Mathematics is the art of abstraction, of putting basic theories into more general frameworks to gain insights that would otherwise be hard or impossible to see. James Clerk Maxwell recognized how many years ago that some vector fields are to be understood in relation to curves i.e. the natural relation is the curve integral and some are to be understood as flux through a surface. This thought paves the way to a new paradigm which presents itself in differential forms which are a classic subject of differential geometry. Differential forms were recognized later by many to be the more suitable objects to study in fields like electromagnetism [**<empty citation>**] and fluid dynamics.

This thesis is about the problem of finding a static solution to Maxwell's equations. This would be the magnetic field $\mathbf{B}$ s.t. for a given current source $\mathbf{J}$

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= \mathbf{J} \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

A fascinating realization is that the well-posedness of this equation actually depends on fundamental topological quantities of the domain. If it is simply connected then there exists a unique solution.

This problem arises as a part of the ??? code where it is posed on the exterior of a toroidal domain with Neumann boundary conditions $\mathbf{B} \cdot \mathbf{n} = 0$. This is obviously not a simpliy connected domain and as an additional constraint we are given the curve integral

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0$$

with $C_0 \in \mathbb{R}$ and Γ being the curve that goes around the torus in poloidal direction (see **??**). It is a fascinating PDE for the reason that it combines the differential equations with this curve integral constraint which gives this problem a strong topological flavour. For a student familiar with standard PDEs, it begs the question how these notions can be combined to investigate this problem.

To come back to the statement from the beginning, the answer is provided by changing the point to a more general one, namely to differential geometry which gives a much more general framework in which basic vector calculus can be embedded. One standard example of this is the beautiful Stokes theorem which generalizes the Gauss divergence theorem and Stokes' from vector calculus into the language of differential forms.

Differential forms are especially useful when we start integrating them. Whitney derives in his classic book [**t**]hat differential forms are the actual objects suited to be integrated over manifolds like curves and surfaces which comes back to what Maxwell said about the natural integration of vector fields either over a curve or a surface. Another recommended exposition about this topic is Terence Tao's paper [**<empty citation>**].

This integration of differential forms over manifolds turns out to provide a beautiful synergy between the three fields, calculus, differential geometry and topology. De Rham's famous theorem which relies on Stokes theorem for gives a very simple and yet deep relationship between the singular homology – a popular tool in algebraic topology to study topological quanities – and the integration of differential forms.

For a mathematician interested in analysis and motivated by the beauty of abstraction these reasons should already be sufficient to start learning and investigating differential forms and as a necessary fundament differential geometry. We will give a short exposition of these topics in the first sections of this thesis and then apply these ideas to the magnetostatic problem above.

This has been done before on bounded domains, but we want to do it on unbounded domains. Traditionally, the theory of PDEs on unbounded domains is sparse in comparison to bounded ones. But unbounded domains can change the situation drastically. Even one of the simplest most basic problems in the topic of PDEs, the Laplace equation with Dirichlet boundary conditions, does not have a unique solution anymore when posed on an unbounded domains.

Combining the tools from singular homology, differential geometry and functional analysis we will the existence and uniqueness of the magnetostatic problem under suitable assumptions.

The second part of this thesis will then be concerned with the numerical approximation. As is often done in numerical analysis – especially in a master's thesis – we will simplify the problem. to bounded domains in 2D, but we keep the curve integral constraint and the non-trivial topology of the domain. As explained above, the natural way to study an equation like this is in the language of differential forms. In the last decade, there has been a lot of research done about the combination of that which has been formulated in abstract form in the Acta Numerica paper from Arnold, Falk and Winther [**<empty citation>**] who coined the term Finite Element Exterior Calculus (FEEC). This is the perfect tool for our purposes. We will study a variational formulation of the problem and will strongy rely on the tools from FEEC.

This work will be split between the study of well-posedness of the magnetostatic problem in chapter **??** and then the numerics in chapter

In chapter will first spend a lot of time introducing the necessary tools for the proof of well-posedness of the magnetostatic problelm in 3D. We start with differential forms in Sec. ?? from the starting point of multilinear algebra. Afterwards we will give a short introduction on singular homology in Sec. which concludes in the de Rham isomorphism and then we will talk about unbounded operators and Hilbert complexes. After applying some regularity results in ?? we will put all this together in Sec. to prove the existence and uniqueness of the magnetostatic problem und suitable suitable assumptions.

In Chapter ??, we derive a variational formulation of a 2D version of the magnetostatic problem in Sec.. Then we give a short overview of the discrete Hilbert complexes which are fundamental to FEEC theory before applying it to our new variational formulation to obtain the well-posedness and a-priori estimate in Sec.. In the end, we will explain how this problem is actually implemented (Sec. ??) and provide some numerical examples in Sec ??.