx \quad \forall u_1 \in C_0^\infty \tag{4.1.1}$$ {eq:adjoint_grad

so we recognize that $-(\operatorname{div}^* v)_1$ is the weak derivative of w.r.t. the first coordinate i.e. $-\partial_1 v$ and analogous for the other coordinates so we recognize $\operatorname{div}^* = -\operatorname{grad}$. We further see that the domain s.t. (4.1.1) is fulfilled is $H^1(\Omega)$ by definition. That means we showed

$$(\operatorname{div}, C_0^\infty)^* = (-\operatorname{grad}, H^1). \tag{4.1.2}$$ {eq:adjoint_grad

We know from Cor. 4.1.6 that its adjoint $(-\operatorname{grad}, H^1)$ is closed and we can conclude from Prop. 4.1.3 that $H^1(\Omega)$ is in fact a Hilbert space when using the graph norm which is the H^1-norm here. This provides an alternative way to derive this basic result.

Assume from now on that our domain Ω is a Lipschitz domain with compact boundary $\partial\Omega$. Note that $\Omega \subseteq \mathbb{R}^3$ can be unbounded. In the case of an exterior domain, i.e. when Ω is the complement of a compact set, this condition is trivially fulfilled.

Let us denote the $C_b^k(\overline{\Omega})$ the space of k times continuously differentiable functions with bounded support in $\overline{\Omega}$. In contrast to $C_0^k(\Omega)$ these functions are not necessarily zero on the boundary. For $u \in C_b^1(\Omega)$, $\mathbf{v} \in C_b^1(\Omega;\mathbb{R}^3)$ we have the integration-by-parts formula

$$\int_\Omega u \operatorname{div} \mathbf{v}\, dx = -\int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds \tag{4.1.3}$$ {eq:integration_

where $\mathbf{n}$ is the unit normal which exists almost everywhere on the boundary of a Lipschitz domain. This formula is usually stated only for bounded domains (see [10, Cor. 3.20]), but when $\partial\Omega$ is compact we can simply reduce the domain because the supports of u and $\mathbf{v}$ are both bounded. This is done using a standard cut-off argument. To be precise, let $B_R \subseteq \mathbb{R}^3$ be the open ball around the origin with radius R large enought s.t. $\partial\Omega \subseteq B_R$ and $\operatorname{supp} u \subseteq B_R$, $\operatorname{supp} \mathbf{v} \subseteq B_R$. Define the reduced domain $\Omega_R := \Omega \cap B_R$. Then because u and $\mathbf{v}$ are both zero on ∂B_R

$$\begin{aligned}\int_\Omega u \operatorname{div} \mathbf{v}\, dx &= \int_{\Omega_R} u \operatorname{div} \mathbf{v}\, dx \\ &= -\int_{\Omega_R} \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds + \int_{\partial B_R} u|_{\partial B_R} \mathbf{v}|_{\partial B_R} \cdot \mathbf{n}\, ds \\ &= -\int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds\end{aligned}$$

as claimed. This formula motivates us to define the divergence in a weak sense.

Definition 4.1.10 (Weak divergence)**.** Let $\Omega \subseteq \mathbb{R}^3$ be open. For $\mathbf{v} \in L^2(\Omega)^3$ we define the space

$$H(\operatorname{div}; \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^3 \mid \exists \sigma \in L^2(\Omega)\, \forall \phi \in C_0^\infty(\Omega) : \int_\Omega \sigma\, \phi\, dx = -\int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi\, dx \right\}$$

Then we call the σ in the definition the *weak divergence of* $\mathbf{v}$ denoted by $\operatorname{div} \mathbf{v}$.

Using basic arguments from Sobolev theory this defines $\operatorname{div} \mathbf{v}$ uniquely almost everywhere.

We will frequently leave out the reference to the domain in the space definition. We can immediately recognize from the definition that

$$(\operatorname{div}, H(\operatorname{div})) = (-\operatorname{grad}, C_0^\infty)^*. \tag{4.1.4}$$ {eq:adjoint_grad

The next thing we want to talk about is the trace of $H(\operatorname{div})$. We will use the well-known properties of the trace in H^1. Let us recall quickly the definition of fractional order Sobolev spaces (cf. [**<empty citation>**]). Let $U \subseteq \mathbb{R}^n$ be an open domain and take $m \in \mathbb{N}$ and $s \in [0, 1)$ and $1 < p < \infty$. For a multiindex $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ we define $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$ and denote

$$\partial_{\boldsymbol{\alpha}} \phi = \partial_{\alpha_1} \partial_{\alpha_2} \ldots \partial_{\alpha_n} \phi. \tag{4.1.5}$$

Then we define fractional Sobolev norm

$$\|u\|_{W^{m+s,p}} = \left\{ \|u\|_{W^{m,p}}^p + \sum_{|\boldsymbol{\alpha}|=m} \int_U \int_U \frac{|\partial_{\boldsymbol{\alpha}} u(x) - \partial_{\boldsymbol{\alpha}} u(y)|^p}{|x-y|^{n+sp}} \, dx dy \right\}^{1/p} \tag{4.1.6}$$

{eq:fractional_s

and the corresponding space $W^{m+s,p}(U) = \{u \in W^{m,p}(U) \mid \|u\|_{W^{m+s,p}} < \infty\}$ which is a Banach space [10, p.42]. We denote $H^{m+s}(U) = W^{m+s,2}(U)$ as in the integer case. We will need the fractional Sobolev space over the boundary of a Lipschitz domain U. We assume again that the boundary is compact. We define fractional order Sobolev spaces by taking the integrals of (4.1.6) over the boundary instead i.e. for $m = 0$

$$\|u\|_{W^{s,p}(\partial\Omega)} = \left\{ \int_{\partial\Omega} |u|^p \, ds + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n-1+sp}} \, ds(x) ds(y) \right\}^{1/p}.$$

Then it is a well-known result that $H^1(\Omega)$ has the trace operator $\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ which is surjective [2, p.28]. We are trying to find something analogous for $H(\operatorname{div})$.

Remark 4.1.11. The H^1 trace operator is usually defined only for bounded Lipschitz domains (see [10, Thm. 3.9]), but it can be extended easily to unbounded Lipschitz domains with compact boundary considering only values in some bounded subdomain. To make this precise take an open ball centered at the origin B_r s.t. $\partial\Omega \subseteq B_r$. We denote $|\cdot|$ as the standard Euclidian norm and define

$$\chi_R(x) := \begin{cases} 1 & |x| \leq R \\ R + 1 - |x| & R < |x| < R + 1 \\ 0 & \text{else} \end{cases}$$

Let us denote $\Omega_{R+1} := \Omega \cap B_{R+1}$ To define the trace of a function $u \in H^1(\Omega)$ take the trace of $\chi_R u$ – which has compact support in $\overline{B}_{R+1}$ – on Ω_R and restrict it to $\partial\Omega$ i.e. $\operatorname{tr} u = \big(\operatorname{tr}_{\Omega_R} u \chi_R\big)|_{\partial\Omega}$. Then $tr : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective and with C_R being the continuity constant of the trace on Ω_R

$$\begin{aligned} \|\operatorname{tr} u\|_{H^{1/2}(\partial\Omega)} &= \|\operatorname{tr}_{\Omega_R} u \, \chi_R\|_{H^{1/2}(\partial\Omega_R)} \leq C_R \|u \, \chi_R\|_{H^1(\Omega_R)} = C_R \|u \, \chi_R\|_{H^1(\Omega)} \\ &\leq \sqrt{3} C_R \|u\|_{H^1(\Omega)} \end{aligned}$$

and thus the trace is bounded as well. The last inequality follows from the expression of grad χ_R and some standard estimates.

We will start with the following abstract lemma. As is standard, for a bounded linear functional l on a normed space V we write $\langle l, v\rangle_{V'\times V} = l(v)$ or just $\langle l, v\rangle$.

{lem:dual_map_of

Lemma 4.1.12. *Let X, Y be Banach spaces and let $\gamma : X \to Y$ be a linear bounded surjection with kernel Z. Then the dual map is defined as*

$$\gamma' : Y' \to X', l \mapsto l \circ \gamma$$

or with product notation

$$\langle \gamma' \, l, x\rangle_{X'\times X} = \langle l, \gamma \, x\rangle_{Y'\times Y}.$$

This dual map is then a bounded injection with image being the annihilator of Z which is defined as $\{f \in X' \mid f|_Z \equiv 0\}$. In other words, if we have $f \in X'$ with $\langle f, z\rangle = 0$ for all $z \in Z$ then there exists a unique $g_f \in Y'$ s.t. $\langle f, x\rangle_{X'\times X} = \langle g, \gamma \, x\rangle_{Y'\times Y}$.

Proof. Since γ is bounded it is obviously a closed and densely defined unbounded operator. Then we know from [4, Thm. 2.20] that γ' is injective. Thm. 2.19 in the same reference then gives us that $\operatorname{im} \gamma'$ is the annihilator of Z. The dual map is actually the adjoint in the generalized sense on Banach spaces which is how the adjoint is defined in this reference. Then the annihilator corresponds in that notation to "$Z^\perp$", but we will not use this notation because we only introduced the adjoint for Hilbert spaces. □

We now want to apply this lemma to the trace. We follow the standard notation and write $H^{-1/2}$ for the dual space of $H^{1/2}$

{prop:existence_

Proposition 4.1.13. *Let $l \in H^1(\Omega)'$ s.t. $\langle l, \sigma\rangle = 0$ for all $\sigma \in H^1_0(\Omega)$. Then there exists a unique $g_l \in H^{-1/2}(\partial\Omega)$ s.t.*

$$\langle l, u\rangle_{H^1(\Omega)'\times H^1(\Omega)} = \langle g_l, \operatorname{tr} u\rangle_{H^{1/2}(\partial\Omega)'\times H^{1/2}(\partial\Omega)}, \quad \forall u \in H^1(\Omega). \tag{4.1.7}$$ {eq:functional_a

Also, there exist positive constants C_1 and C_2 s.t.

$$C_1 \|g_l\|_{H^{-1/2}(\partial\Omega)} \le \|l\|_{H^1(\Omega)'} \le C_2 \|g_l\|_{H^{-1/2}(\partial\Omega)}.$$

Proof. Let us check the conditions of the lemma. We have the trace operator $\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ as a linear bounded surjection. So the conditions of Lemma 4.1.12 are fulfilled and (4.1.7) follows. Using the fact that the H^1 trace is bounded we can estimate

$$|\langle l, u\rangle| = |\langle g_l, \operatorname{tr} u\rangle| \le C_2 \|g_l\|_{H^{-1/2}} \|u\|_{H^1}$$

for some constant $C_2 > 0$ and thus $\|l\|_{H(\Omega)'} \leq C_2 \|g_l\|_{H^{-1/2}}$. For the inverse inequality we take some $\mu \in H^{1/2}$. If we can find a $f \in H^1$ s.t. $\operatorname{tr} f = \mu$ and

$$\|f\|_{H^1} \leq 1/C_1 \|\mu\|_{H^{1/2}} \tag{4.1.8}$$ {eq:existence_gl

holds then we can conclude

$$|\langle g_l, \mu \rangle| = |\langle l, f \rangle| \leq \|l\|_{H^{-1}} \|f\|_{H^1} \leq \|l\|_{H^{-1}} 1/C_1 \|\mu\|_{H^{1/2}}$$

and thus $\|g_l\|_{H^{-1/2}} \leq 1/C_1 \|l\|_{H^{-1}}$.

We have to show that such an f exists. This is true for bounded domains (cf. [5, Thm. 3.10]). To generalize it to unbounded domains with compact boundary, take a open ball B_R centered at the origin with radius R large enough s.t. $\partial\Omega \subseteq B_R$. Then define $\Omega_R = \Omega \cap B_R$. Take f_R as a function s.t. $\operatorname{tr} f_R = \mu$ on $\partial\Omega$ and $\operatorname{tr} f_R = 0$ on ∂B_R and $\|\operatorname{tr} f_R\|_{H^{1/2}(\Omega_R)} \leq C_R \|f_R\|_{H^1(\Omega_R)}$. Because $\operatorname{tr} f_R = 0$ on ∂B_R we have $\|\operatorname{tr} f_R\|_{H^{1/2}(\partial\Omega_R)} = \|\operatorname{tr} f_R\|_{H^{1/2}(\partial\Omega)}$ Because $f = 0$ on ∂B_R we can extend define $\bar{f}_R \in H^1(\Omega)$ by extending f by zero outside of B_R and $\|\bar{f}_R\|_{H^1(\Omega)} = \|\bar{f}_R\|_{H^1(\Omega_R)}$ so we have found f s.t. (4.1.8) holds which completes the proof. □

We can now define a trace operator for $H(\operatorname{div})$.

{thm:trace_of_hd

Theorem 4.1.14 (Trace of $H(\operatorname{div})$). *The operator* $\mathbf{v} \mapsto \langle \mathbf{v}|_{\partial\Omega} \cdot \mathbf{n}, \cdot \rangle_{L^2(\partial\Omega)} \in H^{-1/2}(\partial\Omega)$ *on* $C_b^1(\overline{\Omega})$ *can be extended to an operator* $\gamma_n : H(\operatorname{div}; \Omega) \to H^{-1/2}$ *s.t.*

$$\int_\Omega \operatorname{div} \mathbf{v} u \, dx = -\int_\Omega \mathbf{v} \cdot \operatorname{grad} u \, dx + \langle \gamma_n \mathbf{v}, \operatorname{tr} u \rangle$$

Also, analogous to the standard trace theorem, we have

$$\|\gamma_n v\|_{H^{-1/2}(\Omega)} \leq \|v\|_{H(\operatorname{div};\Omega)}.$$

Proof. Let us define the linear functional $l_\mathbf{v} \in H^1(\Omega)'$ for a fixed $\mathbf{v} \in H(\operatorname{div})$ as

$$\langle l_\mathbf{v}, u \rangle := \int_\Omega u \operatorname{div} \mathbf{v} \, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v} \, dx.$$

Using Cauchy-Schwarz we get

$$\langle l_\mathbf{v}, u \rangle \leq \|\mathbf{v}\|_{H(\operatorname{div})} \|u\|_{H^1}$$

and so $l_\mathbf{v}$ is indeed a linear bounded functional on H^1.

Take $u \in H_0^1(\Omega)$. Let $(u^k)_{k\in\mathbb{N}} \subseteq C_0^\infty$ s.t. $u^k \xrightarrow{H^1} u$ which is possible due to density. $\langle l_{\mathbf{v}}, u^k\rangle = 0$ due to the definition of the weak divergence and thus $\langle l_{\mathbf{v}}, u\rangle = 0$ by continuity.

We are in the situation of Prop.4.1.13 and we find $\gamma_n \mathbf{v} \in H^{-1/2}(\partial\Omega)$ s.t.

$$\int_\Omega u \operatorname{div} \mathbf{v} \, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v} \, dx = \langle \gamma_n \mathbf{v}, u \rangle \quad \forall u \in H^1(\Omega).$$

The linearity of γ_n is trivial to see. For the boundedness, using Cauchy-Schwarz

$$\langle \gamma_n \mathbf{v}, \operatorname{tr} u\rangle = \int_\Omega u \operatorname{div} \mathbf{v} \, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v} \, dx \leq \|\mathbf{v}\|_{H(\mathrm{div})} \|u\|_{H^1}$$

and so $\|\gamma_n \mathbf{v}\|_{H^{-1/2}} \leq \|\mathbf{v}\|_{H(\mathrm{div})}$ which proves that γ_n is indeed a bounded linear operator. The integration by parts formula is fulfilled by construction.

Let us now prove the case when $\mathbf{v} \in C_b^1(\overline{\Omega}; \mathbb{R}^3)$. We know the integration-by-parts formula holds for all $u \in H^1$ as explained above. We have

$$\langle \gamma_n \mathbf{v}, \operatorname{tr} u\rangle = \int_\Omega u \operatorname{div} \mathbf{v} \, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v} \, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \operatorname{tr} u \, ds.$$

Due to the surjectivity of the trace of H^1 we get

$$\langle \gamma_n \mathbf{v}, \cdot\rangle = \langle \mathbf{v} \cdot n, \cdot\rangle_{L^2(\partial\Omega)}.$$

So in this sense the operator $\mathbf{v} \mapsto \langle \mathbf{v} \cdot n, \cdot\rangle_{L^2(\partial\Omega)} \in H^{-1/2}$ extends to the bounded operator $\gamma_n : H(\mathrm{div}) \to H^{-1/2}$. □

This theorem motivates us to recognize the normal trace as the natural trace on $H(\mathrm{div})$. Now we can define

$$H_0(\mathrm{div}; \Omega) := \{\mathbf{v} \in H(\mathrm{div}) \mid \gamma_n \mathbf{v} = 0\}.$$

We know that H^1 contains all smooth functions $C_b^\infty(\overline{\Omega})$ which are dense in L^2. This follows easily from the fact that $C^\infty(\overline{\Omega})$ is dense in H^1 if Ω is bounded. Analogously, it can be shown that $C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\mathrm{div})$ as well by using the fact that $C^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\mathrm{div})$ if Ω is bounded ([10, Thm. 3.22]). Hence, both of (grad, H^1) and $(\mathrm{div}, H(\mathrm{div}))$ are densely defined and their adjoints are well-defined and we will compute them next. To do so, we need the following theorem.

Theorem 4.1.15 (Surjectivity of γ_n)**.** *The operator $\gamma_n : H(\mathrm{div}; \Omega) \to H^{-1/2}(\partial\Omega)$ is surjective.*

Proof. Take $g \in H^{-1/2}(\Omega)$. As in previous proofs, take the open ball B_R centered at the origin with radius R large enough s.t. $\partial\Omega \subseteq B_R$ and define $\Omega_R = \Omega \cap B_R$. This is obviously a bounded Lipschitz domain with boundary $\partial\Omega_R = \partial\Omega \dot\cup \partial B_R$. Then we take u_R as the solution of the problem

$$\begin{aligned} -\Delta u_R + u_R &= 0 \text{ in } \Omega_R \\ u_R &= g \text{ on } \partial\Omega \text{ and} \\ u_R &= 0 \text{ on } \partial B_R. \end{aligned}$$

which reads in variational formulation

$$\int_{\Omega_R} \operatorname{grad} u_R \cdot \operatorname{grad} \phi \, dx + \int_{\Omega_R} u_R \, \phi \, dx = \langle g, \operatorname{tr} \phi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \quad \forall \phi \in H^1(\Omega_R).$$

This problem has a unique solution since Ω_R is bounded (see [10, Thm. 3.12]). Take $\mathbf{v} := \operatorname{grad} u_R$. Then we see from the variational formulation by choosing $\phi \in C_0^\infty(\Omega_R)$ that $u_R = \operatorname{div} \mathbf{v}$. And then define $\bar{u}_R$ as the extension by zero outside of B_R which is in H^1 because $u_R = 0$ on ∂B_R and take $\bar{\mathbf{v}} = \operatorname{grad} \bar{u}_R$. Then $\operatorname{div} \bar{\mathbf{v}} = \bar{u}_R$ and for any $\phi \in H^1(\Omega)$

$$\begin{aligned} \langle \gamma_n \mathbf{v}, \operatorname{tr} \phi \rangle &= \int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi \, dx + \int_\Omega \operatorname{div} \mathbf{v} \, \phi \, dx \\ &= \int_\Omega \operatorname{grad} \bar{u}_R \cdot \operatorname{grad} \phi dx + \int_\Omega \bar{u}_R \, \phi dx \\ &= \int_{\Omega_R} \operatorname{grad} u_R \cdot \operatorname{grad} \phi dx + \int_{\Omega_R} u_R \, \phi dx \\ &= \langle g, \operatorname{tr} \phi \rangle. \end{aligned}$$

Since the trace on H^1 is surjective we get $\gamma_n \mathbf{v} = g$. □

Now we can compute the adjoints of the operators grad and div without boundary conditions.

{thm:adjoints_gr

Theorem 4.1.16.

$$(-\operatorname{div}, H_0(\operatorname{div})) = (\operatorname{grad}, H^1)^* \tag{4.1.9}$$
$$(-\operatorname{grad}, H_0^1) = (\operatorname{div}, H(\operatorname{div}))^* \tag{4.1.10}$$

Proof. If we take $\mathbf{v} \in H_0(\operatorname{div})$ then we immediately get from the integration by parts formula

$$\langle -\operatorname{div} \mathbf{v}, u \rangle = \langle \mathbf{v}, \operatorname{grad} u \rangle \quad \forall u \in H^1$$

and so $\mathbf{v} \in D((\operatorname{grad}, H^1)^*)$ and $\operatorname{grad}^* \mathbf{v} = -\operatorname{div} \mathbf{v}$. The inner product is the L^2 inner product and we will from now denote the L^2 inner product just by $\langle \cdot, \cdot \rangle$. Vice versa, take $\mathbf{v} \in D((\operatorname{grad}, H^1)^*)$. Then there exists $\sigma \in L^2$ s.t.

$$\langle \mathbf{v}, \operatorname{grad} u \rangle = \langle \sigma, u \rangle$$

for any $u \in H^1$. By choosing $u \in C_0^\infty$ we see $\mathbf{v} \in H(\operatorname{div})$ and $\sigma = -\operatorname{div} \mathbf{v}$. Now taking again $u \in H^1$ arbitrary,

$$0 = \langle \mathbf{v}, \operatorname{grad} u \rangle + \langle \operatorname{div} \mathbf{v}, u \rangle = \langle \gamma_n \mathbf{v}, \operatorname{tr} u \rangle$$

and since tr is surjective we get $\gamma_n \mathbf{v} = 0$ i.e. $\mathbf{v} \in H_0(\operatorname{div})$ and we have proven the first equality.

Take $u \in D((\operatorname{div}, H(\operatorname{div}))^*)$. Then we already see

$$\int_\Omega \operatorname{div}^* u \cdot \boldsymbol{\phi} \, dx = \int_\Omega u \operatorname{div} \boldsymbol{\phi} \, dx \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3)$$

and we can conclude due to (4.1.2) that $u \in H^1$ and $\operatorname{div}^* u = -\operatorname{grad} u$. Then

$$0 = \langle \operatorname{grad} u, \mathbf{v} \rangle + \langle u, \operatorname{div} \mathbf{v} \rangle = \langle \gamma_n \mathbf{v}, \operatorname{tr} u \rangle.$$

Now we use the surjectivity of γ_n and see that $\langle g, \operatorname{tr} u \rangle = 0$ for all $g \in H^{-1/2}(\partial\Omega)$ which implies $\operatorname{tr} u = 0$ and thus $u \in H_0^1$. The other direction follows again from the integration by parts formula immediately and we get $D((\operatorname{div}, H(\operatorname{div}))^*) = H_0^1$ and the claim follows. □

This theorem provides us with an easy computation of the adjoints of the grad and div with homogeneous boundary conditions.

Corollary 4.1.17.

$$(-\operatorname{div}, H(\operatorname{div})) = (\operatorname{grad}, H_0^1)^* \tag{4.1.11}$$

$$(-\operatorname{grad}, H^1) = (\operatorname{div}, H_0(\operatorname{div}))^* \tag{4.1.12}$$

Proof. $(\operatorname{div}, H(\operatorname{div}))$ is the adjoint of the densely defined operator $(-\operatorname{grad}, C_0^\infty)$ and hence a closed operator. It is obviously densely defined and thus we can use previous results,

$$(\operatorname{div}, H(\operatorname{div})) \overset{\textit{Prop.} 4.1.8}{=} (\operatorname{div}, H(\operatorname{div}))^{**} \overset{\textit{Thm.} 4.1.16}{=} (-\operatorname{grad}, H_0^1)^*.$$

The adjoint of $(\operatorname{div}, H_0(\operatorname{div}))$ is computed completely analogously. □

This completes the computation of the adjoints of grad and div.Because $(\operatorname{div}, H(\operatorname{div}))$ is a closed and densely defined operator by taking the graph inner product on $H(\operatorname{div})$

$$\langle \mathbf{v}, \mathbf{w} \rangle_{H(\operatorname{div})} := \langle \mathbf{v}, \mathbf{w} \rangle + \langle \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w} \rangle$$

and the induced norm we know from Prop. 4.1.3 that $H(\operatorname{div})$ becomes a Hilbert space.

Now let us turn our attention to the remaining fundamental differential operator curl. Recall that we have for $\mathbf{u}, \mathbf{v} \in C^1(\overline{\Omega})$ the integration-by-parts formula

$$\int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{u} \, dx = \int_\Omega \operatorname{curl} \mathbf{v} \cdot \mathbf{u} \, dx + \int_{\partial\Omega} \mathbf{v} \times n \cdot \mathbf{u} \, ds \tag{4.1.13}$$ {eq:integration_

if Ω is bounded. Just as before this integration by parts formula can be extended to Ω unbounded when $\partial\Omega$ is compact. We define as before

Definition 4.1.18. Let $\Omega \subseteq \mathbb{R}^3$ be an open domain. Then we define

$$H(\operatorname{curl}; \Omega) := \{\mathbf{v} \in L^2(\Omega; \mathbb{R}^3) \mid \exists \mathbf{w} \in L^2(\Omega; \mathbb{R}^3) \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3) \int_\Omega \mathbf{w} \cdot \boldsymbol{\phi} \, dx = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\phi} \, dx\}$$

andd we denote $\mathbf{w} = \operatorname{curl} \mathbf{v}$.

From the definition we can see

$$(\operatorname{curl}, H(\operatorname{curl})) = (\operatorname{curl}, C_0^\infty)^*.$$

Following analogous arguments as above, we obtain the following

{thm:trace_hcurl

Theorem 4.1.19. *Let Ω be Lipschitz domain with compact boundary $\partial\Omega$. Then the operator $\mathbf{v} \mapsto \langle \mathbf{v}|_{\partial\Omega} \times \mathbf{n}, \cdot \rangle_{L^2(\partial\Omega)}$ from $C_b^1(\overline{\Omega})$ extends to a bounded linear operator $\gamma_\tau : H(\operatorname{curl}) \to H^{-1/2}(\partial\Omega; \mathbb{R}^3)$ s.t. the integration-by-parts formula*

$$\int_\Omega \operatorname{curl} \mathbf{v} \cdot \mathbf{u} \, dx = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{u} \, dx + \langle \gamma_\tau \mathbf{v}, \mathbf{u} \rangle$$

for all $\mathbf{v} \in H(curl), \mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$ is satisfied and there exists $C > 0$ s.t. $\|\gamma_\tau \mathbf{v}\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^3)} \leq C \|\mathbf{v}\|_{H(\operatorname{curl})}$.

Analogous to before, we define

$$H_0(\operatorname{curl}) := \{\mathbf{v} \in H(\operatorname{curl}) \mid \gamma_\tau \mathbf{v} = 0\}.$$

Using this definition we can compute the adjoint.

Theorem 4.1.20. *We have the following adjoints:*

$$(\mathrm{curl}, H_0(\mathrm{curl})) = (\mathrm{curl}, H(\mathrm{curl}))^*$$
$$(\mathrm{curl}, H(\mathrm{curl})) = (\mathrm{curl}, H_0(\mathrm{curl}))^*$$

Proof. Take $\mathbf{v} \in D((\mathrm{curl}, H(\mathrm{curl}))^*)$.

$$\int_\Omega \mathrm{curl}^* \mathbf{v} \cdot \boldsymbol{\phi} \, dx = \int_\Omega \mathbf{v} \cdot \mathrm{curl} \, \boldsymbol{\phi} \, dx \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3)$$

and thus $\mathrm{curl}^* \mathbf{v} = \mathrm{curl} \, \mathbf{v}$ and we have

$$\langle \mathrm{curl} \, \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w} \rangle \tag{4.1.14}$$ {eq:adjoint_curl

for all $\mathbf{w} \in H(\mathrm{curl})$. That means that for $\mathbf{w} \in H^1(\Omega; \mathbb{R}^3)$

$$\langle \gamma_\tau \mathbf{v}, \mathrm{tr} \, \mathbf{w} \rangle = \langle \mathrm{curl} \, \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w} \rangle = 0$$

and thus $\gamma_\tau \mathbf{v} = 0$ i.e. $\mathbf{v} \in H_0(\mathrm{curl})$.

Vice versa, take $\mathbf{v} \in H_0(\mathrm{curl})$. Then we need to show that 4.1.14 is fulfilled and we would like to use the integration-by-parts formula for that. However, this formula only holds for $\mathbf{w} \in H^1(\Omega; \mathbb{R}^3)$. We need to use the fact that $C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\mathrm{curl})$. Analogous to what we have done before, the argument follows easily from the fact that for bounded domains U $C^\infty(\overline{U}; \mathbb{R}^3)$ is dense in $H(\mathrm{curl}; U)$. The statement for bounded domains is proven in [10, Lemma 3.27]. The proof is quite technical and we will not present it here. But using the density it is clear that $H^1(\Omega; \mathbb{R}^3)$ is also dense in $H(\mathrm{curl})$ since it contains all smooth functions with bounded support. Take $\mathbf{w} \in H(\mathrm{curl})$ arbitrary and $(\mathbf{w}^k)_{k \in \mathbb{N}} \subseteq H^1(\Omega; \mathbb{R}^3)$. s.t. $\mathbf{w}^k \xrightarrow{H(\mathrm{curl})} \mathbf{w}$. Because $\gamma_\tau \mathbf{v} = 0$ we have

$$\langle \mathrm{curl} \, \mathbf{v}, \mathbf{w}^{\mathbf{k}} \rangle = \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w}^{\mathbf{k}} \rangle$$

Now taking the limits on both sides and using the fact that $\mathbf{w}$ was arbitrary we get $\mathbf{v} \in D((\mathrm{curl}, H(\mathrm{curl}))^*)$ and we obtain the first equality.

The second equality follows from the fact that $(\mathrm{curl}, H(\mathrm{curl}))$ is a closed and densely defined operator and thus

$$(\mathrm{curl}, H(\mathrm{curl})) = (\mathrm{curl}, H(\mathrm{curl}))^{**} = (\mathrm{curl}, H_0(\mathrm{curl}))^*.$$

□

With this, we computed the adjoints of the most important differential operators that will be needed in the following section.

4.2 Hilbert complexes

Now we will combine the idea of cochain complexes from Section 3 with unbounded operators. We will then together with previous results of this section derive the Hodge decomposition and apply it to the three dimensional case.

Recall that a cochain complex is in full generality a sequence of groups $(G^i)_{i\in\mathbb{Z}}$ and group homomorphims $f^i : G^i \to G^{i+1}$ s.t. $f^{i+1} \circ f^i = 0$.

Definition 4.2.1 (Hilbert complex)**.** A Hilbert complex is a sequence of real Hilbert spaces $(W^k)_{k\in\mathbb{Z}}$ and a sequence of closed, densely defined unbounded operators $d^k : W^k \to W^{k+1}$ with domain $V^k \subseteq W^k$ s.t. $d^{k+1} \circ d^k = 0$.

We denote $\mathfrak{Z}^k := \ker d^k$ and $\mathfrak{B}^k := \operatorname{im} d^{k-1}$. Then it follows from the definition that $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$.

To be precise, a Hilbert complex is in fact not a cochain complex in the exact sense because it is not defined on the whole space, but it is if we look at the operators defined on their domains V^k instead.

Because unbounded operators are bounded w.r.t. the graph norm, the restriction of the operators to their domain, $d^k : V^k \to V^{k+1}$, are bounded operators when we use the graph norm on V^k. Because we assume d^k to be closed we know from Prop. 4.1.3 that V^k are Hilbert spaces w.r.t. the graph norm $\|\cdot\|_{V^k}$. So we see that d^k together with V^k is also a Hilbert complex which we call *domain complex*. In this Hilbert complex all operators are bounded. Notice because the operators are defined on the whole Hilbert space V^k this fits the definition of a cochain complex since vector spaces with summation are groups and the d^k are linear mappings and hence group homomorphisms.

Now let us investigate the adjoints of the operators in a Hilbert complex. Since we assume the operators to be closed and densely defined the adjoints exist and we denote with $d_k^* : W^k \to W^{k-1}$ the adjoint of d^k. Due to Prop. 4.1.7 we know that the adjoints are also closed and densely defined. We denote $V_k^* := D(d_k^*)$, $\mathfrak{Z}_k^* := \ker d_k^*$ and $\mathfrak{B}_k^* := \operatorname{im} d_k^*$. We will frequently leave out the indices from now on.

We can now apply Prop. 4.1.9 to this construction. Then we observe

$$\begin{aligned} \mathfrak{B}^{\perp} &= \mathfrak{Z}^*, \\ \mathfrak{Z}^{\perp} &= \overline{\mathfrak{B}^*}, \\ \mathfrak{B}^{*\perp} &= \mathfrak{Z} \text{ and} \\ \mathfrak{Z}^{*\perp} &= \overline{\mathfrak{B}^*}. \end{aligned}$$

Now recall the basic fact from Hilbert space theory that in any Hilbert space if we have any two subspaces $V \subseteq W$ then taking the orthogonal complements reverses the inclusion i.e. $V^\perp \supseteq W^\perp$. Then we get

$$\mathfrak{B}^* \subseteq \overline{\mathfrak{B}^*} = \mathfrak{Z}^\perp \subseteq \mathfrak{B}^\perp = \mathfrak{Z}^*. \tag{4.2.1}$$ {eq:image_kernel

So we recognize that $d_k^* : W^k \to W^{k-1}$ form a structure very similar to a Hilbert complex with the only difference being that the indices of the spaces decrease. We call this complex the *dual complex* of the Hilbert complex.

Definition 4.2.2. We call a $v \in V^k \cap V_k^*$ harmonic if $d^k v = 0$ and $d_k^* v = 0$. Denote the space of harmonic elements as $\mathfrak{H}^k$.

We can rewrite this as $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$. Using (4.2.1) we can write this as

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{B}_k^{*,\perp} \cap \mathfrak{Z}_k^*.$$

Now we can formulate the most important result of this chapter.

{thm:hodge_decom

Theorem 4.2.3 (Hodge decomposition)**.** *Let $d^k : W^k \to W^{k+1}$ form a Hilbert complex. Then we have*

$$\mathfrak{Z}^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \text{ and}$$
$$\mathfrak{Z}_k^* = \overline{\mathfrak{B}_k^*} \overset{\perp}{\oplus} \mathfrak{H}^k.$$

We obtain the Hodge decomposition of the space W^k

$$W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}_k^*}.$$

Proof. Let us first prove $\mathfrak{Z}^k \subseteq \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. Take $z \in \mathfrak{Z}^k$ arbitrary. From basic Hilbert theory we know that $W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{B}^{k,\perp}$. So we find $z = z_1 + z_2$ with $z_1 \in \overline{\mathfrak{B}^k}$ and $z_2 \in \mathfrak{B}^{k,\perp}$. Because $\mathfrak{Z}^k$ is closed $z_1 \in \overline{\mathfrak{B}^k} \subseteq \mathfrak{Z}^k$ and thus $z_2 = z - z_1 \in \mathfrak{Z}^k$ as well i.e. $z_2 \in \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{H}^k$ and so $z \in \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. The other inclusion is obvious since $\mathfrak{H}^k = \mathfrak{B}^{k,\perp} \cap \mathfrak{Z}^k$. The proof for the second equality is completely analogous.

For the Hodge decomposition, since $\mathfrak{Z}^k$ is closed

$$W^k = \mathfrak{Z}^k \overset{\perp}{\oplus} \mathfrak{Z}^{k,\perp} = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}_k^*}.$$

□

The Hodge decomposition is a very powerful tool whenever deal with Hilbert complexes. Another important result will be the Poincaré inequality in the case where $\mathfrak{B}^k$ is closed.

{thm:poincare_in

Theorem 4.2.4 (Poincaré inequality). *For any k, if $\mathfrak{B}^{k+1}$ is closed in V then there exists a $c_{P,k} > 0$ s.t.*

$$\|z\|_V \leq c_{P,k}\|dz\|, \quad \forall z \in \mathfrak{Z}^{k,\perp_V}$$

where $\perp_V$ denotes the orthogonal complement w.r.t. the V-inner product.

Proof. This follows directly form the Banach inverse theorem. Because $\mathfrak{B}^{k+1}$ is closed it is itself a Hilbert space. If we restrict d^k to $\mathfrak{Z}^{k,\perp_V}$ it is injective and so $d^k|_{\mathfrak{Z}^{k,\perp_V}} : \mathfrak{Z}^{k,\perp_V} \to \mathfrak{B}^{k+1}$ is a bounded isomorphism between Hilbert spaces and we can apply the Banach inverse theorem to obtain the inverse denoted by slight abuse of notation as d^{-1} which is also bounded. So for any $z \in \mathfrak{Z}^{k,\perp_V}$

$$\|z\|_V = \|d^{-1}dz\|_V \leq c_{P,k}\|dz\|_V = c_{P,k}\|dz\|.$$

□

4.2.1 L^2 de Rham complex in 3D

{sec:l2_de_rham_

Let us investigate the situation for the differential operators grad, div and curl. All the necessary ingredients were already proven in Sec. 4.1.1. Let Ω be a Lipschitz domain of $\mathbb{R}^3$ with compact boundary $\partial\Omega$.

We take $W^0 = W^3 = L^2(\Omega)$ and $W^1 = W^2 = L^2(\Omega; \mathbb{R}^3)$ and we set all other W^k to zero in order to obtain a sequence. Then we choose the operators $d^0 = \mathrm{grad}$, $d^1 = \mathrm{curl}$, $d^2 = \mathrm{div}$ and for the domains we choose $V^0 = H^1(\Omega)$, $V^1 = H(\mathrm{curl}; \Omega)$, $V^2 = H(\mathrm{div}; \Omega)$ and $V^3 = L^2(\Omega) = W^3$. As before, we will leave out the reference to the domain Ω now. All other d^k are just zero. The resulting domain complex is then

$$0 \to H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0 \tag{4.2.2}$$ {eq:primal_de_rh

All these operators are closed and densely defined. It remains to show that $d^{k+1} \circ d^k = 0$. If $k < 0$ or $k > 2$ this is clear from the definition. Then we have from the definition of these operators for any $u \in H^1$ and $\mathbf{v} \in C_0^\infty(\Omega; \mathbb{R}^3)$

$$\int_\Omega \mathrm{curl}\,\mathrm{grad}\, u \cdot \mathbf{v}\, dx = \int_\Omega \mathrm{grad}\, u \cdot \mathrm{curl}\, \mathbf{v}\, dx = -\int_\Omega u\ \mathrm{div}\,\mathrm{curl}\, \mathbf{v}\, dx = 0$$

which implies that $\mathrm{curl}\,\mathrm{grad}\, u = 0$. $\mathrm{div}\,\mathrm{curl} = 0$ is proven completely analogously. So (4.2.2) is indeed a Hilbert complex.

{thm:closed_rang

Theorem 4.2.5. *If Ω is bounded then* $\operatorname{grad} H^1$, $\operatorname{curl} H(\operatorname{curl})$ *and* $\operatorname{div} H(\operatorname{div})$ *are closed subspaces of $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^3)$ respectively.*

Proof. See [2, p.38]. □

Having closed range should not be confused with the operators being closed. The mentioned differential operators are closed even on unbounded domains.

The resulting dual domain complex is

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} H_0(\operatorname{div}) \xleftarrow{\operatorname{curl}} H_0(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H_0^1 \leftarrow 0.$$

For the harmonic elements – which are scalar and vector fields here – we obtain

$$\begin{aligned}
\mathfrak{H}^0 &= \{u \in H^1 \mid \operatorname{grad} u = 0\},\\
\mathfrak{H}^1 &= \{\mathbf{u} \in H(\operatorname{curl}) \cap H_0(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\},\\
\mathfrak{H}^2 &= \{\mathbf{u} \in H_0(\operatorname{curl}) \cap H(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\} \text{ and}\\
\mathfrak{H}^3 &= \{u \in H_0^1 \mid \operatorname{grad} u = 0\} = \{0\}.
\end{aligned}$$

For the last equality we used the fact that $\operatorname{grad} u = 0$ implies that u is constant almost everywhere with possibly different constants for different path components of Ω. But because we have homogeneous boundary conditions we get $u = 0$. Note that $\mathfrak{H}^1$ and $\mathfrak{H}^2$ are very similar. The only difference are the boundary conditions. $\mathbf{u} \in \mathfrak{H}^2 \subseteq H_0(\operatorname{curl})$ means that the generalized tangential trace $\gamma_\tau \mathbf{u}$ is zero. If $\mathbf{u} \in \mathfrak{H}^1 \subseteq H_0(\operatorname{div})$ then the generalized normal trace $\gamma_n \mathbf{u}$ vanishes.

This gives us the Hodge decomposition in the 3D case.

{thm:hodge_decom

Theorem 4.2.6 (Hodge decomposition in 3D)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with compact boundary. Then we have the following decompositions of the kernels*

$$\begin{aligned}
\{\mathbf{u} \in H(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{grad} H^1} \overset{\perp}{\oplus} \mathfrak{H}^1\\
\{\mathbf{u} \in H(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2\\
\{\mathbf{u} \in H_0(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} H_0(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^1\\
\{\mathbf{u} \in H_0(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{div} H_0(\operatorname{div})} \overset{\perp}{\oplus} \mathfrak{H}^2.
\end{aligned}$$

We can express $L^2(\Omega)$ as

$$\begin{aligned} L^2(\Omega) &= \overline{\operatorname{div} H_0(\operatorname{div})} \overset{\perp}{\oplus} \{v \in H^1 \mid \operatorname{grad} v = 0\} \\ &= \overline{\operatorname{div} H(\operatorname{div})}. \end{aligned}$$

and for vector valued functions

$$\begin{aligned} L^2(\Omega; \mathbb{R}^3) &= \overline{\operatorname{grad} H(\operatorname{grad})} \overset{\perp}{\oplus} \mathfrak{H}^1 \overset{\perp}{\oplus} \overline{\operatorname{curl} H_0(\operatorname{curl})} \\ &= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2 \overset{\perp}{\oplus} \overline{\operatorname{grad} H_0^1}. \end{aligned}$$

Proof. This is just an application of the general Hodge decomposition Thm. 4.2.3 combined with what we derived above. □

Remark 4.2.7. Alternatively, we could have chosen the sequence with zero boundary conditions as the primal sequence i.e.

$$0 \to H_0^1 \xrightarrow{\operatorname{grad}} H_0(\operatorname{curl}) \xrightarrow{\operatorname{curl}} H_0(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2 \to 0$$

Then we can follow the exact same arguments to get the dual sequence

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} H(\operatorname{div}) \xleftarrow{\operatorname{curl}} H(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H^1 \leftarrow 0.$$

5 Existence and uniqueness of solutions

{sec:existence_a

In this section, we will apply the developed theory of the preceding chapters to prove the existence and uniqueness of the magnetostatic problem on exterior domains. But at first, we have to properly formulate the problem.

$\Omega \subseteq \mathbb{R}^3$ is an exterior domain which means that that our domain $\Omega \subseteq \mathbb{R}^3$ is the complement of a compact set. Furthermore, we assume its boundary to be Lipschitz. Note that $\partial\Omega$ is compact in this setting. The main motivation for this problem is the special case of Ω being the complement of a torus. This is also the motivation behind the topological assumption that we will give. It might be useful to keep this example in mind.

The condition $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is replaced with $\mathbf{B} \in H_0(\operatorname{div})$ i.e. $\gamma_n B = 0$ where γ_n is the generalized normal trace. We assume additionally that $\mathbf{B} \in H(\operatorname{curl})$. This means that the divergence and curl are meant in the weak sense.

Recall that another condition was the piecewise smooth curve integral around the torus. Recalling section 3 we replace the curve γ formally with a smooth 1-chain, which is essentially the same, so we will denote by slight abuse of notation as γ as well. We emphasize it only to make it mathematically rigorous and so we can talk about integrals of 1-forms on it later. Note that smooth chain means here that the chosen singular simplices are smooth. The curve γ is only piecewise smooth in general.

So we arrive at the following problem that we want to investigate:

{prob:magnetosta

Problem 5.0.1. Find $\mathbf{B} \in H_0(\text{div};\Omega) \cap H(\text{curl})$ s.t.

$$\text{curl } \mathbf{B} = 0, \tag{5.0.1}$$

$$\text{div } \mathbf{B} = 0 \text{ in } \Omega \text{ and} \tag{5.0.2}$$

$$\int_\gamma \mathbf{B} \cdot dl = C_0. \tag{5.0.3}$$

Of course in order for the curve integral constraint to be well-defined we need to check the regularity of solutions. Then using the tools we developed in the previous sections we will proof existence and uniqueness.

5.1 Regularity of solutions

{sec:regularity_

We will rely on standard regularity results about elliptic systems of the following form. Take $A_{ij}^{\alpha\beta} \in \mathbb{R}$ for $i,\ j,\ \alpha,\ \beta = 1,2,3$ and $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Then we have systems of the form

$$-\sum_{\alpha,\beta,j} \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta B_j) = f_i - \sum_\alpha \partial_\alpha F_i^\alpha \tag{5.1.1}$$ {eq:elliptic_syst

with data $f_i, F_i^\alpha \in L^2(\Omega)$. We call this system *elliptic* if A satisfies the Legendre condition i.e.

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^{3\times 3} \tag{5.1.2}$$ {eq:legendre_cond

with $c > 0$. $|\xi|$ is here the Frobenius norm, but technically the chosen norm is irrelevant due to all norms on $\mathbb{R}^{3\times 3}$ being equivalent.

We call a open $Q \subseteq \Omega$ *precompact* if $\overline{Q} \subseteq \Omega$ is compact where the closure is taken in the topology of $\mathbb{R}^3$ and we write $Q \subset\subset \Omega$. This can be understood as Q being bounded and having positive distance from the boundary. Recall the definition of

$$H_{loc}^k(\Omega) := \{u \in L^2 \mid \forall Q \subset\subset \Omega : u \in H^k(Q)\}.$$

for $k \in \mathbb{N}$. Analogous to $H^k(\Omega;\mathbb{R}^3)$, $H^k_{loc}(\Omega;\mathbb{R}^3)$ means that all the component are in $H^k_{loc}(\Omega)$. We will frequently leave out the reference to the domain if it is clear.

We then call $\mathbf{B} = (B_1, B_2, B_3)^\top \in H^1_{loc}(\Omega;\mathbb{R}^3)$ a weak solution of (5.1.1) if

$$\int_\Omega \sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij} \partial_\beta B_j \partial_\alpha \varphi_i dx = \int_\Omega \left\{ \sum_i f_i \varphi_i + \sum_{\alpha,i} F^\alpha_i \partial_\alpha \varphi_i \right\} dx \qquad (5.1.3)$$

{eq:weak_ellipti

for all $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^\top \in C^1_0(\Omega;\mathbb{R}^3)$. This formulation is taken from [1, Sec. 1.3]. At first we will slightly modify the notion of weak solution.

{prop:weak_solut

Proposition 5.1.1. *(5.1.3) is fulfilled for all $\boldsymbol{\varphi} \in C^1_0(\Omega;\mathbb{R}^3)$ if and only if it is fulfilled for $\boldsymbol{\varphi} \in C^\infty_0(\Omega;\mathbb{R}^3)$.*

Proof. This follows by a simple density argument. Assume that (5.1.3) is fulfilled for all test functions in $C^\infty_0(\Omega;\mathbb{R}^3)$. Now take $\boldsymbol{\varphi} \in C^1_0(\Omega;\mathbb{R}^3)$ arbitrary. Because $\boldsymbol{\varphi} \in H^1_0(\Omega;\mathbb{R}^3)$ and $C^\infty_0(\Omega;\mathbb{R}^3)$ is dense in $H^1_0(\Omega;\mathbb{R}^3)$ we can find a sequence $(\boldsymbol{\varphi}^{(l)})_{l\in\mathbb{N}} \subseteq C^\infty_0(\Omega;\mathbb{R}^3)$ s.t. $\boldsymbol{\varphi}^{(l)} \to \boldsymbol{\varphi}$ in $H^1(\Omega;\mathbb{R}^3)$. Thus the partial derivatives converge in $L^2(\Omega)$ and we get

$$\int_\Omega \sum_{i,j,\alpha,\beta} A^{\alpha,\beta}_{ij} \partial_\beta B_j \partial_\alpha \varphi_i dx = \sum_{i,j,\alpha,\beta} A^{\alpha,\beta}_{ij} \int_\Omega \partial_\beta B_j \lim_{l\to\infty} \partial_\alpha \varphi^{(l)}_i dx$$

$$\overset{L^2 \text{ limit}}{=} \lim_{l\to\infty} \int_\Omega \left\{ \sum_i f_i \varphi^{(l)}_i + \sum_{\alpha,i} F^\alpha_i \partial_\alpha \varphi^{(l)}_i \right\} dx = \int_\Omega \left\{ \sum_i f_i \varphi_i + \sum_{\alpha,i} F^\alpha_i \partial_\alpha \varphi_i \right\} dx.$$

Since $\boldsymbol{\varphi} \in C^1_0(\Omega;\mathbb{R}^3)$ was arbitrary the first direction of the equivalence is proved. The other direction is trivial. □

So we see that in the case of constant coefficents we can consider just smooth compactly supported functions as test functions. Next, we will state the crucial result about the regularity of elliptic systems which will give us the desired regularity of solutions of our system. This is Theorem 2.13 and Remark 2.16 in [1] in slightly less generality and for 3D.

{thm:regularity_

Theorem 5.1.2. *Let Ω be an open domain in $\mathbb{R}^3$. Let A satisfy the Legendre condition (5.1.2). Then for every $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$ weak solution in the sense of (5.1.3) with $\mathbf{f} \in H^k_{loc}(\Omega;\mathbb{R}^3)$ and $F \in H^{k+1}_{loc}(\Omega;\mathbb{R}^{3\times3})$ we have $\mathbf{B} \in H^{k+2}_{loc}(\Omega;\mathbb{R}^3)$.*

{cor:smooth_solu

Corollary 5.1.3. *If under the assumptions of the previous theorem we consider the homogeneous problem, i.e.*

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \partial_\beta B_j \partial_\alpha \varphi_i dx = 0$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$*, then* $\mathbf{B} \in C^\infty(\Omega;\mathbb{R}^3)$*.*

Proof. Here we have $F = 0$ and $\mathbf{f} = 0$. Thus, $\mathbf{f} \in H^k_{loc}(\Omega;\mathbb{R}^3)$ for any $k \in \mathbb{N}$. Take any $Q \subseteq \Omega$ pre-compact. Then we know from Thm. 5.1.2. that $\mathbf{B} \in H^{k+2}_{loc}(\Omega;\mathbb{R}^3)$ and thus $\mathbf{B} \in H^{k+2}(Q;\mathbb{R}^3)$. Therefore, we can apply the standard Sobolev embedding theorem locally to get $\mathbf{B} \in C^l(Q;\mathbb{R}^3)$ for any $l \in \mathbb{N}$ and hence $\mathbf{B} \in C^\infty(\Omega;\mathbb{R}^3)$ since Q pre-compact was arbitrary. □

It should be noted that this does not guarantee us any regularity on the boundary.

Before we can apply this result, we have to check whether a solution of our problem $\mathbf{B}$ is actually in $H^1_{loc}(\Omega;\mathbb{R}^3)$.

Theorem 5.1.4. *Assume* $\mathbf{B} \in H(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega)$*. Then* $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$*.* {thm:solution_in

Note that we did not assume $\mathbf{B}$ to be a solution.

Proof. We know that for a function $\mathbf{u} \in H_0(\mathrm{curl};U) \cap H(\mathrm{div};U)$ for some smooth domain U we have $\mathbf{u} \in H^1(U;\mathbb{R}^3)$ (cf. [10, Remark 3.48]). Our Ω is just assumed to be Lipschitz so we can not apply this result directly.

Take $Q \subset\subset \Omega$ open and pre-compact. Then we can find an open cover of $\overline{Q}$ with a finite set of open balls $\{K_i\}_{i=1}^N$ s.t. $K_i \subseteq \Omega$ and

$$\overline{Q} \subseteq \bigcup_{i=1}^N K_i.$$

As an open cover of a compact set we can find a smooth partition of unity $\{\chi_i\}_{i=1}^N$ subordinate to $\{K_i\}_{i=1}^N$. $(\mathbf{B}\chi_i)|_{K_i} \in H_0(\mathrm{curl};K_i) \cap H(\mathrm{div};K_i)$ and thus $(\mathbf{B}\chi_i)|_{K_i} \in H^1(K_i;\mathbb{R}^3)$ by the above mentioned result. Also because $\mathbf{B}\chi_i$ has compact support in K_i we can extend it by zero to obtain $\mathbf{B}\chi_i \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ where we abused the notation by denoting the extension the same. Whence,

$$\mathbf{B}|_Q = (\sum_{i=1}^N \chi_i|_Q)\mathbf{B}|_Q = \sum_{i=1}^N (\chi_i \mathbf{B})|_Q \in H^1(Q;\mathbb{R}^3)$$

i.e. $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$. □

The following lemma is a reformulation of the differential operator $\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl}$ which will be needed when we write our magnetostatic problem in the above standard elliptic form.

{lem:graddiv_cur

Lemma 5.1.5. *Let* $\mathbf{F} \in H^2_{loc}(\Omega;\mathbb{R}^3)$*. Then*

$$\operatorname{grad}\operatorname{div}\mathbf{F} - \operatorname{curl}\operatorname{curl}\mathbf{F} = \begin{pmatrix}\Delta F_1\\ \Delta F_2\\ \Delta F_3\end{pmatrix}.$$

Proof. By a simple calculation and changing the order of differentiation

$$\operatorname{grad}\operatorname{div}\mathbf{F} = \begin{pmatrix}\partial_1^2 F_1 + \partial_1\partial_2 F_2 + \partial_1\partial_3 F_3\\ \partial_1\partial_2 F_2 + \partial_2^2 F_2 + \partial_2\partial_3 F_3\\ \partial_1\partial_3 F_1 + \partial_2\partial_3 F_2 + \partial_3^2 F_3\end{pmatrix}$$

and

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{curl}\begin{pmatrix}\partial_2 F_3 - \partial_3 F_2\\ \partial_3 F_1 - \partial_1 F_3\\ \partial_1 F_2 - \partial_2 F_1\end{pmatrix} = \begin{pmatrix}\partial_2(\partial_1 F_2 - \partial_2 F_1) - \partial_3(\partial_3 F_1 - \partial_1 F_3)\\ \partial_3(\partial_2 F_3 - \partial_3 F_2) - \partial_1(\partial_1 F_2 - \partial_2 F_1)\\ \partial_1(\partial_3 F_1 - \partial_1 F_3) - \partial_2(\partial_2 F_3 - \partial_3 F_2)\end{pmatrix}$$
$$= \begin{pmatrix}\partial_1\partial_2 F_2 - \partial_2^2 F_1 - \partial_3^2 F_1 + \partial_1\partial_3 F_3\\ \partial_2\partial_3 F_3 - \partial_3^2 F_2 - \partial_1^2 F_2 + \partial_1\partial_2 F_3\\ \partial_1\partial_3 F_3 - \partial_3^2 F_2 - \partial_2^2 F_3 + \partial_2\partial_3 F_2\end{pmatrix}$$

and so by subtracting the two expressions

$$\operatorname{grad}\operatorname{div}\mathbf{F} - \operatorname{curl}\operatorname{curl}\mathbf{F} = \begin{pmatrix}\partial_1^2 F_1 + \partial_2^2 F_1 + \partial_3^2 F_1\\ \partial_1^2 F_2 + \partial_2^2 F_2 + \partial_3^2 F_3\\ \partial_1^2 F_3 + \partial_2^2 F_3 + \partial_3^2 F_3\end{pmatrix} = \begin{pmatrix}\Delta F_1\\ \Delta F_2\\ \Delta F_3\end{pmatrix}.$$

□

We want to rewrite this system in the expression of the elliptic system (5.1.1). We can rewrite the Laplacian

$$-\Delta F_i = -\sum_{\alpha=1}^{3}\partial_\alpha\partial_\alpha F_i = -\sum_{\alpha,\beta=1}^{3}\partial_\alpha\delta_{\alpha,\beta}\partial_\beta F_i = -\sum_{\alpha,\beta,j=1}^{3}\partial_\alpha\delta_{\alpha,\beta}\delta_{ij}\partial_\beta F_j$$

with δ_{ij} being the Kronecker delta, so we get $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. We have to check that the resulting differential operator is indeed elliptic, but this trivial because for any $(\xi_\alpha^i)_{1\le i,\alpha\le 3}$ we get

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j = \sum_{\alpha,\beta,i,j}\delta_{ij}\delta_{\alpha\beta}\xi_\alpha^i\xi_\beta^j = \sum_{\alpha,i}(\xi_\alpha^i)^2 = |\xi|^2$$

so the Legendre condition (5.1.2) is fulfilled and the resulting system is elliptic. The weak formulation is

$$\int_\Omega \sum_{\alpha,\beta,i,j} \delta_{ij}\delta_{\alpha\beta}\partial_\beta B_j \partial_\alpha \varphi_i dx = \sum_{i=1}^{3} \int_\Omega \operatorname{grad} B_i \cdot \operatorname{grad} \varphi_i dx.$$

Here we can assume $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$ due to Prop. 5.1.1.

{thm:smoothness_

Theorem 5.1.6 (Smoothness of solutions)**.** *Let* $\Omega \subseteq \mathbb{R}^3$ *open and* $\mathbf{B} \in H(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega)$ *and*

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= 0, \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

Then $\mathbf{B}$ *is smooth i.e. in* $C^\infty(\Omega;\mathbb{R}^3)$*.*

Proof. Take $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$. Then

$$\begin{aligned} 0 = \int_\Omega \operatorname{div} \mathbf{B} \operatorname{div} \boldsymbol{\varphi} + \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \boldsymbol{\varphi} dx &= -\int_\Omega \mathbf{B} \cdot (\operatorname{grad} \operatorname{div} \boldsymbol{\varphi} - \operatorname{curl} \operatorname{curl} \boldsymbol{\varphi}) dx \\ \overset{Lemma 5.1.5}{=} -\int_\Omega \mathbf{B} \cdot \begin{pmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \Delta\varphi_3 \end{pmatrix} &= \sum_{i=1}^{3} \int_\Omega \operatorname{grad} B_i \cdot \operatorname{grad} \varphi_i dx. \end{aligned}$$

Note that the last integration by parts is well defined because $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$ according to Thm. 5.1.4. So $\mathbf{B}$ is a weak solution of the elliptic system given by $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. Because we look at the homogenous problem our right hand side is obviously smooth and thus $\mathbf{B}$ is smooth as well due to Cor. 5.1.3. $\square$

Remark 5.1.7. This also works for any Ω open without regularity assumption on the boundary.

5.2 Existence and uniqueness

The curve integral condition is closely linked to the topology of our domain which we will have to use in our proof. This will rely on the tools of homology from Sec. 3. Because of this connection if we want the curve integral to give us uniqueness of the solution we need to assume certain topological properties. In our case, this will be the condition that our first homology group is generated by the chain that we are integrating over i.e.

$$H_1(\Omega) = \mathbb{Z}[\gamma].$$

With this assumption, we will first prove a existence and uniqueness result on the level of singular cohomology. This will lead to an analogous result for the de Rham cohomology. Once we have proven these propositions we can move on to prove existence and uniqueness of Problem 5.0.1.

{prop:uniqueness

Proposition 5.2.1. *Assume that $H_1(\Omega) = \mathbb{Z}[\gamma]$ i.e. the homology class of the closed 1-chain γ is a generator of the first homology group. Then we have the following:*

(i) For any $C_0 \in \mathbb{R}$ there exists a closed 1-cochain $F \in Z^1(\Omega)$ with $F(\gamma) = C_0$,

(ii) any other $G \in Z^1(\Omega)$ with $G(\gamma) = C_0$ is in the same cohomology class i.e. $[F] = [G]$

i.e. the cochain is unique up to cohomology.

Proof. **Proof of (i)** Because $[\gamma]$ is a generator of the homology group we obtain a homomorphism $\hat{F} \in \mathrm{Hom}(H_1(\Omega), \mathbb{R})$ by fixing $\hat{F}([\gamma]) = C_0$.

This determines the other values. Recall the isomorphism $\beta : H^1(\Omega) \to \mathrm{Hom}(H_1(\Omega); \mathbb{R})$ from the universal coefficent theorem (3.2.1) with $\beta([f])([c]) = f(c)$ (Note that $H^1(\Omega)$ is an abuse of notation because we mean here the first cohomology group of Ω and not the Sobolev space. In the context of singular homology this will always be the case.) Then we know that there exists a $[F] \in H^1(\Omega)$ with $\beta([F]) = \hat{F}$ because β is a isomorphism. So we obtain

$$F(\gamma) = \beta([F])([\gamma]) = \hat{F}([\gamma]) = C_0.$$

Proof of (ii) Take $[c] \in H_1(\Omega)$ arbitrary. Then there exists $n \in \mathbb{Z}$ s.t. $[c] = n[\gamma]$. Using β from (3.2.1) We have

$$\beta([F])([c]) = \beta([F])(n[\gamma]) = n\beta([F])([\gamma]) = n\,F(\gamma) = n\,G(\gamma) = \beta([G])([c])$$

and thus $\beta([F]) = \beta([G])$. Because β is an isomorphism we arrive at $[F] = [G]$. $\square$

This abstract topological result can now be linked to the differential forms via the de Rham isomorphism from Thm. 3.3.2. We will formulate it in a way that demonstrates the connection of differential forms and cochains.

{cor:existence_u

Corollary 5.2.2. *Assume $H_1(\Omega) = \mathbb{Z}[\gamma]$ as above. Then*

(i) For any $C_0 \in \mathbb{R}$ there exists a closed smooth 1-form $\theta \in \mathfrak{Z}^1(\Omega)$ with

$$I(\theta)(\gamma) = \int_\gamma \theta = C_0$$

(ii) any other $\eta \in \mathfrak{Z}^1(\Omega)$ *with*

$$I(\eta)(\gamma) = \int_\gamma \eta = C_0$$

is in the same cohomology class of $H^1_{dR}(\Omega)$ *i.e.* $[\eta] = [\theta]$*.*

Proof. **Proof of (i)** Recall from Sec. 3.3 that the integration of differential forms over chains induces an isomorphism on cohomology $[I] : H^1_{dR}(\Omega) \to H^1(\Omega)$ which we call de Rham isomorphism. We know from Prop. 5.2.1 that there exists $F \in H^1(\Omega)$ s.t. $F(\gamma) = C_0$. The surjectivity of the de Rham isomorphism now gives us $[\theta] \in H^1(\Omega)$ s.t.

$$[I(\theta)] = [I]([\theta]) = [F]$$

i.e.

$$I(\theta) = F + \partial^0 J$$

with $J \in C^0$ (here of course the zero cochains not continuous functions). Then,

$$I(\theta)(\gamma) = F(\gamma) + \partial^0 J(\gamma) = C_0 + J(\partial_1 \gamma) \overset{\gamma \text{ closed}}{=} C_0.$$

Proof of (ii) We have $I(\eta)$ is a 1-cochain with $I(\eta)(\gamma) = C_0$. Thus, we can apply Prop. 5.2.1 to get

$$[I]([\eta]) = [I(\eta)] = [I(\theta)] = [I]([\theta]).$$

Because $[I]$ is an isomorphism we can conclude $[\eta] = [\theta]$. □

{lem:1form_with_

Lemma 5.2.3. *Let* Ω *be an exterior domain i.e.* $\mathbb{R}^3 \setminus \Omega$ *is compact and assume that the first singular homology group* $H_1(\Omega)$ *is generated by* $[\gamma]$ *i.e.* $H_1(\Omega) = \mathbb{Z}[\gamma]$*. Take now a smooth* 1*-form* $\theta \in C^\infty\Lambda^1(\Omega)$*. Then there exists a* $\hat{\theta} \in C^\infty_b\Lambda^1(\Omega)$ *i.e. smooth with bounded support s.t.* $\theta = \hat{\theta} + d\mu$ *for some* $\mu \in C^\infty\Lambda^0(\Omega)$

Proof. Take a ball K_R centered at the origin with radius R large enough s.t. $\mathbb{R}^3 \setminus \Omega \subseteq K_R$. Denote $A := \mathbb{R}^3 \setminus \overline{K}_R$ and define

$$C^\infty\Lambda^k(\Omega, A) := \{\omega \in C^\infty\Lambda^k(\Omega) \mid \omega(x) = 0 \quad \forall x \in A\}.$$

for $k \in \mathbb{N}$. Note that this defines a cochain complex with the exterior derivative d. We denote the resulting cohomology groups as $H^1_{dR}(\Omega, A)$

Now recall the definition of a exact sequence from Def. 3.2.3. Denote $\iota : C^\infty\Lambda^k(\Omega, A) \hookrightarrow C^\infty\Lambda^k(\Omega)$ the inclusion operator and $\mathscr{R} : C^\infty\Lambda^k(\Omega) \to C^\infty\Lambda^k(A)$ the restriction operator. Then it is obvious to see that ι and $\mathscr{R}$ are cochain maps i.e. they commute with the exterior derivative and we also recognize that the sequence

$$0 \to C^\infty\Lambda^k(\Omega, A) \xrightarrow{\iota} C^\infty\Lambda^k(\Omega) \xrightarrow{\mathscr{R}} C^\infty\Lambda^k(A) \to 0$$

is an exact sequence. In this situation we can use Thm. V.5.6 from [3] to get a long exact sequence on the level of cohomology of which we take the following partial sequence

$$H^1_{dR}(\Omega, A) \xrightarrow{[\iota]} H^1_{dR}(\Omega) \xrightarrow{[\mathscr{R}]} H^1_{dR}(A).$$

We now need information about the homology group of A. We will only sketch the argument since it requires some notions that we did not introduce. A is a so called deformation retract of the sphere $\mathbb{S}_t$ around the origin with sufficiently large radius t. This can be seen by following the argument of Example I.14.7 in [3]. This implies that A is homotopy equivalent to $\mathbb{S}_t$ and thus the homology groups are isomorphic according to Prop. IV.6.3 in the same reference. The first homology group of the unit sphere $\mathbb{S}^2$ is zero and thus it is easy to check that $H_1(\mathbb{S}_t) = 0$ as well. This implies due to the universal coefficient theorem that $H^1(A) = 0$ and then $H^1_{dR}(A) = 0$ with the de Rham isomorphism.

The exactness now implies that $[\iota]$ is sujective. Thus there exists a closed $\hat{\theta} \in C^\infty\Lambda^1(\Omega, A)$ s.t. $[\theta] = [\iota]([\hat{\theta}]) = [\hat{\theta}]$ which is equivalent to the existence of a $\mu \in C^\infty\Lambda^0(\Omega)$ s.t.

$$\theta = \hat{\theta} + d\mu.$$

Since $\hat{\theta}$ has support in $\overline{K}_R$ the result follows. □

{thm:existence}

Theorem 5.2.4 (Existence of solution). *Let $\Omega \subseteq \mathbb{R}^3$ be such that $\mathbb{R}^3 \setminus \Omega$ is compact. For the topology, we require that $H_1(\Omega) = \mathbb{Z}[\gamma]$ for a closed smooth 1-chain γ. Assume further that there exists an ϵ-neighborhood*

$$\Omega_\epsilon := \{x \in \mathbb{R}^3 \mid d(x, \Omega) < \epsilon\}$$

s.t. $H_1(\Omega_\epsilon) = \mathbb{Z}[\gamma]$ as well. Then there exists a solution to Problem 5.0.1.

Let us say a view words about the topological assumption regarding Ω_ϵ. This just means that we can slightly increase the domain without changing the first homology group. As an example, think again of a torus in $\mathbb{R}^3$. Assuming the torus has non-empty interior we can slightly reduce the poloidal radius without changing the topology of its exterior domain.

Proof. At first we want to find a smooth differential 1-form $\theta \in C_b^\infty\Lambda^1(\overline{\Omega})$ with the desired curve integral. In order to do that we will increase the domain slightly. We start by referring to Cor. 5.2.2 to get a smooth closed differential 1-form $\tilde{\theta} \in \Lambda^1(\Omega_\epsilon)$ with

$$\int_\gamma \tilde{\theta} = C_0. \tag{5.2.1}$$

{eq:integral_the

Now we use Lemma 5.2.3 on Ω_ϵ to get a $\hat{\theta}$ which vanishes outside of a sufficiently large ball with a $\mu \in C^\infty\Lambda^0(\Omega_\epsilon)$ s.t. $\hat{\theta} + d\mu = \tilde{\theta}$. Notice that

$$\int_\gamma \tilde{\theta} = \int_\gamma \hat{\theta} = C_0.$$

since γ is closed.

We now refer back to Sec. 2.3 and change back to vector proxies. Let $\tilde{\boldsymbol{\phi}}$ be the vector proxy of $\hat{\theta}$ i.e. by recalling the musical isomorphism from Sec. 2.3 $\hat{\theta}^\sharp = \tilde{\boldsymbol{\phi}} \in C^\infty(\Omega_\epsilon; \mathbb{R}^3)$. Because $\hat{\theta}$ is closed and (5.2.1) holds we obtain the corresponding properties of $\tilde{\boldsymbol{\phi}}$ using Example 2.4.2 and the connection of the curl with the exterior derivative,

$$\int_\gamma \tilde{\boldsymbol{\phi}} \cdot dl = \int_\gamma \hat{\theta} = C_0$$
$$\operatorname{curl} \tilde{\boldsymbol{\phi}} = (\star d\boldsymbol{\phi}^\flat)^\sharp = (\star d\hat{\theta})^\sharp = 0.$$

We define $\boldsymbol{\phi} \in C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ by restricting $\tilde{\boldsymbol{\phi}}$ to $\overline{\Omega}$.

Since $\boldsymbol{\phi}$ is curl-free and in $L^2(\Omega)$ we can use the Hodge decomposition to find $\mathbf{B} = \boldsymbol{\phi} - \lim_{i\to\infty} \operatorname{grad} \psi_i$ with $\psi_i \in H^1(\Omega)$ (here $H^1(\Omega)$ is the Sobolev space and not the cochain cohomology).

Define $\Omega_R := \Omega \cap K_R$ with K_R the ball around the origin with radius R large enough s.t. $\mathbb{R}^3 \setminus \Omega \subseteq K_R$ and $\gamma \subseteq K_R$. We know from Thm. 4.2.5 that $\operatorname{grad} H^1(\Omega_R)$ is closed in L^2 because Ω_R is bounded. So we get that

$$\lim_{i\to\infty} \operatorname{grad} \psi_i|_{\Omega_R} = \operatorname{grad} \psi_R$$

with some $\psi_R \in H^1(\Omega_R)$. We also know that $\mathbf{B}$ is smooth because it is curl and divergence free from Thm. 5.1.6. Because $\mathbf{B}$ and $\boldsymbol{\phi}$ are smooth ψ_R must be smooth as well and so we have

$$\int_\gamma \mathbf{B} \cdot dl = \int_\gamma \boldsymbol{\phi} \cdot dl.$$

Thus we see that $\mathbf{B}$ has the desired curve integral. Since $\mathbf{B} \in \mathfrak{H}^1 \subseteq H_0(\operatorname{div})$ it has zero normal trace is curl as well as divergence free. Hence, $\mathbf{B}$ is a solution to our problem 5.0.1. $\square$

In the proof of uniqueness we will use the following lemma.

{lem:gradient_se

Lemma 5.2.5. *Let* $\phi \in L^2_{loc}(\Omega)$ *with* $\operatorname{grad}\phi \in L^2(\Omega;\mathbb{R}^3)$. *Then there exists a sequence* $(\phi_i)_{i\in\mathbb{N}} \subseteq H^1(\Omega)$ *s.t.* $\operatorname{grad}\phi_i \to \operatorname{grad}\phi$ *in* $L^2(\Omega;\mathbb{R}^3)$.

Proof. Take K_R the open ball around the origin with R large enough s.t. $\mathbb{R}^3 \setminus (K_R) \subseteq \Omega$. Define $\Omega_R := K_R \cap \Omega$. Ω_R is a bounded Lipschitz domain. Then we can define $\phi_R := \phi|_{\Omega_R}$.

We want to show $\phi_R \in H^1(\Omega_R)$. We alredy know that $\operatorname{grad}\phi_R \in L^2(\Omega,\mathbb{R}^3)$. Since $L^2(\Omega_R;\mathbb{R}^3) \subseteq H^{-1}(\Omega_R;\mathbb{R}^3)$ we can conclude that $\phi_R \in L^2$ because Ω_R is bounded and Lipschitz ([10, Lemma 3.11]) and thus $\phi_R \in H^1(\Omega_R)$. This essentially means that if $\phi \notin L^2(\Omega)$ it is due to its behavious at infinity.

Since Ω_R is bounded and Lipschitz we can find an extension $\bar{\phi}_R \in H^1(\mathbb{R}^3)$ s.t. $\bar{\phi}_R|_{\Omega_R} = \phi_R$ (cf. [9, Sec. 1.5.1]). So we can now define

$$\bar{\phi} := \begin{cases} \phi & \text{in } \Omega \\ \bar{\phi}_R & \text{in } \Omega^c. \end{cases}$$

Then $\bar{\phi} \in L^2_{loc}(\mathbb{R}^3)$ and $\operatorname{grad}\bar{\phi} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Then there exists a sequence $(\phi_l)_{l\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^3)$ s.t. $\operatorname{grad}\phi_l \to \operatorname{grad}\bar{\phi}$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ (cf. [12, Lemma 1.1]). By restricting ϕ_l to Ω we obtain the result. □

Theorem 5.2.6. *Let the same assumptions hold as in Thm. 5.2.4. Then the solution of the problem is unique.*

Proof. Let $\mathbf{B}$ and $\tilde{\mathbf{B}}$ both be solutions and denote with ω and $\tilde{\omega}$ the corresponding 1-forms i.e. $\omega = \mathbf{B}^\flat$ and $\tilde{\omega} = \tilde{\mathbf{B}}^\flat$. So we have $I(\omega)(\gamma) = I(\tilde{\omega})(\gamma) = C_0$ since

$$\int_\gamma \omega = \int_\gamma \mathbf{B}^\flat = \int_\gamma \mathbf{B}\cdot dl = \int_\gamma \tilde{\mathbf{B}}\cdot dl = \int_\gamma \tilde{\mathbf{B}}^\flat = \int_\gamma \tilde{\omega}$$

Then we know from Cor. 5.2.2 that ω and $\tilde{\omega}$ are in the same cohomology class in H^1_{dR}. So there exists a 0-form i.e. $\mu \in C^\infty(\Omega)$ s.t. $\omega - \tilde{\omega} = d\mu$. By applying $\sharp$ on both sides

$$\mathbf{B} - \tilde{\mathbf{B}} = \omega^\sharp - \tilde{\omega}^\sharp = (d\mu)^\sharp = \operatorname{grad}\mu.$$

However, μ need not be in L^2 since Ω is unbounded. But we know that $\operatorname{grad}\mu \in L^2(\Omega)^3$ and $\mu \in L^2_{loc}(\Omega)$. Here we can now apply Lemma 5.2.5 and conclude

$$\mathbf{B} - \tilde{\mathbf{B}} \in \overline{\operatorname{grad} H^1(\Omega)}.$$

Now remembering the Hodge decomposition in the 3D case (Thm. 4.2.6) we know

$$\mathbf{B} - \tilde{\mathbf{B}} \in \overline{\operatorname{grad} H^1(\Omega)}^{\perp}$$

because $\mathbf{B}, \tilde{\mathbf{B}} \in \mathfrak{H}^1$. Thus, $\mathbf{B} = \tilde{\mathbf{B}}$ which concludes the proof of uniqueness. $\square$