Contents

Chapter 1

Introduction

{chap:introduc

Mathematics is the art of abstraction, of putting basic theories into more general frameworks to gain insights that would otherwise be hard or impossible to see. James Clerk Maxwell recognized that some vector fields are to be understood in relation to curves and some to be understood in reference to a surface. This thought paves the way to a new level of abstraction which presents itself in differential forms which are a classic subject of differential geometry. Differential forms have since been sucessfully used for electromagnetic theory (see [24]). This connection to differential forms and their deep relation to geometry and topology will be one major thing we investigate. These are indeed relations between different fields of mathematics which would be hard to reach without this overarching abstract theory.

To become more specific, this thesis is about the problem of finding a static solution to Maxwell's equations i.e. we look for the magnetic field $\mathbf{B}$ s.t. for a given current source $\mathbf{J}$

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= \mathbf{J} \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

A fascinating realization is that the well-posedness of this equation actually depends on fundamental topological quantities of the domain. If it is simply connected then there exists a unique solution, but otherwise additional constraints might be necessary.

Our main motivation behind studying this problem arises as part of the search for plasma equilibria where we have to compute the magnetic field outside of the plasma. In this situation we know that the magnetic field lines are orthogonal to the plasma surface which translates to imposing Neumann boundary conditions $\mathbf{B}\cdot\mathbf{n} = 0$. The complement of a toroidal domain is obviously not simply connected and so we require an additional constraint. This is given by prescribing the total

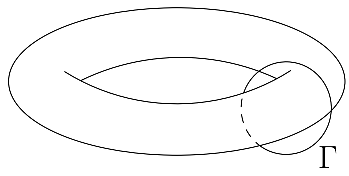

Figure 1.0.1: Sketch of a feasible domain with curve Γ. The domain of interest would be the complement of the torus. {fig:exterior_

current flowing through the cross section of the torus. After using Stokes theorem for surfaces, this gives us

$$\int_\Gamma \mathbf{B} \cdot d\ell = \frac{4\pi\mu_0}{c} =: C_0$$

with Γ being the curve that goes around the torus in poloidal direction (see Fig. 1.0.1). We assume here that Γ has positive distance from the torus and $\mathbf{J}$ is zero between Γ and the torus.

The resulting partial differential equation is fascinating for the reason that it combines the differential equations with this curve integral constraint which gives this problem a strong topological flavour. For a student familiar with standard PDEs, it begs the question how these notions can be combined to investigate this problem.

To come back to the statement from the beginning, the answer is provided by changing the context to a more general one, namely to differential geometry. This field gives a much more general framework in which basic vector calculus can be embedded. One standard example of this is Stokes' theorem which generalizes the Gauss divergence theorem and Stokes' theorem from vector calculus into the language of differential forms.

Differential forms prove especially useful for integration. Whitney derives in his classic book [25] why differential forms are the correct objects to be integrated over manifolds like curves and surfaces which comes back to Maxwell's statement about the relation of vector fields either to a curve or a surface. Another recommended exposition about this topic is Terence Tao's short paper [23].

The integration of differential forms over manifolds turns out to provide a beautiful synergy between the three fields calculus, differential geometry and topology. De Rham's famous theorem which relies on Stokes theorem gives a very simple and yet deep relationship between the singular homology – a popular tool in algebraic topology to study topological quanities – and the integration of differential forms. We recall it in Sec. 2.2.3.

For a mathematician interested in analysis and motivated by the beauty of abstraction these reasons should already be sufficient to start learning and investigating differential forms and – as a necessary fundament – differential geometry. We will give a short exposition of these topics in the first sections of this thesis and then apply these ideas to the magnetostatic problem above.

We focus on the homogeneous problem with $\mathbf{J} = 0$. Existence and uniqueness have been proven before on bounded domains (see [18, Thm. 5.4]), yet an exterior domain is always unbounded. Traditionally, the theory of PDEs on unbounded domains is sparse in comparison to bounded ones. But unbounded domains can change the situation drastically. Even one of the simplest most basic problems in the topic of PDEs, the Laplace equation with homogeneous Dirichlet boundary conditions, does not have a unique classical solution (i.e. a solution using strong derivatives) anymore when posed on unbounded domains. As a counterexample, pose the problem on the domain of positive real numbers. Then, any linear function is a solution.

Combining the tools from singular homology, differential geometry and functional analysis we will prove the existence and uniqueness of the magnetostatic problem under suitable assumptions.

The second part of this thesis will then be concerned with the numerical approximation. As is often done in numerical analysis – especially in a master's thesis – we will simplify the problem. We will investigate the 2D magnetostatic problem on bounded domains, but we keep the curve integral constraint and the non-trivial topology of the domain. The ideas from differential geometry and differential forms can be applied in finite elements which has been formulated in the Acta Numerica paper from Arnold, Falk and Winther [3] who coined the term Finite Element Exterior Calculus (FEEC). This will prove to be the right tool for our purposes. We will study a variational formulation of the problem and will strongy rely on the tools from FEEC.

This work will be split between the study of well-posedness of the magnetostatic problem in Chapter 2 and then the numerics in Chapter 3.

In Chapter 2, we will spend the first sections on introducing the necessary tools for the proof of well-posedness of the magnetostatic problem in 3D. We start with differential forms in Sec. 2.1 from the starting point of multilinear algebra. Afterwards we will give a short introduction on singular homology in Sec. 2.2 which concludes in the de Rham isomorphism and then we will talk about unbounded operators and Hilbert complexes. After applying some regularity results in Sec. 2.4.1, we will put all this together in Sec. 2.4 to prove the existence and uniqueness of the magnetostatic problem und suitable suitable assumptions.

In Chapter 3, we derive a variational formulation of a 2D version of the magnetostatic problem in Sec. 3.1.1. Then we give a short overview of the discrete Hilbert complexes which are fundamental to FEEC theory before applying it to our new variational formulation to obtain the well-posedness and a-priori estimate in Sec. 3.2. In the end, we will explain how this problem is actually implemented

(Sec. 3.3) and provide some numerical examples in Sec 3.4.

Chapter 2

Existence and uniqueness

{chap:existenc

Our goal is to study the homogeneous magnetostatic problem on the exterior of a toroidal Lipschitz domain. That means for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have that $\mathbb{R}^3 \setminus \Omega$ is the closure of a toroidal set. At the moment, toroidal is a very heuristic term, but we will later specify the exact topological assumption for the proof in Sec. 2.4. We also need a piecewise smooth closed curve Γ around the torus (see Fig. 1.0.1) because we want to use the curve integral along this Γ as an additional constraint.

Let $\mathbf{B}$ be the magnetic field. We obtain the following problem: Find a vector field $\mathbf{B}$ on Ω s.t.

$$\operatorname{curl} \mathbf{B} = 0, \tag{2.0.1}$$

$$\operatorname{div} \mathbf{B} = 0 \text{ in } \Omega \tag{2.0.2}$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and} \tag{2.0.3}$$

{eq:pointwise_

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0 \tag{2.0.4}$$

{eq:first_curv

where $\mathbf{n}$ is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$.

In order to show existence and uniqueness, we first have to find a proper formulation of this problem. This requires appropriate assumptions on the domain Ω and restating the problem in terms of the correct Sobolev spaces.

We will first spend a lot of work on introducing the tools needed for the proof. In Section 2.1, we will define differential forms with all necessary operations on them. Section 2.2 gives a very short overview of the basics of singular homology from algebraic topology. This will be needed to properly formulate and treat the curve integral condition described above. It will culminate in de Rham's theorem which can be seen as the connection between differential forms and singular homology. In Section 2.3, the focus lies on unbounded operators and the concept of a Hilbert complex which will be a fundamental tool in our proofs of existence and uniqueness. At last, we will use the developed theory in Section 2.4 to formulate the problem rigorously and proof existence and uniqueness under reasonable assumptions.

Throughout the first sections, we will provide definitions, basic results and show the proofs of many of them if they are not too long or trivial. Since all the presented topics are vast these sections should be seen as a very basic introduction, but references to the literature will be given.

2.1 Differential forms

{sec:different

We will introduce differential forms on manifolds. We start with the abstract formulation of alternating forms on finite dimensional real vector spaces in Section 2.1.1. After a short introduction of the essential notions from differential geometry in Section 2.1.2, we will finally define differential forms and their mathematical operations in Section 2.1.3. The integration of differential forms will be developed in its own Section 2.1.4 at the end.

2.1.1 Alternating forms

{sec:alternati

Understanding alternating forms is essential in understanding differential forms since these provide us with an alternating form at any point of a manifold. That is the reason why we will properly introduce the concept first. For the introduction of alternating forms, we follow the short section in Arnold's book [2, Sec. 6.1.] combine it with material from [4, Sec. V.1]. However, more arguments and additional details are provided especially in Sec. 2.1.1 about scalar and vector proxies.

Basic definitions

Definition 2.1.1 (Alternating k-linear form)**.** Let V be a real vector space with $\dim V = n$. We call $\omega : V^k \to \mathbb{R}$ *k-linear form* if it is linear in every argument. A k-linear form is *alternating* if the sign switches when two arguments are exchanged, i.e.

$$\begin{aligned}&\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k),\\ &\quad \text{for } 1 \leq i < j \leq k, \quad v_1, ..., v_k \in V.\end{aligned}$$

We denote the space of alternating k-linear forms on V as $\mathrm{Alt}^k\, V$. For the special case $k = 0$ we define $\mathrm{Alt}^0\, V := \mathbb{R}$.

Let $\mathcal{S}_k$ be the set of all permutations $\{1, 2, ..., k\} \to \{1, 2, ..., k\}$. A permutation that only exchanges two numbers is called transposition and the transposition that exchanges i and j with $i \neq j$ is denoted by (i, j). Every permutation can be written as the composition of transpositions. This decomposition into transpositions is not unique. Take a permutation $\pi \in \mathcal{S}_k$ and decompose it into transpositions

$$\pi = \tau_p \circ \tau_{p-1} \circ ... \circ \tau_1.$$

The sign $\operatorname{sgn}(\pi)$ of a permutation $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ is equal to $(-1)^p$. Even though the decomposition is not unique, the parity of the number of transpositions is always the same and hence the sign is well-defined. For example, the permutation $(1, 2, 3, 4) \mapsto (2, 3, 1, 4)$ can be built by performing the transpositions $(1, 2)$ and $(1, 3)$ so $\operatorname{sgn}(\pi) = (-1)^2 = 1$.

Going back to alternating forms, this means for any permutation $\pi : \{1, 2, ..., k\} \to \{1, 2, ..., k\}$ and $\omega \in \operatorname{Alt}^k V$

$$\omega(v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(k)}) = \operatorname{sgn}(\pi)\, \omega(v_1, v_2, ..., v_k).$$

Definition 2.1.2 (Wedge product)**.** For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \operatorname{Alt}^{k+l} V$

$$\begin{aligned}&(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l})\\ &= \sum_{\pi} \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, ..., v_{\pi(k)})\, \nu(v_{\pi(k+1)}, ..., v_{\pi(k+l)})\end{aligned}$$

where we sum over all permutations $\pi \in \mathcal{S}_{k+l}$ s.t. $\pi(1) < ... < \pi(k)$ and $\pi(k + 1) < ... < \pi(k + l)$.

Let us mention some important properties of the wedge product. It is associative, but not commutative. For $\omega \in \operatorname{Alt}^k V$, $\mu \in \operatorname{Alt}^l V$ we have

$$\omega \wedge \mu = (-1)^{kl} \mu \wedge \omega. \tag{2.1.1}$$

{eq:commutativ

We denote the dual space of V as V'. Recalling the definition of the sign of a permutation $\pi \in \mathcal{S}_k$, we get for linear forms $\omega_1, \omega_2, ..., \omega_k \in V'$

$$\omega_{\pi(1)} \wedge \omega_{\pi(2)} \wedge ... \wedge \omega_{\pi(k)} = \operatorname{sgn}(\pi)\, \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_k$$

and if a linear form appears twice then the expression is zero.

For $\omega_1, ..., \omega_k \in \operatorname{Alt}^1 V = V'$, $k \leq n$, we can compute their wedge product with the formula ([4, p.260])

$$\omega_1 \wedge ... \wedge \omega_k(v_1, ..., v_k) = \det\left(\langle \omega_s, v_t \rangle\right)_{1 \leq s,t \leq n}. \tag{2.1.2}$$

{eq:wedge_prod

This formula can be easily proven by induction using the definition of the wedge product and the determinant.

We will use the dual inner product notation, i.e. for $\ell \in V'$ we denote

$$\langle \ell, v \rangle_{V' \times V} := \ell(v)$$

and we will usually not write the subscript if it is clear that the dual product is used. Let $\{b_i\}_{i=1}^n$ be any basis of V and $\{b^i\}_{i=1}^n$ the correspoding dual basis, i.e. $b^i \in V'$, $\langle b^i, b_j \rangle = \delta_{ij}$ for $i, j = 1, 2, ..., n$. Then

$$\{b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} | \, 1 \leq i_1 < ... < i_k \leq n\}$$

is a basis of $\mathrm{Alt}^k\ V$. In particular, $\dim\ \mathrm{Alt}^k\ V = \binom{n}{k}$.

Assume now that we are given an inner product $\langle \cdot, \cdot \rangle_V$ on V, where we will usually leave out the subscript if it is clear what space we mean. Recall the Riesz isomorphism $\Phi : V \to V'$ defined by

$$\langle \Phi v, w \rangle = \langle v, w \rangle.$$

Then we obtain an inner product on the dual space V' by using the Riesz isomorphism Φ

$$\langle \Phi v, \Phi w \rangle_{V'} := \langle v, w \rangle_V$$

which makes the Riesz isomorphism an isometry.

Now we can define an inner product on $\mathrm{Alt}^k\ V$ by defining

$$\langle b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}, b^{j_1} \wedge ... \wedge b^{j_k} \rangle_{\mathrm{Alt}^k\ V} := \det \left(\langle b^{i_k}, b^{i_l} \rangle_V \right)_{1 \leq k, l \leq n} \qquad (2.1.3)$$ {eq:inner_prod

which is then extended to all of $\mathrm{Alt}^k\ V$ by linearity. We denote with $|\cdot|_{\mathrm{Alt}^k\ V}$ the induced norm. For an orthonormal basis $u_1, ..., u_n$ the corresponding basis $u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}$, $1 \leq i_1 < ... < i_k \leq n$ is an orthonormal basis of $\mathrm{Alt}^k\ V$. Next, we want to introduce the *pullback* as a natural linear mapping between spaces of alternating forms.

Definition 2.1.3. Let V and W be finite-dimensional real vector spaces with ordered bases $(b_i)_{i=1}^n$ and $(c_j)_{j=1}^m$ respectively. We write a basis in round brackets $(\cdot)$ if it is ordered. Let $L \in \mathcal{L}(V, W)$ where $\mathcal{L}(V, W)$ is the space of linear mappings from V to W. For $\omega \in \mathrm{Alt}^k\ W$ we define the pullback $L^*\omega \in \mathrm{Alt}^k\ V$ via

$$(L^*\omega)(v_1, ..., v_k) = \omega(L\, v_1, ..., L\, v_k).$$

It is then easy to see that L^* is a linear mapping from $\mathrm{Alt}^k\ W$ to $\mathrm{Alt}^k\ V$ and from the definitions of the wedge product and the pullback it is obvious that

$$L^*(\omega \wedge \nu) = L^*\omega \wedge L^*\nu \quad \forall \omega \in \mathrm{Alt}^k\ W,\ \nu \in \mathrm{Alt}^l\ W.$$

Proposition 2.1.4. *Let $A \in \mathbb{R}^{m\times n}$ be the matrix representation of L in the above bases i.e. $L\, b_i = \sum_{j=1}^{m} A_{ji} c_j$. Then the basis representation of the pullback is*

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1\leq i_1<...<i_k\leq n} \det A_{(j_1,...,j_k),(i_1,...,i_k)}\, b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \tag{2.1.4}$$

{eq:basis_repr

where $A_{(j_1,...,j_k),(i_1,...,i_k)}$ is the matrix we get by choosing the rows $j_1, ..., j_k$ and the columns i_1, ... , i_k.

Proof. Because $\{b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \mid 1 \leq i_1 < ... < i_k \leq n\}$ and $\{c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k} \mid 1 \leq j_1 < ... < j_k \leq m\}$ are bases for $\mathrm{Alt}^k\, V$ and $\mathrm{Alt}^k\, W$ respectively, we can find $\lambda_{i_1\cdots i_k} \in \mathbb{R}$ s.t.

$$L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}) = \sum_{1\leq i_1<...<i_k\leq n} \lambda_{i_1\cdots i_k} b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k} \tag{2.1.5}$$

{eq:basis_repr

Now recall the formula for the wedge product of 1-forms $\nu_i \in \mathrm{Alt}^1\, V = V'$

$$\nu_1 \wedge ... \wedge \nu_k(v_1, ..., v_k) = \det \big(\langle \nu_s, v_t\rangle\big)_{1\leq s,t\leq n}.$$

Fix now $1 \leq l_1 < ... < l_k \leq n$. Then we get from this formula $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}(b_{l_1}, ..., b_{l_k}) = 1$ i.i.f. $(i_1, ..., i_k) = (l_1, ..., l_k)$. Here it is important to remember that these indices are ordered. Plugging this into (2.1.5) gives us

$$\begin{aligned}
\lambda_{l_1\cdots l_k} &= L^*(c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k})(b_{l_1}, ..., b_{l_k}) \\
&= c^{j_1} \wedge c^{j_2} \wedge ... \wedge c^{j_k}(L\, b_{l_1}, ..., L\, b_{l_k}) \\
&= \det \Big(\Big\langle c^{j_s}, \sum_{r_t=1}^{m} A_{r_t,l_t} c_{r_t}\Big\rangle\Big)_{1\leq s,t\leq k} \\
&= \det \Big(\sum_{r_t=1}^{m} A_{r_t,l_t} \delta_{j_s,r_t}\Big)_{1\leq s,t\leq k} \\
&= \det \big(A_{j_s,l_t}\big)_{1\leq s,t\leq k} \\
&= \det A_{(j_1,...,j_k),(l_1,...,l_k)}.
\end{aligned}$$

□

We want to emphasize the special case of the pullback of an alternating n-linear form with $m = n$. So take $\omega \in \mathrm{Alt}^n\, W$. Then we know that $\dim \mathrm{Alt}^n\, W = \binom{n}{n} = 1$ and so $\omega = \lambda\, c^1 \wedge ... \wedge c^n$ for some $\lambda \in \mathbb{R}$. There only remains one summand in (2.1.4) and we obtain for this special case

$$L^*\omega = \lambda \det A\, b^1 \wedge ... \wedge b^n \tag{2.1.6}$$

{eq:pullback_a

We want to examine $\mathrm{Alt}^n\ V$ a bit closer. $\mathrm{Alt}^n\ V$ is one-dimensional and so we can choose a basis by fixing any non-zero element. We want to choose one specific element called the *volume form* which will play a crucial role when we define integration on a manifold in Sec. 2.1.4. We also need it to define the Hodge star operator below.

The choice of this volume form will depend on the orientation. We say that two ordered bases of V have the same orientation if the change of basis has positive determinant. That divides the ordered bases into two classes with different orientation. We choose one of these classes and call these bases positively oriented. In $\mathbb{R}^n$, the convention is to define the class as positively oriented which includes the standard orthonormal basis.

Definition 2.1.5 (Volume form)**.** Let $(b_i)_{i=1}^n$ be any positively oriented basis. Let G be the Gramian matrix, i.e. $G_{ij} = \langle b_i, b_j \rangle$ which is always a symmetric positive definite matrix. Then we define the *volume form*

$$\mathrm{vol} := \sqrt{\det G}\, b^1 \wedge b^2 \wedge ... \wedge b^n.$$

We have the following defining property of the volume form.

Proposition 2.1.6. *For any ordered orthonormal basis $(u_i)_{i=1}^n$ we have*

$$\mathrm{vol}(u_1,\, u_2, ...,\, u_n) = (-1)^s$$

with $s = 0$ if $(u_1, ..., u_n)$ has the same orientation as $(b_i)_{i=1}^n$ and $s = 1$ otherwise.

Proof. Let us define the matrix $B \in \mathbb{R}^{n\times n}$, $B_{k,i} = \langle b_i, u_k \rangle_V$ which is just the change of basis matrix from $(b_i)_{i=1}^n$ to $(u_i)_{i=1}^n$. Using basic linear algebra, we get $G = B^\top B$ and $\sqrt{\det G} = (-1)^s \det B$. Let now Ψ be the linear map with $\Psi b_i = u_i$. In the basis $(b_i)_{i=1}^n$, this has the matrix representation B^{-1} and so by using (2.1.6) we get

$$\begin{aligned}
\mathrm{vol}(u_1,\, u_2, ...,\, u_n) &= \sqrt{\det G}\, b^1 \wedge ... \wedge b^n(\Psi b_1, ..., \Psi b_n) \\
&= (-1)^s \det B\, \Psi^*(b^1 \wedge ... \wedge b^n)(b_1, ..., b_n) \\
&= (-1)^s \det B\, \det B^{-1}\, (b^1 \wedge ... \wedge b^n)(b_1, ..., b_n) = (-1)^s.
\end{aligned}$$

□

This property also defines the volume form uniquely so it is independent of the chosen basis. It only depends on the orientation. It also shows that vol is non-zero and thus

$$\mathrm{Alt}^n\ V = \mathrm{span}\{\mathrm{vol}\}.$$

Note that if we choose $\{b_i\}_i$ to be an orthonormal basis to begin with, the Gramian matrix is just the identity and $\mathrm{vol} = b^1 \wedge ... \wedge b^n$. Especially in the case of $\mathbb{R}^n$ if we denote the standard basis by $\{e_i\}_{i=1}^n$ and the resulting dual basis as $\{e^i\}_{i=1}^n$ then

$$\mathrm{vol} = e^1 \wedge ... \wedge e^n.$$

We will from now on assume that we fixed an orientation on V and with it the volume form vol. Using the resulting volume form on V we can now define the *Hodge star operator* with the following proposition.

Proposition 2.1.7. *There exists an isomorphism* $\star : \mathrm{Alt}^k\ V \to \mathrm{Alt}^{n-k}\ V$ *s.t.*

$$\omega \wedge \mu = \langle \star\omega, \mu \rangle_{\mathrm{Alt}^{n-k}\ V} \mathrm{vol} \quad \forall \mu \in \mathrm{Alt}^{n-k}\ V. \tag{2.1.7}$$ {eq:hodge_star_

We call this isomorphism the Hodge star operator.

Proof. Let us define $\mathrm{vol}' \in (\mathrm{Alt}^n\ V)'$ via

$$\langle \mathrm{vol}', \mathrm{vol} \rangle = 1.$$

Recall that $\mathrm{Alt}^n\ V$ is one-dimensional and so this defines vol' uniquely. Let us fix $\omega \in \mathrm{Alt}^k\ V$. Then we take the following linear form on $\mathrm{Alt}^{n-k}\ V$

$$\mu \mapsto \langle \mathrm{vol}', \omega \wedge \mu \rangle.$$

and define $\star\omega$ as the Riesz representative of this linear form that means we have $\langle \mathrm{vol}', \omega \wedge \mu \rangle = \langle \star\omega, \mu \rangle_{\mathrm{Alt}^{n-k}\ V}$ for all $\mu \in \mathrm{Alt}^{n-k}\ V$ which is equivalent to (2.1.7). Checking linearity is trivial and will be omitted.

It is also clear from the uniqueness of the Riesz representative that $\star\omega$ is uniquely determined by the above condition and thus $\star$ is injective. Since $\dim \mathrm{Alt}^k\ V = \binom{n}{k} = \binom{n}{n-k} = \dim \mathrm{Alt}^{n-k}\ V$ the injectivity implies surjectivity and $\star$ is an isomorphism. $\square$

Let us collect some important properties of the Hodge star operator.

{prop:properti

Proposition 2.1.8. *Let* $(u_i)_{i=1}^n$ *be an ordered orthonormal basis of* V. *Let* $\{i_1, ..., i_n\} = \{1, ..., n\}$, $i_1 < i_2 < ... < i_k$ *and* $i_{k+1} < ... < i_n$. *Then*

$$\star(u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}) = \mathrm{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n} \tag{2.1.8}$$ {eq:hodge_star_

where $\mathrm{sgn}(i_1, i_2, ..., i_n)$ *is the sign of the permutation* $j \mapsto i_j$. *In particular,* $\star$ *is an isometry since it maps orthonormal bases to orthonormal bases. Furthermore,*

(i) $\quad \star \star \omega = (-1)^{k(n-k)} \omega \quad \forall \omega \in \mathrm{Alt}^k\ V$

(ii) $\quad \omega \wedge \star\nu = \langle \omega, \nu \rangle_{\mathrm{Alt}^k\ V} \mathrm{vol} \quad \forall \omega, \nu \in \mathrm{Alt}^k\ V.$

Proof. Take $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ with $1 \leq j_{k+1} < ... < j_n \leq n$. Then

$$\begin{aligned}&\langle \operatorname{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\mathrm{Alt}^{n-k} V} \\ &= \operatorname{sgn}(i_1, i_2, ..., i_n) \det \left(\langle u^{i_s}, u^{j_t} \rangle \right)_{k+1 \leq s,t \leq n} \\ &= \operatorname{sgn}(i_1, i_2, ..., i_n) \det \left(\delta_{i_s, j_t} \right)_{k+1 \leq s,t \leq n}.\end{aligned}$$

In the last step we used the fact that u^i are orthonormal since the u_i are. Now observe due to the ordering that $\det \left(\delta_{i_s, j_t} \right)_{k+1 \leq s,t \leq n} = 1$ i.i.f. $i_s = j_s$ for $s = k+1, ..., n$ and is zero otherwise.

For the wedge product we get

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} = \begin{cases} \operatorname{sgn}(i_1, ..., i_n) \ \mathrm{vol}, & \text{if } (i_{k+1}, ..., i_n) = (j_{k+1}, ..., j_n) \\ 0, & \text{otherwise.} \end{cases}.$$

Comparing both expressions we just proved

$$\begin{aligned}&u^{i_1} \wedge ... \wedge u^{i_k} \wedge u^{j_{k+1}} \wedge ... \wedge u^{j_n} \\ &= \langle \operatorname{sgn}(i_1, i_2, ..., i_n) u^{i_{k+1}} \wedge ... \wedge u^{i_n}, u^{j_{k+1}} \wedge ... \wedge u^{j_n} \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol}\,.\end{aligned}$$

Because the $u^{j_{k+1}} \wedge ... \wedge u^{j_n}$ for $j_{k+1} < ... < j_n$ are a basis of $\mathrm{Alt}^{n-k}\ V$, we can deduce that

$$u^{i_1} \wedge ... \wedge u^{i_k} \wedge \nu = \langle \operatorname{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n}, \nu \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol}\,.$$

and thus $\star(u^{i_1} \wedge ... \wedge u^{i_k}) = \operatorname{sgn}(i_1, i_2, ..., i_n)\, u^{i_{k+1}} \wedge ... \wedge u^{i_n}$ as claimed. We see from this that $\star$ maps the orthonormal basis of $\mathrm{Alt}^k\ V$ to an orthonormal basis of $\mathrm{Alt}^{n-k}\ V$ and is thus an isometry.

The other two claims follow from that easily. For any $\omega \in \mathrm{Alt}^k\ V$, $\mu \in \mathrm{Alt}^{n-k}\ V$

$$\begin{aligned}\langle \star \star \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol} &= \star\omega \wedge \mu = (-1)^{k(n-k)} \mu \wedge \star\omega \\ &= (-1)^{k(n-k)} \langle \star\mu, \star\omega \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} \\ &= \langle (-1)^{k(n-k)} \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol}\,.\end{aligned}$$

Then the first claim follows since $\mu \in \mathrm{Alt}^k\ V$ was arbitrary.

For the second claim, take $\omega, \nu \in \mathrm{Alt}^k\ V$, then

$$\omega \wedge \star\nu = \langle \star\nu, \star\omega \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} = \langle \omega, \nu \rangle_{\mathrm{Alt}^k V} \mathrm{vol}\,.$$

□

In particular in $\mathbb{R}^3$, we have $\star\star = \mathrm{Id}$. Notice also that we always have $\star 1 = \mathrm{vol}$.

Let us quickly derive the expression in any basis for the Hodge star applied to linear forms which we will need later. Let $\omega = \sum_i \omega_i b^i \in \mathrm{Alt}^1\ V = V'$. Let us denote $g^{ij} = \langle b^i, b^j \rangle$ and $G_{ij} = \langle b_i, b_j \rangle$ again the Gramian matrix. Then we claim

$$\star\omega = \sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^n \tag{2.1.9}$$

{eq:hodge_star_

where $\widehat{b^j}$ means that it is left out. The proof is very simple in this case. Let ν denote the expression on the right hand side of (2.1.9). For any $1 \leq l \leq n$ we get

$$\begin{aligned}
\nu \wedge b^l &= \left(\sqrt{\det G} \sum_{i,j=1}^{n} \omega_i (-1)^{j-1} g^{ij} b^1 \wedge b^2 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^n \right) \wedge b^l \\
&= \sqrt{\det G} \sum_{j=1}^{n} \langle b^j, \sum_{i=1}^{n} \omega_i b^i \rangle (-1)^{j-1} b^1 \wedge b^2 \wedge \ldots \wedge \widehat{b^j} \wedge \ldots \wedge b^n \wedge b^l \\
&= \sqrt{\det G} \langle b^l, \omega \rangle (-1)^{(l-1)} (-1)^{(n-l)} b^1 \wedge b^2 \wedge \ldots \wedge b^l \wedge \ldots \wedge b^n \\
&= \langle (-1)^{n-1} \omega, b^l \rangle \,\mathrm{vol}\,.
\end{aligned}$$

where the inner product is always in the corresponding space of alternating forms. In the second step, we used that if $j \neq l$ then one basis element must appear twice in the wedge product which is then zero. If $j = l$ then $\mathrm{sgn}(1, 2, \ldots, \hat{l}, \ldots, n, l) = (-1)^{n-l}$ because we need $(n-l)$ transpostions to bring the indices into order. So we proved $(-1)^{n-1}\omega = \star\nu$ and thus $\star\omega = (-1)^{n-1} \star\star\nu = \nu$. This shows also that the explicit expression of the Hodge star becomes much more complicated if the basis is not orthonormal.

Scalar and Vector proxies

{sec:scalar_an

We want to relate alternating maps to elements of the vector space V itself or to scalars. Let us start with the easiest case. $\mathrm{Alt}^0\ V$ are already scalars by definition. Now we can use the Hodge star operator which is an isometry $\star : \mathrm{Alt}^0\ V \to \mathrm{Alt}^n\ V$ with $\star c = c\,\mathrm{vol}$. For $\omega = \lambda\,\mathrm{vol} \in \mathrm{Alt}^n\ V$ we call then $\star^{-1}\omega = \lambda \in \mathbb{R}$ the *scalar proxy* of ω.

Next, we will move on to $\mathrm{Alt}^1\ V$ and $\mathrm{Alt}^{n-1}\ V$. Let $\Phi : V \to V'$ denote the Riesz isomorphism which is an isometry. Because $V' = \mathrm{Alt}^1\ V$, this gives us the correspondence of vectors and linear forms. Now we can once again use the Hodge star and obtain the isometry $\star\Phi : V \to \mathrm{Alt}^{n-1}\ V$ to find the connection between vectors and $(n-1)$-forms. For $\omega \in \mathrm{Alt}^1\ V = V'$ we then call $\Phi^{-1}\omega \in V$ the *vector proxy* of ω. For $\nu \in \mathrm{Alt}^{n-1}\ V$ the vector proxy is $\Phi^{-1} \star^{-1} \nu \in V$.

These ways to identify alternating forms gives us the ability to look at the notions defined above in the context of scalars and vectors. Let us look at the wedge product. We have for $v, w \in V$

$$\Phi v \wedge \star \Phi w = \langle \Phi v, \Phi w \rangle_{V'} \operatorname{vol} = \langle v, w \rangle_V \operatorname{vol}$$

which means that the wedge product of a linear form and an alternating $(n-1)$-linear form corresponds in proxies to the inner product.

In the case of $V = \mathbb{R}^3$ with the standard basis vectors e_1, e_2 and e_3 denote the resulting elements of the dual basis with e^1, e^2 and e^3 respectively. Take $v = v_1 e_1 + v_2 e_2 + v_3 e_3 \in \mathbb{R}^3$ and recall that for a orthonormal basis the Riesz isomorphism maps basis elements to their dual basis elements i.e. $\Phi e_i = e^i$. Hence, we get $\Phi v = v_1 e^1 + v_2 e^2 + v_3 e^3$. Take another $w \in \mathbb{R}^3$. Then using the $e^i \wedge e^j = -e^j \wedge e^i$ we get

$$\Phi v \wedge \Phi w = \star \Phi (v \times w). \tag{2.1.10}$$

{eq:cross_prod

That means in $3D$ in terms of vector proxies, the wedge product of two linear forms corresponds to the cross product. Note that (2.1.10) is formulated without using a specific basis and can therefore be computed using any basis, i.e. if we have $v = \tilde{v}_1 b_1 + \tilde{v}_2 b_2 + \tilde{v}_3 b_3$ and analogous for w we could still calculate the cross product directly as

$$v \times w = \Phi^{-1} \star \big(\Phi v \wedge \Phi w \big).$$

Although it has to be considered that the Riesz isomorphism does not map basis elements b_i to their respective dual basis elements b^i if the basis is not orthonormal. Instead we have

$$\Phi b_i = \sum_{i=1}^{n} \langle b_j, b_i \rangle b^j,$$

i.e. it has the Gramian matrix G as basis representation. This is easy to see. Let $\Phi b_i = \sum_j \lambda_j b^j$. Then

$$\lambda_j = \langle \Phi b_i, b_j \rangle = \langle b_i, b_j \rangle.$$

As derived above, the Hodge star is not as trivial to compute either.

Similarly, we want to explore the pullback in terms of vector proxies as well. These will be important in the next section when we talk about the pullback of differential forms and apply these to the tranformation of integrals. In order to avoid complicated computations we will stick to orthonormal bases. Let $\{b_i\}_{i=1}^n$ be an orthonormal basis of V and $\{c_j\}_{j=1}^m$ be an orthonormal basis of W. Let

$L : V \to W$ again be a linear map and A be the basis representation of it w.r.t. the two bases given, i.e. $Lb_i = \sum_j A_{ji} c_j$. Then we get the pullback of linear forms in terms of vector proxies as $\Phi_V^{-1} L^* \Phi_W : W \to V$. Recall formula (2.1.6) which gives us for the pullback of one forms

$$L^* c^j = \sum_{i=1}^{n} A_{j,i} b^i \tag{2.1.11}$$ {eq:pullback_l

so the matrix representation w.r.t. the given dual bases is $A^\top$. Since we have $\Phi_V b_i = b^i$ and analogous for c^j, the matrix representation of the Riesz isomorphism is just the identity. In total, the basis representation of $\Phi_V^{-1} L^* \Phi_W$ is $A^\top$.

For $m = n$, let us look at the pullback of alternating $(n-1)$-linear maps. In terms of vector proxies this can then be expressed as $\Phi_V^{-1} \star^{-1} L^* \star \Phi_W$. Note that we used the same symbol $\star$, but it is once applied in W and then the inverse in V. It can be shown with the same ideas and (2.1.8) that the matrix representation is the adjugate matrix ad(A) defined as

$$\text{ad}(A)_{ij} = (-1)^{i+j} \det A_{-j,-i}$$

where $A_{-j,-i}$ is the matrix without the j-th row and i-th column. If A is invertible then $(\det A)\, A^{-1} = \text{ad}(A)$.

The pullback of n-linear alternating forms in terms of scalar proxies is $\star^{-1} L^* \star$. Again in the case of $n = m$ and an orthonormal basis we get for $\lambda \in \mathbb{R}$

$$\begin{aligned} \star^{-1} L^* \star \lambda &= \star^{-1} L^* (\lambda\, c^1 \wedge c^2 \wedge ... \wedge c^n) \\ &= \star^{-1} \lambda \det A\, b^1 \wedge b^2 \wedge ... \wedge b^n = \lambda \det A. \end{aligned}$$

2.1.2 Basics from differential geometry

{sec:different

Before we define differential forms, let us start by revising some basics from differential geometry. We follow the approach from [4, Sec. II] for the most part. At first, we will define a manifold and then tangent spaces and vector fields.

Manifolds

In order to formulate the definition of a manifold, let us recall the definition of a topological space.

Definition 2.1.9 (Topological space)**.** A topological space is a set X together with a collection of subsets of X denoted by $\mathcal{T}$ s.t.

- $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

- For $\{U_i \in \mathcal{T} \mid i \in \mathcal{I}\}$ with any index set $\mathcal{I}$, $\bigcup_{i\in\mathcal{I}} U_i \in \mathcal{T}$ and
- $\varnothing, X \in \mathcal{T}$.

The sets contained in $\mathcal{T}$ are called *open*.

For example, a metric space together with its usual open sets is a topological space. A well known example of topologies which do not arise from a metric are the weak and weak-$\star$ topology on infinite dimensional spaces (see [5, Ch. 3]).

Definition 2.1.10 (Second countable topological space)**.** Let $(X, \mathcal{T})$ be a topological space. Then we call $\mathcal{B} \subseteq \mathcal{T}$ a basis for the topology of X if every open set (i.e. every set in $\mathcal{T}$) is a union of sets in $\mathcal{B}$. If a topological space has a countable basis it is called *second countable*.

$\mathbb{R}^n$ with the standard norm is an example of a second countable topological space. Consider the countable set of balls $\{B_r(x) \mid 0 < r \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ where $B_r(x)$ are the balls with center x and radius r. Then it is trivial to show that any open set in $\mathbb{R}^n$ can be written as a union of these balls. Hence, $\mathbb{R}^n$ is second countable.

Definition 2.1.11 (Hausdorff space)**.** Let $(X, \mathcal{T})$ be a topological space. We call $(X, \mathcal{T})$ *Hausdorff* if we can separate any two different points of X with disjoint neighborhoods. That means for any $x, y \in X$, $x \neq y$ there are $U_x, U_y \in \mathcal{T}$ s.t. $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \varnothing$.

Example 2.1.12. Let (X, d) be a metric space and take $x, y \in X$, $x \neq y$. Then for the distance $\delta := d(x, y) > 0$ we can choose the open balls around x and y with radius $\delta/2$, denoted by $B_{\delta/2}(x)$ and $B_{\delta/2}(y)$. These are open and obviously disjoint. Therefore, any metric space is Hausdorff.

Definition 2.1.13 (Manifold)**.** A n-dimensional C^α manifold M, $\alpha \in \mathbb{N}$ (in this thesis the natural numbers start with zero), is a second countable Hausdorff space M with an open cover $\{U_i\}_{i\in I}$ and a collection of maps called *charts* $\boldsymbol{\phi}_i$, $i \in I$ for some index set I s.t.

- $\boldsymbol{\phi}_i : U_i \to V_i \subseteq \mathbb{R}^n$ are homeomorphisms,
- for two charts $\boldsymbol{\phi}_i$, $\boldsymbol{\phi}_j$ the *change of coordinates* or *chart transition* $\boldsymbol{\phi}_j \circ \boldsymbol{\phi}_i^{-1}$: $\boldsymbol{\phi}_i(U_i \cap U_j) \to \boldsymbol{\phi}_j(U_i \cap U_j)$ is a C^α diffeomorphism. If $\alpha = 0$ it is a homeomorphism.

When we write $(U_i, \boldsymbol{\phi}_i)$ we mean the chart $\boldsymbol{\phi}_i$ with domain U_i. We call $\{(U_i, \boldsymbol{\phi}_i)\}_{i\in I}$ an *atlas* of the manifold.

Vector valued quantities like charts will be denoted in bold symbols throughout this thesis. The charts provide us with *local coordinates* $x_k : U_i \to \mathbb{R}^n$, $k = 1, ..., n$ with

$$\mathbf{x}(p) = (x_1(p), ..., x_n(p))^\top = (\phi_1(p), ..., \phi_n(p))^\top = \boldsymbol{\phi}(p) \in \mathbb{R}^n.$$

Let us take a look at an example. Take any open domain $\Omega \subseteq \mathbb{R}^n$. Then we can choose the identity as a chart. Since $\mathbb{R}^n$ with the standard topology is Hausdorff and second countable, the same reasoning applies to open subdomains. Hence, Ω is an n-dimensional C^∞ manifold.

Proposition 2.1.14. *Every manifold has a countable atlas.*

Proof. Take any atlas $\{(U_i, \boldsymbol{\phi}_i)\}_{i \in I}$. Because the manifold is second countable there exists a countable basis $\mathcal{B} = \{B_0, B_1, ...\}$ of the topology. Every basis of a topological space is an open cover which is easily checked. Define $\mathcal{A} = \{k \in \mathbb{N} \mid B_k \subseteq U_i \text{ for some } i \in I\}$. Then for every $k \in \mathcal{A}$ choose $i_k \in I$ s.t. $B_k \subseteq U_{i_k}$. Then $\{U_{i_k}\}_{k \in \mathcal{A}}$ is an open cover and thus $\{(U_{i_k}, \boldsymbol{\phi}_{i_k})\}_{k \in \mathcal{A}}$ is a countable atlas. □

Due to this proposition we will from now on assume that all the chosen atlases are countable. Note that here the second countability is essential. Some authors do not require this in the defintion of a manifold and then this proposition might not hold (cf. [10, 1.A.2]). Another important property is the existence of a partition of unity on M assuming M is smooth which we will not prove.

Theorem 2.1.15 (Partition of unity)**.** *Let M be a smooth (i.e. C^∞) manifold with atlas $\{U_i, \boldsymbol{\phi}_i)\}_{i=0}^N$, $N \leq \infty$. Then there exists a smooth partition of unity subordinate to the open cover $\{U_i\}_{i=0}^N$. That means there exists a family of non-negative smooth functions $\{\chi_i\}_{i=0}^N$ s.t.* $\operatorname{supp} \chi_i \subseteq U_i$ *and* $\sum_{i=0}^N \chi_i(p) = 1$ *for every* $p \in M$.

These partitions of unity are typically used to extend a construction that is done locally to the entire manifold as we will see later.

Let us denote $\mathbb{R}_- := \{x \in \mathbb{R} \mid x \leq 0\}$ and equip $\mathbb{R}_- \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ with the subspace topology i.e. a set $V \subseteq \mathbb{R}_- \times \mathbb{R}^{n-1}$ is open i.i.f. there exists an open set $V' \subseteq \mathbb{R}^n$ s.t. $V = V' \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$. This means, e.g., that $B_1(0) \cap \mathbb{R}_- \times \mathbb{R}^{n-1}$ is open which is not an open set in the standard topology of $\mathbb{R}^n$.

{def:manifold_

Definition 2.1.16 (Manifold with boundary)**.** We call M a *manifold with boundary* if the charts $\boldsymbol{\phi}_i$, $i \in I$, are homeomorphisms from M into $\mathbb{R}_- \times \mathbb{R}^{n-1}$ endowed with the subspace topology. The *boundary* ∂M are the points that get mapped to $\{0\} \times \mathbb{R}^{n-1}$ by the charts, i.e. $\partial M = \bigcup_{i \in I} \boldsymbol{\phi}_i^{-1}(\{0\} \times \mathbb{R}^{n-1})$.

Remark 2.1.17. We should note the relationship between the boundary in the usual topological sense and the above definition of a boundary of a manifold. Let

$\Omega \subseteq \mathbb{R}^n$ be an open C^1 domain. Then $\partial\Omega$ in the usual topological sense is $\overline{\Omega} \setminus \Omega$. However, if we see Ω as a manifold with boundary then the boundary $\partial\Omega$ in the sense of Def. 2.1.16 is empty.

Another important difference is that in the case of manifold $\partial\partial M = \varnothing$, but for the general topological boundary that is not the case. If we have, e.g., a single point $\{x\} \subseteq \mathbb{R}^n$ then $\partial\{x\} = \{x\}$.

Tangent spaces

{sec:tangent_s

In the remainder of the section, we want to investigate tangent spaces on a manifold which require some regularity to be defined. Therefore, we will assume $\alpha > 0$ i.e. our manifolds are differentiable.

Remark 2.1.18. There is also the concept of Lipschitzian manifolds. Using Rademachers theorem, some of the structures that are introduced below can be extended with some care. See [20] for a detailed discussion.

Definition 2.1.19 (Orientation of a manifold)**.** We call an atlas *oriented* if the Jacobians of the coordinate changes have positive determinant. A manifold that can be equipped with an oriented atlas is called *orientable*.

The next important concept we will recall are tangent spaces. It should be noted that there are different definitions of tangent space, but these lead to isomorphic notions (see e.g. [10, Sec. 1.B]).

Definition 2.1.20. Let M, N be an n- and m-dimensional manifold with or without boundary and a function $F : M \to N$. Take $p \in M$ and let $(\boldsymbol{\phi}, U)$ and $(\boldsymbol{\psi}, V)$ be charts at p and $F(p)$ with local coordinates $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ respectively. Then we call the function

$$\bar{F}(x_1, ..., x_n) = \boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1}(x_1, ..., x_n)$$

F *expressed in local coordinates*, which depends on the chosen local coordinates.

For a point $p \in M$ and neighborhoods $U \subseteq M$ of p and $V \subseteq N$ of $F(p)$, let us take local coordinate charts $(\boldsymbol{\phi}, U)$ and $(\boldsymbol{\psi}, V)$ and resulting local coordinates $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ respectively and assume M and N to be at least C^1. We call a function $F : U \to N$ differentiable at p if the expression in local coordinates is differentiable i.e. if $\boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1}$ is differentiable at $\boldsymbol{\phi}(p)$. We define its Jacobian $DF(p) := D(\boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1})(\boldsymbol{\phi}(p))$ and we denote

$$\frac{\partial F_j}{\partial x_i}(p) = \frac{\partial(y_j \circ F \circ \boldsymbol{\phi}^{-1})}{\partial x_i}(\boldsymbol{\phi}(p)) \tag{2.1.12}$$

{eq:derivative_

It is important to notice, that the values of the Jacobian and the derivatives depend on the chosen representation. However, the definition of differentiability is independent of the chart. Let $(\tilde{U}, \tilde{\boldsymbol{\phi}})$ with $p \in \tilde{U}$ be another chart and analogous $(\tilde{V}, \tilde{\boldsymbol{\psi}})$ at $F(p)$. Then

$$\tilde{\boldsymbol{\psi}} \circ F \circ \tilde{\boldsymbol{\phi}}^{-1} = \tilde{\boldsymbol{\psi}} \circ \boldsymbol{\psi}^{-1} \circ \boldsymbol{\psi} \circ F \circ \boldsymbol{\phi}^{-1} \circ \boldsymbol{\phi} \circ \tilde{\boldsymbol{\phi}}^{-1}$$

and since the chart transitions are at least C^1-diffeomorphisms by definition $\tilde{\boldsymbol{\psi}} \circ F \circ \tilde{\boldsymbol{\phi}}^{-1}$ is differentiable as well. These types of definitions via local charts on a manifold are frequent in differential geometry. This is a proper definition if it is independent of the chosen chart. Because we do not want to bother with the technicalities of differential geometry too much we will very often leave out these types of proofs.

Definition 2.1.21 (Tangent space)**.** Let $I \subseteq \mathbb{R}$ be an interval containing 0 and $\gamma : I \to M$ be a differentiable curve with $\gamma(0) = p \in M$. If 0 is on the boundary of I then we take the one sided derivative. Let $f : M \to \mathbb{R}$ be a differentiable function. Let $(\boldsymbol{\phi}, U)$ be a local chart. We define the directional derivative $D_\gamma(f) := \frac{d}{dt} f(\gamma(t))|_{t=0}$. We call the functional $D_\gamma : C^1(U) \to \mathbb{R}$ a *tangent vector*. The real vector space of all tangent vectors at p is called the *tangent space* and denoted by T_pM.

This begs the question why T_pM is actually a vector space. Let $(U, \boldsymbol{\phi})$ again be a local chart at p. We can express a tangent vector D_γ in local coordinates by

$$D_\gamma(f) = \frac{d}{dt} f(\gamma(t))\big|_{t=0} = \frac{d}{dt}(f \circ \boldsymbol{\phi}^{-1} \circ \boldsymbol{\phi})\big(\gamma(t)\big)\big|_{t=0} \tag{2.1.13}$$

$$= \sum_{i=1}^{k} \frac{\partial(f \circ \boldsymbol{\phi}^{-1})}{\partial x_i}\big(\boldsymbol{\phi}(p)\big)\,(\phi_i \circ \gamma)'(0) = \Big(\sum_{i=1}^{k} v_i \frac{\partial}{\partial x_i}\Big|_p\Big)(f) \tag{2.1.14}$$

by taking $v_i = (\phi_i \circ \gamma)'(0)$. Thus, we can express

$$D_\gamma = \sum_{i=1}^{k} (\phi_i \circ \gamma)'(0)\, \frac{\partial}{\partial x_i}\Big|_p .$$

We will now leave out the reference to p in the partial derivative. For the other direction take $\mathbf{v} = (v_1, v_2, ..., v_n)^\top \in \mathbb{R}^n$. We now want to find a differentiable curve γ s.t. $\gamma(0) = p$ and $D_\gamma = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$. For that define $\gamma(t) := \boldsymbol{\phi}^{-1}(\boldsymbol{\phi}(p) + \mathbf{v}t)$. Then

$$D_\gamma = \sum_{i=1}^{n} (\phi_i \circ \gamma)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} (\phi_i(p) + v_i t)'(0) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}.$$

This shows that T_pM is a linear space and

$$T_pM = \text{span}\left\{ \frac{\partial}{\partial x_i}\bigg|_p \right\}_{i=1}^n .$$

In order to show that this is a basis, we have to prove linear independence. Assume we have $\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} = 0$. Then $\phi_j \circ \boldsymbol{\phi}^{-1}(\mathbf{x}) = x_j$ for $\mathbf{x} \in \boldsymbol{\phi}(U)$ and $1 \leq j \leq n$ which implies

$$0 = \left(\sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} \right)(\phi_j) = \sum_{i=1}^n \lambda_i \frac{\partial(\phi_j \circ \boldsymbol{\phi}^{-1})}{\partial x_i}(\boldsymbol{\phi}(p)) = \lambda_j$$

so $\frac{\partial}{\partial x_i}$ are linearly independent and thus a basis of T_pM. To summarize, we have shown

Proposition 2.1.22. *The set of tangent vectors at a point $p \in M$ is a vector space. The derivatives w.r.t. the local coordinates $\frac{\partial}{\partial x_i}$ are tangent vectors and*

$$T_pM = \text{span}\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n .$$

Now that we introduced tangent spaces we will define the most important mapping between them.

Definition 2.1.23 (Differential)**.** Let M, N be an n and m dimensional manifold respectively. Take $p \in M$ and let $F : M \to N$ be differentiable at p. Then we define the differential of F at point p,

$$F_{*,p} : T_pM \to T_{F(p)}N, D_\gamma \mapsto D_{F\circ\gamma}.$$

Proposition 2.1.24. *Let $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ be local coordinates at p and $F(p)$ of the charts $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and let $\frac{\partial}{\partial x_i}|_p$ and $\frac{\partial}{\partial y_j}|_{F(p)}$ be the resulting bases for the tangent spaces at T_pM and $T_{F(p)}N$ respectively. Then the resulting matrix representation of the differential $F_{*,p} : T_pM \to T_{F(p)}N$ is the Jacobian as defined at (2.1.12) i.e.*

$$F_{*,p}\left(\frac{\partial}{\partial x_i}\bigg|_p \right) = \sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}\bigg|_{F(p)}.$$

Proof. We will omitt the reference to the points of the partial derivatives. We choose $\gamma = \boldsymbol{\phi}^{-1}(\boldsymbol{\phi}(p) + e_i t)$. Then $D_\gamma = \frac{\partial}{\partial x_i}$ and by applying the chain rule and the above definitions, we compute

$$
\begin{aligned}
F_*\left(\frac{\partial}{\partial x_i}\right)(f) &= D_{F\circ\gamma}(f) = \frac{d}{dt}\big(f(F\circ\gamma(t))\big)|_{t=0} \\
&= \frac{d}{dt}\big(f\circ\boldsymbol{\psi}^{-1}\circ\boldsymbol{\psi}\circ F\circ\boldsymbol{\phi}^{-1}\circ\boldsymbol{\phi}\circ\gamma(t)\big)|_{t=0} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\partial f\circ\boldsymbol{\psi}^{-1}}{\partial y_j}\Big(\boldsymbol{\psi}\big(F(p)\big)\Big)\ \frac{\partial(\boldsymbol{\psi}\circ F\circ\boldsymbol{\phi}^{-1})_j}{\partial x_i}\big(\boldsymbol{\phi}(p)\big)(x_i\circ\gamma)'(0) \\
&= \sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial f}{\partial y_j}\big(F(p)\big) = \left(\sum_{j=1}^{n}\frac{\partial F_j}{\partial x_i}(p)\frac{\partial}{\partial y_j}\right)(f).
\end{aligned}
$$

□

For everything we did above, we fixed one chart $(U,\boldsymbol{\phi})$ at p. Now the question arises what happens when we choose a different chart $(\tilde{U},\tilde{\boldsymbol{\phi}})$ instead. Going through the same steps we end up with another basis $\{\frac{\partial}{\partial\tilde{x}_j}\}_{j=1}^n$ of the tangent space T_pM which are the derivatives w.r.t. the chart $(\tilde{U},\tilde{\boldsymbol{\phi}})$. Let us compute the change of basis. Using the chain rule we can easily compute that

$$
\frac{\partial(f\circ\boldsymbol{\phi}^{-1})}{\partial x_i}\big(\boldsymbol{\phi}(p)\big) = \sum_{j=1}^{n}\frac{\partial(f\circ\tilde{\boldsymbol{\phi}}^{-1})}{\partial y_j}\big(\tilde{\boldsymbol{\phi}}(p)\big)\ \frac{\partial(\tilde{\boldsymbol{\phi}}\circ\boldsymbol{\phi}^{-1})_j}{\partial x_i}\big(\boldsymbol{\phi}(p)\big)
$$

and we recognize that the change of basis matrix is the Jacobian of the chart transition $D(\tilde{\boldsymbol{\phi}}\circ\boldsymbol{\phi}^{-1})(\boldsymbol{\phi}(p))$.

A *vector field* X maps every point p to a tangent vector in the corresponding tangent space i.e. by using local coordinates

$$
X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i}
$$

with $X_i(p)\in\mathbb{R}$. The regularity of a vector field is defined via the regularity of its coefficient e.g. a vector field is differentiable if all its coefficients are. We must be careful though since we require higher regularity of the manifold for this to be well-defined. Assume X is differentiable i.e. X_i are differentiable. Let $(\tilde{U},\tilde{\boldsymbol{\phi}})$ be another local chart. From the change of basis of the tangent space we know that

$$
X(p) = \sum_{i=1}^{n} X_i(p)\frac{\partial}{\partial x_i} = \sum_{i,j=1}^{n} X_i(p)\frac{\partial(\tilde{\boldsymbol{\phi}}\circ\boldsymbol{\phi})_j}{\partial x_i}\big(\boldsymbol{\phi}(p)\big)\frac{\partial}{\partial y_j}
$$

and we see the coefficients w.r.t. to the new basis are

$$\tilde{X}_j = \sum_{i=1}^{n} X_i(p) \frac{\partial(\tilde{\boldsymbol{\phi}} \circ \boldsymbol{\phi})_j}{\partial x_i} \big(\boldsymbol{\phi}(p)\big)$$

So we need $\frac{\partial(\tilde{\boldsymbol{\phi}} \circ \boldsymbol{\phi})_j}{\partial x_i}$ to be differentiable as well which means that we want the manifold to be at least C^2.

We should briefly mention the question of orientation of the tangent spaces. Recall that we partitioned the bases of a finite-dimensional real vector in two orientations. We say that two bases have the same orientation if the change of basis matrix has positive determinant. We want to choose an orientation on the tangent spaces consistently which will be crucial when defining the volume form below.

Assume the manifold M is orientable and we have chosen an oriented atlas. For any tangent space T_pM with local coordinates x_i near p, we define the resulting basis $\frac{\partial}{\partial x_i}$ as positively oriented. This fixes the orientation of the vector space. We have to show that this is well-defined. But this is clear since for a different chart at p from the oriented atlas resulting in a different basis elements $\frac{\partial}{\partial y_j}$, we know that the change of basis is the Jacobian of the chart transition. But the Jacobian of the chart transition has positive determinant by definition and so the basis $\frac{\partial}{\partial y_j}$ is positively oriented as well.

2.1.3 Differential forms

{sec:different

Now that we introduced the necessary objects from differential geometry, we can finally define differential forms on manifolds and investigate operations on them. In particular, we will transfer many notions from alternating multilinear forms to differential forms.

Definition 2.1.25 (Differential form)**.** A differential k-form ω maps any point $p \in M$ to an alternating k-linear form $\omega_p \in \operatorname{Alt}^k T_pM$. We denote the space of differential k-forms on M as $\Lambda^k M$.

Let T_p^*M be the dual space of T_pM which is usually called *cotangent space*. As before let us choose a local chart $\boldsymbol{\phi} : U \to \mathbb{R}^n$ with $p \in U$ and define $\frac{\partial}{\partial x_i}|_p$ the basis for the tangent space. Denote the corresponding dual basis as dx^i, $i = 1, ..., n$, i.e. $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. From the consideration about alternating maps from Section 2.1.1, we can now write any $\omega \in \Lambda^k M$ with

$$\omega_p = \sum_{1 \le i_1 < ... < i_k \le n} a_{i_1,...,i_k}(p)\, dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k}$$

with $a_{i_1,...,i_k}(p) \in \mathbb{R}$. The regularity of differential forms is then defined via the regularity of these coefficients i.e. we call a differential form smooth if all the $a_{i_1,...,i_k}$ are smooth and we call a differential form differentiable if all the $a_{i_1,...,i_k}$ are differentiable and so on. These definitions of regularity again require the manifold to be sufficiently regular as well.

We denote the space of smooth differential k-forms as $C^\infty \Lambda^k M$ and analogous for other regularity. $C_c^\infty \Lambda^k(M)$ are the smooth differential forms with compact support where the closure is w.r.t. the topology on M. Note that here the topology is crucial because for a manifold with boundary $\omega \in C_c^\infty \Lambda^k(M)$ are not necessarily zero on the boundary.

In order to define the Hodge star and an inner product on differential forms we need that T_pM is an inner product space. A Riemannian metric gives us at every point $p \in M$ a symmetric, positive definite bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$.

Definition 2.1.26 (Riemannian metric)**.** A Riemannian metric g maps every point $p \in M$ to a symmetric, positive definite bilinear form $g_p : T_p \times T_p \to \mathbb{R}$. After choosing local coordinates $\{x_i\}_{i=1}^n$, we frequently denote $g_{p,ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. We also require that for C^α vector fields X and Y the map $g(X,Y)$ is also C^α . A manifold with a Riemannian metric is called *Riemannian manifold*.

From now on, we will leave out the reference to the point p where appropriate. The Riemannian metric provides us with the inner product on every tangent space T_pM.

As explained in the section about alternating multilinear forms, T_p^*M is also an inner product space where the inner product is defined via the Riesz isomorphism. We denote this inner product as $\langle \cdot, \cdot \rangle_{T_p^*M}$. Just as defined in (2.1.3) we also obtain an inner product on $\mathrm{Alt}^k\, T_pM$.

We will from now on assume that M is an oriented Riemannian manifold of sufficient regularity and denote the Riemannian metric by g. Sufficient regularity means here that M is regular enough s.t. all definitions and operations are well-defined. Let $p \in M$ and T_pM be the tangent space at the point p. Due to our assumptions on M, this is an oriented inner product space of dimension n and we can apply all of the constructions from the previous chapter. We will define the volume form, vector and scalar proxies and finally pullbacks of differential forms.

Let us fix a point p and a chart ϕ at this point with local coordinates denoted by x_i, $i = 1, ..., n$. The resulting Gramian matrix is $(G_p)_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. So we have a volume form vol on M

$$\mathrm{vol}_p = \sqrt{\det G_p}\, dx^1 \wedge ... \wedge dx^n.$$

The volume form then depends on the chosen orientation, but not on the chosen local coordinates.

Now that we have a volume form we can apply the Hodge star operator pointwise on differential forms to get $\star : \Lambda^k(M) \to \Lambda^{n-k}(M)$ i.e. for a differential form $\omega \in \Lambda^k M$, $(\star\omega)_p = \star\omega_p$. We do the same for the wedge product $\wedge : \Lambda^k M \times \Lambda^l M \to \Lambda^{k+l} M$,

$$(\omega \wedge \nu)_p = \omega_p \wedge \nu_p.$$

Recall from Prop. 2.1.8 that we then have for $\omega, \nu \in \Lambda^k M$ and $\mu \in \Lambda^{n-k} M$

$$\begin{aligned}
\omega_p \wedge \mu_p &= \langle \star\omega_p, \mu_p \rangle_{\mathrm{Alt}^{n-k}\, T_pM} \operatorname{vol}_p \\
\star \star \omega &= (-1)^{k(n-k)} \omega \\
\omega_p \wedge \star\nu_p &= \langle \omega_p, \nu_p \rangle_{\mathrm{Alt}^{k}\, T_pM} \operatorname{vol}.
\end{aligned}$$

In order for the Hodge star to be well-defined the assumption of an orientation on our manifold is crucial.

We want to apply two important concepts from the previous section about alternating maps – vector proxies and pullbacks – to differential forms. The following is based on [2, Ch. 6], but many details have been added and additional examples are given. Recall, that for a real n-dimensional vector space V we had two ways to identify a vector $v \in V$ with an alternating map. Either as a linear form Φv where Φ is the Riesz isomorphism or as a $(n-1)$-linear alternating map $\star\Phi v$.

We can now identify every vector field with a 1-form or an $(n-1)$-form. $p \mapsto \Phi_{T_pM} X(p)$ defines a 1-form and $p \mapsto \star\Phi_{T_pM} X(p)$ gives us an $(n-1)$-form. In differential geometry, the usual notation is $\Phi_{T_pM} X(p) = X^\flat(p)$. The inverse of $^\flat$ is the $^\sharp$ operator. The isomorphisms $^\flat$ and $^\sharp$ are fittingly called *musical isomorphisms*. With these musical isomorphisms we can identify X with the 1-form $X^\flat$ or the $(n-1)$-form $\star X^\flat$. Vice versa, we find for $\omega \in \Lambda^1 M$ the *vector proxy* $\omega^\sharp$ and for an $(n-1)$-form $\nu \in \Lambda^{n-1} M$ $(\star^{-1}, \nu)^\sharp$.

Recall that the matrix representation of the Riesz isomorphism is the Gramian matrix $G = (g_{ij})_{1 \le i,j \le n}$. So if we have a vector field $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and define $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$. Then we can compute the associated 1-form

$$X^\flat = \sum_{j=1}^{n} (G\mathbf{X})_j dx^j = \sum_{i,j=1}^{n} g_{ij} X_i dx^j.$$

If the basis of the tangent space $\frac{\partial}{\partial x_i}$ are orthonormal then we have $(\frac{\partial}{\partial x_i})^\flat = dx^i$ because the Riesz isomorphism maps basis elements to their dual basis elements in this case.

Next, let us have a look at how we can extend pullbacks to differential forms. Recall again that for a linear map $L : V \to W$ with an n-dimensional real vector space V and an m-dimensional real vector space W we define its pullback $L^* : \mathrm{Alt}^k\, W \to \mathrm{Alt}^k\, V$ via

$$L^*\omega(v_1, ..., v_k) = \omega(Lv_1, ..., Lv_k).$$

We wish to use the analogous idea with differential forms.

Definition 2.1.27. Let M and N be sufficiently smooth, oriented Riemannian manifolds. For $\omega \in \Lambda^k N$ and a differentiable map $F : M \to N$ we define the pullback $F^*\omega$ as

$$(F^*\omega)_p = (F_{*,p})^*\omega_{F(p)},$$

i.e. for all $v_1, ..., v_k \in T_pM$

$$(F^*\omega)_p(v_1, ..., v_k) = \omega_{F(p)}(F_{*,p}v_1, ..., F_{*,p}v_k).$$

As an immediate consequence of this definition, we inherit from the corresponding fact about alternating forms that the pullback commutes with the wedge product i.e. for $\omega \in \Lambda^k M$, $\nu \in \Lambda^l M$

$$F^*(\omega \wedge \nu) = F^*\omega \wedge F^*\nu$$

Now we can use the vector proxies and connect it to what we have done for alternating maps above.

{prop:pullback_

Proposition 2.1.28. *Let X be a vector field on N. Take $p \in M$ and local coordinates $\{x_i\}_{i=1}^n$ at p and $\{y_j\}_{j=1}^m$ at $F(p)$. Then we can write $X = \sum_{j=1}^m X_i \frac{\partial}{\partial x_i}$ and define $\mathbf{X} = (X_1, ..., X_n)^\top$. We assume that the bases $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ and $\{\frac{\partial}{\partial y_j}\}_{j=1}^m$ of T_pM and $T_{F(p)}N$ are orthonormal w.r.t. the Riemannian metrics on M and N respectively. Then we obtain for the pullback in terms of vector proxies of 1-forms*

$$(F^*X^\flat)^\sharp(p) = \sum_{j=1}^n \Big(DF(p)^\top \mathbf{X}\big(F(p)\big)\Big)_j \frac{\partial}{\partial y_j}.$$

If we assume $m = n$ additionally, then

$$(\star^{-1}F^* \star X^\flat)^\sharp = \sum_{j=1}^n \Big(\mathrm{ad}\,\big(DF(p)\big)\mathbf{X}\Big)_j \frac{\partial}{\partial y_j}$$

where $\mathrm{ad}(DF(p))$ is the adjugate matrix of $DF(p)$.

Proof. The resulting follows essentially immediately from applying the corresponding results for alternating maps. Applying the definition of $^\flat$ and $^\sharp$

$$(F^* X^\flat)^\sharp(p) = \Phi^{-1}_{T_pM} F^*_{*,p} \Phi_{T_{F(p)}N} X(p) = \sum_{j=1}^{n} \Big(DF(p)^\top \mathbf{X}\big(F(p)\big)\Big)_j \frac{\partial}{\partial y_j}$$

where we used in the last step the matrix representation of the pullback of vector proxies of linear forms from (2.1.11). The analogous reasoning works for vector proxies of $(n-1)$-forms i.e. the second claim. $\square$

Let $M = \hat{\Omega} \subseteq \mathbb{R}^n$ and $N = \Omega \subseteq \mathbb{R}^m$ be open domains and assume $F : \hat{\Omega} \to \Omega$ is a diffeomorphism. Then we can choose the identity as a chart and obtain the bases $\frac{\partial}{\partial \hat{x}_i}$ and $\frac{\partial}{\partial x_i}$ of the tangent spaces. Using this basis, we can identify a vector field $\sum_{i=1}^m X_i \frac{\partial}{\partial x_i}$ with $\mathbf{X} = (X_1, ..., X_m)^\top$. We then can interpret $\mathbf{X}$ either as a vector proxy of a 1- or $(n-1)$-form and obtain the pullbacks

$$\mathcal{P}^1_F \mathbf{X}(\hat{x}) := DF(\hat{x})^\top \mathbf{X}\big(F(\hat{x})\big) \text{ and} \tag{2.1.15}$$ {eq:pullback_v

$$\mathcal{P}^{n-1}_F \mathbf{X}(\hat{x}) := \big(\det DF(\hat{x})\big)\, DF(\hat{x})^{-1} \mathbf{X}(F(\hat{x})) \tag{2.1.16}$$ {eq:piola_tran

where we used the fact that F is a diffeomorphism. Hence, its Jacobian is invertible and then $\operatorname{ad} DF(\hat{x}) = \big(\det DF(\hat{x})\big)\, DF(\hat{x})^{-1}$. (2.1.16) is widely known as the Piola transformation [8, Def. 9.8].

{rem:vector_pr

Remark 2.1.29. Notice that for $n = 2$, this gives us two ways to identify a vector field with a 1-form. For a vector field $X = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2}$, $X^\flat$ and $\star X^\flat$ are both 1-forms. Depending on the choice, this results in one of the pullbacks of Prop. 2.1.28 and consequently either the pullback (2.1.15) or (2.1.16).

Now let us move on to scalar proxies. Therefore, let $\rho : \Omega \to \mathbb{R}$ be a scalar field i.e. a 0-form. In this case, we have the simple expression for the pullback $F^*\rho = \rho \circ F$.

But ρ could also be the scalar proxy of the n-form $\star\rho = \rho \operatorname{vol}$ with $\operatorname{vol} = dx^1 \wedge dx^2 \wedge ... \wedge dx^n$ being the volume form on Ω. On $\hat{\Omega}$ we denote the volume form $\widehat{\operatorname{vol}} = d\hat{x}^1 \wedge d\hat{x}^2 \wedge ... \wedge d\hat{x}^n$. Then in scalar proxies we obtain the pullback

$$\begin{aligned}
(\mathcal{P}^n_F \rho)(\hat{x}) &= (\star^{-1} F^* \star \rho)_{\hat{x}} = \star^{-1} (F^* \star \rho)_{\hat{x}} = \star^{-1} (F_{*,\hat{x}})^* (\star\rho)_{F(\hat{x})} \\
&= \star^{-1} (F_{*,\hat{x}})^* \rho\big(F(\hat{x})\big)\, dx^1 \wedge ... \wedge dx^n \\
&= \star^{-1} \rho\big(F(\hat{x})\big) \det DF(\hat{x})\, d\hat{x}^1 \wedge ... \wedge d\hat{x}^n \\
&= (\rho \circ F)(\hat{x}) \big(\det DF(\hat{x})\big) \star^{-1} \widehat{\operatorname{vol}} = (\rho \circ F)(\hat{x}) \det DF(\hat{x}).
\end{aligned}$$

This is strikingly similar to the integrand in the standard transformation of integrals formula which will become crucial in the next section where we talk about the integration of differential forms.

The next part of this section will be concerned with introducing a derivative for differential forms.

Definition 2.1.30 (Exterior derivative)**.** Let $\omega \in C^1\Lambda^k M$ be given in local coordinates as

$$\omega_p = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1,\ldots,i_k}(p)\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

Then we define the exterior derivative $d : C^1\Lambda^k M \to C^0\Lambda^{k+1}M$ as

$$(d\omega)_p = \sum_{1 \le i_1 < \ldots < i_k \le n} \sum_{i=1}^{n} \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_i}(p)\, dx^i \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

We remind that the derivative of $a_{i_1,\ldots,i_k}$ is meant w.r.t. to local coordinates as defined at (2.1.12).

It can be shown that $d\omega$ is independent of the chosen coordinates. We call a differential form $\omega \in \Lambda^k M$ *closed* if $d\omega = 0$ and *exact* if there exists $\nu \in \Lambda^{k-1}$ s.t. $d\nu = \omega$. We denote the closed k-forms as $\mathfrak{Z}^k$ and the exact k-forms as $\mathfrak{B}^k$. We have $d \circ d = 0$ and thus $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$.

Let us mention some important properties of the exterior derivative. The relation to the wedge product is described by a Leibniz-type formula. Let $\omega \in C^1\Lambda^k M$ and $\nu \in C^1\Lambda^l M$. Then

$$d(\omega \wedge \nu) = d\omega \wedge \nu + (-1)^k\, \omega \wedge d\nu. \tag{2.1.17}$$ {eq:leibniz_fo

The exterior derivative commutes with the pullback, i.e. for manifolds M and N a differentiable mapping $F : M \to N$ and $\omega \in \Lambda^k N$, we have $dF^*\omega = F^* d\omega$. In terms of proxies this is related to very interesting results as we will see later.

Let us investigate the exterior derivative in the case when $M = \Omega \subseteq \mathbb{R}^n$ is an open subdomain. It turns out that by using scalar and vector proxies as introduced above we can identify the exterior derivative with well-known differential operators. We will use standard Euclidian coordinates meaning that our tangent basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ is orthonormal.

Let us start with a differentiable function $f : \Omega \to \mathbb{R}$ i.e. f is a 0-form. Then

$$(df)^\sharp = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx^i \right)^\sharp = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

which we identify with $\operatorname{grad} f$. In other words, the vector proxy of the exterior derivative of a 0-form corresponds to the gradient.

Let $X = \sum_i X_i \frac{\partial}{\partial x_i}$ be a differentiable vector field on Ω and take $\mathbf{X} = (X_1, X_2, ..., X_n)^\top : \Omega \to \mathbb{R}^n$. X is the vector proxy of the $(n-1)$-form $\star X^\flat$. Then

$$\begin{aligned}\star^{-1} d \star X^\flat &= \star^{-1} d \star \sum_{i=1}^{n} X_i \, dx^i = \star^{-1} d \sum_{i=1}^{n} X_i (-1)^{i-1} \, dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ &= \star^{-1} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} (-1)^{i-1} \, dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n \\ &= \star^{-1} \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} (-1)^{2(i-1)} \, \mathrm{vol} = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = \operatorname{div} \mathbf{X}.\end{aligned}$$

The hat symbol used for $\widehat{dx^i}$ means that this term is left out. So in the case of a $(n-1)$-form, the exterior derivative corresponds to the divergence.

In the case $n = 3$, if we identify $\mathbf{X}$ with a 1-form, then we obtain by using similar computations that $(\star dX^\flat)^\sharp$ corresponds to $\operatorname{curl} \mathbf{X}$. So in 3D we can identify all the exterior derivatives with known differential operators and thereby putting them into a more general framework.

A very nice conclusion can be seen directly from the above computations. The expression on the left hand side does not use any coordinates. Hence, we can use any coordinate system we want and can then compute the divergence in any coordinates we need which we will do next. Note however that the computations are more cumbersome when the bases are not orthonormal.

So let $\mathbf{X} : \Omega \to \mathbb{R}^n$ be differentiable and define the vector field $\sum_i X_i \frac{\partial}{\partial x_i}$. We computed above that

$$\operatorname{div} \mathbf{X} \, \mathrm{vol} = d \star X^\flat.$$

Now let us express $\mathbf{X}$ using different coordinates. Let $\boldsymbol{\phi} : \Omega \to \hat{\Omega} \subseteq \mathbb{R}^n$ be a diffeomorphism. This defines our new coordinates. Assume that

$$\mathbf{X} = \sum_{j=1}^{n} \tilde{X}_j \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_j}.$$

Then by using $\boldsymbol{\phi}$ as a chart

$$X = \sum_i X_i \frac{\partial}{\partial x_i} = \sum_{ij} \tilde{X}_j \frac{\partial (\boldsymbol{\phi}^{-1})_i}{\partial \hat{x}_j} \frac{\partial}{\partial x_i} = \sum_j \tilde{X}_j \frac{\partial}{\partial \hat{x}_j}$$

Let $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_n)^\top$ and

$$G_{kl} = g\Big(\frac{\partial}{\partial \hat{x}_k}, \frac{\partial}{\partial \hat{x}_l}\Big) = \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_k} \cdot \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \hat{x}_l}$$

Then we use the expression of the Hodge star operator for one forms (2.1.9) to compute

$$
\begin{aligned}
(\operatorname{div}\mathbf{X})\,\mathrm{vol} &= d\star X^\flat = d\star\sum_{j=1}^n (G\tilde{\mathbf{X}})_j\, d\hat{x}^j \\
&= d\sum_{j=1}^n\sum_{k=1}^n (G\tilde{\mathbf{X}})_j \sqrt{\det G} g^{jk}(-1)^{k-1} d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge\widehat{d\hat{x}^k}\wedge\ldots\wedge d\hat{x}^n \\
&= \sum_{k=1}^n \frac{\partial(\sqrt{\det G}\sum_{j=1}^n g^{jk}(G\tilde{\mathbf{X}})_j)}{\partial\hat{x}_k}(-1)^{k-1} d\hat{x}^k\wedge d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge\widehat{d\hat{x}^k}\wedge\ldots\wedge d\hat{x}^n \\
&= \sum_{k=1}^n \frac{\partial(\sqrt{\det G}\,\tilde{\mathbf{X}})_k}{\partial\hat{x}_k}(-1)^{2(k-1)} d\hat{x}^1\wedge d\hat{x}^2\wedge\ldots\wedge d\hat{x}^k\wedge\ldots\wedge d\hat{x}^n \\
&= \left[\frac{1}{\sqrt{\det G}}\sum_{k=1}^n \frac{\partial(\sqrt{\det G}\,\tilde{\mathbf{X}})_k}{\partial\hat{x}_k}\right]\mathrm{vol}
\end{aligned}
$$

where we used that $(G^{-1})_{ij} = \langle dx^i, dx^j\rangle_{T_p^*M} = g^{ij}$ and we find the well-known expression for the divergence in general coordinates

$$
\operatorname{div}\mathbf{X} = \frac{1}{\sqrt{\det G}}\sum_{k=1}^n \frac{\partial(\sqrt{\det G}\,\tilde{X}_k)}{\partial\hat{x}_k}.
$$

As mentioned above, let us investigate the consequences of the commutativity of the exterior derivative and the pullback in terms of vector and scalar proxies. Let $F:\widehat{\Omega}\to\Omega$ be a diffeomorphism with $\widehat{\Omega},\Omega\subseteq\mathbb{R}^n$ with volume forms $\widehat{\mathrm{vol}}$ and vol. Let us again consider Euclidian coordinates on Ω and $\widehat{\Omega}$. Then observe

$$
\begin{aligned}
\mathcal{P}_F^n(\operatorname{div}\mathbf{X})(\hat{x})\widehat{\mathrm{vol}}_{\hat{x}} &= (\operatorname{div}\mathbf{X})(F(\hat{x}))\det DF(\hat{x})\widehat{\mathrm{vol}}_{\hat{x}} = (F^*(\operatorname{div}\mathbf{X}))_{\hat{x}}\wedge(F^*\,\mathrm{vol})_{\hat{x}} \\
&= \big(F^*\big(\operatorname{div}\mathbf{X}\wedge\mathrm{vol}\big)\big)_{\hat{x}} = \big(F^*\big(\operatorname{div}\mathbf{X}\,\mathrm{vol}\big)\big)_{\hat{x}} \\
&= \big(F^* d\star X^\flat\big)_{\hat{x}} = \big(dF^*\star X^\flat\big)_{\hat{x}}\,.
\end{aligned}
$$

where we used that F^* commutes with the pullback and the wedge product. Note that the wedge product is just the scalar multiplication if one of the factors is a 0-form. Take $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$. Then we know from Prop. 2.1.28 and (2.1.16) by defining $(\hat{X}_1,\hat{X}_2,...,\hat{X}_n)^\top = \hat{\mathbf{X}} = \mathcal{P}_F^{n-1}\mathbf{X}$ and $\hat{X} = \sum_{i=1}^n \hat{X}_i\frac{\partial}{\partial\hat{x}_i}$

$$
dF^*\star X^\flat = d\star\star^{-1}F^*\star X^\flat = d\star\hat{X}^\flat = \widehat{\operatorname{div}}\hat{\mathbf{X}}\,\widehat{\mathrm{vol}}
$$

and we recognize

$$
\mathcal{P}_F^n(\operatorname{div}\mathbf{X}) = \widehat{\operatorname{div}}\mathcal{P}_F^{n-1}\mathbf{X}.
$$

To summarize the situation in 3D,

{prop:pullback_

Proposition 2.1.31. *Let $F : \hat{\Omega} \to \Omega$ be a diffeomorphism, $\rho \in C^1(\Omega)$ and $\mathbf{X} \in C^1(\Omega; \mathbb{R}^3)$. In scalar and vector proxies for 3D, we obtain the following expressions for pullbacks*

$$\begin{aligned}
(\mathcal{P}_F^0 \rho)(\hat{x}) &= \rho\big(F(\hat{x})\big) \\
(\mathcal{P}_F^1 \mathbf{X})(\hat{x}) &= DF(\hat{x})^\top \mathbf{X}\big(F(\hat{x})\big) \\
(\mathcal{P}_F^2 \mathbf{X})(\hat{x}) &= \det DF(\hat{x}) DF(\hat{x})^{-1} \mathbf{X}\big(F(\hat{x})\big) \\
(\mathcal{P}_F^3 \rho)(\hat{x}) &= \det DF(\hat{x}) \rho\big(F(\hat{x})\big)
\end{aligned}$$

and then the commuting properties

$$\begin{aligned}
\widehat{\mathrm{grad}} \mathcal{P}_F^0 \rho &= \mathcal{P}_F^1 (\mathrm{grad}\, \rho) \\
\widehat{\mathrm{curl}} \mathcal{P}_F^1 \mathbf{X} &= \mathcal{P}_F^2 (\mathrm{curl}\, \mathbf{X}) \\
\widehat{\mathrm{div}} \mathcal{P}_F^2 \mathbf{X} &= \mathcal{P}_F^3 (\mathrm{div}\, \mathbf{X}).
\end{aligned}$$

We proved the last statement and the other two can be proven analogously. These commuting properties are useful for applications in finite elements (see e.g. [8, Sec. 14.3]) and we will use them later when we define finite elements for approximating solutions of the magnetostatic problem in Sec. 3.3.2.

2.1.4 Integration of differential forms

{sec:integrati

Differential k-forms can be integrated over k-dimensional manifolds and the properties of this operation, in particular Stokes' theorem, are one main motivation for working with differential forms in the first place. In many books, integration is only defined for smooth and compactly supported forms (see e.g. [4, Sec. V.3]). This approach is easier, but slightly unsatisfying since then we can not put the integration of forms into the framework of standard Lebesgue integration theory and it is not possible to define L^p and Sobolev spaces of differential forms properly. Thus, even though we will actually only integrate smooth differential forms over compact manifolds in the proofs of existence and uniqueness, we want to introduce the integration more generally. For details about the more general theory of functional spaces on manifolds, see [15, Sec. 10.2.4]. For this section, basic knowledge about measure and Lebesgue integration theory is required.

Throughout this section, we assume that M is a smooth orientable Riemannian manifold of dimension n. We assume that we have chosen a countable oriented atlas $\{(U_i, \boldsymbol{\phi}_i)\}_{i=1}^N$ with $N \le \infty$. Many of the statements in this section can be proven using straightforward computations which will be omitted most of the time. Before we can talk about integration of differential forms let us first investigate the integration of functions on a manifold.

Integration of functions on a manifold

We want to define integration in the framework of usual measure and integration theory which means defining it as a Lebesgue integral w.r.t. a measure on M that we have to define first along with a σ-algebra on M.

It is well known that the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^n$ is generated by all open sets. This idea can be applied to any topological space X by defining the Borel σ-algebra $\mathcal{B}(X)$ as the σ-algebra generated by all open sets as well. So we can simply use the topology on M to define our σ-algebra $\mathcal{B}(M)$. Because we know that all our charts $\boldsymbol{\phi}_i : U_i \to \mathbb{R}^n$ are homeomorphisms it is straightforward to show that a set $E \in \mathcal{B}(M)$ i.i.f. $\boldsymbol{\phi}_i(E \cap U_i) \subseteq \mathbb{R}^n$ is Borel-measurable for all i.

Now, we need to define a measure on the manifold. The following motivation is taken from [10, 3.H.2]. In Euclidian space $\mathbb{R}^n$, we use the standard Lebesgue measure that gives volume one to the unit cube. If we now take any vectors $v_1, v_2, ..., v_n \in \mathbb{R}^n$ then the parallelepiped spanned by these vectors has volume $\det(\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n) = \sqrt{\det\left(\langle \mathbf{v}_i, \mathbf{v}_j \rangle\right)_{1\le i,j\le n}}$ where we used the standard inner product on $\mathbb{R}^n$.

If we now extend this idea to Riemannian manifolds then we are interested in the parallelepiped spanned by the tangent vectors $\frac{\partial}{\partial x_i}$ in the tangent space T_pM with $p \in M$ which would then have volume $\sqrt{\det\left(g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)_{1\le i,j\le n}} = \sqrt{\det G_p}$. Then we define for a Borel-measurable $E \subseteq U$ where U is the domain of a chart $\boldsymbol{\phi} : U \to \mathbb{R}^n$ the measure

$$V(E) := \int_{\boldsymbol{\phi}(E)} \sqrt{\det G_{\boldsymbol{\phi}^{-1}(x)}}\, dx$$

which we will now extend globally to the whole manifold. We do this by "gluing" the local measures together using a partition of unity subordinate to our open cover. Let $\{\chi_i\}_{i=1}^N$, $N \le \infty$, be the partition of unity subordinate to the open cover $\{U_i\}_{i=0}^N$ given by the atlas. Let $\mathbf{x}^{(i)} = (x_1^{(i)}, x_2^{(i)}, ..., x_n^{(i)})^\top$ be the local coordinates of the chart $(U_i, \boldsymbol{\phi}_i)$ and then the $G_p^{(i)}$ be the resulting Gramian matrix of the resulting basis of the tangent space T_pM. Then we define the *Riemannian measure* for any $E \in \mathcal{B}(M)$

$$V(E) := \sum_{i=1}^{\infty} \int_{\boldsymbol{\phi}_i(U_i \cap E)} \chi_i\big(\boldsymbol{\phi}_i^{-1}(x)\big) \sqrt{\det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)}}\, dx \in [0, \infty] \qquad (2.1.18)$$ {eq:riemannian_

It can be shown that (2.1.18) is independent of the chosen oriented atlas using the transformation behaviour of $G^{(i)}$. But the orientation is crucial for it to be well-defined.

Proposition 2.1.32. *The Riemannian measure is independent of the chosen partition of unity and atlas if it is oriented the same.*

Proof. Assume w.l.o.g. that $N = \infty$ i.e. we have a countably infinite atlas. Let $\{(V_j, \boldsymbol{\psi}_j)\}_{j=0}^{\infty}$ be a different atlas with the same orientation and $\{\rho_j\}_{j=0}^{\infty}$ be a partition of unity subordinate to it. If we now define the Riemannian measure (2.1.18) using this atlas, then

$$\begin{aligned}
&\sum_{j=0}^{\infty} \int_{\boldsymbol{\psi}_j(E\cap V_j)} \rho_j\big(\boldsymbol{\psi}_j^{-1}(y)\big)\sqrt{\det G^{(j)}_{\boldsymbol{\psi}^{-1}(y)}}\, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\boldsymbol{\psi}_j(E\cap V_j\cap U_i)} \chi_i\big(\boldsymbol{\psi}_j^{-1}(y)\big)\rho_j\big(\boldsymbol{\psi}_j^{-1}(y)\big)\sqrt{\det G^{(j)}_{\boldsymbol{\psi}_j^{-1}(y)}}\, dy \\
&= \sum_{i,j=0}^{\infty} \int_{\boldsymbol{\phi}_i(E\cap V_j\cap U_i)} \chi_i\big(\boldsymbol{\phi}_i^{-1}(x)\big)\rho_j\big(\boldsymbol{\phi}_i^{-1}(x)\big)\sqrt{\det G^{(j)}_{\boldsymbol{\phi}_i^{-1}(x)}}\, \det D(\boldsymbol{\psi}_j\circ\boldsymbol{\phi}_i^{-1})\big(\boldsymbol{\phi}_i(x)\big)\, dx
\end{aligned} \tag{2.1.19}$$

{eq:double_sum

where we used the fact that the χ_i sum up to one for the first equality and a simple tranformation of integral in the second equality using the chart transition $\boldsymbol{\psi}_j\circ\boldsymbol{\phi}_i^{-1}$. Since all summands are non-negative we can change the order of summation as we like. Then using the change of basis for the tangent space derived in Sec. 2.1.2 we get for $p \in U_i \cap V_j$

$$\begin{aligned}
(G_p^{(j)})_{kl} &= g_p\Big(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\Big) \\
&= \sum_{r,s=0}^{n} \frac{\partial(\boldsymbol{\phi}_i\circ\boldsymbol{\psi}_j^{-1})_r}{\partial y_k}\big(\boldsymbol{\psi}(p)\big)\, \frac{\partial(\boldsymbol{\phi}_i\circ\boldsymbol{\psi}_j^{-1})_s}{\partial y_l}\big(\boldsymbol{\psi}(p)\big)\, g_p\Big(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_s}\Big) \\
&= \left(D(\boldsymbol{\phi}_i\circ\boldsymbol{\psi}_j^{-1})\big(\boldsymbol{\psi}(p)\big)^{\top} G_p^{(i)} D(\boldsymbol{\phi}_i\circ\boldsymbol{\psi}_j^{-1})\big(\boldsymbol{\psi}(p)\big)\right)_{kl}
\end{aligned}$$

and thus

$$\begin{aligned}
\det G^{(j)}_{\boldsymbol{\phi}^{-1}(x)} &= \det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)} \left(\det D\big(\boldsymbol{\phi}_i\circ\boldsymbol{\psi}_j^{-1}\big)\big(\boldsymbol{\psi}_j(\boldsymbol{\phi}_i^{-1}(x))\big)\right)^2 \\
&= \det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(x)} \left(\det D(\boldsymbol{\psi}_j\circ\boldsymbol{\phi}_i^{-1})(x)^{-1}\right)^2.
\end{aligned}$$

Plugging this into (2.1.19) the determinant of the chart transition cancels out because $\det D(\boldsymbol{\psi}_j\circ\boldsymbol{\phi}_i^{-1})(x)^{-1} > 0$. We can integrate over $\boldsymbol{\phi}_i(E\cap U_i)$ instead of

$\phi_i(E \cap V_j \cap U_i)$ without changing the integral because $\operatorname{supp} \rho_j \subseteq V_j$.

$$\begin{aligned}
&\sum_{i=0}^{\infty} \int_{\phi_i(E\cap U_i)} \chi_i(\phi_i^{-1}(x)) \sum_{j=0}^{\infty} \rho_j(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}_{\phi_i^{-1}(x)}}\, dx \\
&= \sum_{i=0}^{\infty} \int_{\phi_i(E\cap U_i)} \chi_i(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}_{\phi_i^{-1}(x)}}\, dx \\
&= V(E).
\end{aligned}$$

□

Now that we have the measure space $(M, \mathcal{B}(M), V)$ we can define integration in the usual Lebesgue way. It is easily shown that a function $f : M \to \mathbb{R}$ is measurable i.i.f. $f \circ \phi_i^{-1}$ is measurable for every chart ϕ_i. It is then simply an application of the definition of Lebesgue integration to show that for any measurable $f \geq 0$ we can express the integration as

$$\int_M f\, dV = \sum_{i=1}^{\infty} \int_{\phi_i(U_i)} \chi_i(\phi_i^{-1}(x)) f(\phi_i^{-1}(x)) \sqrt{\det G^{(i)}(\phi_i^{-1}(x))} dx.$$

By introducing the integral as a Lebesgue integral w.r.t. the Riemannian measure we inherit the theoretical framework of Lebesgue integration. For example, we know that the spaces $L^p(M, V)$ for $1 \leq p < \infty$, i.e. the p-integrable real-valued functions w.r.t. the Riemannian measure, are Banach spaces.

Integration of differential forms

{sec:integrati

Once again, we should first ask ourselves what a measurable differential form should be. We know that we can express our differential form locally for $p \in M$ using the local coordinates as

$$\omega_p = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1,\ldots,i_k}(p)\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}.$$

In the spirit of the definition of differentiability in Sec. 2.1.2, we call an $\omega \in \Lambda^k M$ measurable if for every chart ϕ the coefficient functions of the differential form expressed in local coordinates are measurable. This notion is once again independent of the chosen coordinates which is easily shown using the fact that the change of basis matrix from dx^i to dy^j, if we choose different local coordinates x_i and y_j with charts (ϕ, U) and (ψ, V), is $D(\psi \circ \phi^{-1})^\top$ which is smooth.

Remark 2.1.33. A more principled approach would be to apply the standard definition of measurability to sections of vector bundles which we did not define in this thesis. This leads to the analogous definition in more generality (see [15, Def. 10.2.30]).

Next, we will define integration of an n-form over an n dimensional manifold. At first, we do so for an open set $U \subseteq \mathbb{R}^n$. This is the simplest example of an n-dimensional manifold where we only have one chart which is the identity and the local coordinates are just our standard coordinates which we denote by z_i, $i = 1, ..., n$ and the resulting basis of the tangent space $\frac{\partial}{\partial z_i}$. Let ω be a measurable n-form on U so we can write

$$\omega_z = f(z)\, dz^1 \wedge dz^2 \wedge ... \wedge dz^n$$

for $z \in U$ with $f : U \to \mathbb{R}$ being Borel-measurable. We can now simply define

$$\int_U \omega = \int_U f(z)\, dz^1 dz^2 ... dz^n.$$

With this definition at hand we can now extend this definition to the manifold M. As it is often done in differential geometry we will work locally first and then extend this construction globally by using a partition of unity.

Let $(U, \boldsymbol{\phi})$ be a chart on M and assume $\operatorname{supp} \omega \subseteq U$. Then $(\boldsymbol{\phi}^{-1})^* \omega$ is a n-form on $\boldsymbol{\phi}(U) \subseteq \mathbb{R}^n$ and we can apply our prior definition. So now we just define

$$\int_M \omega := \int_{\boldsymbol{\phi}(U)} (\boldsymbol{\phi}^{-1})^* \omega.$$

Now let us move on to the global definition. Let $\{(U_i, \boldsymbol{\phi}_i)\}_{i=1}^{\infty}$ be an oriented atlas and let $\{\chi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to it. Then $\operatorname{supp} \chi_i \omega \subseteq U_i$ and we define

$$\int_M \omega := \sum_{i=1}^{\infty} \int_M \chi_i \omega.$$

The well-definedness will be proven below.

Let us now investigate how the integration of functions and of differential forms are related to each other. For the chart $(\boldsymbol{\phi}_i, U_i)$ we denote the local coordinates as $x_k^{(i)}$, $k = 1, ..., n$ and the basis of 1-forms as $dx_{(i)}^k$. We know that for $p \in U_i$ we can write the volume form as

$$\mathrm{vol}_p = \sqrt{\det G_p^{(i)}}\, dx_{(i)}^1 \wedge dx_{(i)}^2 \wedge ... \wedge dx_{(i)}^n.$$

where $G_{kl}^{(i)} = g(\frac{\partial}{\partial x_k^{(i)}}, \frac{\partial}{\partial x_l^{(i)}})$.

Because $(\boldsymbol{\phi}_i^{-1})^* dx_{(i)}^k = dz^k$ – which follows directly from the definition – and the pullback commutes with the wedge product we have

$$\big((\boldsymbol{\phi}_i^{-1})^* \,\mathrm{vol}\big)_z = \sqrt{\det G^{(i)}(\boldsymbol{\phi}_i^{-1}(z))}\, dz^1 \wedge dz^2 \wedge ... \wedge dz^n.$$

That means for a n-form $f\,\mathrm{vol}$ we can write the integral as

$$\int_M f\,\mathrm{vol} = \sum_{i=1}^{\infty} \int_{\boldsymbol{\phi}_i(U_i)} \chi_i(\boldsymbol{\phi}_i^{-1}(z)) f(\boldsymbol{\phi}_i^{-1}(z)) \sqrt{\det G^{(i)}_{\boldsymbol{\phi}_i^{-1}(z)}}\, dz^1 dz^2 \cdots dz^n$$

and we see

$$\int_M f dV = \int_M f\,\mathrm{vol}$$

with the two different notions of integration. This also proves that the integral of differential n-forms is independent of the chosen coordinates as long as the orientation is respected. So we see that the the two definitions are essentially equivalent. The big advantage of considering these two approaches is that we know the integration of differential forms is within the framework of Lebesgue integration. It is then also clear how integrability for n-forms should be defined. We call an n-form $f\,\mathrm{vol}$ integrable if f is integrable w.r.t. the Riemannian measure.

Stokes' theorem and integration by parts

One of the most important results about the integration of differential forms is Stokes' theorem which we will state in this section. From it, we will obtain an integration by parts formula.

Before we do so, we have to check how to define the restriction of a differential form to a submanifold $N \subseteq M$. A submanifold N is just a manifold contained in M. As a further restriction, we always assume that we use the subspace topology on a submanifold i.e. a set $U \subseteq N$ is open i.i.f. there exists an open $U' \subseteq M$ s.t. $U = U' \cap N$. We now define the restriction with the inclusion $\iota : N \hookrightarrow M$. For a smooth differential form $\omega \in C^\infty \Lambda^k M$ we define the restriction via the pullback of the inclusion operator i.e. $\iota^* \omega \in C^\infty \Lambda^k N$. For a k-dimensional submanifold N we then denote the integration over N of a k-form ω as

$$\int_N \iota^* \omega =: \int_N \omega.$$

{ex:integratio

Example 2.1.34. Let us investigate the integral of a 1-form over a curve. Let $\boldsymbol{\gamma} : (0,1) \to \Omega$ be a smooth curve in the domain $\Omega \subseteq \mathbb{R}^n$. Denote $\Gamma = \boldsymbol{\gamma}\big((0,1)\big)$ and assume that $\boldsymbol{\gamma}' \neq 0$ and $\boldsymbol{\gamma} : (0,1) \to \Gamma$ is bijective. That means especially that Γ does not intersect itself. Then Γ is a manifold and the chart is $\boldsymbol{\gamma}^{-1}$. Let us calculate the Jacobian of the inclusion ι using the definition from (2.1.12). Since it is the most convenient, we take the identity as chart on Ω. Take $\boldsymbol{\gamma}(t) \in \Gamma$. Then

$$\frac{\partial \iota_i}{\partial t}\big(\boldsymbol{\gamma}(t)\big) = \frac{\partial (\mathrm{Id} \circ \iota \circ \boldsymbol{\gamma})_i}{\partial t}(t) = \frac{\partial \gamma_i}{\partial t}(t)$$

which means

$$D\iota(\boldsymbol{\gamma}(t)) = \boldsymbol{\gamma}'(t)^\top$$

This is then the matrix representation of the differential ι_* at $\boldsymbol{\gamma}(t)$. Now take an at least continuous 1-form $\omega = \sum_{i=1}^n \omega_i \, dz^i$ on Ω. Then we know from the matrix representation of the pullback derived in Sec. 2.1.3 that

$$\iota^*\omega(t) = \sum_{i=1}^n \boldsymbol{\gamma}_i'(t) \, \omega_i(\boldsymbol{\gamma}(t)) \, dt$$

and so

$$\int_\Gamma \omega = \int_0^1 \sum_{i=1}^n \boldsymbol{\gamma}_i'(t) \, \omega_i\big(\boldsymbol{\gamma}(t)\big) \, dt.$$

Let us now look at this in terms of vector proxies. Take a vector field $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$ on Ω. Because we are using Euclidian coordinates we get

$$X^\flat = \sum_{i=1}^n X_i dz^i$$

and thus by denoting $\mathbf{X} = (X_1, X_2, ..., X_n)^\top$

$$\int_\Gamma X^\flat = \int_\Gamma \mathbf{X} \cdot d\ell$$

where the right hand side is the notation for the standard curve integral in $\mathbb{R}^n$. So we can conclude that, interpreted in vector proxies for a domain of $\mathbb{R}^n$, the integral of a 1-form corresponds to the usual curve integral.

Theorem 2.1.35 (Stokes)**.** *Let M be a smooth oriented manifold with boundary ∂M. Let ω be a smooth compactly supported $(n-1)$-form. Then we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. See [4, Sec. V.4]. □

Remember that compactly supported is meant in the topology on a manifold with boundary which means that ω can be non-zero on ∂M. This theorem gives us a relation of the boundary and the exterior derivative which will be crucial in the topological context of differential forms that we will investigate in Sec. 2.2.3.

We can also derive a form of the integration by parts formula from it. Let $\omega \in C_c^\infty \Lambda^k M$ and $\mu \in C_c^\infty \Lambda^{n-k-1} M$. Then $\omega \wedge \mu \in C_c^\infty \Lambda^{n-1} M$. Recall the Leibniz rule for the exterior derivative

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^k \omega \wedge d\mu.$$

By integrating both sides over M and applying Stokes' theorem we obtain

$$\int_{\partial M} \omega \wedge \mu = \int_M d\omega \wedge \mu + (-1)^k \int_M \omega \wedge d\mu.$$

By using vector and scalar proxies as in Sec. 2.1.3 this can be used along with the expression of the restriction of a $(n-1)$-form to prove the well-known formula

$$\int_\Omega u \operatorname{div} \mathbf{F} \, dx = -\int_\Omega \operatorname{grad} u \cdot \mathbf{F} \, dx + \int_{\partial\Omega} u \, \mathbf{F} \cdot \mathbf{n} \, ds$$

where $\mathbf{n}$ is the unit normal, assuming $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain and $\mathbf{F}$ and u are continuously differentiable. In 3D, using the restriction of a 1-form, one can show

$$\int_\Omega \mathbf{v} \cdot \operatorname{curl}\mathbf{F} \, dx = \int_\Omega \operatorname{curl}\mathbf{v} \cdot \ \mathbf{F} \, dx - \int_{\partial\Omega} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{F} \, ds$$

assuming $\mathbf{v}$ is a continuously differentiable vector field. But because these integration-by-parts formulas are well known we will not prove them here again.

2.2 Singular homology

{sec:singular_

The curve integral constraint from the magnetostatic problem is very topological in nature and strongly related to the topology of the domain. In order to deal with this constraint and obtain the desired existence and uniqueness we require some tools from algebraic topology which we will introduce in this section. This material is taken from [4] where a lot more details and results can be found. This section requires some basic knowledge about algebra and group theory.

2.2.1 Homology groups

Denote with $\mathbb{R}^\infty$ the vector space of all real-valued sequences. Let $e_i \in \mathbb{R}^\infty$ for $i \in \mathbb{N}$ denote the sequences that are zero for every index unequal to i and 1 for the index i. Note that the natural numbers start at zero in this thesis. Then we define the standard k-simplex Δ_k as

$$\Delta_k := \Big\{ \sum_{i=0}^k \lambda_i e_i \mid \sum_{i=0}^k \lambda_i = 1, \ 0 \le \lambda_i \le 1 \Big\} = \operatorname{conv}\{e_0, \ldots, e_k\}.$$

where conv is the usual convex combination.

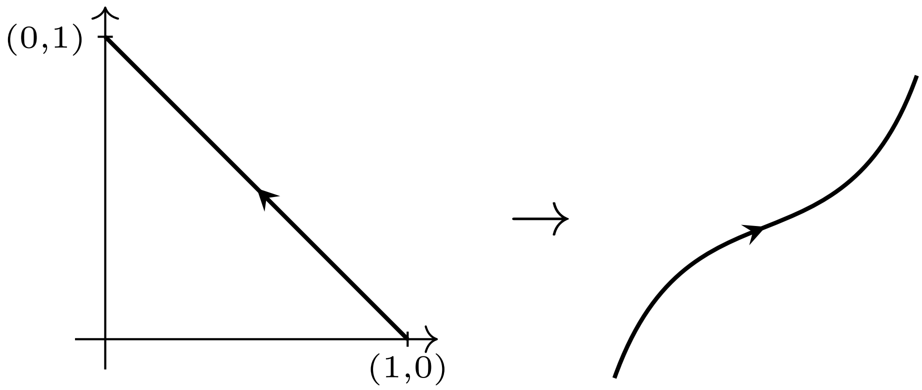

Figure 2.2.1: Standard 1-simplex (left) and a possible 1-simplex in the 2D plane i.e. choosing $X = \mathbb{R}^2$. {fig:1simplex}

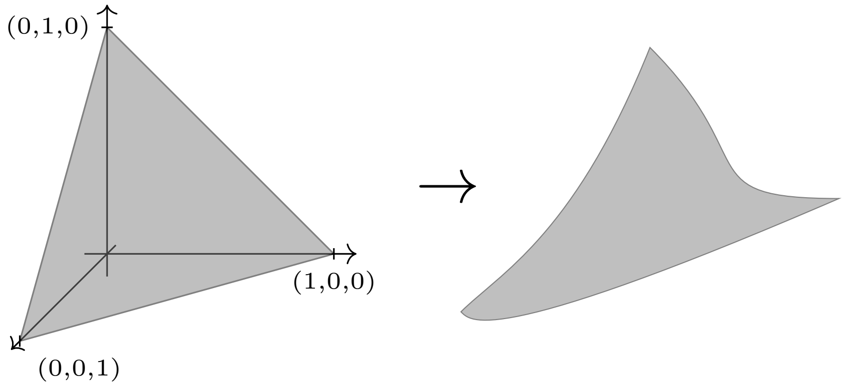

Figure 2.2.2: The standard 2-simplex and a 2-simplex in the 2D plane i.e. taking $X = \mathbb{R}^2$. {fig:2simplex}

Definition 2.2.1 (k-simplex)**.** Let X be a topological space. Then a *singular k-simplex* is a continuous map $\sigma_k : \Delta_k \to X$. We will frequently leave out the term 'singular' and refer to them just as k-simplices. We will very often also call the geometric object in our topological space i.e. $\sigma_k(\Delta_k)$ the singular k-simplex. It should be clear from the context whether we mean the function or the subset.

See Fig. 2.2.1 and Fig. 2.2.2 for the standard 1- and 2-simplex and a possible singular simplex in the plane respectively. As the term 'singular' implies, these simplices can be degenerated. For example, σ_k could just be constant, so the object in the topological space corresponding to the k-simplex is just a point.

We can now introduce an algebraic structure by looking at finite formal sums of the form

$$\sum_{\sigma \text{ } k\text{-simplex}} n_\sigma \sigma.$$

These formal sums form an abelian group which we refer to as the *singular k-chain group $C_k(X)$*. The next step is to introduce an important homomorphism between these groups called the *boundary*.

Definition 2.2.2. Let $\mathbf{v}_0, ..., \mathbf{v}_k \in \mathbb{R}^n$. We define *affine singular k-simplex* as a special singular k-simplex denoted by

$$[\mathbf{v}_0, ..., \mathbf{v}_k] : \Delta_k \to \mathbb{R}^n, \sum_{i=0}^{k} \lambda_i e_i \mapsto \sum_{i=0}^{k} \lambda_i \mathbf{v}_i.$$

As in the general case, the image can be a degenerated simplex in $\mathbb{R}^n$ since the $\mathbf{v}_i$ are not assumed to be affine independent.

We call the affine singular simplex

$$[e_0, ..., \hat{e}_i, ..., e_k] : \Delta_{k-1} \to \Delta_k \tag{2.2.1}$$ {eq:face_map}

the *i-th face map* which we denote by F_i^k and sometimes we will not write the superindex k. The ˆ means this vertex is left out. Here we tacitly used the natural inclusion $\mathbb{R}^{k+1} \subseteq \mathbb{R}^\infty$ so we have $\Delta_k \subseteq \mathbb{R}^{k+1}$. But this is just a way of representation. With the face map we can now define the boundary homomorphim.

Definition 2.2.3 (Boundary)**.** For a singular k-simplex $\sigma : \Delta_k \to X$ we define its i-th face $\sigma^{(i)} := \sigma \circ F_i^k$ which is a $(k-1)$-simplex. We then define the *boundary* of σ as $\partial_k \sigma := \sum_{i=0}^{k} (-1)^i \sigma^{(i)}$. We extend this to a homomorphism between the chain groups

$$\partial_k : C_k(X) \to C_{k-1}(X), \sum_\sigma n_\sigma \sigma \mapsto \sum_\sigma n_\sigma \partial_k \sigma.$$

In the case of $k = 0$, we set $\partial_0 = 0$.

We will frequently leave out the subscript and just write ∂ for the boundary if it is clear from the context. A straightforward computation (cf. [4, Lemma 1.6]) shows the important property

$$\partial_k \circ \partial_{k+1} = 0$$

which implies that $\operatorname{im} \partial_k \subseteq \ker \partial_{k-1}$ is a subgroup. We call a chain $c \in C_k(X)$ *k-cycle* or *closed* if $\partial_k c = 0$ and we call it *k-boundary* or *exact* if $c \in \operatorname{im} \partial_{k+1}$. Denote the group of k-cycles as $Z_k(X)$ and the k-boundaries as $B_k(X)$. Since we are in the abelian setting this motivates us to define the resulting factor groups.

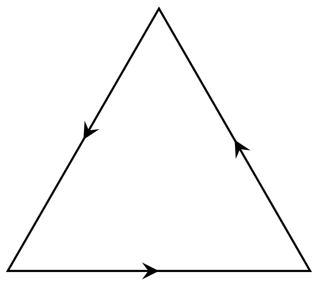

Figure 2.2.3: A simplex with boundary. The boundary is the 1-chain going around the simplex in the given direction. {fig:simplex_w

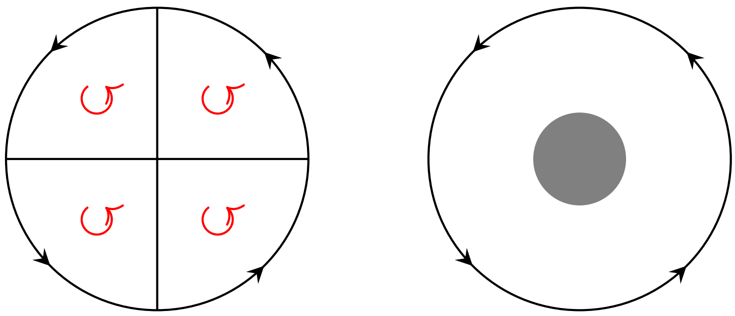

Figure 2.2.4: The 1-chain around the circle on the left is exact since it is the boundary of the 2-chain formed by the sum of the four segments in the middle. The red circular arrows show the orientation. The gray circle on the right represents a whole in the domain and then the same 1-chain around it is still closed, but not exact anymore.

Definition 2.2.4 (Homology groups)**.** We define the *k-th homology group* of the topological space X as the factor group

$$H_k(X) := {}^{Z_k(X)}\!\big/\!{}_{B_k(X)}.$$

We denote the elements of the homology groups i.e. the equivalence classes of a k-cycle c as $[c] \in H_k(X)$.

If the k-th homology group is finitely generated then the rank, i.e. the number of generators, is called the *k-th Betti number*. These Betti numbers are fundamental properties of the topological space. For example, the zeroth Betti number corresponds to the number of path-components of the space. In 3 dimensions, the first Betti number of a compact domain corresponds to the number of "holes" and the second Betti number to the number of enclosed "voids" in the domain [2, p.14]. E.g. a filled torus has the zeroth Betti number one, the first Betti number also equal to one and the second equal to zero which can be proven using the Meyer-Vietoris sequence (see [4, Sec. IV.18]). We will not go into this in further detail since we do not want to dwelve too deeply into algebraic topology.

This construction can be put in an abstract algebraic framework in the following way. We call a collection of abelian groups C_i, $i \in \mathbb{Z}$, a graded group. Together with a collection of homomorphims $\partial_i : C_i \to C_{i-1}$ called *differentials* s.t. $\partial_{i-1} \circ \partial_i = 0$ this is called a *chain complex* which we will denote by C_*.

Example 2.2.5. If we set $C_k(X) = 0$ for $k < 0$ then the groups of k-chains with the boundary operator form a chain complex .

Completely analogous to above, we can define the homology groups of a abstract chain complex

$$H_k(C_*) := {}^{\ker \partial_k}\!/_{\operatorname{im} \partial_{k+1}}.$$

Definition 2.2.6 (Chain map)**.** Let A_* and B_* be chain complexes. With a slight abuse of notation let us denote the differentials of both chain complexes just by ∂. Then a *chain map* $f : A^* \to B^*$ is a collection of homomorphisms $f_i : A_i \to B_i$ s.t. $\partial \circ f_{i-1} = f_i \circ \partial$ i.e. the following diagram is commutative

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & A_{i+1} \arrow[r, "\partial"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\partial"] \arrow[d, "f_i"] & \dots \\
\dots \arrow[r, "\partial"] & B_{i+1} \arrow[r, "\partial"] & B_i \arrow[r, "\partial"] & \dots
\end{tikzcd}$$

We will leave out the indices most of the time if it is clear what we mean. The crucial property of these chain maps is that they induce homomorphisms of the homology groups denoted as

$$[f_i] : H_i(A_*) \to H_i(B_*),\ [f_i]([a]) = [f_i(a)]$$

2.2.2 Cohomology groups

{sec:cohomology

Let us start with the abstract definition of a cochain complex. Let $\{C^i\}_{i\in\mathbb{N}}$ be a collection of abelian groups and homomorphisms $\partial^i : C^i \to C^{i+1}$ with $\partial^{i+1} \circ \partial^i = 0$ called *codifferentials*. Then we call this sequence a *cochain complex*. The only difference to chain complexes is that the index increases when applying the codifferential. Hence, they are basically the same from an algebraic point of view. By convention, we use superindices for anything that is related to cochain complexes.

We define *cochain maps* completely analogous to chain maps i.e. cochain maps commute with the codifferential.

The main motivation for cochain complexes comes from the *singular cochain complexes* that we will introduce next. Let G be any abelian group and X be a topological space as before. Then we define the group of k*-cochains* $C^k(X;G)$ by

$$C^k(X;G) := \mathrm{Hom}(C_k(X),\, G)$$

i.e. the group of all homomorphisms from k-chains $C_k(X)$ to G. Just as for chains we now introduce a homomorphism between the groups of cochains which transforms this into a cochain complex.

Definition 2.2.7 (Coboundary)**.** We define the operator $\partial^k : C^k(X;G) \to C^{k+1}(X;G)$ via

$$(\partial^k f)(c) := f(\partial_{k+1} c).$$

for a $(k+1)$-chain c. We call a cochain $f \in C^k(X;G)$ *closed* if $\partial^k f = 0$ and we call f *exact* if there is a $g \in C^{k-1}(X;G)$ s.t. $f = \partial^{k-1} g$. As for the boundary map we will frequently leave away the superscript if the context is clear.

Notice that this naming is analogous to the naming for closed and exact forms. This is no coincidence as we will see in the Sec. 2.2.3.

From the definition it is obvious that $\partial^{k+1} \circ \partial^k = 0$ and thus we have indeed a cochain complex which we call *singular cochain complex*. If there is no confusion with the general notion of cochain complex we will leave away the term 'singular'.

Definition 2.2.8 (Singular cochain cohomology)**.** Denote the closed k-cochains as $Z^k(X;G)$ and the exact ones with $B^k(X;G)$. We then define the *cochain cohomology groups* $H^k(X;G)$ as

$$H^k(X;G) := {Z^k(X;G)}\big/{B^k(X;G)}.$$

Note that in the case of $G = \mathbb{R}$ this becomes a vector space. Now of course there is the question how the homology and cohomology groups are related to each other. This question is answered by the *universal coefficient theorem*. But before we can formulate it we have to introduce exact sequences.

{def:exact_seq

Definition 2.2.9 (Exact sequence)**.** Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. Then this sequence of homomorphisms is called *exact* if im $f_{i-1} =$ ker f_i.

The universal coefficient theorem in the case of homology states that the sequence

$$0 \to \mathrm{Ext}(H_{k-1}(K), G) \to H^k(K;G) \xrightarrow{\beta} \mathrm{Hom}(H_k(K), G) \to 0 \qquad (2.2.2)$$ {eq:univeral_c

is exact. β is defined via

$$\beta([F])([c]) := F(c). \tag{2.2.3}$$ {eq:isomorphis

The definition of Ext can be found in [4], but it does not matter for our purpose because from now on we will assume $G = \mathbb{R}$ and in this case $\operatorname{Ext}(H_{k-1}(X), \mathbb{R}) = 0$. This follows from the fact that $\mathbb{R}$ is a divisible and hence injective abelian group. The definition of these terms and their connections used can also be found in [4, Sec. V.6]. However, we will not dwelve into the algebraic background further. In the case of $G = \mathbb{R}$, we can conclude from the exactness of the above short sequence that $\ker \beta = 0$ and $\operatorname{im} \beta = \operatorname{Hom}(H_k(X), \mathbb{R})$. So β is an isomorphism and provides us with the link between the singular chains and cochains.

2.2.3 De Rham's theorem

{sec:de_rhams_

It turns out that the singular cochain cohomology is closely related to the cohomology of differential forms, the *de Rham cohomology* which is introduced next.

Let M be a smooth n-dimensional manifold. We will use the notation introduced in Sec. 2.1.3. Then the smooth differential forms $C^\infty \Lambda^k M$ together with the exterior derivative give us a cochain complex due to the property $d \circ d = 0$. This cochain complex is called *de Rham complex* . Note that we have slightly more structure here since the $C^\infty \Lambda^k M$ are vector spaces and the exterior derivative a linear map i.e. a vector space homomorphism. Let us denote the exterior derivative on the space of k-forms as d^k. Then recall the notation

$$\mathfrak{B}^k(M) := \operatorname{im} d^{k-1}$$
$$\mathfrak{Z}^k(M) := \ker d^k$$

which are the exact and closed forms respectively. We define $d^k = 0$ for $k < 0$ and $k > n$. We then define the *de Rham cohomology group*

$$H^k_{dR}(M) := \mathfrak{Z}^k(M) \big/ \mathfrak{B}^k(M)$$

It turns out that the de Rham complex is closely related with the singular cochain complex which is the topic of this section.

Let us recall Stokes' theorem first which said that for a k-form $\omega \in C^\infty_c \Lambda^k M$ on a smooth k-dimensional oriented manifold M we have

$$\int_M d\omega = \int_{\partial M} \omega.$$

The following details are taken from Section V.5 and V.9 of [4]. We will only focus on the main ideas and avoid dwelving into the technical details. The interested reader can find more arguments in the given reference.

Let σ be a smooth k-simplex i.e. $\sigma : \Delta_k \to M$ is smooth. We will solely focus on smooth simplices from now on. Let $C_k^{\text{smooth}}(M)$ be the abelian group generated by smooth k-simplices and then $H^k_{\text{smooth}}(M;\mathbb{R})$ be the cochain groups constructed analogous to the standard case. Then, in fact, $H^k_{\text{smooth}}(M;\mathbb{R})$ and $H^k(M;\mathbb{R})$ are isomorphic. This is the reason why it is sufficient for this section to deal with smooth simplices only and we will, by abuse of notation, refer to them as $C_k(M)$ and the homology groups as $H_k(M)$ etc.

We now define the integral over a k-simplex as

$$\int_\sigma \omega = \int_{\Delta_k} \sigma^*\omega$$

and then the integral over a k-chain $c = \sum_\sigma n_\sigma \sigma$

$$\int_c \omega := \sum_\sigma n_\sigma \int_\sigma \omega.$$

This motivates us to introduce the homomorphism $I : C^\infty\Lambda^k(M) \to \text{Hom}(C_k;\mathbb{R})$ defined by

$$I(\omega)(c) = \int_c \omega.$$

Remark 2.2.10. There are some technical details that we will not discuss in detail here, but that should be mentioned. First, Δ_k is not a manifold. The $(k-2)$-skeleton are the boundaries of the faces i.e. the corners for $k = 2$ and the edges for $k = 3$. If we remove the $(k-2)$-skeleton then Δ_k is a manifold with boundary. But since this is a null-set w.r.t. the full simplex and the boundary as well, this does not matter for our arguments. Second, because we are integrating over the Δ_k their orientation is important and has to be chosen consistently. We will not present the details here. The only important fact is that the simplices are oriented in a way s.t.

$$\int_{[e_0,\dots,\widehat{e_i},\dots,e_k]} \nu = (-1)^i \int_{\Delta_{k-1}} F_i^*\nu \tag{2.2.4}$$ {eq:integral_b

for any integrable $(k-1)$-form on the face $[e_0, ..., \widehat{e_i}, ..., e_k]$ where the F_i are the face maps defined at (2.2.1) (see Fig. 2.2.3 for an illustration). Please consult [2, Sec. V.5] for more details.

Now we get from Stokes' theorem for a k-simplex σ

$$\begin{aligned} I(d\omega)(\sigma) &= \int_\sigma d\omega = \int_{\Delta_k} \sigma^* d\omega = \int_{\Delta_k} d\sigma^*\omega = \int_{\partial\Delta_k} \sigma^*\omega \\ &= \sum_{i=0}^{k} \int_{[e_0,\dots,\widehat{e_i},\dots,e_k]} \sigma^*\omega \overset{(2.2.4)}{=} \sum_{i=0}^{k} (-1)^i \int_{\Delta_{k-1}} F_i^*\sigma^*\omega \\ &= \sum_{i=0}^{k} (-1)^i \int_{\Delta_{k-1}} (\sigma \circ F_i)^*\omega = \sum_{i=0}^{k} (-1)^i \int_{\sigma\circ F_i} \omega \\ &= I(\omega)(\partial\sigma) = \partial\big(I(\omega)\big)(\sigma). \end{aligned}$$

So we obtain

$$I(d\omega) = \partial\big(I(\omega)\big).$$

This means that I is a cochain map and thus induces a homomorphism on cohomology

$$[I] : H^k_{dR}(M) \to H^k(M;\mathbb{R}).$$

Using the notation and the definition of this map we can now formulate de Rham's theorem which will become very important later when proving existence and uniqueness in Sec. 2.4.

{thm:de_rhams_

Theorem 2.2.11 (De Rham's theorem)**.** *Let M be a smooth orientable manifold. Then $[I] : H^k_{dR}(M) \to H^k(M;\mathbb{R})$ is an isomorphism.*

Proving this result requires more tools than we introduced so we will just state it here. Even though it is a very short statement, it has deep implications. It implies that the de Rham cohomology reflects fundamental topological properties of the manifold. It provides a bridge between the two fields of differential geometry and algebraic topology.

2.3 Hilbert complexes

{sec:hilbert_c

We will now move away from geometry and topology to functional analysis. A crucial tool for the proof will be the *Hodge decomposition* in 3D which relies on unbounded operators and Hilbert complexes. These will be introduced in this section. This is essentially a recollection of the parts of chapter 3 and 4 of Arnold's book [2] that we will need.

We start by introducing the concept of unbounded operators on real Hilbert spaces and the adjoint of these. Then we will apply this theory to the differential operators grad, curl and div in 3D. In the second part, we will introduce Hilbert complexes which combine the idea of cochain complexes and unbounded operators on Hilbert spaces. This will lead to the Hodge decomposition which is an important tool that we will need in the proof of existence and uniqueness.

Throughout this section it will be assumed that the reader is familiar with functional analysis, especially Hilbert spaces, and basic knowledge about Sobolev spaces. We will focus on real spaces exclusively.

2.3.1 Unbounded operators

{sec:unbounded_

We will provide the basic definitions about unbounded operators and propositions about those. After defining unbounded operators we will talk about closed and densely defined operators mainly and the adjoint. Most of the proofs are very short and we will be able to do them in detail.

Definition 2.3.1 (Unbounded operators)**.** Let X and Y be Hilbert spaces. Then we call a linear mapping $T : D(T) \to Y$ with a subspace $D(T) \subseteq Y$ an *unbounded operator* from X to Y. We call $D(T)$ the *domain* of T.

We will talk about an unbounded operator $T : X \to Y$ which means that T is not necessarily defined on all of X.

Note that this definition generalizes the standard operator. In particular, it includes the case when T is in fact bounded which can be slightly confusing, but we will stick to this common naming convention.

The domain is a crucial property of unbounded operators. We will sometimes denote the unbounded operator as the tuple $(T, D(T))$. If $D(T)$ is dense in X we call T *densely defined*. We say that two unbounded operators T and S from X to Y are equal if $D(T) = D(S)$ and $Tx = Sx$ for all $x \in D(T)$.

An easy example of an unbounded densely defined operator is the classical gradient with the domain $C_0^1(\Omega) \subseteq L^2(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open i.e. here we have $X = L^2(\Omega)$ and $Y = L^2(\Omega; \mathbb{R}^n)$. In short, grad is an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^1(\Omega)$. We could then denote it as $(\mathrm{grad}, C_0^1(\Omega))$. This also shows that the choice of domain is not unique. We could have instead chosen e.g. the different unbounded operator $(\mathrm{grad}, C_0^\infty(\Omega))$. Another example is the weak gradient with domain $H^1(\Omega)$ i.e. $(\mathrm{grad}, H^1(\Omega))$. All of these unbounded operators mentioned are densely defined.

As for bounded operators we define the kernel or null space of an unbounded operator

$$\ker T = \{x \in D(T) \mid Tx = 0\}$$

and the image or range

$$\operatorname{im} T = \{Tx \mid x \in D(T)\}.$$

The only difference to keep in mind is that the unbounded operators are not defined on the whole X in general.

Recall that the graph of a function $f : X \to Y$ is defined as $\{(x, f(x)) \in X \times Y \mid x \in X\}$. Analogously, the graph of an unbounded operator T is

$$\Gamma(T) := \{(x, Tx) \mid x \in D(T)\}$$

which is obviously a subspace of $X \times Y$.

We define the *graph inner product* on $D(T)$ as

$$\langle x, z\rangle_{D(T)} := \langle x, z\rangle_X + \langle Tx, Tz\rangle_Y, \quad x, z \in D(T).$$

It is easy to show that this is indeed an inner product. We will call its induced norm the *graph norm*

$$\|x\|_{D(T)} = \sqrt{\|x\|_X^2 + \|Tx\|_Y^2}, \quad x \in D(T).$$

Even though this defines a norm, $D(T)$ might not be a Hilbert space because it is in general not complete w.r.t. this norm. Consider for example the unbounded operator $\operatorname{grad} : L^2(\Omega) \to L^2(\Omega; \mathbb{R}^n)$ with domain $C_0^\infty(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ open. The graph norm is then

$$\|\phi\|_{D(\operatorname{grad})} = \sqrt{\|\phi\|_{L^2(\Omega)}^2 + \|\operatorname{grad}\phi\|_{L^2(\Omega)}^2}, \quad \phi \in C_0^\infty(\Omega)$$

which is just the standard H^1-norm. But it is well-known that $C_0^\infty(\Omega)$ is in fact not closed w.r.t. this norm and thus not complete since the completion of it is the space $H_0^1(\Omega)$ i.e. the Sobolev space with zero trace on the boundary. In Prop. 2.3.3, we will provide a sufficient and necessary condition for the domain to be a Hilbert space when the graph norm is used.

The well-known closed graph theorem for bounded operators states that a linear operator from X to Y defined on all of X (in contrast to unbounded operators in general) is bounded i.i.f. its graph is closed in $X \times Y$ w.r.t. the norm $\|(x, y)\|_{X\times Y} = \sqrt{\|x\|_X^2 + \|y\|_Y^2}$. This motivates the following definition.

Definition 2.3.2 (Closed operator)**.** We call an unbounded operator $T : X \to Y$ *closed* if its graph $\Gamma(T)$ is closed w.r.t. the norm $\|\cdot\|_{X\times Y}$.

That means if we have a closed operator T and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \xrightarrow{X} x$ and $Tx_n \xrightarrow{Y} y$ for some $x \in X$ and $y \in Y$. Then $(x_n, Tx_n) \xrightarrow{X\times Y} (x, y)$ and since T is closed $(x, y) \in \Gamma(T)$ i.e. $x \in D(T)$ and $Tx = y$. This is just a rephrasing of the definition essentially so this characterizes closed operators equivalently.

{prop:closed_o

Proposition 2.3.3. *An unbounded operator T is closed i.i.f. its domain $D(T)$, endowed with the graph inner product, is a Hilbert space.*

Proof. As mentioned above, the graph inner product is in fact an inner product on $D(T)$. So we have to show completeness. Assume that T is closed and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ that is Cauchy w.r.t. the graph norm. This implies that (x_n) must be Cauchy w.r.t. the X-norm and (Tx_n) must be Cauchy w.r.t. the Y-norm so both sequences are convergent. Because X and Y are Hilbert spaces there exists $x \in X$ s.t. $x_n \to x$ and $y \in Y$ s.t. $Tx_n \to y$. Because T is closed we know $x \in D(T)$ so $D(T)$ is complete.

For the other direction, assume $D(T)$ is complete and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $x_n \to x \in X$ and $Tx_n \to y$ for some $y \in Y$. Because both sequences are convergent they are both Cauchy and thus (x_n) is Cauchy w.r.t. the graph norm. Due to the completeness of $D(T)$ that implies that $x \in D(T)$ and $x_n \xrightarrow{D(T)} x$ and

$$\|x_n - x\|^2_{D(T)} = \|x_n - x\|^2_X + \|Tx_n - Tx\|^2_Y \to 0$$

so $Tx_n \to Tx$ and thus $Tx = y$ which proves that T is closed. $\square$

As an example, take the unbounded operator $(\mathrm{grad}, H^1(\Omega))$ i.e. the weak gradient as an unbounded operator from $L^2(\Omega)$ to $L^2(\Omega;\mathbb{R}^n)$ with domain $D(\mathrm{grad}) = H^1(\Omega)$. Then we described above that the graph norm here is just the H^1-norm. It is well-known that $H^1(\Omega)$ is a Hilbert space. Therefore Prop. 2.3.3 tells us that $(\mathrm{grad}, H^1(\Omega))$ is a closed operator in contrast to $(\mathrm{grad}, C_0^\infty(\Omega))$ as described above.

The adjoint of bounded operators can be generalized to unbounded operators as well. Let us derive this step by step. Assume $T : X \to Y$ is a densely defined unbounded operator. Let us fix a $y \in Y$ and look at the linear functional $\ell : D(T) \to \mathbb{R}$ given by

$$\ell(x) = \langle y, Tx\rangle_Y.$$

This functional is not necessarily bounded w.r.t. the norm on X. But if it is i.e. if $\ell \in D(T)'$ then because $D(T)$ is dense in X we can extend it to a $\bar{\ell} \in X'$. Let $v \in X$ be its Riesz representative. That means we have

$$\langle v, x\rangle_X = \langle \ell, x\rangle_{X'\times X} = \langle y, Tx\rangle_Y \quad \forall x \in D(T).$$

$\langle \ell, x\rangle_{X'\times X}$ is the usual duality pairing and we will frequenty just write $\langle l, x\rangle$. Then we define $v = T^*y$ and recognize this as the defining property of the adjoint and define

$$D(T^*) := \{y \in Y \mid \exists c_y \in \mathbb{R} : \langle y, Tx\rangle_Y \le c_y\|x\|_X \quad \forall x \in X\}.$$

It is easy to check that this is a linear subspace of Y.

Proposition 2.3.4. *$T^* : Y \to X$ is a linear unbounded operator with domain $D(T^*)$.*

Proof. Note first that T^*y is well-defined for $y \in D(T^*)$ since the T^*y is the Riesz representative of $x \mapsto \langle y, Tx\rangle_Y$ and the Riesz representative is well-known to be unique.

We only have to show that T^* is linear. Take $y_1, y_2 \in D(T^*)$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\langle T^*(y_1 + \lambda y_2), x\rangle_X &= \langle y_1 + \lambda y_2, Tx\rangle_Y = \langle T^*y_1, x\rangle + \lambda\langle T^*y_2, x\rangle \\ &= \langle T^*y_1 + \lambda T^*y_2, x\rangle_X.\end{aligned}$$

for all $x \in D(T)$. Because $D(T)$ is dense in X this implies $T^*(y_1 + \lambda y_2) = T^*y_1 + \lambda T^*y_2$. □

We would like to proof whether T^* is itself densely defined or closed. This can be done in an elegant way by investigating the graphs of T and T^*. But $\Gamma(T)$ is a subspace of $X \times Y$ and $\Gamma(T^*)$ of $Y \times X$. To compare the two, we introduce a rotation operator. For any real vector spaces V and W we define the rotation operator

$$R_{V\times W} : V \times W \to W \times V, (v, w) \mapsto (-w, v).$$

It is obvious that $R_{V\times W}$ is an isometry when V and W are normed spaces and we have $R_{W,V}R_{V,W}Z = Z$ for any subspace $Z \subseteq V \times W$. By using this rotation operator, we can formulate the following lemma.

{lem:rotated_g

Lemma 2.3.5. *Let T be a densely defined unbounded operator from X to Y. Then we have*

$$\begin{aligned}\Gamma(T)^\perp &= R_{Y,X}\Gamma(T^*) \text{ and} \\ \overline{\Gamma(T)} &= \big(R_{Y,X}\Gamma(T^*)\big)^\perp.\end{aligned}$$

Proof. $(x, y) \in \Gamma(T)^\perp$ holds i.i.f.

$$0 = \langle (x, y), (v, Tv)\rangle_{X\times Y} = \langle x, v\rangle_X + \langle y, Tv\rangle_Y \quad \forall v \in D(T).$$

i.e.

$$\langle -x, v\rangle_X = \langle y, Tv\rangle_Y, \quad \forall v \in D(T).$$

This is just equivalent to saying that $-x = T^*y$ i.e.

$$(x, y) = (-T^*y, y) \in R_{Y,X}\Gamma(T^*)$$

which proves the first equality.

For the second equivalence recall the basic fact from Hilbert space theory that for any subspace of a Hilbert space V, $(V^{\perp})^{\perp} = \overline{V}$. Hence, applying the orthogonal complement to both sides of the first equality gives us the second one. □

{cor:adjoint_o

Corollary 2.3.6. *The adjoint T^* of a densely defined operator T is closed.*

Proof. Recall another basic fact from Hilbert space theory that the orthogonal complement of a space is always closed. So we know from the first equality that $R_{Y,X}\Gamma(T^*)$ is closed. Since $R_{Y,X}$ is an isometry we conclude that $\Gamma(T^*)$ is closed and thus T^* a closed unbounded operator. □

{prop:adjoint_

Proposition 2.3.7. *Let T be a densely defined and closed unbounded operator. Then T^* is also densely defined and closed.*

Proof. We know from the previous corollary that T^* is closed. In order to prove density, once again recall a fact from Hilbert space theory that a subspace is dense i.i.f. its orthogonal complement is zero. So take $y \in D(T^*)^{\perp}$ arbitrary. We now have to show that $y = 0$ to complete the proof.

$$0 = \langle y, w\rangle_Y = \langle 0, -T^*w\rangle_X + \langle y, w\rangle_Y = \langle (0,y), (-T^*w, w)\rangle_{X\times Y} \quad \forall w \in D(T^*)$$

which just means

$$(0,y) \in \big(R_{Y,X}\Gamma(T^*)\big)^{\perp} \overset{\text{Lemma 2.3.5}}{=} \overline{\Gamma(T)} = \Gamma(T).$$

In the last line we used the fact, that T is closed. Thus $y = T0 = 0$ which concludes the proof. □

{prop:T_starst

Proposition 2.3.8. *If T is a closed and densely defined operator, then $T^{**} = T$.*

Proof. This is another application of Lemma 2.3.5. Because T is closed, $\Gamma(T) = \big(R_{Y,X}\Gamma(T^*)\big)^{\perp}$. From the previous proposition, we know that T^* is a closed and densely defined operator as well and so $\Gamma(T^*) = \big(R_{X,Y}\Gamma(T^{**})\big)^{\perp}$. T^{**} is closed and hence its graph as well, so from the properties of the rotation we know that

$$\begin{aligned}\Gamma(T^{**}) = \overline{\Gamma(T^{**})} = \Gamma(T^{**})^{\perp\perp} &= \big(R_{Y,X}R_{X,Y}\Gamma(T^{**})\big)^{\perp\perp} \\ &= \Big(R_{Y,X}\big(R_{X,Y}\Gamma(T^{**})\big)^{\perp}\Big)^{\perp} = \Big(R_{Y,X}\Gamma(T^*)\Big)^{\perp} = \Gamma(T)\end{aligned}$$

Thus T and T^{**} have the same graph, which means that they are equal. □

We will now take a closer look at the kernels and images of unbounded operators. Let us first notice a very clear result. If T is a closed unbounded operator then its kernel $\ker T$ is closed. This follows indeed from the definition. But this is not true for the image $\operatorname{im} T$. Let us take $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{im} T$ with $y_n \to y$. If we now take the sequence $(x_n)_{n\in\mathbb{N}} \subseteq D(T)$ s.t. $Tx_n = y_n$, we do not know if (x_n) converges or whether the limit is in $D(T)$ if it does converge. A very simple example is the inclusion operator $\iota : H^1(\Omega) \to L^2(\Omega)$. This is actually a bounded operator and hence closed since

$$\|\iota f\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)} \leq \|f\|_{H^1(\Omega)},$$

but its range $H^1(\Omega)$ is not closed in $L^2(\Omega)$.

Let us summarize the following relationships between the images and kernels of closed densely defined operators and their adjoints.

{prop:kernel_i

Proposition 2.3.9. *Let $T : X \to Y$ be a closed densely defined operator. Then*

$$\begin{array}{lll} \text{(i)} & (\operatorname{im} T)^\perp & = \ker T^* \\ \text{(ii)} & (\ker T)^\perp & = \overline{\operatorname{im}(T^*)} \\ \text{(iii)} & (\operatorname{im} T^*)^\perp & = \ker T \\ \text{(iv)} & (\ker T^*)^\perp & = \overline{\operatorname{im}(T)} \end{array}$$

Proof. We will once again rely on Lemma 2.3.5 about the rotated graph. We will start with (iii)

$$x \in \ker T \Leftrightarrow (x,0) \in \Gamma(T) \overset{T \text{ closed}}{=} \overline{\Gamma(T)} = \big(R_{Y,X}\Gamma(T^*)\big)^\perp.$$

This is equivalent to saying that for any $y \in D(T^*)$ we have

$$0 = \langle (x,0), (-T^*y, y)\rangle_{X\times Y} = \langle x, -T^*y\rangle_X$$

which just means $x \in (\operatorname{im} T^*)^\perp$ and we proved (iii).

(ii) follows from that immediately by taking the orthogonal complement on both sides.

From Prop. 2.3.7, we know that T^* is closed and densely defined because T is. So the completely analogous reasoning with the roles of T and T^* exchanged gives us (i) and taking the orthogonal complement again proves (iv). □

2.3.2 Adjoints of differential operators in 3D

{sec:adjoints_

Let us investigate the situation in 3D with the common differential operators curl, grad and div on a domain $\Omega \subseteq \mathbb{R}^3$. At first, we only assume Ω to be open, but we will later introduce some assumptions on the boundary of it.

We will follow [2, Sec. 3.4] for the most part. However, all the results in Arnold's book are provided for bounded domains only which is insufficient for our situation since we want to cover the case where Ω is unbounded with compact boundary. Thus, we will generalize the proofs provided in the reference to this case which always involves applying some additional cut-off type argument.

Take the unbounded operator $\operatorname{div} : L^2(\Omega;\mathbb{R}^3) \to L^2(\Omega)$ with the smooth compactly supported vector valued functions $C_0^\infty(\Omega;\mathbb{R}^3)$ as domain i.e. $(\operatorname{div}, C_0^\infty(\Omega;\mathbb{R}^3))$ which we will from now on denote as $(\operatorname{div}, C_0^\infty)$. At first, we recognize that the operator $(\operatorname{div}, C_0^\infty(\Omega;\mathbb{R}^3))$ is densely defined and thus the adjoint exists. When we take the adjoint of it then we know for $v \in D((\operatorname{div}, C_0^\infty)^*)$

$$\int_\Omega \operatorname{div}^* v \cdot \mathbf{u}\, dx = \int_\Omega v \operatorname{div} \mathbf{u}\, dx \quad \forall \mathbf{u} \in C_0^\infty(\Omega;\mathbb{R}^3).$$

As before, vector valued quantites are written in bold. Now if we take $\mathbf{u} = (u_1, 0, 0)^\top$ then

$$\int_\Omega (\operatorname{div}^* v)_1\, u_1\, dx = \int_\Omega v\, \partial_1 u_1\, dx \quad \forall u_1 \in C_0^\infty. \tag{2.3.1}$$ {eq:adjoint_gr

From this, we recognize that $-(\operatorname{div}^* v)_1$ is the weak derivative w.r.t. the first coordinate i.e. $-\partial_1 v$ and analogous for the other coordinates so we recognize $\operatorname{div}^* = -\operatorname{grad}$. We further see that the domain s.t. (2.3.1) is fulfilled is $H^1(\Omega)$ by definition. That means we showed

$$(\operatorname{div}, C_0^\infty)^* = (-\operatorname{grad}, H^1). \tag{2.3.2}$$ {eq:adjoint_gr

We know from Cor. 2.3.6 that the adjoint $(-\operatorname{grad}, H^1)$ is closed and we can conclude from Prop. 2.3.3 that $H^1(\Omega)$ is in fact a Hilbert space when using the graph norm which is the H^1-norm here. This provides an alternative way to derive this basic result.

Assume from now on that our domain Ω is a Lipschitz domain with compact boundary $\partial\Omega$. Note that $\Omega \subseteq \mathbb{R}^3$ can be unbounded. In the case of an exterior domain, i.e. when Ω is the complement of a compact set with Lipschitz boundary, this condition is trivially fulfilled.

Let us denote with $C_b^k(\overline{\Omega})$ the space of k times continuously differentiable functions with bounded support in $\overline{\Omega}$. In contrast to $C_0^k(\Omega)$ these functions are not necessarily zero on the boundary. For $u \in C_b^1(\Omega)$, $\mathbf{v} \in C_b^1(\Omega;\mathbb{R}^3)$ we have the integration-by-parts formula

$$\int_\Omega u \operatorname{div} \mathbf{v}\, dx = -\int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds \tag{2.3.3}$$ {eq:integratio

where $\mathbf{n}$ is the unit normal which exists almost everywhere on the boundary of a Lipschitz domain. This formula is usually stated only for bounded domains (see [19, Cor. 3.20]), but when $\partial\Omega$ is compact we can simply reduce the domain because the supports of u and $\mathbf{v}$ are both bounded. This is done using a standard cut-off argument. To be precise, let $B_R \subseteq \mathbb{R}^3$ be the open ball around the origin with radius R large enought s.t. $\partial\Omega \subseteq B_R$ and $\operatorname{supp} u \subseteq B_R$, $\operatorname{supp} \mathbf{v} \subseteq B_R$. Define the reduced domain $\Omega_R := \Omega \cap B_R$. Then because u and $\mathbf{v}$ are both zero on ∂B_R

$$\begin{aligned}\int_\Omega u \,\operatorname{div} \mathbf{v}\, dx &= \int_{\Omega_R} u \,\operatorname{div} \mathbf{v}\, dx \\ &= -\int_{\Omega_R} \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds + \int_{\partial B_R} u|_{\partial B_R} \mathbf{v}|_{\partial B_R} \cdot \mathbf{n}\, ds \\ &= -\int_{\Omega} \operatorname{grad} u \cdot \mathbf{v}\, dx + \int_{\partial\Omega} u|_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, ds\end{aligned}$$

as claimed. This formula motivates us to define the divergence in a weak sense.

Definition 2.3.10 (Weak divergence)**.** Let $\Omega \subseteq \mathbb{R}^3$ be open. For $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ we define the space

$$H(\operatorname{div};\Omega) := \Big\{ \mathbf{v} \in L^2(\Omega;\mathbb{R}^3) \mid \exists \sigma \in L^2(\Omega)\, \forall \phi \in C_0^\infty(\Omega) : \int_\Omega \sigma\, \phi\, dx = -\int_\Omega \mathbf{v} \cdot \operatorname{grad} \phi\, dx \Big\}$$

Then we call the σ in the definition the *weak divergence* of $\mathbf{v}$ denoted by $\operatorname{div} \mathbf{v}$.

Using basic arguments from Sobolev theory this defines $\operatorname{div} \mathbf{v}$ uniquely almost everywhere.

We will frequently leave out the reference to the domain in the space definition. We can immediately recognize from the definition that

$$(\operatorname{div}, H(\operatorname{div})) = (-\operatorname{grad}, C_0^\infty)^*. \tag{2.3.4}$$ {eq:adjoint_gr

The next thing we want to talk about is the trace of $H(\operatorname{div})$. We will use the well-known properties of the trace in H^1. Let us recall quickly the definition of fractional order Sobolev spaces (cf. [19, Sec. 3.2]). Let $U \subseteq \mathbb{R}^n$ be an open domain. Take $m \in \mathbb{N}$, $s \in [0,1)$ and $1 < p < \infty$. For a multiindex $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ we define $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$ and denote

$$\partial_{\boldsymbol{\alpha}} \phi = \partial_{\alpha_1} \partial_{\alpha_2} \dots \partial_{\alpha_n} \phi. \tag{2.3.5}$$

Then we define the fractional Sobolev norm

$$\|u\|_{W^{m+s,p}} = \left\{ \|u\|^p_{W^{m,p}} + \sum_{|\boldsymbol{\alpha}|=m} \int_U \int_U \frac{|\partial_{\boldsymbol{\alpha}} u(x) - \partial_{\boldsymbol{\alpha}} u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \right\}^{1/p} \tag{2.3.6}$$ {eq:fractional_

and the corresponding space $W^{m+s,p}(U) = \{u \in W^{m,p}(U) \mid \|u\|_{W^{m+s,p}} < \infty\}$ which is a Banach space [19, p.42]. We denote $H^{m+s}(U) = W^{m+s,2}(U)$ as in the integer case. We will need the fractional Sobolev space over the boundary of a Lipschitz domain U. We assume again that the boundary is compact. We define fractional order Sobolev spaces by taking the integrals of (2.3.6) over the boundary instead i.e. for $m = 0$

$$\|u\|_{W^{s,p}(\partial\Omega)} = \left\{ \int_{\partial\Omega} |u|^p \, ds + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n-1+sp}} \, ds(x) ds(y) \right\}^{1/p}.$$

Then it is a well-known result that $H^1(\Omega)$ has the trace operator $\mathrm{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ which is surjective [2, p.28]. We are trying to find something analogous for $H(\mathrm{div})$.

Remark 2.3.11. The H^1 trace operator is usually defined only for bounded Lipschitz domains (see [19, Thm. 3.9]), but it can be extended easily to unbounded Lipschitz domains with compact boundary considering only values in some bounded subdomain. To make this precise take an open ball B_R centered at the origin with radius R s.t. $\partial\Omega \subseteq B_R$. We denote $|\cdot|$ as the standard Euclidian norm and define

$$\chi_R(x) := \begin{cases} 1 & |x| \leq R, \\ R + 1 - |x| & R < |x| < R+1, \\ 0 & \text{else.} \end{cases}$$

Let us denote $\Omega_{R+1} := \Omega \cap B_{R+1}$ To define the trace of a function $u \in H^1(\Omega)$ take the trace of $\chi_R u$ – which has compact support in $\overline{B}_{R+1}$ – on Ω_R and restrict it to $\partial\Omega$ i.e. $\mathrm{tr}\, u = \big(\mathrm{tr}_{\Omega_R} u\chi_R\big)|_{\partial\Omega}$. Then $tr : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective and with C_R being the continuity constant of the trace on Ω_R

$$\begin{aligned} \|\mathrm{tr}\, u\|_{H^{1/2}(\partial\Omega)} &= \|\mathrm{tr}_{\Omega_R} u\, \chi_R\|_{H^{1/2}(\partial\Omega_R)} \leq C_R \|u\, \chi_R\|_{H^1(\Omega_R)} = C_R \|u\, \chi_R\|_{H^1(\Omega)} \\ &\leq \sqrt{3} C_R \|u\|_{H^1(\Omega)} \end{aligned}$$

and thus the trace is bounded as well. The last inequality follows from the expression of $\mathrm{grad}\, \chi_R$ and some standard estimates.

We will start with the following abstract lemma. As is standard for a bounded linear functional ℓ on a normed space V, we write $\langle \ell, v\rangle_{V'\times V} = \ell(v)$ or just $\langle \ell, v\rangle$.

Lemma 2.3.12. *Let X, Y be Banach spaces and let $\gamma : X \to Y$ be a linear bounded surjection with kernel Z. Then the dual map is defined as* {lem:dual_map_

$$\gamma' : Y' \to X', \ell \mapsto \ell \circ \gamma$$

or in product notation

$$\langle \gamma' \, \ell, x \rangle_{X' \times X} = \langle \ell, \gamma \, x \rangle_{Y' \times Y}.$$

This dual map is then a bounded injection with image being the annihilator of Z which is defined as $\{f \in X' \mid f|_Z \equiv 0\}$. In other words, if we have $f \in X'$ with $\langle f, z \rangle = 0$ for all $z \in Z$ then there exists a unique $g_f \in Y'$ s.t. $\langle f, x \rangle_{X' \times X} = \langle g_f, \gamma \, x \rangle_{Y' \times Y}$.

Proof. Since γ is bounded it is obviously a closed and densely defined unbounded operator. Then we know from [5, Thm. 2.20] that γ' is injective. Thm. 2.19 in the same reference then gives us that $\operatorname{im} \gamma'$ is the annihilator of Z. The dual map is actually the adjoint in the generalized sense on Banach spaces which is how the adjoint is defined in this reference. Then the annihilator corresponds in that notation to "$Z^{\perp}$", but we will not use this notation because we only introduced the adjoint for Hilbert spaces. □

We now want to apply this lemma to the trace. We follow the standard notation and write $H^{-1/2}$ for the dual space of $H^{1/2}$

Proposition 2.3.13. *Let $\ell \in H^1(\Omega)'$ s.t. $\langle \ell, \sigma \rangle = 0$ for all $\sigma \in H_0^1(\Omega)$. Then there exists a unique $g_\ell \in H^{-1/2}(\partial\Omega)$ s.t.* {prop:existenc

$$\langle \ell, u \rangle_{H^1(\Omega)' \times H^1(\Omega)} = \langle g_\ell, \operatorname{tr} u \rangle_{H^{1/2}(\partial\Omega)' \times H^{1/2}(\partial\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2.3.7}$$ {eq:functional_

Also, there exist positive constants C_1 and C_2 s.t.

$$C_1 \|g_\ell\|_{H^{-1/2}(\partial\Omega)} \leq \|\ell\|_{H^1(\Omega)'} \leq C_2 \|g_\ell\|_{H^{-1/2}(\partial\Omega)}.$$

Proof. Let us check the conditions of the Lemma 2.3.12. We have the trace operator $\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ as a linear bounded surjection. So the conditions of Lemma 2.3.12 are fulfilled and (2.3.7) follows. Using the fact that the H^1 trace is bounded we can estimate

$$|\langle \ell, u \rangle| = |\langle g_\ell, \operatorname{tr} u \rangle| \leq C_2 \|g_\ell\|_{H^{-1/2}} \, \|u\|_{H^1}$$

for some constant $C_2 > 0$ and thus $\|\ell\|_{H^1(\Omega)'} \leq C_2\|g_\ell\|_{H^{-1/2}}$. For the inverse inequality, we take some $\mu \in H^{1/2}$. If we can find a $f \in H^1$ s.t. $\operatorname{tr} f = \mu$ and

$$\|f\|_{H^1} \leq 1/C_1\|\mu\|_{H^{1/2}} \tag{2.3.8}$$ {eq:existence_

holds then we can conclude

$$|\langle g_\ell, \mu\rangle| = |\langle \ell, f\rangle| \leq \|\ell\|_{H^1(\Omega)}\|f\|_{H^1} \leq \|\ell\|_{H^1(\Omega)}1/C_1\|\mu\|_{H^{1/2}}$$

and thus $\|g_\ell\|_{H^{-1/2}} \leq 1/C_1\|\ell\|_{H^1(\Omega)}$.

We have to show that such an f exists. This is true for bounded domains (cf. [8, Thm. 3.10]). To generalize it to unbounded domains with compact boundary, take an open ball B_R centered at the origin with radius R large enough s.t. $\partial\Omega \subseteq B_R$. Then define $\Omega_R = \Omega \cap B_R$. Take f_R as a function s.t. $\operatorname{tr} f_R = \mu$ on $\partial\Omega$, $\operatorname{tr} f_R = 0$ on ∂B_R and $\|f_R\|_{H^1(\Omega_R)} \leq C_R\|\operatorname{tr} f_R\|_{H^{1/2}(\Omega_R)}$ for some $C_R > 0$ independent of μ. Because $\operatorname{tr} f_R = 0$ on ∂B_R we have $\|\operatorname{tr} f_R\|_{H^{1/2}(\partial\Omega_R)} = \|\operatorname{tr} f_R\|_{H^{1/2}(\partial\Omega)}$ and we can define $\bar{f}_R \in H^1(\Omega)$ by extending f by zero outside of B_R so that $\|\bar{f}_R\|_{H^1(\Omega)} = \|f_R\|_{H^1(\Omega_R)}$. To summarize,

$$\|\bar{f}_R\|_{H^1(\Omega)} = \|f_R\|_{H^1(\Omega_R)} \leq C_R\|\operatorname{tr} f_R\|_{H^{1/2}(\Omega_R)} = C_R\|\operatorname{tr} f_R\|_{H^{1/2}(\partial\Omega)} = C_R\|\mu\|_{H^{1/2}(\partial\Omega)}$$

so (2.3.8) holds with $f = \bar{f}_R$ which completes the proof. $\square$

We can now define a trace operator for $H(\operatorname{div})$.

{thm:trace_of_

Theorem 2.3.14 (Trace of $H(\operatorname{div})$). *The operator $\mathbf{v} \mapsto \langle \mathbf{v}|_{\partial\Omega} \cdot \mathbf{n}, \cdot\rangle_{L^2(\partial\Omega)} \in H^{-1/2}(\partial\Omega)$ on $C_b^1(\overline{\Omega})$ can be extended to an operator $\gamma_n : H(\operatorname{div};\Omega) \to H^{-1/2}$ s.t.*

$$\int_\Omega \operatorname{div}\mathbf{v}\, u\, dx = -\int_\Omega \mathbf{v}\cdot \operatorname{grad} u\, dx + \langle \gamma_n\mathbf{v}, \operatorname{tr} u\rangle$$

Also, analogous to the standard trace theorem, we have

$$\|\gamma_n\mathbf{v}\|_{H^{-1/2}(\Omega)} \leq \sqrt{2}\|\mathbf{v}\|_{H(\operatorname{div};\Omega)}.$$

Proof. Let us define the linear functional $\ell_\mathbf{v} \in H^1(\Omega)'$ for a fixed $\mathbf{v} \in H(\operatorname{div})$ as

$$\langle \ell_\mathbf{v}, u\rangle := \int_\Omega u\ \operatorname{div}\mathbf{v}\, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx.$$

Using Cauchy-Schwarz we get

$$\langle \ell_\mathbf{v}, u\rangle \leq \sqrt{2}\|\mathbf{v}\|_{H(\operatorname{div})}\|u\|_{H^1}$$

and so $\ell_{\mathbf{v}}$ is indeed a linear bounded functional on H^1.

Take $u \in H^1_0(\Omega)$. Let $(u^k)_{k\in\mathbb{N}} \subseteq C_0^\infty$ s.t. $u^k \xrightarrow{H^1} u$ which is possible due to density. $\langle \ell_{\mathbf{v}}, u^k\rangle = 0$ which follows directly from the definition of the weak divergence and thus $\langle \ell_{\mathbf{v}}, u\rangle = 0$ by continuity.

We are in the situation of Prop. 2.3.13 and we find $\gamma_n \mathbf{v} \in H^{-1/2}(\partial\Omega)$ s.t.

$$\int_\Omega u \operatorname{div} \mathbf{v}\, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx = \langle \gamma_n \mathbf{v}, \operatorname{tr} u\rangle \quad \forall u \in H^1(\Omega).$$

The linearity of γ_n is trivial to see. For the boundedness, using Cauchy-Schwarz

$$\langle \gamma_n \mathbf{v}, \operatorname{tr} u\rangle = \int_\Omega u \operatorname{div} \mathbf{v}\, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx \leq \sqrt{2}\|\mathbf{v}\|_{H(\operatorname{div})}\|u\|_{H^1}$$

and so $\|\gamma_n \mathbf{v}\|_{H^{-1/2}} \leq \sqrt{2}\|\mathbf{v}\|_{H(\operatorname{div})}$ which proves that γ_n is indeed a bounded linear operator.

Let us now look at the case when $\mathbf{v} \in C_b^1(\overline{\Omega}; \mathbb{R}^3)$. We know the integration-by-parts formula holds for all $u \in H^1$ as explained above. We have

$$\langle \gamma_n \mathbf{v}, \operatorname{tr} u\rangle = \int_\Omega u \operatorname{div} \mathbf{v}\, dx + \int_\Omega \operatorname{grad} u \cdot \mathbf{v}\, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \operatorname{tr} u\, ds.$$

Due to the surjectivity of the trace of H^1 we get

$$\langle \gamma_n \mathbf{v}, \cdot\rangle = \langle \mathbf{v} \cdot n, \cdot\rangle_{L^2(\partial\Omega)}.$$

So in this sense the operator $\mathbf{v} \mapsto \langle \mathbf{v} \cdot n, \cdot\rangle_{L^2(\partial\Omega)} \in H^{-1/2}$ extends to the bounded operator $\gamma_n : H(\operatorname{div}) \to H^{-1/2}$. □

This theorem motivates us to recognize the normal trace as the natural trace on $H(\operatorname{div})$. Now we can define

$$H_0(\operatorname{div}; \Omega) := \{\mathbf{v} \in H(\operatorname{div}) \mid \gamma_n \mathbf{v} = 0\}.$$

We know that H^1 contains all smooth functions $C_b^\infty(\overline{\Omega})$ which are dense in L^2. This follows easily from the fact that $C^\infty(\overline{\Omega})$ is dense in H^1 if Ω is bounded. Analogously, it can be shown that $C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\operatorname{div})$ as well by using the fact that $C^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\operatorname{div})$ if Ω is bounded ([19, Thm. 3.22]). Hence, both of $(\operatorname{grad}, H^1)$ and $(\operatorname{div}, H(\operatorname{div}))$ are densely defined so their adjoints exist and we will compute them next. To do so, we need the following theorem.

Theorem 2.3.15 (Surjectivity of γ_n)**.** *The operator $\gamma_n : H(\operatorname{div}; \Omega) \to H^{-1/2}(\partial\Omega)$ is surjective.*

Proof. Take $g \in H^{-1/2}(\Omega)$. As in previous proofs, take the open ball B_R centered at the origin with radius R large enough s.t. $\partial\Omega \subseteq B_R$ and define $\Omega_R = \Omega \cap B_R$. This is obviously a bounded Lipschitz domain with boundary $\partial\Omega_R = \partial\Omega \,\dot\cup\, \partial B_R$. Then we take u_R as the solution of the problem

$$\begin{aligned} -\Delta u_R + u_R &= 0 \text{ in } \Omega_R, \\ u_R &= g \text{ on } \partial\Omega \text{ and} \\ u_R &= 0 \text{ on } \partial B_R. \end{aligned}$$

which reads in variational formulation

$$\int_{\Omega_R} \operatorname{grad} u_R \cdot \operatorname{grad} \phi \, dx + \int_{\Omega_R} u_R \, \phi \, dx = \langle g, \operatorname{tr} \phi \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \quad \forall \phi \in H^1(\Omega_R).$$

This problem has a unique solution since Ω_R is bounded (see [19, Thm. 3.12]). Take $\mathbf{v} := \operatorname{grad} u_R$. Then we see from the variational formulation by choosing $\phi \in C_0^\infty(\Omega_R)$ that $u_R = \operatorname{div} \mathbf{v}$. Then define $\bar{u}_R$ as the extension by zero outside of B_R which is in H^1 because $u_R = 0$ on ∂B_R and take $\bar{\mathbf{v}} = \operatorname{grad} \bar{u}_R$. Then $\operatorname{div} \bar{\mathbf{v}} = \bar{u}_R$ and for any $\phi \in H^1(\Omega)$

$$\begin{aligned} \langle \gamma_n \bar{\mathbf{v}}, \operatorname{tr} \phi \rangle &= \int_\Omega \bar{\mathbf{v}} \cdot \operatorname{grad} \phi \, dx + \int_\Omega \operatorname{div} \bar{\mathbf{v}} \, \phi \, dx \\ &= \int_\Omega \operatorname{grad} \bar{u}_R \cdot \operatorname{grad} \phi \, dx + \int_\Omega \bar{u}_R \, \phi \, dx \\ &= \int_{\Omega_R} \operatorname{grad} u_R \cdot \operatorname{grad} \phi \, dx + \int_{\Omega_R} u_R \, \phi \, dx \\ &= \langle g, \operatorname{tr} \phi \rangle. \end{aligned}$$

Since the trace on H^1 is surjective we get $\gamma_n \mathbf{v} = g$. $\square$

Now we can compute the adjoints of the operators grad and div without boundary conditions.

{thm:adjoints_

Theorem 2.3.16.

$$(-\operatorname{div}, H_0(\operatorname{div})) = (\operatorname{grad}, H^1)^* \tag{2.3.9}$$

$$(-\operatorname{grad}, H_0^1) = (\operatorname{div}, H(\operatorname{div}))^* \tag{2.3.10}$$

Proof. If we take $\mathbf{v} \in H_0(\operatorname{div})$ then we immediately get from the integration by parts formula

$$\langle -\operatorname{div} \mathbf{v}, u \rangle = \langle \mathbf{v}, \operatorname{grad} u \rangle \quad \forall u \in H^1$$

and so $\mathbf{v} \in D((\operatorname{grad}, H^1)^*)$ and $\operatorname{grad}^* \mathbf{v} = -\operatorname{div} \mathbf{v}$. Here we use the L^2 inner product and we will from now denote the L^2 inner product just by $\langle \cdot, \cdot \rangle$. Vice versa, take $\mathbf{v} \in D((\operatorname{grad}, H^1)^*)$. Then there exists $\sigma \in L^2$ s.t.

$$\langle \mathbf{v}, \operatorname{grad} u \rangle = \langle \sigma, u \rangle$$

for any $u \in H^1$. By choosing $u \in C_0^\infty$ we see $\mathbf{v} \in H(\operatorname{div})$ and $\sigma = -\operatorname{div} \mathbf{v}$. Now taking again $u \in H^1$ arbitrary,

$$0 = \langle \mathbf{v}, \operatorname{grad} u \rangle + \langle \operatorname{div} \mathbf{v}, u \rangle = \langle \gamma_n \mathbf{v}, \operatorname{tr} u \rangle$$

and since tr is surjective we get $\gamma_n \mathbf{v} = 0$ i.e. $\mathbf{v} \in H_0(\operatorname{div})$ and we have proven the first equality.

Take $u \in D((\operatorname{div}, H(\operatorname{div}))^*)$. Then we already see

$$\int_\Omega \operatorname{div}^* u \cdot \boldsymbol{\phi} \, dx = \int_\Omega u \operatorname{div} \boldsymbol{\phi} \, dx \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3)$$

and we can conclude due to (2.3.2) that $u \in H^1$ and $\operatorname{div}^* u = -\operatorname{grad} u$. Then

$$0 = \langle \operatorname{grad} u, \mathbf{v} \rangle + \langle u, \operatorname{div} \mathbf{v} \rangle = \langle \gamma_n \mathbf{v}, \operatorname{tr} u \rangle.$$

Now we use the surjectivity of γ_n and see that $\langle g, \operatorname{tr} u \rangle = 0$ for all $g \in H^{-1/2}(\partial\Omega)$ which implies $\operatorname{tr} u = 0$ and thus $u \in H_0^1$. The other direction follows again from the integration by parts formula immediately and we get $D((\operatorname{div}, H(\operatorname{div}))^*) = H_0^1$ and the claim follows. $\square$

This theorem provides us with an easy computation of the adjoints of the grad and div with homogeneous boundary conditions.

Corollary 2.3.17.

$$(-\operatorname{div}, H(\operatorname{div})) = (\operatorname{grad}, H_0^1)^* \tag{2.3.11}$$

$$(-\operatorname{grad}, H^1) = (\operatorname{div}, H_0(\operatorname{div}))^* \tag{2.3.12}$$

Proof. $(\operatorname{div}, H(\operatorname{div}))$ is the adjoint of the densely defined operator $(-\operatorname{grad}, C_0^\infty)$ and hence a closed operator. It is obviously densely defined and thus we can use previous results,

$$(\operatorname{div}, H(\operatorname{div})) \overset{Prop. 2.3.8}{=} (\operatorname{div}, H(\operatorname{div}))^{**} \overset{Thm. 2.3.16}{=} (-\operatorname{grad}, H_0^1)^*.$$

The adjoint of $(\operatorname{div}, H_0(\operatorname{div}))$ is computed completely analogously. $\square$

This completes the computation of the adjoints of grad and div. Because $(\operatorname{div}, H(\operatorname{div}))$ is a closed and densely defined operator by taking the graph inner product on $H(\operatorname{div})$

$$\langle \mathbf{v}, \mathbf{w} \rangle_{H(\operatorname{div})} := \langle \mathbf{v}, \mathbf{w} \rangle + \langle \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w} \rangle$$

and the induced norm, we know from Prop. 2.3.3 that $H(\operatorname{div})$ becomes a Hilbert space.

Remark 2.3.18. Notice that in the above arguments we never used the fact that we are working in three dimensions. Thus, all the arguments can be generalized to n dimensions without problems.

Now let us turn our attention to the remaining fundamental differential operator curl. For $\mathbf{u}, \mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^3)$ we have the well-known integration-by-parts formula

$$\int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{u} \, dx = \int_\Omega \operatorname{curl} \mathbf{v} \cdot \mathbf{u} \, dx + \int_{\partial\Omega} \mathbf{v} \times \mathbf{n} \cdot \mathbf{u} \, ds \qquad (2.3.13)$$ {eq:integratio

if Ω is bounded and Lipschitz. Just as before, this integration by parts formula can be extended to Ω unbounded when $\partial\Omega$ is compact. We define as before

Definition 2.3.19. Let $\Omega \subseteq \mathbb{R}^3$ be an open domain. Then we define

$$H(\operatorname{curl}; \Omega) := \Big\{ \mathbf{v} \in L^2(\Omega; \mathbb{R}^3) \mid \exists \mathbf{w} \in L^2(\Omega; \mathbb{R}^3) \, \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3) : \int_\Omega \mathbf{w} \cdot \boldsymbol{\phi} \, dx = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \boldsymbol{\phi} \, dx \Big\}$$

and we denote $\mathbf{w} = \operatorname{curl} \mathbf{v}$.

From the definition we can see

$$(\operatorname{curl}, H(\operatorname{curl})) = (\operatorname{curl}, C_0^\infty)^*.$$

Following analogous arguments as above, we obtain the following

{thm:trace_hcu

Theorem 2.3.20. *Let Ω be a Lipschitz domain with compact boundary $\partial\Omega$. Then the operator $\mathbf{v} \mapsto \langle \mathbf{v}|_{\partial\Omega} \times \mathbf{n}, \cdot \rangle_{L^2(\partial\Omega)}$ defined on $C_b^1(\overline{\Omega}; \mathbb{R}^3)$ extends to a bounded linear operator $\gamma_\tau : H(\operatorname{curl}) \to H^{-1/2}(\partial\Omega; \mathbb{R}^3)$ s.t. the integration-by-parts formula*

$$\int_\Omega \operatorname{curl} \mathbf{v} \cdot \mathbf{u} \, dx = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{u} \, dx + \langle \gamma_\tau \mathbf{v}, \mathbf{u} \rangle$$

for all $\mathbf{v} \in H(\operatorname{curl}), \mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$ is satisfied and there exists $C > 0$ s.t. $\|\gamma_\tau \mathbf{v}\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^3)} \leq C \|\mathbf{v}\|_{H(\operatorname{curl})}$.

Analogous to before, we define

$$H_0(\mathrm{curl}) := \{\mathbf{v} \in H(\mathrm{curl}) \mid \gamma_\tau \mathbf{v} = 0\}.$$

Using this definition we can compute the adjoint.

Theorem 2.3.21. *We have the adjoints*

$$(\mathrm{curl}, H_0(\mathrm{curl})) = (\mathrm{curl}, H(\mathrm{curl}))^* \text{ and}$$
$$(\mathrm{curl}, H(\mathrm{curl})) = (\mathrm{curl}, H_0(\mathrm{curl}))^*.$$

Proof. Take $\mathbf{v} \in D((\mathrm{curl}, H(\mathrm{curl}))^*)$.

$$\int_\Omega \mathrm{curl}^* \mathbf{v} \cdot \boldsymbol{\phi} \, dx = \int_\Omega \mathbf{v} \cdot \mathrm{curl} \, \boldsymbol{\phi} \, dx \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega; \mathbb{R}^3)$$

and thus $\mathrm{curl}^* \mathbf{v} = \mathrm{curl} \, \mathbf{v}$ and we have

$$\langle \mathrm{curl} \, \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w} \rangle \tag{2.3.14}$$ {eq:adjoint_cu

for all $\mathbf{w} \in H(\mathrm{curl})$. That means that for $\mathbf{w} \in H^1(\Omega; \mathbb{R}^3)$

$$\langle \gamma_\tau \mathbf{v}, \mathrm{tr} \, \mathbf{w} \rangle = \langle \mathrm{curl} \, \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w} \rangle = 0$$

and thus $\gamma_\tau \mathbf{v} = 0$ i.e. $\mathbf{v} \in H_0(\mathrm{curl})$.

Vice versa, take $\mathbf{v} \in H_0(\mathrm{curl})$. Then we need to show that (2.3.14) is fulfilled and we would like to use the integration-by-parts formula for that. However, this formula only holds for $\mathbf{w} \in H^1(\Omega; \mathbb{R}^3)$. We need to use the fact that $C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ is dense in $H(\mathrm{curl})$. Analogous to what we have done before, the argument follows easily from the fact that for bounded domains U, $C^\infty(\overline{U}; \mathbb{R}^3)$ is dense in $H(\mathrm{curl}; U)$. The statement for bounded domains is proven in [19, Lemma 3.27]. The proof is quite technical and we will not present it here. But using the density it is clear that $H^1(\Omega; \mathbb{R}^3)$ is also dense in $H(\mathrm{curl})$ since it contains all smooth functions with bounded support. Take $\mathbf{w} \in H(\mathrm{curl})$ arbitrary and $(\mathbf{w}^k)_{k \in \mathbb{N}} \subseteq H^1(\Omega; \mathbb{R}^3)$ s.t. $\mathbf{w}^k \xrightarrow{H(\mathrm{curl})} \mathbf{w}$. Because $\gamma_\tau \mathbf{v} = 0$ we have

$$\langle \mathrm{curl} \, \mathbf{v}, \mathbf{w}^k \rangle = \langle \mathbf{v}, \mathrm{curl} \, \mathbf{w}^k \rangle$$

Now taking the limits on both sides and using the fact that $\mathbf{w}$ was arbitrary we get $\mathbf{v} \in D((\mathrm{curl}, H(\mathrm{curl}))^*)$ and we obtain the first equality.

The second equality follows from the fact that $(\mathrm{curl}, H(\mathrm{curl}))$ is a closed and densely defined operator and thus

$$(\mathrm{curl}, H(\mathrm{curl})) = (\mathrm{curl}, H(\mathrm{curl}))^{**} = (\mathrm{curl}, H_0(\mathrm{curl}))^*.$$

□

With this, we computed the adjoints of the most important differential operators that will be needed in the following section.

2.3.3 Hilbert complexes

{sec:hilbert_c

Now we will combine the idea of cochain complexes from Section 2.2 with unbounded operators. We will derive the Hodge decomposition with the theory of unbounded operators of Sec. 2.3.1 and apply it to the three dimensional case using the results of Sec. 2.3.2 about the differential operators in 3D.

Recall that a cochain complex is in full generality a sequence of groups $(G^i)_{i\in\mathbb{Z}}$ and group homomorphims $f^i : G^i \to G^{i+1}$ s.t. $f^{i+1} \circ f^i = 0$.

Definition 2.3.22 (Hilbert complex)**.** A Hilbert complex is a sequence of real Hilbert spaces $(W^k)_{k\in\mathbb{Z}}$ and a sequence of closed, densely defined unbounded operators $d^k : W^k \to W^{k+1}$ with domain $V^k \subseteq W^k$ s.t. $d^{k+1} \circ d^k = 0$.

We denote $\mathfrak{Z}^k := \ker d^k$ and $\mathfrak{B}^k := \operatorname{im} d^{k-1}$. Then it follows from the definition that $\mathfrak{B}^k \subseteq \mathfrak{Z}^k$.

To be precise, a Hilbert complex is in fact not a cochain complex in the exact sense because the homomorphims d^k are not defined on the whole space, but it is if we look at the operators defined on their domains V^k instead.

Because unbounded operators are bounded w.r.t. the graph norm, the restriction of the operators to their domain, $d^k : V^k \to V^{k+1}$, are bounded operators when we use the graph norm on V^k. Because we assume d^k to be closed we know from Prop. 2.3.3 that V^k are Hilbert spaces w.r.t. the graph norm $\|\cdot\|_{V^k}$. So we see that d^k together with V^k is also a Hilbert complex which we call *domain complex*. In this Hilbert complex, all operators are bounded. Notice since the operators are defined on the whole Hilbert space V^k this fits the definition of a cochain complex, because vector spaces with summation are groups and the d^k are linear mappings and hence group homomorphisms.

Now let us investigate the adjoints of the operators in a Hilbert complex. Since we assume the operators to be closed and densely defined the adjoints exist and we denote with $d_k^* : W^k \to W^{k-1}$ the adjoint of d^k. Due to Prop. 2.3.7 we know that the adjoints are also closed and densely defined. We denote $V_k^* := D(d_k^*)$, $\mathfrak{Z}_k^* := \ker d_k^*$ and $\mathfrak{B}_k^* := \operatorname{im} d_k^*$. We will frequently leave out the indices from now on.

We can apply Prop. 2.3.9 to this construction. Then we observe

$$\begin{aligned} \mathfrak{B}^\perp &= \mathfrak{Z}^*, \\ \mathfrak{Z}^\perp &= \overline{\mathfrak{B}^*}, \\ \mathfrak{B}^{*\perp} &= \mathfrak{Z} \text{ and} \\ \mathfrak{Z}^{*\perp} &= \overline{\mathfrak{B}^*}. \end{aligned}$$

Now recall the basic fact from Hilbert space theory that in any Hilbert space if we have any two subspaces $V \subseteq W$ then taking the orthogonal complements reverses the inclusion i.e. $V^\perp \supseteq W^\perp$. Then we get

$$\mathfrak{B}^* \subseteq \overline{\mathfrak{B}^*} = \mathfrak{Z}^\perp \subseteq \mathfrak{B}^\perp = \mathfrak{Z}^*. \tag{2.3.15}$$ {eq:image_kern

We call

$$\dots \xrightarrow{d^*_{k+2}} V^*_{k+1} \xrightarrow{d^*_{k+1}} V^*_k \xrightarrow{d^*_k} V^*_{k-1} \xrightarrow{d^*_{k-1}} \dots$$

the *dual complex* of the Hilbert complex.

Definition 2.3.23. We call a $v \in V^k \cap V^*_k$ *harmonic form* if $d^k v = 0$ and $d^*_k v = 0$. Denote the space of harmonic forms as $\mathfrak{H}^k$.

We can rewrite this as $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}^*_k$. Using (2.3.15,

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{B}^{*,\perp}_k \cap \mathfrak{Z}^*_k.$$

Now we can formulate the most important result of this chapter.

{thm:hodge_dec

Theorem 2.3.24 (Hodge decomposition)**.** *Let $d^k : W^k \to W^{k+1}$ form a Hilbert complex. Then we have*

$$\mathfrak{Z}^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \text{ and}$$
$$\mathfrak{Z}^*_k = \overline{\mathfrak{B}^*_k} \overset{\perp}{\oplus} \mathfrak{H}^k.$$

We obtain the Hodge decomposition of the space W^k

$$W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}^*_k}.$$

Proof. Let us first prove $\mathfrak{Z}^k \subseteq \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. Take $z \in \mathfrak{Z}^k$ arbitrary. From basic Hilbert theory we know that $W^k = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{B}^{k,\perp}$. So we find $z = z_1 + z_2$ with $z_1 \in \overline{\mathfrak{B}^k}$ and $z_2 \in \mathfrak{B}^{k,\perp}$. Because $\mathfrak{Z}^k$ is closed $z_1 \in \overline{\mathfrak{B}^k} \subseteq \mathfrak{Z}^k$ and thus $z_2 = z - z_1 \in \mathfrak{Z}^k$ as well i.e. $z_2 \in \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{H}^k$ and so $z \in \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k$. The other inclusion is obvious since $\mathfrak{H}^k = \mathfrak{B}^{k,\perp} \cap \mathfrak{Z}^k$. The proof for the second equality is completely analogous.

For the Hodge decomposition, since $\mathfrak{Z}^k$ is closed

$$W^k = \mathfrak{Z}^k \overset{\perp}{\oplus} \mathfrak{Z}^{k,\perp} = \overline{\mathfrak{B}^k} \overset{\perp}{\oplus} \mathfrak{H}^k \overset{\perp}{\oplus} \overline{\mathfrak{B}^*_k}.$$

□

The Hodge decomposition is a very powerful tool whenever we deal with Hilbert complexes. Another important result will be the Poincaré inequality in the case where $\mathfrak{B}^k$ is closed.

{thm:poincare_

Theorem 2.3.25 (Poincaré inequality)**.** *For any k, if $\mathfrak{B}^{k+1}$ is closed in V then there exists a $c_{P,k} > 0$ s.t.*

$$\|z\|_V \leq c_{P,k} \|dz\|, \quad \forall z \in \mathfrak{Z}^{k,\perp_V}$$

where $\perp_V$ denotes the orthogonal complement w.r.t. the V-inner product.

Proof. This follows directly form the Banach inverse theorem. Because $\mathfrak{B}^{k+1}$ is closed it is itself a Hilbert space. If we restrict d^k to $\mathfrak{Z}^{k,\perp_V}$ it is injective and so $d^k|_{\mathfrak{Z}^{k,\perp_V}} : \mathfrak{Z}^{k,\perp_V} \to \mathfrak{B}^{k+1}$ is a bounded isomorphism between Hilbert spaces and we can apply the Banach inverse theorem to obtain the inverse denoted by slight abuse of notation as d^{-1} which is also bounded. So for any $z \in \mathfrak{Z}^{k,\perp_V}$

$$\|z\|_V = \|d^{-1} dz\|_V \leq c_{P,k} \|dz\|_V = c_{P,k} \|dz\|.$$

□

Just as with other objects, we will frequently leave out the index and denote the Poincaré constant $c_{P,k}$ just as c_P.

L^2 de Rham complex in 3D

{sec:l2_de_rhar

Let us investigate the situation for the differential operators grad, div and curl. All the necessary ingredients were already proven in Sec. 2.3.2. Let Ω be a Lipschitz domain of $\mathbb{R}^3$ with compact boundary $\partial\Omega$.

We take $W^0 = W^3 = L^2(\Omega)$, $W^1 = W^2 = L^2(\Omega; \mathbb{R}^3)$ and we set all other W^k to zero in order to obtain a sequence. Then we choose the operators $d^0 = \mathrm{grad}$, $d^1 = \mathrm{curl}$, $d^2 = \mathrm{div}$ and for the domains we choose $V^0 = H^1(\Omega)$, $V^1 = H(\mathrm{curl}; \Omega)$, $V^2 = H(\mathrm{div}; \Omega)$ and $V^3 = L^2(\Omega) = W^3$. As before, we will leave out the reference to the domain Ω now. All other d^k are just zero. The resulting domain complex is then

$$0 \to H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0 \tag{2.3.16}$$ {eq:primal_de_

All these operators are closed and densely defined. It remains to show that $d^{k+1} \circ d^k = 0$. If $k < 0$ or $k > 2$ this is clear. Then we have from the definition of the weak grad and curl for any $u \in H^1$ and $\mathbf{v} \in C_0^\infty(\Omega; \mathbb{R}^3)$

$$\int_\Omega \mathrm{curl}\,\mathrm{grad}\, u \cdot \mathbf{v}\, dx = \int_\Omega \mathrm{grad}\, u \cdot \mathrm{curl}\, \mathbf{v}\, dx = -\int_\Omega u \ \mathrm{div}\,\mathrm{curl}\, \mathbf{v}\, dx = 0$$

which implies that $\mathrm{curl}\,\mathrm{grad}\, u = 0$. $\mathrm{div}\,\mathrm{curl} = 0$ is proven completely analogously. So (2.3.16) is indeed a Hilbert complex.

{thm:closed_ra

Theorem 2.3.26. *If Ω is bounded then* $\operatorname{grad} H^1$, $\operatorname{curl} H(\operatorname{curl})$ *and* $\operatorname{div} H(\operatorname{div})$ *are closed subspaces of $L^2(\Omega)$ and $L^2(\Omega;\mathbb{R}^3)$ respectively.*

Proof. See [2, p.38]. □

Having closed range should not be confused with the operators being closed. The mentioned differential operators are closed even on unbounded domains.

The resulting dual domain complex is

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} H_0(\operatorname{div}) \xleftarrow{\operatorname{curl}} H_0(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H_0^1 \leftarrow 0.$$

For the harmonic forms – which are scalar and vector fields here – we obtain

$$\begin{aligned}
\mathfrak{H}^0 &= \{u \in H^1 \mid \operatorname{grad} u = 0\},\\
\mathfrak{H}^1 &= \{\mathbf{u} \in H(\operatorname{curl}) \cap H_0(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\},\\
\mathfrak{H}^2 &= \{\mathbf{u} \in H_0(\operatorname{curl}) \cap H(\operatorname{div}) \mid \operatorname{curl} \mathbf{u} = 0, \operatorname{div} \mathbf{u} = 0\} \text{ and}\\
\mathfrak{H}^3 &= \{u \in H_0^1 \mid \operatorname{grad} u = 0\} = \{0\}.
\end{aligned}$$

For the last equality we used the fact that $\operatorname{grad} u = 0$ implies that u is constant almost everywhere with possibly different constants for different path components of Ω. But because we have homogeneous boundary conditions we get $u = 0$. Note that $\mathfrak{H}^1$ and $\mathfrak{H}^2$ are very similar. The only difference are the boundary conditions. $\mathbf{u} \in \mathfrak{H}^2 \subseteq H_0(\operatorname{curl})$ means that the generalized tangential trace $\gamma_\tau \mathbf{u}$ is zero. If $\mathbf{u} \in \mathfrak{H}^1 \subseteq H_0(\operatorname{div})$ then the generalized normal trace $\gamma_n \mathbf{u}$ vanishes.

This gives us the Hodge decomposition in the 3D case.

{thm:hodge_dec

Theorem 2.3.27 (Hodge decomposition in 3D)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with compact boundary. Then we have the following decompositions of the kernels*

$$\begin{aligned}
\{\mathbf{u} \in H(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{grad} H^1} \overset{\perp}{\oplus} \mathfrak{H}^1\\
\{\mathbf{u} \in H(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2\\
\{\mathbf{u} \in H_0(\operatorname{div}) \mid \operatorname{div} \mathbf{u} = 0\} &= \overline{\operatorname{curl} H_0(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^1\\
\{\mathbf{u} \in H_0(\operatorname{curl}) \mid \operatorname{curl} \mathbf{u} = 0\} &= \overline{\operatorname{div} H_0(\operatorname{div})} \overset{\perp}{\oplus} \mathfrak{H}^2.
\end{aligned}$$

We can express $L^2(\Omega)$ as

$$\begin{aligned}
L^2(\Omega) &= \overline{\operatorname{div} H_0(\operatorname{div})} \overset{\perp}{\oplus} \{v \in H^1 \mid \operatorname{grad} v = 0\}\\
&= \overline{\operatorname{div} H(\operatorname{div})}.
\end{aligned}$$

and for vector valued functions

$$
\begin{aligned}
L^2(\Omega;\mathbb{R}^3) &= \overline{\operatorname{grad} H(\operatorname{grad})} \overset{\perp}{\oplus} \mathfrak{H}^1 \overset{\perp}{\oplus} \overline{\operatorname{curl} H_0(\operatorname{curl})} \\
&= \overline{\operatorname{curl} H(\operatorname{curl})} \overset{\perp}{\oplus} \mathfrak{H}^2 \overset{\perp}{\oplus} \overline{\operatorname{grad} H_0^1}.
\end{aligned}
$$

Proof. This is just an application of the general Hodge decomposition Thm. 2.3.24 combined with what we derived above. □

Remark 2.3.28. Alternatively, we could have chosen the sequence with zero boundary conditions as the primal sequence i.e.

$$0 \to H_0^1 \xrightarrow{\operatorname{grad}} H_0(\operatorname{curl}) \xrightarrow{\operatorname{curl}} H_0(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2 \to 0$$

Then we can follow the exact same arguments to get the dual sequence

$$0 \leftarrow L^2 \xleftarrow{-\operatorname{div}} H(\operatorname{div}) \xleftarrow{\operatorname{curl}} H(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H^1 \leftarrow 0.$$

2.4 Existence and uniqueness of solutions

{sec:existence_

In this section, we will apply the developed theory of the preceding sections to prove the existence and uniqueness of the magnetostatic problem on exterior domains. But at first, we have to properly formulate the problem.

$\Omega \subseteq \mathbb{R}^3$ is an exterior domain which means that our domain $\Omega \subseteq \mathbb{R}^3$ is the complement of a compact set. Furthermore, we assume its boundary to be Lipschitz. Note that $\partial\Omega$ is compact in this setting. The main motivation for this problem is the special case of Ω being the complement of a torus. This is also the motivation behind the topological assumption that we will give. It might be useful to keep this example in mind (see Fig. 1.0.1).

The condition $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$ from (2.0.3) is replaced with $\mathbf{B} \in H_0(\operatorname{div})$ i.e. $\gamma_n \mathbf{B} = 0$ where γ_n is the generalized normal trace. We assume additionally that $\mathbf{B} \in H(\operatorname{curl})$.

Recall as another condition we had equation (2.0.4) which was the curve integral along the piecewise smooth curve around the torus. Recalling Section 2.2, we replace the curve Γ with a smooth 1-chain, which will play the same role, so we will denote it, by slight abuse of notation, as Γ as well. We emphasize it to make it mathematically rigorous and so we can talk about integrals of 1-forms on it later. Note that smooth chain means here that the chosen singular simplices are smooth as explained in Sec. 2.2.3. The curve Γ is only piecewise smooth in general.

So we arrive at the following problem that we want to investigate:

{prob:magnetos

Problem 2.4.1. Find $\mathbf{B} \in H_0(\mathrm{div};\Omega) \cap H(\mathrm{curl})$ s.t.

$$\mathrm{curl}\ \mathbf{B} = 0, \tag{2.4.1}$$

$$\mathrm{div}\ \mathbf{B} = 0 \text{ in } \Omega \text{ and} \tag{2.4.2}$$

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0. \tag{2.4.3}$$

Of course in order for the curve integral constraint to be well-defined we need to check the regularity of solutions. Then using the tools we developed in the previous sections, we will proof existence and uniqueness.

2.4.1 Regularity of solutions

{sec:regularit

We will rely on standard regularity results about elliptic systems. Take $A_{ij}^{\alpha\beta} \in \mathbb{R}$ for i, j, α, $\beta = 1, 2, 3$ and $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Then we look at systems of the form

$$-\sum_{\alpha,\beta,j} \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta B_j) = f_i - \sum_\alpha \partial_\alpha F_i^\alpha \tag{2.4.4}$$ {eq:elliptic_s

with data $f_i, F_i^\alpha \in L^2(\Omega)$. We call this system *elliptic* if A satisfies the Legendre condition, i.e.

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \geq c|\xi|^2, \quad \forall \xi \in \mathbb{R}^{3\times 3} \tag{2.4.5}$$ {eq:legendre_c

with $c > 0$. $|\xi|$ is here the Frobenius norm, but technically the chosen norm is irrelevant due to all norms on $\mathbb{R}^{3\times 3}$ being equivalent.

We call an open $Q \subseteq \Omega$ *precompact* if $\overline{Q} \subseteq \Omega$ is compact where the closure is taken in the topology of $\mathbb{R}^3$ and we write $Q \subset\subset \Omega$. This can be understood as Q being bounded and having positive distance from the boundary. Recall the definition of

$$H_{loc}^k(\Omega) := \{u \in L^2 \mid \forall Q \subset\subset \Omega : u \in H^k(Q)\}.$$

for $k \in \mathbb{N}$. Analogous to $H^k(\Omega;\mathbb{R}^3)$, $H_{loc}^k(\Omega;\mathbb{R}^3)$ means that all components are in $H_{loc}^k(\Omega)$. We will frequently leave out the reference to the domain if it is clear.

We then call $\mathbf{B} = (B_1, B_2, B_3)^\top \in H_{loc}^1(\Omega;\mathbb{R}^3)$ a weak solution of (2.4.4) if

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\, \partial_\beta B_j\, \partial_\alpha \varphi_i\, dx = \int_\Omega \left\{ \sum_i f_i \varphi_i + \sum_{\alpha,i} F_i^\alpha\, \partial_\alpha \varphi_i \right\} dx \tag{2.4.6}$$ {eq:weak_ellip

for all $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)^\top \in C_0^1(\Omega;\mathbb{R}^3)$. This formulation is taken from [1, Sec. 1.3]. At first we will slightly modify the notion of weak solution.

{prop:weak_sol

Proposition 2.4.2. *(2.4.6) is fulfilled for all* $\boldsymbol{\varphi} \in C_0^1(\Omega;\mathbb{R}^3)$ *if and only if it is fulfilled for* $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$.

Proof. This follows by a simple density argument. Assume that (2.4.6) is fulfilled for all test functions in $C_0^\infty(\Omega;\mathbb{R}^3)$. Now take $\boldsymbol{\varphi} \in C_0^1(\Omega;\mathbb{R}^3)$ arbitrary. Because $\boldsymbol{\varphi} \in H_0^1(\Omega;\mathbb{R}^3)$ and $C_0^\infty(\Omega;\mathbb{R}^3)$ is dense in $H_0^1(\Omega;\mathbb{R}^3)$ we can find a sequence $(\boldsymbol{\varphi}^{(l)})_{l\in\mathbb{N}} \subseteq C_0^\infty(\Omega;\mathbb{R}^3)$ s.t. $\boldsymbol{\varphi}^{(l)} \to \boldsymbol{\varphi}$ in $H^1(\Omega;\mathbb{R}^3)$. Thus the partial derivatives converge in $L^2(\Omega)$ and we get

$$\int_\Omega \sum_{i,j,\alpha,\beta} A_{ij}^{\alpha,\beta}\,\partial_\beta B_j\,\partial_\alpha \varphi_i\,dx = \sum_{i,j,\alpha,\beta} A_{ij}^{\alpha,\beta}\int_\Omega \partial_\beta B_j\,\lim_{l\to\infty}\partial_\alpha\varphi_i^{(l)}\,dx$$

$$\overset{L^2\text{ limit}}{=} \lim_{l\to\infty}\int_\Omega \left\{\sum_i f_i\varphi_i^{(l)} + \sum_{\alpha,i} F_i^\alpha\,\partial_\alpha\varphi_i^{(l)}\right\}dx = \int_\Omega \left\{\sum_i f_i\varphi_i + \sum_{\alpha,i} F_i^\alpha\,\partial_\alpha\varphi_i\right\}dx.$$

Since $\boldsymbol{\varphi} \in C_0^1(\Omega;\mathbb{R}^3)$ was arbitrary the first direction of the equivalence is proved. The other direction is trivial. □

So we see that in the case of constant coefficients we can consider just smooth compactly supported functions as test functions. Next, we will state the crucial result about the regularity of elliptic systems which will give us the desired regularity of solutions of our system. This is Theorem 2.13 and Remark 2.16 in [1] in slightly less generality and for 3D.

{thm:regularit

Theorem 2.4.3. *Let* Ω *be an open domain in* $\mathbb{R}^3$. *Let* A *satisfy the Legendre condition (2.4.5). Then for every* $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$ *weak solution in the sense of (2.4.6) with* $\mathbf{f} \in H^k_{loc}(\Omega;\mathbb{R}^3)$ *and* $F \in H^{k+1}_{loc}(\Omega;\mathbb{R}^{3\times 3})$ *we have* $\mathbf{B} \in H^{k+2}_{loc}(\Omega;\mathbb{R}^3)$.

{cor:smooth_so

Corollary 2.4.4. *If under the assumptions of the previous theorem we consider the homogeneous problem, i.e.*

$$\int_\Omega \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\partial_\beta B_j\,\partial_\alpha\varphi_i\,dx = 0$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega;\mathbb{R}^3)$*, then* $\mathbf{B} \in C^\infty(\Omega;\mathbb{R}^3)$.

Proof. Here we have $F = 0$ and $\mathbf{f} = 0$. Thus, $\mathbf{f} \in H^k_{loc}(\Omega;\mathbb{R}^3)$ for any $k \in \mathbb{N}$. Take any $Q \subseteq \Omega$ pre-compact. Then we know from Thm. 2.4.3. that $\mathbf{B} \in H^{k+2}_{loc}(\Omega;\mathbb{R}^3)$ and thus $\mathbf{B} \in H^{k+2}(Q;\mathbb{R}^3)$. Therefore, we can apply the standard Sobolev embedding theorem locally to get $\mathbf{B} \in C^l(Q;\mathbb{R}^3)$ for any $l \in \mathbb{N}$ and hence $\mathbf{B} \in C^\infty(\Omega;\mathbb{R}^3)$ since Q pre-compact was arbitrary. □

It should be noted that this does not guarantee us any regularity on the boundary.

Before we can apply this result, we have to check whether a solution of our problem $\mathbf{B}$ is actually in $H^1_{loc}(\Omega;\mathbb{R}^3)$.

{thm:solution_

Theorem 2.4.5. *Assume* $\mathbf{B} \in H(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega)$. *Then* $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$.

Note that we did not assume $\mathbf{B}$ to be a solution.

Proof. We know that for a function $\mathbf{u} \in H_0(\mathrm{curl};U) \cap H(\mathrm{div};U)$ for some smooth domain U we have $\mathbf{u} \in H^1(U;\mathbb{R}^3)$ (cf. [19, Remark 3.48]). Our domain Ω is just assumed to be Lipschitz so we can not apply this result directly.

Take $Q \subset\subset \Omega$ open and pre-compact. Then we can find an open cover of $\overline{Q}$ with a finite set of open balls $\{K_i\}_{i=1}^N$ s.t. $K_i \subseteq \Omega$ and

$$\overline{Q} \subseteq \bigcup_{i=1}^N K_i.$$

As an open cover of a compact set, we can find a smooth partition of unity $\{\chi_i\}_{i=1}^N$ subordinate to $\{K_i\}_{i=1}^N$. $(\mathbf{B}\chi_i)|_{K_i} \in H_0(\mathrm{curl};K_i) \cap H(\mathrm{div};K_i)$ and thus $(\mathbf{B}\chi_i)|_{K_i} \in H^1(K_i;\mathbb{R}^3)$ by the above mentioned result. Also because $\mathbf{B}\chi_i$ has compact support in K_i we can extend it by zero to obtain $\mathbf{B}\chi_i \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ where we abused the notation by denoting the extension the same. Whence,

$$\mathbf{B}|_Q = \Big(\sum_{i=1}^N \chi_i|_Q\Big)\mathbf{B}|_Q = \sum_{i=1}^N (\chi_i\mathbf{B})|_Q \in H^1(Q;\mathbb{R}^3)$$

i.e. $\mathbf{B} \in H^1_{loc}(\Omega;\mathbb{R}^3)$. □

The following lemma is a reformulation of the differential operator $\mathrm{grad}\,\mathrm{div} - \mathrm{curl}\,\mathrm{curl}$ which will be needed when we write our magnetostatic problem in the above standard elliptic form.

{lem:graddiv_c

Lemma 2.4.6. *Let* $\mathbf{F} \in H^2_{loc}(\Omega;\mathbb{R}^3)$. *Then*

$$\mathrm{grad}\,\mathrm{div}\,\mathbf{F} - \mathrm{curl}\,\mathrm{curl}\,\mathbf{F} = \begin{pmatrix}\Delta F_1\\ \Delta F_2\\ \Delta F_3\end{pmatrix}.$$

Proof. By a simple calculation and changing the order of differentiation

$$\mathrm{grad}\,\mathrm{div}\,\mathbf{F} = \begin{pmatrix}\partial_1^2 F_1 + \partial_1\partial_2 F_2 + \partial_1\partial_3 F_3\\ \partial_1\partial_2 F_1 + \partial_2^2 F_2 + \partial_2\partial_3 F_3\\ \partial_1\partial_3 F_1 + \partial_2\partial_3 F_2 + \partial_3^2 F_3\end{pmatrix}$$

and

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{curl}\begin{pmatrix}\partial_2 F_3 - \partial_3 F_2\\ \partial_3 F_1 - \partial_1 F_3\\ \partial_1 F_2 - \partial_2 F_1\end{pmatrix} = \begin{pmatrix}\partial_2(\partial_1 F_2 - \partial_2 F_1) - \partial_3(\partial_3 F_1 - \partial_1 F_3)\\ \partial_3(\partial_2 F_3 - \partial_3 F_2) - \partial_1(\partial_1 F_2 - \partial_2 F_1)\\ \partial_1(\partial_3 F_1 - \partial_1 F_3) - \partial_2(\partial_2 F_3 - \partial_3 F_2)\end{pmatrix}$$

$$= \begin{pmatrix}\partial_1\partial_2 F_2 - \partial_2^2 F_1 - \partial_3^2 F_1 + \partial_1\partial_3 F_3\\ \partial_2\partial_3 F_3 - \partial_3^2 F_2 - \partial_1^2 F_2 + \partial_1\partial_2 F_1\\ \partial_1\partial_3 F_3 - \partial_1^2 F_3 - \partial_2^2 F_3 + \partial_2\partial_3 F_2\end{pmatrix}$$

and so by subtracting the two expressions

$$\operatorname{grad}\operatorname{div}\mathbf{F} - \operatorname{curl}\operatorname{curl}\mathbf{F} = \begin{pmatrix}\partial_1^2 F_1 + \partial_2^2 F_1 + \partial_3^2 F_1\\ \partial_1^2 F_2 + \partial_2^2 F_2 + \partial_3^2 F_3\\ \partial_1^2 F_3 + \partial_2^2 F_3 + \partial_3^2 F_3\end{pmatrix} = \begin{pmatrix}\Delta F_1\\ \Delta F_2\\ \Delta F_3\end{pmatrix}.$$

$\square$

We want to rewrite this system in the expression of the elliptic system (2.4.4). In order to do so, we rewrite the Laplacian

$$-\Delta F_i = -\sum_{\alpha=1}^{3}\partial_\alpha\partial_\alpha F_i = -\sum_{\alpha,\beta=1}^{3}\partial_\alpha\delta_{\alpha,\beta}\partial_\beta F_i = -\sum_{\alpha,\beta,j=1}^{3}\partial_\alpha\delta_{\alpha,\beta}\delta_{ij}\partial_\beta F_j$$

with δ_{ij} being the Kronecker delta, so we get $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. We have to check that the resulting differential operator is indeed elliptic, but this is trivial because for any $(\xi_\alpha^i)_{1\le i,\alpha\le 3}$ we get

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}\xi_\alpha^i\,\xi_\beta^j = \sum_{\alpha,\beta,i,j}\delta_{ij}\delta_{\alpha\beta}\,\xi_\alpha^i\,\xi_\beta^j = \sum_{\alpha,i}(\xi_\alpha^i)^2 = |\xi|^2$$

so the Legendre condition (2.4.5) is fulfilled and the resulting system is elliptic. The left hand side of the weak formulation is

$$\int_\Omega \sum_{\alpha,\beta,i,j}\delta_{ij}\delta_{\alpha\beta}\,\partial_\beta B_j\,\partial_\alpha\varphi_i\,dx = \sum_{i=1}^{3}\int_\Omega \operatorname{grad} B_i\cdot\operatorname{grad}\varphi_i\,dx.$$

Here we can assume $\boldsymbol{\varphi}\in C_0^\infty(\Omega;\mathbb{R}^3)$ due to Prop. 2.4.2.

{thm:smoothnes

Theorem 2.4.7 (Smoothness of solutions)**.** *Let* $\Omega\subseteq\mathbb{R}^3$ *open and* $\mathbf{B}\in H(\operatorname{div};\Omega)\cap H(\operatorname{curl};\Omega)$ *and*

$$\operatorname{curl}\mathbf{B} = 0,$$
$$\operatorname{div}\mathbf{B} = 0.$$

Then $\mathbf{B}$ *is smooth i.e. in* $C^\infty(\Omega;\mathbb{R}^3)$.

Proof. Take $\boldsymbol{\varphi} \in C_0^\infty(\Omega; \mathbb{R}^3)$. Then

$$0 = \int_\Omega \operatorname{div} \mathbf{B} \operatorname{div} \boldsymbol{\varphi} + \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \boldsymbol{\varphi} \, dx = -\int_\Omega \mathbf{B} \cdot (\operatorname{grad} \operatorname{div} \boldsymbol{\varphi} - \operatorname{curl} \operatorname{curl} \boldsymbol{\varphi}) \, dx$$

$$\overset{\text{Lemma 2.4.6}}{=} -\int_\Omega \mathbf{B} \cdot \begin{pmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \Delta\varphi_3 \end{pmatrix} = \sum_{i=1}^{3} \int_\Omega \operatorname{grad} B_i \cdot \operatorname{grad} \varphi_i \, dx.$$

Note that the last integration by parts is well defined because $\mathbf{B} \in H^1_{loc}(\Omega; \mathbb{R}^3)$ according to Thm. 2.4.5. So $\mathbf{B}$ is a weak solution of the elliptic system given by $A_{ij}^{\alpha\beta} = \delta_{ij}\delta_{\alpha\beta}$. Because we look at the homogenous problem our right hand side is obviously smooth and thus $\mathbf{B}$ is smooth as well due to Cor. 2.4.4. $\square$

2.4.2 Existence and uniqueness

The curve integral condition is closely linked to the topology of our domain which we will have to use in our proof. This will rely on the tools of homology from Sec. 2.2. Because of this connection if we want the curve integral to give us uniqueness of the solution we need to assume certain topological properties. In our case, this will be the condition that our first homology group is generated by the chain that we are integrating over i.e.

$$H_1(\Omega) = \mathbb{Z}[\Gamma]. \tag{2.4.7}$$ {eq:gamma_gene

With this assumption, we will first prove an existence and uniqueness result on the level of singular cohomology. This will lead to an analogous result for the de Rham cohomology. Once we have proven these propositions, we can move on to prove existence and uniqueness of Problem 2.4.1.

Remark 2.4.8. In the case where Ω is the exterior of a torus and Γ is the curve that goes around it (cf. Fig. 1.0.1), (2.4.7) is satisfied. This can be proven using tools from algebraic topology like the Meyer-Vietoris sequence, but it is intuitively clear that the curve can not be the boundary of a surface or a 2-chain and so it is closed, but not exact. Also – if we argue heuristically – any closed 1-chain outside the torus is contractible and hence not exact and any other curve around the torus can be ”moved and continously deformed” to become equal to Γ. ”Moving and deforming” can be translated to using homotopies and then this argument can be formulated rigorously. For more details, please consult [4, Sec. IV].

{prop:uniquene

Proposition 2.4.9. *Assume that $H_1(\Omega) = \mathbb{Z}[\Gamma]$ i.e. the homology class of the closed 1-chain Γ is a generator of the first homology group. Then we have the* *following:*

(i) *For any $C_0 \in \mathbb{R}$ there exists a closed 1-cochain $F \in Z^1(\Omega)$ with $F(\Gamma) = C_0$,*

(ii) *any other $G \in Z^1(\Omega)$ with $G(\Gamma) = C_0$ is in the same cohomology class i.e. $[F] = [G]$*

i.e. the cochain is unique up to cohomology.

Proof. **Proof of (i)** Because $[\Gamma]$ is a generator of the homology group we obtain a homomorphism $\hat{F} \in \mathrm{Hom}(H_1(\Omega), \mathbb{R})$ by fixing $\hat{F}([\Gamma]) = C_0$. This determines the other values. Recall the isomorphism $\beta : H^1(\Omega) \to \mathrm{Hom}(H_1(\Omega); \mathbb{R})$ from the universal coefficient theorem (2.2.2) with $\beta([f])([c]) = f(c)$ (Note that $H^1(\Omega)$ is an abuse of notation because we mean here the first cohomology group of Ω and not the Sobolev space. In the context of singular homology, this will always be the case.) Then we know that there exists a $[F] \in H^1(\Omega)$ with $\beta([F]) = \hat{F}$ because β is an isomorphism. So we obtain

$$F(\Gamma) = \beta([F])([\Gamma]) = \hat{F}([\Gamma]) = C_0.$$

Proof of (ii) Take $[c] \in H_1(\Omega)$ arbitrary. Then there exists $n \in \mathbb{Z}$ s.t. $[c] = n[\Gamma]$. Using β from (2.2.2), we have

$$\beta([F])([c]) = \beta([F])(n[\Gamma]) = n\beta([F])([\Gamma]) = n\,F(\Gamma) = n\,G(\Gamma) = \beta([G])([c])$$

and thus $\beta([F]) = \beta([G])$. Because β is an isomorphism we arrive at $[F] = [G]$. □

This abstract topological result can now be linked to the differential forms via the de Rham isomorphism from Thm. 2.2.11. We will formulate it in a way that demonstrates the connection of differential forms and cochains.

{cor:existence_

Corollary 2.4.10. *Assume $H_1(\Omega) = \mathbb{Z}[\Gamma]$ as above. Then*

(i) *For any $C_0 \in \mathbb{R}$ there exists a closed smooth 1-form $\theta \in \mathfrak{Z}^1(\Omega)$ with*

$$I(\theta)(\Gamma) = \int_\Gamma \theta = C_0$$

(ii) *any other $\eta \in \mathfrak{Z}^1(\Omega)$ with*

$$I(\eta)(\Gamma) = \int_\Gamma \eta = C_0$$

is in the same cohomology class of $H^1_{dR}(\Omega)$ i.e. $[\eta] = [\theta]$.

Proof. **Proof of (i)** Recall from Sec. 2.2.3 that the integration of differential forms over chains induces an isomorphism on cohomology $[I] : H^1_{dR}(\Omega) \to H^1(\Omega)$ which we call de Rham isomorphism. We know from Prop. 2.4.9 that there exists $F \in H^1(\Omega)$ s.t. $F(\Gamma) = C_0$. The surjectivity of the de Rham isomorphism now gives us $[\theta] \in H^1(\Omega)$ s.t.

$$[I(\theta)] = [I]([\theta]) = [F]$$

i.e.

$$I(\theta) = F + \partial^0 J$$

with $J \in C^0$ (here of course the zero cochains not continuous functions). Then,

$$I(\theta)(\Gamma) = F(\Gamma) + \partial^0 J(\Gamma) = C_0 + J(\partial_1 \Gamma) \overset{\Gamma \text{ closed}}{=} C_0.$$

Proof of (ii) We have $I(\eta)$ is a 1-cochain with $I(\eta)(\Gamma) = C_0$. Thus, we can apply Prop. 2.4.9 to get

$$[I]([\eta]) = [I(\eta)] = [I(\theta)] = [I]([\theta]).$$

Because $[I]$ is an isomorphism we can conclude $[\eta] = [\theta]$. □

{lem:1form_wit

Lemma 2.4.11. *Let Ω be an exterior domain i.e. $\mathbb{R}^3 \setminus \Omega$ is compact and assume that the first singular homology group $H_1(\Omega)$ is generated by $[\Gamma]$ i.e. $H_1(\Omega) = \mathbb{Z}[\Gamma]$. Take now a smooth 1-form $\theta \in C^\infty\Lambda^1(\Omega)$. Then there exists a $\hat{\theta} \in C^\infty_b\Lambda^1(\Omega)$ i.e. smooth with bounded support s.t. $\theta = \hat{\theta} + d\mu$ for some $\mu \in C^\infty\Lambda^0(\Omega)$.*

Proof. Take a ball K_R centered at the origin with radius R large enough s.t. $\mathbb{R}^3 \setminus \Omega \subseteq K_R$. Denote $A := \mathbb{R}^3 \setminus \overline{K}_R$ and define

$$C^\infty\Lambda^k(\Omega, A) := \{\omega \in C^\infty\Lambda^k(\Omega) \mid \omega(x) = 0 \quad \forall x \in A\}.$$

for $k \in \mathbb{N}$. Note that these spaces define a cochain complex with the exterior derivative d. We denote the resulting cohomology groups as $H^1_{dR}(\Omega, A)$

Now recall the definition of an exact sequence from Def. 2.2.9. Denote $\iota : C^\infty\Lambda^k(\Omega, A) \hookrightarrow C^\infty\Lambda^k(\Omega)$ the inclusion operator and $\mathscr{R} : C^\infty\Lambda^k(\Omega) \to C^\infty\Lambda^k(A)$ the restriction operator. Then it is obvious to see that ι and $\mathscr{R}$ are cochain maps i.e. they commute with the exterior derivative and we also recognize that the sequence

$$0 \to C^\infty\Lambda^k(\Omega, A) \xrightarrow{\iota} C^\infty\Lambda^k(\Omega) \xrightarrow{\mathscr{R}} C^\infty\Lambda^k(A) \to 0$$

is an exact sequence. In this situation, we can use Thm. IV.5.6 from [4] to get a long exact sequence on the level of cohomology of which we take the following partial exact sequence

$$H^1_{dR}(\Omega, A) \xrightarrow{[\iota]} H^1_{dR}(\Omega) \xrightarrow{[\mathscr{R}]} H^1_{dR}(A).$$

We now need information about the homology group of A. We will only sketch the argument since it requires some notions that we did not introduce. The sphere around the origin $\mathbb{S}_r^2$ with sufficiently large radius r is a so called deformation retract of A. This can be seen by following the argument of Example I.14.7 in [4]. This implies that A is homotopy equivalent to $\mathbb{S}_r^2$ and thus the homology groups are isomorphic according to Prop. IV.6.3 in the same reference. The first homology group of the unit sphere $\mathbb{S}^2$ is zero and thus it is easy to check that $H_1(\mathbb{S}_r^2) = 0$ as well. This implies due to the universal coefficient theorem that $H^1(A) = 0$ and then $H^1_{dR}(A) = 0$ with the de Rham isomorphism.

The exactness now implies that $[\iota]$ is surjective. Thus, there exists a closed $\hat{\theta} \in C^\infty\Lambda^1(\Omega, A)$ s.t. $[\theta] = [\iota]([\hat{\theta}]) = [\hat{\theta}]$ which is equivalent to the existence of a $\mu \in C^\infty\Lambda^0(\Omega)$ s.t.

$$\theta = \hat{\theta} + d\mu.$$

Since $\hat{\theta}$ has support in $\overline{K}_R$ the result follows. $\square$

{thm:existence}

Theorem 2.4.12 (Existence of solution)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be such that $\mathbb{R}^3 \setminus \Omega$ is compact. For the topology, we require that $H_1(\Omega) = \mathbb{Z}[\Gamma]$ for a closed smooth 1-chain Γ. Assume further that there exists an ϵ-neighborhood*

$$\Omega_\epsilon := \{x \in \mathbb{R}^3 \mid d(x, \Omega) < \epsilon\}$$

s.t. $H_1(\Omega_\epsilon) = \mathbb{Z}[\Gamma]$ as well. Then there exists a solution to Problem 2.4.1.

Let us say a view words about the topological assumption regarding Ω_ϵ. This just means that we can slightly increase the domain without changing the first homology group. As an example, think again of a torus in $\mathbb{R}^3$. Assuming the torus has non-empty interior we can slightly reduce the poloidal radius without changing the topology of its exterior domain.

Proof. At first, we want to find a smooth differential 1-form $\theta \in C_b^\infty\Lambda^1(\overline{\Omega})$ with the desired curve integral. In order to do that we will increase the domain slightly. We start by referring to Cor. 2.4.10 to get a smooth closed differential 1-form $\tilde{\theta} \in \Lambda^1(\Omega_\epsilon)$ with

$$\int_\Gamma \tilde{\theta} = C_0. \tag{2.4.8}$$

{eq:integral_t

Now we use Lemma 2.4.11 on Ω_ϵ to get a $\hat{\theta}$ which vanishes outside of a sufficiently large ball with a $\mu \in C^\infty\Lambda^0(\Omega_\epsilon)$ s.t. $\hat{\theta} + d\mu = \tilde{\theta}$. Notice that

$$\int_\Gamma \tilde{\theta} = \int_\Gamma \hat{\theta} = C_0. \tag{2.4.9}$$

{eq:curve_inte

since Γ is closed.

We now refer to Sec. 2.1.3 and change back to vector proxies. Let $\tilde{\boldsymbol{\phi}}$ be the vector proxy of $\hat{\theta}$, i.e. by recalling the musical isomorphism from Sec. 2.1.3, $\hat{\theta}^\sharp = \tilde{\boldsymbol{\phi}} \in C^\infty(\Omega_\epsilon; \mathbb{R}^3)$. Because $\hat{\theta}$ is closed and (2.4.9) holds we obtain the corresponding properties of $\tilde{\boldsymbol{\phi}}$ using Example 2.1.34 and the connection of the curl with the exterior derivative,

$$\int_\Gamma \tilde{\boldsymbol{\phi}} \cdot d\ell = \int_\Gamma \hat{\theta} = C_0$$
$$\operatorname{curl} \tilde{\boldsymbol{\phi}} = (\star d\tilde{\boldsymbol{\phi}}^\flat)^\sharp = (\star d\hat{\theta})^\sharp = 0.$$

We define $\boldsymbol{\phi} \in C_b^\infty(\overline{\Omega}; \mathbb{R}^3)$ by restricting $\tilde{\boldsymbol{\phi}}$ to $\overline{\Omega}$.

Since $\boldsymbol{\phi}$ is curl-free and in $L^2(\Omega)$ we can use the Hodge decomposition (Thm. 2.3.27) to find $\mathbf{B} \in \mathfrak{H}^1$ s.t. $\mathbf{B} = \boldsymbol{\phi} - \lim_{i\to\infty} \operatorname{grad} \psi_i$ with $\psi_i \in H^1(\Omega)$ (here $H^1(\Omega)$ is the Sobolev space and not the cochain cohomology group).

Define $\Omega_R := \Omega \cap K_R$ with K_R the ball around the origin with radius R large enough s.t. $\mathbb{R}^3 \setminus \Omega \subseteq K_R$ and $\Gamma \subseteq K_R$. We know from Thm. 2.3.26 that $\operatorname{grad} H^1(\Omega_R)$ is closed in L^2 because Ω_R is bounded. So we get that

$$\lim_{i\to\infty} \operatorname{grad} \psi_i|_{\Omega_R} = \operatorname{grad} \psi_R$$

with some $\psi_R \in H^1(\Omega_R)$. We also know that $\mathbf{B}$ is smooth because it is curl and divergence free from Thm. 2.4.7. Because $\mathbf{B}$ and $\boldsymbol{\phi}$ are smooth ψ_R must be smooth as well and so we have

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_\Gamma \boldsymbol{\phi} \cdot d\ell.$$

Thus we see that $\mathbf{B}$ has the desired curve integral. Since $\mathbf{B} \in \mathfrak{H}^1 \subseteq H_0(\operatorname{div})$ it has zero normal trace and is curl as well as divergence free. Hence, $\mathbf{B}$ is a solution of Problem 2.4.1. $\square$

In the proof of uniqueness we will use the following lemma.

{lem:gradient_

Lemma 2.4.13. *Let Ω be a Lipschitz domain and $\phi \in L^2_{loc}(\Omega)$ with $\operatorname{grad} \phi \in L^2(\Omega; \mathbb{R}^3)$. Then there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^1(\Omega)$ s.t. $\operatorname{grad} \phi_i \to \operatorname{grad} \phi$ in $L^2(\Omega; \mathbb{R}^3)$.*

Proof. Take K_R the open ball around the origin with R large enough s.t. $\mathbb{R}^3 \setminus K_R \subseteq \Omega$. Define $\Omega_R := K_R \cap \Omega$. Ω_R is a bounded Lipschitz domain. Then we can define $\phi_R := \phi|_{\Omega_R}$.

We want to show $\phi_R \in H^1(\Omega_R)$. We already know that $\operatorname{grad} \phi_R \in L^2(\Omega; \mathbb{R}^3)$. Since $L^2(\Omega_R; \mathbb{R}^3) \subseteq H^{-1}(\Omega_R; \mathbb{R}^3)$ we can conclude that $\phi_R \in L^2$ because Ω_R is bounded and Lipschitz ([19, Lemma 3.11]) and thus $\phi_R \in H^1(\Omega_R)$. This essentially means that if $\phi \notin L^2(\Omega)$ it is due to its behaviour at infinity.

Since Ω_R is bounded and Lipschitz we can find an extension $\bar{\phi}_R \in H^1(\mathbb{R}^3)$ s.t. $\bar{\phi}_R|_{\Omega_R} = \phi_R$ (cf. [17, Sec. 1.5.1]). So we can now define

$$\bar{\phi} := \begin{cases} \phi & \text{in } \Omega \\ \bar{\phi}_R & \text{in } \Omega^c. \end{cases}$$

Then $\bar{\phi} \in L^2_{loc}(\mathbb{R}^3)$ and $\operatorname{grad} \bar{\phi} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. Then there exists a sequence $(\phi_l)_{l \in \mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^3)$ s.t. $\operatorname{grad} \phi_l \to \operatorname{grad} \bar{\phi}$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ (cf. [22, Lemma 1.1]). By restricting ϕ_l to Ω we obtain the result. $\square$

Theorem 2.4.14. *Let the same assumptions hold as in Thm. 2.4.12. Then the solution of the problem is unique.*

Proof. Let $\mathbf{B}$ and $\tilde{\mathbf{B}}$ both be solutions and denote with ω and $\tilde{\omega}$ the corresponding 1-forms i.e. $\omega = \mathbf{B}^\flat$ and $\tilde{\omega} = \tilde{\mathbf{B}}^\flat$. So we have $I(\omega)(\Gamma) = I(\tilde{\omega})(\Gamma) = C_0$ since

$$\int_\Gamma \omega = \int_\Gamma \mathbf{B}^\flat = \int_\Gamma \mathbf{B} \cdot d\ell = \int_\Gamma \tilde{\mathbf{B}} \cdot d\ell = \int_\Gamma \tilde{\mathbf{B}}^\flat = \int_\Gamma \tilde{\omega}$$

Then we know from Cor. 2.4.10 that ω and $\tilde{\omega}$ are in the same cohomology class in H^1_{dR}. So there exists a 0-form i.e. $\mu \in C^\infty(\Omega)$ s.t. $\omega - \tilde{\omega} = d\mu$. By applying $\sharp$ on both sides

$$\mathbf{B} - \tilde{\mathbf{B}} = \omega^\sharp - \tilde{\omega}^\sharp = (d\mu)^\sharp = \operatorname{grad} \mu.$$

However, μ need not be in L^2 since Ω is unbounded. But we know that $\operatorname{grad} \mu \in L^2(\Omega; \mathbb{R}^3)$ and $\mu \in L^2_{loc}(\Omega)$. Here we can now apply Lemma 2.4.13 and conclude

$$\mathbf{B} - \tilde{\mathbf{B}} \in \overline{\operatorname{grad} H^1(\Omega)}.$$

Remembering the Hodge decomposition in the 3D case (Thm. 2.3.27), we know

$$\mathbf{B} - \tilde{\mathbf{B}} \in \overline{\operatorname{grad} H^1(\Omega)}^\perp$$

because $\mathbf{B}, \tilde{\mathbf{B}} \in \mathfrak{H}^1$. Thus, $\mathbf{B} = \tilde{\mathbf{B}}$ which concludes the proof of uniqueness. $\square$

Chapter 3

Numerical approximation in 2D

{chap:approxim

This chapter, which will be the second main part of this thesis, is devoted to the numerical approximation of the magnetostatic problem in 2D which can be derived from a special case of the standard magnetostatic problem. Instead of the exterior of a toroidal domain as in the first part, we will pose the problem on an "annulus like" domain which will be defined more exactly. A curve integral will again be given as an additional constraint and we will investigate the idea to incorporate it using an integration-by-parts approach which is easily applicable to the finite element approximation that we will use.

We start in Sec. **??** by deriving a variational formulation of the magnetostatic problem, including the alternative description of the curve integral constraint, and prove well-posedness of the resulting formulation. In Sec. **??** the discretization of the problem will be described including a prove ofwell-posedness and of an a-priori estimate. The implementation will be explained in Sec. **??** and numerical examples given in Sec.**??** which confirm our theoretical predictions.

Prerequisites for this chapter are familiarity with finite element theory and basic knowledge of functional analysis and Sobolev spaces.

3.1 Variational formulation of the magnetostatic problem in 2D

For simplicity, we will now turn to the 2D case and we assume that our open domain will be bounded and Lipschitz. This involves introducing a different Hilbert complex with other differential operators. Then we derive the 2D magnetostatic problem from the three-dimensional one. We will assume our domain Ω to have an "annulus like" form which we will clarify in more rigour. In order for the 2D magnetostatic problem to be well-posed, we require an additional constraint that will again be a curve integral. We will investigate an alternative way to represent this curve integral which will turn out to be easily suitable to be included in our numerical approximation. Prerequisite for this section are knowledge about basic functional analysis and fundamentals of finite element theory. We will also depend on the notions introduced in Sec. 2.3.3, in particular the Hodge decomposition in the general case.

3.1.1 The $\mathbf{curl}$-div Hilbert complex

{sec:variation

We start with the introduction of the relevant differential operators and the resulting 2D Hilbert complex. We will then explain what domains we will consider and state the 2D magnetostatic problem in strong form.

We define the scalar curl for $\mathbf{v} \in C^1(\Omega; \mathbb{R}^2)$ as

$$\operatorname{curl} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

Additionally, we have the vector-valued curl, denoted in bold, defined for $v \in C^1(\Omega)$

$$\mathbf{curl}\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

In contrast to chapter 2, we will from now on denote vector valued operators in bold, i.e. $\mathbf{curl}$ and $\mathbf{grad}$. The cross product for 2D reads for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$,

$$\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1 = \mathbf{a} \cdot \mathbf{R}_{-\pi/2} \mathbf{b}.$$

where $\mathbf{R}_{-\pi/2}$ is the rotation in clockwise direction by $\pi/2$.

A straightforward calculation shows that the following integration-by-parts formula holds for $u \in C^1(\overline{\Omega})$, $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^2)$, assuming Ω is Lipschitz and bounded

$$\int_\Omega \mathbf{curl}\, u \cdot \mathbf{v}\, dx = \int_\Omega u \operatorname{curl} \mathbf{v}\, dx + \int_{\partial\Omega} u\, \mathbf{v} \times \mathbf{n}\, d\ell \tag{3.1.1}$$ {eq:2D_integra

where $\mathbf{n}$ is the outward unit normal of Ω. Analogous to what we did in Sec. 2.3.2, we can now extend this definition in the weak sense. First, notice that $\mathbf{curl}\, u = R_{-\pi/2}\, \mathbf{grad}\, u$ and thus $\mathbf{curl}$ is well-defined on H^1. We define

$$H(\operatorname{curl}; \Omega) = \left\{ \mathbf{v} \in L^2 \mid \exists w \in L^2 : \int_\Omega w\phi\, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \phi\, dx \quad \forall \phi \in C_0^\infty \right\}$$

and w in the definition – which is uniquely determined – coincides with $\operatorname{curl} \mathbf{v}$ in distributional sense.

Using the notation of unbounded operators introduced in Sec. 2.3.1, this is equivalent to $(\mathrm{curl}, H(\mathrm{curl})) = (\mathbf{curl}, C_0^\infty)^*$. Analogous to Section 2.3.2, it is then possible to extend the operator

$$\mathbf{v} \mapsto \langle \mathbf{v} \times \mathbf{n}, \cdot \rangle_{L^2(\partial\Omega)} \in H^{-1/2}(\partial\Omega) \tag{3.1.2}$$

defined on $C^1(\Omega, \mathbb{R}^2)$ to an operator γ_τ defined on $H(\mathrm{curl})$ s.t. for any $u \in H^1(\Omega)$, $\mathbf{v} \in H(\mathrm{curl})$ the integration by parts formula

$$\langle \mathbf{curl}\, u, \mathbf{v} \rangle = \langle u, \mathrm{curl}\, \mathbf{v} \rangle + \langle \gamma_\tau \mathbf{v}, \mathrm{tr}\, u \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}.$$

holds. From now on, we will leave out the subindex of the duality inner product. Also analogous to the 3D case, we can define

$$H_0(\mathrm{curl}) := \{\mathbf{v} \in H(\mathrm{curl}) \mid \gamma_\tau \mathbf{v} = 0\}$$

and can then compute the adjoints analogously to what we did in Section 2.3.2,

$$\begin{aligned} (\mathrm{curl}, H_0(\mathrm{curl})) &= (\mathbf{curl}, H^1)^* \\ (\mathrm{curl}, H(\mathrm{curl})) &= (\mathbf{curl}, H_0^1)^*. \end{aligned}$$

Notice that $\mathrm{div}\, \mathbf{curl} = 0$ and so we have the following 2D Hilbert complex

$$0 \to H_0^1 \xrightarrow{\mathbf{curl}} H_0(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0. \tag{3.1.3}$$ {eq:2D_hilbert_

and the dual complex

$$0 \leftarrow L^2 \xleftarrow{\mathrm{curl}} H(\mathrm{curl}) \xleftarrow{-\,\mathbf{grad}} H^1 \leftarrow 0$$

We use the notation introduced in Sec. 2.3 for general Hilbert complexes i.e. $V^0 = H_0^1$, $V^1 = H_0(\mathrm{div})$, $V^2 = L^2$, $V_0^* = L^2$, $V_1^* = H(\mathrm{curl})$ $V_2^* = H^1$, $d^0 = \mathbf{curl}$, $d^1 = \mathrm{div}$ and we set $d^k = 0$ for the remaining $k \in \mathbb{Z}$. d_k^* is the adjoint of d^k. Also we remind of the notation $\mathfrak{B}^k$ for the image of the differential operator, $\mathfrak{B}_k^*$ for the image of the adjoint and analogous $\mathfrak{Z}^k$ for the kernel and $\mathfrak{Z}_k^*$ for the kernel of the adjoint.

Remark 3.1.1. Since we are working only on bounded domains in this and the coming sections, the Hilbert complex is closed, i.e. all the images of the differential operators are closed subspaces w.r.t. the V-norm. This was stated in Thm. 2.3.26 for the three-dimensional case, but it holds for 2D as well. The reason is that these are both special cases for the analogous result for the exterior derivative on Riemannian manifolds in arbitrary dimensions (see [2, Sec. 6.2.6]). This means in particular that we can use the Poincaré inequality (Thm. 2.3.25).

3.1.2 Strong formulation of the 2D magnetostatic problem

The 2D magnetostatic problem will be derived from a special case of the 3D problem. Then the type of domains considered will be clarified and the strong formulation stated at the end.

Assume that our current source $\mathbf{J}$ is pointing in z-direction i.e. $\mathbf{J} = J\mathbf{e}_3$. Further assume that there is a $\tilde{\Omega}$ s.t. $\Omega = \tilde{\Omega} \times \mathbb{R}$. If B_3 does not change in z-direction we get that

$$0 = \operatorname{div} \mathbf{B} = \partial_x B_1 + \partial_y B_2 = \operatorname{div} \tilde{\mathbf{B}}.$$

where $\tilde{\mathbf{B}} = (B_1, B_2)^\top$. The third component of the equation $\operatorname{curl} \mathbf{B} = \mathbf{J}$ from the magnetostatic problem in three dimensions reads

$$J = \partial_x B_2 - \partial_y B_1 = \operatorname{curl} \tilde{\mathbf{B}}$$

The unit outer normal of Ω is zero in z-direction and thus $\tilde{\mathbf{B}}$ satisfies the boundary condition

$$0 = \mathbf{B} \cdot \mathbf{n} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{n}}$$

with $\tilde{\mathbf{n}} = (n_1, n_2)^\top$ being the outer unit normal $\tilde{\Omega}$.

Now we will abuse notation and refer to $\tilde{\mathbf{B}}$ as $\mathbf{B}$, $\tilde{\mathbf{n}}$ as $\mathbf{n}$ and $\tilde{\Omega}$ as Ω. Let $J \in L^2$ be given. Then we see that $\mathbf{B}$ must fulfill the following equations

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= J, \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

Depending on the domain, this problem is in general not well-posed – just as the problem in 3D – and requires an additional constraint. Let us now make certain restrictions on what type of domain we will consider.

From now on, we assume that the space of harmonic forms $\mathfrak{H}^1$ has dimension one and that our domain is encompassed by two disjoint closed curves, i.e. we have curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$ s.t.

$$\partial\Omega_{in} \,\dot{\cup}\, \partial\Omega_{out} = \partial\Omega.$$

Let now Γ be a closed curve in Ω that goes around the hole in the middle, i.e. the area surrounded by Γ contains $\partial\Omega_{in}$. Denote its parametrization with $\boldsymbol{\gamma}$: $[0, |\Gamma|] \to \Omega$ s.t. $|\boldsymbol{\gamma}'(t)| = 1$ and assume that $\boldsymbol{\gamma}$ is bijective i.e. the curve does not intersect itself. We assume that Γ has positive distance from $\partial\Omega_{in}$. We do not assume anything like that for the exterior boundary i.e. Γ can touch or be

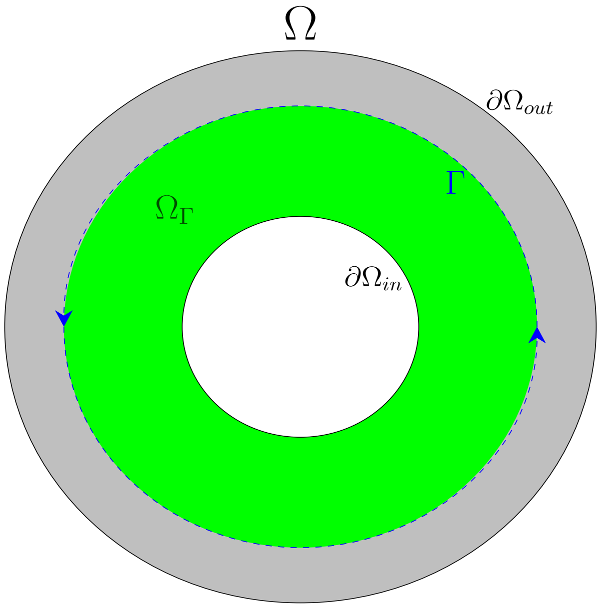

Figure 3.1.1: A simple example for a domain Ω as described in the text. The boundary is given by two disjoint curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$. The curve Γ is parametrized in anticlockwise direction and Ω_Γ is the area enclosed by Γ and $\partial\Omega_{in}$. {fig:annulus_d

identical to $\partial\Omega_{out}$. We then denote the area that is enclosed by Γ and $\partial\Omega_{in}$ as Ω_Γ (cf. Fig. 3.1.1).

From now on, our domain Ω is always assumed to be of that kind. We will later make further restrictions on what types of domain we will consider that will be suitable for discretization (see Assumption 3.3.2).

We add the curve integral along Γ, which we assume to be well-defined, as an additional constraint. So in total, we obtain the following problem.

{prob:2d_magne

Problem 3.1.2 (2D magnetostatic problem)**.** Assume Ω and the curve Γ are of the form as described above. Given $J \in L^2(\Omega)$ and $C_0 \in \mathbb{R}$, find $\mathbf{B} \in H_0(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega)$ s.t.

$$\begin{aligned} \mathrm{curl}\,\mathbf{B} &= J, \\ \mathrm{div}\,\mathbf{B} &= 0, \\ \int_\Gamma \mathbf{B} \cdot d\ell &= C_0 \end{aligned}$$

Another option for the additional constraint would be an orthogonality constraint as discussed in [11, Sec. 3.5].

3.1.3 Mixed formulation

In order to solve this problem numerically using finite elements, we have to choose a suitable variational formulation of the problem. This variational formulation will be stated without the curve integral constraint and then we will show the equivalence with the strong formulation.

Ignoring the curve integral at first, we will use the following. We choose a non-zero harmonic form $\mathbf{p} \in \mathfrak{H}^1$ and have $J \in L^2$. Then the problem is: Find $\sigma \in H_0^1$, $B \in H_0(\mathrm{div})$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\,\tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \tag{3.1.4}$$ {eq:first_eq_m

$$\langle \mathbf{curl}\,\sigma, \mathbf{v} \rangle + \langle \mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathbf{v} \rangle + \lambda \langle \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in H_0(\mathrm{div}) \tag{3.1.5}$$ {eq:second_eq_

As before, the inner product without subscript denotes the L^2 inner product and $\|\cdot\|$ the L^2-norm. Here the curve integral condition is missing. It is difficult to include the curve integral condition directly when solving this system numerically. So we will replace it below in Sec. 3.1.4.

Even though this formulation appears more complicated in comparison to the first two equations of the 2D magnetostatic problem (Problem 3.1.2), it will turn out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will first investigate the formulation without curve integral.

{prop:equivale

Proposition 3.1.3. *For any $J \in L^2$, (3.1.4) and (3.1.5) hold i.i.f. $\sigma = 0$, $\lambda = 0$,* $\operatorname{curl} \mathbf{B} = J$ *and* $\operatorname{div} \mathbf{B} = 0$ *i.e.* $\mathbf{B}$ *solves the 2D magnetostatic problem (Problem 3.1.2) without the additional curve integral constraint.*

Proof. Assume $(\sigma, \mathbf{B}, \lambda)$ is a solution of (3.1.4) and (3.1.5). Then the first equation is

$$\langle \sigma + J, \tau \rangle = \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle \quad \forall \tau \in H_0^1$$

which is equivalent to $\mathbf{B} \in H(\operatorname{curl})$ and $J + \sigma = \operatorname{curl} \mathbf{B}$.

Now assume additionally, that (3.1.5) holds. Then by choosing $\mathbf{v} = \mathbf{p} \in \mathfrak{H}^1$, we get $\operatorname{div} \mathbf{p} = 0$ from the definition of the harmonic forms and $\mathfrak{H}^1 \perp \mathbf{curl}\, H_0^1$ from the Hodge decomposition and thus

$$\langle \mathbf{curl}\, \sigma, \mathbf{p} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{p} \rangle + \lambda \langle \mathbf{p}, \mathbf{p} \rangle = \lambda \langle \mathbf{p}, \mathbf{p} \rangle = 0$$

and so $\lambda = 0$. Then we can choose $\mathbf{v} = \mathbf{curl}\, \sigma$ to get

$$\langle \mathbf{curl}\, \sigma, \mathbf{curl}\, \sigma \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{curl}\, \sigma \rangle + \lambda \langle \mathbf{p}, \mathbf{curl}\, \sigma \rangle = \|\mathbf{curl}\, \sigma\|^2 = 0.$$

Because $\sigma \in H_0^1$ this gives us $\sigma = 0$. Also we have then $J = \operatorname{curl} \mathbf{B}$. At last, we choose $\mathbf{v} = \mathbf{B}$ which gives us $\operatorname{div} \mathbf{B} = 0$ and thus we proved the first direction.

The other implication is clear i.e. if $\mathbf{B} \in H(\operatorname{curl}) \cap H_0(\operatorname{div})$ with $\operatorname{curl} \mathbf{B} = J$ and $\operatorname{div} \mathbf{B} = 0$, $\sigma = 0$ and $\lambda = 0$ then the variational formulation clearly holds. □

Notice that the variable λ is not necessary for this variational formulation, but we will need it later, since we will add another equation representing the curve integral constraint and, for that purpose, we need another variable to have the same number of unknowns and equations. If we now add the same additional constraint to both formulations of the problem then they will remain equivalent.

3.1.4 Curve integral constraint

{sec:curve_int

We still need to find a good way to include the curve integral constraint from Problem 3.1.2 in our formulation. Instead of incorporating it directly, we will substitute it with another equation. We will first derive this equation as an immediate consequence of the integration by parts formula (3.1.1) and then state the final variational formulation of the 2D magnetostatic problem which we will investigate in the coming sections.

Because $\mathbf{n}$ is the unit outward normal of Ω_Γ and $\boldsymbol{\gamma}$ the parametrization of Γ we know that $\mathbf{n} \perp \boldsymbol{\gamma}'$ and

$$\mathbf{B} \times \mathbf{n} = (B_1 n_2 - B_2 n_1) = \mathbf{B} \cdot \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} = -\mathbf{B} \cdot \mathbf{R}_{\pi/2}\mathbf{n}.$$

Now we assume again that $|\boldsymbol{\gamma}'| = 1$ and that Γ does not intersect itself. Then $\mathbf{R}_{\pi/2}\mathbf{n}$ is either $\boldsymbol{\gamma}'$ or $-\boldsymbol{\gamma}'$. Assume w.l.o.g. that $\mathbf{R}_{\pi/2}\mathbf{n} = \boldsymbol{\gamma}'$ and thus

$$\mathbf{B} \times \mathbf{n} = -\mathbf{B} \cdot \boldsymbol{\gamma}'$$

and so the curve integral becomes

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_0^{|\Gamma|} \mathbf{B}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = -\int_\Gamma \mathbf{B} \times \mathbf{n}\, d\ell.$$

Choose $\psi \in H^1$ s.t.

$$\psi = 0 \text{ on } \partial\Omega_{in},\ \psi = 1 \text{ on } \Gamma \text{ and } \psi \equiv 1 \text{ in } \Omega \setminus \Omega_\Gamma. \tag{3.1.6}$$

{eq:conditions_

Then we observe

$$\begin{aligned}
\int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx &= \int_{\Omega_\Gamma} \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx \\
= \int_{\Omega_\Gamma} \psi\, J\, dx + \int_{\partial\Omega_\Gamma} \psi\, \mathbf{B} \times \mathbf{n}\, d\ell &= \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Gamma \mathbf{B} \cdot d\ell
\end{aligned}$$

and we can rewrite the curve integral as

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx. \tag{3.1.7}$$

{eq:rewrite_cu

So if the curve integral

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0$$

is given and we can compute $\int_{\Omega_\Gamma} \psi\, J\, dx$ we can add the equation

$$\langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1 \tag{3.1.8}$$

{eq:variationa

with

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0$$

to our system.

From the above derivations it is then clear that for $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0 \Leftrightarrow \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1.$$

This is the motivation to add the right equation to our system instead of the curve integral since it is much easier to enforce numerically.

However, ψ is of course not uniquely determined by the conditions above so we have to check what happens when we use a different ψ respecting the conditions (3.1.6). Assume first that $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$. Then we know from (3.1.7) that

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx = C_0 = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\, \tilde{\psi} \cdot \mathbf{B}\, dx$$

which immediately implies that, if we define

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0,$$

then we know

$$\langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1 \Leftrightarrow \langle \mathbf{curl}\, \tilde{\psi}, \mathbf{B} \rangle = \tilde{C}_1 \tag{3.1.9}$$ {eq:equivalenc

so the choice of ψ does not matter in this case as long as the right hand side is computed accordingly.

However, $\mathbf{B}$ might not be $C^1(\overline{\Omega}; \mathbb{R}^2)$. But assume, that we are still given $C_0 \in \mathbb{R}$. Then the question arises if the choice of ψ matters. This is not the case as the following proposition shows.

{prop:differen

Proposition 3.1.4. *Let ψ and $\tilde{\psi}$ both fulfill the assumptions given by (3.1.6) and $(\sigma, \mathbf{B}, \lambda)$ be a solution of (3.1.4) and (3.1.5). Then*

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\, \tilde{\psi} \cdot \mathbf{B}\, dx. \tag{3.1.10}$$ {eq:different_

Proof. We know that $\psi = \tilde{\psi} = 0$ on $\partial\Omega_{in}$. Also, both are equal to one on Γ and between Γ and $\partial\Omega_{out}$ and so we also know that $\psi = \tilde{\psi} = 1$ on $\partial\Omega_{out}$. Hence, $\psi - \tilde{\psi} \in H_0^1(\Omega)$ and $\psi - \tilde{\psi} \equiv 0$ in $\Omega \setminus \Omega_\Gamma$. Recall that when $(\sigma, \mathbf{B}, \lambda)$ solves (3.1.4) and (3.1.5) then $\sigma = 0$ and $\lambda = 0$ as proven in Prop. 3.1.3. So because $\mathbf{B}$ solves (3.1.4) we get

$$\int_\Omega \mathbf{curl}(\psi - \tilde{\psi}) \cdot \mathbf{B}\, dx = \int_\Omega (\psi - \tilde{\psi})\, J\, dx = \int_{\Omega_\Gamma} (\psi - \tilde{\psi})\, J\, dx.$$

□

That means if we are just given C_0 and define

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0$$

as before then the equivalence (3.1.9) holds as well, even if $\mathbf{B}$ is not continuously differentiable.

There is an important invariance property of the curve integral that we would like to have for its replacement as well. Take two closed curves Γ_1 and Γ_2 with the same assumption as before, i.e. parametrized counterclockwise and not intersecting themselves. We obtain Ω_{Γ_1} and Ω_{Γ_2} defined as before. If $J = 0$ in Ω_{Γ_1} and Ω_{Γ_2}, i.e. it is zero between $\partial\Omega_{in}$ and both curves, then we have

$$\int_{\Gamma_1} \mathbf{B} \cdot d\ell = \int_{\Gamma_2} \mathbf{B} \cdot d\ell. \tag{3.1.11}$$

We would like this to be preserved for our replacement of the curve integral which is indeed the case.

{prop:differen

Proposition 3.1.5. *Let Γ_1 and Γ_2 be chosen as just described and ψ_1 and ψ_2 accordingly. Let $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\operatorname{div})$ and $\lambda \in \mathbb{R}$ be a solution of (3.1.4) and (3.1.5) with $J = 0$ in Ω_{Γ_1} and Ω_{Γ_2}. Then $C_1 = -C_0$ and*

$$\langle \mathbf{curl}\, \psi_1, \mathbf{B} \rangle = \langle \mathbf{curl}\, \psi_2, \mathbf{B} \rangle.$$

Proof. $C_1 = -C_0$ is obvious due to $J = 0$ in in Ω_{Γ_1} and Ω_{Γ_2}. Notice that even if we have different curves Γ_1 and Γ_2 the fact that ψ_1 and ψ_2 are constant one between the curve and $\partial\Omega_{out}$ implies that $\psi_1 - \psi_2 = 0$ on $\partial\Omega_{out}$. From the assumptions (3.1.6), we also get $\psi_1 - \psi_2 = 0$ on $\partial\Omega_{in}$ so in total $\psi_1 - \psi_2 \in H_0^1$. Recall that if $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$. So because $\mathbf{B}$ solves (3.1.4) and (3.1.5) with $J = 0$, this means

$$\langle \mathbf{curl}(\psi_1 - \psi_2), \mathbf{B} \rangle = 0.$$

□

In order to get a variational formulation to study theoretically, we multiply (3.1.8) with an arbitrary $\mu \in \mathbb{R}$. In conclusion, we have the following variational problem:

{prob:magnetos

Problem 3.1.6. Let $J \in L^2$, $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Find $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\operatorname{div})$, $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \qquad \forall \tau \in H_0^1, \tag{3.1.12}$$ {eq:first_eq_m

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle = 0 \qquad \forall \mathbf{v} \in H_0(\operatorname{div}), \tag{3.1.13}$$ {eq:second_eq_

$$\mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = \mu C_1 \qquad \forall \mu \in \mathbb{R}, \tag{3.1.14}$$

which is the variational formulation of the magnetostatic problem with curve integral constraint (Problem 3.1.2). We will study the well-posedness of this formulation next.

3.1.5 Well-posedness of the magnetostatic system

{sec:well-pose

The well-posedness is based on the well-known Banach-Nečas-Babuška (BNB) theorem concerning general variational problems of the following form: Find $x \in X$ s.t.

$$a(x, y) = \ell(y) \quad \forall y \in Y \tag{3.1.15}$$

{eq:general_va

where X and Y are Banach spaces, a is a bilinear form and $\ell \in Y'$. The BNB-theorem then answers the question of well-posedness, i.e. if there exists a unique solution and if we can find a stability estimate. The following formulation is from [9, Sec. 25.3] in the real case.

{thm:BNB}

Theorem 3.1.7 (BNB)**.** *Let X be a Banach space and Y be a reflexive Banach space. Let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and $\ell \in Y'$. Then a problem of the form (3.1.15) is well-posed i.i.f. the following two criteria are fulfilled*

$$(1) \quad \inf_{x \in X} \sup_{y \in Y} \frac{|a(x, y)|}{\|x\|_X \|y\|_Y} =: \gamma > 0 \tag{3.1.16}$$

{eq:BNB1}

$$(2) \quad \textit{for any } y \in Y \textit{ if } a(x, y) = 0 \textit{ for every } x \in X\textit{, then } y = 0. \tag{3.1.17}$$

{eq:BNB2}

We then obtain the stability estimate for a solution x

$$\|x\|_X \leq \frac{1}{\gamma} \|\ell\|_{Y'}.$$

Note that (3.1.16) is equivalent to the fact that for any $x \in X \setminus \{0\}$ there exists $y \in Y \setminus \{0\}$ s.t. $a(x, y) \geq \gamma \|x\|_X \|y\|_Y$.

Since we are dealing with Hilbert spaces only we can utilize the following proposition to prove it (see [8, Rem. 25.14]).

{prop:T_coerci

Proposition 3.1.8 (T-coercivity)**.** *Let X and Y be Hilbert spaces. Then* (3.1.16) *and* (3.1.17) *hold, if there exists a bounded bijective operator $T : X \to Y$ and $\eta > 0$ s.t.*

$$a(x, Tx) \geq \eta \|x\|_X^2 \quad \forall x \in X. \tag{3.1.18}$$

{eq:T_coercivi

Then γ from (3.1.16) *can be chosen as $\eta / \|T\|_{\mathcal{L}(X,Y)}$.*

Proof. For any $x \in X$, by taking $y = Tx \in Y$ and using the boundedness of T we have

$$a(x, T(x)) \geq \eta \|x\|^2 \geq \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}} \|x\|_X \|y\|_Y$$

and thus (3.1.16) holds with $\gamma = \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}}$.

For (3.1.17) assume that we have $y \in Y$ s.t. $a(x, y) = 0$ for all $x \in X$.

$$0 = a(T^{-1}y, TT^{-1}y) \geq \eta \|T^{-1}y\|_X^2$$

so $T^{-1}y = 0$ and thus $y = 0$. ☐

Remark 3.1.9. The other direction is also true i.e. if (3.1.16) and (3.1.17) are fulfilled we can construct a T with the desired properties.

Note also that when we have found T s.t. (3.1.18) holds then it must be injective. This is because if $Tx = 0$ for any $x \in X$ then $x = 0$ must hold.

The next step is to put our formulation of Problem 3.1.6 into this general framework. To this end, we define $X := H_0^1 \times H_0(\mathrm{div}) \times \mathbb{R}$ and for $(\sigma, \mathbf{B}, \lambda) \in X$

$$\|(\sigma, \mathbf{B}, \lambda)\|_X := \sqrt{\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2}.$$

Notice that X is then a Hilbert space with inner product

$$\langle (\sigma, \mathbf{B}, \lambda), (\tau, \mathbf{v}, \mu) \rangle_X = \langle \sigma, \tau \rangle_{H^1} + \langle \mathbf{B}, \mathbf{v} \rangle_{H(\mathrm{div})} + \lambda\mu.$$

For the formulation of the problem, we define the bilinear form $a : X \times X \to \mathbb{R}$

$$\begin{aligned} a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = {} & \langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle + \langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle \\ & + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle. \end{aligned} \tag{3.1.19}$$

{eq:definition_

and

$$\ell(\tau, \mathbf{v}, \mu) = -\langle J, \tau \rangle - \mu C_1$$

which are bounded due to the Cauchy-Schwarz inequality. Then Problem 3.1.6 is equivalent to the following: Find $(\sigma, \mathbf{B}, \lambda) \in X$ s.t.

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \ell(\tau, \mathbf{v}, \mu) \quad \forall (\tau, \mathbf{v}, \mu) \in X.$$

Note that the bilinear form a is not symmetric.

The next step is to show important properties of $\mathbf{curl}\, \psi$ assuming ψ has been chosen as described above.

{prop:psi_zero_

Proposition 3.1.10. *Under the given assumptions on ψ, $\mathbf{curl}\,\psi \in H_0(\mathrm{div})$.*

Proof. We need to show $0 = \gamma_n \mathbf{curl}\,\psi$. The integration by parts from Thm. 2.3.14 gives

$$\langle \gamma_n \mathbf{curl}\,\psi, \mathrm{tr}\, u\rangle = \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\, u\, dx + \int_\Omega \mathrm{div}\,\mathbf{curl}\,\psi\, u\, dx$$

where the last term vanishes. Take now $\phi \in C^1(\overline{\Omega})$ arbitrary. Then we take $\phi_1 \in C^1(\overline{\Omega})$ s.t. $\phi_1 = \phi$ in a neighborhood of $\partial\Omega_{in}$ and zero near $\partial\Omega_{out}$. Analogously, take $\phi_2 \in C^1(\overline{\Omega})$ s.t. $\phi_2 = \phi$ in a neighborhood of $\partial\Omega_{out}$ and zero near $\partial\Omega_{in}$. Here we used the fact that the two parts of the boundary are disjoint and have positive distance from one another. Then also $\mathrm{tr}\,\phi = \mathrm{tr}\,\phi_1 + \mathrm{tr}\,\phi_2$. We use the integration-by-parts formula for the divergence again, $\mathrm{div}\,\mathbf{curl} = 0$ and the integration-by-parts formula for the curl

$$\begin{aligned}
&\langle \gamma_n \mathbf{curl}\,\psi, \mathrm{tr}\,\phi\rangle = \langle \gamma_n \mathbf{curl}\,\psi, \mathrm{tr}\,\phi_1\rangle + \langle \gamma_n \mathbf{curl}\,\psi, \mathrm{tr}\,\phi_2\rangle \\
&= \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\,\phi_1\, dx + \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\,\phi_2\, dx \\
&= \int_\Omega \psi\ \mathrm{curl}\,\mathbf{grad}\,\phi_1\, dx + \langle \gamma_\tau \mathbf{grad}\,\phi_1, \psi\rangle + \int_\Omega \psi\ \mathrm{curl}\,\mathbf{grad}\,\phi_2\, dx + \langle \gamma_\tau \mathbf{grad}\,\phi_2, \psi\rangle.
\end{aligned}$$

Here only the boundary terms remain because $\mathrm{curl}\,\mathbf{grad} = 0$. Now remember that because $\phi_j \in C^1(\overline{\Omega})$, $\langle \gamma_\tau \mathbf{grad}\,\phi_j, \psi\rangle = \langle \mathbf{grad}\,\phi_j \times \mathbf{n}, \psi\rangle_{L^2(\partial\Omega)}$. So

$$\langle \gamma_\tau \mathbf{grad}\,\phi_1, \psi\rangle = \langle \mathbf{grad}\,\phi_1 \times \mathbf{n}, \psi\rangle_{L^2(\partial\Omega)} = 0$$

because ϕ_1 is zero near $\partial\Omega_{out}$ and ψ is zero on $\partial\Omega_{in}$. For the remaining term,

$$\begin{aligned}
\langle \gamma_\tau \mathbf{grad}\,\phi_2, \psi\rangle &= \int_{\partial\Omega_{out}} \psi\, \mathbf{grad}\,\phi_2 \times \mathbf{n}\, d\ell = \int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_2 \times \mathbf{n}\, d\ell \\
&= -\int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_2 \cdot d\ell
\end{aligned}$$

and then we know from basic vector calculus because $\partial\Omega_{out}$ is closed

$$\int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_1 \cdot d\ell = 0$$

and so in conclusion, $\gamma_n \mathbf{curl}\,\psi = 0$. $\square$

The last equation in Problem 3.1.6 is used to determine the harmonic part of the solution. This implies that we would like $\mathbf{curl}\,\psi$ to have non-vanishing harmonic part. This is indeed true.

Proposition 3.1.11. *Let $P_{\mathfrak{H}^1} : L^2 \to \mathfrak{H}^1$ be the orthogonal projection onto the harmonic forms. Then with ψ satisfying (3.1.6) we have $P^1_{\mathfrak{H}} \operatorname{\mathbf{curl}} \psi \neq 0$.*

Proof. Since $\operatorname{div} \operatorname{\mathbf{curl}} \psi = 0$ and $\operatorname{\mathbf{curl}} \psi \in H_0(\operatorname{div})$, we know that

$$\operatorname{\mathbf{curl}} \psi \in \mathfrak{Z}^1 = \mathfrak{B}^1 \overset{\perp}{\oplus} \mathfrak{H}^1$$

using the Hodge decomposition (cf. Thm. 2.3.24). Assume for contradiction that $\operatorname{\mathbf{curl}} \psi \in \mathfrak{B}^1$, i.e. there exists $\psi_0 \in H^1_0$ s.t. $\operatorname{\mathbf{curl}} \psi_0 = \operatorname{\mathbf{curl}} \psi$. Since $\mathbf{curl}$ is just the rotated gradient we would get that $\mathbf{grad}(\psi - \psi_0) = 0$ and thus $\psi - \psi_0$ is constant almost everywhere. But this is a contradiction since $\operatorname{tr} \psi_0$ is zero on $\partial\Omega_{in}$ and $\partial\Omega_{out}$, but $\operatorname{tr} \psi = 0$ on $\partial\Omega_{in}$ and $\operatorname{tr} \psi = 1$ on $\partial\Omega_{in}$. Thus $\operatorname{\mathbf{curl}} \psi \notin \mathfrak{B}^1$ and the claim follows. □

Now we can apply the Hodge decomposition of $\ker \operatorname{div}$ on $\operatorname{\mathbf{curl}} \psi$ and obtain the following

Corollary 3.1.12. *Let $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Then there exists $\psi_0 \in H^1_0$ and $c_\psi \in \mathbb{R} \setminus \{0\}$ s.t.*

$$\operatorname{\mathbf{curl}} \psi = \operatorname{\mathbf{curl}} \psi_0 + c_\psi \mathbf{p}. \tag{3.1.20}$$ {eq:decomposit

Here it is important to remember that we assumed $\mathfrak{H}^1$ to be one-dimensional. Because we can choose $\mathbf{p}$ we can assume w.l.o.g. that $c_\psi > 0$ and we will do so from now on. Also we can prove that the harmonic part is independent of the chosen ψ.

{prop:harmonic_

Proposition 3.1.13. *Choose ψ and $\tilde{\psi}$ according to (3.1.6). Then*

$$P_{\mathfrak{H}^1} \operatorname{\mathbf{curl}} \psi = P_{\mathfrak{H}^1} \operatorname{\mathbf{curl}} \tilde{\psi}.$$

In particular, if we decompose them as in (3.1.20) then $c_\psi = c_{\tilde{\psi}}$.

Proof. Take $\mathbf{p} \in \mathfrak{H}^1$ arbitrary. Then using the same argument as in the proof of Prop. 3.1.4 we get $\psi - \tilde{\psi} \in H^1_0$ and because $\mathfrak{H}^1 \perp \mathfrak{B}^1$

$$\langle \mathbf{curl}(\psi - \tilde{\psi}), \mathbf{p} \rangle = 0. \tag{3.1.21}$$

Because $\mathbf{p} \in \mathfrak{H}^1$ was arbitrary the claim follows. □

As stated in the proof of the Poincare inequality (Thm. 2.3.25), $\mathbf{curl}|_{\mathfrak{Z}^\perp} : \mathfrak{Z}^\perp \to \mathfrak{B}^1$ is bijective and since it is bounded w.r.t. the V-norm – which is the H^1-norm here – due to the Banach inverse theorem it is invertible and we denote this inverse $\mathbf{curl}^{-1}$. This is a slight abuse of notation since it is not really the inverse of the full $\mathbf{curl}$.

Let $P_{\mathfrak{B}}$ be the L^2-orthogonal projection onto $\mathfrak{B}^j$, we then denote $\mathbf{v}_{\mathfrak{B}} = P_{\mathfrak{B}}\mathbf{v}$ for any $\mathbf{v} \in L^2$ and analogous for $\mathfrak{H}^j$ and $\mathfrak{B}^*_j$. In order to prove the T-coercivity, we need the following lemma.

{lem:T_for_T_c

Lemma 3.1.14. *Take* $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ *with* $c_\psi > 0$, $\mathbf{p} \in \mathfrak{H}^1$ *and* $\|\mathbf{p}\| = 1$. *Define* $T : X \to X$ *as*

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}).$$

with $\alpha < 0$ *and* $\beta > 0$. *Then* T *is bounded and surjective.*

Proof. The boundedness is clear since all operators used in the definition are bounded w.r.t. the norm of their domains. From the Poincaré inequality, we know that $\|\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}\| \leq c_P \|\mathbf{B}_{\mathfrak{B}}\|$ and so

$$\begin{aligned}
&\|T(\sigma, \mathbf{B}, \lambda)\|_X^2 \\
&= \|\sigma - \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + \|\mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}\|_{H(\mathrm{div})}^2 + \left(\alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right)^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^4}\|\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 \\
&\quad + 2\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}\|_{H(\mathrm{div})}^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq C_T\left(\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2\right)
\end{aligned}$$

with

$$C_T := \max\left\{5, \frac{2}{c_P^2} + 3 + 2\alpha^2, 3\beta^2 + \frac{2}{c_\psi^2}\right\}. \qquad (3.1.22)$$ {eq:bound_on_n

So T is bounded and $\|T\|_{\mathcal{L}(X,X)} \leq \sqrt{C_T}$.

Take $(\tau, \mathbf{v}, \mu) \in X$ arbitrary. In order to prove surjectivity, we will split up $\mathbf{v} = \mathbf{v}_{\mathfrak{B}} + \mathbf{v}_{\mathfrak{H}} + \mathbf{v}_{\mathfrak{B}^*}$ using the Hodge decomposition and choose

$$\sigma = \left(1 + \frac{1}{c_P^2}\right)^{-1}\left(\tau + \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{v}_{\mathfrak{B}}\right) \text{ and } \mathbf{B}_{\mathfrak{B}} = \mathbf{v}_{\mathfrak{B}} - \mathbf{curl}\,\sigma.$$

So

$$\begin{aligned}
\sigma - \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}} &= \sigma - \frac{1}{c_P^2}(\mathbf{curl}^{-1}\,\mathbf{v}_{\mathfrak{B}} - \sigma) \\
&= \left(1 + \frac{1}{c_P^2}\right)\sigma - \frac{1}{c_P^2}\,\mathbf{curl}^{-1}\,\mathbf{v}_{\mathfrak{B}} = \tau.
\end{aligned}$$

We simply choose $\mathbf{B}_{\mathfrak{B}^*} = \mathbf{v}_{\mathfrak{B}^*}$. For the harmonic part take κ_v s.t. $\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p}$. Let us look at the system

$$\begin{pmatrix} 1 & \beta \\ \alpha & 1/c_\psi \end{pmatrix} \begin{pmatrix} \kappa_B \\ \lambda \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \mu \end{pmatrix}$$

Now since $c_\psi > 0$ and $\alpha < 0$, $\beta > 0$ we get $1/c_\psi - \alpha\beta \neq 0$ and the system has a solution. Choose $\mathbf{B}_{\mathfrak{H}} = \kappa_B \mathbf{p}$. Then we see

$$\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p} = \mathbf{p}(\kappa_B + \beta\lambda) = \mathbf{B}_{\mathfrak{H}} + \beta\lambda\mathbf{p}$$

and

$$\mu = \alpha\kappa_B + \frac{\lambda}{c_\psi} = \alpha\kappa_B \|\mathbf{p}\|^2 + \frac{\lambda}{c_\psi} = \alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi}.$$

By combining all that we arrive at $T(\sigma, \mathbf{B}, \lambda) = (\tau, \mathbf{v}, \mu)$. $\square$

We assume from now on that we have always chosen $\mathbf{p}$ in a way s.t. c_ψ – as defined in the previous lemma – is positive and $\mathbf{p}$ has norm one. This comes down to choosing $\mathbf{p}$ with the correct sign and normalizing it. Now we can use the T-coercivity (Prop. 3.1.8) to prove the inf-sup condition and thus well-posedness of our formulation.

Theorem 3.1.15. *Let $\mathbf{curl}\,\psi_0 + c_\psi \mathbf{p} = \mathbf{curl}\,\psi$ and assume we have chosen the sign of $\mathbf{p}$ s.t. $c_\psi > 0$. Then take $c_1 > 0$ s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$ (e.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$ would be a valid choice). Define $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ and $\alpha = -\frac{c_\psi}{4c_1^2 c_P^2}$. Then the bilinear form a defined at (3.1.19) satisfies the inf-sup condition i.e. (3.1.16) and (3.1.17) with $\gamma \geq \eta/\sqrt{C_T}$, C_T from (3.1.22) and*

$$\eta := \min\left\{ \frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2} \right\}.$$

Proof. Choose $(\sigma, \mathbf{B}, \lambda) \in X$ arbitrary and define $\rho := \mathbf{curl}^{-1}\,\mathbf{B}_{\mathfrak{B}}$. We take T as in (3.1.14),

$$T(\sigma, \mathbf{B}, \lambda) = \left(\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right)$$

Then T is surjective due to Lemma 3.1.14. Note

$$\langle \mathbf{B}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2 \langle \frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2$$

where we used in the last equality that $\frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}$ is either $+\mathbf{p}$ or $-\mathbf{p}$ because $\mathfrak{H}^1$ is assumed to be one-dimensional. We split up $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ to get

$$
\begin{aligned}
&a(\sigma, \mathbf{B}, \lambda; T(\sigma, \mathbf{B}, \lambda)) \\
&= \langle \sigma, \sigma - \frac{1}{c_P^2}\rho\rangle - \langle \mathbf{B}, \mathbf{curl}\,\sigma - \frac{1}{c_P^2}\mathbf{curl}\,\rho\rangle + \langle \mathbf{curl}\,\sigma, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle \\
&\quad + \langle \operatorname{div}\mathbf{B}, \operatorname{div}\mathbf{curl}\,\sigma + \operatorname{div}\mathbf{B} + \beta\lambda \operatorname{div}\mathbf{p}\rangle \\
&\quad + \langle \lambda\mathbf{p}, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle - \Big(\alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi}\Big)\langle \mathbf{B}, \mathbf{curl}\,\psi\rangle \\
&= \|\sigma\|^2 - \frac{1}{c_P^2}\langle \sigma, \rho\rangle + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\beta - \alpha c_\psi\|\mathbf{B}_{\mathfrak{H}}\|^2 \\
&\quad - \alpha\langle \mathbf{p}, \mathbf{B}_{\mathfrak{H}}\rangle\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle - \frac{\lambda}{c_\psi}\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle
\end{aligned}
$$

Due to the Poincaré inequality

$$
\|\rho\| \le \|\rho\|_{H^1} \overset{\text{Poincaré}}{\le} c_P\|\mathbf{curl}\,\rho\| = c_P\|\mathbf{B}_{\mathfrak{B}}\|.
$$

Using ϵ-Young combined with Cauchy-Schwarz inequality several times we obtain the lower bound.

$$
\begin{aligned}
&\|\sigma\|^2 - \left(\frac{1}{2}\|\sigma\|^2 + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2}{2c_P^2}\right) + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 \\
&\quad + \lambda^2\beta - \alpha c_\psi\|\mathbf{B}_{\mathfrak{H}}\|^2 - \left(\frac{\epsilon_1\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2}{2} + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2\epsilon_1}\right) \\
&\quad - \left(\frac{\lambda^2}{2\epsilon_2 c_\psi^2} + \frac{\epsilon_2\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2}\right)
\end{aligned}
$$

Choose $\epsilon_1 = 4c_1^2c_P^2$ to get

$$
\begin{aligned}
&\frac{1}{2}\|\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{1}{2\epsilon_2 c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(-\alpha c_\psi - \frac{4c_1^2c_P^2\alpha^2}{2}\right) - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2} - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\epsilon_2\|\mathbf{curl}\,\psi_0\|^2}{2}
\end{aligned}
$$

Now choose $\epsilon_2 = \frac{1}{4c_1^2c_P^2}$, plug in the definition of α and use $\|\mathbf{curl}\,\psi_0\| \le c_1$ to get the next lower bound

$$
\begin{aligned}
&\frac{1}{2}\|\sigma\|^2 + \|\mathbf{B}_{\mathfrak{B}}\|^2\left(\frac{1}{2c_P^2} - \frac{1}{8c_P^2} - \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2}\right) + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{4c_1^2c_P^2}{2c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(\frac{c_\psi^2}{4c_1^2c_P^2} - \frac{c_1^2c_P^2c_\psi^2}{8c_1^4c_P^4}\right)
\end{aligned}
$$

and finally by using the Poincaré inequality $\|\mathbf{B}_{\mathfrak{B}^*}\| \leq c_P \|\operatorname{div} \mathbf{B}\|$ and $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ we obtain the next bound

$$\frac{1}{2}\|\sigma\|^2 + \frac{1}{4c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}^*}\|^2 + \frac{1}{2}\|\operatorname{div}\mathbf{B}\|^2 + \frac{c_1^2 c_P^2}{c_\psi^2}\lambda^2 + \frac{c_\psi^2}{8c_1^2 c_P^2}\|\mathbf{B}_{\mathfrak{H}}\|^2.$$
$$\geq \eta\|(\sigma, \mathbf{B}, \lambda)\|_X^2$$

where we chose

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2}\right\}$$

to obtain the T-coercivity. We can then choose γ from (3.1.16) as $\eta/\|T\|_{\mathcal{L}(X,X)}$ and then use C_T from (3.1.22) to get a lower bound

$$\gamma \geq \frac{\eta}{\sqrt{C_T}}.$$

□

Corollary 3.1.16 (Well-posedness)**.** *The variational formulation of the magnetostatic problem* (Problem 3.1.6) *is well-posed. For a solution* $(\sigma, \mathbf{B}, \lambda) \in X$ *we have the stability estimate*

$$\|\mathbf{B}\| = \|(\sigma, \mathbf{B}, \lambda)\|_X \leq \frac{\|J\| + |C_1|}{\gamma}.$$

Proof. Recall that when $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$ and $\operatorname{div}\mathbf{B} = 0$ which implies the first equality. The statement follows immediately from the previous theorem and Thm. 3.1.7 and the fact that

$$|\ell(\tau, \mathbf{v}, \mu)| = |-\langle J, \tau\rangle - C_1\mu| \leq (\|J\| + |C_1|)\|(\tau, \mathbf{v}, \mu)\|_X$$

and thus $\|\ell\|_{X'} \leq \|J\| + |C_1|$. □

Remark 3.1.17. Note that $1/c_\psi$ terms arise in the stability constant $1/\gamma$. This is not surprising since the term $\langle \mathbf{curl}\,\psi, \mathbf{B}\rangle$ will not enforce the harmonic part if the harmonic part of $\mathbf{curl}\,\psi$ would be zero because $\mathbf{B}_{\mathfrak{H}}$ will disappear from the formulation. But we know from Prop. 3.1.13 that the harmonic part of $\mathbf{curl}\,\psi$ is actually independent of the choice of ψ. If we take $c_1 = \|\mathbf{curl}\,\psi\| + 1$ then we could expect stability issues if this value becomes very large. But this should not be the case for a reasonable choice of ψ.

3.2 Discretized magnetostatic problem

{sec:discretiz

In order to approximate solutions of the 2D magnetostatic problem, we want to use finite elements. A very typical question that arises for any discretization of a model is what notions of the continuous model are represented in the discretized one. In our case, a fundamental structure of our problem is the Hilbert complex that we introduced in Section 3.1.1. We would like to represent it in our discretization, leading to the discrete Hilbert complex. This section follows Sec. 5.2 in Arnold's book [2]. We start with reviewing the theory behind the discretization of general Hilbert complexes before applying this theory to our problem. Once we discretized it, we will utilize the inf-sup condition proven in Section 3.1.1 to prove well-posedness of the discrete formulation and derive a quasi-optimal error estimate.

3.2.1 Discrete Hilbert complex

{sec:discrete_

Let us at first stick to the general situation. We assume that we have a Hilbert complex (W^k, d^k) with its corresponding domain complex (V^k, d^k) and dual complex (V_k^*, d_k^*) for $k \in \mathbb{Z}$. For this chapter, we will only need a short subsequence

$$V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1}$$

for some fixed k and $j \in \{k-1, k, k+1\}$ will always be the index to refer to all three spaces. We will sometimes leave out the indices if the meaning is clear from the context.

Let us assume that we have finite dimensional subspaces $V_h^j \subseteq V^j$. As usual in numerical analysis, $h > 0$ stands loosely for the fineness of our discretization, e.g. the grid size or the maximal diameter of the elements in the mesh. Then we define completely analogous to the continuous case,

$$\begin{aligned} \mathfrak{Z}_h^j &:= \{v_h \in V_h^j \mid d^j v_h = 0\} = \ker d^j \cap V_h^j \\ \mathfrak{B}_h^j &:= \{d^j v_h \mid v_h \in V_h^{j-1}\}. \end{aligned}$$

We can now also define the discrete harmonic forms. This time, the situation is slightly different however. We will not use the adjoint d_j^* to define it. Instead,

$$\mathfrak{H}_h^j := \{v_h \in \mathfrak{Z}_h^j \mid v_h \perp \mathfrak{B}_h^j\} = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp}.$$

Notice that we have $\mathfrak{Z}_h^j \subseteq \mathfrak{Z}^j$ and $\mathfrak{B}_h^j \subseteq \mathfrak{B}^j$, but due to $\mathfrak{B}_h^{j,\perp} \supseteq \mathfrak{B}^{j,\perp}$ we have in general

$$\mathfrak{H}_h^j = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp} \not\subseteq \mathfrak{Z}^j \cap \mathfrak{B}^{j,\perp} = \mathfrak{H}^j.$$

We will later investigate the difference between the space of the discrete and continuous harmonic forms more closely.

There are three crucial assumptions that we will need to prove stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want the discrete spaces V_h^j to approximate the continuous ones V^j.

{ass:convergen

Assumption 3.2.1. For the discrete spaces, we require $V_h^j \subseteq V^j$ and

$$\lim_{h \to 0} \inf_{v_h \in V_h^j} \|w - v_h\|_V = 0, \quad \forall w \in V^j.$$

This is usually satisfied for a reasonable choice of finite element space. The next property is more restrictive and is a compatibility condition between the spaces.

Assumption 3.2.2. For $j = k-1, k$,

$$dV_h^j \subseteq V_h^{j+1}.$$

This means that we cannot simply use arbitrary discrete subspaces independent from one another. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. 2.3.3 directly to it. Let us do that.

Denote the restriction of d^j to V_h^j as d_h^j. Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \to V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to d_h^j the adjoint $d_{j,h}^*$ is not the restriction of the adjoint d_j^*. In general, $V_h^j \not\subseteq V_j^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h^j$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1} \tag{3.2.1}$$ {eq:discrete_h

and its dual complex

$$V_h^{k-1} \xleftarrow{d_{k,h}^*} V_h^k \xleftarrow{d_{k+1,h}^*} V_h^{k+1}$$

From the general Hilbert complex theory (cf. Thm. 2.3.24), we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \overset{\perp}{\oplus} \mathfrak{H}_h^j \overset{\perp}{\oplus} \mathfrak{B}_{jh}^*.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. We will investigate the question how well the discrete harmonic forms approximate the continuous ones more thoroughly later.

Assumption 3.2.3. There exist *bounded cochain projections* $\Pi_h^j : V^j \to V_h^j$. This is a projection that is a cochain map in the sense of cochain complexes (see Sec. 2.2.2) i.e. the following diagram commutes:

$$\begin{array}{ccccc} V^{k-1} & \xrightarrow{d^{k-1}} & V^k & \xrightarrow{d^k} & V^{k+1} \\ \downarrow{\scriptstyle \Pi_h^{k-1}} & & \downarrow{\scriptstyle \Pi_h^k} & & \downarrow{\scriptstyle \Pi_h^{k+1}} \\ V_h^{k-1} & \xrightarrow{d^{k-1}} & V_h^k & \xrightarrow{d^k} & V_h^{k+1} \end{array}$$

Π_h^j are either bounded in the V- or in the W-norm.

Quite often we will leave out the index and just say the discrete Hilbert complex admits a V- or W-bounded cochain projection Π_h. The cochain projection will play an important role in the stablity of the discrete system. Thanks to the commuting property, if Π_h^j are W-bounded then they are V-bounded. In this section, we will make the weaker assumption of V-boundedness.

The fact that Π_h is a V-bounded projection immediately allows a quasi optimal estimate. For any $v \in V^j$, we can take $w_h \in V_h^j$ arbitrary and then by using the triangular inequality

$$\begin{aligned} \|v - \Pi_h^j v\|_V &= \|v - w_h + \Pi_h^j(w_h - v)\|_V & (3.2.2) \\ &\leq \|I - \Pi_h^j\|_{\mathcal{L}(V,V)} \|v - w_h\|_V. & (3.2.3) \end{aligned}$$ {eq:bound_proj

From now on we will denote the operator norm $\|\cdot\|_{\mathcal{L}(V,V)}$ by slight abuse of notation as $\|\cdot\|_V$. Since w_h was arbitrary we can take the infimum over $w_h \in V_h^k$ and obtain a quasi optimal estimate.

Let us now answer the question about the difference between discrete and continuous harmonic forms. In order to do that, we need some way to measure the "difference" between two subspaces.

For a general metric space (X, d), we will use the standard notation $d(x, M) := \inf_{m \in M} d(x, m)$ for $x \in X$ and $M \subseteq X$. If we are dealing with a normed space then we take the metric induced by the norm.

Definition 3.2.4 (Gap between subspaces)**.** For a Banach space W with subspaces Z_1 and Z_2 let S_1 and S_2 be the unit spheres in Z_1 and Z_2 respectively i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$ and analogous for S_2. Then we define the gap between these subspaces as

$$\mathrm{gap}(Z_1, Z_2) = \max\left\{ \sup_{z_1 \in S_1} d(z_1, Z_2), \sup_{z_2 \in S_2} d(z_2, Z_1) \right\}$$

This definition is from [13, Ch.4 §2.1] and defines a metric on the set of closed subspaces of W. If W is a Hilbert space – as it is throughout this section – and Z_1 and Z_2 are closed then the $\operatorname{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|_{\mathcal{L}(W,W)}$ i.e. the difference in operator norm of the orthogonal projections onto Z_1 and Z_2. This gives us a measure of distance between spaces which we can now apply to the question about the difference between discrete and continuous harmonic forms.

Proposition 3.2.5 (Gap between harmonic forms)**.** *Assume that the discrete complex (3.2.1) admits a V-bounded cochain projection Π_h. Then*

$$\|(I - P_{\mathfrak{H}_h^k})q\|_V \leq \|(I - \Pi_h^k)q\|_V, \qquad \forall q \in \mathfrak{H}^k \tag{3.2.4}$$

{eq:difference_

$$\|(I - P_{\mathfrak{H}^k})q_h\|_V \leq \|(I - \Pi_h^k)P_{\mathfrak{H}^k}q_h\|_V, \quad \forall q_h \in \mathfrak{H}_h^k \tag{3.2.5}$$

and

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q \in \mathfrak{H}, \|q\|=1} \|(I - \Pi_h^k)q\|_V$$

Proof. See [2, Thm. 5.2]. □

The following proposition clarifies how close a discrete harmonic form can be chosen. But in order to prove it, we will need a small lemma.

{lem:distance_

Lemma 3.2.6. *Let W be a Banach space, $Z \subseteq W$ a closed subspace. Denote $S_Z := \{z \in Z \mid \|z\|_W = 1\}$. Then for any $w \in W$ with $\|w\|_W = 1$ we get*

$$d(w, S_Z) \leq 2\, d(w, Z)$$

Proof. See [13, Ch.4 §2, (2.13)]. □

{prop:choice_o

Proposition 3.2.7. *Take $p \in \mathfrak{H}^k$ with $\|p\| = 1$. Then we can choose $p_h \in \mathfrak{H}_h^k$ with $\|p_h\| = 1$ s.t.*

$$\|p - p_h\| \leq 2\, \|I - \Pi_h^k\|_V \inf_{v_h \in V_h} \|p - v_h\|_V.$$

Proof. Notice that since $\mathfrak{H}^k \subseteq \mathfrak{Z}^k$ we always have $\|q\|_V = \|q\|$ for all $q \in \mathfrak{H}^k$. The same is true for $\mathfrak{H}_h^k$. Denote $S_h := \{q_h \in \mathfrak{H}_h^k \mid \|q_h\| = 1\}$. Since S_h is closed we can find $p_h \in S_h$ s.t.

$$\|p_h - p\| = \inf_{q_h \in S_h} \|q_h - p\|.$$

The right hand side can be estimated using (3.2.4) and then the quasi optimal bound for the projection derived at (3.2.3).

$$\begin{aligned}\inf_{q_h \in S_h} \|q_h - p\|_V = \inf_{q_h \in S_h} \|q_h - p\| &\overset{\text{Lem. 3.2.6}}{\leq} 2 \inf_{q_h \in \mathfrak{H}_h} \|q_h - p\| = 2\, \|P_{\mathfrak{H}_h} p - p\| \\ &\overset{(3.2.4)}{\leq} 2\, \|\Pi_h^k p - p\|_V \leq 2\, \|I - \Pi_h^k\|_{\mathcal{L}(V,V)} \inf_{v_h \in V_h} \|p - v_h\|_V\end{aligned}$$

which gives us the estimate. □

If we have the standard situation that we have for $v \in V^j$

$$\|v - \Pi_h^j v\|_V \leq C\|v\|_V h^s \tag{3.2.6}$$ {eq:standard_e

for some generic constant $C > 0$ independent of v and the mesh size h and some $s > 0$ then we can improve the estimate to

$$\|p_h - p\| \leq Ch^s$$

by applying it in the last estimate of the proof.

Also if we assume $\|\Pi_h\| \leq c_\Pi$ for h small enough and $c_\Pi > 0$ independent of h then Assumption 3.2.1 implies

$$p_h \xrightarrow{V} p \text{ as } h \to 0.$$

{thm:dim_of_di

Theorem 3.2.8 (Dimension of $\mathfrak{H}_h^k$)**.** *Assume that we have a finite-dimensional subcomplex with a V-bounded cochain projection. Assume further, that*

$$\|q - \Pi_h^k q\| < \|q\|, \quad \forall q \in \mathfrak{H}^k \setminus \{0\}. \tag{3.2.7}$$ {eq:assumption_

Then $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ are isomorphic. In particular, $\dim \mathfrak{H}^k = \dim \mathfrak{H}_h^k$.

Proof. See [2, Thm 5.1] and the explanation after the proof. □

If we assume a standard error estimate for the projection as (3.2.6) then (3.2.7) is fulfilled if $Ch^s < 1$ which will be true for h small enough.

{prop:gap_image

Proposition 3.2.9. *Assuming again a finite-dimensional subcomplex with a V-bounded cochain projection, we can bound the gap between the image spaces*

$$\operatorname{gap}(\mathfrak{B}_h^j, \mathfrak{B}^j) \leq \|I - \Pi_h^j\|_V \sup_{\substack{z \in \mathfrak{B}^j \\ \|z\|=1}} \inf_{v_h \in V_h^j} \|z - v_h\|_V.$$

Proof. At first note, that for any $z \in \mathfrak{B}$, $\|z\|_V = \|z\|$. Since $\mathfrak{B}_h \subseteq \mathfrak{B}$, $d(z_h, \mathfrak{B}) = 0$ for any $z_h \in \mathfrak{B}_h$.

Take $z \in \mathfrak{B}^j$ arbitrary, i.e. there exists $w \in V^{j-1}$ s.t. $dw = z$. Since Π_h is a cochain projection, we thus have

$$\Pi_h z = \Pi_h dw = d\Pi_h w \in \mathfrak{B}_h^j$$

so Π_h maps $\mathfrak{B}^j$ into $\mathfrak{B}_h^j$. Putting all that together,

$$d(z, \mathfrak{B}_h^j) = \inf_{z_h \in \mathfrak{B}_h} \|z - z_h\| \leq \|z - \Pi_h z\| = \|z - \Pi_h z\|_V \overset{(3.2.3)}{\leq} \|I - \Pi_h\|_V \inf_{v_h \in V_h} \|z - v_h\|_V$$

and then the claim follows from the definition of the gap. □

Proposition 3.2.10 (Discrete Poincare inequality)**.** *Assume that we have a V-bounded cochain projection Π_h for the discrete Hilbert complex. Then*

$$\|v_h\|_V \leq c_{P,h}\|dv_h\|, \quad \forall v_h \in \mathfrak{Z}_h^{k,\perp} \cap V_h$$

with $c_{P,h} := c_P\|\Pi_h\|_V$ and c_P being the Poincare constant from Thm. 2.3.25.

Proof. The proof is from [2, Thm. 5.3] with some additional details. This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k,\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}^{k,\perp} \cap V^k) = \mathfrak{B}^{k+1} \supseteq \mathfrak{B}_h^{k+1}$ we find $z \in \mathfrak{Z}^{k,\perp} \cap V^k$ s.t. $dz = dv_h$. We can apply now the continuous Poincare inequality (Thm. 2.3.25) to get $\|z\|_V \leq c_P\|dz\|_V = c_P\|dv_h\|_V$. We use the fact that Π_h is a cochain map and a projection:

$$d\Pi_h^k z = \Pi_h^{k+1}dz = \Pi_h^{k+1}dv_h = dv_h$$

For the last equality, we need the fact that we have a discrete complex, i.e. $d^kV_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \Pi_h z) = 0$, i.e. $v_h - \Pi_h z \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v_h, v_h - \Pi_h z\rangle = \langle v_h, v_h - \Pi_h z\rangle + \langle dv_h, d(v_h - \Pi_h z)\rangle = \langle v_h, v_h - \Pi_h z\rangle_V,$$

so $v_h - \Pi_h z$ is V orthogonal to v_h. So

$$\begin{aligned}\|v_h\|_V^2 &= \langle v_h, \Pi_h^k z\rangle_V + \langle v_h, v_h - \Pi_h^k z\rangle_V = \langle v_h, \Pi_h^k z\rangle_V \leq \|v_h\|_V\|\Pi_h\|_V\|z\|_V \\ &\overset{\text{Poincare ineq.}}{\leq} \|v_h\|_V c_P\|\Pi_h\|_V\|dv_h\|_V\end{aligned}$$

□

Notice that if we assume that $\lim_{h\to 0}\|\Pi_h\|_V = 1$ (which is true if e.g. the standard estimate (3.2.6) is fulfilled) then $c_{P,h} \to c_P$ for $h \to 0$.

In conclusion, we obtain a discrete version of the Hilbert complex where the harmonic forms are accurately represented if h is small enough.

3.2.2 Discretized magnetostatic problem

Let us apply the theory of discrete Hilbert complexes to the 2D Hilbert complex (3.1.3). We assume that we have finite dimensional subspaces $V_h^0 \subseteq H_0^1$, $V_h^1 \subseteq H_0(\mathrm{div})$ and $V_h^2 \subseteq L^2$ that approximate the full spaces in the sense of Assumption 3.2.1 and

$$V_h^0 \xrightarrow{\mathbf{curl}} V_h^1 \xrightarrow{\mathrm{div}} V_h^2$$

and the dual complex

$$V_h^0 \xleftarrow{\widetilde{\mathrm{curl}}_h} V_h^1 \xleftarrow{-\widetilde{\mathbf{grad}}_h} V_h^2$$

where $\widetilde{\mathrm{curl}}_h$ is the adjoint of $\mathbf{curl}_h$ and can thus be seen as a weak approximation of curl and analogous for $\widetilde{\mathbf{grad}}_h$ and div. As in the continuous case, we assume that $\dim \mathfrak{H}_h^1 = 1$ which is not unreasonable thanks to Thm. 3.2.8 for $h > 0$ small enough.

For our domain, we assume from now on that Ω is suitable for discretization in the sense that the functions in the discrete spaces and the continuous ones are both defined on it. What that means exactly depends on the chosen discretization and we will explain it later when we go into more detail about the actual implementation (see Assumption 3.3.2).

The discretized version of the strong formulation of the magnetostatic problem (Problem 3.1.2) then states: Find $\mathbf{B}_h \in V_h^1$ s.t.

$$\widetilde{\mathrm{curl}}_h \mathbf{B}_h = J \text{ and } \mathrm{div}\, \mathbf{B}_h = 0$$

plus the additional curve integral constraint. If $J \notin V_h^0$ then this clearly does not have a solution. Note that the divergence is enforced strongly while the curl is only enforced weakly.

As explained in Sec. 3.1.4, we will substitute the curve integral constraint from Problem 3.1.2 with (3.1.8). This gives us the following discrete formulation. Choose $\mathbf{p}_h \in \mathfrak{H}_h^1$ s.t. $\|\mathbf{p}_h\| = 1$.

{prob:magnetos

Problem 3.2.11. Find $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned}
\langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle &= -\langle J, \tau_h \rangle && \forall \tau_h \in V_h^0, \\
\langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \mathrm{div}\, \mathbf{B}_h, \mathrm{div}\, \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle &= 0 && \forall \mathbf{v}_h \in V_h^1, \\
\mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle &= \mu C_1 && \forall \mu \in \mathbb{R}.
\end{aligned}$$

Here we assume for simplicity that $\mathbf{curl}\, \psi \in V_h^1$. Since we can choose ψ and we assume to be on a discretizable domain this is not unreasonable. In practice, $\mathbf{p}_h$ is computed numerically before assembling the system.

We define $X_h := V_h^0 \times V_h^1 \times \mathbb{R}$. Note that this trial and test space is indeed conforming i.e. $X_h \subseteq X$, but we choose the discrete bilinear form $a_h : X_h \times X_h \to \mathbb{R}$,

$$\begin{aligned}
&a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) \\
&= \langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle + \langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \mathrm{div}\, \mathbf{B}_h, \mathrm{div}\, \mathbf{v}_h \rangle \\
&\quad + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle
\end{aligned}$$

with $\mathbf{p}_h \in \mathfrak{H}_h^1$, so the resulting bilinear forms are different since we have $\mathbf{p}_h$ instead of $\mathbf{p}$. We can then write the discrete problem in standard form: Find $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ s.t.

$$a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) = \ell(\tau_h, \mathbf{v}_h, \mu) \quad \forall (\tau_h, \mathbf{v}_h, \mu) \in X_h.$$

For simplicity, we assume for the theoretical considerations that we can compute all inner products exactly and that C_1 is given exactly as well. That also means that the right hand side ℓ is the same for the continuous and discrete problem, i.e.

$$\ell(\tau_h, \mathbf{v}_h, \mu) = -\langle J, \tau_h \rangle - \mu C_1.$$

Because we have transfered the continuous structures to the discrete case we can apply the same arguments as in Sec. 3.1.5.

Theorem 3.2.12 (Well-posedness of the discrete problem)**.** *For the following assumptions we always tacitly require that $h > 0$ is small enough. We assume that the Hilbert complex admits uniformly V-bounded cochain projections Π_h i.e. there exists $c_\Pi > 0$ independent of h s.t. $\|\Pi_h\|_{\mathcal{L}(V,V)} \leq c_\Pi$. We assume that $\mathbf{curl}\,\psi \in V_h^1$. Then we find $\psi_{0,h} \in V_h^0$ and $c_{\psi,h} > 0$ s.t. $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_{0,h} + c_{\psi,h}\mathbf{p}_h$ where $c_{\psi,h}$ does not depend on the choice of ψ. We also assume that (3.2.7) holds and thus $\dim \mathfrak{H}_h^1 = \dim \mathfrak{H}^1 = 1$ and we choose $\mathbf{p}_h$ according to Prop. 3.2.7. Then the discrete variational problem (Problem 3.2.11) is well-posed, i.e. there exists a unique solution $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ and we have the stability estimate*

$$\|\mathbf{B}_h\| \leq \frac{\|J\| + |C_1|}{\gamma_h}.$$

where γ_h has the same expression as γ except $c_{P,h}$ instead of c_P and $c_{\psi,h}$ instead of c_ψ. Furthermore if $c_{P,h} \to c_P$ then $\gamma_h \to \gamma$ for $h \to 0$.

Proof. By following the exact same arguments as in Sec. 3.1.5, we can prove the well-posedness through the BNB-theorem. We also get that $c_{\psi,h}$ does not depend on the choice of ψ by following the same argument as in Prop. 3.1.13 using the discrete spaces. However, we have to argue why $c_{\psi,h} > 0$ if $c_\psi > 0$. Notice since $\|\mathbf{p}_h\| = \|\mathbf{p}\| = 1$ and $\dim \mathfrak{H}^1 = \dim \mathfrak{H}_h^1 = 1$ we have

$$c_\psi = \langle \mathbf{p}, \mathbf{curl}\,\psi \rangle \text{ and } c_{\psi,h} = \langle \mathbf{p}_h, \mathbf{curl}\,\psi \rangle.$$

So

$$|c_{\psi,h} - c_\psi| = |\langle \mathbf{p} - \mathbf{p}_h, \mathbf{curl}\,\psi \rangle| \leq \|\mathbf{curl}\,\psi\| \|\mathbf{p} - \mathbf{p}_h\|.$$

and, because we chose $\mathbf{p}_h$ as described in Prop. 3.2.7 and assume Π_h to be uniformly bounded, we have $\|\mathbf{p} - \mathbf{p}_h\| \to 0$ as $h \to 0$ and thus we obtain $c_{\psi,h} \to c_\psi$ for $h \to 0$ and we can assume $c_{\psi,h} > 0$ for h small enough.

The next question is why we can choose c_1 for the discrete case just as in the continuous one. Remember that $c_1 > 0$ was chosen s.t. $\|\mathbf{curl}\,\psi_0\| \le c_1$. Choose now w.l.o.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$. Then it would be clear, that we can choose the same c_1 if $\|\mathbf{curl}\,\psi_{0,h}\| \to \|\mathbf{curl}\,\psi_0\|$. This is indeed true:

$$\begin{aligned}&\big|\|\mathbf{curl}\,\psi_{0,h}\| - \|\mathbf{curl}\,\psi_0\|\big| \le \|\mathbf{curl}\,\psi_{0,h} - \mathbf{curl}\,\psi_0\| = \|P_{\mathfrak{B}^1_h}\,\mathbf{curl}\,\psi - P_{\mathfrak{B}^1}\,\mathbf{curl}\,\psi\| \\ &\le \|P_{\mathfrak{B}^1_h} - P_{\mathfrak{B}^1}\|\|\mathbf{curl}\,\psi\| = \operatorname{gap}(\mathfrak{B}^1_h, \mathfrak{B}^1)\|\mathbf{curl}\,\psi\| \to 0\end{aligned}$$

where we use Prop. 3.2.9 combined with the uniform boundedness of Π_h to obtain convergence.

If $c_{P,h} \to c_P$ then $\gamma_h \to \gamma$ is clear because γ depends continuously on c_ψ and c_P. $\square$

Finally, we want to use the inf-sup condition to derive an a-priori error estimate. In order to do so, we have to consider the fact that the bilinear forms are different in the discrete and continuous case. So we will use the following lemma.

{lem:a_priori_

Lemma 3.2.13. *Let $x \in X$ be a solution of a general variational problem of the form (3.1.15) and $x_h \in X_h$ be a solution of the discretized version, i.e for $X_h \subseteq X$ and $Y_h \subseteq Y$ finite-dimensional subspaces*

$$a_h(x_h, y_h) = \ell(y_h) \quad \forall y_h \in Y_h.$$

Assume that an inf-sup condition holds for the discrete problem with constant γ_h. Define $\delta_h(x) \in Y'$ as

$$\langle \delta_h(x), y\rangle_{Y' \times Y} := a(x,y) - a_h(x,y).$$

and assume

$$\|a_h\| := \sup_{x \in X, y \in Y} \frac{a_h(x,y)}{\|x\|_X\,\|y\|_Y} < \infty.$$

Then

$$\|x - x_h\|_X \le \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in X_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

Proof. Take $z_h \in X_h$ arbitrary. Then with the triangular inequality

$$\|x - x_h\|_X \le \|x - z_h\|_X + \|x_h - z_h\|_X. \tag{3.2.8}$$ {eq:a_priori:t

We now have to bound the last term on the right hand side. Assume w.l.o.g. that $x_h - z_h$ is not zero. Then from the inf-sup condition we can find $y_h \in Y_h \setminus \{0\}$ s.t.

$$\begin{aligned}
&\gamma_h \|x_h - z_h\|_X \|y_h\|_Y \leq a_h(x_h - z_h, y_h) \\
&= a_h(x - z_h, y_h) + a_h(x_h, y_h) - a(x, y_h) + a(x, y_h) - a_h(x, y_h) \\
&= a_h(x - z_h, y_h) + \langle \delta_h(x), y_h \rangle_{Y' \times Y} \\
&\leq \|a_h\| \|x - z_h\|_X \|y_h\|_Y + \|\delta_h(x)\|_{Y'} \|y_h\|_Y
\end{aligned}$$

In the third step, we used the fact that x and x_h are solutions and the discrete problem has the same right hand side as the continuous one. So we can bound $\|x_h - z_h\|_X$ by

$$\|x_h - z_h\|_X \leq \frac{\|a_h\|}{\gamma_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

Plugging this into (3.2.8) and taking the infimum over $z_h \in X_h$, we get

$$\|x - x_h\| \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in V_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

□

We now have to apply this lemma to the magnetostatic formulation.

Theorem 3.2.14 (Quasi optimal a-priori estimate)**.** *Let $(\sigma, \mathbf{B}, \lambda) \in X$ be the exact solution of Problem 3.1.6 and $(\sigma_h, \mathbf{B}_h, \lambda_h) \in X_h$ the solution of the discrete Problem 3.2.11. Then*

$$\|\mathbf{B} - \mathbf{B}_h\| \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{\mathbf{z}_h \in V_h^1} \|\mathbf{B} - \mathbf{z}_h\|_{H(\mathrm{div})}$$

Proof. At first recall that if $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$, $\lambda = 0$ and $\operatorname{div} \mathbf{B} = 0$. So $\|(\sigma, \mathbf{B}, \lambda)\|_X = \|\mathbf{B}\|$ and analogous for $(\sigma_h, \mathbf{B}_h, \lambda_h)$. Also recognize then for any $y = (\tau, \mathbf{v}, \mu) \in X$

$$\langle \delta_h(x), y \rangle = \lambda \langle \mathbf{p}, \mathbf{v} \rangle - \lambda \langle \mathbf{p}_h, \mathbf{v} \rangle = 0.$$

Thus the estimate follows immediately from Lemma 3.2.13. □

One more thing to investigate is the substituted curve integral constraint in the discrete setting. For all the analysis above, we fixed ψ before which raises the question if the choice of ψ matters. We know that in the continuous case it does not (cf. Prop. 3.1.4 and the explanation before). It turns out that the analogous result holds with the same argument if we replace $\mathbf{B}$ with a solution of the first two equations of the discrete formulation $\mathbf{B}_h$ and choose the discrete space V_h^0 instead of H_0^1 (recall that we assume $\psi \in V_h^0$). We use the discrete formulation of the problem, i.e. the first two equations of Problem 3.2.11, and obtain

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}_h\, dx = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\, \tilde{\psi} \cdot \mathbf{B}_h\, dx \tag{3.2.9}$$

and thus an equivalent system by choosing

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0.$$

Here we always assumed that we can compute all the inner products exactly. In summary, the choice of ψ does not matter for the discrete system either.

Another important property of our new curve integral constraint was the invariance if $J = 0$, i.e. if we have two different curves Γ_1 and Γ_2 with ψ_1 and ψ_2 then we would like

$$\langle \mathbf{curl}\, \psi_1, \mathbf{B}_h \rangle = \langle \mathbf{curl}\, \psi_2, \mathbf{B}_h \rangle$$

to hold. This was true in the continuous case (see Prop. 3.1.5) and it is also true for the discrete problem. Again, the argument is completely identical by using a solution $\mathbf{B}_h$ of the discrete system (Prob. 3.2.11) instead and V_h^0 instead of H_0^1.

3.3 Implementation on a single patch domain

{sec:implement.

We start with a very short introduction on Splines and their tensor product space which will be our choice of basis on the reference domain. Then we choose our degrees of freedom and basis dual to them. We will work at first on a reference domain and then transfer it to the physical domain using pushforwards. In the end, we will explain how the homogeneous boundary conditions are enforced and state the final system to be solved.

3.3.1 Splines

For the finite element spaces, we will use the pushforwards of tensor product splines defined on a rectangular reference domain $\hat{\Omega}$. This section is a recollection of [11, Sec. 4.2] since we use the same method as presented in this paper. We start with a very brief introduction of B-Splines and the spaces of tensor products of these.

We choose a knot sequence $\boldsymbol{\xi} = (\xi_i)_{i=0}^{n+p}$ with $\xi_0 \le \xi_1 \le ... \le \xi_{n+p}$. B-Splines B_i^p of degree $p \ge 0$ for $i = 0, ..., n-1$ are defined recursively as

$$B_i^p(x) := \frac{x - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(x) + \frac{\xi_{i+p+1} - x}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(x)$$

and

$$B_i^0(x) := \begin{cases} 1, & \text{if } x \in [\xi_i, \xi_{i+1}), \\ 0, & \text{else.} \end{cases}$$

We choose two types of sequences, periodic and non-periodic ones. On an interval $[a, b]$, let $a = x_0 < x_1 < ... < x_N = b$ be our grid. We will stick to the equidistant case i.e. h will be our grid size $x_{i+1} - x_i$. For the non-periodic case, we choose an *open* knot sequence by $\xi_0 = ... = \xi_p = x_0$, $\xi_{p+k} = x_k$ for $k = 0, 1, ..., N$ and $\xi_n = \xi_{n+1} = ... = \xi_{n+p} = x_N$, i.e.

$$\boldsymbol{\xi} = (\underbrace{x_0, x_0, ..., x_0}_{p+1 \text{ times}}, x_1, x_2, ..., x_{N-1}, \underbrace{x_N, x_N, ..., x_N}_{p+1 \text{ times}})$$

and for the periodic case $\xi_0 = x_0 - ph, \xi_1 = x_0 - (p-1)h, ..., \xi_p = x_0, \xi_{p+k} = x_k$ for $k = 0, ..., n$ and $\xi_{n+k} = x_N + kh$ for $k = 0, ..., p$,

$$\begin{aligned} \boldsymbol{\xi} = (&x_0 - ph, x_0 - (p-1)h, ..., x_0 - h, x_0, x_1, x_2, ..., x_{N-1}, x_N, \\ &x_N + h, x_N + 2h, ..., x_N + ph). \end{aligned}$$

In the periodic case, the splines at the beginning and end of the interval are extended periodically. In both cases, we have $n + p + 1 = N + 2p + 1$ knots in our knot sequence $\boldsymbol{\xi}$ and hence obtain $n = N + p$ splines. We then define the spline space $\mathbb{S}^p(\boldsymbol{\xi}) := \text{span}\{B_i^p\}_{i=0}^{n-1}$ which has dimension $N + p$.

Note that all the knot multiplicities in the interior are one and thus our spline space has maximal regularity which implies that it is equal to the piecewise polynomial space

$$\mathbb{S}^p(\boldsymbol{\xi}) = \{v \in C^{p-1} \mid v|_{[x_j, x_{j+1})} \in \mathbb{P}_p([x_j, x_{j+1})) \text{ for } j \in [N-1]\}.$$

where $\mathbb{P}_p([x_j, x_{j+1}))$ are the polynomials of degree p on $[x_j, x_{j+1})$ and we use the standard notation $[K] = \{0, 1, ..., K\}$ for $K \in \mathbb{N}$.

This construction can now be generalized to 2D by using a tensor product approach. Denote $a_1 = x_0 < x_1 < x_2 < ... < x_N = b_1$ the grid in x-direction and $a_2 = y_0 < y_1 < y_2 < ... < y_N = b_2$ in y-direction and assume that we have the same number of grid points in both dimensions for simplicity. Using the same definition of knot sequence as above gives us the knot sequence $\boldsymbol{\xi}$ in x-direction and $\boldsymbol{\eta}$ in y-direction. The resulting splines are denoted as $B^{q_1}_{i,\xi}$, $i = 0, ..., n_1 = N + q_1$, and $B^{q_2}_{j,\eta}$, $j = 0, ..., N + q_2$, and the spaces $\mathbb{S}^{q_1}(\boldsymbol{\xi})$ and $\mathbb{S}^{q_2}(\boldsymbol{\eta})$ respectively.

We use the notation with $\mathbf{q} \in \{p-1, p\}^2$, $n_1 = N + q_1$, $n_2 = N + q_2$ and we define for $\mathbf{i} \in [n_1 - 1] \times [n_2 - 1]$ the *tensor product spline*

$$B^{\mathbf{q}}_{\mathbf{i}}(x, y) = B^{q_1}_{i_1,\xi}(x) B^{q_2}_{i_2,\eta}(y)$$

and $\mathbb{S}^{\mathbf{q}}(\xi, \eta)$ the span of these tensor product splines. We will from now on leave out the reference to the knot sequences and assume them to be fixed. The spline spaces used in the tensor product can also be periodic or be periodic in one direction and non-periodic in the other which will the case in our implementation later.

Then we obtain the following discrete Hilbert complex on our reference domain $\hat{\Omega}$

$$\mathbb{S}^{p,p} \xrightarrow{\mathbf{curl}} \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \xrightarrow{\operatorname{div}} \mathbb{S}^{p-1,p-1}$$

and we denote

$$\hat{V}^0_h = \mathbb{S}^{p,p}, \ \hat{V}^1_h = \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \text{ and } \hat{V}^2_h = \mathbb{S}^{p-1,p-1}. \tag{3.3.1}$$ {eq:discrete_s

It is well-known that if we have a function $\mathbf{v}$ that is piecewise smooth then $\mathbf{v} \in H(\operatorname{div})$ i.i.f. the normal trace across element interfaces agrees. Analogously, a piecewise smooth $\tau \in H^1$ i.i.f. the values agree on the interfaces. We will always assume $p \geq 2$ and thus know that all our tensor splines are at least continuous globally and so $\hat{V}^0_h \subseteq H^1(\hat{\Omega})$ and $\hat{V}^1_h \subseteq H(\operatorname{div}, \hat{\Omega})$ as desired.

3.3.2 Basis and degrees of freedom

{sec:basis_and_

Let us now investigate the degrees of freedom and the corresponding basis functions. At first, we will define these only on the reference domain $\hat{\Omega}$. Then we will use pullbacks and pushforwards to transfer them from the reference domain to the physical domain Ω. We will also use this opportunity to specify more precisely, what domains we consider such that this approach works.

We now fix p and get $n = N + p$. Here it is crucial to remember that we assumed to have the same number of grid points in both dimensions. We will use geometric degrees of freedom, i.e. each degree of freedom can be associated with some geometrical element of our grid. We define *Greville points* by

$$\zeta^x_i := \frac{\xi_{i+1} + ... + \xi_{i+p}}{p},$$

i.e. the knot averages for $i = 0, ..., n-1$ and analogous ζ_j^y in y-direction using the knot sequence $\boldsymbol{\eta}$. Then the spline interpolation at these points is well-defined (see [16, Sec. 3.3.1]). Note that in the periodic case some Greville points lie outside of the grid, but we assume the function that is interpolated to be periodic as well, so the values at the Greville points are well-defined.

This gives us the following geometric elements nodes, edges and cells

$$\begin{aligned}
\hat{\mathtt{n}}_{\mathbf{i}} &:= (\zeta_{i_1}^x, \zeta_{i_2}^y), \quad \mathbf{i} \in \mathcal{M}^0 \\
\hat{\mathtt{e}}_{d,\mathbf{i}} &:= [\hat{\mathtt{n}}_{\mathbf{i}}, \hat{\mathtt{n}}_{\mathbf{i}+\mathbf{e}_d}], \quad (d, \mathbf{i}) \in \mathcal{M}^1 \\
\hat{\mathtt{c}}_{\mathbf{i}} &:= [\hat{\mathtt{e}}_{1,\mathbf{i}}, \hat{\mathtt{e}}_{1,\mathbf{i}+\mathbf{e}_1}] = [\zeta_{i_1}^x, \zeta_{i_1+1}^x] \times [\zeta_{i_2}^y, \zeta_{i_2+1}^y], \quad \mathbf{i} \in \mathcal{M}^2
\end{aligned}$$

with $[\cdot]$ being the convex hull. As before, $\mathbf{e}_d$ for $d = 1, 2$ is the standard basis vector of $\mathbb{R}^2$. The set of multiindices are defined as

$$\begin{aligned}
\mathcal{M}^0 &:= [n-1]^2 \\
\mathcal{M}^1 &:= \{(d, \mathbf{i}) \mid d \in \{1, 2\}, \mathbf{i} \in [0, n-1]^2, i_d < n-1\} \\
\mathcal{M}^2 &:= [n-2]^2
\end{aligned}$$

Now that we have defined the geometric elements we define the corresponding degrees of freedom. We define $\mathbf{e}_d^\perp$ as $\mathbf{R}_{\pi/2}\mathbf{e}_d$, i.e. the rotation by $\pi/2$ in counter clockwise direction, which leads to $\mathbf{e}_1^\perp = \mathbf{e}_2$ and $\mathbf{e}_2^\perp = -\mathbf{e}_1$. The degrees of freedom are then

$$\begin{aligned}
\hat{\sigma}_{\mathbf{i}}^0(v) &:= v(\hat{\mathtt{n}}_{\mathbf{i}}), \quad \mathbf{i} \in \mathcal{M}^0 \\
\hat{\sigma}_{d,\mathbf{i}}^1(\mathbf{v}) &:= \int_{\hat{\mathtt{e}}_{d,\mathbf{i}}} \mathbf{v} \cdot \mathbf{e}_d^\perp, \quad (d, \mathbf{i}) \in \mathcal{M}^1 \\
\hat{\sigma}_{\mathbf{i}}^2(v) &:= \int_{\hat{\mathtt{c}}_{\mathbf{i}}} v, \quad \mathbf{i} \in \mathcal{M}^2
\end{aligned}$$

These degrees of freedom are unisolvent on $\hat{V}_h^\ell$ i.e. with $N_\ell = |\mathcal{M}^\ell|$ and some ordering $\mu_0, \mu_1, ..., \mu_{N_\ell}$ of the indices of $\mathcal{M}^\ell$, denoting $\sigma_{\mu_i}^\ell = \sigma_i^\ell$, we define

$$\hat{\boldsymbol{\sigma}}^\ell := (\hat{\sigma}_0^\ell, \hat{\sigma}_1^\ell, ..., \hat{\sigma}_{N_\ell}^\ell)^\top : \hat{V}_h^\ell \to \mathbb{R}^{N_\ell},$$

which is bijective, and we can thus define our basis functions $\hat{\Lambda}_\mu^\ell$, $\mu \in \mathcal{M}^\ell$, (denoted in bold if vector valued) as the basis which is dual to the degrees of freedom in the sense

$$\hat{\sigma}_\mu^\ell(\hat{\Lambda}_\nu^\ell) = \delta_{\mu,\nu} \quad \forall \mu, \nu \in \mathcal{M}^\ell.$$

Remark 3.3.1. For implementational purposes, these basis functions dual to the degrees of freedom are not necessarily the best option. For the computation of mass matrices etc. it is more convenient to use the B-splines directly due to their local support and fast computation. We will not go too deeply into the details of implementation however. More details about the use of B-splines and the connection with the basis Λ^ℓ_μ can be found in [11, Sec. 4.8]

The question is now on what function spaces these degrees of freedom are defined. We note first that the standard choice with $\hat{V}^0 = H^1(\hat{\Omega})$, $\hat{V}^1 = H(\mathrm{div}, \hat{\Omega})$ and $\hat{V}^2 = L^2(\hat{\Omega})$ can not work because, e.g., the evaluation at point values is generally not well-defined for H^1-functions in 2D. Thus, we need to choose function spaces with higher regularity or integrability.

Let us define the spaces

$$\begin{aligned} W^1_{1,2}(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_1\partial_2 v \in L^1(\hat{\Omega})\} \\ W^1_d(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_d v \in L^1(\hat{\Omega})\} \text{ for } d \in \{1, 2\} \end{aligned}$$

and then denote $\hat{U}^0_{L^1} := W^1_{1,2}(\hat{\Omega})$, $\hat{U}^1_{L^1} := W^1_1(\hat{\Omega}) \times W^1_2(\hat{\Omega})$ and $\hat{U}^2_{L^1} := L^1(\hat{\Omega})$. Then the degrees of freedom σ^ℓ_μ are well-defined on $\hat{U}^\ell := \hat{U}^\ell_{L^1} \cap \hat{V}^\ell$.

We thus obtained the degrees of freedom and basis functions on the reference domain $\hat{\Omega}$. The idea to continue is now to define the basis functions on the physical domain Ω by a pushforward of the basis functions from the reference domain. We will now clarify what types of domain we will consider for the discretization. Then we will define the pushforward as the inverse of the pullback and apply it to the basis functions and degrees of freedom to transfer them to the physical domain.

{ass:discretiz

Assumption 3.3.2. There exists a rectangular reference domain $\hat{\Omega} = [a_1, b_1] \times [0, T]$, $a_1 < b_1$, $T > 0$ and an orientation preserving diffeomorphism $\mathbf{F}$ defined on $\hat{\Omega}$, which is T-periodic in the second variable, such that $\Omega = F(\hat{\Omega})$.

We have the tensor grid on the reference domain

$$\{a_1 = x_0 < x_1 < ... < x_N = b_1\} \times \{0 = y_0 < y_1 < ... < y_N = T\}.$$

For the curve Γ, we assume that there exists $\hat{\Gamma} \subseteq \hat{\Omega}$ that goes along the edges of the grid such that $\Gamma = \mathbf{F}(\hat{\Gamma})$.

This is usually referred as the *single patch case* since we only use one mapping from the reference domain. We see that this assumption is made so that notions from the physical domain are well-represented on the reference domain. Notice that under this assumption the preimage of Ω_Γ corresponds to a subgrid on the reference domain.

We already introduced the pullback in Prop. 2.1.31 in the 3D case. Now, we define the pullbacks in 2D

$$\begin{aligned}
&\mathcal{P}^0_{\mathbf{F}} : v \mapsto \hat{v} := v \circ \mathbf{F} \\
&\mathcal{P}^1_{\mathbf{F}} : \mathbf{v} \mapsto \hat{\mathbf{v}} := (\det D\mathbf{F})\, D\mathbf{F}^{-1}(\mathbf{v} \circ \mathbf{F}) \\
&\mathcal{P}^2_{\mathbf{F}} : v \mapsto \hat{v} := (\det D\mathbf{F})(v \circ \mathbf{F})
\end{aligned}$$

which map functions on the physical domain Ω to functions on the reference domain $\hat{\Omega}$. Then we have the commuting properties

$$\begin{aligned}
\widehat{\mathbf{curl}}\mathcal{P}^0_{\mathbf{F}} v &= \mathcal{P}^1_{\mathbf{F}}\, \mathbf{curl}\, v \\
\widehat{\operatorname{div}}\mathcal{P}^1_{\mathbf{F}} \mathbf{v} &= \mathcal{P}^2_{\mathbf{F}} \operatorname{div} \mathbf{v}
\end{aligned}$$

when the $\mathbf{curl}$ and divergence are well-defined.

Remark 3.3.3. Just as in Sec. 2.1.3, these pullbacks can be derived from the pullbacks of differential forms. Keep in mind however, that in 2D there are two different ways to identify a 1-form with a vector field (cf. Remark 2.1.29). This leads to different pullbacks. We have chosen it here s.t. we have the commuting property with the differential operators in our Hilbert complex (3.1.3).

We now define the pushforwards as $\mathcal{F}^\ell := (\mathcal{P}^\ell_{\mathbf{F}})^{-1}$ which then read

$$\begin{aligned}
&\mathcal{F}^0 : \hat{v} \mapsto v := \hat{v} \circ \mathbf{F}^{-1} \\
&\mathcal{F}^1 : \hat{\mathbf{v}} \mapsto \mathbf{v} := (\det D\mathbf{F}^{-1})\, D\mathbf{F}(\hat{\mathbf{v}} \circ \mathbf{F}^{-1}) \\
&\mathcal{F}^2 : \hat{v} \mapsto v := (\det D\mathbf{F}^{-1})(\hat{v} \circ \mathbf{F}^{-1}).
\end{aligned}$$

By applying these pushforwards we get the basis functions on the physical domain

$$\Lambda^\ell_\mu := \mathcal{F}^\ell \hat{\Lambda}^\ell_\mu$$

and then

$$\bar{V}^\ell_h := \operatorname{span}\{\Lambda^\ell_\mu \mid \mu \in \mathcal{M}^\ell\}$$

are our discrete spaces without boundary conditions.

Using the geometric degrees of freedom on the reference domain, we can now construct the corresponding degrees of freedom on the physical domain as

$$\sigma^\ell_\mu := \hat{\sigma}^\ell_\mu \circ \mathcal{P}^\ell_F$$

Then we have by construction $\sigma^\ell_\mu(\Lambda^\ell_\nu) = \delta_{\mu,\nu}$ for all $\mu, \nu \in \mathcal{M}^\ell$.

The degrees of freedom then also correspond to geometric objects in the physical domain. The reference grid gets mapped to the physical domain which gives us the geometric objects there. We define the mapped nodes in the physical domain as

$$\mathtt{n}_{\mathbf{i}} := \mathbf{F}(\hat{\mathtt{n}}_{\mathbf{i}}),$$

and analogous edges $\mathtt{e}_{d,\mathbf{i}}$, $(d, \mathbf{i}) \in \mathcal{M}^1$, and cells $\mathtt{c}_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{M}^2$. σ^0 corresponds to point values at the nodes, σ^1 to the fluxes through the edges and σ^2 to the integral over the mapped cells.

Remark 3.3.4. If we would take the now obvious choice of $v \mapsto \sum_{\mu \in \mathcal{M}^0} \sigma^0_\mu(v)\Lambda^0_\mu \in \bar{V}^0_h$ and analogous for the other function spaces, the result would be a cochain projection, but it would not be bounded and thus insufficient for the theory introduced in Sec. 3.2.1.

However, there are L^2-stable quasi-interpolants which also commute with the differential operators (see [6, Sec. 4]) and provide us with the stable cochain projections that we need for the theory.

3.3.3 Building the discrete system

{sec:assemblin

Now that we have specified the basis and degrees of freedom, we will talk about how we will build the system to solve the magnetostatic problem on Ω. In particular, we will explain how we will enforce homogeneous boundary conditions since this is the way the magnetostatic problem was posed. To achieve that, we define the boundary projections and reformulate the discrete problem with them. When this is done, we will write the equations in matrix form to see what system is solved in practice.

Recall that $\mathbf{F}$, the parametrization of the physical domain, is assumed to be periodic in the second variable and so the "north" and "south" edge of the rectangle $\hat{\Omega}$ do not correspond to a boundary of the physical domain Ω. This motivates the (slight abuse of) notation

$$\partial\hat{\Omega} := \{a, b\} \times [0, T].$$

Let us define the sets of boundary indices $\mathcal{B}^\ell$ which correspond to the multi-indices of the geometric elements on the boundary, i.e.

$$\begin{aligned} \mathcal{B}^0 &:= \{\mathbf{i} \in \mathcal{M}^0 \mid \hat{\mathtt{n}}_{\mathbf{i}} \in \partial\hat{\Omega}\}, \\ \mathcal{B}^1 &:= \{(d, \mathbf{i}) \in \mathcal{M}^1 \mid \hat{\mathtt{e}}_{d,\mathbf{i}} \subseteq \partial\hat{\Omega}\}. \end{aligned}$$

Since $V_h^2 \subseteq L^2$ we do not require any boundary conditions there. For $\ell = 1, 2$, we define the spaces

$$V_h^\ell := \{v_h \in \bar{V}_h^\ell \mid \sigma_\mu^\ell(v_h) = 0 \quad \forall \mu \in \mathcal{B}^\ell\} = \operatorname{span}\{\Lambda_\mu^\ell \mid \mu \in \mathcal{M}^\ell \setminus \mathcal{B}^\ell\}.$$

For these spaces it holds then

$$\begin{aligned} V_h^0 &= \bar{V}_h^0 \cap H_0^1(\Omega) \\ V_h^1 &= \bar{V}_h^1 \cap H_0(\operatorname{div}, \Omega) \end{aligned}$$

and so we see that the homogeneous boundary conditions simply correspond to the boundary degrees of freedom being zero. This means for V_h^0 we require the nodal values on the boundary to vanish and for V_h^1 zero flux through the boundary edges.

This motivates us to define the projections $P_h^\ell : \bar{V}_h^\ell \to \bar{V}_h^\ell$ which set the boundary degrees of freedom to zero and are thus a projection onto V_h^ℓ. The matrix representation $\mathbb{P}^\ell$ is then simply $(\mathbb{P}^\ell)_{\mu,\nu} = 1$ i.i.f. $\mu = \nu$ and μ does not correspond to a geometric element on the boundary. They are easily constructed by taking the identity matrix and setting the diagonal entries to zero that belong to boundary degrees of freedom. These matrices are obviously symmetric.

We now reformulate the discrete system using these projections. We apply the boundary projections to all functions involved and then add the boundary penalization terms. With boundary penalities the discrete system is: Find $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned} \langle (I - P_h^0)\sigma_h, (I - P_h^0)\tau_h\rangle + \langle P_h^0\sigma_h, P_h^0\tau_h\rangle & \\ - \langle P_h^1\mathbf{B}_h, \mathbf{curl}\, P_h^0\tau_h\rangle &= -\langle J, P_h^0\tau_h\rangle \quad \forall \tau_h \in \bar{V}_h^0, \end{aligned} \tag{3.3.2}$$

{eq:system_wit

$$\begin{aligned} \langle \mathbf{curl}\, P_h^0\sigma_h, P_h^1\mathbf{v}_h\rangle + \langle (I - P_h^1)\mathbf{B}_h, (I - P_h^1)\mathbf{v}_h\rangle & \\ + \langle \operatorname{div} P_h^1\mathbf{B}_h, \operatorname{div} P_h^1\mathbf{v}_h\rangle + \langle \lambda\mathbf{p}_h, P_h^1\mathbf{v}_h\rangle &= 0 \quad \forall \mathbf{v}_h \in \bar{V}_h^1, \end{aligned} \tag{3.3.3}$$

{eq:system_wit

$$\langle \mathbf{curl}\, \psi, P_h^1\mathbf{B}_h\rangle = C_1. \tag{3.3.4}$$

{eq:system_wit

Since we apply the projection everywhere it is then easy to show that $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ solve (3.3.2)-(3.3.4) i.i.f. $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $(\sigma_h, \mathbf{B}_h, \lambda)$ solves the system with homogeneous discrete spaces (Prob. 3.2.11). So the two formulations are equivalent.

We will from now on often use implicitly some flattening of the multiindices in $\mathcal{M}^\ell$. Define $\mathbb{M}^\ell$ as the mass matrix of $\bar{V}_h^\ell$ i.e. $\mathbb{M}_{ij}^\ell = \langle \Lambda_i^\ell, \Lambda_j^\ell\rangle$. We define the matrix $\mathbb{D}$ as the matrix representation of the divergence restricted to $\bar{V}_h^1$, i.e. $\operatorname{div}|_{\bar{V}_h^1} : \bar{V}_h^1 \to \bar{V}_h^2$. Analogously $\mathbb{C}$ is the matrix representation of $\mathbf{curl}$. Then we have, as mentioned before, the matrix representation of the boundary projections

$\mathbb{P}^\ell$ and $\mathbb{I}^\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ is the identity matrix. We denote the vector of coefficients of a function with an underline, e.g. $\underline{\sigma} \in \mathbb{R}^{N_0}$ is the vector of coefficients of σ in the basis Λ^0_μ, $\mu \in \mathcal{M}^0$. $\underline{\mathbf{B}}, \underline{\mathbf{p}} \in \mathbb{R}^{N_1}$ are the coefficients of $\mathbf{B}_h$ and $\mathbf{p}_h$ in the basis $\mathbf{\Lambda}^1_\mu$, $\mu \in \mathcal{M}^1$. So rewriting the equations (3.3.2)-(3.3.4) in matrix-vector form gives us

$$\underline{\tau}^\top (\mathbb{I}^0 - \mathbb{P}^0)\mathbb{M}^0(\mathbb{I}^0 - \mathbb{P}^0)\underline{\sigma} + \underline{\tau}^\top \mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \underline{\tau}^\top \mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \underline{\tau}^\top \underline{\tilde{J}} \quad \forall \underline{\tau} \in \mathbb{R}^{N_0}$$
$$\underline{\mathbf{v}}^\top \mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + \underline{\mathbf{v}}^\top (\mathbb{I}^1 - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1 - \mathbb{P}^1)\underline{\mathbf{B}}$$
$$+\underline{\mathbf{v}}^\top \mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \underline{\mathbf{v}}^\top \mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0 \quad \forall \underline{\mathbf{v}} \in \mathbb{R}^{N_1}$$
$$\underline{\psi}^\top \mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1$$

where $\underline{\tilde{J}} = (\langle J, \Lambda^0_i\rangle)_{i=1}^{N_0}$ which gives us the final system to be solved

$$(\mathbb{I}^0 - \mathbb{P}^0)^\top\mathbb{M}^0(\mathbb{I}^0 - \mathbb{P}^0)\underline{\sigma} + \mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \underline{\tilde{J}}$$
$$\mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + (\mathbb{I} - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I} - \mathbb{P}^1)\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0$$
$$\underline{\psi}^\top \mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1$$

Now we will explain how the discrete harmonic form is computed. In order to do that, we will characterize harmonic forms in a different way.

Proposition 3.3.5. *Define the penalized discrete Hodge Laplacian operator* $\bar{L}^1_h : \bar{V}^1_h \to \bar{V}^1_h$ *as*

$$\bar{L}^1_h = -\widetilde{\mathbf{grad}}_h \operatorname{div} P^1_h + \widetilde{\operatorname{curl}}_h \mathbf{curl}\, P^0_h + (I - P^1_h)^*(I - P^1_h)$$

where $\widetilde{\mathbf{grad}}_h$ *and* $\widetilde{\operatorname{curl}}_h$ *are the adjoints of* $\operatorname{div} P^1_h$ *and* $\mathbf{curl}\, P^0_h$ *respectively. Then*

$$\ker \bar{L}^1_h = \mathfrak{H}^1_h.$$

Note that the spaces in the discrete Hilbert complex (3.2.1) have homogeneous boundary conditions and thus this is also true for the spaces $\mathfrak{H}^j_h$ and $\mathfrak{B}^j_h$ and $\mathfrak{B}^*_{jh}$ by definition.

Proof. See [7, Thm. 3.2] □

Notice that for any linear operator $\phi : V \to W$ between finite dimensional inner product spaces V and W with matrix representation A, the matrix representation of the adjoint ϕ^* is $G_V^{-1}A^\top G_W$ where G_V and G_W are the Gramian matrices of the chosen bases in V and W respectively. So the penalized discrete Hodge Laplacian has then the matrix representation

$$(\mathbb{M}^1)^{-1}\mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1 + \mathbb{C}\mathbb{P}^0(\mathbb{M}^0)^{-1}\mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1 + (\mathbb{M}^1)^{-1}(\mathbb{I}^1 - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1 - \mathbb{P}^1)$$

and we compute the coefficients $\underline{\mathbf{p}}$ of the discrete harmonic form by computing an element of the kernel of this matrix. One can also multiply it with $\mathbb{M}^1$ from the left which does not change the kernel, but avoids having to compute $(\mathbb{M}^1)^{-1}$.

Remark 3.3.6. We acknowledge the fact that computing the inverse of the mass matrix $\mathbb{M}^0$ is a computational bottleneck of this implementation. The inverse of the mass matrix is in general dense and is thus problematic memory-wise. One way to avoid this would be to use multiple patches instead and then use a CONGA approach described in detail in [11]. Then the mass matrices are block diagonal and inverting them much less costly.

3.4 Numerical examples

{sec:numerical_

We will consider two numerical examples, the simulation of the induced magnetic field induced by the current through a wire, where the exact solution is given by the Biot-Savart law, and a manufactured solution. The Biot-Savart solution will be approximated on a standard annulus and a "distorted annulus" and the manufactured solution will be approximated on the standard annulus. In both cases, we will pose the problem with two different curves and investigate the convergence of the error.

3.4.1 Magnetic field induced current through wire

As a first simple numerical test, we consider a standard example from magnetostatics which is the magnetic field induced by a current through an infinitely long, straight wire with radius zero. The *Biot-Savart law* can be used to compute it. Let the wire be equal to the z-axis and I be the electrical current flowing through it. With $\boldsymbol{\ell}(s) = s\mathbf{e}_3$, $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$,

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\ell' \times (x - \ell(s))}{|x - \ell(s)|^3}\, ds = \frac{\mu_0 I}{4\pi |x|^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

For convenience, we pick now $I = \frac{2\pi}{\mu_0}$ to get

$$\mathbf{B}(x) = \frac{2}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

We will only focus on the first two components since our approximation is in 2D. We choose as our domain of computation Ω the annulus with inner radius 1 and outer radius 2. We will investigate two different curves. The curve Γ_1 will be the parametrization of the circle with radius 1.5 in anticlockwise direction, which

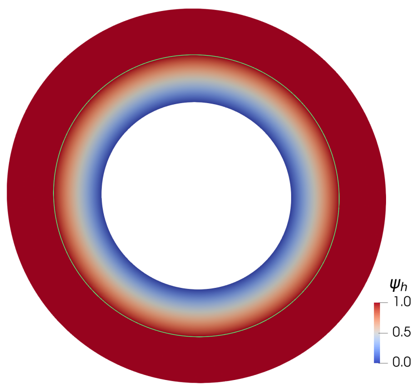

Figure 3.4.1: A possible choice of ψ on an annulus. Γ is here in the middle of the annulus (marked in green). $\psi = 0$ at the inner boundary and $\psi = 1$ at Γ and between Γ and the outer boundary {fig:psi_annul

gives us the curve integral

$$C_0 = \int_{\Gamma_1} \mathbf{B} \cdot d\ell = 4\pi.$$

Γ_2 will be the exterior boundary of the annulus, i.e. the circle with radius 2. Because $J = 0$ on our domain the curve integral along Γ_2 is the same and hence $C_1 = C_0$. The reference domain $\hat{\Omega} = [0,1] \times [0, 2\pi]$ and the mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} (\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

Then we choose ψ_1 as a simple interpolation from the inner boundary to the curve Γ, i.e. in logical coordinates

$$\psi(\hat{x}_1, \hat{x}_2) = \begin{cases} 2\hat{x}_1 & \text{for } \hat{x}_1 \leq 0.5 \\ 1 & \text{else.} \end{cases} \tag{3.4.1}$$ {eq:linear_int

This fulfills all the criteria we had for ψ_1 (cf. (3.1.6)) for Γ_1 (see Fig. 3.4.1).

For Γ_2, we choose ψ_2 as the solution of the Laplace problem with boundary conditions $\psi_2 = 0$ on $\partial\Omega_{in}$ and $\psi_2 = 1$ on $\partial\Omega_{out}$ (see Fig. 3.4.2)

For the implementation, we are using the PSYDAC library (`https://github.com/pyccel/psydac`) which is an open source Python 3 library for isogeometric analysis (see [12] for more details). See Fig. 3.4.3 for the solution where we chose 64 cells for our grid on the reference domain in both dimensions and $p = 2$. Note that $p = 2$ refers to the choice of our spaces from (3.3.1) and hence

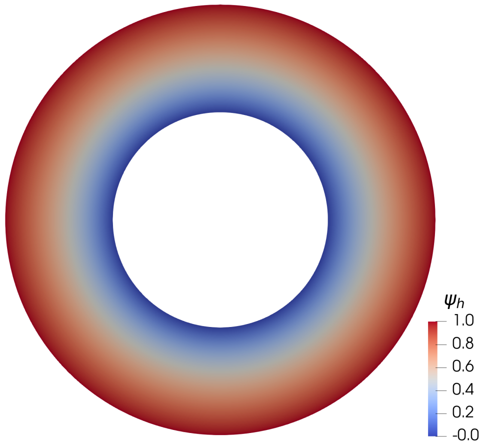

Figure 3.4.2: ψ as the solution of a Poisson problem with $\psi = 0$ on the interior and $\psi = 1$ on the exterior boundary. Γ is here equal to $\partial\Omega_{out}$. {fig:psi_poiss

the spline space for approximating the magnetic field will be $\mathbb{S}_{2,1} \times \mathbb{S}_{1,2}$ so the lower spline degree will be one. The pushforward spline space achieves then an approximation error of h^{p+1} [14, Ch.4, (4.48)] if the solution is at least H^{p+1}-regular. Thus the expected rate of convergence is two in our case which can be observed (see Fig. 3.4.4). As predicted by the theory (cf. Prop. 3.1.5), the errors for Γ_1 and Γ_2 are equal up to rounding errors.

Now we will change the domain where the problem is posed, but we will stick to the Biot-Savart setting. This means we still assume the wire goes through the origin in z-direction, which results in the same solution, but we will approximate it on a "distorted annulus" (see 3.4.5) using the same reference domain, but the different mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} 3(\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

Note that we do not have the homogeneous boundary condition $\mathbf{B} \cdot \mathbf{n} = 0$ for this domain anymore. Thus we have to deal with the boundary conditions by using a standard lifting approach which means we take an interpolation of the boundary conditions $\mathbf{B}_{h,g}$ which we compute before and then split $\mathbf{B}_h = \mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ with $\mathbf{B}_{h,0} \in V_h^1$. Then we substitute $P_h^1\mathbf{B}_h$ in (3.3.2)-(3.3.4) with $\mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ and put the terms with $\mathbf{B}_{h,g}$ on the right hand side. This leads to the same left hand side, but as right hand side we get $-\langle J, \tau_h\rangle + \langle \mathbf{B}_{h,g}, \mathbf{curl}\, P_h^0\tau_h\rangle$ in the first equation, $-\langle \operatorname{div} \mathbf{B}_{h,g}, \operatorname{div} P_h^1\mathbf{v}_h\rangle$ in the second and $C_1 - \langle \mathbf{curl}\,\psi, \mathbf{B}_{h,g}\rangle$ in the third. In practice, since we know the exact solution $\mathbf{B}$, we simply take $\mathbf{B}_{h,g} := (I - P_h^1)\mathbf{B}$. The remaining steps of assembling the system are completely analogous

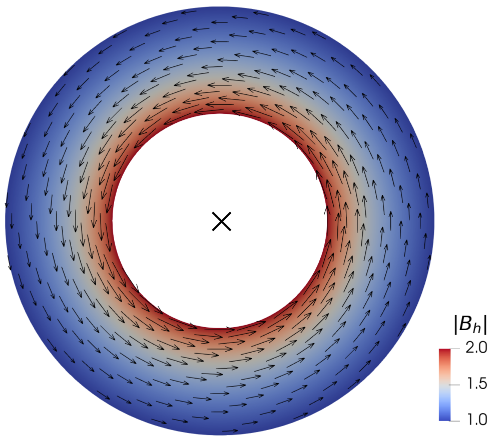

Figure 3.4.3: Solution of magnetostatic problem induced by current I through a wire on an annulus for 64 grid cells on the reference domain in both dimensions and spline degree $p = 2$. The wire goes through the origin in z-direction towards the point of view. {fig:biot_sava

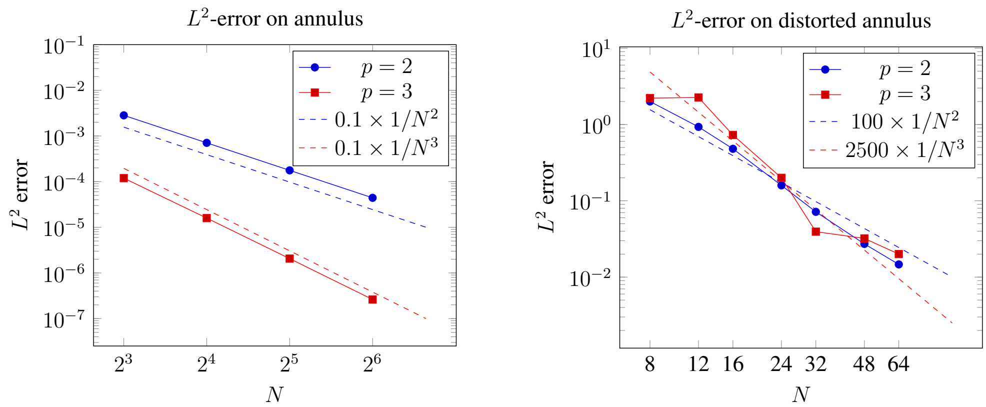

Figure 3.4.4: Convergence analysis for the approximation of the Biot-Savart solution with $p = 2$ and $p = 3$ on the annulus domain and the distorted annulus domain. For the solution with $p = 3$, the number of grid points in $\hat{x}_2$-direction is $N/2$ to avoid memory issues. {fig:convergen

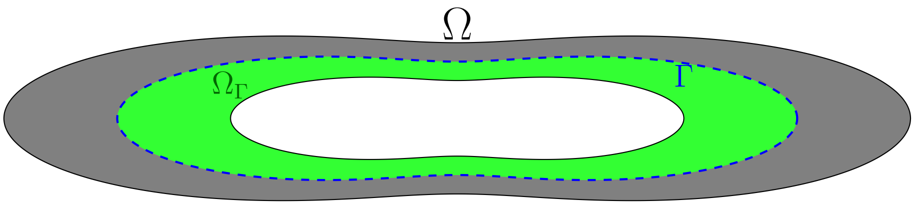

Figure 3.4.5: The distorted annulus domain {fig:distorted_

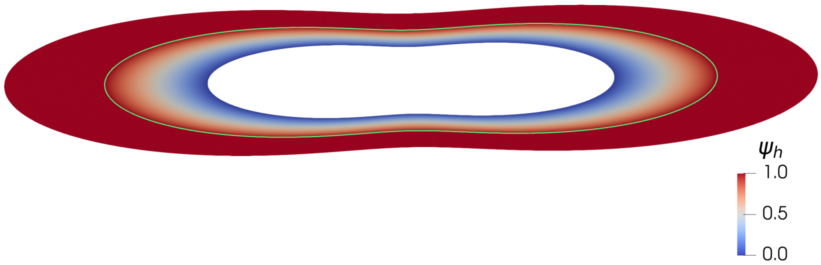

Figure 3.4.6: ψ on the distorted annulus domain. Γ (green) is again in the middle between the interior and exterior boundary. {fig:psi_disto

to Sec. 3.3.3. We solve this system to obtain $B_{h,0}$ and then add the boundary interpolation to obtain the solution.

For ψ, we use the exact same definition as in (3.4.1) which results in a different curve due to the new domain parametrization (see Fig. 3.4.6). J is of course still zero and the curve integral does not change either. We obtain again second order convergence for $p = 2$ (see Fig. 3.4.4), but the error is several orders of magnitude larger compared to the normal annulus.

3.4.2 Manufactured solution

As an example with a non-vanishing J, we will use a manufactured solution. Take $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$. It is easy to see that $\operatorname{div} \mathbf{B} = 0$ and $\mathbf{B} \cdot \mathbf{n} = 0$ for the annulus. This results in

$$J(x) = 4|x|^2 - 12|x| + 8.$$

We will pose this problem on the standard annulus domain from before with the same curves Γ_1 and Γ_2. Since J is not zero this leads to different curve integrals

$$\int_{\Gamma_1} \mathbf{B} \cdot d\ell = \frac{9\pi}{8} \quad \text{and} \quad \int_{\Gamma_2} \mathbf{B} \cdot d\ell = 0.$$

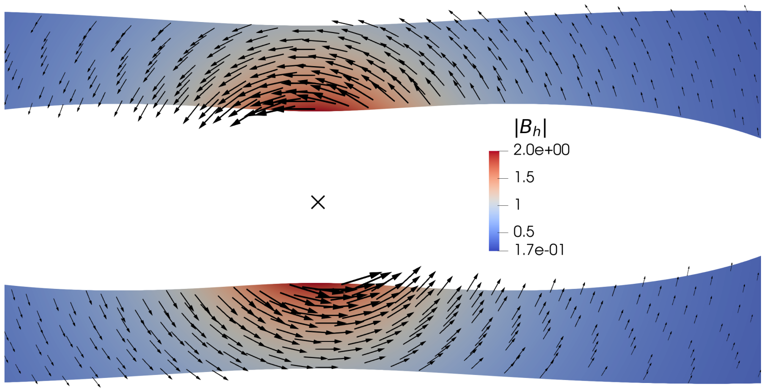

Figure 3.4.7: Solution of the Biot-Savart law on the distorted annulus with 64 cells in both dimensions and $p = 2$. The wire is in the centre pointing towards the point of view. {fig:distorted_

We choose ψ_1 and ψ_2 as for the Biot-Savart problem. We observe the same convergence rate (Fig. 3.4.8), but we recognize that the errors are slightly different and so the choice of curve integral and ψ matters.

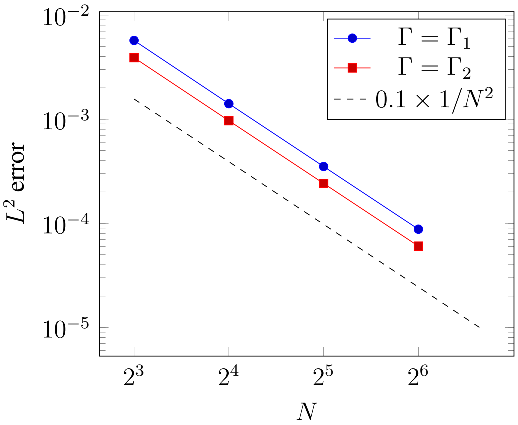

Figure 3.4.8: Convergence analysis for the manufactured solution $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$ with different choices of ψ corresponding to different curves Γ_1 as before. {fig:convergen

Chapter 4

Conclusion and Further Directions

{chap:conclusi

We developed lots of theoretical tools in the first chapter. We introduced differential forms, singular homology and Hilbert complexes to prove the existence and uniqueness of the magnetostatic problem on the exterior domain of a torus where we reformulated the problem using the assumption that the first homology group being generated by the homology class of the curve that we are integrating over. We utilized existence and uniqueness results on the level of homology and applied the de Rham isomorphism to obtain analogous results for differential forms. These were then used together with the Hodge decomposition to prove existence and uniqueness.

In the second part, we looked at the 2D magnetostatic problem with curve integral constraint. We derived a convenient variational formulation and proved well-posedness as well as an a-priori estimate. Numerical experiments confirmed that the method works as expected.

One immediate possibility to extend these results would be to look for solutions on domain with more complicated topologies and different topological constraints. For example, the first Betti number, i.e. the number of generators of the first homology group, could be increased and pose the problem with more curve integrals. This generalization should be straightforward.

We only showed the existence and uniqueness of the homogeneous magnetostatic problem with $\mathbf{J} = 0$. Of course, the uniqueness for $\mathbf{J} \neq 0$ follows from that immediately because the problem is linear. With the existence however, one has to be careful. The main issue is the application of the regularity results, but this would be possible if the assumptions on $\mathbf{J}$ are chosen carefully.

Another way to go would be to generalize the magnetostatic problem itself and pose it to find a differential form with exterior derivative and codifferential (see [2, Sec. 6.2.6]) equal to zero. This has been done, see e.g. [21, Chap. 3]. One would have to be careful then to obtain the regularity and density results that we relied on which will most likely come down to certain regularity assumptions on the manifold itself. This generalization would certainly not be easy, but would be a way to put this result into the general framework of differential forms together with the exterior derivative and the codifferential. To come back to the very first sentence of this thesis, this would fit in a way to the core of mathematics: The beauty of abstraction.

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