Our goal is to study the homogeneous magnetostatic problem on the exterior domain of a triangulated torus. That means that for the unbounded domain $\Omega \subseteq \mathbb{R}^3$ we have $\mathbb{R}^3 \setminus \Omega$ is a triangulated torus. We also need a piecewise straight (i.e. triangulated) closed curve around the torus. (TBD: Define the "triangulated torus" more rigorous)

Let $B$ be a magnetic field on the domain $\Omega$. We the have the following boundary value problem:

$$\text{curl } B = 0, \tag{1}$$
$$\text{div } B = 0 \text{ in } \Omega \tag{2}$$
$$B \cdot n = 0 \text{ on } \partial\Omega \text{ and} \tag{3}$$
$$\int_\gamma B \cdot dl = C_0 \tag{4}$$

where $n$ is the outward normal vector field on $\partial\Omega$ and $C_0 \in \mathbb{R}$. We want to prove existence and uniqueness of solutions. In order to do so we will need to introduce Sobolev spaces of differential forms and basics from simplicial topology.

At first, let us introduce some basic notions about differential forms. We follow the brief introduction given by Arnold (cf. [1, Sec. 6.1]), but less details will be given. Let $V$ be a real vector space with $\dim V = n$ and $\text{Alt}^k V$ be the space of $k$-alternating maps from $V^k$ to $\mathbb{R}$. For $\omega \in \text{Alt}^k V$, $\mu \in \text{Alt}^l V$ we define the wedge product $\omega \wedge \mu \in \text{Alt}^{k+l} V$

$$(\omega \wedge \mu)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) = \sum_{\substack{i_1 < ... < i_k \\ i_{k+1} < ... < i_{k+l}}} \text{sgn}(i_1, ..., i_{k+l}) \omega(v_{i_1}, ..., v_{i_k}) \nu(v_{i_{k+1}}, ..., v_{i_{k+l}})$$

where $\text{sgn}(i_1, ..., i_{k+l})$ is the sign of the permutation $(1, ..., k+l) \mapsto (i_1, ..., i_{k+l})$. This definition is not very intuitive. TBD: Examples in 3D.

Let $\{u_i\}_{i=1}^n$ be any basis of $V$ and $\{u^i\}_{i=1}^n$ the correspoding dual basis. Then

$$\{u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k} \,|\, 1 \le i_1 < ... < i_k \le n\}$$

is a basis of $\text{Alt}^k V$. In particular, $\dim \text{Alt}^k V = \binom{n}{k}$.

Given a inner product $\langle \cdot, \cdot \rangle_V$ on $V$ we obtain an inner product on $\text{Alt}^k V$ by defining

$$\langle u^{i_1} \wedge u^{i_2} \wedge ... \wedge u^{i_k}, u^{j_1} \wedge ... \wedge u^{j_k} \rangle_{\text{Alt}^k V} := \det\left[(\langle u_{i_k}, u_{i_l} \rangle_V)_{1 \le k, l \le n}\right]$$

which is then extended to all of $\mathrm{Alt}^k V$ by linearity. We denote with $\|\cdot\|_{\mathrm{Alt}^k V}$ the induced norm. From this definition it follows directly that for a orthonormal basis $b_1$,..., $b_n$ the corresponding basis $b^{i_1} \wedge b^{i_2} \wedge ... \wedge b^{i_k}$, $1 \le i_1 < ... < i_k \le n$ is an orthonormal basis of $\mathrm{Alt}^k V$.

$\mathrm{Alt}^n V$ is one-dimensional and so we have to choose a basis. We say that two orthonormal bases of $V$ have the same orientation if the change of basis has positive determinant. That divides the orthonormal bases into two classes with different orientation. We choose one of these classes and call these orthonormal bases positively oriented. Take $\omega \in \mathrm{Alt}^n V$. Then $\omega(b_1, ..., b_n)$ is the same for any positively oriented orthornormal basis. We now define the *volume form* $\mathrm{vol} \in \mathrm{Alt}^n v$ by requiring it to be 1 on all positively oriented orthonormal bases. Using this volume form we can now define the *Hodge star operator* $* : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$ via the property

$$\omega \wedge \mu = \langle *\omega, \mu \rangle_{\mathrm{Alt}^{n-k} V} \mathrm{vol} \quad \forall \omega \in \mathrm{Alt}^k V,\ \mu \in \mathrm{Alt}^{n-k} V.$$

The Hodge star is an isometry, we have $** = (-1)^{k(n-k)}\mathrm{Id}$ and

$$\omega \wedge *\mu = \langle \omega, \mu \rangle_{\mathrm{Alt}^k V} \mathrm{vol} \quad \forall \omega, \mu \in \mathrm{Alt}^k V.$$

In particular in $\mathbb{R}^3$, we have $** = \mathrm{Id}$ i.e. $*$ is self-inverse.

Now we will move on to differential forms. We will mostly deal with the case $V = \mathbb{R}^n$ and denote $\mathrm{Alt}^k \mathbb{R}^n$ just as $\mathrm{Alt}^k$. Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and denote the space of differential forms of degree $k$ on $\Omega$ as $\Lambda^k(\Omega)$. We extend the Hodge star operator to differential forms $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ simply by applying it pointwise.

Then we define the $L_p$-norm of a $k$-form $\omega$ for $1 \le p < \infty$ as (cf. [3])

$$\|\omega\|_{L_p^k(\Omega)} := \left( \int_\Omega \|\omega\|_{\mathrm{Alt}^k}^p \right)^{1/p}$$

and for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x \in \Omega} \|\omega(x)\|_{\mathrm{Alt}^k}.$$

$L_p^k(\Omega)$ are the spaces of $k$-forms s.t. the corresponding $L_p$-norm is finite. For $p = 2$ we obtain a Hilbert space (cf. [1, Sec. 6.2.6]) with the $L_2$ inner product

$$\langle \omega, \nu \rangle := \int_\Omega \langle \omega, \nu \rangle_{\mathrm{Alt}^k}. \tag{5}$$

**Proposition 1.** *The Hodge star operator $* : L_2^k(\Omega) \to L_2^{n-k}(\Omega)$ is a Hilbert space isometry.*

*Proof.* This follows directly from the definition of the inner product (5) and the fact that $*$ is an isometry when applied to alternating forms $\mathrm{Alt}^k$. $\square$

Our next goal is to extend the exterior derivative $d$ of smooth differential forms in the weak sense (cf. [3]). Let $\mathring{d} : L_2^k(\Omega) \to L_2^{k+1}(\Omega)$ be the exterior derivative as an unbounded operator with domain $D(\mathring{d}) = C_0^\infty\Lambda^k(\Omega)$ which are the smooth compactly supported differential forms of degree $k$. Analogous, let $\mathring{\delta} : L_2^k(\Omega) \to L_2^{k-1}(\Omega)$ be the codifferential operator $\mathring{\delta} := (-1)^{n(k-1)+1} * \mathring{d} *$ also with domain $C_0^\infty\Lambda^k(\Omega)$.

Then the exterior derivative $d\omega \in L_{k+1}^p(\Omega)$ is defined as the unique $(k+1)$-form in $L_{k+1}^p(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge *\phi = \int_\Omega \omega \wedge *\mathring{\delta}\phi \quad \forall \phi \in C_0^\infty\Lambda^k(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$\begin{aligned} W_p^k(\Omega) &= \left\{\omega \in L_p^k(\Omega) \,|\, d\omega \in L_p^{k+1}(\Omega)\right\}, \\ W_{p,loc}^k(\Omega) &= \left\{\omega\ k\text{-form} \,|\, \omega|_A \in W_p^k(A) \text{ for every open } A \subseteq \Omega \text{ s.t. } \overline{A} \subseteq \Omega \text{ is compact}\right\}. \end{aligned}$$

For $\omega \in W_p^k(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W_p^k} := \left(\|\omega\|_{L_p^k}^p + \|d\omega\|_{L_p^k}^p\right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W_\infty^k} := \max\left\{\|\omega\|_{L_\infty^k}, \|d\omega\|_{L_\infty^k}\right\}.$$

**Definition 1** ($L^p$-cohomology)**.** We define the following subspaces of $W_p^k(\Omega)$, $1 \le p \le \infty$:

$$\begin{aligned} \mathfrak{B}_k &:= dW_p^{k-1}(\Omega) \text{ and} \\ \mathfrak{Z}_k &:= \{\omega \in W_p^k(\Omega) |\, d\omega = 0\}. \end{aligned}$$

We call the $k$-forms in $\mathfrak{B}_k$ exact and the forms in $\mathfrak{Z}_k$ closed. Because $d \circ d = 0$ we always have $\mathfrak{B}_k \subseteq \mathfrak{Z}_k$. Then we define the de Rham- or $L^p$-cohomology space $H_{p,dR}^k(\Omega)$ as the quotient space

$$H_{p,dR}^k(\Omega) := \mathfrak{Z}_k/\mathfrak{B}_k.$$

We want to examine the Hilbert space $L_2^k(\Omega)$ more closely (see [1, Sec. 6.2.6] for more details). We denote $H^k(d;\Omega) := W_2^k(\Omega)$. If the domain is clear we will leave it out. Note that the above definition of the exterior derivative is in the Hilbert space setting equivalent to defining $d$ as the adjoint of $\mathring{\delta}$.

Now we just define $\delta := (-1)^{n(k-1)+1} * d*$ as in the smooth setting. We will show that this is the adjoint of $\mathring{d}$. Denote with $D(\mathring{d}^*) \subseteq L_2^{k+1}(\Omega)$ the domain of the adjoint. Define

$$H^k(\delta;\Omega) := \{\omega \in L_2^k(\Omega) \mid *\omega \in H^{n-k}(d)\}.$$

Now take $\omega \in H^{k+1}(\delta)$ and $\phi \in C_0^\infty\Lambda^k(\Omega)$. Then

$$\begin{aligned}
&\langle \delta\omega, \phi\rangle = (-1)^{nk+1}\langle *d*\omega, \phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}\langle d*\omega, *\phi\rangle = (-1)^{nk+1}(-1)^{k(n-k)}\langle *\omega, \mathring{\delta}*\phi\rangle \\
&= (-1)^{nk+1}(-1)^{k(n-k)}(-1)^{n(n-k-1)+1}\langle *\omega, *\mathring{d}**\phi\rangle \\
&= (-1)^{n(n-1)+2}(-1)^{k(n-k)}\langle \omega, \mathring{d}**\phi\rangle \\
&= \langle \omega, \mathring{d}\phi\rangle
\end{aligned}$$

where we used repeatedly that $*$ is an isometry and $** = (-1)^{k(n-k)}\mathrm{Id}$. This shows that $H^{k+1}(\delta) \subseteq D(\mathring{d}^*)$ and that $\mathring{d}^*\omega = \delta\omega$. Now for the other inclusion assume that $\omega \in D(\mathring{d}^*)$ and take $\phi \in C_0^\infty\Lambda^{n-k}(\Omega)$ arbitrary.

$$\langle *\omega, \mathring{\delta}\phi\rangle = \pm\langle \omega, \mathring{d}*\phi\rangle = \pm\langle \mathring{d}^*\omega, *\phi\rangle = \pm\langle *\mathring{d}^*\omega, \phi\rangle.$$

Here we use $\pm$ to mean that we choose the sign correctly, s.t. all the operations are correct. Then by choosing the sign appropriately we find that $\pm *\mathring{d}^*\omega = d*\omega$ and therefore $*\omega \in H^{n-k-1}(d)$ so we proved $D(\mathring{d}^*) \subseteq H^{k+1}(\delta)$ and we are done.

We then define additionally the space $\mathring{H}^k(d;\Omega)$ as the closure of $C_0^\infty\Lambda^k(\Omega) \subseteq H^k(d;\Omega)$ w.r.t. the $H^k(d)$-norm i.e. $\mathring{H}^k(d;\Omega)$ corresponds to $k$-forms in $H^k(d;\Omega)$ being zero on the boundary. TBD: There are several different ways to characterize zero boundary conditions in the $L^2$ setting. We have to choose the one that works best. Then we define the spaces

$$\begin{aligned}
H_0^k(d;\Omega) &:= \{\omega \in H^k(d;\Omega) | d\omega = 0\} \\
\mathring{H}_0^k(d;\Omega) &:= \{\omega \in \mathring{H}^k(d;\Omega) | d\omega = 0\}
\end{aligned}$$

i.e. the spaces of closed forms. We will use the analogous definition for $H_0^k(\delta;\Omega)$ and $\mathring{H}_0^k(\delta;\Omega)$ which we call coclosed forms. We then define the spaces of harmonic forms

$$\mathring{H}_0^k(d,\delta;\Omega) := \{\omega \in \mathring{H}^k(d;\Omega) \mid d\omega = 0, \delta\omega = 0\}.$$

With this one can prove the Hodge decomposition ([1, Lemma 1])

$$L_2^k(\Omega) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \overset{\perp}{\oplus} \overline{\delta H^{k+1}(\delta)} \tag{6}$$

and furthermore for the closed and coclosed forms respectively,

$$\mathring{H}_0^k(d) = \overline{d\mathring{H}^{k-1}(d)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta) \tag{7}$$

$$H_0^k(\delta) = \overline{\delta H^{k+1}(\delta)} \overset{\perp}{\oplus} \mathring{H}_0^k(d,\delta). \tag{8}$$

# 1 Tools from Algebraic Topology

In order to deal with the topological aspects of the problem we need to use some results from algebraic topology. This material is taken from Chapter 4 and 5 of Bredon's "Topology and Geometry" [2] where a lot more details and results can be found. We start with the basic definitions from homology.

**Definition 2** (Standard simplex)**.** Let $\ell^2$ be the space of real square-summable sequences and $e_1$, $e_2$, ... be the standard orthonormal basis. Then the standard simplex $\Delta_k$ is defined as the convex combination of $e_1$, $e_2$, ..., $e_k$ i.e.

$$\Delta_k := \left\{ \sum_{i=0}^{k} \lambda_i e_i | \sum_{i=0}^{k} \lambda_i = 1,\, 0 \leq \lambda_i \leq 1 \right\}.$$

**Definition 3** ((Singular) affine simplex)**.** Given points $v_0$, $v_1$, ..., $v_k \in \mathbb{R}^n$ we define the map

$$[v_0, ..., v_k] : \Delta_k \to \mathbb{R}^n, \quad \sum_{i=0}^{k} \lambda_i e_i \mapsto \sum_{i=0}^{k} \lambda_i v_i.$$

This map is called *affine singular simplex.* If the $v_i$ are additionally affine independent we call it *affine simplex.*

This map is well defined because the $\lambda_i$ are the barycentric coordinates and for every $x \in \Delta_k$ the $\lambda_i \in [0, 1]$ s.t. $x = \sum_i \lambda_i e_i$ are uniquely determined.

$[e_0, ..., \hat{e}_i, ..., e_k]$ denotes the affine simplex where the vertex $e_i$ was removed.

**Definition 4** (Face map)**.** The affine simplex $[e_0, ..., \hat{e}_i, ..., e_k] : \Delta_{k-1} \to \Delta_k$ is called the $i$-th face map and is denoted by $F_i^k$.

**Definition 5** (Singular $k$-chain group)**.** Let $X$ be a topological space. Then a *singular $k$-simplex* of $X$ is a continuous map $\sigma_k : \Delta_k \to X$. The *singular $k$-chain group* $\Delta_k(X)$ is the free abelian group based on all singular $k$-simplices i.e. it is the group of all formal finite sums $\sum_\sigma n_\sigma \sigma_k$ with $n_\sigma \in \mathbb{Z}$. We set $\Delta_k(X) = 0$ for $k < 0$

**Definition 6** (Boundary)**.** Let $\sigma : \Delta_k \to X$ be a singular $k$-simplex. Then the *$i$-th face* of $\sigma$ is $\sigma^{(i)} = \sigma \circ F_i^k$. The boundary of $\sigma$ is

$$\partial_k \sigma := \sum_{i=0}^{k} (-1)^i \sigma^{(i)}.$$

This is a $(k-1)$-chain. We extend this map to all $k$-chains by

$$\partial_k : \Delta_k(X) \to \Delta_{k-1}(X),\ \partial_k(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma \partial_k \sigma.$$

We set $\partial_k = 0$ for $k \leq 0$. We have $\partial_k \circ \partial_{k+1} = 0$. The proof is done by direct computation, see [2, Ch. 4, Lemma 1.6]. We call a $k$-chain $c$ a *$k$-cycle* if $\partial_k c = 0$ and we call $c$ a *$k$-boundary* if there exists a $(k+1)$-chain $d$ s.t. $c = \partial_{k+1} d$ or phrased differently

$$\text{"}k\text{-cycles"} = \ker \partial_k =: Z_k(X)$$
$$\text{"}k\text{-boundaries"} = \operatorname{im} \partial_{k+1} =:= B_k(X)$$

We are now ready to define homology groups.

**Definition 7** (Singular homology group)**.** The *$k$-th homology group* $H_k(X)$ is the quotient group

$$H_k(X) := {}^{Z_k(X)}\!\big/_{B_k(X)}.$$

For a $k$-chain $c$ we write $[c]$ for the corresponding homology class.

If $H_k(X)$ is finitely generated then its rank is called the *$k$-th Betti number*. In the cases that we will consider all homology groups will be finitely generated.

**Definition 8** (Simplicial complex)**.**

The homology spaces are independent of the chosen simplicial complex [**empty citation**].

Simplices as defined before are maps which is very abstract. The next definition is what one commonly thinks of as a simplex in $\mathbb{R}^n$.

**Definition 9** (Physical simplex)**.** Let $x_0$, $x_1$, ..., $x_k \in \mathbb{R}^n$ be affine independent. Then

$$[x_0, x_1, ..., x_k] := \operatorname{conv}\{x_0, ..., x_k\}$$

is called physical $k$-simplex.

It should be mentioned that Bredon calls these just affine simplices in [2]. We changed the naming here to make the difference to the affine simplices from above more clear as these are completely different objects.

**Definition 10** (Simplicial complex)**.** A *simplicial complex* $K$ is a collection of phyiscal simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of $\sigma$ is in $K$,
2. $\sigma$, $\tau \in K \Rightarrow \sigma \cap \tau$ is either empty or a face of both $\sigma$ and $\tau$.

We call $|K| := \bigcup\{\sigma | \sigma \in K\}$ the polyhedron of $K$.

For any topological space $X$ a homeomorphism $\tau : |K| \to X$ is called *triangulation* of $X$. Let $\{x_1, x_2, ...\}$ be the vertices in the simplicial complex $K$. We fix an ordering of the vertices for every simplex. That means for any $k$ every $k$ simplex $\sigma$ has a designated representation in the form of

$$\sigma = [x_{i_0}, x_{i_1}, ..., x_{i_k}].$$

In Section 2.1 we will only deal with simplicial complexes instead of chain groups.

**Definition 11** (Cochain group)**.** Let $G$ be a group. Then we define the *$k$-cochain group*

$$C^k(X; G) := \operatorname{Hom}(C_k(X), G)$$

i.e. the group of all homomorphisms from the chain group to $G$.

We use a superindex if an object is related to cochains and we use a subindex if it is related to chains. In this spirit, we define

$$\partial^k : C^k(X; G) \to C^{k+1}(X; G),\ (\partial^k F)(c) := F(\partial_k c).$$

We call cochain $F \in C^k(X; G)$ *closed* if $\partial^k F = 0$ and we call $F$ *exact* if there exists $G \in C^{k+1}(X; G)$ s.t. $F = \partial^{k-1} G$. So in analogy to the boundary map we denote

$$\text{"closed } k\text{-cochains"} = \ker \partial^k =: Z^k(X; G)$$
$$\text{"exact } k\text{-cochains"} = \operatorname{im} \partial^{k-1} =: B^k(X; G).$$

We get immediately from the definition of $\partial^k$ that $\partial^k \circ \partial^{k+1} = 0$, analogous to the boundary of chains.

Analogous to the homology groups, we thus obtain the *cohomology groups*.

**Definition 12** (Singular cohomology groups)**.** We define the *singular cohomology group* as

$$H^k(X;G) := {}^{Z^k(X;G)}\diagup_{B^k(X;G)}.$$

In our situation later, $G$ will always be the real numbers.

Now the question arises how the homology and cohomology groups are related to each other. In order to answer this question, we have to introduce the notion of *exact sequences*.

**Definition 13** (Exact sequence)**.** Let $(G_i)_{i\in\mathbb{Z}}$ be a sequence of groups and $(f_i)_{i\in\mathbb{Z}}$ be a sequence of homomorphisms $f_i : G_i \to G_{i+1}$. We call this sequence of homomorphisms exact if $\operatorname{im} f_{i-1} = \ker f_i$.

Then the *universal coefficient theorem* gives us the following exact sequence

$$0 \to \operatorname{Ext}(H_{k-1}(X), G) \to H^k(X;G) \xrightarrow{\beta} \operatorname{Hom}(H_k(x), G) \to 0$$

with the homomorphism

$$\beta([F])([c]) = F(c).$$

The definition of Ext can be found in [2, Sec. V.6], but the definition does not matter for our purposes because for $G = \mathbb{R}$ we get $\operatorname{Ext}(H_{k-1}(X), G) = 0$. This is because the real numbers are a divisible and hence injective group. The definition of these terms and the necessary connections can be found in [2, p. V.6], but we will not go any further into the algebraic background.

So we obtain the exact sequence

$$0 \to H^k(X;\mathbb{R}) \xrightarrow{\beta} \operatorname{Hom}(H_k(x), \mathbb{R}) \to 0.$$

The exactness gives us $\ker \beta = 0$ and $\operatorname{im} \beta = \operatorname{Hom}(H_k(x), G)$ i.e. $\beta$ is an isomorphism.

As an application let us prove the following proposition that will be used later to prove the uniqueness of the magnetostatic problem.

**Proposition 2.** *Assume that $H_1(X) \cong \mathbb{Z}[\gamma]$ i.e. $[\gamma]$ is a generator of the homology group. Let $F$ and $G$ both be closed 1-chains with $F(\gamma) = G(\gamma)$. Then $[F] = [G]$.*

*Proof.* Take $c$ be any closed 1-chain. Then $[c] = [n\gamma]$ for some $n \in \mathbb{Z}$ and thus

$$\beta([F])([c]) = \beta([F])(n[\gamma]) = n\, F(\gamma) = n\, G(\gamma) = \beta([G])(n[\gamma]) = \beta([G])([c])$$

so $\beta([F]) = \beta([G])$. Because $\beta$ is an isomorphism we obtain $[F] = [G]$. $\square$

In order to show existence and uniqueness of solutions of the magnetostatic problem we rely on a result about the isomorphism of a simplicial cohomology space $H_p^k(K)$ and the $L^p$-cohomology space $H_{p,dR}^k(\overline{\Omega})$. This result was proven in [3]. In the diploma thesis of Nikolai Nowaczyk [4], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [3] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [3]). Therefore, we can apply the result to our case.

# 2 Isomorphism of Cohomology

Before we state the theorem about the isomorphism of we will first formulate a crucial assumption for this result to hold.

Because $\overline{\Omega}$ from our problem is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [3, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1, x_2, ...\}$ and take $f : |K| \to \ell^2$ where $\ell^2$ is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and $f$ is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^*g$ where $g$ is the standard metric in $\ell^2$. Let $\langle\cdot,\cdot\rangle$ be the standard scalar product on $\ell^2$. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, $\sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned} g_S|_x \left( \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_k}{\partial x_j}(x) \\ &= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left( Df(x)^T Df(x) \right)_{ij} \\ &= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle , \end{aligned}$$

where $D$ denotes the Jacobian. (TBD: This Jacobian as written here would technically be in $\mathbb{R}^{\infty \times n}$. Only finitely many lines are non-zero though, but this is not quite rigorous yet. )

We have two crucial assumptions on the triangulation for the result to hold (cf. [3, p.194]). We summarize them under *GKS-condition* named after the three authors of [3].

**Assumption 1** (GKS-condition)**.** We will assume the following on the simplicial complex $K$ and the triangulation $\tau$:

1. The star of every vertex in $K$ contains at most $N$ simplices.

2. For the differential of $\tau$ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1, \ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

where $d$ denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most $N$ simplices, which is fulfilled if we have a shape regular mesh.

Because $\tau$ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

## 2.1 Statement of the Isomorphism

From now on we will assume that the GKS condition is fulfilled.

**Definition 14** (Induced map)**.** Let $V$ and $W$ be real vector spaces, $X \subseteq V$, $Y \subseteq W$ be subspaces. For a linear map $L : V \to W$ with $L(X) \subseteq Y$ we define the induced map

$$[L] : {}^{V}\!/\!_{X} \to {}^{W}\!/\!_{Y},\ [v] \mapsto [Lv].$$

It is easy to check that the induced map is well-defined using the definition of quotient space.

The first isomorphism is induced from a linear mapping from the so called *S-forms* $S_p^k(K)$ to *p-summable k-cochains* $C_p^k(K)$ which will both be defined next.

**Definition 15.** We define the following norm of a $k$-cochain $f$

$$\|f\|_{C_p^k(K)} := \left( \sum_{c\, k\text{-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{ f \ k\text{-cochain} |\ \|f\|_{C_p^k(K)} < \infty \}.$$

The idea is that for a $k$-form $\theta$ we have the cochain $I(\theta) \in C^k(K)$, $I(\theta)(\sigma) = \int_\sigma \theta$ for any $k$-simplex $\sigma$. We want to show that this map induces a isomorphism on cohomology.

At first, we will discuss when and how this integration mapping is well-defined. In order to do so, we will use restriction operatorss that are introduced in [3, p.191]. Let $\sigma, \tau \in K$ be simplices s.t. $\tau < \sigma$ i.e. $\tau$ is a face of $\sigma$. The pullback of the standard inclusion mapping $j_{\tau,\sigma} : \tau \hookrightarrow \sigma$ gives us a restriction $j_{\tau,\sigma}^* : W_\infty^k(\sigma) \to W_\infty^k(\tau)$, but we have to check whether this is well-defined. In order to do so, take some $\eta \in W_\infty^k(\sigma)$. Before restriction we extend $\eta$ to a neighborhood $U \supseteq \sigma$ in the affine hull of $\sigma$ to get $\tilde{\eta} \in W_\infty^k(U)$. Then we set $j_{\tau,\sigma}^*\eta = j_{\tau,U}^*$ which is now well-defined and independent of the extension $\tilde{\eta}$ chosen. It should be stressed that this construction does not work for $W_p^k$ with $p < \infty$.

Now take $\eta \in W_{\infty,loc}^k(\overline{\Omega})$. By applying this restriction operator repeatedly we can now find $\eta_\sigma \in W_\infty^k(\sigma)$ for every $\sigma \in K$. Then we define

$$(I\eta)(\sigma) = \int_\sigma \eta_\sigma \text{ for all } k\text{-simplices } \sigma \in K.$$

So we have established that $I : W^k_{\infty,loc}(\overline{\Omega}) \to C^k(K)$ is well-defined.

In the next step, we will use two operators $\mathscr{R}$ and $\mathscr{A}$ from [3]. The precise definition and details of their construction are not relevant for our purposes because we will only use the following properties (cf. [3, Thm.2]).

**Theorem 1.** *Assume that the triangulation $\tau$ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L^k_{1,loc}(\overline{\Omega}) \to L^k_{1,loc}(\overline{\Omega})$, $\mathscr{A} : L^k_{1,loc}(\overline{\Omega}) \to L^{k-1}_{1,loc}(\overline{\Omega})$ such that*

1. *$\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ for $\omega \in W^k_{1,loc}(\overline{\Omega})$*
2. *for any $1 \leq p \leq \infty$, $\mathscr{R}(W^k_p(\overline{\Omega})) \subseteq S^k_p(\overline{\Omega})$.*

where

$$S^k_p(\overline{\Omega}) = \{\eta \in W^k_{\infty,loc} |\, I\eta \in C^k_p(K)\}.$$

Now define $\bar{I} := I \circ \mathscr{R}$. This gives a well-defined mapping from $W^k_p(\overline{\Omega})$ to $C^k_p(K)$. Following the arguments of [3] and using the properties of $\mathscr{R}$ and $\mathscr{A}$ one can show that the induced homomorphism of cohomologies $[\bar{I}] : H^k_{p,dR}(\overline{\Omega}) \to H^k_p(K)$ is an isomorphism. This isomorphism will play a crucial part in the proof of uniqueness and existence of solutions in the next section.

# 3 Existence and uniqueness of solutions

## 3.1 Reformulation of the problem

We will return now to the magnetostatic problem. In order to use the results above we will reformulate the problem in the notation of differential forms. There are two ways to identify a vector field with a differential form (cf. [1, Table 6.1 and p.70]) either as a 1-form or a 2-form. For a vector field $B$ we define

$$\begin{aligned} F^1\, B &:= B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \text{ and} \\ F^2\, B &:= B_2\, dx_2 \wedge dx_3 - B_2\, dx_1 \wedge dx_3 + B_3\, dx_1 \wedge dx_2 \end{aligned}$$

as the corresponding 1-form and 2-form. Then the exterior derivative is $dF^2\,\omega$ corresponds to the divergence, the codifferential $\delta F^2\,\omega$ corresponds to the curl and the normal component being zero on the boundary corresponds to $\omega \in \mathring{H}^2(d)$.[**empty citation**].

If we then use the association of 3-forms with scalars we have the corresponding boundary value problem without the integral condition for 2-forms: Find $\omega \in \mathring{H}^2(d)$ s.t.

$$\delta\omega = 0, \tag{9}$$

$$d\omega = 0 \text{ in } \Omega. \tag{10}$$

Next, we have to add the integral condition. We use that we are in three dimensions so $** = (-1)^{k(n-k)}\tilde{\nu} = \tilde{\nu}$ [1, p.66] for any $k$-form $\tilde{\nu}$. and observe

$$\begin{aligned} *F^2\, B &= B_1 \, **dx_1 + B_2 \, **dx_2 + B_3 \, **dx_3 = B_1\, dx_1 + B_2\, dx_2 + B_3\, dx_3 \\ &= F^1\, B. \end{aligned}$$

Then we have

$$\int_\gamma *F^2\, B = \int_\gamma F^1\, B = \int_\gamma B \cdot \mathrm{d}l.$$

In the last step we used the fact that the integration of a 1-form over a curve is equivalent to the curve integral of the associated vector field (cf. [1, Sec. 6.2.3]). Hence, we can add the integral condition

$$\int_\gamma *\omega = C_0. \tag{11}$$

However, we have only $\omega \in \mathring{H}^2_0(d,\delta)$ so $*\omega \in H^1(\delta)$ so this integral might not be well defined. In order to deal with this, we will again use the operator $\mathscr{R}$ and $\bar{I}$ from Sec. 2.1. Instead of using (11) directly we use the condition

$$\int_\gamma \mathscr{R} * \omega = C_0.$$

This is equivalent to $(\bar{I} * \omega)(\gamma) = C_0$. We know that this is well-defined.

We want to give some justification about why this is a reasonable extension. For any closed $\eta \in W^1_p(\overline{\Omega})$ we have $\mathscr{R}\eta = \eta - d\mathscr{A}\eta$ from Thm. 1. For smooth $\phi \in \Lambda^0(\overline{\Omega})$ we immediately get from the standard Stoke's theorem that $\int_c d\phi = 0$ for all closed 1-chains $c$. If we assume sufficient regularity on $\mathscr{R}\eta$, $d\mathscr{A}\eta$ and $\eta$ then we would have indeed

$$\int_\gamma \mathscr{R}\eta = \int_\gamma \eta - d\mathscr{A}\eta = \int_\gamma \eta.$$

**Remark 1.** This justification can be done more rigorously with the help of S-forms (cf. [3]) which correspond to $W_{\infty,loc}(\overline{\Omega})$ with an additional decaying condition. Then we obtain that the integral is consistent with the integral on these S-forms.

To summarize we obtain the following problem.

**Problem 1.** Find $\omega \in \mathring{H}^2(d;\Omega)$ s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0 \text{ in } \Omega, \\ (\bar{I}\omega)(\gamma) &= C_0. \end{aligned}$$

We will examine existence and uniqueness of this problem in the next section.

## 3.2 Existence and uniqueness

We start with the following

**Proposition 3.** *Let $\gamma$ be a closed $k$-chain s.t. the homology class $[\gamma]$ spans the homology space $H_c^k$. Then for any $C_0 \in \mathbb{R}\setminus\{0\}$ if we have closed cochains $F, G$ s.t.*

$$F(\gamma) = G(\gamma) = C_0$$

*then $[F] = [G]$ i.e. their cohomology classes are equal.*

*Proof.* From [1, Sec. 2.5] we know that $\dim H^k(K) = \dim H_c^k$. Because $F$ and $G$ are closed we therefore have $\lambda_F, \lambda_G \in \mathbb{R}$ and a cohomology class $[b]$ s.t. $[F] = \lambda_F[b]$ and $[G] = \lambda_G[b]$. This is equivalent to the existence of $(k-1)$-cochains $J_F, J_G$ s.t.

$$F = \lambda_F b + \delta J_F \text{ and } G = \lambda_G b + \delta J_G.$$

so

$$0 \neq \lambda_F b(\gamma) + \delta J_F(\gamma) = F(\gamma) = G(\gamma) = \lambda_G b(\gamma) + \delta J_F(\gamma).$$

Because $\gamma$ is closed we have for any $(k-1)$-chain $J$, $\delta J(\gamma) = J(\partial\gamma) = 0$ and so we arrive at $\lambda_F = \lambda_G$ i.e. $[F] = [G]$. □

We are now able to proof existence of a solution. Take a closed cochain $F \in C_2^1(K)$ s.t. $F(\gamma) = C_0$ and $F(\partial d) = 0$ for any 2-chain $d$ as in Prop. 3. Then we know from Sec. 2.1 that there exists a unique $[\theta] \in \mathscr{H}_2^1(K)$ s.t. $[I]([\theta]) = [F]$. Let us take the 1-form $\eta := \varphi^{-1}\theta \in W_2^1(\omega)$. Then $\int_\gamma \eta = C_0$ holds. Because we need a 2-form we will take a closer look at $*\eta$. We have that the Hodge star operator $* : L_2^1(\Omega) \to L_2^2(\Omega)$ is a Hilbert space isometry.

First we compute

$$\delta * \eta = (-1)^{n(k-1)+1} * d * *\eta = (-1)^{n(k-1)+1} * d\eta = 0.$$

Using the Hodge decomposition for unbounded domains (cf. [5, Lemma 1]) we get therefore that there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq H^3(\delta,\Omega)$ and a harmonic $\omega \in \mathring{H}^2(d) \cap H^3(\delta)$ s.t.

$$*\eta = \lim_{i\to\infty} \delta\phi_i + \omega.$$

where the limit is in $L_2^1(\Omega)$.

We will show that $\omega$ is then a solution. Because $\omega$ is harmonic we already know that $d\omega = 0$, $\delta\omega = 0$ and $\omega \in \mathring{H}^2(d)$. It remains to show that the integral condition (11) is also satisfied.

At first we check that the integral is well-defined. We have $\delta\omega = 0$ which implies

$$d * \omega = * * d * \omega = (-1)^{n(k-1)+1} * \delta\omega = 0$$

so $*\omega$ is a closed 1-form. Therefore the integral is well-defined as shown in Sec. **??**.

Take a ball of radius $R > 0$ around the origin with $R$ large enough s.t. $\gamma \subseteq B_R$. Using the fact that the range of $\delta$ is closed on bounded domains (cf. [5, Lemma 7]) there is some $\phi_R \in H^3(\delta; B_R)$ s.t.

$$*\eta|_{B_R} = \omega|_{B_R} + \delta\phi_R.$$

We then get for the integral condition (11)

$$\begin{aligned}\int_\gamma *\omega &= \int_\gamma *\omega|_{B_R} = \int_\gamma *(*\eta|_{B_R} - \delta\phi_R) = \int_\gamma \eta|_{B_R} - (-1)^{3\cdot 2+1} d * \phi_R \\ &= \int_\gamma \eta|_{B_R} = C_0\end{aligned}$$

so the integral condition (11) is fulfilled and $\omega$ is indeed a solution.
TBD: Put all this in a theorem.

**Theorem 2.** *The solution of the problem is unique.*

*Proof.* Let $\omega, \tilde{\omega}$ both be solutions. Because $*\omega$ and $*\tilde{\omega}$ are closed the cochains $c \mapsto \int_c \mathscr{R} * \omega$ and $c \mapsto \int_c \mathscr{R} * \tilde{\omega}$ are closed.

Due to $\int_\gamma \mathscr{R}*\omega = \int_\gamma \mathscr{R}*\tilde{\omega}$ and the assumption that $[\gamma]$ spans the homology space we have with Prop. 3 $[I(\mathscr{R} * \omega)] = [I(\mathscr{R} * \tilde{\omega})]$ and because $[I]$ is an isomorphism $[\mathscr{R} * \omega] = [\mathscr{R} * \tilde{\omega}]$. Hence,

$$[*\tilde{\omega}] = [\mathscr{R} * \tilde{\omega}] = [\mathscr{R} * \omega] = [*\omega].$$

That is equivalent to the existence of some 0-form $\phi \in H^0(d)$ s.t. $*\omega = *\tilde{\omega} + d\phi$. We continue by applying the Hodge star operator to both sides and use the definition of the codifferential $\delta$:

$$\omega = \tilde{\omega} + *d\phi = \tilde{\omega} + *d**\phi = \tilde{\omega} + (-1)^{(n-k)(k-1)+1}\delta * \phi.$$

Then because $\omega$ and $\tilde{\omega}$ are harmonic we have $\omega, \tilde{\omega} \perp \delta H^3(\delta)$ and therefore

$$\omega = \tilde{\omega}.$$

$\square$

If we now translate this back to standard vector calculus terms we have found the unique solution of the homogeneous magnetostatic on our domain $\Omega$.

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