1 Variational formulation of the magnetostatic problem in 2D

For simplicity we will now turn to the 2D case and we assume that our open domain will be bounded and Lipschitz. This involves to introduce a different Hilbert complex with other differential operators. Then we derive the 2D magnetostatic problem from the three-dimensional one. We will assume our domain Ω to have a "annulus like" form which we will clarify in more rigour. In order for the 2D magnetostatic problem to be well-posed we require an additional constraint that will again be a curve integral. We will investigate an alternative way to represent this curve integral which will turn out to be easily suitable to be included in our numerical approximation.

1.1 The curl-div Hilbert complex

{sec:variational

We start with the introduction of the relevant differential operators and the resulting 2D Hilbert complex. We will then explain what domains we will consider and state the magnetostatic problem in strong form.

We define the scalar curl for $\mathbf{v} \in C^1(\Omega; \mathbb{R}^2)$ as

$$\operatorname{curl} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

Additionally, we have the vector-valued curl, denoted in bold, defined for $v \in C^1(\Omega)$

$$\mathbf{curl}\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

The cross product for 2D reads for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$,

$$\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1.$$

A straightforward calculation shows that the following integration-by-parts formula holds for $u \in C^1(\overline{\Omega})$, $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^2)$, assuming Ω is Lipschitz and bounded

$$\int_\Omega \mathbf{curl}\, u \cdot \mathbf{v}\, dx = \int_\Omega u \operatorname{curl} \mathbf{v}\, dx + \int_{\partial\Omega} u\, \mathbf{v} \times \mathbf{n}\, d\ell \qquad (1.1.1)$$

{eq:2D_integrati

where $\mathbf{n}$ is the outward unit normal of Ω. Analogous to what we did in Sec. ?? we can now extend this definition in the weak sense. First, notice that $\mathbf{curl}\, u = R_{-\pi/2} \operatorname{grad} u$ and thus $\mathbf{curl}$ is well-defined on H^1. For the scalar curl we define

$$H(\text{curl};\Omega) = \{\mathbf{v} \in L^2 \mid \exists w \in L^2 : \int_\Omega w\phi\, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\,\phi\, dx \quad \forall \phi \in C_0^\infty\}$$

and we call denote w in the definition – which is uniquely determined – as $\operatorname{curl} \mathbf{v}$, so in short $(\text{curl}, H(\text{curl})) = (\mathbf{curl}, C_0^\infty)^*$. Analogous to Section **??**, it is then possible to extend the tangential trace to an operator γ_τ defined on $H(\text{curl})$ s.t. for any $u \in H^1(\Omega)$, $\mathbf{v} \in H(\text{curl})$ the integration by parts formula

$$\langle \mathbf{curl}\, u, \mathbf{v}\rangle = \langle u, \operatorname{curl} \mathbf{v}\rangle + \langle \gamma_\tau \mathbf{v}, \operatorname{tr} u\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.$$

holds. From now on we will leave out the subindex of the duality inner product. Also analogous to the 3D case, we can define

$$H_0(\text{curl}) := \{\mathbf{v} \in H(\text{curl}) \mid \gamma_\tau \mathbf{v} = 0\}$$

and can then compute the adjoints – analogous to what we did in **??**

$$\begin{aligned}(\text{curl}, H_0(\text{curl})) &= (\mathbf{curl}, H^1)^* \\ (\text{curl}, H(\text{curl})) &= (\mathbf{curl}, H_0^1)^*.\end{aligned}$$

Notice that $\operatorname{div} \mathbf{curl} = 0$ and so we have the following 2D Hilbert complex

$$H_0^1 \xrightarrow{\mathbf{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L^2. \qquad (1.1.2)$$ {eq:2D_hilbert_c

and the dual complex

$$L^2 \xleftarrow{\text{curl}} H(\text{curl}) \xleftarrow{-\text{grad}} H^1$$

We use the notation introduced in Sec. **??** for general Hilbert complexes i.e. $V^0 = H_0^1$, $V^1 = H_0(\text{div})$, $V_1^* = H(\text{curl})$ and $d^0 = \mathbf{curl}$ and $d^1 = \text{div}$. Also we remind of the notation $\mathfrak{B}^k$ for the image of the differential operator, $\mathfrak{B}_k^*$ for the image of the adjoint and analogous $\mathfrak{Z}^k$ for the kernel.

1.2 Strong formulation of the 2D magnetostatic problem

The 2D magnetostatic problem will be derived from a special case of the 3D problem. Then the type of domains considered will be clarified and the strong formulation stated at the end.

Assume that our current source $\mathbf{J}$ is pointing in z-direction i.e. $\mathbf{J} = J\mathbf{e}_z$. Further assume that there is a $\tilde{\Omega}$ s.t. $\Omega = \tilde{\Omega} \times \mathbb{R}$. Due to symmetry we can then assume further that B_3 does not change in z-direction which implies that

$$0 = \operatorname{div} \mathbf{B} = \partial_x B_1 + \partial_y B_2 = \operatorname{div} \tilde{\mathbf{B}}.$$

where $\tilde{\mathbf{B}} = (B_1, B_2)^\top$. The third component of the equation $\operatorname{curl} \mathbf{B} = \mathbf{J}$ from the magnetostatic problem reads

$$J = \partial_x B_2 - \partial_y B_1 = \operatorname{curl} \tilde{\mathbf{B}}$$

For Ω the unit outer normal is zero in z-direction and thus $\tilde{\mathbf{B}}$ satisfies the boundary condition

$$0 = \mathbf{B} \cdot \mathbf{n} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{n}}$$

with $\tilde{\mathbf{n}} = (n_1, n_2)^\top$ being the outer unit normal $\tilde{\Omega}$.

Now we will abuse notation and refer to $\tilde{\mathbf{B}}$ as $\mathbf{B}$, $\tilde{\mathbf{n}}$ as $\mathbf{n}$ and $\tilde{\Omega}$ as Ω. Let $J \in L^2$ be given. Then we see that $\mathbf{B}$ must fulfill the following equations

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= J, \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

Depending on the domain, this problem is in general not well-posed – just as the problem in 3D – and requires an additional constraint. Let us now make certain restrictions on what type of domain we will consider.

From now on, we assume that the space of harmonic forms $\mathfrak{H}^1$ has dimension one and that our domain is encompassed by two disjoint closed curves (cf. Fig.**??**) i.e. we have curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$ s.t. $\partial\Omega_{in} \dot\cup \partial\Omega_{out}$ is the boundary of Ω. Let now Γ be s.t. it is a closed curve in Ω that goes around the hole in the middle i.e. the area surrounded by Γ contains $\partial\Omega_{in}$. Denote its parametrization with $\boldsymbol{\gamma} : [0, |\Gamma|] \to \Omega$ s.t. $|\boldsymbol{\gamma}'(t)| = 1$ and assume that $\boldsymbol{\gamma}$ is bijective i.e. the curve does not intersect itself. We assume that Γ has positive distance from $\partial\Omega_{in}$. We do not assume anything like that for the exterior boundary i.e. Γ can touch or be identical to $\partial\Omega_{out}$. We then denote the area that is enclosed by Γ and $\partial\Omega_{in}$ as Ω_Γ.

From now on, our domain Ω is always assumed to be of that kind. We will later make further restrictions on what types of domain we will consider that will be suitable for discretization (see **??**).

We add the curve integral along Γ, which we assume to be well-defined, as an additional constraint. So in total, we obtain the following problem.

{prob:2d_magneto

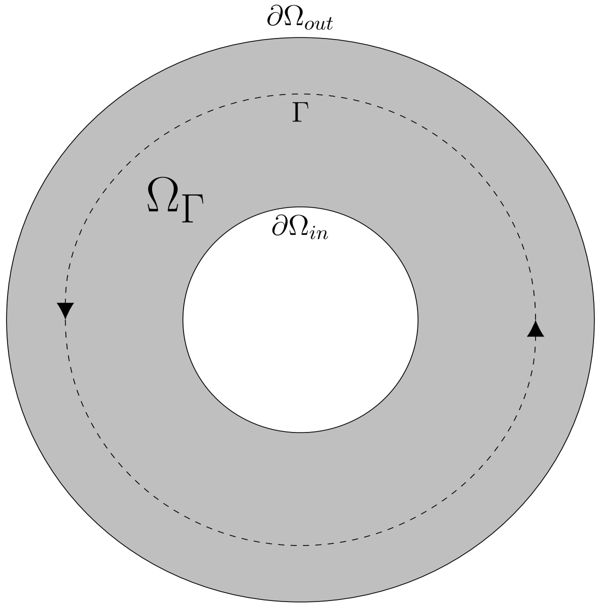

Figure 1: Does this really work? {fig:annulus_dom

Problem 1.2.1 (2D magnetostatic problem)**.** Given $J \in L^2$ and $C_0 \in \mathbb{R}$, find $\mathbf{B} \in H_0(\mathrm{div}) \cap H(\mathrm{curl})$ s.t.

$$\operatorname{curl} \mathbf{B} = J,$$
$$\operatorname{div} \mathbf{B} = 0,$$
$$\int_\Gamma \mathbf{B} \cdot \, dl = C_0$$

Another option for the additional constraint would be an orthogonality constraint as discussed in [2, Sec. 3.5].

1.3 Mixed formulation

In order to solve this problem numerically using finite elements we have to choose a suitable variational formulation of the problem. This variational formulation will be stated – without the curve integral constraint – and then we will show the equivalence with the strong formulation.

Ignoring the curve integral at first, we will use the following. We choose a non-zero harmonic form $\mathbf{p} \in \mathfrak{H}^1$ and have $J \in L^2$. Then the problem is: Find $\sigma \in H_0^1$, $B \in H_0(\mathrm{div})$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \qquad (1.3.1)$$ {eq:first_eq_mix

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{v} \rangle + \lambda \langle \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in H_0(\mathrm{div}) \qquad (1.3.2)$$ {eq:second_eq_mi

As before, the inner product without subscript denotes the L^2 inner product and $\|\cdot\|$ the L^2 norm. Here the curve integral condition is missing. It is difficult to include the curve integral condition directly when solving this system numerically. So we will reformulate it below in Sec. 1.4.

Even though this formulation appears more complicated in comparison to the first two equations of the 2D magnetostatic problem (Problem 1.2.1), it will turn out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will first investigate the formulation without curve integral

Proposition 1.3.1. *For any $J \in L^2$, (1.3.1) and (1.3.2) hold i.i.f. $\sigma = 0$, $\lambda = 0$, $\operatorname{curl} \mathbf{B} = J$ and $\operatorname{div} \mathbf{B} = 0$ i.e. $\mathbf{B}$ solves the 2D magnetostatic problem (Problem 1.2.1) without the additional curve integral constraint.*

Proof. Assume $(\sigma, \mathbf{B}, \lambda)$ is a solution of (1.3.1) and (1.3.2). Then the first equation is

$$\langle \sigma + J, \tau \rangle = \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle \quad \forall \tau \in H_0^1$$

which is equivalent to $\mathbf{B} \in H(\text{curl})$ and $J + \sigma = \operatorname{curl} \mathbf{B}$.

Now assume additionally, that (1.3.2) holds. Then by choosing $\mathbf{v} = \mathbf{p} \in \mathfrak{H}^1$, we get $\operatorname{div} \mathbf{p} = 0$ from the definition of the harmonic forms and $\mathfrak{H}^1 \perp \mathbf{curl}\, H_0^1$ from the Hodge decomposition and thus

$$\langle \mathbf{curl}\, \sigma, \mathbf{p} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{p} \rangle + \lambda \langle \mathbf{p}, \mathbf{p} \rangle = \lambda \langle \mathbf{p}, \mathbf{p} \rangle = 0$$

and so $\lambda = 0$. Then we can choose $\mathbf{v} = \mathbf{curl}\, \sigma$ to get

$$\langle \mathbf{curl}\, \sigma, \mathbf{curl}\, \sigma \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{curl}\, \sigma \rangle + \lambda \langle \mathbf{p}, \mathbf{curl}\, \sigma \rangle = \|\mathbf{curl}\, \sigma\|^2 = 0.$$

Because $\sigma \in H_0^1$ this gives us $\sigma = 0$. Also we have then $J = \operatorname{curl} \mathbf{B}$. At last we choose $\mathbf{v} = \mathbf{B}$ which gives us $\operatorname{div} \mathbf{B} = 0$ and thus we proved the first direction.

The other implication is clear i.e. if $\mathbf{B} \in H(\text{curl}) \cap H_0(\text{div})$ with $\operatorname{curl} \mathbf{B} = J$ and $\operatorname{div} \mathbf{B} = 0$ then the variational formulation clearly holds. □

Notice that the variable λ is not necessary for this variational formulation, but we will need it later, since we will add another equation representing the curve integral constraint and hence we need another variable to have the same number of unknowns and equations. If we now add the same additional constraint to both formulations of the problem then they will remain equivalent.

1.4 Curve integral constraint

{sec:curve_integ

We still need to find a good way to include the curve integral constraint from Problem 1.2.1 in our formulation. Instead of incorporating it directly, we will substitute it with another equation. We will first derive this equation as an immediate consequence of the integration by parts formula (1.1.1) and then state the final variational formulation of the 2D magnetostatic problem which we will investigate in the coming sections.

Because $\mathbf{n}$ is the unit outward normal of Ω_Γ and $\boldsymbol{\gamma}$ the parametrization of Γ that $\mathbf{n} \perp \boldsymbol{\gamma}'$ and

$$\mathbf{B} \times \mathbf{n} = (B_1 n_2 - B_2 n_1) = \mathbf{B} \cdot \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} = -\mathbf{B} \cdot \mathbf{R}_{\pi/2}\mathbf{n}.$$

$\mathbf{R}_{\pi/2}\mathbf{n}$ is either $\boldsymbol{\gamma}'$ or $-\boldsymbol{\gamma}'$. Assume w.l.o.g. that $\mathbf{R}_{\pi/2}\mathbf{n} = \boldsymbol{\gamma}'$ and thus

$$\mathbf{B} \times \mathbf{n} = -\mathbf{B} \cdot \boldsymbol{\gamma}'$$

and so the curve integral becomes

$$\int_\Gamma \mathbf{B} \cdot dl = \int_0^{|\Gamma|} \mathbf{B}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = \int_0^{|\Gamma|} \mathbf{n}(\boldsymbol{\gamma}(t)) \times \mathbf{B}(\boldsymbol{\gamma}(t))\, dt = -\int_\Gamma \mathbf{B} \times \mathbf{n} d\ell.$$

Choose $\psi \in H^1$ s.t. $\psi = 0$ on $\partial\Omega_{in}$, $\psi = 1$ on $\partial\Omega_{out}$ and $\psi \equiv 1$ in $\Omega \setminus \Omega_\Gamma$ Then we observe

$$\int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \psi\, J\, dx + \int_{\partial\Omega} \mathbf{B} \times \mathbf{n}\, dl = \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Gamma \mathbf{B} \cdot dl$$

So if the curve integral

$$\int_\Gamma \mathbf{B} \cdot dl = C_0$$

is given and we can compute $\int_{\Omega_\Gamma} \psi\, J\, dx$ we can add the equation

$$\langle \mathbf{curl}\,\psi, \mathbf{B} \rangle = C_1 \tag{1.4.1}$$ {eq:variational_

with

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0$$

to our system.

From the above derivations it is also clear that for $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$

$$\int_\Gamma \mathbf{B} \cdot dl = C_0 \Leftrightarrow \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1.$$

This is the motivation why it makes sense to add the right equation to our system instead of the curve integral since it is much easier to enforce numerically.

In order to get a variational formulation to study theoretically, we multiply (1.4.1) with an arbitrary $\mu \in \mathbb{R}$. In conclusion, we have the following variational problem:

{prob:magnetosta

Problem 1.4.1. Let $J \in L^2$, $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Find $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\mathrm{div})$, $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle &= -\langle J, \tau \rangle && \forall \tau \in H_0^1, && (1.4.2) \\ \langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle &= 0 && \forall \mathbf{v} \in H_0(\mathrm{div}), && (1.4.3) \\ \mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle &= \mu C_1 && \forall \mu \in \mathbb{R}. && (1.4.4) \end{aligned}$$

{eq:first_eq_mix
{eq:second_eq_mi

which gives us the variational formulation of the magnetostatic problem with curve integral constraint (Problem 1.2.1). We will study the well-posedness of this formulation next.

1.5 Well-posedness of the magnetostatic system

{sec:well-posedn

The well-posedness is based on the well-known Banach-Nečas-Babuška (BNB) theorem concerning general variational problems of the following form: Find $x \in X$ s.t.

$$a(x, y) = \ell(y) \quad \forall y \in Y \qquad (1.5.1)$$

{eq:general_vari

where X and Y are Banach spaces, a is a bilinear form and $\ell \in Y'$. The BNB-theorem then answers the question of well-posedness i.e. if there exists a unique solution and if we can find a stability estimate. This formulation is from [3, Sec. 25.3] in the real case.

{thm:BNB}

Theorem 1.5.1 (BNB)**.** *Let X be a Banach space and Y be a reflexive Banach space. Let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and $\ell \in Y'$. Then a problem of the form (1.5.1) is well-posed i.i.f. the following two criteria are fulfilled*

(1) $$\inf_{x \in X} \sup_{y \in Y} \frac{|a(x, y)|}{\|x\|_X \|y\|_Y} =: \gamma > 0 \qquad (1.5.2)$$ {eq:BNB1}

(2) *for any $y \in Y$ if $a(x, y) = 0$ for every $x \in X$, then $y = 0$.* (1.5.3) {eq:BNB2}

We obtain the stability estimate for a solution x

$$\|x\|_X \leq \frac{1}{\gamma}\|\ell\|_{Y'}.$$

Note that (1.5.2) is equivalent to the fact that for any $x \in X \setminus \{0\}$ there exists $y \in Y \setminus \{0\}$ s.t. $a(x, y) \geq \gamma\|x\|_X\|y\|_Y$.

Since we are dealing with Hilbert spaces only we can utilize the following proposition to prove it (see [3, Rem. 25.14]).

{prop:T_coercivi

Proposition 1.5.2 (T-coercivity)**.** *Let X and Y be Hilbert spaces. Then* (1.5.2) *and* (1.5.3) *hold, if there exists a bounded bijective operator $T : X \to Y$ s.t.*

$$a(x, Tx) \geq \eta\|x\|_X^2 \quad \forall x \in X. \tag{1.5.4}$$

{eq:T_coercivity

Then γ from (1.5.2) *can be chosen as $\eta/\|T\|_{\mathcal{L}(X,Y)}$.*

Proof. For any $x \in X$, by taking $y = Tx \in Y$ and using the boundedness of T we have

$$a(x, T(x)) \geq \eta\|x\|^2 \geq \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}}\|x\|_X\|y\|_Y$$

and thus (1.5.2) holds with $\gamma = \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}}$.

For (1.5.3) assume that we have $y \in Y$ s.t. $a(x, y) = 0$ for all $x \in X$.

$$0 = a(T^{-1}y, TT^{-1}y) \geq \eta\|T^{-1}y\|_X^2$$

so $T^{-1}y = 0$ and thus $y = 0$. □

Remark 1.5.3. The other direction is also true i.e. if (1.5.2) and (1.5.3) are fulfilled we can construct a T with the desired properties.

Note also that when we have found T s.t. (1.5.4) holds then it must be injective. This is because if $Tx = 0$ for any $x \in X$ then $x = 0$ follows from the T-coercivity.

The next step is to put our formulation of Problem 1.4.1 into this general framework. To this end, we define $X := H_0^1 \times H_0(\mathrm{div}) \times \mathbb{R}$ and for $(\sigma, \mathbf{B}, \lambda) \in X$

$$\|(\sigma, \mathbf{B}, \lambda)\|_X := \sqrt{\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2}$$

and the bilinear form $a : X \times X \to \mathbb{R}$

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \langle\sigma, \tau\rangle - \langle\mathbf{B}, \mathbf{curl}\,\tau\rangle + \langle\mathbf{curl}\,\sigma, \mathbf{v}\rangle + \langle\mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathbf{v}\rangle + \langle\lambda\mathbf{p}, \mathbf{v}\rangle - \mu\langle\mathbf{curl}\,\psi, \mathbf{B}\rangle. \tag{1.5.5}$$

{eq:definition_b

and

$$\ell(\tau, \mathbf{v}, \mu) = -\langle J, \tau \rangle - \mu C_1.$$

Then Problem 1.4.1 is equivalent to the following: Find $(\sigma, \mathbf{B}, \lambda) \in X$ s.t.

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \ell(\tau, \mathbf{v}, \mu) \quad \forall (\tau, \mathbf{v}, \mu) \in X.$$

Note that the bilinear form a is not symmetric.

We need to show some things about $\mathbf{curl}\, \psi$.

Proposition 1.5.4. *Under the given assumptions on ψ, $\mathbf{curl}\, \psi \in H_0(\mathrm{div})$.*

Proof. We need to show $0 = \gamma_n \mathbf{curl}\, \psi$. Recall that the definition of

$$\langle \gamma_n \mathbf{curl}\, \psi, \mathrm{tr}\, u \rangle = \int_\Omega \mathbf{curl}\, \psi \cdot \mathrm{grad}\, u \, dx + \int_\Omega \mathrm{div}\, \mathbf{curl}\, \psi \, u \, dx$$

where the last term vanishes. Take now $\phi \in C^1(\overline{\Omega})$ arbitrary. Then we take $\phi_1 \in C^1(\overline{\Omega})$ s.t. $\phi_1 = \phi$ in a neighborhood of $\partial\Omega_{in}$ and zero near $\partial\Omega_{out}$. Then $\phi_2 \in C^1(\overline{\Omega})$ the other way around. Here we used the fact that the two parts of the boundary are disjoint and have positive distance from one another. Then also $\mathrm{tr}\, \phi = \mathrm{tr}\, \phi_1 + \mathrm{tr}\, \phi_2$. We use the integration by parts formula **??**, $\mathrm{div}\, \mathbf{curl} = 0$ and the integration-by-parts formula for the curl

$$\begin{aligned}
&\langle \gamma_n \mathbf{curl}\, \psi, \mathrm{tr}\, \phi \rangle = \langle \gamma_n \mathbf{curl}\, \psi, \mathrm{tr}\, \phi_1 \rangle + \langle \gamma_n \mathbf{curl}\, \psi, \mathrm{tr}\, \phi_2 \rangle \\
&= \int_\Omega \mathbf{curl}\, \psi \cdot \mathrm{grad}\, \phi_1 \, dx + \int_\Omega \mathbf{curl}\, \psi \cdot \mathrm{grad}\, \phi_2 \, dx \\
&= \int_\Omega \psi \cdot \mathrm{curl}\, \mathrm{grad}\, \phi_1 \, dx + \langle \gamma_\tau \mathrm{grad}\, \phi_1, \psi \rangle + \int_\Omega \psi \cdot \mathrm{curl}\, \mathrm{grad}\, \phi_2 \, dx + \langle \gamma_\tau \mathrm{grad}\, \phi_2, \psi \rangle.
\end{aligned}$$

Now remember that because $\phi_j \in C^1(\overline{\Omega})$, $\langle \gamma_\tau \mathrm{grad}\, \phi_j, \psi \rangle = \langle \mathrm{grad}\, \phi_j \times \mathbf{n}, \psi \rangle_{L^2(\partial\Omega)}$. So

$$\langle \gamma_\tau \mathrm{grad}\, \phi_1, \psi \rangle = \langle \mathrm{grad}\, \phi_1 \times \mathbf{n}, \psi \rangle_{L^2(\partial\Omega)} = 0$$

because ϕ_1 is zero near $\partial\Omega_{out}$ and ψ is zero on $\partial\Omega_{in}$. For the remaining term,

$$\langle \gamma_\tau \mathrm{grad}\, \phi_2, \psi \rangle = \int_{\partial\Omega_{out}} \psi \, \mathrm{grad}\, \phi_1 \times \mathbf{n} d\ell = \int_{\partial\Omega_{out}} \mathrm{grad}\, \phi_1 \times \mathbf{n} d\ell = -\int_{\partial\Omega_{out}} \mathrm{grad}\, \phi_1 \cdot d\ell$$

using a counter clockwise parametrization of $\partial\Omega_{out}$ for a parametrization $\mathbf{s}$ i.e. $\mathrm{grad}\, \phi_1 \times \mathbf{n} = -\,\mathrm{grad}\, \phi_1 \cdot \mathbf{R}_{\pi/2}\mathbf{s}'$ and then we know from basic vector calculus because $\partial\Omega_{out}$ is closed

$$\int_{\partial\Omega_{out}} \mathrm{grad}\, \phi_1 \cdot d\ell = 0.$$

and so in conclusion, $\gamma_n \mathbf{curl}\, \psi = 0$. □

Usually the last equation in Problem 1.4.1 is used to determine the harmonic part of the solution. This implies that we would like $\mathbf{curl}\,\psi$ to have non-vanishing harmonic part. This is indeed true.

Proposition 1.5.5. *Let $Q^1_{\mathfrak{H}} : L^2 \to \mathfrak{H}^1$ be the orthogonal projection onto the harmonic forms. Then with ψ defined as above we have $Q^1_{\mathfrak{H}}\,\mathbf{curl}\,\psi \neq 0$.*

Proof. Since $\operatorname{div}\mathbf{curl}\,\psi = 0$ we know that

$$\mathbf{curl}\,\psi \in \mathfrak{Z}^1 = \mathfrak{B}^1 \overset{\perp}{\oplus} \mathfrak{H}^1$$

using the Hodge decomposition (cf. Thm. **??**). Assume for contradiction that $\mathbf{curl}\,\psi \in \mathfrak{B}^1$ i.e. there exists $\psi_0 \in H^1_0$ s.t. $\mathbf{curl}\,\psi_0 = \mathbf{curl}\,\psi$. Since $\mathbf{curl}$ is just the rotated gradient we would get that $\operatorname{grad}(\psi - \psi_0) = 0$ and thus $\psi - \psi_0$ is constant almost everywhere. But this is a contradiction since $\operatorname{tr}\psi_0$ is zero on $\partial\Omega_{in}$ and $\partial\Omega_{out}$, but $\operatorname{tr}\psi = 0$ on $\partial\Omega_{in}$ and $\operatorname{tr}\psi = 1$ on $\partial\Omega_{in}$. Thus $\mathbf{curl}\,\psi \notin \mathfrak{B}^1$ and the claim follows. □

Now we can apply the Hodge decomposition of the ker div on $\mathbf{curl}\,\psi$ and obtain the following

Corollary 1.5.6. *Let $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Then there exists $\psi_0 \in H^1_0$ and $c_\psi \in \mathbb{R} \setminus \{0\}$ s.t.*

$$\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}.$$

Because we can choose $\mathbf{p}$ we can assume w.l.o.g. that $c_\psi > 0$ and we will do so from now on.

As stated in the proof of the Poincare inequality **??** the $\mathbf{curl}|_{\mathfrak{Z}^\perp} : \mathfrak{Z}^\perp \to \mathfrak{B}^1$ is bijective and since it is bounded w.r.t. the V-norm – which is the H^1-norm here – due to Banach inverse theorem it is invertible and we denote this inverse $\mathbf{curl}^{-1}$. This is a slight abuse of notation since it is not really the inverse of the full $\mathbf{curl}$.

Let $Q_{\mathfrak{B}}$ be the L^2-orthogonal projection onto $\mathfrak{B}^j$, we then denote $\mathbf{v}_{\mathfrak{B}} = Q_{\mathfrak{B}}\mathbf{v}$ for any $\mathbf{v} \in L^2$ and analogous for $\mathfrak{H}^j$ and $\mathfrak{B}^*_j$. In order to prove the T-coercivity, we need the following lemma.

{lem:T_for_T_coe

Lemma 1.5.7. *Take $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ with $c_\psi > 0$,$\mathbf{p} \in \mathfrak{H}^1$ and $\|\mathbf{p}\| = 1$. Define $T : X \to X$ as*

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}).$$

with $\alpha < 0$ and $\beta > 0$. Then T is bounded and surjective.

Proof. The boundedness is clear since all operators used in the definition are bounded w.r.t. the norms of their domains. From the Poincaré inequality we know that $\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\| \leq c_P\|\mathbf{B}_{\mathfrak{B}}\|$ and so

$$\begin{aligned}
&\|T(\sigma,\mathbf{B},\lambda)\|_X^2 = \|\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + \|\mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}\|_{H(\mathrm{div})}^2 + \left(\alpha\langle\mathbf{p},\mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right)^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^4}\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}\|_{H(\mathrm{div})}^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq C_T\left(\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2\right)
\end{aligned}$$

with

$$C_T := \max\left\{5, \frac{2}{c_P^2} + 3 + 2\alpha^2, 3\beta^2 + \frac{2}{c_\psi^2}\right\}. \tag{1.5.6}$$

{eq:bound_on_norr

So T is bounded and $\|T\|_{\mathcal{L}(X,X)} \leq \sqrt{C_T}$.

In order to prove surjectivity, we will split up $\mathbf{v} = \mathbf{v}_{\mathfrak{B}} + \mathbf{v}_{\mathfrak{H}} + \mathbf{v}_{\mathfrak{B}^*}$ using the Hodge decomposition. Take $(\tau,\mathbf{v},\mu) \in X$ arbitrary and choose

$$\sigma = (1 + \frac{1}{c_P^2})^{-1}(\tau + \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}}) \text{ and } \mathbf{B}_{\mathfrak{B}} = \mathbf{v}_{\mathfrak{B}} - \mathbf{curl}\,\sigma.$$

So

$$\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}} = \sigma - \frac{1}{c_P^2}(\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}} - \sigma) = (1 + \frac{1}{c_P^2})\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}} = \tau.$$

We simply choose $\mathbf{B}_{\mathfrak{B}^*} = \mathbf{v}_{\mathfrak{B}^*}$. For the harmonic part take κ_v s.t. $\mathbf{v}_{\mathfrak{H}} = \kappa_v\mathbf{p}$. Let us look at the system

$$\begin{pmatrix} 1 & \beta \\ \alpha & 1/c_\psi \end{pmatrix}\begin{pmatrix} \kappa_B \\ \lambda \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \mu \end{pmatrix}$$

Now since $c_\psi > 0$ and $\alpha < 0$, $\beta > 0$ we get $1/c_\psi - \alpha\beta \neq 0$ and the system has a solution. Choose $\mathbf{B}_{\mathfrak{H}} = \kappa_B\mathbf{p}$. Then we see

$$\mathbf{v}_{\mathfrak{H}} = \kappa_v\mathbf{p} = \mathbf{p}(\kappa_B + \beta\lambda) = \mathbf{B}_{\mathfrak{H}} + \beta\lambda\mathbf{p}$$

and

$$\mu = \alpha\kappa_B + \frac{\lambda}{c_\psi} = \alpha\kappa_B\|\mathbf{p}\|^2 + \frac{\lambda}{c_\psi} = \alpha\langle\mathbf{B},\mathbf{p}\rangle + \frac{\lambda}{c_\psi}.$$

By combining all that we arrive at $T(\sigma,\mathbf{B},\lambda) = (\tau,\mathbf{v},\mu)$. ☐

We assume from now on that we have always chosen $\mathbf{p}$ in a way s.t. c_ψ – as defined in the previous lemma – is positive and $\mathbf{p}$ has norm one. This comes down to choosing $\mathbf{p}$ with the correct sign and normalizing it. Now we can use the T-coercivity (Prop. 1.5.2) to prove the inf-sup condition and thus well-posedness of our formulation.

Theorem 1.5.8. *Let $\mathbf{curl}\,\psi_0 + c_\psi \mathbf{p} = \mathbf{curl}\,\psi$ and assume we have chosen the sign of $\mathbf{p}$ s.t. $c_\psi > 0$. Then take $c_1 > 0$ s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$ (e.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$ would be a valid choice). Define $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ and $\alpha = -\frac{c_\psi}{4c_1^2 c_P^2}$. Then the bilinear form a defined at (1.5.5) satisfies the inf-sup condition i.e. (1.5.2) and (1.5.3) with $\gamma \geq \eta/\sqrt{C_T}$ with C_T from (1.5.6) and*

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2}\right\}.$$

Proof. We will use T-coercivity to prove it. Choose $(\sigma, \mathbf{B}, \lambda) \in X$ arbitrary and define $\rho := \mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}$. We take T as in (1.5.7),

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi})$$

Then T is surjective due to Lemma 1.5.7. Note

$$\langle \mathbf{B}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2 \langle \frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2$$

where we used in the last equality that $\frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}$ is either $+\mathbf{p}$ or $-\mathbf{p}$ because $\mathfrak{H}^1$ is assumed to be one-dimensional. We split up $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ to get

$$\begin{aligned}
&a(\sigma, \mathbf{B}, \lambda; T(\sigma, \mathbf{B}, \lambda)) \\
&= \langle \sigma, \sigma - \frac{1}{c_P^2}\rho\rangle - \langle \mathbf{B}, \mathbf{curl}\,\sigma - \frac{1}{c_P^2}\,\mathbf{curl}\,\rho\rangle + \langle \mathbf{curl}\,\sigma, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\,\mathrm{div}\,\mathbf{p}\rangle \\
&\quad + \langle \mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathbf{curl}\,\sigma + \mathrm{div}\,\mathbf{B} + \beta\lambda\,\mathrm{div}\,\mathbf{p}\rangle \\
&\quad + \langle \lambda\mathbf{p}, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle - (\alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi})\langle \mathbf{B}, \mathbf{curl}\,\psi\rangle \\
&= \|\sigma\|^2 - \frac{1}{c_P^2}\langle \sigma, \rho\rangle + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\beta - \alpha c_\psi \|\mathbf{B}_{\mathfrak{H}}\|^2 \\
&\quad - \alpha\langle \mathbf{p}, \mathbf{B}_{\mathfrak{H}}\rangle\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle - \frac{\lambda}{c_\psi}\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle
\end{aligned}$$

Due to the Poincaré inequality

$$\|\rho\| \leq \|\rho\|_{H^1} \overset{\text{Poincaré}}{\leq} c_P\|\mathbf{curl}\,\rho\| = c_P\|\mathbf{B}_{\mathfrak{B}}\|.$$

Using ϵ-Young combined with Cauchy-Schwarz inequality several times we obtain the lower bound.

$$\|\sigma\|^2 - \left(\frac{1}{2}\|\sigma\|^2 + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2}{2c_P^2}\right) + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2$$
$$+ \lambda^2\beta - \alpha c_\psi \|\mathbf{B}_{\mathfrak{H}}\|^2 - \left(\frac{\epsilon_1\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2}{2} + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2\epsilon_1}\right) - \left(\frac{\lambda^2}{2\epsilon_2 c_\psi^2} + \frac{\epsilon_2\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2}\right)$$

Choose $\epsilon_1 = 4c_1^2c_P^2$ to get

$$\frac{1}{2}\|\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{1}{2\epsilon_2 c_\psi^2}\right)$$
$$+ \|\mathbf{B}_{\mathfrak{H}}\|^2\left(-\alpha c_\psi - \frac{4c_1^2c_P^2\alpha^2}{2}\right) - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2} - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\epsilon_2\|\mathbf{curl}\,\psi_0\|^2}{2}$$

Now choose $\epsilon_2 = \frac{1}{4c_1^2c_P^2}$, plug in the definition of α and use $\|\mathbf{curl}\,\psi_0\| \le c_1$ to get the next lower bound

$$\frac{1}{2}\|\sigma\|^2 + \|\mathbf{B}_{\mathfrak{B}}\|^2\left(\frac{1}{2c_P^2} - \frac{1}{8c_P^2} - \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2}\right) + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{4c_1^2c_P^2}{2c_\psi^2}\right)$$
$$+ \|\mathbf{B}_{\mathfrak{H}}\|^2\left(\frac{c_\psi^2}{4c_1^2c_P^2} - \frac{c_1^2c_P^2c_\psi^2}{8c_1^4c_P^4}\right)$$

and finally by using the Poincaré inequality $\|\mathbf{B}_{\mathfrak{B}^*}\| \le c_P\|\mathrm{div}\,\mathbf{B}\|$ and $\beta = \frac{3c_1^2c_P^2}{c_\psi^2}$ we obtain the next bound

$$\frac{1}{2}\|\sigma\|^2 + \frac{1}{4c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}^*}\|^2 + \frac{1}{2}\|\mathrm{div}\,\mathbf{B}\|^2 + \frac{c_1^2c_P^2}{c_\psi^2}\lambda^2 + \frac{c_\psi^2}{8c_1^2c_P^2}\|\mathbf{B}_{\mathfrak{H}}\|^2.$$
$$\ge \eta\|(\sigma, \mathbf{B}, \lambda)\|_X^2$$

where we chose

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2c_P^2}\right\}$$

to obtain the T-coercivity. We can then choose γ from (1.5.2) as $\eta/\|T\|_{\mathcal{L}(X,X)}$ and then use C_T from (1.5.6) to get a lower bound

$$\gamma \ge \frac{\eta}{\sqrt{C_T}}.$$

□

Corollary 1.5.9 (Well-posedness)**.** *The variational formulation of the magnetostatic problem* (Problem 1.4.1) *is well-posed. For a solution* $(\sigma, \mathbf{B}, \lambda) \in X$ *we have the stability estimate*

$$\|\mathbf{B}\|_{H(\mathrm{div})} \leq \frac{\|J\| + |C_1|}{\gamma}.$$

Proof. Recall that when $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$. The statement follows immediately from the previous theorem and Thm. 1.5.1 and the fact that

$$|\ell(\tau, \mathbf{v}, \mu)| = |-\langle J, \tau\rangle - C_1\mu| \leq (\|J\| + |C_1|)\|(\tau, \mathbf{v}, \mu)\|_X$$

and thus $\|\ell\|_{X'} \leq \|J\| + |C_1|$. □

Remark 1.5.10. Note that $1/c_\psi$ terms arise in the stability constant $1/\gamma$. This is not surprising since the term $\langle \mathbf{curl}\,\psi, \mathbf{B}\rangle$ will not enforce the harmonic part if the harmonic part of $\mathbf{curl}\,\psi$ would be zero because $\mathbf{B}_{\mathfrak{H}}$ will disappear from the formulation. So we expect stability issues if the harmonic part is too small. This also forces us to choose ψ with this in mind to obtain a stable system.

2 Discretized magnetostatic problem

In order to approximate solutions of the 2D magnetostatic problem we want to use finite elements. A very typical question that arises for any discretization of a model is what notions of the continuous model are represented in the discretized one. In our case, a fundamental structure of our problem is the Hilbert complex we introdced in Section 1.1. We would like to represent it in our discretization, leading to the discrete Hilbert complex. This section follows Sec. 5.2 in Arnold's book [1]. We start with reviewing the general theory behind the discretization of general Hilbert complexes before applying this theory to our problem. Once we discretized our problem, we will utilize the inf-sup condition proven in Section 1.1 to prove well-posedness of the discrete formulation and derive a quasi-optimal error estimate.

2.1 Discrete Hilbert complex

Let us at first stick to the general situation. We assume that we have a Hilbert complex (W^k, d^k) with its corresponding domain complex (V^k, d^k)

and dual complex (V_k^*, d_k^*) for $k \in \mathbb{Z}$. For this chapter we will only need a short subsequence

$$V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1}$$

for some fixed k and $j \in \{k-1, k, k+1\}$ will always be the index to refer to all three spaces. We will sometimes leave out the indices if the meaning is clear from the context.

Let us assume that we have finite dimensional subspaces $V_h^j \subseteq V^j$. As usual in numerical analysis, $h > 0$ stands loosely for the fineness of our discretization, e.g. the grid size or the maximal diameter of the elements in the mesh. Then we define completely analogous to the continuous case,

$$\begin{aligned}\mathfrak{Z}_h^j &:= \{v_h \in V_h^j \mid d^j v_h = 0\} = \ker d^j \cap V_h^j \\ \mathfrak{B}_h^j &:= \{d^j v_h \mid v_h \in V_h^{j-1}\}.\end{aligned}$$

We can now also define the discrete harmonic forms. Now the situation is slightly different however. We will not use the adjoint d_j^* to define it. Instead,

$$\mathfrak{H}_h^j := \{v \in \mathfrak{Z}_h^j \mid v \perp \mathfrak{B}_h^j\} = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp}.$$

Notice that we have $\mathfrak{Z}_h^j \subseteq \mathfrak{Z}^j$ and $\mathfrak{B}_h^j \subseteq \mathfrak{B}^j$, but due to $\mathfrak{B}_h^{j,\perp} \supseteq \mathfrak{B}^{j,\perp}$ we have in general

$$\mathfrak{H}_h^j = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp} \nsubseteq \mathfrak{Z}^j \cap \mathfrak{B}^{j,\perp} = \mathfrak{H}^j.$$

We will later investigate the difference between the space of discrete and harmonic forms more closely.

There are three crucial properties that are necessary for stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want that the discrete spaces V_h^j approximate the continuous ones V^j. This can be generally summarized that

$$\lim_{h \to 0} \inf_{v_h \in V_h^j} \|w - v_h\|_V = 0, \quad \forall w \in V^j.$$

This is usually satisfied for a reasonable choice of finite element space.

The next property is more restrictive. We require that $dV_h^j \subseteq V_h^{j+1}$. This shows that we cannot simply use arbitrary discrete subspaces independent from one another. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. **??** directly to it. Let us do that.

Denote the restriction of d^j to V_h^j as d_h^j. Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \to V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to d_h^j the adjoint $d_{j,h}^*$ is not the restriction of the adjoint d_j^*. In general, $V_h^j \not\subseteq V_j^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h^j$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1} \tag{2.1.1}$$ {eq:discrete_hil

and its dual complex

$$V_h^{k-1} \xleftarrow{d_{k,h}^*} V_h^k \xleftarrow{d_{k+1,h}^*} V_h^{k+1}$$

From the general Hilbert complex theory (Thm. **??**) we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \overset{\perp}{\oplus} \mathfrak{H}_h^j \overset{\perp}{\oplus} \mathfrak{B}_{jh}^*.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. We will investigate the question how well the discrete harmonic forms approximate the continuous ones more thoroughly later.

The third crucial assumption is the existence of *bounded cochain projections* $\Pi_h^j : V^j \to V_h^j$. This is a projection that is a cochain map in the sense of cochain complexes **??** i.e. the following diagram commutes:

$$\begin{array}{ccccc}
V^{k-1} & \xrightarrow{d^{k-1}} & V^k & \xrightarrow{d^k} & V^{k+1} \\
\downarrow{\scriptstyle \Pi_h^{k-1}} & & \downarrow{\scriptstyle \Pi_h^k} & & \downarrow{\scriptstyle \Pi_h^{k+1}} \\
V_h^{k-1} & \xrightarrow{d^{k-1}} & V_h^k & \xrightarrow{d^k} & V_h^{k+1}
\end{array}$$

Π_h^j are either bounded in the V- or in the W-norm where W-boundedness implies V-boundedness. The cochain projection will play an important role in the stablity of the discrete system. In this section, we will make the weaker assumption of V-boundedness.

The fact that Π_h is a V-bounded projection immediately allows a quasi optimal estimate. For any $v \in V^j$, we can take $w_h \in V_h^j$ arbitrary s.t.

$$\|v - \Pi_h^j v\|_V = \|v - w_h + \Pi_h^j (w_h - v)\|_V \tag{2.1.2}$$

$$\leq \|I - \Pi_h^j\|_{\mathcal{L}(V,V)} \|v - w_h\|_V. \tag{2.1.3}$$ {eq:bound_projec

From now on we will denote the operator norm $\|\cdot\|_{\mathcal{L}(V,V)}$ by slight abuse of notation as $\|\cdot\|_V$. Since w_h was arbitrary we can take the infimum over $w_h \in V_h^k$ and obtain a quasi optimal estimate.

Let us now answer the question about the difference between discrete and continous harmonic forms. In order to do that, we need some way to measure the "difference" between two subspaces.

For a general metric space (X, d), we will use the standard notation $d(x, M) := \inf_{m\in M} d(x, m)$ for $x \in X$ being in a metric space and $M \subseteq X$. If we are dealing with a normed space then we take the metric induced by the norm.

Definition 2.1.1 (Gap between subspaces)**.** For a Banach space W with subspaces Z_1 and Z_2 let S_1 and S_2 be the unit spheres in Z_1 and Z_2 respectively i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$ and analogous for S_2. Then we define the gap between these subspaces as

$$\operatorname{gap}(Z_1, Z_2) = \max\{\sup_{z_1\in S_1} d(z_1, Z_2), \sup_{z_2\in S_2} d(z_2, Z_1)\}$$

This definition is from [4, Ch.4 §2.1] and defines a metric on the set of closed subspaces of W. If W is a Hilbert space – as it is throughout this section – and Z_1 and Z_2 are closed then the $\operatorname{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|_{\mathcal{L}(W,W)}$ i.e. the difference in operator norm of the orthogonal projections onto Z_1 and Z_2. This gives us a measure of distance between spaces which we can now apply to the question about the difference between discrete and continuous harmonic forms.

Proposition 2.1.2 (Gap between harmonic forms)**.** *Assume that the discrete complex (2.1.1) admits a V-bounded cochain projection Π_h. Then*

$$\|(I - P_{\mathfrak{H}_h^k})q\|_V \leq \|(I - \Pi_h^k)q\|_V, \forall q \in \mathfrak{H}^k \tag{2.1.4}$$ {eq:difference_i

$$\|(I - P_{\mathfrak{H}^k})q_h\|_V \leq \|(I - \Pi_h^k)P_{\mathfrak{H}^k}q_h\|_V, \quad \forall q_h \in \mathfrak{H}_h^k \tag{2.1.5}$$

and then

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q\in\mathfrak{H}, \|q\|=1} \|(I - \Pi_h^k)q\|_V$$

Proof. See [1, Thm. 5.2]. □

Do not forget the continuous poincare inequality This then implies that there is a quasi optimal kind of bound of the gap. The following proposition clarifies how close a discrete harmonic form can be chosen. But in order to do that, we will need a small lemma.

{lem:distance_un

Lemma 2.1.3. *Let W be a Banach space, $Z \subseteq W$ a closed subspace. Denote $S_Z := \{z \in Z \mid \|z\|_W = 1\}$. Then for any $w \in W$ with $\|w\|_W = 1$ we get*

$$d(w, S_Z) \leq 2\, d(w, Z)$$

Proof. See [4, Ch.4 §2, (2.13)]. □

{prop:choice_of_

Proposition 2.1.4. *Take $p \in \mathfrak{H}^k$ with $\|p\| = 1$. Then we can choose $p_h \in \mathfrak{H}_h^k$ with $\|p_h\| = 1$ s.t.*

$$\|p - p_h\| \leq 2\, \|I - \Pi_h^k\|_V \inf_{v_h \in V_h} \|p - v_h\|_V.$$

Proof. Notice that since $\mathfrak{H}^k \subseteq \mathfrak{Z}^k$ we always have $\|q\|_V = \|q\|$ for all $q \in \mathfrak{H}^k$. The same is true for $\mathfrak{H}_h^k$. Denote $S_h := \{q_h \in \mathfrak{H}_h^k \mid \|q_h\| = 1\}$. Since S_h is closed we can find $p_h \in S_h$ s.t.

$$\|p_h - p\| = \inf_{q_h \in S_h} \|q_h - p\|$$

The right hand side can be estimated using (2.1.4) and then the quasi optimal bound for the projection derived at (2.1.3).

$$\begin{aligned} \inf_{q_h \in S_h} \|q_h - p\|_V = \inf_{q_h \in S_h} \|q_h - p\| &\overset{Lem.\, 2.1.3}{\leq} 2 \inf_{q_h \in \mathfrak{H}_h} \|q_h - p\| = 2\, \|P_{\mathfrak{H}_h} p - p\| \overset{\textbf{??}}{\leq} 2\, \|\Pi_h^k p - p\|_V \\ &\leq 2\, \|I - \Pi_h^k\|_{\mathcal{L}(V,V)} \inf_{v_h \in V_h} \|p - v_h\|_V \end{aligned}$$

which gives us the estimate. □

If we have the standard situation that we have for $v \in V^j$

$$\|v - \Pi_h^j v\|_V \leq C \|v\|_V h^s \tag{2.1.6}$$ {eq:standard_est

for some generic constant $C > 0$ independent of v and the mesh size h and some $s > 0$ then we can improve the estimate to

$$\|p_h - p\| \leq C h^s$$

by applying it in the last estimate of the proof.

Also if we assume $\|\Pi_h\| \leq c_\Pi$ for h small enough and $c_\Pi > 0$ independent of h then we Assumption **??** applies

$$p_h \xrightarrow{V} p \text{ as } h \to 0.$$

{thm:dim_of_disc

Theorem 2.1.5 (Dimension of $\mathfrak{H}_h^k$). *Assume that we have a finite-dimensional subcomplex with a V-bounded cochain projection. Assume further, that*

$$\|q - \Pi_h^k q\| < \|q\|, \quad \forall q \in \mathfrak{H}^k \setminus \{0\}. \tag{2.1.7}$$

{eq:assumption_s

Then $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ are isomorphic. In particular, $\dim \mathfrak{H}^k = \dim \mathfrak{H}_h^k$.

Proof. See [1, Thm 5.1] and the explanation after the proof. □

If we assume a standard error estimate for the projection as (2.1.6) then (2.1.7) is fulfilled if $Ch^s < 1$ which will be true for h small enough.

{prop:gap_image_

Proposition 2.1.6. *Assuming again a finite-dimensional subcomplex with a V-bounded cochain projection. Then we can bound the gap between the image spaces*

$$\mathrm{gap}(\mathfrak{B}_h^j, \mathfrak{B}^j) \le \|I - \Pi_h^j\|_V \inf_{v_h \in V_h^j} \|z - v_h\|_V.$$

Proof. At first note, that for any $z \in \mathfrak{B}$, $\|z\|_V = \|z\|$. Since $\mathfrak{B}_h \subseteq \mathfrak{B}$, $d(z_h, \mathfrak{B}) = 0$ for any $z_h \in \mathfrak{B}_h$.

Take $z \in \mathfrak{B}^j$ arbitrary i.e. there exists $w \in V^{j-1}$ s.t. $dw = z$. Since Π_h is a cochain projection, we thus have

$$\Pi_h z = \Pi_h dw = d\Pi_h w \in \mathfrak{B}_h^j$$

so Π_h maps $\mathfrak{B}^j$ into $\mathfrak{B}_h^j$. Putting all that together,

$$d(z, \mathfrak{B}_h^j) = \inf_{z_h \in \mathfrak{B}_h} \|z - z_h\| \le \|z - \Pi_h z\| = \|z - \Pi_h z\|_V \overset{(2.1.3)}{\le} \|I - \Pi_h\|_V \inf_{v_h \in V_h} \|z - v_h\|_V$$

□

Proposition 2.1.7 (Discrete Poincare inequality). *Assume that we have a V-bounded cochain projection Π_h for the discrete Hilbert complex. Then*

$$\|v_h\|_V \le c_{P,h} \|dv_h\|, \quad \forall v_h \in \mathfrak{Z}_h^{k\perp} \cap V_h$$

with $c_{P,h} := c_P \|\Pi_h\|_V$ and c_P being the Poincare constant from **??**.

Proof. This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}^{k,\perp} \cap V^k) = \mathfrak{B} \supseteq \mathfrak{B}_h$ we find $z \in \mathfrak{Z}^{k\perp} \cap V_h$ s.t. $dz = dv$. We can apply now the continuous Poincare inequality **??** to get $\|z\|_V \le c_P \|dz\|_V = c_P \|dv_h\|_V$. Now we can combine the different assumptions about the discrete Hilbert complex teo get $v_h - \pi_h z \in V_h^k$. Now we can use the fact that π_h is a cochain map and the fact that π_h is a projection:

$$d\pi_h^k z = \pi_h^{k+1} dz = \pi_h^{k+1} dv_h = dv_h$$

For the last equality we used also the fact that we have a discrete complex i.e. $d^k V_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \pi_h z) = 0$ i.e. $(v_h - \pi_h z) \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v, v_h - \pi_h z\rangle = \langle v, v_h - \pi_h z\rangle + \langle dv, d(v_h - \pi_h z)\rangle = \langle v, v_h - \pi_h z\rangle_V$$

so $v_h - \pi_h z$ is V orthogonal to v_h. So

$$\|v_h\|_V^2 = \langle v_h, \pi_h^k z\rangle_V + \langle v_h, v_h - \pi_h^k z\rangle_V = \langle v_h, \pi_h^k z\rangle_V \le \|\pi_h\|_V \|dv\| \overset{Poincareineq.}{\le} c_P \|\pi_h\|_V \|dv\|_V$$

□

Notice that if we assume that $\lim_{h\to 0}\|\Pi_h\|_V = 1$ (which is true if e.g. the standard estimate (2.1.6) is fulfilled) then $c_{P,h} \to c_P$ for $h \to 0$.

In conclusion, we obtain a discrete version of the Hilbert complex where the harmonic forms are accurately represented if h is small enough.

2.2 Discretized magnetostatic problem

Let us apply the theory of discrete Hilbert complexes to the 2D Hilbert complex (1.1.2). We assume that we have finite dimensional subspaces $V_h^0 \subseteq H_0^1$, $V_h^1 \subseteq H_0(\mathrm{div})$ and $V_h^2 \subseteq L^2$ that approximate the full spaces in the sense of **??** and

$$V_h^0 \xrightarrow{\mathbf{curl}} V_h^1 \xrightarrow{\mathrm{div}} V_h^2$$

and the dual complex

$$V_h^0 \xleftarrow{\widetilde{\mathrm{curl}}_h} V_h^1 \xleftarrow{-\widetilde{\mathrm{grad}}_h} V_h^2$$

where $\widetilde{\mathrm{curl}}_h$ is the adjoint of $\mathbf{curl}_h$ and can thus be seen as a weak approximation of curl and the same for $\widetilde{\mathrm{grad}}_h$. Analogous to the continuous case we assume that $\dim \mathfrak{H}_h^1 = 1$ which is not unreasonable thanks to Thm. 2.1.5 for $h > 0$ small enough.

For our domain, we assume from now on that Ω is suitable for discretization in the sense that the functions in the discrete spaces and the continuous ones are both defined on it. What that means exactly depends on the chosen discretization and we will explain it later in **??** when we go into more detail about the actual implementation. We need a analogous assumption for Ω_Γ.

The discretized version of the strong formulation of the magnetostatic problem (Problem 1.2.1) then states: Find $\mathbf{B}_h \in V_h^1$ s.t.

$$\widetilde{\mathrm{curl}}_h \mathbf{B}_h = J \text{ and } \operatorname{div} \mathbf{B}_h = 0$$

plus the additional curve integral constraint. If $J \notin V_h^0$ then this clearly does not have a solution. Note that the divergence is enforced strongly while the curl is only enforced weakly.

As explained in Sec. 1.4, we will substitute the curve integral constraint from Problem 1.2.1 with (1.4.1). This gives us the following discrete formulation. Choose $\mathbf{p}_h \in \mathfrak{H}_h^1$ s.t. $\|\mathbf{p}_h\| = 1$.

Problem 2.2.1. Find $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $\lambda \in \mathbb{R}$ s.t. {prob:magnetosta

$$\begin{aligned}
\langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle &= -\langle J, \tau_h \rangle \quad \forall \tau_h \in V_h^0, \\
\langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \operatorname{div} \mathbf{B}_h, \operatorname{div} \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle &= 0 \quad \forall \mathbf{v}_h \in V_h^1, \\
\mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle &= \mu C_1 \quad \forall \mu \in \mathbb{R}.
\end{aligned}$$

Here we assume for simplicity that $\mathbf{curl}\, \psi \in V_h^1$. Since we can choose ψ this is not unreasonable What assumptions do we need to suit enforced a "discretized curve" Γ?. In practice, $\mathbf{p}_h$ is computed numerically before assembling the system.

We define $X_h := V_h^0 \times V_h^1 \times \mathbb{R}$. Note that this trial and test space is indeed conforming i.e. $X_h \subseteq X$, but we choose the discrete bilinear form $a_h : X_h \times X_h \to \mathbb{R}$

$$\begin{aligned}
&a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) \\
&= \langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle + \langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \operatorname{div} \mathbf{B}_h, \operatorname{div} \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle.
\end{aligned}$$

with $\mathbf{p}_h \in \mathfrak{H}_h^1$ so the resulting bilinear forms are different since we have $\mathbf{p}_h$ instead of $\mathbf{p}$. So we can write the discrete problem in standard form: Find $\sigma_h, \mathbf{B}_h, \lambda \in X_h$ s.t.

$$a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) = \ell(\tau_h, \mathbf{v}_h, \mu) \quad \forall (\tau_h, \mathbf{v}_h, \mu) \in X_h.$$

For simplicity we assume for the theoretical considerations that we can compute all inner products exactly and that C_1 is given exactly as well. That also means that the right hand side ℓ is the same for the continuous and discrete problem.

Because we have transfered the continuous structures to the discrete case we can apply the same arguments as in Sec. 1.5.

Theorem 2.2.2 (Well-posedness of the discrete problem)**.** *For the following assumptions we always tacitly require that $h > 0$ is small enough. We assume that the Hilbert complex admits uniformly V-bounded cochain projections Π_h i.e. there exists $c_\Pi > 0$ independent of h s.t. $\|\Pi_h\|_{\mathcal{L}(V,V)} \leq c_\Pi$. We assume that $\mathbf{curl}\,\psi \in V_h^1$. Then we find $\psi_{0,h} \in V_h^0$ and $c_{\psi,h} > 0$ s.t. $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_{0,h} + c_{\psi,h}\mathbf{p}_h$. We also assume that (2.1.7) holds and thus $\dim \mathfrak{H}_h^1 = \dim \mathfrak{H}^1 = 1$ and we choose $\mathbf{p}_h$ according to Prop. 2.1.4 Then the discrete variational problem (Problem 2.2.1) is well-posed i.e. there exists a unique solution and for a solution $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ we have the stability estimate*

$$\|\mathbf{B}_h\|_{H(\mathrm{div})} \leq \frac{\|J\| + |C_1|}{\gamma_h}.$$

where γ_h has the same expression as γ except $c_{P,h}$ instead of c_P and $c_{\psi,h}$ instead of c_ψ. Furthermore if $c_{P,h} \to c_P$ then $\gamma_h \to \gamma$ for $h \to 0$.

Proof. By following exactly the same arguments as in Sec. 1.5, we can prove the well-posedness through the BNB-theorem. However, we have to argue why $c_{\psi,h} > 0$ if $c_\psi > 0$. Notice since $\|\mathbf{p}_h\| = \|\mathbf{p}\| = 1$ and $\dim \mathfrak{H}^1 = \dim \mathfrak{H}_h^1 = 1$ we have

$$c_\psi = \langle \mathbf{p}, \mathbf{curl}\,\psi \rangle \text{ and } c_{\psi,h} = \langle \mathbf{p}_h, \mathbf{curl}\,\psi \rangle.$$

So

$$|c_{\psi,h} - c_\psi| = |\langle \mathbf{p} - \mathbf{p}_h, \mathbf{curl}\,\psi \rangle| \leq \|\mathbf{curl}\,\psi\|\|\mathbf{p} - \mathbf{p}_h\|.$$

and so because we chose $\mathbf{p}_h$ as described in Prop. 2.1.4 and assume Π_h being uniformly bounded we have $\|\mathbf{p} - \mathbf{p}_h\| \to 0$ as $h \to 0$ and thus we obtain $c_{\psi,h} \to c_\psi$ for $h \to 0$ and hence we can assume $c_{\psi,h} > 0$ for h small enough.

The next question is why we can choose c_1 for the discrete case just as in the continuous one. Remember that $c_1 > 0$ was chosen s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$. Choose now w.l.o.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$. Then it would be clear, that we can choose the same c_1 if $\|\mathbf{curl}\,\psi_{0,h}\| \to \|\mathbf{curl}\,\psi_0\|$. This is indeed true:

$$\begin{aligned}\big|\|\mathbf{curl}\,\psi_{0,h}\| - \|\mathbf{curl}\,\psi_0\|\big| &\leq \|\mathbf{curl}\,\psi_{0,h} - \mathbf{curl}\,\psi_0\| = \|P_{\mathfrak{B}_h^1}\mathbf{curl}\,\psi - P_{\mathfrak{B}^1}\mathbf{curl}\,\psi\| \\ &\leq \|P_{\mathfrak{B}_h^1} - P_{\mathfrak{B}^1}\|\|\mathbf{curl}\,\psi\| = \mathrm{gap}(\mathfrak{B}_h^1, \mathfrak{B}^1)\|\mathbf{curl}\,\psi\| \to 0\end{aligned}$$

where we use Prop. 2.1.6 combined with the uniform boundedness of Π_h to obtain convergence.

Then $\gamma_h \to \gamma$ if $c_{P,h} \to c_P$ is clear because γ depends continuously on c_ψ and c_P. □

In the situation that we are in we (i.e. conforming subspace but a different bilinear form in the discrete problem) we can formulate the following result.

{lem:a_priori_es

Lemma 2.2.3. *Let $x \in X$ be a solution of a general variational problem of the form (1.5.1) and $x_h \in X_h$ be a solution of the discretized version i.e for $X_h \subseteq X$ and $Y_h \subseteq Y$ finite-dimensional subspaces*

$$a_h(x_h, y_h) = \ell(y_h) \quad \forall y_h \in Y_h.$$

Assume that a inf-sup condition holds for the discrete problem with constant γ_h. Define $\delta_h(x) \in Y'$ as

$$\langle \delta(x), y \rangle_{Y' \times Y} := a(x, y) - a_h(x, y).$$

Then

$$\|x - x_h\|_X \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in X_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

Proof. Take $z_h \in X_h$ arbitrary. Then with the triangular inequality

$$\|x - x_h\|_X \leq \|x - z_h\|_X + \|x_h - z_h\|_X. \tag{2.2.1}$$ {eq:a_priori:tri

We now have to bound the last term on the right hand side. Assume w.l.o.g. that $x_h - z_h$ is not zero. Then from the inf-sup condition we can find $y_h \in Y_h \setminus \{0\}$ s.t.

$$\begin{aligned}
&\gamma_h \|x_h - z_h\|_X \|y_h\|_Y \leq a_h(x_h - z_h, y_h) \\
&= a_h(x - z_h, y_h) + a_h(x_h, y_h) - a(x, y_h) + a(x, y_h) - a_h(x, y_h) \\
&= a_h(x - z_h, y_h) + \langle \delta_h(x), y_h \rangle_{Y' \times Y} \\
&\leq \|a_h\| \|x - z_h\|_X \|y_h\|_Y + \|\delta_h(x)\|_{Y'} \|y_h\|_Y
\end{aligned}$$

In the third step, we used the fact that x and x_h are solutions and the discrete problem has the same right hand side as the continous one. So we can bound $\|x_h - z_h\|_X$ by

$$\|x_h - z_h\|_X \leq \frac{\|a_h\|}{\gamma_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}$$

and plugging this in (2.2.1) and taking the infimum over $z_h \in X_h$ we get

$$\|x - x_h\| \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in V_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

□

We now have to apply this lemma to the magnetostatic formulation.

Theorem 2.2.4 (Quasi optimal a-priori estimate)**.** *Let* $(\sigma, \mathbf{B}, \lambda) \in X$ *the exact solution of Problem 1.4.1 and* $(\sigma_h, \mathbf{B}_h, \lambda_h) \in X_h$ *the solution of the discrete Problem 2.2.1. Then*

$$\|\mathbf{B} - \mathbf{B}_h\|_{H(\mathrm{div})} \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{\mathbf{z}_h \in V_h^1} \|\mathbf{B} - \mathbf{z}_h\|_{H(\mathrm{div})}$$

Proof. At first recall that if $(\sigma, \mathbf{B}, \lambda)$ is solution then $\sigma = 0$ and $\lambda = 0$. So $\|(\sigma, \mathbf{B}, \lambda)\|_X = \|\mathbf{B}\|_{H(\mathrm{div})}$ and analogous for $(\sigma_h, \mathbf{B}_h, \lambda_h)$. Also recognize then for any $y = (\tau, \mathbf{v}, \mu) \in X$

$$\langle \delta_h(x), y \rangle = \lambda \langle \mathbf{p}, \mathbf{v} \rangle - \lambda \langle \mathbf{p}_h, \mathbf{v} \rangle = 0.$$

Thus the estimate follows immediately from Lemma 2.2.3. $\square$

3 Implementation using Pushforward Splines

3.1 Splines

For the discretization we will use the space of pushforward tensor product splines defined on a rectangular reference domain $\hat{\Omega}$. We use geometric degrees of freedom which we will introduce below **??**. This section is a recollection of [2, Sec. 4.2] since we use the same method as presented in this paper and also to fix notation.

We will use two different types knot sequences, non-periodic and periodic ones. We choose a knot sequence $\{\xi_i\}_{i=0}^{n+p}$ with $\xi_0 \leq \xi_1 \leq ...\xi_{n+p}$. We choose two types of sequences.

Let us define $x_0 < x_1 < ... < x_N$ as the physical knots which is our actual grid. We will stick to the equidistant case. Let h be $x_{i+1} - x_i$.

Define $n = N + p$. For the non-periodic case we choose an *open* knot sequence by $\xi_0 = ... = \xi_p = x_0$, $\xi_{p+l} = x_l$ for $l = 0, 1, ..., N$ and $\xi_n = \xi_{n+1} = ... = \xi_{n+p} = x_N$

$$\xi_0 = ... = \xi_p < \xi_{p+1} < ...\xi_n = \xi_{n+1} = ... = \xi_{n+p}$$

and for the periodic case $\xi_0 = x_0 - ph, \xi_1 = x_0 - (p-1)h, ..., \xi_p = x_0$, $\xi_{p+l} = x_l$ and $\xi_{n+l} = x_N + lh$ for $l = 0, ..., p$.

Note that all the knot multiplicities in the interior are one and thus our spline space has maximal regularity. We then define $\mathcal{N}_i^q$ be the normalized

B-spline [**multipatch paper**]. We then define the spline spce $\mathbb{S}_q = \mathbb{S}_q(\boldsymbol{\xi}) = span \mathcal{N}_i^q \mid i = 0, ..., n-1$. Since we have maximal regularity we get that

$$\{v \in C^{q-1} \mid v|_{\xi_{q+j,q+j+1}} \in \mathbb{P}_q\}.$$

$\mathcal{N}_0^{p-1}$ vanishes.

We now can take tensor product of spline spaces. We use the notation with $\mathbf{q} \in \{p-1, p\}^2$ and we define with $\mathbf{i} \in [N_0] \times [N_1]$

$$\mathcal{N}_{\mathbf{i}}^{\mathbf{q}} \mathcal{N}_{i_1}^{q_1} \mathcal{N}_{i_2}^{q_2}.$$

We write this as $\mathbb{S}_{\mathbf{q}} = \mathbb{S}_{q_1} \otimes \mathbb{S}_{q_2}$ The spline spaces used in the tensor product can also be periodic or only one of them can be periodic. On $\hat{\Omega}$ we obtain the following discrete Hilbert complex

$$\mathbb{S}_{p,p} \xrightarrow{\mathbf{curl}} \mathbb{S}_{p-1,p} \xrightarrow{\text{div}} \mathbb{S}_{p-1,p-1}$$

and we denote $\hat{V}^0 = \mathbb{S}_{p,p}$, $\hat{V}^1 = \mathbb{S}_{p-1,p}$ and $\hat{V}^2 = \mathbb{S}_{p-1,p-1}$.

It is well-known that if we have a function $\mathbf{v}$ that is piecewise smooth then $\mathbf{v} \in H(\text{div})$ i.i.f. the normal trace across element interfaces agrees almost everywhere. Analogously for $\tau \in H^1$ i.i.f. the values agree on the interfaces almost everywhere. Since we always have $p \geq 2$ we thus know that all our tensor splines are at least continuous globally and thus we get $\hat{V}_h^j \subseteq \hat{V}^j$ for $j = 0, 1, 2$ as desired.

3.2 Basis and degrees of freedom

Let us now investigate the degrees of freedom. At first, we will work on only on the reference domain and then use a pullback and pushforward to transfer our degrees of freedom and basis functions to the physical domain. We will use geometric degrees of freedom i.e. each degree of freedom can be associated with some geometrical element of our domain. We define Greville points by

$$\zeta_i := \frac{\xi_{i+1} + ... + \xi_{i+p}}{p}$$

i.e. the knot averages for $i = 0, ..., n-1$. Then the spline interpolation at these points is well-defined (see [**isogeometric analysis**]). Note that in the case periodic some Greville points lie outside of the grid. But since the function is periodic it can simply be extended periodically and then interpolated at these points.

This gives us the following geometric elements nodes, edges and cells

$$\begin{aligned}
\hat{\mathtt{n}}_{\mathbf{i}} &:= (\zeta_{i_1}, \zeta_{i_2}), \quad \mathbf{i} \in \mathcal{M}^0 \\
\hat{\mathtt{e}}_{d,\mathbf{i}} &:= [\hat{\mathtt{n}}_{\mathbf{i}}, \hat{\mathtt{n}}_{\mathbf{i}+\mathbf{e}_d}], \quad (d, \mathbf{i}) \in \mathcal{M}^1 \\
\hat{\mathtt{c}}_{\mathbf{i}} := [\hat{\mathtt{e}}_{1,\mathbf{i}}, \hat{\mathtt{e}}_{1,\mathbf{i}+\mathbf{e}_1}] &= [\zeta_{i_1}, \zeta_{i_1+1}] \times [\zeta_{i_2}, \zeta_{i_2+1}], \quad \mathbf{i} \in \mathcal{M}^2
\end{aligned}$$

with $[\cdot]$ being the convex hull. As before, $\mathbf{e}_d$ for $d = 1, 2$ is the standard basis vector of $\mathbb{R}^2$. We define for a $m \in \mathbb{N}$, $[m] := \{0, 1, ..., m\}$. The set of multiindices are defined as

$$\begin{aligned}
\mathcal{M}^0 &:= [n-1]^2 \\
\mathcal{M}^1 &:= \{(d, \mathbf{i} =) \mid \mathbf{i} \in \mathcal{M}^0, d \in \{1, 2\}\} \\
\mathcal{M}^2 &:= [n-2]^2
\end{aligned}$$

Does this stuff go through for periodic case?

Now that we have defined the geometric elements we define the corresponding degrees of freedom as

$$\begin{aligned}
\hat{\sigma}^0_{\mathbf{i}}(v) &:= v(\hat{\mathtt{n}}_{\mathbf{i}}), \quad \mathbf{i} \in \mathcal{M}^0 \\
\hat{\sigma}^2_{\mathbf{i}}(v) &:= \int_{\hat{\mathtt{c}}_{\mathbf{i}}} v, \quad \mathbf{i} \in \mathcal{M}^2
\end{aligned}$$

We define $\mathbf{e}_d^\perp$ as $\mathbf{R}_{\pi/2}\mathbf{e}_d$ i.e. the rotation by $\pi/2$ in counter clockwise direction i.e. $\mathbf{e}_1^\perp = \mathbf{e}_2$ and $\mathbf{e}_2^\perp = -\mathbf{e}_1$.

These degrees of freedome are unisolvent i.e. with $N_l = |\mathcal{M}^l|$ with some ordering $\mu_0, \mu_1, ..., \mu_{N_l}$ of the indices of $\mathcal{M}^l$ we define

$$\boldsymbol{\sigma}^l := (\sigma^l_{\mu_0}, \sigma^l_{\mu_1}, ..., \sigma^l_{\mu_{N_l}})^\top : \hat{V}^l_h \to \mathbb{R}^{N_l}$$

which is bijective. And we can thus define our basis functions $\hat{\Lambda}^l_\mu$, $\mu \in \mathcal{M}^l$ as the basis which is dual to the degrees of freedom in the sense

$$\hat{\sigma}^l_\mu(\hat{\Lambda}^l_\nu) = \delta_{\mu,\nu} \quad \forall \mu, \nu \in \mathcal{M}^l.$$

The question is now on what function spaces these degrees of freedom are defined. We note first that the standard choice as described above with $\hat{V}^0 = H^1_0(\hat{\Omega})$ $\hat{V}^1 = H_0(\mathrm{div})$ and $\hat{V}^2 = L^2(\hat{\Omega})$ can not work because the evaluation at point values is not well-defined for H^1 in 2D since it can not be embedded into the continuous functions. Thus, we need to choose function spaces with higher regularity or integrability.

Let us define the spaces

$$W^1_{1,2}(\hat{\Omega}) := \{v \in L^1(\hat{\Omega}) \mid \partial_1\partial_2 v \in L^1(\hat{\Omega})\}$$
$$W^1_d(\hat{\Omega}) := \{v \in L^1(\hat{\Omega}) \mid \partial_d v \in L^1(\hat{\Omega})\}$$

Why do we set the sequence to the spaces with L^1 sub instead of the intersection?

We thus obtained the degrees of freedom and basis functions on the reference domain $\hat{\Omega}$. The idea to continue is now to define the basis functions on the physical domain Ω by a pushforward of the basis functions on the reference domain. We will now clarify what types of domain we will consider for the discretization. Then we will define the pushforward as the inverse of the pullback and apply it to the basis functions and degrees of freedom to transfer them to the physical domain.

We already introduced the pullback in Sec.**??**. In the case of two dimensions

The pushforward is the inverse of the pullback introduce in Sec. **??**. Here it is in the 2D setting, but it works completely analogous as introduced there.

We will stick to the single patch case meaning that there is a diffeomorphism $\mathbf{F} : \hat{\Omega} \to \Omega$. Then we define the pullbacks

$$\mathcal{P}^0_F : v \mapsto \hat{v} := v \circ \mathbf{F}$$
$$\mathcal{P}^1_F : \mathbf{v} \mapsto \hat{\mathbf{v}} := \det D\mathbf{F}\, D\mathbf{F}^{-1}(\mathbf{v} \circ \mathbf{F})$$
$$\mathcal{P}^2_{\mathbf{F}} : v \mapsto \hat{v} := (\det DF)(v \circ \mathbf{F})$$

which map functions on the physical domain Ω to functions on the reference domain $\hat{\Omega}$. Then we have the commuting properties

$$\widehat{\mathbf{curl}}\mathcal{P}^0_F v = \mathcal{P}^1_F \,\mathbf{curl}\, v$$
$$\widehat{\mathrm{div}}\mathcal{P}^1_F \mathbf{v} = \mathcal{P}^1_F \,\mathrm{div}\, v$$

These are easy to prove by adapting the arguments of Sec. **??** i.e. we can use the corresponding operators on differential forms.

Using the pullbacks we define the pushforwards as $\mathcal{F}^l := (\mathcal{P}^l_F)^{-1}$ and then we get the basis functions on the physical domain

$$\Lambda^l_\mu := \mathcal{F}^l \hat{\Lambda}^l_\mu$$

and then

$$V^l_h := \mathrm{span}\{\Lambda^l_\mu \mid \mu \in \mathcal{M}^l\}$$

are our discrete spaces. Approximation properties???

Using the geometric degrees from **??** we can now construct the corresponding by

$$\sigma^l_\mu := \hat{\sigma}^l_\mu \circ \mathcal{P}^l_F$$

Then we have by construction that $\sigma^l_\mu(\Lambda^l_\nu) = \delta_{\mu,\nu}$.

We can thus define the projection operators

$$\pi^l_h : U^l \to V^l_h, v \mapsto \sum_{\mu \in \mathcal{M}^l} \sigma^l_\mu \Lambda^l_\mu.$$

Then these degrees of freedom commute with the corresponding differential operators as desired from **??** i.e. the diagram

$$\begin{tikzcd} H^1_0 \arrow[r, "\mathbf{curl}"] \arrow[d, "\Pi^0_h"] & H_0(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^1_h"] & L^2 \arrow[d, "\Pi^2_h"] \\ V^0_h \arrow[r, "\mathbf{curl}"] & V^1_h \arrow[r, "\mathrm{div}"] & V^2_h \end{tikzcd}$$

and is thus a cochain projection. They also correspond to geometric elements. σ^0 corresponds to point values in the physical domain, σ^1 to the fluxes through the image of edges and σ^2 to the integral over the mapped cells.

Remark 3.2.1. The cochain projections we use here are in fact not V-bounded i.e. neither Π^0_h is bounded H^1 nor Π^1_h in $H(\mathrm{div})$. This is a drawback of the geometric projections and means that – strictly speaking – we can not apply the theoretical results to them. We will use them anyway for the practical implementation since they are easier to implement than other options e.g. quasi-interpolants (see [**<empty citation>**]).

3.3 Building the discrete system

This very simple geometric interpretation gives us the ability to enforce the boundary condition directly by setting the corresponding degrees of freedom to zero.

For $V^0_h \subseteq H^1_0$ we want the trace on the boundary to vanish which means that we set the values at the boundary nodes to zero. Thus, for $\mathtt{n}_\mathbf{i}$ on the boundary we want $\sigma^0_\mathbf{i}(v) = 0$.

For $V^1_h \subseteq H_0(\mathrm{div})$ we want to have the normal trace zero. So when $\mathtt{e}_{d,\mathbf{i}}(\mathbf{v})$ is boundary edge we require This is then achieved when $\sigma^1_{d,\mathbf{i}}(\mathbf{v}) = 0$.

We now define the spaces $\bar{V}^l_h$ which are the correspoding spaces without any boundary conditions. Then we define the projections $P^l_h : \bar{V}^l_h \to \bar{V}^l_h$ which set the boundary degrees of freedom to zero. They have a very simple matrix representation $\mathbb{P}^l_h$ $(\mathbb{P}^l_h)_{\mu,\nu} = 1$ i.i.f. $\mu = \nu$ and μ does not correspond to a geometric element on the boundary. They are easily constructed by taking the identity matrix and setting the diagonal entries to zero that belong to boundary degrees of freedom.

We now reformulate the discrete system using these projections. We apply them to all functions involved and then penalize the boundary. With boundary penalities the discrete system is: Find $\sigma_h \in \bar{V}^0_h$, $\mathbf{B}_h \in \bar{V}^1_h$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned}
\langle (I-P^0_h)\sigma_h, (I-P^0_h)\tau_h\rangle + \langle P^0_h\sigma_h, P^0_h\tau_h\rangle - \langle P^1_h\mathbf{B}_h, \mathrm{curl} P^0_h\tau_h\rangle &= -\langle J, P^0_h\tau_h\rangle \quad \forall \tau_h \in \bar{V}^0_h,\\
\langle \mathrm{curl} P^0_h\sigma_h, P^1_h\mathbf{v}_h\rangle + \langle (I-P^1_h)\mathbf{B}_h, (I-P^1_h)\mathbf{v}_h\rangle &\\
+\langle \operatorname{div} P^1_h\mathbf{B}_h, \operatorname{div} P^1_h\mathbf{v}_h\rangle + \langle \lambda\mathbf{p}_h, P^1_h\mathbf{v}_h\rangle &= 0 \quad \forall \mathbf{v}_h \in \bar{V}^1_h,\\
\mu\langle \mathbf{curl}\,\psi, P^1_h\mathbf{B}_h\rangle &= \mu C_1 \quad \forall \mu \in \mathbb{R}.
\end{aligned}$$

Since we apply the projection everywhere it is then easy to show that $\sigma_h \in \bar{V}^0_h$, $\mathbf{B}_h \in \bar{V}^1_h$ and $\lambda \in \mathbb{R}$ solve **??** i.i.f. $\sigma_h \in V^0_h$, $\mathbf{B}_h \in V^1_h$ and it solves the system with homogeneous discrete spaces **??**. So the two formulations are equivalent.

In the matrix formulation. Define $\mathbb{M}^l$ as the mass matrix of V^l_h i.e. $\mathbb{M}_{ij} = \langle \Lambda^l_i, \Lambda^l_j\rangle$ where we used some flattening of the multiindices in $\mathcal{M}^l$. We also define the matrix $\mathbb{D}$ the matrix representation of the divergence applied to V^1_h i.e. $\operatorname{div}|_{V^1_h} : V^1_h \to V^2_h$. Analogously $\mathbb{C}$ is the matrix representation of **curl**. Then we have, as mentioned before, the matrix representation of the boundary projections $\mathbb{P}^l$. Overall we can rewrite the linear system like this. We denote the vector of coefficients of a function in bold with an underline e.g. $\underline{\boldsymbol{\sigma}}$ is the vector of coefficients of σ in the basis Λ^0_μ $\mu \in \mathcal{M}^0$ and $\underline{\mathbf{B}}$ is are the coefficients of $\mathbf{B}$ in the basis Λ^1_μ $\mu \in \mathcal{M}^1$. using some flattening again.

$$\begin{aligned}
\underline{\boldsymbol{\tau}}^\top(I-\mathbb{P}^0_h)\top\mathbb{M}^0(I-\mathbb{P}^0_h)\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\tau}}^\top\mathbb{P}^0_h\mathbb{M}^0\mathbb{P}^0_h\underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{\tau}}^\top\mathbb{P}^0_h\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1_h\underline{\mathbf{B}} &= \underline{\boldsymbol{\tau}}^\top\underline{\tilde{\mathbf{J}}} \quad \forall\underline{\boldsymbol{\tau}} \in \mathbb{R}^{N_0}\\
\underline{\mathbf{v}}^\top\mathbb{P}^1_h\mathbb{M}^1\mathbb{C}\mathbb{P}^0_h\underline{\boldsymbol{\sigma}} + \underline{\mathbf{v}}^\top(\mathbb{P}^1)^\top\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \underline{\mathbf{v}}^\top(\mathbb{P}^1_h)^\top\mathbb{M}^1\underline{\mathbf{v}}^\top(\mathbb{P}^1)\mathbb{M}^1\underline{\mathbf{p}} &= 0 \quad \forall\underline{\mathbf{v}} \in \mathbb{R}^{N_1}\\
\underline{\boldsymbol{\psi}}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1_h\underline{\mathbf{B}} &= C_1
\end{aligned}$$

where $\underline{\tilde{J}}_i = \langle J, \Lambda^0_i\rangle$ which gives us the final system to be solved

$$\begin{aligned}
(I-\mathbb{P}^0_h)\top\mathbb{M}^0(I-\mathbb{P}^0_h)\underline{\boldsymbol{\sigma}} + \mathbb{P}^0_h\mathbb{M}^0\mathbb{P}^0_h\underline{\boldsymbol{\sigma}} + \mathbb{P}^0_h\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1_h\underline{\mathbf{B}} &= \underline{\tilde{\mathbf{J}}}\\
\mathbb{P}^1_h\mathbb{M}^1\mathbb{C}\mathbb{P}^0_h\underline{\boldsymbol{\sigma}} + (\mathbb{P}^1)^\top\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + (\mathbb{P}^1_h)^\top\mathbb{M}^1(\mathbb{P}^1)\mathbb{M}^1\underline{\mathbf{p}} &= 0\\
\underline{\boldsymbol{\psi}}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1_h\underline{\mathbf{B}} &= C_1
\end{aligned}$$

Now we will explain how the discrete harmonic form is computed. Notice first that for any linear operator ϕ between finite dimensional inner product spaces V and W with matrix representation A we get the matrix representation of adjoint ϕ^* has matrix representation $G_V^{-1}A^\top G_W$ where G_V and G_W are the gramian matrices of the chosen bases in V and W respectively.

We compute $\underset{\sim}{\mathbf{p}_h}$ as an element of the kernel of the discrete Hodge Laplacian operator $-\mathrm{grad}_h\,\mathrm{div} + \mathbf{curl}\,\widetilde{\mathrm{curl}}_h$ which has matrix representation $(\mathbb{M}^1)^{-1}\mathbb{D}^\top\mathbb{M}^1\mathbb{D}+\mathbb{C}(\mathbb{M}^0)^{-1}\mathbb{C}^\top\mathbb{M}^1$.

Remark 3.3.1. For implementational purposes, these dual basis functions are not necessarily the best option. For the computation of mass matrices etc. it is more convenient to use the normalized B-splines directly due to their local support and fast computation. We will not go to deep into the details of implementation however. More details about the use of B-splines and the connection with the basis Λ^l_μ can be found in [2, Sec. 4.8]

4 Numerical examples

As a first simple numerical example we consider a standard example from magnetostatics which is the magnetic field induced by an infinitely long, wire with radius zero. The *Biot-Savart law* can be used to compute it. Let the wire be equal to the z-axis and I be the electrical current flowing through it. $\boldsymbol{\ell}(s) = s\mathbf{e}_3$, $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_{\infty}^{\infty} \frac{\ell' \times (x-\ell(s))}{|x-\ell(s)|^3}\,ds = \frac{\mu_0 I}{4\pi|x|^2}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}$$

for convenience we pick now $I = \frac{2\pi}{\mu}$ to get

$$\mathbf{B}(x) = \frac{2}{|x|^2}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}.$$

We choose as our domain of computation Ω as the annulus with inner radius 1 and outer radius 2. We choose as our curve Γ the parametrization of the circle with radius 1.5 in anticlockwise direction. We obtain the curve integral

$$C_0 = \int_\Gamma \mathbf{B}\cdot d\ell = 4\pi$$

$J = 0$ on our domain and hence $C_1 = C_0$.

The reference domain $\hat{\Omega} = [0,1] \times [0,2\pi]$ and the mapping

$$F(\hat{x}) = \begin{pmatrix} (\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

Then we choose ψ as a simple interpolation i.e. in logical coordinates $\begin{cases} 2\hat{x}_1 & \text{for } \hat{x}_1 \leq 0.5 \\ 1 & \text{else.} \end{cases}$ This fulfills all the criteria we had for ψ for this given Γ.

We give as another example by choosing now the different mapping

$$F(\hat{x}) = \begin{pmatrix} (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

which results in a kind of "distorted annulus"(see **??**). Note that now for this domain we do not have $\mathbf{B} \cdot \mathbf{n}$ anymore. Thus we have to deal with the boundary conditions by using a lifting approach. This means we take an interpolation of the boundary conditions $\mathbf{B}_{h,g}$ which we compute before and then split $\mathbf{B}_h = \mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ and put the terms with $\mathbf{B}_{h,g}$ on the right hand side.

As an example with a non-vanishing J will use a manufactured solution. Let $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$. It is easy to see that $\mathbf{B} \cdot \mathbf{n} = 0$. Then the resulting

$$J(x) = 4|x|^2 - 12|x| + 8.$$

We will pose this problem on the annulus domain from before, but we will use two different choices for Γ. Γ_1 will be the same Γ as before, but Γ_2 will be the parametrization of the bouter boundary of Ω i.e. it is equal to $\partial\Omega_{out}$ (cf Fig.**??**).

For Γ_1 we will choose ψ as before. For Γ_2 we will choose ψ as the solution of the Laplace problem with boundary condition $\psi = 0$ on $\partial\Omega_{in}$ and $\psi = 0$ on $\partial\Omega_{out}$.