Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain and Alt^k the space of alternating k-linear maps on $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle_{\mathrm{Alt}^k}$ be the standard inner product of alterating k-linear maps and $\|\cdot\|_{\mathrm{Alt}^k}$ be the induced norm. Then we define the L^p-norm of a k-form ω for $1 \leq p < \infty$ as (cf. [Gol88])

$$\|\omega\|_{L^p\Lambda^k(\Omega)} := \left(\int_\Omega \|\omega\|^p_{\mathrm{Alt}^k} \right)^{1/p}$$

and for for $p = \infty$ as

$$\operatorname*{ess\,sup}_{x \in \Omega} \|\omega(x)\|_{\mathrm{Alt}^k}.$$

For $p = 2$ we obtain a Hilbert space (cf. [Arn18, Sec. 6.2.6]) with the L^2 inner product

$$\langle \omega, \nu \rangle := \int_\Omega \omega \wedge *\nu.$$

Then $L^p\Lambda^k(\Omega)$ are the spaces of k-forms s.t. the corresponding L^p-norm is finite.

Our next goal is to extend the exterior derivative d of smooth differential forms in the weak sense (cf. [Gol88]). This is done analogous to the definition of the usual weak derivative. The exterior derivative $d\omega \in L^p\Lambda^{k+1}(\Omega)$ is defined as the unique $(k+1)$-form in $L^p\Lambda^{k+1}(\Omega)$ s.t.

$$\int_\Omega d\omega \wedge \phi = (-1)^{k+1} \int_\Omega \omega \wedge d\phi \quad \forall \phi \in C_0^\infty \Lambda^{n-k-1}(\Omega).$$

Just as in the usual Sobolev setting we define the following spaces:

$$\begin{aligned} W^k_p(\Omega) &= \left\{ \omega \in L^p\Lambda^k(\Omega) \,|\, d\omega \in L^p\Lambda^{k+1}(\Omega) \right\}, \\ W^k_{p,loc}(\Omega) &= \left\{ \omega \; k\text{-form} \,|\, \omega|_A \in W^k_\infty(A) \text{ for every } A \subseteq M \text{ compact} \right\}. \end{aligned}$$

For $\omega \in W^k_p(\Omega)$ for $p < \infty$ we define the norm

$$\|\omega\|_{W^k_p} := \left(\|\omega\|^p_{L^k_p} + \|d\omega\|^p_{L^k_p} \right)^{1/p}$$

and for $p = \infty$

$$\|\omega\|_{W^k_p} := \max \left\{ \|\omega\|_{L^k_\infty}, \|d\omega\|_{L^k_\infty} \right\}.$$

We want to examine the Hilbert space $L^2\Lambda^k(\Omega)$ more closely (see [Arn18, Sec. 6.2.6] for more details). Because $d : L^2\Lambda^k(\Omega) \to L^2\Lambda^{k+1}(\Omega)$ is an unbounded, densely defined operator with domain $C_0^\infty \Lambda^k(\Omega)$ the adjoint δ is well-defined and is equal to the codifferential operator of differential forms. We then define additionally the space $\mathring{W}_2^k(\Omega)$ as the closure of $C_0^\infty(\Omega) \subseteq W_2^k(\Omega)$ w.r.t. the $W_2^k(\Omega)$-norm.

Now we can formulate our boundary value problem. Let $\Omega \subseteq \mathbb{R}^n$ from now on be a an exterior polyhedral domain of a compact set i.e. $\mathbb{R}^n \setminus \Omega$ is a compact polyhedron. We then have the following boundary value problem: For a fixed $C_0 \in \mathbb{R}$ and closed bounded k-chain γ find $\omega \in \mathring{W}_2^k(\Omega)$ s.t.

$$\begin{aligned} d\omega &= 0, \\ \delta\omega &= 0 \text{ and} \\ \int_\gamma \omega &= C_0. \end{aligned}$$

Because we consider polyhedral domains we assume that γ consists of finitely many k-simplices and that the cohomology class $[\gamma]$ is a generator of the simplicial homology group. The well-definedness of the integral over γ will be discussed later. Our goal will be to show existence and uniqueness of solutions. In order to achieve this, we rely on a result about the isomorphism of a simplicial cohomology space $H_p^k(K)$ which will be defined below and the L^p-cohomology space $H_{p,dR}^k(\overline{\Omega})$ (dR short for de Rham). This result was proven in [Gol88]. In the diploma thesis of Nikolai Nowaczyk [Now11], which mostly is based on this paper, many additional details can be found. The result will be presented in the next section. It should be noted that even though the results in [Gol88] are proved explicitely for smooth manifolds without boundary the results can be extended to Lipschitz manifolds with boundary (see the proof of Theorem 2 and the remark at the end in [Gol88]). Therefore, we can apply the result to our case.

1 Isomorphism of Cohomology

1.1 Assumptions

In order to formulate the assumptions necessary for the result from [Gol88] to work we will define some basic things from simplicial topology theory. More details and references can be found in [Bre13, Chapter 4.21].

Definition 1 (Affine simplex)**.** Let $x_1, x_2, ..., x_n$ be affine independent. Then

$$[x_1, x_2, ..., x_n] := \operatorname{conv}\{x_1, ..., x_n\}$$

is called an affine simplex.

Definition 2 (Simplicial complex)**.** A *simplicial complex* K is a collection of affine simplices s.t.

1. $\sigma \in K \Rightarrow$ any face of σ is in K,
2. $\sigma,\, \tau \in K \Rightarrow \sigma \cap \tau$ is in K.

We call $|K| := \bigcup\{\sigma | \sigma \in K\}$ the polyhedron of K.

For any topological space X a homeomorphism $\tau : |K| \to X$ is called *triangulation* of X.

Because $\overline{\Omega}$ is itself a polyhedron we can assume that $\overline{\Omega}$ and $|K|$ are equal as subsets of $\mathbb{R}^n$ and we can simply use the identity as triangulation. However, we will use different metrics on $|K|$ and $\overline{\Omega}$. We use the Euclidian metric on $\overline{\Omega}$ and we use the standard simplicial metric on $|K|$ (cf. [Gol88, p.191]). This metric is defined as follows:

Choose some numbering of the vertices $\{x_1,\, x_2, ...\}$ and take $f : |K| \to \ell^2$ where ℓ^2 is the Hilbert space of real-valued square-summable sequences s.t. $f(x_i) = e_i$ with $e_i \in \ell^2$ being the standard unit vectors and f is affine on every simplex. This mapping is unique.

Then we define the metric on $|K|$ as the pullback $g_S = f^*g$ where g is the standard metric in ℓ^2. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on ℓ^2. Then for $x \in |K|$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, $\sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \in T_x|K|$ we have

$$\begin{aligned}
g_S|_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n w_j \frac{\partial}{\partial x_j} \right) &= \left\langle \sum_{k=1}^\infty \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) \frac{\partial}{\partial y_k}, \sum_{l=1}^\infty \sum_{j,l=1}^n w_j \frac{\partial f_l}{\partial x_j}(x) \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k,l=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \left\langle \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l} \right\rangle \\
&= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i \frac{\partial f_k}{\partial x_i}(x) w_j \frac{\partial f_l}{\partial x_j}(x) \\
&= \sum_{i,j=1}^n \sum_{k=1}^\infty v_i w_j \left(Df(x)^T Df(x) \right)_{ij} \\
&= v^T Df(x)^T Df(x) w = \langle Df(x)v, Df(x)w \rangle,
\end{aligned}$$

where D denotes the Jacobian.

We have two crucial assumptions on the triangulation for the result to hold (cf. [Gol88, p.194]). We summarize them under *GKS-condition* named after the three authors of the [Gol88].

Assumption 1 (GKS-condition)**.** We will assume the following on the simplicial complex K and the triangulation τ:

1. The star of every vertex in K contains at most N simplices.

2. For the differential of τ we have constants $C_1, C_2 > 0$ s.t.

$$\|d\tau|_x\| < C_1, \ \|d\tau^{-1}|_{\tau(x)}\| < C_2,$$

   where d denotes the differential in the sense of differential geometry and the norm is the operator norm w.r.t. the metrics on $|K|$ and $\overline{\Omega}$.

The first assumption is equivalent to every vertex being contained in at most N simplices, which is fulfilled if we have a shape regular mesh.

Because τ is just the identity in our case the second assumption says that for every $x \in |K|$

$$\sup_{v \neq 0} \frac{\|v\|}{\sqrt{g_S|_x(v,v)}} = \sup_{v \neq 0} \frac{\|v\|}{\|Df(x)v\|} < C_1$$

and analogously

$$\sup_{v \neq 0} \frac{\|Df(x)v\|}{\|v\|} < C_1.$$

1.2 Statement of the Isomorphism

The isomorphism of the cohomology spaces from [Gol88] uses several mappings between different cohomology spaces. The first isomorphism is induced from a linear mapping between from the so called *S-forms* $S_p^k(K)$ to *p-summable k-cochains* $C_p^k(K)$ which will both be defined next.

Definition 3. We define the following norm of a k-cochain f

$$\|f\|_{C_p^k(K)} := \left(\sum_{c\, k\text{-chain}} |f(c)|^p \right)^{1/p}.$$

and the space of *p-summable k-cochains*

$$C_p^k(K) := \{f\ k\text{-cochain} |\ \|f\|_{C_p^k(K)} < \infty\}.$$

Take $\tau, \sigma \in K$ s.t. τ is a face of σ which we write as $\tau < \sigma$. It can be shown that the standard embedding $j : \tau \hookrightarrow \sigma$ induces an restriction operator $j^*_{\sigma,\tau} : W^*_\infty(\sigma) \to W^*_\infty(\tau)$ which is bounded (cf [Gol88, p.191]).

Definition 4 (S-forms)**.** Let

$$\theta = \{\theta(\sigma) \in W^k_\infty(\sigma) | \sigma \in K\}$$

be a collection of differential k-forms. We call θ S-form of degree k if we have for all for simplices $\mu < \sigma$

$$j^*_{\sigma,\mu}\theta(\sigma) = \theta(\mu).$$

We denote with $S^k(K)$ the space of all S-forms of degree k over the chain complex K. For $\theta \in S^k(K)$ we define $d\theta := \{d\theta(\sigma)|\sigma \in K\} \in S^{k+1}(K)$. $S^*(K)$ is the resulting cochain complex.

For $\theta \in S^k(K)$ we now define the norm

$$\|\theta\|_{S_p(K)} := \left(\sum_{\sigma \in K} \|\theta(\sigma)\|^p_{W^k_\infty(\sigma)} \right)^{1/p}.$$

$S^k_p(K)$ are the S-forms of degree k s.t. this norm is finite.

Using integration we can define define the homomorphism (see [Gol88, p.191])

$$I : S^k_p(K) \to C^k_p(K),\ I(\theta)(\sigma) = \int_\sigma \theta(\sigma) \text{ for } \sigma \in K.$$

With the exterior derivative d on S-forms as defined above we define

$$\mathcal{Z}^k_p := \{\theta \in S^k_p(K)|\, d\theta = 0\}$$
$$\mathcal{B}^k_p := dS^k_p(K)$$

and then the resulting cohomology space

$$\mathscr{H}^k_p(K) := {}^{\mathcal{Z}^k_p}\!/_{\mathcal{B}^k_p}.$$

We denote the standard cochain cohomology as $H^k_p(K)$. Then we have that the integration mapping $I : S^k_p(K) \to C^k_p(K)$ induces an isomorphism on the cohomologies i.e. $[I] : \mathscr{H}^k_p(K) \to H^k_p(K)$ is an isomorphism of vector spaces (see Theorem 1 in [Gol88] and the proof thereof).

Then we define

$$\varphi_\tau : W^k_{\infty,loc}(M) \to S^k(K),\ \omega \mapsto \{\tau|^*_\sigma\omega | \sigma \in K\}.$$

This is a well-defined vector space isomorphism ([Gol88, p.191]). This way we can identify $W^k_{\infty,loc}(M)$ with $S^k(K)$. Using the isomorphism φ_τ we now define $S^k_p(M) := (\varphi_\tau)^{-1}S^k_p(K)$.

We then have $S^k_p(M) \subseteq W^k_p(M)$ and the inclusion induces an isomorphism on cohomology [Gol88, Lemma 4, Corollary].

Above, we defined the integral operator I for $S^k_p(K)$ which can be therefore be applied on $S^k_p(M)$ as well. If we fix now a closed finite k-chain γ. Then $I(\cdot)(\gamma) = \int_\gamma$ becomes a functional on $S^k_p(M)$, but is is a-priori not clear how to extend this to $W^k_p(M)$. We know that $\int_\gamma d\eta = 0$ for $\eta \in S^k_p(M)$ because otherwise I would not induce a isomorphism on cohomology. We extend this now by setting $\int_\gamma d\nu = 0$ for all $\nu \in W^{k-1}_p(M)$. We have to check whether this is consistent with the definition above. Let $\nu \in W^k_p(M)$ s.t. $d\nu \in S^k_p(M)$. Let $A \subseteq M$ be a bounded neighborhood of γ. We can then find $\tilde{\nu}$ s.t. $\tilde{\nu} \in W_q(A)$ for any $q > 1$ and $d\tilde{\nu} = d\nu$ [Sch06, Thm 3.1.1]. We can then apply Stoke's theorem [GKS82, Thm. 9] to get $\int_\gamma d\nu = 0$. This shows consistency.

In the second part of [Gol88] they construct the operators $\mathscr{R}$ and $\mathscr{A}$. The precise definition and construction of these operators is not relevant for our purposes because we will only use the following properties (cf. [Gol88, Thm.2]).

Theorem 1. *Assume that the triangulation τ fulfills the GKS-condition. Then there exist linear mappings $\mathscr{R} : L^k_{1,loc} \to L^k_{1,loc}$, $\mathscr{A} : L^k_{1,loc} \to L^{k-1}_{1,loc}$ such that*

1. *$\mathscr{R}\omega - \omega = d\mathscr{A}\omega + \mathscr{A}d\omega$ for $\omega \in W^k_{1,loc}(M)$*
2. *for any $1 \leq p \leq \infty$, $\mathscr{R}(W^k_p(M)) \subseteq S^k_p(M)$.*

We can now use this operator $\mathscr{R}$ to define $\int_\gamma \omega$ for closed $\omega \in W^k_p(M)$ as

$$\int_\gamma \omega := \int_\gamma \mathscr{R}\omega.$$

This is consistent because if $\omega \in S^k_p(M)$ closed then due to Thm. 1

$$\int_\gamma \mathscr{R}\omega = \int_\gamma \omega + d\mathscr{A}\omega + \mathscr{A}d\omega = \int_\gamma \omega.$$

2 Existence and uniqueness of solutions

2.1 Existence

Returning now back to the problem, we are now able to proof existence of a solution. Take a closed cochain $F \in C_p^k(K)$ s.t. $F(\gamma) = C_0$ and $F(\partial d) = 0$ for (k-1)-chains d. Then we know from ??? that there exists a unique $[\theta] \in \mathscr{H}_p^k(K)$ s.t. $[I]([\theta]) = [F]$. Let us take $\eta := \varphi_\tau^{-1}\theta$. Then $\int_\gamma \eta = C_0$ holds. If we now take the Hodge decomposition $L_2^k(M) = \bar{\mathfrak{B}}^{\mathfrak{k}} \bigoplus \mathcal{H}^k \bigoplus \bar{\mathfrak{B}}_{\mathfrak{k}}^*$ and define ω as the projection of η onto the harmonic forms $\mathcal{H}^k$. Then we know that $d\omega = 0$, $\delta\omega = 0$ and $\operatorname{tr}\omega = 0$. So we only have to show that

$$\int_\gamma \omega = C_0.$$

We know from the Hodge decomposition that there exists a sequence $(\phi_i)_{i\in\mathbb{N}} \subseteq L_2^{k-1}(M)$ s.t. $\omega = \eta - \lim_{i\to\infty} d\phi_i$. Let now be $R > 0$ large enough s.t. $\gamma \subseteq B_R$. Then we know that $dW_2^{k-1}(B_R)$ is closed in $L_2^k(B_R)$. Therefore there exists $\phi_R \in W_2^{k-1}(B_R)$ s.t. $\lim_{i\to\infty} d\phi_i|_{B_R} = d\phi_R$. So we have $\omega|_{B_R} = \eta|_{B_R} - d\phi_R$ and

$$\int_\gamma \omega = \int_\gamma \omega|_{B_R} = \int_\gamma \eta|_{B_R} = C_0.$$

This proves existence.

2.2 Uniqueness

The first step is to show that the cochain chosen in the proof of existence is in fact unique if restricted to closed chains.

Proposition 1. *Let γ be a k-chain s.t. the homology class $[\gamma]$ is a generator of the homology group. Assume for some $C_0 \in \mathbb{R}$ there exist cochains $F, G \in C_p^k(K)$ s.t.*

$$F(\gamma) = C_0 \text{ and } F(\partial d) = 0 \text{ for all } (k-1)\text{-chains } d$$

and the same for G. Then the restriction of F and G to closed chains is the same.

Proof. Take any closed k-chain c. Because γ is the generator of the homology group we have $n \in \mathbb{Z}$ s.t. $[c] = [n\,\gamma]$ where $[\cdot]$ is the corresponding homology class. That means that we have some $(k-1)$-chain d s.t. $c = n\,\gamma + \partial d$. Using the properties of F and G,

$$F(c) = F(n\,\gamma + \partial d) = nF(\gamma) = n\,C_0.$$

Because the same computation is valid for G $F(c) = G(c)$ follows. $\square$

Theorem 2. *Assume that a co-chain as in Prop. 2.2 exists. Then the solution of the problem is unique.*

Proof. Let $\omega, \tilde{\omega}$ both be solutions. Because $\int_\gamma \omega = \int_\gamma \tilde{\omega}$ and ω and $\tilde{\omega}$ are closed we have due to Prop. 2.2 that $\int_c \omega = \int_c \tilde{\omega}$ for any closed k-chain c. So we have for the induced homomorphism $[I]([\mathscr{R}\omega]) = [I]([\mathscr{R}\tilde{\omega}])$ and therefore due to the isomorphism of cohomology $[\mathscr{R}\omega] = [\mathscr{R}\tilde{\omega}]$. Hence,

$$[\tilde{\omega}] = [\mathscr{R}\tilde{\omega}] = [\mathscr{R}\omega] = [\omega]$$

we get the equality of the cohomology classes.

That is equivalent to the existence of some $(k-1)$-form $\phi \in W_2^{k-1}(\overline{\Omega})$ s.t. $\omega = \tilde{\omega} + d\phi$. Then because ω and $\tilde{\omega}$ are harmonic we have $\omega, \tilde{\omega} \perp dW_2^{k-1}(\overline{\Omega})$ and therefore

$$\omega = \tilde{\omega}.$$

$\square$