

Let  $\Omega \subseteq \mathbb{R}^n$  be a Lipschitz domain and  $\text{Alt}^k$  the space of alternating  $k$ -linear maps on  $\mathbb{R}^n$ . Let  $\langle \cdot, \cdot \rangle_{\text{Alt}^k}$  be the standard inner product of alternating  $k$ -linear maps and  $\|\cdot\|_{\text{Alt}^k}$  be the induced norm. Then we define the  $L^p$ -norm of a  $k$ -form  $\omega$  for  $1 \leq p < \infty$  as (cf. [4])

$$\|\omega\|_{L_p^k(\Omega)} := \left( \int_{\Omega} \|\omega\|_{\text{Alt}^k}^p \right)^{1/p}$$

and for  $p = \infty$  as

$$\text{ess sup}_{x \in \Omega} \|\omega(x)\|_{\text{Alt}^k}.$$

For  $p = 2$  we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the  $L^2$  inner product

$$\langle \omega, \nu \rangle := \int_{\Omega} \omega \wedge * \nu.$$

$L_p^k(\Omega)$  are the spaces of  $k$ -forms s.t. the corresponding  $L^p$ -norm is finite.

Our next goal is to extend the exterior derivative  $d$  of smooth differential forms in the weak sense (cf. [4]). This is done analogous to the definition of the usual weak derivative. The exterior derivative  $d\omega \in L^p \Lambda^{k+1}(\Omega)$  is defined as the unique  $(k+1)$ -form in  $L^p \Lambda^{k+1}(\Omega)$  s.t.

$$\int_{\Omega} d\omega \wedge \phi = (-1)^{k+1} \int_{\Omega} \omega \wedge d\phi \quad \forall \phi \in C_0^\infty \Lambda^{n-k-1}(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$\begin{aligned} W_p^k(\Omega) &= \{ \omega \in L^p \Lambda^k(\Omega) \mid d\omega \in L^p \Lambda^{k+1}(\Omega) \}, \\ W_{p,loc}^k(\Omega) &= \{ \omega \text{ } k\text{-form} \mid \omega|_A \in W_\infty^k(A) \text{ for every } A \subseteq M \text{ compact} \}. \end{aligned}$$

For  $\omega \in W_p^k(\Omega)$  for  $p < \infty$  we define the norm

$$\|\omega\|_{W_p^k} := \left( \|\omega\|_{L_p^k}^p + \|d\omega\|_{L_p^k}^p \right)^{1/p}$$

and for  $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max \{ \|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k} \}.$$

**Definition 1** ( $L^p$ -cohomology). We define the following subspaces of  $W_p^k(\Omega)$ ,  $1 \leq p \leq \infty$ :

$$\begin{aligned} \mathfrak{B}_k &:= dW_p^{k-1}(\Omega) \text{ and} \\ \mathfrak{Z}_k &:= \{ \omega \in W_p^k(\Omega) \mid d\omega = 0 \}. \end{aligned}$$

We call the  $k$ -forms in  $\mathfrak{B}_k$  exact and the forms in  $\mathfrak{Z}_k$  closed. Then we define the de Rham- or  $L^p$ -cohomology space  $H_{p,dR}^k(\Omega)$  as the quotient space

$$H_{p,dR}^k(\Omega) := \mathfrak{Z}_k / \mathfrak{B}_k.$$

We want to examine the Hilbert space  $L^2\Lambda^k(\Omega)$  more closely (see [1, Sec. 6.2.6] for more details). Because  $d : L^2\Lambda^k(\Omega) \rightarrow L^2\Lambda^{k+1}(\Omega)$  is an unbounded, densely defined operator with domain  $C_0^\infty\Lambda^k(\Omega)$  the adjoint  $\delta$  is well-defined and is equal to the codifferential operator of differential forms. We then define additionally the space  $\mathring{W}_2^k(\Omega)$  as the closure of  $C_0^\infty(\Omega) \subseteq W_2^k(\Omega)$  w.r.t. the  $W_2^k(\Omega)$ -norm.

Now we can formulate our boundary value problem. Let  $\Omega \subseteq \mathbb{R}^n$  from now on be a an exterior polyhedral domain of a compact set i.e.  $\mathbb{R}^n \setminus \Omega$  is a compact polyhedron. We then have the following boundary value problem: For a fixed  $C_0 \in \mathbb{R}$  and closed bounded  $k$ -chain  $\gamma$  find  $\omega \in \mathring{W}_2^k(\Omega)$  s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0 \text{ and} \\ \int_\gamma \omega &= C_0. \end{aligned}$$

Because we consider polyhedral domains we assume that  $\gamma$  consists of finitely many  $k$ -simplices and that the cohomology class  $[\gamma]$  is a generator of the simplicial homology group. The well-definedness of the integral over  $\gamma$  will be discussed later. Our goal will be to show existence and uniqueness of solutions. In order to achieve this, we rely on a result about the isomorphism of a simplicial cohomology space  $H_p^k(K)$  which will be defined below and the  $L^p$ -cohomology space  $H_{p,dR}^k(\overline{\Omega})$  ( $dR$  short for de Rham). This result was proven in [4]. In the diploma thesis of Nikolai Nowaczyk [5], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [4] are proved explicitly for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [4]). Therefore, we can apply the result to our case.

# 1 Isomorphism of Cohomology

## 1.1 Assumptions

In order to formulate the assumptions necessary for the result from [4] to work we will define some basic things from simplicial topology theory. More details and references can be found in [2, Chapter 4.21].

**Definition 2** (Affine simplex). Let  $x_1, x_2, \dots, x_n$  be affine independent. Then

$$[x_1, x_2, \dots, x_n] := \text{conv} \{x_1, \dots, x_n\}$$

is called an affine simplex.

**Definition 3** (Simplicial complex). A *simplicial complex*  $K$  is a collection of affine simplices s.t.

1.  $\sigma \in K \Rightarrow$  any face of  $\sigma$  is in  $K$ ,
2.  $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$  is in  $K$ .

We call  $|K| := \bigcup \{\sigma \mid \sigma \in K\}$  the polyhedron of  $K$ .

For any topological space  $X$  a homeomorphism  $\tau : |K| \rightarrow X$  is called *triangulation* of  $X$ .

Because  $\bar{\Omega}$  is itself a polyhedron we can assume that  $\bar{\Omega}$  and  $|K|$  are equal as subsets of  $\mathbb{R}^n$  and we can simply use the identity as triangulation. However, we will use different metrics on  $|K|$  and  $\bar{\Omega}$ . We use the Euclidian metric on  $\bar{\Omega}$  and we use the standard simplicial metric on  $|K|$  (cf. [4, p.191]). This metric is defined as follows:

Choose some numbering of the vertices  $\{x_1, x_2, \dots\}$  and take  $f : |K| \rightarrow \ell^2$  where  $\ell^2$  is the Hilbert space of real-valued square-summable sequences s.t.  $f(x_i) = e_i$  with  $e_i \in \ell^2$  being the standard unit vectors and  $f$  is affine on every simplex. This mapping is unique.

Then we define the metric on  $|K|$  as the pullback  $g_S = f^*g$  where  $g$  is the standard metric in  $\ell^2$ . Let  $\langle \cdot, \cdot \rangle$  be the standard scalar product on  $\ell^2$ . Then

for  $x \in |K|$  and  $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x |K|$  we have

$$\begin{aligned}
g_S|_x \left( \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^{\infty} \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^{\infty} \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k,l=1}^{\infty} v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k=1}^{\infty} v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \\
&= \sum_{i,j=1}^n \sum_{k=1}^{\infty} v_i w_j (Df(x)^T Df(x))_{ij} \\
&= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle,
\end{aligned}$$

where  $D$  denotes the Jacobian.

We have two crucial assumptions on the triangulation for the result to hold (cf. [4, p.194]). We summarize them under *GKS-condition* named after the three authors of the [4].

**Assumption 1** (GKS-condition). We will assume the following on the simplicial complex  $K$  and the triangulation  $\tau$ :

1. The star of every vertex in  $K$  contains at most  $N$  simplices.
2. For the differential of  $\tau$  we have constants  $C_1, C_2 > 0$  s.t.

$$\|d\tau|_x\| < C_1, \quad \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where  $d$  denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on  $|K|$  and  $\overline{\Omega}$ .

The first assumption is equivalent to every vertex being contained in at most  $N$  simplices, which is fulfilled if we have a shape regular mesh.

Because  $\tau$  is just the identity in our case the second assumption says that for every  $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v, v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

## 1.2 Statement of the Isomorphism

The isomorphism of the cohomology spaces from [4] uses several mappings between different cohomology spaces. The first isomorphism is induced from a linear mapping between the so called *S-forms*  $S_p^k(K)$  to *p-summable k-cochains*  $C_p^k(K)$  which will both be defined next.

**Definition 4.** We define the following norm of a *k-cochain*  $f$

$$\|f\|_{C_p^k(K)} := \left( \sum_{c \text{ k-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{f \text{ k-cochain} \mid \|f\|_{C_p^k(K)} < \infty\}.$$

Take  $\tau, \sigma \in K$  s.t.  $\tau$  is a face of  $\sigma$  which we write as  $\tau < \sigma$ . It can be shown that the standard embedding  $j : \tau \hookrightarrow \sigma$  induces an restriction operator  $j_{\sigma, \tau}^* : W_\infty^*(\sigma) \rightarrow W_\infty^*(\tau)$  which is bounded (cf [4, p.191]).

**Definition 5** (S-forms). Let

$$\theta = \{\theta(\sigma) \in W_\infty^k(\sigma) \mid \sigma \in K\}$$

be a collection of differential *k-forms*. We call  $\theta$  S-form of degree *k* if we have for all for simplices  $\mu < \sigma$

$$j_{\sigma, \mu}^* \theta(\sigma) = \theta(\mu).$$

We denote with  $S^k(K)$  the space of all S-forms of degree *k* over the chain complex  $K$ . For  $\theta \in S^k(K)$  we define  $d\theta := \{d\theta(\sigma) \mid \sigma \in K\} \in S^{k+1}(K)$ .  $S^*(K)$  is the resulting cochain complex.

For  $\theta \in S^k(K)$  we now define the norm

$$\|\theta\|_{S_p^k(K)} := \left( \sum_{\sigma \in K} \|\theta(\sigma)\|_{W_\infty^k(\sigma)}^p \right)^{1/p}.$$

$S_p^k(K)$  are the S-forms of degree *k* s.t. this norm is finite.

Using integration we can define the homomorphism (see [4, p.191])

$$I : S_p^k(K) \rightarrow C_p^k(K), \quad I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K.$$

With the exterior derivative  $d$  on S-forms as defined above we define

$$\begin{aligned}\mathcal{Z}_p^k &:= \{\theta \in S_p^k(K) \mid d\theta = 0\} \\ \mathcal{B}_p^k &:= dS_p^k(K)\end{aligned}$$

and then the resulting cohomology space

$$\mathcal{H}_p^k(K) := \mathcal{Z}_p^k / \mathcal{B}_p^k.$$

We denote the standard cochain cohomology as  $H_p^k(K)$ . Then we have that the integration mapping  $I : S_p^k(K) \rightarrow C_p^k(K)$  induces an isomorphism on the cohomologies i.e.  $[I] : \mathcal{H}_p^k(K) \rightarrow H_p^k(K)$  is an isomorphism of vector spaces (see Theorem 1 in [4] and the proof thereof).

The next step is to obtain an isomorphism between the cohomology of S-forms  $\mathcal{H}_p^k(K)$  and the  $L^p$  cohomology  $H_{p,dR}^k(\overline{\Omega})$ . At first, we define

$$\varphi : W_{\infty,loc}^k(M) \rightarrow S^k(K), \quad \omega \mapsto \{\omega|_{\sigma} \mid \sigma \in K\}.$$

This is a well-defined vector space isomorphism ([4, p.191]). This way we can identify  $W_{\infty,loc}^k(M)$  with  $S^k(K)$ . Using the isomorphism  $\varphi$  we now define  $S_p^k(M) := \varphi^{-1}S_p^k(K)$ . It can be shown that  $S_p^k(M) \subseteq W_p^k(M)$ . Let  $\iota : S_p^k(M) \hookrightarrow W_p^k(M)$  be the inclusion operator. The inclusion induces an isomorphism on cohomology [4, Lemma 4, Corollary] i.e.  $[\iota] : \mathcal{H}_p^k(K) \rightarrow H_{p,dR}^k(\overline{\Omega})$  is an isomorphism.

In conclusion, we get the following isomorphisms of cohomologies:

$$H_{p,dR}^k(\overline{\Omega}) \xrightarrow{[\iota]^{-1}} \mathcal{H}_p^k(K) \xrightarrow{[I]} H_p^k(K).$$

## 2 Existence and uniqueness of solutions

### 2.1 Well-definedness of the integral constraint

In the problem, we have the additional constraint

$$\int_{\gamma} \omega = C_0.$$

for some  $C_0 \in \mathbb{R}$  and a closed bounded  $k$ -chain  $\gamma$ . However, we only assume  $\omega \in \mathring{W}_2^k(\Omega)$  so we have to check if and how this can be well defined.

Above, we introduced the integral operator  $I$  for  $S_p^k(K)$  which can therefore be applied on  $\omega \in S_p^k(M)$  as

$$I(\omega) := I(\varphi(\omega)).$$

If we fix now the closed  $k$ -chain  $\gamma$  then  $I(\cdot)(\gamma) = \int_\gamma$  becomes a functional on  $S_p^k(\overline{\Omega})$ , but it is a-priori not clear how to extend this to closed forms in  $W_p^k(\overline{\Omega})$ .

We know that  $\int_\gamma d\eta = 0$  for  $\eta \in S_p^{k-1}(\overline{\Omega})$  because otherwise  $I$  would not induce an isomorphism on cohomology. We extend this now by setting  $\int_\gamma d\nu = 0$  for all  $\nu \in W_p^{k-1}(\overline{\Omega})$ . We have to check whether this is consistent with the definition above i.e. we have to show that if  $d\nu \in S_p^k(M)$  for some  $\nu \in W_p^{k-1}(\overline{\Omega})$  then it must follow from the previous definition of the integral that indeed

$$\int_\gamma d\nu = 0$$

holds. Let  $A \subseteq M$  be a bounded neighborhood of  $\gamma$ . We can then find  $\tilde{\nu}$  s.t.  $\tilde{\nu} \in W_q^{k-1}(A)$  for any  $q > 1$  and  $d\tilde{\nu} = d\nu$  [6, Thm 3.1.1]. Now it is possible to apply Stoke's theorem [3, Thm. 9] to get  $\int_\gamma d\nu = 0$  and consequently we have shown consistency.

In order to extend the functional  $\int_\gamma$  further we will use two operators  $\mathcal{R}$  and  $\mathcal{A}$  which are constructed in the second section of [4]. The precise definition and details of their construction are not relevant for our purposes because we will only use the following properties (cf. [4, Thm.2]).

**Theorem 1.** *Assume that the triangulation  $\tau$  fulfills the GKS-condition. Then there exist linear mappings  $\mathcal{R} : L_{1,loc}^k \rightarrow L_{1,loc}^k$ ,  $\mathcal{A} : L_{1,loc}^k \rightarrow L_{1,loc}^{k-1}$  such that*

1.  $\mathcal{R}\omega - \omega = d\mathcal{A}\omega + \mathcal{A}d\omega$  for  $\omega \in W_{1,loc}^k(\overline{\Omega})$
2. for any  $1 \leq p \leq \infty$ ,  $\mathcal{R}(W_p^k(\overline{\Omega})) \subseteq S_p^k(\overline{\Omega})$ .

We can now use the operator  $\mathcal{R}$  to define  $\int_\gamma \omega$  for closed  $\omega \in W_p^k(M)$  as

$$\int_\gamma \omega := \int_\gamma \mathcal{R}\omega.$$

This is consistent with the curve integral for S-forms because if  $\omega \in S_p^k(M)$  closed then due to Thm. 1

$$\int_\gamma \mathcal{R}\omega = \int_\gamma \omega + d\mathcal{A}\omega + \mathcal{A}d\omega = \int_\gamma \omega.$$

## 2.2 Existence and uniqueness

We start with the following

**Proposition 1.** *Let  $\gamma$  be a  $k$ -chain s.t. the homology class  $[\gamma]$  is a generator of the homology group. For any  $C_0 \in \mathbb{R}$  there exist a cochain  $F \in C_p^k(K)$  s.t.*

$$F(\gamma) = C_0 \text{ and } F(\partial d) = 0 \text{ for all } (k-1)\text{-chains } d.$$

*Furthermore, if we have another  $G \in C_p^k(K)$  with these two properties then*

$$F(c) = G(c) \text{ for all closed } k\text{-chains } c.$$

*Proof.* At first, we construct such a  $k$ -cochain  $F$ . Note that we assume  $[\gamma]$  to be a generator of the homology group. Using this we define  $\tilde{F} : H_p^k(K) \rightarrow \mathbb{R}$  by setting  $\tilde{F}([\gamma]) = C_0$ . Let  $Z_p^k \subseteq C_p^k(K)$  denote the space of closed  $k$ -cochains. Then we define the linear map  $\pi : Z_p^k \rightarrow H_p^k(K)$ ,  $f \mapsto [f]$  and  $F := \tilde{F} \circ \pi$ . This  $F$  has then obviously the desired properties.

Next, we show uniqueness on closed chains. Take any closed  $k$ -chain  $c$ . Because  $\gamma$  is the generator of the homology group we have  $\lambda \in \mathbb{R}$  s.t.  $[c] = [\lambda \gamma]$  where  $[\cdot]$  is the corresponding homology class. That means that we have some  $(k-1)$ -chain  $d$  s.t.  $c = \lambda \gamma + \partial d$ . Using the properties of  $F$  and  $G$ ,

$$F(c) = F(\lambda \gamma + \partial d) = \lambda F(\gamma) = \lambda C_0.$$

Because the same computation is valid for  $G$ ,  $F(c) = G(c)$  follows.  $\square$

We are now able to proof existence of a solution. Take such a closed cochain  $F \in C_p^k(K)$  s.t.  $F(\gamma) = C_0$  and  $F(\partial d) = 0$  for  $(k-1)$ -chains  $d$ . Then we know from Sec.1.2 that there exists a unique  $[\theta] \in \mathcal{H}_p^k(K)$  s.t.  $[I]([\theta]) = [F]$ . Let us take  $\eta := \varphi^{-1}\theta$ . Then  $\int_\gamma \eta = C_0$  holds. If we now take the Hodge decomposition  $L_2^k(\bar{\Omega}) = \bar{\mathfrak{B}}^\sharp \oplus \mathcal{H}^k \oplus \bar{\mathfrak{B}}_\sharp^*$  and define  $\omega$  as the projection of  $\eta$  onto the harmonic forms  $\mathcal{H}^k$ . Then we know that  $d\omega = 0$ ,  $\delta\omega = 0$  and  $\text{tr } \omega = 0$ . So we only have to show that

$$\int_\gamma \omega = C_0.$$

We know from the Hodge decomposition that there exists a sequence  $(\phi_i)_{i \in \mathbb{N}} \subseteq L_2^{k-1}(M)$  s.t.  $\omega = \eta - \lim_{i \rightarrow \infty} d\phi_i$ . Let now be  $R > 0$  large enough s.t.  $\gamma \subseteq B_R$ . Then we know that  $dW_2^{k-1}(B_R)$  is closed in  $L_2^k(B_R)$ .



[empty citation] Therefore there exists  $\phi_R \in W_2^{k-1}(B_R)$  s.t.  $\lim_{i \rightarrow \infty} d\phi_i|_{B_R} = d\phi_R$ . So we have  $\omega|_{B_R} = \eta|_{B_R} - d\phi_R$  and

$$\int_{\gamma} \omega = \int_{\gamma} \omega|_{B_R} = \int_{\gamma} \eta|_{B_R} = C_0.$$

This proves existence.

**Theorem 2.** *The solution of the problem is unique.*

*Proof.* Let  $\omega, \tilde{\omega}$  both be solutions. Because  $\int_{\gamma} \omega = \int_{\gamma} \tilde{\omega}$  and  $\omega$  and  $\tilde{\omega}$  are closed we have due to Prop. 1 that  $\int_c \omega = \int_c \tilde{\omega}$  for any closed  $k$ -chain  $c$ . So we have for the induced homomorphism  $[I](\mathcal{R}\omega) = [I](\mathcal{R}\tilde{\omega})$  and because  $[I]$  is an isomorphism  $[\mathcal{R}\omega] = [\mathcal{R}\tilde{\omega}]$ . Hence,

$$[\tilde{\omega}] = [\mathcal{R}\tilde{\omega}] = [\mathcal{R}\omega] = [\omega]$$

we get the equality of the cohomology classes.

That is equivalent to the existence of some  $(k-1)$ -form  $\phi \in W_2^{k-1}(\bar{\Omega})$  s.t.  $\omega = \tilde{\omega} + d\phi$ . Then because  $\omega$  and  $\tilde{\omega}$  are harmonic we have  $\omega, \tilde{\omega} \perp dW_2^{k-1}(\bar{\Omega})$  and therefore

$$\omega = \tilde{\omega}.$$

□

## References

- [1] Douglas N Arnold. *Finite Element Exterior Calculus*. SIAM, 2018.
- [2] Glen E Bredon. *Topology and geometry*. Vol. 139. Graduate Texts in Mathematics. Springer, 2013.
- [3] VM Gol'dshtein, VI Kuz'minov, and IA Shvedov. "Integration of differential forms of the classes  $W^* p, q$ ". In: *Siberian Mathematical Journal* 23.5 (1982), pp. 640–653.
- [4] Gol'dshtein, V.M., Kuz'minov, V.I. & Shvedov, I.A. "De Rham isomorphism of the  $L_p$ -cohomology of noncompact Riemannian manifolds". In: *Sib Math J* 29 322.10 (1988), pp. 190–197.
- [5] N Nowaczyk. "The de Rham Isomorphism and the  $L_p$ -Cohomology of non-compact Riemannian Manifolds". Friedrich-Wilhelms-Universität Bonn, 2011. URL: [https://nikno.de/index.php/publications/%20\(last%20visited:%2016.11.2022\)](https://nikno.de/index.php/publications/%20(last%20visited:%2016.11.2022)).
- [6] Günter Schwarz. *Hodge Decomposition—A method for solving boundary value problems*. Springer, 2006.