This chapter, which will be the second main part of this thesis, is devoted to the numerical approximation of the magnetostatic problem in 2D which can be derived from a special case of the standard magnetostatic problem. Instead of the exterior of a toroidal domain as in the first part, we will pose the problem on an "annulus like" domain which will be defined more exactly. A curve integral will again be given as an additional constraint and we will investigate the idea to incorporate it using an integration-by-parts approach which is easily applicable to the finite element approximation that we will use.

We start in Sec. 0.1 by deriving a variational formulation of the 2D magnetostatic problem, including the alternative description of the curve integral constraint, and prove well-posedness of the resulting formulation. In Sec. 0.2, the discretization of the problem will be described including a proof of well-posedness and of an a-priori estimate. The implementation will be explained in Sec. 0.3 and numerical examples given in Sec.0.4 which confirm our theoretical predictions.

Prerequisites for this chapter are familiarity with finite element theory and basic knowledge of functional analysis and Sobolev spaces.

## 0.1 Variational formulation of the magnetostatic problem in 2D

For simplicity, we will now turn to the 2D case and we assume that our open domain will be bounded and Lipschitz. This involves introducing a different Hilbert complex with other differential operators. Then we derive the 2D magnetostatic problem from the three-dimensional one. We will assume our domain $\Omega$ to have an "annulus like" form which we will clarify in more rigour. In order for the 2D magnetostatic problem to be well-posed, we require an additional constraint that will again be a curve integral. We will investigate an alternative way to represent this curve integral which will turn out to be easily suitable to be included in our numerical approximation. Prerequisite for this section are knowledge about basic functional analysis and fundamentals of finite element theory. We will also depend on the notions introduced in Sec. **??**, in particular the Hodge decomposition in the general case.

### 0.1.1 The $\mathrm{curl}$-$\mathrm{div}$ Hilbert complex

We start with the introduction of the relevant differential operators and the resulting 2D Hilbert complex. We will then explain what domains we will consider and state the 2D magnetostatic problem in strong form.

We define the scalar curl for $\mathbf{v} \in C^1(\Omega; \mathbb{R}^2)$ as

$$\operatorname{curl} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

Additionally, we have the vector-valued curl, denoted in bold, defined for $v \in C^1(\Omega)$

$$\mathbf{curl}\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

In contrast to chapter **??**, we will from now on denote vector valued operators in bold, i.e. $\mathbf{curl}$ and $\mathbf{grad}$. The cross product for 2D reads for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$,

$$\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1 = \mathbf{a} \cdot \mathbf{R}_{-\pi/2} \mathbf{b}.$$

where $\mathbf{R}_{-\pi/2}$ is the rotation in clockwise direction by $\pi/2$.

A straightforward calculation shows that the following integration-by-parts formula holds for $u \in C^1(\overline{\Omega})$, $\mathbf{v} \in C^1(\overline{\Omega}; \mathbb{R}^2)$, assuming $\Omega$ is Lipschitz and bounded

$$\int_\Omega \mathbf{curl}\, u \cdot \mathbf{v}\, dx = \int_\Omega u \operatorname{curl} \mathbf{v}\, dx + \int_{\partial\Omega} u\, \mathbf{v} \times \mathbf{n}\, d\ell \tag{0.1.1}$$

where $\mathbf{n}$ is the outward unit normal of $\Omega$. Analogous to what we did in Sec. **??**, we can now extend this definition in the weak sense. First, notice that $\mathbf{curl}\, u = R_{-\pi/2}\, \mathbf{grad}\, u$ and thus $\mathbf{curl}$ is well-defined on $H^1$. We define

$$H(\operatorname{curl}; \Omega) = \left\{ \mathbf{v} \in L^2 \mid \exists w \in L^2 : \int_\Omega w\phi\, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \phi\, dx \quad \forall \phi \in C_0^\infty \right\}$$

and $w$ in the definition – which is uniquely determined – coincides with $\operatorname{curl} \mathbf{v}$ in distributional sense.

Using the notation of unbounded operators introduced in Sec. **??**, this is equivalent to $(\operatorname{curl}, H(\operatorname{curl})) = (\mathbf{curl}, C_0^\infty)^*$. Analogous to Section **??**, it is then possible to extend the operator

$$\mathbf{v} \mapsto \langle \mathbf{v} \times \mathbf{n},\, \cdot\, \rangle_{L^2(\partial\Omega)} \in H^{-1/2}(\partial\Omega) \tag{0.1.2}$$

defined on $C^1(\Omega; \mathbb{R}^2)$ to an operator $\gamma_\tau$ defined on $H(\operatorname{curl})$ s.t. for any $u \in H^1(\Omega)$, $\mathbf{v} \in H(\operatorname{curl})$ the integration by parts formula

$$\langle \mathbf{curl}\, u, \mathbf{v} \rangle = \langle u, \operatorname{curl} \mathbf{v} \rangle + \langle \gamma_\tau \mathbf{v}, \operatorname{tr} u \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}.$$

holds. From now on, we will leave out the subindex of the duality inner product. Also analogous to the 3D case, we can define

$$H_0(\operatorname{curl}) := \{ \mathbf{v} \in H(\operatorname{curl}) \mid \gamma_\tau \mathbf{v} = 0 \}$$

and can then compute the adjoints analogously to what we did in Section **??**,

$$
\begin{aligned}
(\mathrm{curl}, H_0(\mathrm{curl})) &= (\mathbf{curl}, H^1)^* \\
(\mathrm{curl}, H(\mathrm{curl})) &= (\mathbf{curl}, H^1_0)^*.
\end{aligned}
$$

Notice that $\operatorname{div} \mathbf{curl} = 0$ and so we have the following 2D Hilbert complex

$$0 \to H^1_0 \xrightarrow{\mathbf{curl}} H_0(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2 \to 0. \tag{0.1.3}$$

and the dual complex

$$0 \leftarrow L^2 \xleftarrow{\mathrm{curl}} H(\mathrm{curl}) \xleftarrow{-\,\mathbf{grad}} H^1 \leftarrow 0$$

We use the notation introduced in Sec. **??** for general Hilbert complexes i.e. $V^0 = H^1_0$, $V^1 = H_0(\mathrm{div})$, $V^2 = L^2$, $V^*_0 = L^2$, $V^*_1 = H(\mathrm{curl})$ $V^*_2 = H^1$, $d^0 = \mathbf{curl}$, $d^1 = \mathrm{div}$ and we set $d^k = 0$ for the remaining $k \in \mathbb{Z}$. $d^*_k$ is the adjoint of $d^k$. Also we remind of the notation $\mathfrak{B}^k$ for the image of the differential operator, $\mathfrak{B}^*_k$ for the image of the adjoint and analogous $\mathfrak{Z}^k$ for the kernel and $\mathfrak{Z}^*_k$ for the kernel of the adjoint.

**Remark 0.1.1.** Since we are working only on bounded domains in this and the coming sections, the Hilbert complex is closed, i.e. all the images of the differential operators are closed subspaces w.r.t. the $V$-norm. This was stated in Thm. **??** for the three-dimensional case, but it holds for 2D as well. The reason is that these are both special cases for the analogous result for the exterior derivative on Riemannian manifolds in arbitrary dimensions (see [1, Sec. 6.2.6]). This means in particular that we can use the Poincaré inequality (Thm. **??**).

### 0.1.2 Strong formulation of the 2D magnetostatic problem

The 2D magnetostatic problem will be derived from a special case of the 3D problem. Then the type of domains considered will be clarified and the strong formulation stated at the end.

Assume that our current source $\mathbf{J}$ is pointing in $z$-direction i.e. $\mathbf{J} = J\mathbf{e}_3$. Further assume that there is a $\tilde{\Omega}$ s.t. $\Omega = \tilde{\Omega} \times \mathbb{R}$. If $B_3$ does not change in $z$-direction we get that

$$0 = \operatorname{div} \mathbf{B} = \partial_x B_1 + \partial_y B_2 = \operatorname{div} \tilde{\mathbf{B}}.$$

where $\tilde{\mathbf{B}} = (B_1, B_2)^\top$. The third component of the equation $\operatorname{curl} \mathbf{B} = \mathbf{J}$ from the magnetostatic problem in three dimensions reads

$$J = \partial_x B_2 - \partial_y B_1 = \operatorname{curl} \tilde{\mathbf{B}}$$

The unit outer normal of $\Omega$ is zero in $z$-direction and thus $\tilde{\mathbf{B}}$ satisfies the boundary condition

$$0 = \mathbf{B} \cdot \mathbf{n} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{n}}$$

with $\tilde{\mathbf{n}} = (n_1, n_2)^\top$ being the outer unit normal $\tilde{\Omega}$.

Now we will abuse notation and refer to $\tilde{\mathbf{B}}$ as $\mathbf{B}$, $\tilde{\mathbf{n}}$ as $\mathbf{n}$ and $\tilde{\Omega}$ as $\Omega$. Let $J \in L^2$ be given. Then we see that $\mathbf{B}$ must fulfill the following equations

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= J, \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

Depending on the domain, this problem is in general not well-posed – just as the problem in 3D – and requires an additional constraint. Let us now make certain restrictions on what type of domain we will consider.

From now on, we assume that the space of harmonic forms $\mathfrak{H}^1$ has dimension one and that our domain is encompassed by two disjoint closed curves, i.e. we have curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$ s.t.

$$\partial\Omega_{in} \,\dot{\cup}\, \partial\Omega_{out} = \partial\Omega.$$

Let now $\Gamma$ be a closed curve in $\Omega$ that goes around the hole in the middle, i.e. the area surrounded by $\Gamma$ contains $\partial\Omega_{in}$. Denote its parametrization with $\boldsymbol{\gamma} : [0, |\Gamma|] \to \Omega$ s.t. $|\boldsymbol{\gamma}'(t)| = 1$ and assume that $\boldsymbol{\gamma}$ is bijective i.e. the curve does not intersect itself. We assume that $\Gamma$ has positive distance from $\partial\Omega_{in}$. We do not assume anything like that for the exterior boundary i.e. $\Gamma$ can touch or be identical to $\partial\Omega_{out}$. We then denote the area that is enclosed by $\Gamma$ and $\partial\Omega_{in}$ as $\Omega_\Gamma$ (cf. Fig. 0.1.1).

From now on, our domain $\Omega$ is always assumed to be of that kind. We will later make further restrictions on what types of domain we will consider that will be suitable for discretization (see Assumption 0.3.2).

We add the curve integral along $\Gamma$, which we assume to be well-defined, as an additional constraint. So in total, we obtain the following problem.

**Problem 0.1.2** (2D magnetostatic problem)**.** Assume $\Omega$ and the curve $\Gamma$ are of the form as described above. Given $J \in L^2(\Omega)$ and $C_0 \in \mathbb{R}$, find $\mathbf{B} \in H_0(\mathrm{div}; \Omega) \cap H(\mathrm{curl}; \Omega)$ s.t.

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= J, \\ \operatorname{div} \mathbf{B} &= 0, \\ \int_\Gamma \mathbf{B} \cdot d\ell &= C_0. \end{aligned}$$

Another option for the additional constraint would be an orthogonality constraint as discussed in [5, Sec. 3.5].

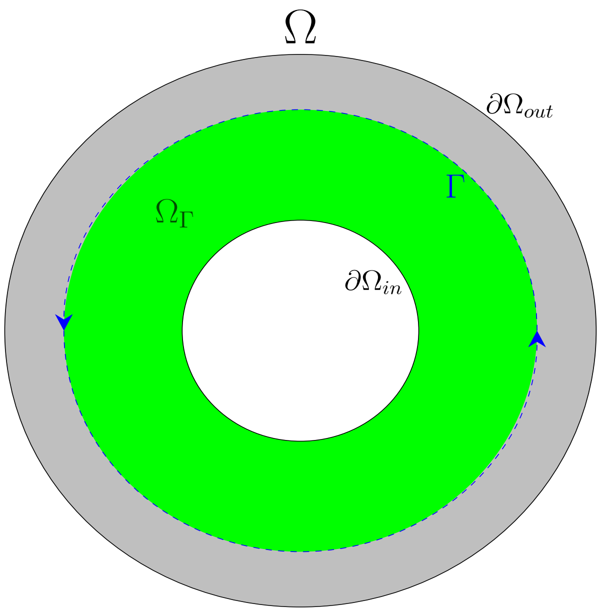


**Figure 0.1.1:** A simple example for a domain $\Omega$ as described in the text. The boundary is given by two disjoint curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$. The curve $\Gamma$ is parametrized in anticlockwise direction and $\Omega_\Gamma$ is the area enclosed by $\Gamma$ and $\partial\Omega_{in}$.

### 0.1.3 Mixed formulation

In order to solve this problem numerically using finite elements, we have to choose a suitable variational formulation. This variational formulation will be stated without the curve integral constraint and then we will show the equivalence with the strong formulation.

Ignoring the curve integral at first, we will use the following. We choose a non-zero harmonic form $\mathbf{p} \in \mathfrak{H}^1$ and have $J \in L^2$ . Then the problem is: Find $\sigma \in H_0^1$, $B \in H_0(\mathrm{div})$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \tag{0.1.4}$$

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \lambda \langle \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in H_0(\mathrm{div}) \tag{0.1.5}$$

As before, the inner product without subscript denotes the $L^2$ inner product and $\|\cdot\|$ the $L^2$-norm. Here the curve integral condition is missing. It is difficult to include the curve integral condition directly when solving this system numerically. So we will replace it below in Sec. 0.1.4.

Even though this formulation appears more complicated in comparison to the first two equations of the 2D magnetostatic problem (Problem 0.1.2), it will turn out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will first investigate the formulation without curve integral.

**Proposition 0.1.3.** *For any $J \in L^2$, (0.1.4) and (0.1.5) hold iif $\sigma = 0$, $\lambda = 0$, $\operatorname{curl} \mathbf{B} = J$ and $\operatorname{div} \mathbf{B} = 0$, i.e. $\mathbf{B}$ solves the 2D magnetostatic problem (Problem 0.1.2) without the additional curve integral constraint.*

*Proof.* Assume $(\sigma, \mathbf{B}, \lambda)$ is a solution of (0.1.4) and (0.1.5). Then the first equation is

$$\langle \sigma + J, \tau \rangle = \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle \quad \forall \tau \in H_0^1$$

which is equivalent to $\mathbf{B} \in H(\operatorname{curl})$ and $J + \sigma = \operatorname{curl} \mathbf{B}$.

Now assume additionally, that (0.1.5) holds. Then by choosing $\mathbf{v} = \mathbf{p} \in \mathfrak{H}^1$, we get $\operatorname{div} \mathbf{p} = 0$ from the definition of the harmonic forms and $\mathfrak{H}^1 \perp \mathbf{curl}\, H_0^1$ from the Hodge decomposition and thus

$$\langle \mathbf{curl}\, \sigma, \mathbf{p} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{p} \rangle + \lambda \langle \mathbf{p}, \mathbf{p} \rangle = \lambda \langle \mathbf{p}, \mathbf{p} \rangle = 0$$

and so $\lambda = 0$. Then we can choose $\mathbf{v} = \mathbf{curl}\, \sigma$ to get

$$\langle \mathbf{curl}\, \sigma, \mathbf{curl}\, \sigma \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{curl}\, \sigma \rangle + \lambda \langle \mathbf{p}, \mathbf{curl}\, \sigma \rangle = \|\mathbf{curl}\, \sigma\|^2 = 0.$$

Because $\sigma \in H_0^1$ this gives us $\sigma = 0$. Also we have then $J = \operatorname{curl} \mathbf{B}$. At last, we choose $\mathbf{v} = \mathbf{B}$ which gives us $\operatorname{div} \mathbf{B} = 0$ and thus we proved the first direction.

The other implication is clear, i.e. if $\mathbf{B} \in H(\operatorname{curl}) \cap H_0(\operatorname{div})$ with $\operatorname{curl} \mathbf{B} = J$ and $\operatorname{div} \mathbf{B} = 0$, $\sigma = 0$ and $\lambda = 0$, then the variational formulation clearly holds. $\square$

Notice that the variable $\lambda$ is not necessary for this variational formulation, but we will need it later, since we will add another equation representing the curve integral constraint and, for that purpose, we need another variable to have the same number of unknowns and equations. If we now add the same additional constraint to both formulations of the problem, then they will remain equivalent.

### 0.1.4 Curve integral constraint

We still need to find a good way to include the curve integral constraint from Problem 0.1.2 in our formulation. Instead of incorporating it directly, we will substitute it with another equation. We will first derive this equation as an immediate consequence of the integration by parts formula (0.1.1) and then state the final variational formulation of the 2D magnetostatic problem which we will investigate in the coming sections.

Let $\mathbf{n}$ be the unit outward normal of $\Omega_\Gamma$ and $\boldsymbol{\gamma}$ the parametrization of $\Gamma$. Then we know that $\mathbf{n} \perp \boldsymbol{\gamma}'$ and

$$\mathbf{B} \times \mathbf{n} = (B_1 n_2 - B_2 n_1) = \mathbf{B} \cdot \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} = -\mathbf{B} \cdot \mathbf{R}_{\pi/2}\mathbf{n}.$$

Now we assume again that $|\boldsymbol{\gamma}'| = 1$ and that $\Gamma$ does not intersect itself. Then $\mathbf{R}_{\pi/2}\mathbf{n}$ is either $\boldsymbol{\gamma}'$ or $-\boldsymbol{\gamma}'$. Assume w.l.o.g. that $\mathbf{R}_{\pi/2}\mathbf{n} = \boldsymbol{\gamma}'$ and thus

$$\mathbf{B} \times \mathbf{n} = -\mathbf{B} \cdot \boldsymbol{\gamma}'$$

and so the curve integral becomes

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_0^{|\Gamma|} \mathbf{B}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = -\int_\Gamma \mathbf{B} \times \mathbf{n}\, d\ell.$$

Choose $\psi \in H^1$ s.t.

$$\psi = 0 \text{ on } \partial\Omega_{in},\ \psi = 1 \text{ on } \Gamma \text{ and } \psi \equiv 1 \text{ in } \Omega \setminus \Omega_\Gamma. \tag{0.1.6}$$

We call such a $\psi$ *admissible for* $\Gamma$. Then we observe

$$\begin{aligned} &\int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx \\ &= \int_{\Omega_\Gamma} \psi\, J\, dx + \int_{\partial\Omega_\Gamma} \psi\, \mathbf{B} \times \mathbf{n}\, d\ell = \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Gamma \mathbf{B} \cdot d\ell \end{aligned}$$

and we can rewrite the curve integral as

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx. \tag{0.1.7}$$

So if the curve integral

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0$$

is given and we can compute $\int_{\Omega_\Gamma} \psi\, J\, dx$ we can add the equation

$$\langle \mathbf{curl}\,\psi, \mathbf{B} \rangle = C_1 \tag{0.1.8}$$

with

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0$$

to our system.

From the above derivations it is then clear that for $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0 \Leftrightarrow \langle \mathbf{curl}\,\psi, \mathbf{B} \rangle = C_1.$$

This is the motivation to add the right equation to our system instead of the curve integral since it is much easier to enforce numerically.

However, $\psi$ is of course not uniquely determined by (0.1.6) so we have to check what happens when we use a different admissible $\psi$ respecting these conditions. Assume first that $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$. Then we know from (0.1.7) that

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx = C_0 = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\,\tilde{\psi} \cdot \mathbf{B}\, dx$$

which immediately implies that, if we define

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0,$$

then we know

$$\langle \mathbf{curl}\,\psi, \mathbf{B} \rangle = C_1 \Leftrightarrow \langle \mathbf{curl}\,\tilde{\psi}, \mathbf{B} \rangle = \tilde{C}_1 \tag{0.1.9}$$

so the choice of $\psi$ does not matter in this case as long as the right hand side is computed accordingly.

However, $\mathbf{B}$ might not be $C^1(\overline{\Omega}; \mathbb{R}^2)$. But assume, that we are still given $C_0 \in \mathbb{R}$. Then the question arises if the choice of $\psi$ matters. This is not the case as the following proposition shows.

**Proposition 0.1.4.** *Let $\psi$ and $\tilde{\psi}$ both fulfill the assumptions given by (0.1.6), i.e. they are both admissible for $\Gamma$, and $(\sigma, \mathbf{B}, \lambda)$ be a solution of (0.1.4) and (0.1.5). Then*

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\,\tilde{\psi} \cdot \mathbf{B}\, dx. \tag{0.1.10}$$

*Proof.* We know that $\psi = \tilde{\psi} = 0$ on $\partial\Omega_{in}$. Also, both are equal to one on $\Gamma$ and between $\Gamma$ and $\partial\Omega_{out}$ and so we also know that $\psi = \tilde{\psi} = 1$ on $\partial\Omega_{out}$. Hence, $\psi - \tilde{\psi} \in H_0^1(\Omega)$ and $\psi - \tilde{\psi} \equiv 0$ in $\Omega \setminus \Omega_\Gamma$. Recall that when $(\sigma, \mathbf{B}, \lambda)$ solves (0.1.4) and (0.1.5) then $\sigma = 0$ and $\lambda = 0$ as proven in Prop. 0.1.3. So because $\mathbf{B}$ solves (0.1.4) we get

$$\int_\Omega \mathbf{curl}(\psi - \tilde{\psi}) \cdot \mathbf{B}\, dx = \int_\Omega (\psi - \tilde{\psi})\, J\, dx = \int_{\Omega_\Gamma} (\psi - \tilde{\psi})\, J\, dx.$$

$\square$

That means if we are just given $C_0$ and define

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0$$

as before, then the equivalence (0.1.9) holds as well, even if $\mathbf{B}$ is not continuously differentiable.

There is an important invariance property of the curve integral that we would like to have for its replacement as well. Take two closed curves $\Gamma_1$ and $\Gamma_2$ with the same assumption as before, i.e. parametrized counterclockwise and not intersecting themselves. We obtain $\Omega_{\Gamma_1}$ and $\Omega_{\Gamma_2}$ defined as before. If $J = 0$ in $\Omega_{\Gamma_1}$ and $\Omega_{\Gamma_2}$, i.e. it is zero between $\partial\Omega_{in}$ and both curves, then we have

$$\int_{\Gamma_1} \mathbf{B} \cdot d\ell = \int_{\Gamma_2} \mathbf{B} \cdot d\ell. \tag{0.1.11}$$

We would like this to be preserved for our replacement of the curve integral which is indeed the case.

**Proposition 0.1.5.** *Let $\Gamma_1$ and $\Gamma_2$ be chosen as just described and admissible $\psi_1$ and $\psi_2$ respectively. Let $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\mathrm{div})$ and $\lambda \in \mathbb{R}$ be a solution of (0.1.4) and (0.1.5) with $J = 0$ in $\Omega_{\Gamma_1}$ and $\Omega_{\Gamma_2}$. Then $C_1 = -C_0$ and*

$$\langle \mathbf{curl}\, \psi_1, \mathbf{B} \rangle = \langle \mathbf{curl}\, \psi_2, \mathbf{B} \rangle.$$

*Proof.* $C_1 = -C_0$ is obvious due to $J = 0$ in in $\Omega_{\Gamma_1}$ and $\Omega_{\Gamma_2}$. Notice that even if we have different curves $\Gamma_1$ and $\Gamma_2$ the fact that $\psi_1$ and $\psi_2$ are constant one between the curve and $\partial\Omega_{out}$ implies that $\psi_1 - \psi_2 = 0$ on $\partial\Omega_{out}$. From the admissibility (0.1.6), we also get $\psi_1 - \psi_2 = 0$ on $\partial\Omega_{in}$ so in total $\psi_1 - \psi_2 \in H_0^1$. Recall that if $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$. So because $\mathbf{B}$ solves (0.1.4) and (0.1.5), $J = 0$ in $\Omega_{\Gamma_1} \cup \Omega_{\Gamma_2}$ and $\psi_1 = \psi_2 = 1$ in $\Omega \setminus (\Omega_{\Gamma_1} \cup \Omega_{\Gamma_2})$, this means

$$\langle \mathbf{curl}(\psi_1 - \psi_2), \mathbf{B} \rangle = \int_{\Omega \setminus (\Omega_{\Gamma_1} \cup \Omega_{\Gamma_2})} (\psi_1 - \psi_2)\, J\, dx = 0.$$

□

In order to get a variational formulation to study theoretically, we multiply (0.1.8) with an arbitrary $\mu \in \mathbb{R}$. In conclusion, we have the following variational problem:

**Problem 0.1.6.** Let $J \in L^2$, $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Find $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\mathrm{div})$, $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \qquad \forall \tau \in H_0^1, \tag{0.1.12}$$

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle = 0 \qquad \forall \mathbf{v} \in H_0(\mathrm{div}), \tag{0.1.13}$$

$$\mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = \mu C_1 \qquad \forall \mu \in \mathbb{R}, \tag{0.1.14}$$

which is the variational formulation of the magnetostatic problem with curve integral constraint (Problem 0.1.2). We will study the well-posedness of this formulation next.

### 0.1.5 Well-posedness of the magnetostatic system

The well-posedness is based on the well-known Banach-Nečas-Babuška (BNB) theorem concerning general variational problems of the following form: Find $x \in X$ s.t.

$$a(x, y) = \ell(y) \quad \forall y \in Y \tag{0.1.15}$$

where $X$ and $Y$ are Banach spaces, $a$ is a bilinear form and $\ell \in Y'$. The BNB-theorem then answers the question of well-posedness, i.e. if there exists a unique solution and if we can find a stability estimate. The following formulation is from [4, Sec. 25.3] in the real case.

**Theorem 0.1.7** (BNB). *Let $X$ be a Banach space and $Y$ be a reflexive Banach space. Let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and $\ell \in Y'$. Then a problem of the form (0.1.15) is well-posed iif the following two criteria are fulfilled*

$$(1) \quad \inf_{x\in X} \sup_{y\in Y} \frac{|a(x, y)|}{\|x\|_X \|y\|_Y} =: \gamma > 0 \tag{0.1.16}$$

(2) *for any $y \in Y$ if $a(x, y) = 0$ for every $x \in X$, then $y = 0$.* (0.1.17)

*We then obtain the stability estimate for a solution $x$*

$$\|x\|_X \leq \frac{1}{\gamma} \|\ell\|_{Y'}.$$

Note that (0.1.16) is equivalent to the fact that for any $x \in X \setminus \{0\}$ there exists $y \in Y \setminus \{0\}$ s.t. $a(x, y) \geq \gamma \|x\|_X \|y\|_Y$.

Since we are dealing with Hilbert spaces only we can utilize the following proposition to prove it (see [4, Rem. 25.14]).

**Proposition 0.1.8** ($T$-coercivity). *Let $X$ and $Y$ be Hilbert spaces. Then* (0.1.16) *and* (0.1.17) *hold, if there exists a bounded bijective operator $T : X \to Y$ and $\eta > 0$ s.t.*

$$a(x, Tx) \geq \eta \|x\|_X^2 \quad \forall x \in X. \tag{0.1.18}$$

*Then $\gamma$ from* (0.1.16) *can be chosen as $\eta / \|T\|_{\mathcal{L}(X,Y)}$.*

*Proof.* For any $x \in X$, by taking $y = Tx \in Y$ and using the boundedness of $T$, we have

$$a(x, T(x)) \geq \eta \|x\|^2 \geq \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}} \|x\|_X \|y\|_Y$$

and thus (0.1.16) holds with $\gamma = \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}}$.

For (0.1.17) assume that we have $y \in Y$ s.t. $a(x, y) = 0$ for all $x \in X$.

$$0 = a(T^{-1}y, TT^{-1}y) \geq \eta \|T^{-1}y\|_X^2$$

so $T^{-1}y = 0$ and thus $y = 0$. $\square$

**Remark 0.1.9.** The other direction is also true, i.e. if (0.1.16) and (0.1.17) are fulfilled, we can construct a $T$ with the desired properties.

Note also that when we have found $T$ s.t. (0.1.18) holds then it must be injective. This is because if $Tx = 0$ for any $x \in X$ then $x = 0$ must hold.

The next step is to put our formulation of Problem 0.1.6 into this general framework. To this end, we define $X := H_0^1 \times H_0(\mathrm{div}) \times \mathbb{R}$ and for $(\sigma, \mathbf{B}, \lambda) \in X$

$$\|(\sigma, \mathbf{B}, \lambda)\|_X := \sqrt{\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2}.$$

Notice that $X$ is then a Hilbert space with inner product

$$\langle (\sigma, \mathbf{B}, \lambda), (\tau, \mathbf{v}, \mu) \rangle_X = \langle \sigma, \tau \rangle_{H^1} + \langle \mathbf{B}, \mathbf{v} \rangle_{H(\mathrm{div})} + \lambda\mu.$$

For the formulation of the problem, we define the bilinear form $a : X \times X \to \mathbb{R}$

$$\begin{aligned} a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle + \langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle \\ + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle \end{aligned} \tag{0.1.19}$$

and

$$\ell(\tau, \mathbf{v}, \mu) = -\langle J, \tau \rangle - \mu C_1$$

which are bounded due to the Cauchy-Schwarz inequality. Then Problem 0.1.6 is equivalent to the following: Find $(\sigma, \mathbf{B}, \lambda) \in X$ s.t.

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \ell(\tau, \mathbf{v}, \mu) \quad \forall (\tau, \mathbf{v}, \mu) \in X.$$

Note that the bilinear form $a$ is not symmetric.

The next step is to show important properties of $\mathbf{curl}\, \psi$ assuming $\psi$ is admissible for $\Gamma$ i.e. (0.1.6) holds.

**Proposition 0.1.10.** *Under the given assumptions on $\psi$, $\mathbf{curl}\,\psi \in H_0(\mathrm{div})$.*

*Proof.* We need to show $0 = \gamma_n \,\mathbf{curl}\,\psi$. The integration by parts from Thm. **??** gives

$$\langle \gamma_n \,\mathbf{curl}\,\psi, \mathrm{tr}\, u\rangle = \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\, u \, dx + \int_\Omega \mathrm{div}\,\mathbf{curl}\,\psi \, u \, dx$$

where the last term vanishes. Take now $\phi \in C^1(\overline{\Omega})$ arbitrary. Then we take $\phi_1 \in C^1(\overline{\Omega})$ s.t. $\phi_1 = \phi$ in a neighborhood of $\partial\Omega_{in}$ and zero near $\partial\Omega_{out}$. Analogously, take $\phi_2 \in C^1(\overline{\Omega})$ s.t. $\phi_2 = \phi$ in a neighborhood of $\partial\Omega_{out}$ and zero near $\partial\Omega_{in}$. Here we used the fact that the two parts of the boundary are disjoint and have positive distance from one another. Then also $\mathrm{tr}\,\phi = \mathrm{tr}\,\phi_1 + \mathrm{tr}\,\phi_2$. We use the integration-by-parts formula for the divergence again, $\mathrm{div}\,\mathbf{curl} = 0$ and the integration-by-parts formula for the curl

$$\begin{aligned}
&\langle \gamma_n \,\mathbf{curl}\,\psi, \mathrm{tr}\,\phi\rangle = \langle \gamma_n \,\mathbf{curl}\,\psi, \mathrm{tr}\,\phi_1\rangle + \langle \gamma_n \,\mathbf{curl}\,\psi, \mathrm{tr}\,\phi_2\rangle \\
&= \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\,\phi_1 \, dx + \int_\Omega \mathbf{curl}\,\psi \cdot \mathbf{grad}\,\phi_2 \, dx \\
&= \int_\Omega \psi \ \mathrm{curl}\,\mathbf{grad}\,\phi_1 \, dx + \langle \gamma_\tau \,\mathbf{grad}\,\phi_1, \psi\rangle + \int_\Omega \psi \ \mathrm{curl}\,\mathbf{grad}\,\phi_2 \, dx + \langle \gamma_\tau \,\mathbf{grad}\,\phi_2, \psi\rangle.
\end{aligned}$$

Here only the boundary terms remain because $\mathrm{curl}\,\mathbf{grad} = 0$. Now remember that because $\phi_j \in C^1(\overline{\Omega})$, $\langle \gamma_\tau \,\mathbf{grad}\,\phi_j, \psi\rangle = \langle \mathbf{grad}\,\phi_j \times \mathbf{n}, \psi\rangle_{L^2(\partial\Omega)}$. So

$$\langle \gamma_\tau \,\mathbf{grad}\,\phi_1, \psi\rangle = \langle \mathbf{grad}\,\phi_1 \times \mathbf{n}, \psi\rangle_{L^2(\partial\Omega)} = 0$$

because $\phi_1$ is zero near $\partial\Omega_{out}$ and $\psi$ is zero on $\partial\Omega_{in}$. For the remaining term,

$$\begin{aligned}
\langle \gamma_\tau \,\mathbf{grad}\,\phi_2, \psi\rangle &= \int_{\partial\Omega_{out}} \psi \,\mathbf{grad}\,\phi_2 \times \mathbf{n} \, d\ell = \int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_2 \times \mathbf{n} \, d\ell \\
&= -\int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_2 \cdot d\ell
\end{aligned}$$

and then we know from basic vector calculus because $\partial\Omega_{out}$ is closed

$$\int_{\partial\Omega_{out}} \mathbf{grad}\,\phi_1 \cdot d\ell = 0$$

and so in conclusion, $\gamma_n \,\mathbf{curl}\,\psi = 0$. □

The last equation in Problem 0.1.6 is used to determine the harmonic part of the solution. This implies that we would like $\mathbf{curl}\,\psi$ to have non-vanishing harmonic part. This is indeed true.

**Proposition 0.1.11.** *Let $P_{\mathfrak{H}^1} : L^2 \to \mathfrak{H}^1$ be the orthogonal projection onto the harmonic forms. Then with $\psi$ being admissible for $\Gamma$, i.e. fulfilling conditions (0.1.6), we have $P^1_{\mathfrak{H}} \operatorname{\mathbf{curl}} \psi \neq 0$.*

*Proof.* Since $\operatorname{div} \operatorname{\mathbf{curl}} \psi = 0$ and $\operatorname{\mathbf{curl}} \psi \in H_0(\operatorname{div})$, we know that

$$\operatorname{\mathbf{curl}} \psi \in \mathfrak{Z}^1 = \mathfrak{B}^1 \overset{\perp}{\oplus} \mathfrak{H}^1$$

using the Hodge decomposition (cf. Thm. **??**). Assume for contradiction that $\operatorname{\mathbf{curl}} \psi \in \mathfrak{B}^1$, i.e. there exists $\psi_0 \in H^1_0$ s.t. $\operatorname{\mathbf{curl}} \psi_0 = \operatorname{\mathbf{curl}} \psi$. Since $\operatorname{\mathbf{curl}}$ is just the rotated gradient we would get that $\operatorname{\mathbf{grad}}(\psi - \psi_0) = 0$ and thus $\psi - \psi_0$ is constant almost everywhere. But this is a contradiction since $\operatorname{tr} \psi_0$ is zero on $\partial\Omega_{in}$ and $\partial\Omega_{out}$, but $\operatorname{tr} \psi = 0$ on $\partial\Omega_{in}$ and $\operatorname{tr} \psi = 1$ on $\partial\Omega_{in}$. Thus $\operatorname{\mathbf{curl}} \psi \notin \mathfrak{B}^1$ and the claim follows. □

Now we can apply the Hodge decomposition of $\ker \operatorname{div}$ on $\operatorname{\mathbf{curl}} \psi$ and obtain the following

**Corollary 0.1.12.** *Let $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Then there exists $\psi_0 \in H^1_0$ and $c_\psi \in \mathbb{R} \setminus \{0\}$ s.t.*

$$\operatorname{\mathbf{curl}} \psi = \operatorname{\mathbf{curl}} \psi_0 + c_\psi \mathbf{p}. \tag{0.1.20}$$

Here it is important to remember that we assumed $\mathfrak{H}^1$ to be one-dimensional. Because we can choose $\mathbf{p}$, we can assume w.l.o.g. that $c_\psi > 0$ and we will do so from now on. Also we can prove that the harmonic part is independent of the chosen $\psi$.

**Proposition 0.1.13.** *Choose $\psi$ and $\tilde{\psi}$ admissible for $\Gamma$ according to (0.1.6). Then*

$$P_{\mathfrak{H}^1} \operatorname{\mathbf{curl}} \psi = P_{\mathfrak{H}^1} \operatorname{\mathbf{curl}} \tilde{\psi}.$$

*In particular, if we decompose them as in (0.1.20) then $c_\psi = c_{\tilde{\psi}}$.*

*Proof.* Take $\mathbf{p} \in \mathfrak{H}^1$ arbitrary. Then using the same argument as in the proof of Prop. 0.1.4, we get $\psi - \tilde{\psi} \in H^1_0$ and because $\mathfrak{H}^1 \perp \mathfrak{B}^1$

$$\langle \operatorname{\mathbf{curl}}(\psi - \tilde{\psi}), \mathbf{p} \rangle = 0. \tag{0.1.21}$$

Because $\mathbf{p} \in \mathfrak{H}^1$ was arbitrary the claim follows. □

As stated in the proof of the Poincare inequality (Thm. **??**), $\operatorname{\mathbf{curl}}|_{\mathfrak{Z}^\perp} : \mathfrak{Z}^\perp \to \mathfrak{B}^1$ is bijective and since it is bounded w.r.t. the $V$-norm – which is the $H^1$-norm here – due to the Banach inverse theorem it is invertible and we denote this inverse

$\mathbf{curl}^{-1}$. This is a slight abuse of notation since it is not really the inverse of the full $\mathbf{curl}$.

Let $P_{\mathfrak{B}}$ be the $L^2$-orthogonal projection onto $\mathfrak{B}^j$, we then denote $\mathbf{v}_{\mathfrak{B}} = P_{\mathfrak{B}}\mathbf{v}$ for any $\mathbf{v} \in L^2$ and analogous for $\mathfrak{H}^j$ and $\mathfrak{B}^*_j$. In order to prove the $T$-coercivity, we need the following lemma.

**Lemma 0.1.14.** *Take* $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ *with* $c_\psi > 0$, $\mathbf{p} \in \mathfrak{H}^1$ *and* $\|\mathbf{p}\| = 1$. *Define* $T : X \to X$ *as*

$$T(\sigma, \mathbf{B}, \lambda) = \left(\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}, \alpha\langle\mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right).$$

*with* $\alpha < 0$ *and* $\beta > 0$. *Then* $T$ *is bounded and surjective.*

*Proof.* The boundedness is clear since all operators used in the definition are bounded w.r.t. the norm of their domains. From the Poincaré inequality, we know that $\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\| \le c_P\|\mathbf{B}_{\mathfrak{B}}\|$ and so

$$\begin{aligned}
&\|T(\sigma, \mathbf{B}, \lambda)\|_X^2 \\
&= \|\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + \|\mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}\|_{H(\mathrm{div})}^2 + \left(\alpha\langle\mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right)^2 \\
&\le 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^4}\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 \\
&\quad + 2\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\le 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}\|_{H(\mathrm{div})}^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\le C_T\left(\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2\right)
\end{aligned}$$

with

$$C_T := \max\left\{5, \frac{2}{c_P^2} + 3 + 2\alpha^2, 3\beta^2 + \frac{2}{c_\psi^2}\right\}. \tag{0.1.22}$$

So $T$ is bounded and $\|T\|_{\mathcal{L}(X,X)} \le \sqrt{C_T}$.

Take $(\tau, \mathbf{v}, \mu) \in X$ arbitrary. In order to prove surjectivity, we will split up $\mathbf{v} = \mathbf{v}_{\mathfrak{B}} + \mathbf{v}_{\mathfrak{H}} + \mathbf{v}_{\mathfrak{B}^*}$ using the Hodge decomposition and choose

$$\sigma = \left(1 + \frac{1}{c_P^2}\right)^{-1}\left(\tau + \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}}\right) \text{ and } \mathbf{B}_{\mathfrak{B}} = \mathbf{v}_{\mathfrak{B}} - \mathbf{curl}\,\sigma.$$

So

$$\sigma - \frac{1}{c_P^2} \mathbf{curl}^{-1} \mathbf{B}_{\mathfrak{B}} = \sigma - \frac{1}{c_P^2} (\mathbf{curl}^{-1} \mathbf{v}_{\mathfrak{B}} - \sigma)$$
$$= \Big(1 + \frac{1}{c_P^2}\Big)\sigma - \frac{1}{c_P^2} \mathbf{curl}^{-1} \mathbf{v}_{\mathfrak{B}} = \tau.$$

We simply choose $\mathbf{B}_{\mathfrak{B}^*} = \mathbf{v}_{\mathfrak{B}^*}$. For the harmonic part take $\kappa_v$ s.t. $\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p}$. Let us look at the system

$$\begin{pmatrix} 1 & \beta \\ \alpha & 1/c_\psi \end{pmatrix} \begin{pmatrix} \kappa_B \\ \lambda \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \mu \end{pmatrix}$$

Now since $c_\psi > 0$ and $\alpha < 0$, $\beta > 0$ we get $1/c_\psi - \alpha\beta \neq 0$ and the system has a solution. Choose $\mathbf{B}_{\mathfrak{H}} = \kappa_B \mathbf{p}$. Then we see

$$\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p} = \mathbf{p}(\kappa_B + \beta\lambda) = \mathbf{B}_{\mathfrak{H}} + \beta\lambda\mathbf{p}$$

and

$$\mu = \alpha\kappa_B + \frac{\lambda}{c_\psi} = \alpha\kappa_B \|\mathbf{p}\|^2 + \frac{\lambda}{c_\psi} = \alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi}.$$

By combining all that we arrive at $T(\sigma, \mathbf{B}, \lambda) = (\tau, \mathbf{v}, \mu)$. □

We assume from now on that we have always chosen $\mathbf{p}$ in a way s.t. $c_\psi$ – as defined in the previous lemma – is positive and $\mathbf{p}$ has norm one. This comes down to choosing $\mathbf{p}$ with the correct sign and normalizing it. Now we can use the T-coercivity (Prop. 0.1.8) to prove the inf-sup condition and thus well-posedness of our formulation.

**Theorem 0.1.15.** *Let $\psi$ be admissible for $\Gamma$ and $\mathbf{curl}\,\psi_0 + c_\psi \mathbf{p} = \mathbf{curl}\,\psi$ and assume we have chosen the sign of $\mathbf{p}$ s.t. $c_\psi > 0$. Then take $c_1 > 0$ s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$ (e.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$ would be a valid choice). Define $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ and $\alpha = -\frac{c_\psi}{4c_1^2 c_P^2}$. Then the bilinear form $a$ defined at (0.1.19) satisfies the inf-sup condition, i.e. (0.1.16) and (0.1.17) with $\gamma \geq \eta/\sqrt{C_T}$, $C_T$ from (0.1.22) and*

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2}\right\}.$$

*Proof.* Choose $(\sigma, \mathbf{B}, \lambda) \in X$ arbitrary and define $\rho := \mathbf{curl}^{-1} \mathbf{B}_{\mathfrak{B}}$. We take $T$ as in (0.1.14),

$$T(\sigma, \mathbf{B}, \lambda) = \Big(\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\Big)$$

Then $T$ is surjective due to Lemma 0.1.14. Note

$$\langle \mathbf{B}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2 \langle \frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2$$

where we used in the last equality that $\frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}$ is either $+\mathbf{p}$ or $-\mathbf{p}$ because $\mathfrak{H}^1$ is assumed to be one-dimensional. We split up $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ to get

$$\begin{aligned}
&a(\sigma, \mathbf{B}, \lambda; T(\sigma, \mathbf{B}, \lambda)) \\
&= \langle \sigma, \sigma - \frac{1}{c_P^2}\rho\rangle - \langle \mathbf{B}, \mathbf{curl}\,\sigma - \frac{1}{c_P^2}\mathbf{curl}\,\rho\rangle + \langle \mathbf{curl}\,\sigma, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle \\
&\quad + \langle \operatorname{div}\mathbf{B}, \operatorname{div}\mathbf{curl}\,\sigma + \operatorname{div}\mathbf{B} + \beta\lambda \operatorname{div}\mathbf{p}\rangle \\
&\quad + \langle \lambda\mathbf{p}, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle - \Big(\alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi}\Big)\langle \mathbf{B}, \mathbf{curl}\,\psi\rangle \\
&= \|\sigma\|^2 - \frac{1}{c_P^2}\langle \sigma, \rho\rangle + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\beta - \alpha c_\psi \|\mathbf{B}_{\mathfrak{H}}\|^2 \\
&\quad - \alpha\langle \mathbf{p}, \mathbf{B}_{\mathfrak{H}}\rangle\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle - \frac{\lambda}{c_\psi}\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle
\end{aligned}$$

Due to the Poincaré inequality

$$\|\rho\| \leq \|\rho\|_{H^1} \overset{\text{Poincaré}}{\leq} c_P\|\mathbf{curl}\,\rho\| = c_P\|\mathbf{B}_{\mathfrak{B}}\|.$$

Using $\epsilon$-Young combined with Cauchy-Schwarz inequality several times, we obtain the lower bound.

$$\begin{aligned}
&\|\sigma\|^2 - \left(\frac{1}{2}\|\sigma\|^2 + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2}{2c_P^2}\right) + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 \\
&\quad + \lambda^2\beta - \alpha c_\psi\|\mathbf{B}_{\mathfrak{H}}\|^2 - \left(\frac{\epsilon_1\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2}{2} + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2\epsilon_1}\right) \\
&\quad - \left(\frac{\lambda^2}{2\epsilon_2 c_\psi^2} + \frac{\epsilon_2\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2}\right)
\end{aligned}$$

Choose $\epsilon_1 = 4c_1^2c_P^2$ to get

$$\begin{aligned}
&\frac{1}{2}\|\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{1}{2\epsilon_2 c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(-\alpha c_\psi - \frac{4c_1^2c_P^2\alpha^2}{2}\right) - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2c_P^2} - \|\mathbf{B}_{\mathfrak{B}}\|^2\frac{\epsilon_2\|\mathbf{curl}\,\psi_0\|^2}{2}
\end{aligned}$$

Now choose $\epsilon_2 = \frac{1}{4c_1^2 c_P^2}$, plug in the definition of $\alpha$ and use $\|\mathbf{curl}\,\psi_0\| \leq c_1$ to get the next lower bound

$$\frac{1}{2}\|\sigma\|^2 + \|\mathbf{B}_{\mathfrak{B}}\|^2 \left(\frac{1}{2c_P^2} - \frac{1}{8c_P^2} - \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2 c_P^2}\right) + \|\mathbf{curl}\,\sigma\|^2 + \|\operatorname{div}\mathbf{B}\|^2$$
$$+ \lambda^2 \left(\beta - \frac{4c_1^2 c_P^2}{2c_\psi^2}\right) + \|\mathbf{B}_{\mathfrak{H}}\|^2 \left(\frac{c_\psi^2}{4c_1^2 c_P^2} - \frac{c_1^2 c_P^2 c_\psi^2}{8c_1^4 c_P^4}\right)$$

and finally by using the Poincaré inequality $\|\mathbf{B}_{\mathfrak{B}^*}\| \leq c_P \|\operatorname{div}\mathbf{B}\|$ and $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ we obtain the next bound

$$\frac{1}{2}\|\sigma\|^2 + \frac{1}{4c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}^*}\|^2 + \frac{1}{2}\|\operatorname{div}\mathbf{B}\|^2$$
$$+ \frac{c_1^2 c_P^2}{c_\psi^2}\lambda^2 + \frac{c_\psi^2}{8c_1^2 c_P^2}\|\mathbf{B}_{\mathfrak{H}}\|^2 \geq \eta\|(\sigma, \mathbf{B}, \lambda)\|_X^2$$

where we chose

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2}\right\}$$

to obtain the $T$-coercivity. We can then choose $\gamma$ from (0.1.16) as $\eta/\|T\|_{\mathcal{L}(X,X)}$ and then use $C_T$ from (0.1.22) to get a lower bound

$$\gamma \geq \frac{\eta}{\sqrt{C_T}}.$$

□

**Corollary 0.1.16** (Well-posedness)**.** *The variational formulation of the magnetostatic problem* (Problem 0.1.6) *is well-posed. For a solution* $(\sigma, \mathbf{B}, \lambda) \in X$*, we have the stability estimate*

$$\|\mathbf{B}\| = \|(\sigma, \mathbf{B}, \lambda)\|_X \leq \frac{\|J\| + |C_1|}{\gamma}.$$

*Proof.* Recall that when $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$ and $\operatorname{div}\mathbf{B} = 0$ which implies the first equality. The statement follows immediately from the previous theorem, Thm. 0.1.7 and the fact that

$$|\ell(\tau, \mathbf{v}, \mu)| = |-\langle J, \tau\rangle - C_1\mu| \leq (\|J\| + |C_1|)\|(\tau, \mathbf{v}, \mu)\|_X$$

and thus $\|\ell\|_{X'} \leq \|J\| + |C_1|$. □

**Remark 0.1.17.** Note that $1/c_\psi$ terms arise in the stability constant $1/\gamma$. This is not surprising since the term $\langle \mathbf{curl}\,\psi, \mathbf{B}\rangle$ will not enforce the harmonic part if the harmonic part of $\mathbf{curl}\,\psi$ would be zero because $\mathbf{B}_{\mathfrak{H}}$ will disappear from the formulation. But we know from Prop. 0.1.13 that the harmonic part of $\mathbf{curl}\,\psi$ is actually independent of the choice of $\psi$. If we take $c_1 = \|\mathbf{curl}\,\psi\| + 1$ then we could expect stability issues if this value becomes very large. But this should not be the case for a reasonable choice of $\psi$.

## 0.2 Discretized magnetostatic problem

In order to approximate solutions of the 2D magnetostatic problem, we want to use finite elements. A very typical question that arises for any discretization of a model is what notions of the continuous model are represented in the discretized one. In our case, a fundamental structure of our problem is the Hilbert complex that we introduced in Section 0.1. We would like to represent it in our discretization, which leads us to the discrete Hilbert complex. This section follows Sec. 5.2 in Arnold's book [1]. We start with reviewing the theory behind the discretization of general Hilbert complexes before applying this theory to our problem. Once we discretized it, we will utilize the inf-sup condition proven in Section 0.1 to prove well-posedness of the discrete formulation and derive a quasi-optimal error estimate.

### 0.2.1 Discrete Hilbert complex

Let us at first stick to the general situation. We assume that we have a Hilbert complex $(W^k, d^k)$ with its corresponding domain complex $(V^k, d^k)$ and dual complex $(V_k^*, d_k^*)$ for $k \in \mathbb{Z}$. For this section, we will only need a short subsequence

$$V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1}$$

for some fixed $k$ and $j \in \{k-1, k, k+1\}$ will always be the index to refer to all three spaces. We will sometimes leave out the indices if the meaning is clear from the context.

Let us assume that we have finite dimensional subspaces $V_h^j \subseteq V^j$. As usual in numerical analysis, $h > 0$ stands loosely for the fineness of our discretization, e.g. the grid size or the maximal diameter of the elements in the mesh. Then we define completely analogous to the continuous case,

$$\begin{aligned}
\mathfrak{Z}_h^j &:= \{v_h \in V_h^j \mid d^j v_h = 0\} = \ker d^j \cap V_h^j \\
\mathfrak{B}_h^j &:= \{d^j v_h \mid v_h \in V_h^{j-1}\}.
\end{aligned}$$

We can now also define the discrete harmonic forms. This time, the situation is slightly different however. We will not use the adjoint $d_j^*$ to define it. Instead,

$$\mathfrak{H}_h^j := \{v_h \in \mathfrak{Z}_h^j \mid v_h \perp \mathfrak{B}_h^j\} = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp}.$$

Notice that we have $\mathfrak{Z}_h^j \subseteq \mathfrak{Z}^j$ and $\mathfrak{B}_h^j \subseteq \mathfrak{B}^j$, but due to $\mathfrak{B}_h^{j,\perp} \supseteq \mathfrak{B}^{j,\perp}$ we have in general

$$\mathfrak{H}_h^j = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp} \not\subseteq \mathfrak{Z}^j \cap \mathfrak{B}^{j,\perp} = \mathfrak{H}^j.$$

We will later investigate the difference between the spaces of discrete and continuous harmonic forms more closely.

There are three crucial assumptions that we will need to prove stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want the discrete spaces $V_h^j$ to approximate the continuous ones $V^j$.

**Assumption 0.2.1.** For the discrete spaces, we require $V_h^j \subseteq V^j$ and

$$\lim_{h\to 0} \inf_{v_h \in V_h^j} \|w - v_h\|_V = 0, \quad \forall w \in V^j.$$

This is usually satisfied for a reasonable choice of finite element space. The next property is more restrictive and is a compatibility condition between the spaces.

**Assumption 0.2.2.** For $j = k-1, k$,

$$dV_h^j \subseteq V_h^{j+1}.$$

This means that we cannot simply use arbitrary discrete subspaces independent from one another. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. **??** directly to it. Let us do that.

Denote the restriction of $d^j$ to $V_h^j$ as $d_h^j$. Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \to V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to $d_h^j$ the adjoint $d_{j,h}^*$ is not the restriction of the adjoint $d_j^*$. In general, $V_h^j \not\subseteq V_j^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h^j$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1} \tag{0.2.1}$$

and its dual complex

$$V_h^{k-1} \xleftarrow{d^*_{k,h}} V_h^k \xleftarrow{d^*_{k+1,h}} V_h^{k+1}$$

From the general Hilbert complex theory (cf. Thm. **??**), we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \overset{\perp}{\oplus} \mathfrak{H}_h^j \overset{\perp}{\oplus} \mathfrak{B}^*_{jh}.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. We will investigate the question how well the discrete harmonic forms approximate the continuous ones more thoroughly later.

**Assumption 0.2.3.** There exist *bounded cochain projections* $\Pi_h^j : V^j \to V_h^j$. This is a projection that is a cochain map in the sense of cochain complexes (see Sec. **??**), i.e. the following diagram commutes:

$$\begin{array}{ccccc} V^{k-1} & \xrightarrow{d^{k-1}} & V^k & \xrightarrow{d^k} & V^{k+1} \\ \downarrow{\scriptstyle \Pi_h^{k-1}} & & \downarrow{\scriptstyle \Pi_h^k} & & \downarrow{\scriptstyle \Pi_h^{k+1}} \\ V_h^{k-1} & \xrightarrow{d^{k-1}} & V_h^k & \xrightarrow{d^k} & V_h^{k+1} \end{array}$$

$\Pi_h^j$ are either bounded in the $V$- or in the $W$-norm.

Quite often we will leave out the index and just say the discrete Hilbert complex admits a $V$- or $W$-bounded cochain projection $\Pi_h$. The cochain projection will play an important role in the stablity of the discrete system. Thanks to the commuting property, if $\Pi_h^j$ are $W$-bounded then they are $V$-bounded. In this section, we will make the weaker assumption of $V$-boundedness.

The fact that $\Pi_h$ is a $V$-bounded projection immediately allows a quasi optimal estimate. For any $v \in V^j$, we can take $w_h \in V_h^j$ arbitrary and then by using the triangular inequality

$$\begin{aligned} \|v - \Pi_h^j v\|_V &= \|v - w_h + \Pi_h^j(w_h - v)\|_V && (0.2.2) \\ &\le \|I - \Pi_h^j\|_{\mathcal{L}(V,V)} \|v - w_h\|_V. && (0.2.3) \end{aligned}$$

From now on we will denote the operator norm $\|\cdot\|_{\mathcal{L}(V,V)}$ by slight abuse of notation as $\|\cdot\|_V$. Since $w_h$ was arbitrary we can take the infimum over $w_h \in V_h^k$ and obtain a quasi optimal estimate.

Let us now answer the question about the difference between discrete and continuous harmonic forms. In order to do that, we need some way to measure the "difference" between two subspaces.

For a general metric space $(X, d)$, we will use the standard notation $d(x, M) := \inf_{m \in M} d(x, m)$ for $x \in X$ and $M \subseteq X$. If we are dealing with a normed space then we take the metric induced by the norm.

**Definition 0.2.4** (Gap between subspaces)**.** For a Banach space $W$ with subspaces $Z_1$ and $Z_2$ let $S_1$ and $S_2$ be the unit spheres in $Z_1$ and $Z_2$ respectively, i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$ and analogous for $S_2$. Then we define the gap between these subspaces as

$$\operatorname{gap}(Z_1, Z_2) = \max \left\{ \sup_{z_1 \in S_1} d(z_1, Z_2), \sup_{z_2 \in S_2} d(z_2, Z_1) \right\}$$

This definition is from [7, Ch.4 §2.1] and defines a metric on the set of closed subspaces of $W$. If $W$ is a Hilbert space – as it is throughout this section – and $Z_1$ and $Z_2$ are closed then the $\operatorname{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|_{\mathcal{L}(W,W)}$, i.e. the difference in operator norm of the orthogonal projections onto $Z_1$ and $Z_2$. This gives us a measure of distance between spaces which we can now apply to the question about the difference between discrete and continuous harmonic forms.

**Proposition 0.2.5** (Gap between harmonic forms)**.** *Assume that the discrete complex (0.2.1) admits a $V$-bounded cochain projection $\Pi_h$. Then*

$$\|(I - P_{\mathfrak{H}_h^k})q\|_V \leq \|(I - \Pi_h^k)q\|_V, \qquad \forall q \in \mathfrak{H}^k \tag{0.2.4}$$

$$\|(I - P_{\mathfrak{H}^k})q_h\|_V \leq \|(I - \Pi_h^k)P_{\mathfrak{H}^k}q_h\|_V, \quad \forall q_h \in \mathfrak{H}_h^k \tag{0.2.5}$$

*and*

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q \in \mathfrak{H}, \|q\|=1} \|(I - \Pi_h^k)q\|_V.$$

*Proof.* See [1, Thm. 5.2]. □

The following proposition clarifies how close a discrete harmonic form can be chosen. But in order to prove it, we will need a small lemma.

**Lemma 0.2.6.** *Let $W$ be a Banach space, $Z \subseteq W$ a closed subspace. Denote $S_Z := \{z \in Z \mid \|z\|_W = 1\}$. Then for any $w \in W$ with $\|w\|_W = 1$ we get*

$$d(w, S_Z) \leq 2\, d(w, Z)$$

*Proof.* See [7, Ch.4 §2, (2.13)]. □

**Proposition 0.2.7.** *Take $p \in \mathfrak{H}^k$ with $\|p\| = 1$. Then we can choose $p_h \in \mathfrak{H}_h^k$ with $\|p_h\| = 1$ s.t.*

$$\|p - p_h\| \leq 2\, \|I - \Pi_h^k\|_V \inf_{v_h \in V_h} \|p - v_h\|_V.$$

*Proof.* Notice that since $\mathfrak{H}^k \subseteq \mathfrak{Z}^k$ we always have $\|q\|_V = \|q\|$ for all $q \in \mathfrak{H}^k$. The same is true for $\mathfrak{H}_h^k$. Denote $S_h := \{q_h \in \mathfrak{H}_h^k \mid \|q_h\| = 1\}$. Since $S_h$ is closed we can find $p_h \in S_h$ s.t.

$$\|p_h - p\| = \inf_{q_h \in S_h} \|q_h - p\|.$$

The right hand side can be estimated using (0.2.4) and then the quasi optimal bound for the projection derived at (0.2.3).

$$\begin{aligned}\inf_{q_h \in S_h} \|q_h - p\|_V = \inf_{q_h \in S_h} \|q_h - p\| &\overset{\text{Lem. 0.2.6}}{\leq} 2 \inf_{q_h \in \mathfrak{H}_h} \|q_h - p\| = 2\, \|P_{\mathfrak{H}_h} p - p\| \\ &\overset{(0.2.4)}{\leq} 2\, \|\Pi_h^k p - p\|_V \leq 2\, \|I - \Pi_h^k\|_{\mathcal{L}(V,V)} \inf_{v_h \in V_h} \|p - v_h\|_V\end{aligned}$$

which gives us the estimate. $\square$

Also if we assume $\|\Pi_h\| \leq c_\Pi$ for $h$ small enough and $c_\Pi > 0$ independent of $h$, then Assumption 0.2.1 implies

$$p_h \xrightarrow{V} p \text{ as } h \to 0.$$

**Theorem 0.2.8** (Dimension of $\mathfrak{H}_h^k$)**.** *Assume that we have a finite-dimensional subcomplex with a $V$-bounded cochain projection. Assume further, that*

$$\|q - \Pi_h^k q\| < \|q\|, \quad \forall q \in \mathfrak{H}^k \setminus \{0\}. \tag{0.2.6}$$

*Then $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ are isomorphic. In particular, $\dim \mathfrak{H}^k = \dim \mathfrak{H}_h^k$.*

*Proof.* See [1, Thm 5.1] and the explanation after the proof. $\square$

**Proposition 0.2.9** (Discrete Poincare inequality)**.** *Assume that we have a $V$-bounded cochain projection $\Pi_h$ for the discrete Hilbert complex. Then*

$$\|v_h\|_V \leq c_{P,h} \|dv_h\|, \quad \forall v_h \in \mathfrak{Z}_h^{k,\perp} \cap V_h$$

*with $c_{P,h} := c_P \|\Pi_h\|_V$ and $c_P$ being the Poincare constant from Thm.* **??**.

*Proof.* The proof is from [1, Thm. 5.3] with some additional details. This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k,\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}^{k,\perp} \cap V^k) = \mathfrak{B}^{k+1} \supseteq \mathfrak{B}_h^{k+1}$ we find $z \in \mathfrak{Z}^{k,\perp} \cap V^k$ s.t. $dz = dv_h$. We can apply now the continuous Poincare inequality (Thm. **??**) to get $\|z\|_V \leq c_P \|dz\|_V = c_P \|dv_h\|_V$. We use the fact that $\Pi_h$ is a cochain map and a projection:

$$d\Pi_h^k z = \Pi_h^{k+1} dz = \Pi_h^{k+1} dv_h = dv_h$$

For the last equality, we need the fact that we have a discrete complex, i.e. $d^k V_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \Pi_h z) = 0$, i.e. $v_h - \Pi_h z \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v_h, v_h - \Pi_h z \rangle = \langle v_h, v_h - \Pi_h z \rangle + \langle dv_h, d(v_h - \Pi_h z) \rangle = \langle v_h, v_h - \Pi_h z \rangle_V,$$

so $v_h - \Pi_h z$ is $V$ orthogonal to $v_h$. So

$$\begin{aligned} \|v_h\|_V^2 &= \langle v_h, \Pi_h^k z \rangle_V + \langle v_h, v_h - \Pi_h^k z \rangle_V = \langle v_h, \Pi_h^k z \rangle_V \leq \|v_h\|_V \|\Pi_h\|_V \|z\|_V \\ &\overset{\text{Poincare ineq.}}{\leq} \|v_h\|_V c_P \|\Pi_h\|_V \|dv_h\|_V \end{aligned}$$

□

In conclusion, we obtain a discrete version of the Hilbert complex where the harmonic forms are accurately represented if $h$ is small enough.

### 0.2.2 Discretized magnetostatic problem

Let us apply the theory of discrete Hilbert complexes to the 2D Hilbert complex (0.1.3). We assume that we have finite dimensional subspaces $V_h^0 \subseteq H_0^1$, $V_h^1 \subseteq H_0(\mathrm{div})$ and $V_h^2 \subseteq L^2$ that approximate the full spaces in the sense of Assumption 0.2.1 and

$$V_h^0 \xrightarrow{\mathbf{curl}} V_h^1 \xrightarrow{\mathrm{div}} V_h^2$$

and the dual complex

$$V_h^0 \xleftarrow{\widetilde{\mathrm{curl}}_h} V_h^1 \xleftarrow{-\widetilde{\mathbf{grad}}_h} V_h^2$$

where $\widetilde{\mathrm{curl}}_h$ is the adjoint of $\mathbf{curl}_h$ and can thus be seen as a weak approximation of curl and analogous for $\widetilde{\mathbf{grad}}_h$ and div. As in the continuous case, we assume that $\dim \mathfrak{H}_h^1 = 1$ which is not unreasonable thanks to Thm. 0.2.8 for $h > 0$ small enough.

For our domain, we assume from now on that $\Omega$ is suitable for discretization in the sense that the functions in the discrete spaces and the continuous ones are both defined on it. What that means exactly depends on the chosen discretization and we will explain it later when we go into more detail about the actual implementation (see Assumption 0.3.2).

The discretized version of the strong formulation of the magnetostatic problem (Problem 0.1.2) then states: Find $\mathbf{B}_h \in V_h^1$ s.t.

$$\widetilde{\mathrm{curl}}_h \mathbf{B}_h = J_h \text{ and } \operatorname{div} \mathbf{B}_h = 0$$

with $J_h \in V_h^0$ plus the additional curve integral constraint. Note that the divergence is enforced strongly while the curl is only enforced weakly.

As explained in Sec. 0.1.4, we will substitute the curve integral constraint from Problem 0.1.2 with (0.1.8). This gives us the following discrete formulation. Choose $\mathbf{p}_h \in \mathfrak{H}_h^1$ s.t. $\|\mathbf{p}_h\| = 1$.

**Problem 0.2.10.** Let $J \in L^2$, $C_1 \in \mathbb{R}$. Find $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle &= -\langle J, \tau_h \rangle && \forall \tau_h \in V_h^0, \\ \langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \operatorname{div} \mathbf{B}_h, \operatorname{div} \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle &= 0 && \forall \mathbf{v}_h \in V_h^1, \\ \mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle &= \mu C_1 && \forall \mu \in \mathbb{R}. \end{aligned}$$

In practice, $\mathbf{p}_h$ is computed numerically before assembling the system. Now we make some assumptions on $\psi$. Assume there exists a discrete space $\bar{V}_h^0 \supseteq V_h^0$ s.t. $\bar{V}_h^0 \cap H_0^1 = V_h^0$. $\bar{V}_h^0$ will correspond to the discrete space without boundary conditions.

**Definition 0.2.11.** We call $\psi$ *discretely admissible* for $\Gamma$ if it is admissible for $\Gamma$, i.e. fulfilling (0.1.6), and additionally

$$\psi \in \bar{V}_h^0 \text{ and } \mathbf{curl}\, \psi \in V_h^1. \tag{0.2.7}$$

This has the consequence that for $\psi, \tilde{\psi} \in \bar{V}_h^0$ being discretely admissible for $\Gamma$ and $\tilde{\Gamma}$ respectively, then $\psi - \tilde{\psi} \in H_0^1$ and thus

$$\psi - \tilde{\psi} \in \bar{V}_h^0 \cap H_0^1 = V_h^0. \tag{0.2.8}$$

Also, we can apply the discrete Hodge decomposition, since $\mathbf{curl}\, \psi \in V_h^1$, to get $\psi_{0,h} \in V_h^0$, $c_{\psi,h} \in \mathbb{R}$ s.t.

$$\mathbf{curl}\, \psi = \mathbf{curl}\, \psi_{0,h} + c_{\psi,h} \mathbf{p}_h. \tag{0.2.9}$$

We define $X_h := V_h^0 \times V_h^1 \times \mathbb{R}$. Note that this trial and test space is indeed conforming, i.e. $X_h \subseteq X$, but we choose the discrete bilinear form $a_h : X_h \times X_h \to \mathbb{R}$,

$$\begin{aligned} &a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) \\ &= \langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle + \langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \operatorname{div} \mathbf{B}_h, \operatorname{div} \mathbf{v}_h \rangle \\ &\quad + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle \end{aligned}$$

with $\mathbf{p}_h \in \mathfrak{H}_h^1$, so the resulting bilinear forms are different since we have $\mathbf{p}_h$ instead of $\mathbf{p}$. We can then write the discrete problem in standard form: Find $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ s.t.

$$a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) = \ell(\tau_h, \mathbf{v}_h, \mu) \quad \forall (\tau_h, \mathbf{v}_h, \mu) \in X_h.$$

For simplicity, we assume for the theoretical considerations that we can compute all inner products exactly and that $C_1$ is given exactly as well. That also means that the right hand side $\ell$ is the same for the continuous and discrete problem, i.e.

$$\ell(\tau_h, \mathbf{v}_h, \mu) = -\langle J, \tau_h \rangle - \mu C_1.$$

Because we have transfered the continuous structures to the discrete case we can apply the same arguments as in Sec. 0.1.5.

**Theorem 0.2.12** (Well-posedness of the discrete problem)**.** *For the following assumptions we always tacitly require that $h > 0$ is small enough. We assume that the Hilbert complex admits uniformly $V$-bounded cochain projections $\Pi_h$, i.e. there exists $c_\Pi > 0$ independent of $h$ s.t. $\|\Pi_h\|_{\mathcal{L}(V,V)} \leq c_\Pi$. Assume $\psi$ to be discretely admissible for $\Gamma$ (cf. Def. 0.2.11). Split $\psi$ according to (0.2.9). Then $c_{\psi,h} > 0$ does not depend on the choice of $\psi$. Choose $c_{1,h} > 0$ s.t. $\|\mathbf{curl}\, \psi_{0,h}\| \leq c_{1,h}$. We also assume that (0.2.6) holds and thus $\dim \mathfrak{H}_h^1 = \dim \mathfrak{H}^1 = 1$ and we choose $\mathbf{p}_h$ according to Prop. 0.2.7. Then the discrete variational problem (Problem 0.2.10) is well-posed, i.e. there exists a unique solution $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ and we have the stability estimate*

$$\|\mathbf{B}_h\| \leq \frac{\|J\| + |C_1|}{\gamma_h}.$$

*where $\gamma_h$ has the same expression as $\gamma$ except $c_{P,h}$ instead of $c_P$, $c_{\psi,h}$ instead of $c_\psi$ and $c_{1,h}$ instead of $c_1$.*

*Proof.* By following the exact same arguments as in Sec. 0.1.5, we can prove the well-posedness through the BNB-theorem. We also get that $c_{\psi,h}$ does not depend on the choice of $\psi$ by following the same argument as in Prop. 0.1.13 using the discrete spaces. However, we have to argue why $c_{\psi,h} > 0$ if $c_\psi > 0$. Notice since $\|\mathbf{p}_h\| = \|\mathbf{p}\| = 1$ and $\dim \mathfrak{H}^1 = \dim \mathfrak{H}^1_h = 1$ we have

$$c_\psi = \langle \mathbf{p}, \mathbf{curl}\,\psi\rangle \text{ and } c_{\psi,h} = \langle \mathbf{p}_h, \mathbf{curl}\,\psi\rangle.$$

So

$$|c_{\psi,h} - c_\psi| = |\langle \mathbf{p} - \mathbf{p}_h, \mathbf{curl}\,\psi\rangle| \leq \|\mathbf{curl}\,\psi\|\|\mathbf{p} - \mathbf{p}_h\|.$$

and, because we chose $\mathbf{p}_h$ as described in Prop. 0.2.7 and assume $\Pi_h$ to be uniformly bounded, we have $\|\mathbf{p} - \mathbf{p}_h\| \to 0$ as $h \to 0$ and thus we obtain $c_{\psi,h} \to c_\psi$ for $h \to 0$ and we can assume $c_{\psi,h} > 0$ for $h$ small enough.

□

Finally, we want to use the inf-sup condition to derive an a-priori error estimate. In order to do so, we have to consider the fact that the bilinear forms are different in the discrete and continuous case. So we will use the following lemma.

**Lemma 0.2.13.** *Let $x \in X$ be a solution of a general variational problem of the form (0.1.15) and $x_h \in X_h$ be a solution of the discretized version, i.e for $X_h \subseteq X$ and $Y_h \subseteq Y$ finite-dimensional subspaces*

$$a_h(x_h, y_h) = \ell(y_h) \quad \forall y_h \in Y_h.$$

*Assume that an inf-sup condition holds for the discrete problem with constant $\gamma_h$. Define $\delta_h(x) \in Y'$ as*

$$\langle \delta_h(x), y\rangle_{Y' \times Y} := a(x, y) - a_h(x, y).$$

*and assume*

$$\|a_h\| := \sup_{x \in X, y \in Y} \frac{a_h(x, y)}{\|x\|_X \, \|y\|_Y} < \infty.$$

*Then*

$$\|x - x_h\|_X \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in X_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

*Proof.* Take $z_h \in X_h$ arbitrary. Then with the triangular inequality

$$\|x - x_h\|_X \leq \|x - z_h\|_X + \|x_h - z_h\|_X. \tag{0.2.10}$$

We now have to bound the last term on the right hand side. Assume w.l.o.g. that $x_h - z_h$ is not zero. Then from the inf-sup condition we can find $y_h \in Y_h \setminus \{0\}$ s.t.

$$\begin{aligned}
&\gamma_h \|x_h - z_h\|_X \|y_h\|_Y \le a_h(x_h - z_h, y_h) \\
&= a_h(x - z_h, y_h) + a_h(x_h, y_h) - a(x, y_h) + a(x, y_h) - a_h(x, y_h) \\
&= a_h(x - z_h, y_h) + \langle \delta_h(x), y_h \rangle_{Y' \times Y} \\
&\le \|a_h\| \|x - z_h\|_X \|y_h\|_Y + \|\delta_h(x)\|_{Y'} \|y_h\|_Y
\end{aligned}$$

In the third step, we used the fact that $x$ and $x_h$ are solutions and the discrete problem has the same right hand side as the continuous one. So we can bound $\|x_h - z_h\|_X$ by

$$\|x_h - z_h\|_X \le \frac{\|a_h\|}{\gamma_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

Plugging this into (0.2.10) and taking the infimum over $z_h \in X_h$, we get

$$\|x - x_h\| \le \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in V_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

□

We now have to apply this lemma to the magnetostatic formulation.

**Theorem 0.2.14** (Quasi optimal a-priori estimate)**.** *Let the same assumptions hold as for Thm. 0.2.12 and let $(\sigma, \mathbf{B}, \lambda) \in X$ be the exact solution of Problem 0.1.6 and $(\sigma_h, \mathbf{B}_h, \lambda_h) \in X_h$ the solution of the discrete Problem 0.2.10. Then*

$$\|\mathbf{B} - \mathbf{B}_h\| \le \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{\mathbf{z}_h \in V_h^1} \|\mathbf{B} - \mathbf{z}_h\|_{H(\mathrm{div})}$$

*Proof.* At first, recall that if $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$, $\lambda = 0$ and $\operatorname{div} \mathbf{B} = 0$. So $\|(\sigma, \mathbf{B}, \lambda)\|_X = \|\mathbf{B}\|$ and analogous for $(\sigma_h, \mathbf{B}_h, \lambda_h)$. Also recognize then for any $y = (\tau, \mathbf{v}, \mu) \in X$

$$\langle \delta_h(x), y \rangle = \lambda \langle \mathbf{p}, \mathbf{v} \rangle - \lambda \langle \mathbf{p}_h, \mathbf{v} \rangle = 0.$$

Thus the estimate follows immediately from Lemma 0.2.13. □

One more thing to investigate is the substituted curve integral constraint in the discrete setting. For all the analysis above, we fixed $\psi$ before which raises the question if the choice of $\psi$ matters. We know that in the continuous case it does not (cf. Prop. 0.1.4 and the explanation before). It turns out that the analogous result holds with the same argument if we replace $\mathbf{B}$ with a solution of the first two equations of the discrete formulation $\mathbf{B}_h$ and choose the discrete space $V_h^0$ instead of $H_0^1$ where we also need (0.2.8). We use the discrete formulation of the problem, i.e. the first two equations of Problem 0.2.10, and obtain

$$\int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}_h\, dx = \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - \int_\Omega \mathbf{curl}\, \tilde{\psi} \cdot \mathbf{B}_h\, dx, \quad (0.2.11)$$

assuming $\psi$ is discretely admissible for $\Gamma$, and thus an equivalent system by choosing

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0 \text{ and } \tilde{C}_1 := \int_{\Omega_\Gamma} \tilde{\psi}\, J\, dx - C_0.$$

Here we always assumed that we can compute all the inner products exactly. In summary, the choice of $\psi$ does not matter for the discrete system either.

Another important property of our new curve integral constraint was the invariance if $J = 0$ between the curves and the inner boundary, i.e. if we have two different curves $\Gamma_1$ and $\Gamma_2$ with $\psi_1$ and $\psi_2$ discretely admissible for them and $J = 0$ in $\Omega_{\Gamma_1} \cup \Omega_{\Gamma_1}$, then we would like

$$\langle \mathbf{curl}\, \psi_1, \mathbf{B}_h \rangle = \langle \mathbf{curl}\, \psi_2, \mathbf{B}_h \rangle$$

to hold. This was true in the continuous case (see Prop. 0.1.5) and it is also true for the discrete problem. Again, the argument is completely identical by using a solution $\mathbf{B}_h$ of the discrete system (Prob. 0.2.10) instead, $V_h^0$ instead of $H_0^1$ and (0.2.8).

## 0.3 Implementation on a single patch domain

We start with a very short introduction on Splines and their tensor product space which will be our choice of basis on the reference domain. Then we choose our degrees of freedom and basis dual to them. We will work at first on a reference domain and then transfer it to the physical domain using pushforwards. In the end, we will explain how the homogeneous boundary conditions are enforced and state the final system to be solved.

### 0.3.1 Splines

For the finite element spaces, we will use the pushforwards of tensor product splines defined on a rectangular reference domain $\hat{\Omega}$. This section is a recollection of [5, Sec. 4.2] since we use the same method as presented in this paper. We start with a very brief introduction of B-Splines and the spaces of tensor products of these.

We choose a knot sequence $\boldsymbol{\xi} = (\xi_i)_{i=0}^{n+p}$ with $\xi_0 \leq \xi_1 \leq ... \leq \xi_{n+p}$. B-Splines $B_i^p$ of degree $p \geq 0$ for $i = 0, ..., n-1$ are defined recursively as

$$B_i^p(x) := \frac{x - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(x) + \frac{\xi_{i+p+1} - x}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(x)$$

and

$$B_i^0(x) := \begin{cases} 1, & \text{if } x \in [\xi_i, \xi_{i+1}), \\ 0, & \text{else.} \end{cases}$$

We choose two types of sequences, periodic and non-periodic ones. On an interval $[a, b]$, let $a = x_0 < x_1 < ... < x_N = b$ be our grid. We will stick to the equidistant case, i.e. $h$ will be our grid size $x_{i+1} - x_i$. For the non-periodic case, we choose an *open* knot sequence by $\xi_0 = ... = \xi_p = x_0$, $\xi_{p+k} = x_k$ for $k = 0, 1, ..., N$ and $\xi_n = \xi_{n+1} = ... = \xi_{n+p} = x_N$, i.e.

$$\boldsymbol{\xi} = (\underbrace{x_0, x_0, ..., x_0}_{p+1 \text{ times}}, x_1, x_2, ..., x_{N-1}, \underbrace{x_N, x_N, ..., x_N}_{p+1 \text{ times}})$$

and for the periodic case $\xi_0 = x_0 - ph, \xi_1 = x_0 - (p-1)h, ..., \xi_p = x_0, \xi_{p+k} = x_k$ for $k = 0, ..., n$ and $\xi_{n+k} = x_N + kh$ for $k = 0, ..., p$,

$$\begin{aligned} \boldsymbol{\xi} = (&x_0 - ph, x_0 - (p-1)h, ..., x_0 - h, x_0, x_1, x_2, ..., x_{N-1}, x_N, \\ &x_N + h, x_N + 2h, ..., x_N + ph). \end{aligned}$$

In the periodic case, the splines at the beginning and end of the interval are extended periodically. In both cases, we have $n + p + 1 = N + 2p + 1$ knots in our knot sequence $\boldsymbol{\xi}$ and hence obtain $n = N + p$ splines. We then define the spline space $\mathbb{S}^p(\boldsymbol{\xi}) := \operatorname{span}\{B_i^p\}_{i=0}^{n-1}$ which has dimension $N + p$.

Note that all the knot multiplicities in the interior are one and thus our spline space has maximal regularity which implies that it is equal to the piecewise polynomial space

$$\mathbb{S}^p(\boldsymbol{\xi}) = \{v \in C^{p-1} \mid v|_{[x_j, x_{j+1})} \in \mathbb{P}_p([x_j, x_{j+1})) \text{ for } j \in [N-1]\}.$$

where $\mathbb{P}_p([x_j, x_{j+1}))$ are the polynomials of degree $p$ on $[x_j, x_{j+1})$ and we use the standard notation $[K] = \{0, 1, ..., K\}$ for $K \in \mathbb{N}$.

This construction can now be generalized to 2D by using a tensor product approach. Denote $a_1 = x_0 < x_1 < x_2 < ... < x_N = b_1$ the grid in $x$-direction and $a_2 = y_0 < y_1 < y_2 < ... < y_N = b_2$ in $y$-direction and assume that we have the same number of grid points in both dimensions for simplicity. Using the same definition of knot sequence as above gives us the knot sequence $\boldsymbol{\xi}$ in $x$-direction and $\boldsymbol{\eta}$ in $y$-direction. The resulting splines are denoted as $B^{q_1}_{i,\xi}$, $i = 0, ..., n_1 - 1 = N + q_1 - 1$, and $B^{q_2}_{j,\eta}$, $j = 0, ..., N + q_2 - 1$, and the spaces $\mathbb{S}^{q_1}(\boldsymbol{\xi})$ and $\mathbb{S}^{q_2}(\boldsymbol{\eta})$ respectively.

We use the notation with $\mathbf{q} \in \{p-1, p\}^2$, $n_1 = N + q_1$, $n_2 = N + q_2$ and we define for $\mathbf{i} \in [n_1 - 1] \times [n_2 - 1]$ the *tensor product spline*

$$B^{\mathbf{q}}_{\mathbf{i}}(x, y) = B^{q_1}_{i_1,\xi}(x) B^{q_2}_{i_2,\eta}(y)$$

and $\mathbb{S}^{\mathbf{q}}(\xi, \eta)$ the span of these tensor product splines. We will from now on leave out the reference to the knot sequences and assume them to be fixed. The spline spaces used in the tensor product can also be periodic or be periodic in one direction and non-periodic in the other which will the case in our implementation later.

Then we obtain the following discrete Hilbert complex on our reference domain $\hat{\Omega}$

$$\mathbb{S}^{p,p} \xrightarrow{\mathbf{curl}} \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \xrightarrow{\text{div}} \mathbb{S}^{p-1,p-1}$$

and we denote

$$\hat{V}^0_h = \mathbb{S}^{p,p}, \ \hat{V}^1_h = \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \text{ and } \hat{V}^2_h = \mathbb{S}^{p-1,p-1}. \tag{0.3.1}$$

It is well-known that for a piecewise smooth function $\mathbf{v}$, $\mathbf{v} \in H(\text{div})$ iif the normal trace across element interfaces agrees. Analogously, a piecewise smooth $\tau \in H^1$ iif the values agree on the interfaces. We will always assume $p \geq 2$ and thus know that all our tensor splines are at least continuous globally and so $\hat{V}^0_h \subseteq H^1(\hat{\Omega})$ and $\hat{V}^1_h \subseteq H(\text{div}, \hat{\Omega})$ as desired.

### 0.3.2 Basis and degrees of freedom

Let us now investigate the degrees of freedom and the corresponding basis functions. At first, we will define these only on the reference domain $\hat{\Omega}$. Then we will use pullbacks and pushforwards to transfer them from the reference domain to the physical domain $\Omega$. We will also use this opportunity to specify more precisely, what domains we consider such that this approach works.

We now fix $p$ and get $n = N + p$. Here it is crucial to remember that we assumed to have the same number of grid points in both dimensions. We will use geometric degrees of freedom, i.e. each degree of freedom can be associated with some geometrical element of our grid. We define *Greville points* by

$$\zeta_i^x := \frac{\xi_{i+1} + ... + \xi_{i+p}}{p},$$

i.e. the knot averages for $i = 0, ..., n-1$ and analogous $\zeta_j^y$ in $y$-direction using the knot sequence $\boldsymbol{\eta}$. Then the spline interpolation at these points is well-defined (see [9, Sec. 3.3.1]). Note that in the periodic case some Greville points lie outside of the grid, but we assume the function that is interpolated to be periodic as well, so the values at the Greville points are well-defined.

This gives us the following geometric elements nodes, edges and cells

$$\begin{aligned}
\hat{\mathtt{n}}_{\mathbf{i}} &:= (\zeta_{i_1}^x, \zeta_{i_2}^y), \quad \mathbf{i} \in \mathcal{M}^0,\\
\hat{\mathtt{e}}_{d,\mathbf{i}} &:= [\hat{\mathtt{n}}_{\mathbf{i}}, \hat{\mathtt{n}}_{\mathbf{i}+\mathbf{e}_d}], \quad (d, \mathbf{i}) \in \mathcal{M}^1,\\
\hat{\mathtt{c}}_{\mathbf{i}} &:= [\hat{\mathtt{e}}_{1,\mathbf{i}}, \hat{\mathtt{e}}_{1,\mathbf{i}+\mathbf{e}_1}] = [\zeta_{i_1}^x, \zeta_{i_1+1}^x] \times [\zeta_{i_2}^y, \zeta_{i_2+1}^y], \quad \mathbf{i} \in \mathcal{M}^2,
\end{aligned}$$

with $[\cdot]$ being the convex hull. As before, $\mathbf{e}_d$ for $d = 1, 2$ is the standard basis vector of $\mathbb{R}^2$. The set of multiindices are defined as

$$\begin{aligned}
&\mathcal{M}^0 := [n-1]^2,\\
&\mathcal{M}^1 := \{(d, \mathbf{i}) \mid d \in \{1, 2\}, \mathbf{i} \in [0, n-1]^2, i_d < n-1\},\\
&\mathcal{M}^2 := [n-2]^2.
\end{aligned}$$

Now that we have defined the geometric elements we define the corresponding degrees of freedom. We define $\mathbf{e}_d^\perp$ as $\mathbf{R}_{\pi/2}\mathbf{e}_d$, i.e. the rotation by $\pi/2$ in counter clockwise direction, which leads to $\mathbf{e}_1^\perp = \mathbf{e}_2$ and $\mathbf{e}_2^\perp = -\mathbf{e}_1$. The degrees of freedom are then

$$\begin{aligned}
\hat{\sigma}_{\mathbf{i}}^0(v) &:= v(\hat{\mathtt{n}}_{\mathbf{i}}), \quad \mathbf{i} \in \mathcal{M}^0,\\
\hat{\sigma}_{d,\mathbf{i}}^1(\mathbf{v}) &:= \int_{\hat{\mathtt{e}}_{d,\mathbf{i}}} \mathbf{v} \cdot \mathbf{e}_d^\perp, \quad (d, \mathbf{i}) \in \mathcal{M}^1,\\
\hat{\sigma}_{\mathbf{i}}^2(v) &:= \int_{\hat{\mathtt{c}}_{\mathbf{i}}} v, \quad \mathbf{i} \in \mathcal{M}^2.
\end{aligned}$$

These degrees of freedom are unisolvent on $\hat{V}_h^\ell$, i.e. with $N_\ell = |\mathcal{M}^\ell|$ and some ordering $\mu_0, \mu_1, ..., \mu_{N_\ell}$ of the indices of $\mathcal{M}^\ell$, denoting $\hat{\sigma}_{\mu_i}^\ell = \hat{\sigma}_i^\ell$, we define

$$\hat{\boldsymbol{\sigma}}^\ell := (\hat{\sigma}_0^\ell, \hat{\sigma}_1^\ell, ..., \hat{\sigma}_{N_\ell}^\ell)^\top : \hat{V}_h^\ell \to \mathbb{R}^{N_\ell},$$

which is bijective, and we can thus define our basis functions $\hat{\Lambda}_\mu^\ell$, $\mu \in \mathcal{M}^\ell$, (denoted in bold if vector valued) as the basis which is dual to the degrees of freedom in the sense

$$\hat{\sigma}_\mu^\ell(\hat{\Lambda}_\nu^\ell) = \delta_{\mu,\nu} \quad \forall \mu, \nu \in \mathcal{M}^\ell.$$

**Remark 0.3.1.** For implementational purposes, these basis functions dual to the degrees of freedom are not necessarily the best option. For the computation of mass matrices etc. it is more convenient to use the B-splines directly due to their local support and fast computation. We will not go too deeply into the details of implementation however. More details about the use of B-splines and the connection with the basis $\Lambda^\ell_\mu$ can be found in [5, Sec. 4.8]

The question is now on what function spaces these degrees of freedom are defined. We note first that the standard choice with $\hat{V}^0 = H^1(\hat{\Omega})$, $\hat{V}^1 = H(\mathrm{div}, \hat{\Omega})$ and $\hat{V}^2 = L^2(\hat{\Omega})$ can not work because, e.g., the evaluation at point values is generally not well-defined for $H^1$-functions in 2D. Thus, we need to choose function spaces with higher regularity or integrability.

Let us define the spaces

$$\begin{aligned} W^1_{1,2}(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_1\partial_2 v \in L^1(\hat{\Omega})\} \\ W^1_d(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_d v \in L^1(\hat{\Omega})\} \quad \text{for } d \in \{1,2\} \end{aligned}$$

and then denote $\hat{U}^0_{L^1} := W^1_{1,2}(\hat{\Omega})$, $\hat{U}^1_{L^1} := W^1_1(\hat{\Omega}) \times W^1_2(\hat{\Omega})$ and $\hat{U}^2_{L^1} := L^1(\hat{\Omega})$. Then the degrees of freedom $\hat{\sigma}^\ell_\mu$ are well-defined on $\hat{U}^\ell := \hat{U}^\ell_{L^1} \cap \hat{V}^\ell$.

We thus obtained the degrees of freedom and basis functions on the reference domain $\hat{\Omega}$. The idea to continue is to define the basis functions on the physical domain $\Omega$ by a pushforward of the basis functions from the reference domain. We will now clarify what types of domain we will consider for the discretization. Then we will define the pushforward as the inverse of the pullback and apply it to the basis functions and degrees of freedom to transfer them to the physical domain.

**Assumption 0.3.2.** There exists a rectangular reference domain $\hat{\Omega} = [a_1, b_1] \times [0, T]$, $a_1 < b_1$, $T > 0$ and an orientation preserving diffeomorphism $\mathbf{F}$ defined on $\hat{\Omega}$, which is $T$-periodic in the second variable, such that $\Omega = F(\hat{\Omega})$.

We have the tensor grid on the reference domain

$$\{a_1 = x_0 < x_1 < ... < x_N = b_1\} \times \{0 = y_0 < y_1 < ... < y_N = T\}.$$

For the curve $\Gamma$, we assume that there exists $\hat{\Gamma} \subseteq \hat{\Omega}$ that goes along the edges of the grid such that $\Gamma = \mathbf{F}(\hat{\Gamma})$.

See Fig. 0.3.1 for an example where $\Omega$ is an annulus and $N = 3$. This is usually referred as the *single patch case* since we only use one mapping from the reference domain. We see that this assumption is made so that notions from the physical domain are well-represented on the reference domain. Notice that under

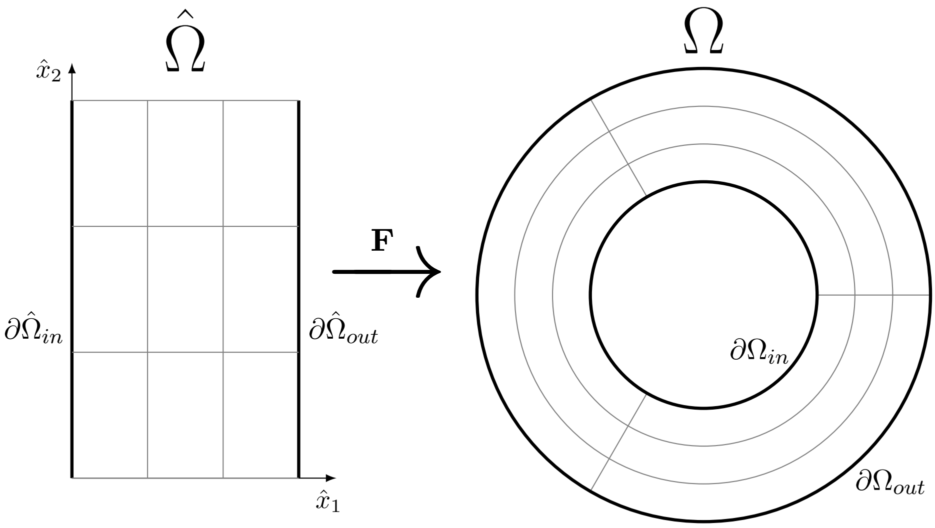


**Figure 0.3.1:** Reference domain $\hat{\Omega}$ and physical domain $\Omega$ with grid. We have three grid cells in both dimensions, i.e. $N = 3$. Here $\hat{\Omega}$ has no boundary at the "northern" and the "southern" edge.

this assumption the preimage of $\Omega_\Gamma$ corresponds to a subgrid on the reference domain.

We already introduced the pullback in Prop. **??** in the 3D case. Now, we define the pullbacks in 2D

$$\begin{aligned}
\mathcal{P}^0_{\mathbf{F}} &: v \mapsto \hat{v} := v \circ \mathbf{F} \\
\mathcal{P}^1_{\mathbf{F}} &: \mathbf{v} \mapsto \hat{\mathbf{v}} := (\det D\mathbf{F})\, D\mathbf{F}^{-1}(\mathbf{v} \circ \mathbf{F}) \\
\mathcal{P}^2_{\mathbf{F}} &: v \mapsto \hat{v} := (\det D\mathbf{F})(v \circ \mathbf{F})
\end{aligned}$$

which map functions on the physical domain $\Omega$ to functions on the reference domain $\hat{\Omega}$. Then we have the commuting properties

$$\begin{aligned}
\widehat{\mathbf{curl}}\mathcal{P}^0_{\mathbf{F}} v &= \mathcal{P}^1_{\mathbf{F}}\, \mathbf{curl}\, v \\
\widehat{\mathrm{div}}\mathcal{P}^1_{\mathbf{F}} \mathbf{v} &= \mathcal{P}^2_{\mathbf{F}}\, \mathrm{div}\, \mathbf{v}
\end{aligned}$$

when the $\mathbf{curl}$ and divergence are well-defined.

**Remark 0.3.3.** Just as in Sec. **??**, these pullbacks can be derived from the pullbacks of differential forms. Keep in mind however, that in 2D there are two different ways to identify a 1-form with a vector field (cf. Remark **??**). This leads to different pullbacks. We have chosen it here s.t. we have the commuting property with the differential operators in our Hilbert complex (0.1.3).

We now define the pushforwards as $\mathcal{F}^\ell := (\mathcal{P}^\ell_{\mathbf{F}})^{-1}$ which then read

$$\begin{aligned}
&\mathcal{F}^0 : \hat{v} \mapsto v := \hat{v} \circ \mathbf{F}^{-1} \\
&\mathcal{F}^1 : \hat{\mathbf{v}} \mapsto \mathbf{v} := (\det D\mathbf{F}^{-1})\, D\mathbf{F}(\hat{\mathbf{v}} \circ \mathbf{F}^{-1}) \\
&\mathcal{F}^2 : \hat{v} \mapsto v := (\det D\mathbf{F}^{-1})(\hat{v} \circ \mathbf{F}^{-1}).
\end{aligned}$$

By applying these pushforwards we get the basis functions on the physical domain

$$\Lambda^\ell_\mu := \mathcal{F}^\ell \hat{\Lambda}^\ell_\mu$$

and then

$$\bar{V}^\ell_h := \mathrm{span}\{\Lambda^\ell_\mu \mid \mu \in \mathcal{M}^\ell\}$$

are our discrete spaces without boundary conditions.

Using the geometric degrees of freedom on the reference domain, we can now construct the corresponding degrees of freedom on the physical domain as

$$\sigma^\ell_\mu := \hat{\sigma}^\ell_\mu \circ \mathcal{P}^\ell_F$$

Then we have by construction $\sigma^\ell_\mu(\Lambda^\ell_\nu) = \delta_{\mu,\nu}$ for all $\mu, \nu \in \mathcal{M}^\ell$.

The degrees of freedom then also correspond to geometric objects in the physical domain. The reference grid gets mapped to the physical domain which gives us the geometric objects there. We define the mapped nodes in the physical domain as

$$\mathtt{n}_{\mathbf{i}} := \mathbf{F}(\hat{\mathtt{n}}_{\mathbf{i}}),$$

and analogous edges $\mathtt{e}_{d,\mathbf{i}}$, $(d, \mathbf{i}) \in \mathcal{M}^1$, and cells $\mathtt{c}_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{M}^2$. $\sigma^0$ corresponds to point values at the nodes, $\sigma^1$ to the fluxes through the edges and $\sigma^2$ to the integral over the mapped cells.

**Remark 0.3.4.** If we would take the now obvious choice of $v \mapsto \sum_{\mu\in\mathcal{M}^0} \sigma^0_\mu(v)\Lambda^0_\mu \in \bar{V}^0_h$ and analogous for the other function spaces, the result would be a cochain projection, but it would not be bounded and thus insufficient for the theory introduced in Sec. 0.2.1.

However, there are $L^2$-stable quasi-interpolants which also commute with the differential operators (see [2, Sec. 4]) and provide us with the stable cochain projections that we need for the theory.

### 0.3.3 Building the discrete system

Now that we have specified the basis and degrees of freedom, we will talk about how we will build the system to solve the magnetostatic problem on $\Omega$. In particular, we will explain how we will enforce homogeneous boundary conditions since this is the way the magnetostatic problem was posed. To achieve that, we define the boundary projections and reformulate the discrete problem with them. When this is done, we will write the equations in matrix form to see what system is solved in practice.

Recall that $\mathbf{F}$, the parametrization of the physical domain, is assumed to be periodic in the second variable and so the "north" and "south" edge of the rectangle $\hat{\Omega}$ do not correspond to a boundary of the physical domain $\Omega$. This motivates the (slight abuse of) notation

$$\partial\hat{\Omega} := \{a, b\} \times [0, T].$$

Let us define the sets of boundary indices $\mathcal{B}^\ell$ which correspond to the multi-indices of the geometric elements on the boundary, i.e.

$$\begin{aligned}
\mathcal{B}^0 &:= \{\mathbf{i} \in \mathcal{M}^0 \mid \hat{\mathtt{n}}_{\mathbf{i}} \in \partial\hat{\Omega}\}, \\
\mathcal{B}^1 &:= \{(d, \mathbf{i}) \in \mathcal{M}^1 \mid \hat{\mathtt{e}}_{d,\mathbf{i}} \subseteq \partial\hat{\Omega}\}.
\end{aligned}$$

Since $V_h^2 \subseteq L^2$, we do not require any boundary conditions there. For $\ell = 1, 2$, we define the spaces

$$V_h^\ell := \{v_h \in \bar{V}_h^\ell \mid \sigma_\mu^\ell(v_h) = 0 \quad \forall \mu \in \mathcal{B}^\ell\} = \operatorname{span}\{\Lambda_\mu^\ell \mid \mu \in \mathcal{M}^\ell \setminus \mathcal{B}^\ell\}.$$

For these spaces it holds then

$$\begin{aligned}
V_h^0 &= \bar{V}_h^0 \cap H_0^1(\Omega) \\
V_h^1 &= \bar{V}_h^1 \cap H_0(\operatorname{div}, \Omega)
\end{aligned}$$

and so we see that the homogeneous boundary conditions simply correspond to the boundary degrees of freedom being zero. This means for $V_h^0$ we require the nodal values on the boundary to vanish and for $V_h^1$ zero flux through the boundary edges.

This motivates us to define the projections $P_h^\ell : \bar{V}_h^\ell \to \bar{V}_h^\ell$ which set the boundary degrees of freedom to zero and are thus a projection onto $V_h^\ell$. The matrix representation $\mathbb{P}^\ell$ is then simply $(\mathbb{P}^\ell)_{\mu,\nu} = 1$ iif $\mu = \nu$ and $\mu$ does not correspond to a geometric element on the boundary. They are easily constructed by taking the identity matrix and setting the diagonal entries to zero that belong to boundary degrees of freedom. These matrices are obviously symmetric.

We now reformulate the discrete system using these projections. We apply the boundary projections to all functions involved and then add the boundary penalization terms. With boundary penalities the discrete system is: Find $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned}\langle (I-P_h^0)\sigma_h, (I-P_h^0)\tau_h\rangle + \langle P_h^0\sigma_h, P_h^0\tau_h\rangle \\ -\langle P_h^1\mathbf{B}_h, \mathbf{curl}\, P_h^0\tau_h\rangle = -\langle J, P_h^0\tau_h\rangle \quad \forall \tau_h \in \bar{V}_h^0,\end{aligned} \tag{0.3.2}$$

$$\begin{aligned}\langle \mathbf{curl}\, P_h^0\sigma_h, P_h^1\mathbf{v}_h\rangle + \langle (I-P_h^1)\mathbf{B}_h, (I-P_h^1)\mathbf{v}_h\rangle \\ +\langle \operatorname{div} P_h^1\mathbf{B}_h, \operatorname{div} P_h^1\mathbf{v}_h\rangle + \langle \lambda\mathbf{p}_h, P_h^1\mathbf{v}_h\rangle = 0 \quad \forall \mathbf{v}_h \in \bar{V}_h^1,\end{aligned} \tag{0.3.3}$$

$$\langle \mathbf{curl}\,\psi, P_h^1\mathbf{B}_h\rangle = C_1. \tag{0.3.4}$$

Since we apply the projection everywhere it is then easy to show that $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ solve (0.3.2)-(0.3.4) iif $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $(\sigma_h, \mathbf{B}_h, \lambda)$ solves the system with homogeneous discrete spaces (Prob. 0.2.10). So the two formulations are equivalent.

We will often from now implicitly use some flattening of the multiindices in $\mathcal{M}^\ell$. Define $\mathbb{M}^\ell$ as the mass matrix of $\bar{V}_h^\ell$, i.e. $\mathbb{M}_{ij}^\ell = \langle \Lambda_i^\ell, \Lambda_j^\ell\rangle$. We define the matrix $\mathbb{D}$ as the matrix representation of the divergence restricted to $\bar{V}_h^1$, i.e. $\operatorname{div}|_{\bar{V}_h^1} : \bar{V}_h^1 \to \bar{V}_h^2$. Analogously $\mathbb{C}$ is the matrix representation of $\mathbf{curl}$. Then we have, as mentioned before, the matrix representation of the boundary projections $\mathbb{P}^\ell$ and $\mathbb{I}^\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ is the identity matrix. We denote the vector of coefficients of a function with an underline, e.g. $\underline{\sigma} \in \mathbb{R}^{N_0}$ is the vector of coefficients of $\sigma$ in the basis $\Lambda_\mu^0$, $\mu \in \mathcal{M}^0$. $\underline{\mathbf{B}}, \underline{\mathbf{p}} \in \mathbb{R}^{N_1}$ are the coefficients of $\mathbf{B}_h$ and $\mathbf{p}_h$ in the basis $\mathbf{\Lambda}_\mu^1$, $\mu \in \mathcal{M}^1$. So rewriting the equations (0.3.2)-(0.3.4) in matrix-vector form gives us

$$\begin{aligned}\underline{\tau}^\top(\mathbb{I}^0-\mathbb{P}^0)\mathbb{M}^0(\mathbb{I}^0-\mathbb{P}^0)\underline{\sigma} + \underline{\tau}^\top\mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \underline{\tau}^\top\mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \underline{\tau}^\top\mathbb{P}^0\underline{\tilde{J}} \quad \forall\underline{\tau} \in \mathbb{R}^{N_0} \\ \underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + \underline{\mathbf{v}}^\top(\mathbb{I}^1-\mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1-\mathbb{P}^1)\underline{\mathbf{B}} \\ +\underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0 \quad \forall\underline{\mathbf{v}} \in \mathbb{R}^{N_1} \\ \underline{\psi}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1\end{aligned}$$

where $\underline{\tilde{J}} = (\langle J, \Lambda_i^0\rangle)_{i=1}^{N_0}$ which gives us the final system to be solved

$$\begin{aligned}(\mathbb{I}^0-\mathbb{P}^0)^\top\mathbb{M}^0(\mathbb{I}^0-\mathbb{P}^0)\underline{\sigma} + \mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \mathbb{P}^0\underline{\tilde{J}} \\ \mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + (\mathbb{I}^1-\mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1-\mathbb{P}^1)\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0 \\ \underline{\psi}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1\end{aligned}$$

Now we will explain how the discrete harmonic form is computed. In order to do that, we will characterize harmonic forms in a different way.

**Proposition 0.3.5.** *Define the penalized discrete Hodge Laplacian operator* $\bar{L}_h^1 : \bar{V}_h^1 \to \bar{V}_h^1$ *as*

$$\bar{L}_h^1 = -\widetilde{\mathbf{grad}}_h \operatorname{div} P_h^1 + \mathbf{curl}\, P_h^0 \widetilde{\operatorname{curl}}_h + (I - P_h^1)^*(I - P_h^1)$$

*where* $\widetilde{\mathbf{grad}}_h$ *and* $\widetilde{\operatorname{curl}}_h$ *are the adjoints of* $\operatorname{div} P_h^1$ *and* $\mathbf{curl}\, P_h^0$ *respectively. Then*

$$\ker \bar{L}_h^1 = \mathfrak{H}_h^1.$$

Note that the spaces in the discrete Hilbert complex (0.2.1) have homogeneous boundary conditions and thus this is also true for the spaces $\mathfrak{H}_h^j$ and $\mathfrak{B}_h^j$ and $\mathfrak{B}_{jh}^*$ by definition.

*Proof.* See [3, Thm. 3.2] □

Notice that for any linear operator $\phi : V \to W$ between finite dimensional inner product spaces $V$ and $W$ with matrix representation $A$, the matrix representation of the adjoint $\phi^*$ is $G_V^{-1} A^\top G_W$ where $G_V$ and $G_W$ are the Gramian matrices of the chosen bases in $V$ and $W$ respectively. So the penalized discrete Hodge Laplacian has then the matrix representation

$$(\mathbb{M}^1)^{-1}\mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1 + \mathbb{C}\mathbb{P}^0(\mathbb{M}^0)^{-1}\mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1 + (\mathbb{M}^1)^{-1}(\mathbb{I}^1 - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1 - \mathbb{P}^1)$$

and we compute the coefficients $\underline{\mathbf{p}}$ of the discrete harmonic form by computing an element of the kernel of this matrix. One can also multiply it with $\mathbb{M}^1$ from the left which does not change the kernel, but avoids having to compute $(\mathbb{M}^1)^{-1}$.

**Remark 0.3.6.** We acknowledge the fact that computing the inverse of the mass matrix $\mathbb{M}^0$ is a computational bottleneck of this implementation. The inverse of the mass matrix is in general dense and is thus problematic memory-wise. One way to avoid this would be to use multiple patches instead and then use a CONGA approach described in detail in [5]. Then the mass matrices are block diagonal and inverting them much less costly.

## 0.4 Numerical examples

We will consider two numerical examples, the simulation of the magnetic field induced by a current through a wire, where the exact solution is given by the Biot-Savart law, and a manufactured solution. The Biot-Savart solution will be approximated on a standard annulus and a ”distorted annulus”. In both cases, we will pose the problem with two different curves and investigate the convergence error.

### 0.4.1 Magnetic field induced by current through a wire

As a first simple numerical test, we consider a standard example from magnetostatics which is the magnetic field induced by a current through an infinitely long, straight wire with radius zero. The *Biot-Savart law* can be used to compute it. Let the wire be equal to the $z$-axis and $I$ be the electrical current flowing through it. With $\boldsymbol{\ell}(s) = s\mathbf{e}_3$, $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$,

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\boldsymbol{\ell}' \times (x - \boldsymbol{\ell}(s))}{|x - \boldsymbol{\ell}(s)|^3}\, ds = \frac{\mu_0 I}{4\pi |x|^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

where $\mu_0$ is the magnetic constant. For convenience, we pick now $I = \frac{2\pi}{\mu_0}$ to get

$$\mathbf{B}(x) = \frac{2}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

We will only focus on the first two components since our approximation is in 2D. We choose as our domain of computation $\Omega$ the annulus with inner radius $1$ and outer radius $2$. We will investigate two different curves. The curve $\Gamma_1$ will be the parametrization of the circle with radius $1.5$ in anticlockwise direction, which gives us the curve integral

$$C_0 = \int_{\Gamma_1} \mathbf{B} \cdot d\ell = 4\pi.$$

$\Gamma_2$ will be the exterior boundary of the annulus, i.e. the circle with radius $2$. Because $J = 0$ on our domain, the curve integral along $\Gamma_2$ is the same and hence $C_1 = C_0$. The reference domain $\hat{\Omega} = [0,1] \times [0, 2\pi]$ and the mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} (\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

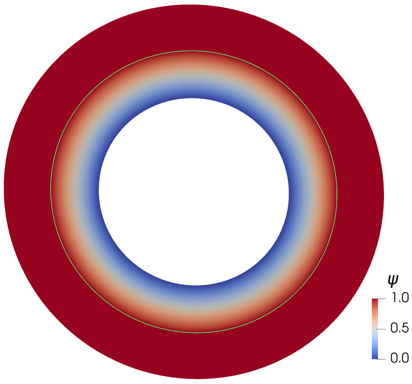


**Figure 0.4.1:** A possible choice of $\psi$ on an annulus. $\Gamma$ is here in the middle of the annulus (marked in green). $\psi = 0$ at the inner boundary and $\psi = 1$ at $\Gamma$ and between $\Gamma$ and the outer boundary

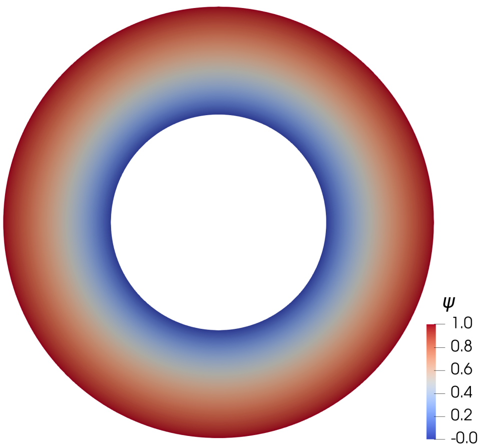


**Figure 0.4.2:** $\psi$ as the solution of a Poisson problem with $\psi = 0$ on the interior and $\psi = 1$ on the exterior boundary. $\Gamma$ is here equal to $\partial\Omega_{out}$.

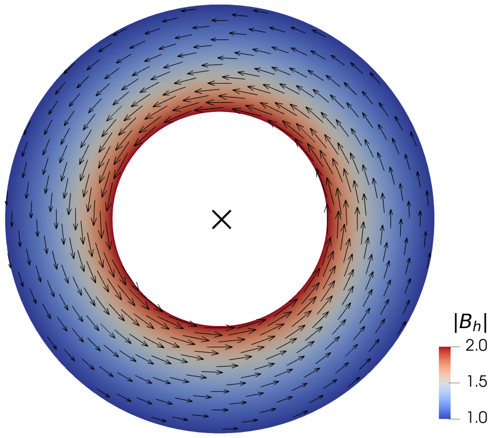


**Figure 0.4.3:** Solution of magnetostatic problem induced by current $I = 2\pi/\mu_0$ through a wire on an annulus for 64 grid cells on the reference domain in both dimensions and spline degree $p = 2$. The wire goes through the origin in $z$-direction towards the point of view.

Then we choose $\psi_1$ as a simple interpolation from the inner boundary to the curve $\Gamma_1$, i.e. in logical coordinates

$$\psi(\hat{x}_1, \hat{x}_2) = \begin{cases} 2\hat{x}_1 & \text{for } \hat{x}_1 \leq 0.5 \\ 1 & \text{else.} \end{cases} \tag{0.4.1}$$

Then $\psi_1$ is admissible for $\Gamma_1$ (cf. (0.1.6)) (see Fig. 0.4.1).

For $\Gamma_2$, we choose $\psi_2$ as the solution of the Laplace problem with boundary conditions $\psi_2 = 0$ on $\partial\Omega_{in}$ and $\psi_2 = 1$ on $\partial\Omega_{out}$ (see Fig. 0.4.2)

For the implementation, we are using the PSYDAC library (`https://github.com/pyccel/psydac`) which is an open source Python 3 library for isogeometric analysis (see [6] for more details). See Fig. 0.4.3 for the solution where we chose 64 cells for our grid on the reference domain in both dimensions for $p = 2$. Note that $p = 2$ refers to the choice of our spaces from (0.3.1) and hence the spline space for approximating the magnetic field will be $\mathbb{S}_{2,1} \times \mathbb{S}_{1,2}$ so the lower spline degree will be one. The pushforward spline space achieves then an approximation error of $h^p$ [8, Ch.4, (4.48)], if the solution is at least $H^{p+1}$-regular, which can be observed (see Fig. 0.4.4). As predicted by the theory (cf. Prop. 0.1.5), the errors for $\Gamma_1$ and $\Gamma_2$ are equal up to rounding errors.

Now we will change the domain where the problem is posed, but we will stick to the Biot-Savart setting. This means we still assume the wire goes through the origin in $z$-direction, which results in the same solution, but we will approximate it on a "distorted annulus" (see 0.4.5) using the same reference domain, but the

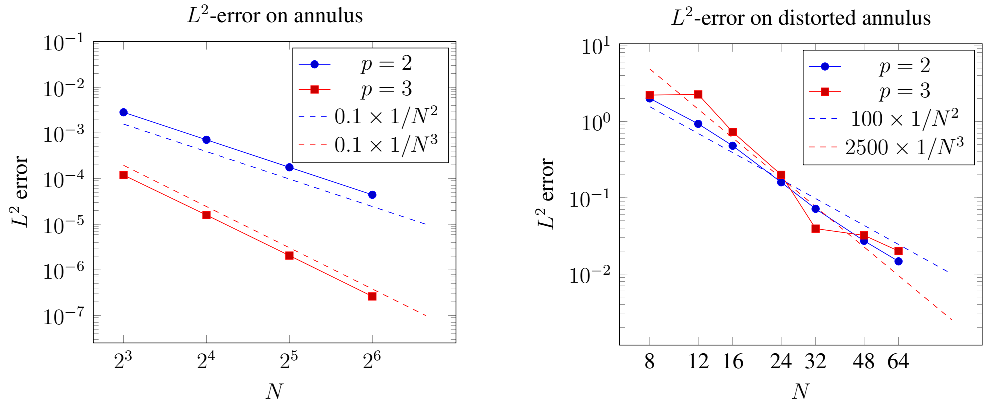


**Figure 0.4.4:** Convergence analysis for the approximation of the Biot-Savart solution with $p = 2$ and $p = 3$ on the annulus domain and the distorted annulus domain. For the solution with $p = 3$, the number of grid points in $\hat{x}_2$-direction is $N/2$ to avoid memory issues.

different mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} 3(\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

Note that we do not have the homogeneous boundary condition $\mathbf{B} \cdot \mathbf{n} = 0$ for this domain anymore. Thus, we have to deal with the boundary conditions by using a standard lifting approach which means we take an interpolation of the boundary conditions $\mathbf{B}_{h,g}$, that we compute before, and then split $\mathbf{B}_h = \mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ with $\mathbf{B}_{h,0} \in V_h^1$. Then we substitute $P_h^1 \mathbf{B}_h$ in (0.3.2)-(0.3.4) with $\mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ and put the terms with $\mathbf{B}_{h,g}$ on the right hand side. This leads to the same left hand side, but as right hand side we get $-\langle J, P_h^0 \tau_h \rangle + \langle \mathbf{B}_{h,g}, \mathbf{curl}\, P_h^0 \tau_h \rangle$ in the first equation, $-\langle \operatorname{div} \mathbf{B}_{h,g}, \operatorname{div} P_h^1 \mathbf{v}_h \rangle$ in the second and $C_1 - \langle \mathbf{curl}\, \psi, \mathbf{B}_{h,g} \rangle$ in the third. In practice, since we know the exact solution $\mathbf{B}$, we simply take $\mathbf{B}_{h,g} := (I - P_h^1)\mathbf{B}$. The remaining steps of assembling the system are completely analogous to Sec. 0.3.3. We solve this system to obtain $B_{h,0}$ and then add the boundary interpolation to obtain the solution.

For $\psi$, we use the exact same definition as in (0.4.1) which results in a different curve due to the new domain parametrization (see Fig. 0.4.6). $J$ is of course still zero and the curve integral does not change either. We obtain again second order convergence for $p = 2$ (see Fig. 0.4.4), but the error is several orders of magnitude larger compared to the normal annulus. For $p = 3$ the convergence rate is at least 2, but the plot does not confirm convergence of order 3 in this case.

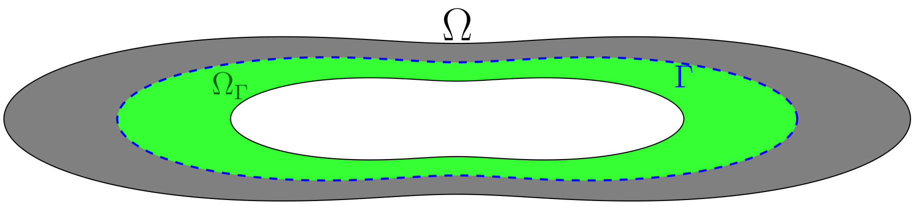


**Figure 0.4.5:** The distorted annulus domain

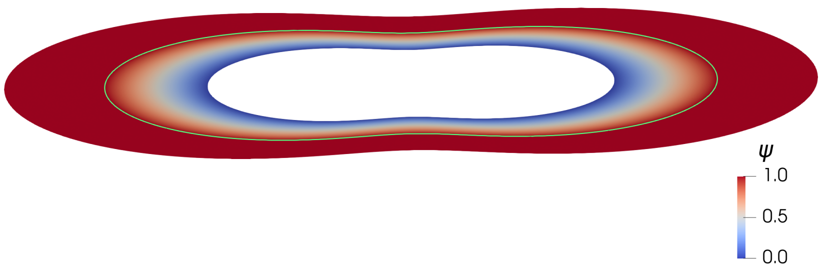


**Figure 0.4.6:** $\psi$ on the distorted annulus domain. $\Gamma$ (green) is again in the middle between the interior and exterior boundary.

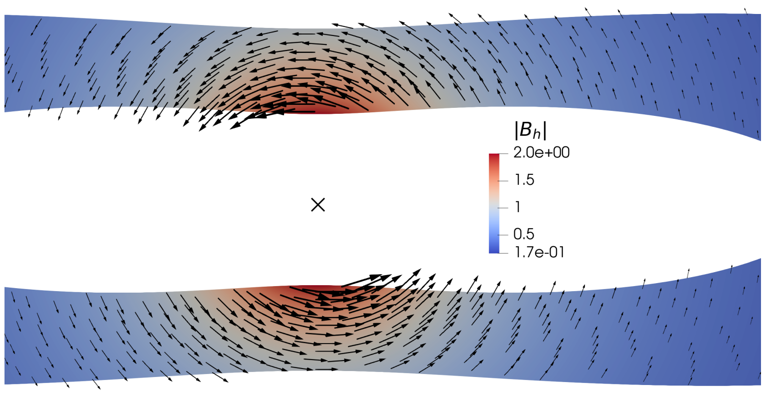


**Figure 0.4.7:** Solution of the Biot-Savart law on the distorted annulus with $64$ cells in both dimensions and $p = 2$. The wire is in the centre pointing towards the point of view.

### 0.4.2 Manufactured solution

As an example with a non-vanishing $J$, we will use a manufactured solution. Take $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$. It is easy to see that $\operatorname{div} \mathbf{B} = 0$ and $\mathbf{B} \cdot \mathbf{n} = 0$ for the annulus. This results in

$$J(x) = 4|x|^2 - 12|x| + 8.$$

We will pose this problem on the standard annulus domain from before with the same curves $\Gamma_1$ and $\Gamma_2$. Since $J$ is not zero this leads to different curve integrals

$$\int_{\Gamma_1} \mathbf{B} \cdot d\ell = \frac{9\pi}{8} \quad \text{and} \quad \int_{\Gamma_2} \mathbf{B} \cdot d\ell = 0.$$

We choose $\psi_1$ and $\psi_2$ as for the Biot-Savart problem. We observe the same convergence rate (Fig. 0.4.8) for $p = 2$, but we recognize that the errors are slightly different and so the choice of curve integral and $\psi$ matters.

For $p = 3$, we observe again second order convergence when $\Gamma_1$ is chosen, but third order convergence for $\Gamma_2$. This is likely because $\psi_1$ is not $H^2$. Since $J \neq 0$, we have to compute

$$\int_{\Omega_{\Gamma_1}} \psi_1 \, J \, dx$$

numerically. In order to do that, we interpolate $\psi_1$. Because $\psi_1$ is not $H^2$-regular, the interpolation error dominates and so we do not obtain third order convergence. This suggests the necessity to choose a $\psi$ smooth enough s.t. it can be approximated sufficiently well.

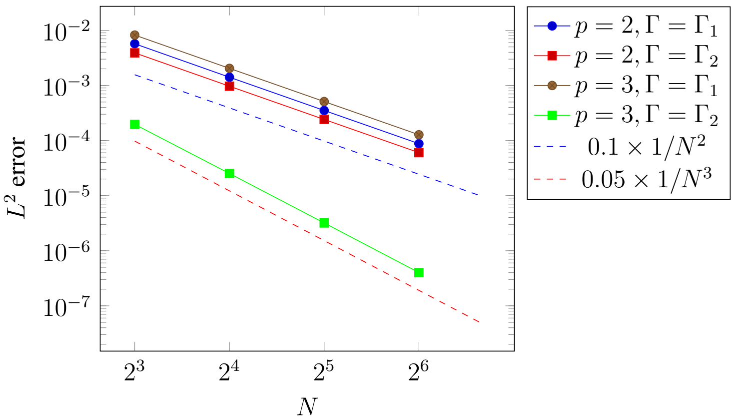


**Figure 0.4.8:** Convergence analysis for the manufactured solution $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$ with different choices of $\psi_i$ corresponding to different curves $\Gamma_i$ as before.