# 1 Variational formulation of the magnetostatic problem in 2D

For simplicity we will now turn to the 2D case and we assume that our open domain will be bounded and Lipschitz. This involves introducing a different Hilbert complex with other differential operators. Then we derive the 2D magnetostatic problem from the three-dimensional one. We will assume our domain $\Omega$ to have a "annulus like" form which we will clarify in more rigour. In order for the 2D magnetostatic problem to be well-posed we require an additional constraint that will again be a curve integral. We will investigate an alternative way to represent this curve integral which will turn out to be easily suitable to be included in our numerical approximation. Prerequisite for this section are knowledge about basic functional analysis and fundamentals of finite element theory. We will also depend on the notions introduced in Sec. **??**, in particular the Hodge decomposition in the general case.

## 1.1 The curl-div Hilbert complex

{sec:variational

We start with the introduction of the relevant differential operators and the resulting 2D Hilbert complex. We will then explain what domains we will consider and state the magnetostatic problem in strong form.

We define the scalar curl for $\mathbf{v} \in C^1(\Omega;\mathbb{R}^2)$ as

$$\operatorname{curl}\mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

Additionally, we have the vector-valued curl, denoted in bold, defined for $v \in C^1(\Omega)$

$$\mathbf{curl}\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}.$$

The cross product for 2D reads for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$,

$$\mathbf{a} \times \mathbf{b} := a_1 b_2 - a_2 b_1 = \mathbf{a} \cdot \mathbf{R}_{-\pi/2}\mathbf{b}.$$

where $\mathbf{R}_{-\pi/2}$ is the rotation in clockwise direction by $\pi/2$.

A straightforward calculation shows that the following integration-by-parts formula holds for $u \in C^1(\overline{\Omega})$, $\mathbf{v} \in C^1(\overline{\Omega};\mathbb{R}^2)$, assuming $\Omega$ is Lipschitz and bounded

$$\int_\Omega \mathbf{curl}\, u \cdot \mathbf{v}\, dx = \int_\Omega u \operatorname{curl}\mathbf{v}\, dx + \int_{\partial\Omega} u\, \mathbf{v} \times \mathbf{n}\, d\ell \tag{1.1.1}$$

{eq:2D_integrati

where $\mathbf{n}$ is the outward unit normal of $\Omega$. Analogous to what we did in Sec. **??** we can now extend this definition in the weak sense. First, notice that $\mathbf{curl}\, u = R_{-\pi/2} \operatorname{grad} u$ and thus $\mathbf{curl}$ is well-defined on $H^1$. For the scalar curl we define

$$H(\operatorname{curl};\Omega) = \{\mathbf{v} \in L^2 \mid \exists w \in L^2 : \int_\Omega w\phi\, dx = \int_\Omega \mathbf{v} \cdot \mathbf{curl}\, \phi\, dx \quad \forall \phi \in C_0^\infty\}$$

and we denote $w$ in the definition – which is uniquely determined – as $\operatorname{curl} \mathbf{v}$, so in short $(\operatorname{curl}, H(\operatorname{curl})) = (\mathbf{curl}, C_0^\infty)^*$. Analogous to Section **??**, it is then possible to extend the tangential trace to an operator $\gamma_\tau$ defined on $H(\operatorname{curl})$ s.t. for any $u \in H^1(\Omega)$, $\mathbf{v} \in H(\operatorname{curl})$ the integration by parts formula

$$\langle \mathbf{curl}\, u, \mathbf{v}\rangle = \langle u, \operatorname{curl} \mathbf{v}\rangle + \langle \gamma_\tau \mathbf{v}, \operatorname{tr} u\rangle_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)}.$$

holds. From now on we will leave out the subindex of the duality inner product. Also analogous to the 3D case, we can define

$$H_0(\operatorname{curl}) := \{\mathbf{v} \in H(\operatorname{curl}) \mid \gamma_\tau \mathbf{v} = 0\}$$

and can then compute the adjoints analogously to what we did in **??**

$$\begin{aligned}(\operatorname{curl}, H_0(\operatorname{curl})) &= (\mathbf{curl}, H^1)^* \\ (\operatorname{curl}, H(\operatorname{curl})) &= (\mathbf{curl}, H_0^1)^*.\end{aligned}$$

Notice that $\operatorname{div} \mathbf{curl} = 0$ and so we have the following 2D Hilbert complex

$$H_0^1 \xrightarrow{\mathbf{curl}} H_0(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2. \tag{1.1.2}$$ {eq:2D_hilbert_c

and the dual complex

$$L^2 \xleftarrow{\operatorname{curl}} H(\operatorname{curl}) \xleftarrow{-\operatorname{grad}} H^1$$

We use the notation introduced in Sec. **??** for general Hilbert complexes i.e. $V^0 = H_0^1$, $V^1 = H_0(\operatorname{div})$, $V^2 = L^2$, $V_0^* = L^2$, $V_1^* = H(\operatorname{curl})$ $V_2^* = H^1$, $d^0 = \mathbf{curl}$, $d^1 = \operatorname{div}$ and we set $d^k = 0$ for the remaining $k \in \mathbb{Z}$. $d_k^*$ is the adjoint of $d^k$. Also we remind of the notation $\mathfrak{B}^k$ for the image of the differential operator, $\mathfrak{B}_k^*$ for the image of the adjoint and analogous $\mathfrak{Z}^k$ for the kernel and $\mathfrak{Z}_k^*$ for the kernel of the adjoint.

## 1.2 Strong formulation of the 2D magnetostatic problem

The 2D magnetostatic problem will be derived from a special case of the 3D problem. Then the type of domains considered will be clarified and the strong formulation stated at the end.

Assume that our current source $\mathbf{J}$ is pointing in $z$-direction i.e. $\mathbf{J} = J\mathbf{e}_3$. Further assume that there is a $\tilde{\Omega}$ s.t. $\Omega = \tilde{\Omega} \times \mathbb{R}$. If $B_3$ does not change in $z$-direction we get that

$$0 = \operatorname{div} \mathbf{B} = \partial_x B_1 + \partial_y B_2 = \operatorname{div} \tilde{\mathbf{B}}.$$

where $\tilde{\mathbf{B}} = (B_1, B_2)^\top$. The third component of the equation $\operatorname{curl} \mathbf{B} = \mathbf{J}$ from the magnetostatic problem reads

$$J = \partial_x B_2 - \partial_y B_1 = \operatorname{curl} \tilde{\mathbf{B}}$$

For $\Omega$ the unit outer normal is zero in $z$-direction and thus $\tilde{\mathbf{B}}$ satisfies the boundary condition

$$0 = \mathbf{B} \cdot \mathbf{n} = \tilde{\mathbf{B}} \cdot \tilde{\mathbf{n}}$$

with $\tilde{\mathbf{n}} = (n_1, n_2)^\top$ being the outer unit normal $\tilde{\Omega}$.

Now we will abuse notation and refer to $\tilde{\mathbf{B}}$ as $\mathbf{B}$, $\tilde{\mathbf{n}}$ as $\mathbf{n}$ and $\tilde{\Omega}$ as $\Omega$. Let $J \in L^2$ be given. Then we see that $\mathbf{B}$ must fulfill the following equations

$$\begin{aligned} \operatorname{curl} \mathbf{B} &= J, \\ \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

Depending on the domain, this problem is in general not well-posed – just as the problem in 3D – and requires an additional constraint. Let us now make certain restrictions on what type of domain we will consider.

From now on, we assume that the space of harmonic forms $\mathfrak{H}^1$ has dimension one and that our domain is encompassed by two disjoint closed curves i.e. we have curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$ s.t. $\partial\Omega_{in} \dot{\cup} \partial\Omega_{out}$ is the boundary of $\Omega$. Let now $\Gamma$ be s.t. it is a closed curve in $\Omega$ that goes around the hole in the middle i.e. the area surrounded by $\Gamma$ contains $\partial\Omega_{in}$. Denote its parametrization with $\boldsymbol{\gamma} : [0, |\Gamma|] \to \Omega$ s.t. $|\boldsymbol{\gamma}'(t)| = 1$ and assume that $\boldsymbol{\gamma}$ is bijective i.e. the curve does not intersect itself. We assume that $\Gamma$ has positive distance from $\partial\Omega_{in}$. We do not assume anything like that for the exterior boundary i.e. $\Gamma$ can touch or be identical to $\partial\Omega_{out}$. We then denote the area that is enclosed by $\Gamma$ and $\partial\Omega_{in}$ as $\Omega_\Gamma$ (cf. Fig. 1.1).

From now on, our domain $\Omega$ is always assumed to be of that kind. We will later make further restrictions on what types of domain we will consider that will be suitable for discretization (see Assumption 3.2.2).

We add the curve integral along $\Gamma$, which we assume to be well-defined, as an additional constraint. So in total, we obtain the following problem.

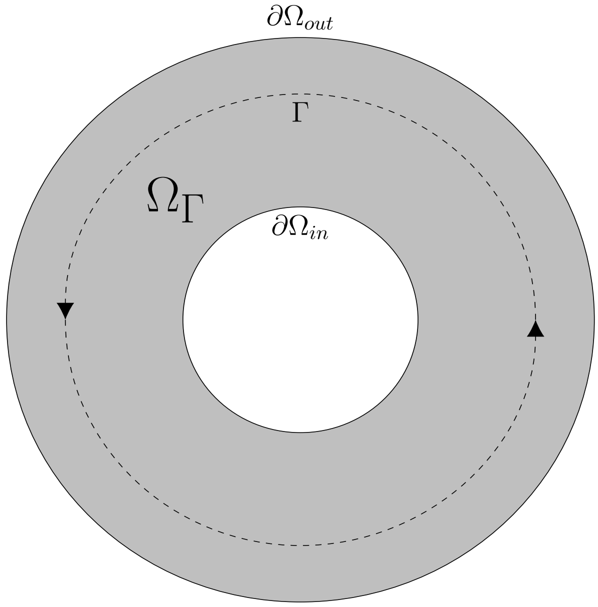



Figure 1.1: A simple example for a domain $\Omega$ as described. The boundary are two disjoint curves $\partial\Omega_{in}$ and $\partial\Omega_{out}$. The curve $\Gamma$ is parametrized in anticlockwise direction and $\Omega_\Gamma$ is the area enclosed by $\Gamma$ and $\partial\Omega_{in}$. {fig:annulus_dom

{prob:2d_magneto

**Problem 1.2.1** (2D magnetostatic problem)**.** Given $J \in L^2$ and $C_0 \in \mathbb{R}$, find $\mathbf{B} \in H_0(\mathrm{div}) \cap H(\mathrm{curl})$ s.t.

$$\begin{aligned} \mathrm{curl}\,\mathbf{B} &= J, \\ \mathrm{div}\,\mathbf{B} &= 0, \\ \int_\Gamma \mathbf{B} \cdot dl &= C_0 \end{aligned}$$

Another option for the additional constraint would be an orthogonality constraint as discussed in [4, Sec. 3.5].

## 1.3 Mixed formulation

In order to solve this problem numerically using finite elements, we have to choose a suitable variational formulation of the problem. This variational formulation will be stated without the curve integral constraint and then we will show the equivalence with the strong formulation.

Ignoring the curve integral at first, we will use the following. We choose a non-zero harmonic form $\mathbf{p} \in \mathfrak{H}^1$ and have $J \in L^2$ . Then the problem is:

Find $\sigma \in H_0^1$, $B \in H_0(\mathrm{div})$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \quad \forall \tau \in H_0^1, \tag{1.3.1}$$

{eq:first_eq_mix

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \lambda \langle \mathbf{p}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in H_0(\mathrm{div}) \tag{1.3.2}$$

{eq:second_eq_mi

As before, the inner product without subscript denotes the $L^2$ inner product and $\|\cdot\|$ the $L^2$ norm. Here the curve integral condition is missing. It is difficult to include the curve integral condition directly when solving this system numerically. So we will replace it below in Sec. 1.4.

Even though this formulation appears more complicated in comparison to the first two equations of the 2D magnetostatic problem (Problem 1.2.1), it will turn out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will first investigate the formulation without curve integral

**Proposition 1.3.1.** *For any $J \in L^2$, (1.3.1) and (1.3.2) hold i.i.f. $\sigma = 0$, $\lambda = 0$, $\mathrm{curl}\, \mathbf{B} = J$ and $\mathrm{div}\, \mathbf{B} = 0$ i.e. $\mathbf{B}$ solves the 2D magnetostatic problem (Problem 1.2.1) without the additional curve integral constraint.*

*Proof.* Assume $(\sigma, \mathbf{B}, \lambda)$ is a solution of (1.3.1) and (1.3.2). Then the first equation is

$$\langle \sigma + J, \tau \rangle = \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle \quad \forall \tau \in H_0^1$$

which is equivalent to $\mathbf{B} \in H(\mathrm{curl})$ and $J + \sigma = \mathrm{curl}\, \mathbf{B}$.

Now assume additionally, that (1.3.2) holds. Then by choosing $\mathbf{v} = \mathbf{p} \in \mathfrak{H}^1$, we get $\mathrm{div}\, \mathbf{p} = 0$ from the definition of the harmonic forms and $\mathfrak{H}^1 \perp \mathbf{curl}\, H_0^1$ from the Hodge decomposition and thus

$$\langle \mathbf{curl}\, \sigma, \mathbf{p} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{p} \rangle + \lambda \langle \mathbf{p}, \mathbf{p} \rangle = \lambda \langle \mathbf{p}, \mathbf{p} \rangle = 0$$

and so $\lambda = 0$. Then we can choose $\mathbf{v} = \mathbf{curl}\, \sigma$ to get

$$\langle \mathbf{curl}\, \sigma, \mathbf{curl}\, \sigma \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{curl}\, \sigma \rangle + \lambda \langle \mathbf{p}, \mathbf{curl}\, \sigma \rangle = \|\mathbf{curl}\, \sigma\|^2 = 0.$$

Because $\sigma \in H_0^1$ this gives us $\sigma = 0$. Also we have then $J = \mathrm{curl}\, \mathbf{B}$. At last we choose $\mathbf{v} = \mathbf{B}$ which gives us $\mathrm{div}\, \mathbf{B} = 0$ and thus we proved the first direction.

The other implication is clear i.e. if $\mathbf{B} \in H(\mathrm{curl}) \cap H_0(\mathrm{div})$ with $\mathrm{curl}\, \mathbf{B} = J$ and $\mathrm{div}\, \mathbf{B} = 0$ then the variational formulation clearly holds. $\square$

Notice that the variable $\lambda$ is not necessary for this variational formulation, but we will need it later, since we will add another equation representing the curve integral constraint and hence we need another variable to have the same number of unknowns and equations. If we now add the same additional constraint to both formulations of the problem then they will remain equivalent.

## 1.4 Curve integral constraint

{sec:curve_integ

We still need to find a good way to include the curve integral constraint from Problem 1.2.1 in our formulation. Instead of incorporating it directly, we will substitute it with another equation. We will first derive this equation as an immediate consequence of the integration by parts formula (1.1.1) and then state the final variational formulation of the 2D magnetostatic problem which we will investigate in the coming sections.

Because $\mathbf{n}$ is the unit outward normal of $\Omega_\Gamma$ and $\boldsymbol{\gamma}$ the parametrization of $\Gamma$ that $\mathbf{n} \perp \boldsymbol{\gamma}'$ and

$$\mathbf{B} \times \mathbf{n} = (B_1 n_2 - B_2 n_1) = \mathbf{B} \cdot \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} = -\mathbf{B} \cdot \mathbf{R}_{\pi/2}\mathbf{n}.$$

$\mathbf{R}_{\pi/2}\mathbf{n}$ is either $\boldsymbol{\gamma}'$ or $-\boldsymbol{\gamma}'$. Assume w.l.o.g. that $\mathbf{R}_{\pi/2}\mathbf{n} = \boldsymbol{\gamma}'$ and thus

$$\mathbf{B} \times \mathbf{n} = -\mathbf{B} \cdot \boldsymbol{\gamma}'$$

and so the curve integral becomes

$$\int_\Gamma \mathbf{B} \cdot d\ell = \int_0^{|\Gamma|} \mathbf{B}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt = -\int_\Gamma \mathbf{B} \times \mathbf{n} d\ell.$$

Choose $\psi \in H^1$ s.t. $\psi = 0$ on $\partial\Omega_{in}$, $\psi = 1$ on $\partial\Omega_{out}$ and $\psi \equiv 1$ in $\Omega \setminus \Omega_\Gamma$
Then we observe

$$\int_\Omega \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \mathbf{curl}\, \psi \cdot \mathbf{B}\, dx = \int_{\Omega_\Gamma} \psi\, J\, dx + \int_{\partial\Omega} \mathbf{B} \times \mathbf{n}\, d\ell = \int_{\Omega_\Gamma} \psi\, J\, dx - \int_\Gamma \mathbf{B} \cdot d\ell$$

So if the curve integral

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0$$

is given and we can compute $\int_{\Omega_\Gamma} \psi\, J\, dx$ we can add the equation

$$\langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1 \qquad (1.4.1)$$ {eq:variational_

with

$$C_1 := \int_{\Omega_\Gamma} \psi\, J\, dx - C_0$$

to our system.

From the above derivations it is also clear that for $\mathbf{B} \in C^1(\overline{\Omega}; \mathbb{R}^2)$

$$\int_\Gamma \mathbf{B} \cdot d\ell = C_0 \Leftrightarrow \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = C_1.$$

This is the motivation to add the right equation to our system instead of the curve integral since it is much easier to enforce numerically.

In order to get a variational formulation to study theoretically, we multiply (1.4.1) with an arbitrary $\mu \in \mathbb{R}$. In conclusion, we have the following variational problem:

{prob:magnetosta

**Problem 1.4.1.** Let $J \in L^2$, $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Find $\sigma \in H_0^1$, $\mathbf{B} \in H_0(\mathrm{div})$, $\lambda \in \mathbb{R}$ s.t.

$$\langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle = -\langle J, \tau \rangle \qquad \forall \tau \in H_0^1, \tag{1.4.2}$$ {eq:first_eq_mix

$$\langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \mathrm{div}\, \mathbf{B}, \mathrm{div}\, \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle = 0 \qquad \forall \mathbf{v} \in H_0(\mathrm{div}), \tag{1.4.3}$$ {eq:second_eq_mi

$$\mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle = \mu C_1 \qquad \forall \mu \in \mathbb{R}. \tag{1.4.4}$$

which gives us the variational formulation of the magnetostatic problem with curve integral constraint (Problem 1.2.1). We will study the well-posedness of this formulation next.

## 1.5 Well-posedness of the magnetostatic system

{sec:well-posedn

The well-posedness is based on the well-known Banach-Nečas-Babuška (BNB) theorem concerning general variational problems of the following form: Find $x \in X$ s.t.

$$a(x, y) = \ell(y) \quad \forall y \in Y \tag{1.5.1}$$ {eq:general_vari

where $X$ and $Y$ are Banach spaces, $a$ is a bilinear form and $\ell \in Y'$. The BNB-theorem then answers the question of well-posedness i.e. if there exists a unique solution and if we can find a stability estimate. The following formulation is from [5, Sec. 25.3] in the real case.

{thm:BNB}

**Theorem 1.5.1** (BNB)**.** *Let $X$ be a Banach space and $Y$ be a reflexive Banach space. Let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and $\ell \in Y'$. Then a problem of the form (1.5.1) is well-posed i.i.f. the following two criteria are fulfilled*

(1) $$\inf_{x \in X} \sup_{y \in Y} \frac{|a(x, y)|}{\|x\|_X \|y\|_Y} =: \gamma > 0 \tag{1.5.2}$$ {eq:BNB1}

(2) *for any $y \in Y$ if $a(x, y) = 0$ for every $x \in X$, then $y = 0$.* (1.5.3) {eq:BNB2}

*We then obtain the stability estimate for a solution $x$*

$$\|x\|_X \leq \frac{1}{\gamma}\|\ell\|_{Y'}.$$

Note that (1.5.2) is equivalent to the fact that for any $x \in X \setminus \{0\}$ there exists $y \in Y \setminus \{0\}$ s.t. $a(x,y) \geq \gamma \|x\|_X \|y\|_Y$.

Since we are dealing with Hilbert spaces only we can utilize the following proposition to prove it (see [5, Rem. 25.14]).

{prop:T_coercivi

**Proposition 1.5.2** ($T$-coercivity)**.** *Let $X$ and $Y$ be Hilbert spaces. Then* (1.5.2) *and* (1.5.3) *hold, if there exists a bounded bijective operator $T : X \to Y$ and $\eta > 0$ s.t.*

$$a(x, Tx) \geq \eta \|x\|_X^2 \quad \forall x \in X. \tag{1.5.4}$$ {eq:T_coercivity

*Then $\gamma$ from* (1.5.2) *can be chosen as $\eta / \|T\|_{\mathcal{L}(X,Y)}$.*

*Proof.* For any $x \in X$, by taking $y = Tx \in Y$ and using the boundedness of $T$ we have

$$a(x, T(x)) \geq \eta \|x\|^2 \geq \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}} \|x\|_X \|y\|_Y$$

and thus (1.5.2) holds with $\gamma = \frac{\eta}{\|T\|_{\mathcal{L}(X,Y)}}$.

For (1.5.3) assume that we have $y \in Y$ s.t. $a(x,y) = 0$ for all $x \in X$.

$$0 = a(T^{-1}y, TT^{-1}y) \geq \eta \|T^{-1}y\|_X^2$$

so $T^{-1}y = 0$ and thus $y = 0$. $\square$

**Remark 1.5.3.** The other direction is also true i.e. if (1.5.2) and (1.5.3) are fulfilled we can construct a $T$ with the desired properties.

Note also that when we have found $T$ s.t. (1.5.4) holds then it must be injective. This is because if $Tx = 0$ for any $x \in X$ then $x = 0$ follows from the $T$-coercivity.

The next step is to put our formulation of Problem 1.4.1 into this general framework. To this end, we define $X := H_0^1 \times H_0(\mathrm{div}) \times \mathbb{R}$ and for $(\sigma, \mathbf{B}, \lambda) \in X$

$$\|(\sigma, \mathbf{B}, \lambda)\|_X := \sqrt{\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2}.$$

Notice that $X$ is then a Hilbert space with inner product

$$\langle(\sigma, \mathbf{B}, \lambda), (\tau, \mathbf{v}, \mu)\rangle_X = \langle \sigma, \tau \rangle_{H^1} + \langle \mathbf{B}, \mathbf{v} \rangle_{H(\mathrm{div})} + \lambda\mu.$$

For the formulation of the problem, we define the bilinear form $a : X \times X \to \mathbb{R}$

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \langle \sigma, \tau \rangle - \langle \mathbf{B}, \mathbf{curl}\, \tau \rangle + \langle \mathbf{curl}\, \sigma, \mathbf{v} \rangle + \langle \operatorname{div} \mathbf{B}, \operatorname{div} \mathbf{v} \rangle + \langle \lambda \mathbf{p}, \mathbf{v} \rangle - \mu \langle \mathbf{curl}\, \psi, \mathbf{B} \rangle. \tag{1.5.5}$$ {eq:definition_b

and

$$\ell(\tau, \mathbf{v}, \mu) = -\langle J, \tau \rangle - \mu C_1.$$

Then Problem 1.4.1 is equivalent to the following: Find $(\sigma, \mathbf{B}, \lambda) \in X$ s.t.

$$a(\sigma, \mathbf{B}, \lambda; \tau, \mathbf{v}, \mu) = \ell(\tau, \mathbf{v}, \mu) \quad \forall (\tau, \mathbf{v}, \mu) \in X.$$

Note that the bilinear form $a$ is not symmetric.

The next step is to show two important properties of $\mathbf{curl}\, \psi$ assuming $\psi$ has been chosen as described above.

**Proposition 1.5.4.** *Under the given assumptions on $\psi$, $\mathbf{curl}\, \psi \in H_0(\operatorname{div})$.*

*Proof.* We need to show $0 = \gamma_n \mathbf{curl}\, \psi$. Recall that the definition of

$$\langle \gamma_n \mathbf{curl}\, \psi, \operatorname{tr} u \rangle = \int_\Omega \mathbf{curl}\, \psi \cdot \operatorname{grad} u \, dx + \int_\Omega \operatorname{div} \mathbf{curl}\, \psi \, u \, dx$$

where the last term vanishes. Take now $\phi \in C^1(\overline{\Omega})$ arbitrary. Then we take $\phi_1 \in C^1(\overline{\Omega})$ s.t. $\phi_1 = \phi$ in a neighborhood of $\partial\Omega_{in}$ and zero near $\partial\Omega_{out}$. Analogously, take $\phi_2 \in C^1(\overline{\Omega})$ s.t. $\phi_2 = \phi$ in a neighborhood of $\partial\Omega_{out}$ and zero near $\partial\Omega_{in}$. Here we used the fact that the two parts of the boundary are disjoint and have positive distance from one another. Then also $\operatorname{tr} \phi = \operatorname{tr} \phi_1 + \operatorname{tr} \phi_2$. We use the integration by parts formula **??**, $\operatorname{div} \mathbf{curl} = 0$ and the integration-by-parts formula for the curl

$$\begin{aligned}
&\langle \gamma_n \mathbf{curl}\, \psi, \operatorname{tr} \phi \rangle = \langle \gamma_n \mathbf{curl}\, \psi, \operatorname{tr} \phi_1 \rangle + \langle \gamma_n \mathbf{curl}\, \psi, \operatorname{tr} \phi_2 \rangle \\
&= \int_\Omega \mathbf{curl}\, \psi \cdot \operatorname{grad} \phi_1 \, dx + \int_\Omega \mathbf{curl}\, \psi \cdot \operatorname{grad} \phi_2 \, dx \\
&= \int_\Omega \psi \cdot \operatorname{curl} \operatorname{grad} \phi_1 \, dx + \langle \gamma_\tau \operatorname{grad} \phi_1, \psi \rangle + \int_\Omega \psi \cdot \operatorname{curl} \operatorname{grad} \phi_2 \, dx + \langle \gamma_\tau \operatorname{grad} \phi_2, \psi \rangle.
\end{aligned}$$

Here only the boundary terms remain because $\operatorname{curl} \operatorname{grad} = 0$. Now remember that because $\phi_j \in C^1(\overline{\Omega})$, $\langle \gamma_\tau \operatorname{grad} \phi_j, \psi \rangle = \langle \operatorname{grad} \phi_j \times \mathbf{n}, \psi \rangle_{L^2(\partial\Omega)}$. So

$$\langle \gamma_\tau \operatorname{grad} \phi_1, \psi \rangle = \langle \operatorname{grad} \phi_1 \times \mathbf{n}, \psi \rangle_{L^2(\partial\Omega)} = 0$$

because $\phi_1$ is zero near $\partial\Omega_{out}$ and $\psi$ is zero on $\partial\Omega_{in}$. For the remaining term,

$$\langle \gamma_\tau \operatorname{grad} \phi_2, \psi \rangle = \int_{\partial\Omega_{out}} \psi \operatorname{grad} \phi_2 \times \mathbf{n} d\ell = \int_{\partial\Omega_{out}} \operatorname{grad} \phi_2 \times \mathbf{n} d\ell = -\int_{\partial\Omega_{out}} \operatorname{grad} \phi_2 \cdot d\ell$$

using a counter clockwise parametrization of $\partial\Omega_{out}$ for a parametrization $\mathbf{s}$ i.e. $\operatorname{grad} \phi_1 \times \mathbf{n} = -\operatorname{grad} \phi_1 \cdot \mathbf{R}_{\pi/2}\mathbf{s}'$ and then we know from basic vector calculus because $\partial\Omega_{out}$ is closed

$$\int_{\partial\Omega_{out}} \operatorname{grad} \phi_1 \cdot d\mathbf{s} = 0.$$

and so in conclusion, $\gamma_n \operatorname{\mathbf{curl}} \psi = 0$. □

Usually the last equation in Problem 1.4.1 is used to determine the harmonic part of the solution. This implies that we would like $\operatorname{\mathbf{curl}} \psi$ to have non-vanishing harmonic part. This is indeed true.

**Proposition 1.5.5.** *Let $P^1_{\mathfrak{H}} : L^2 \to \mathfrak{H}^1$ be the orthogonal projection onto the harmonic forms. Then with $\psi$ defined as above we have $P^1_{\mathfrak{H}} \operatorname{\mathbf{curl}} \psi \neq 0$.*

*Proof.* Since $\operatorname{div} \operatorname{\mathbf{curl}} \psi = 0$ we know that

$$\operatorname{\mathbf{curl}} \psi \in \mathfrak{Z}^1 = \mathfrak{B}^1 \overset{\perp}{\oplus} \mathfrak{H}^1$$

using the Hodge decomposition (cf. Thm. ??). Assume for contradiction that $\operatorname{\mathbf{curl}} \psi \in \mathfrak{B}^1$ i.e. there exists $\psi_0 \in H^1_0$ s.t. $\operatorname{\mathbf{curl}} \psi_0 = \operatorname{\mathbf{curl}} \psi$. Since $\mathbf{curl}$ is just the rotated gradient we would get that $\operatorname{grad}(\psi - \psi_0) = 0$ and thus $\psi - \psi_0$ is constant almost everywhere. But this is a contradiction since $\operatorname{tr} \psi_0$ is zero on $\partial\Omega_{in}$ and $\partial\Omega_{out}$, but $\operatorname{tr} \psi = 0$ on $\partial\Omega_{in}$ and $\operatorname{tr} \psi = 1$ on $\partial\Omega_{in}$. Thus $\operatorname{\mathbf{curl}} \psi \notin \mathfrak{B}^1$ and the claim follows. □

Now we can apply the Hodge decomposition of the $\ker \operatorname{div}$ on $\operatorname{\mathbf{curl}} \psi$ and obtain the following

**Corollary 1.5.6.** *Let $\mathbf{p} \in \mathfrak{H}^1 \setminus \{0\}$. Then there exists $\psi_0 \in H^1_0$ and $c_\psi \in \mathbb{R} \setminus \{0\}$ s.t.*

$$\operatorname{\mathbf{curl}} \psi = \operatorname{\mathbf{curl}} \psi_0 + c_\psi \mathbf{p}.$$

Because we can choose $\mathbf{p}$ we can assume w.l.o.g. that $c_\psi > 0$ and we will do so from now on.

As stated in the proof of the Poincare inequality ?? the $\mathbf{curl}|_{\mathfrak{Z}^\perp} : \mathfrak{Z}^\perp \to \mathfrak{B}^1$ is bijective and since it is bounded w.r.t. the $V$-norm – which is the $H^1$-norm here – due to Banach inverse theorem it is invertible and we denote this inverse $\mathbf{curl}^{-1}$. This is a slight abuse of notation since it is not really the inverse of the full $\mathbf{curl}$.

Let $P_{\mathfrak{B}}$ be the $L^2$-orthogonal projection onto $\mathfrak{B}^j$, we then denote $\mathbf{v}_{\mathfrak{B}} = P_{\mathfrak{B}}\mathbf{v}$ for any $\mathbf{v} \in L^2$ and analogous for $\mathfrak{H}^j$ and $\mathfrak{B}_j^*$. In order to prove the $T$-coercivity, we need the following lemma.

{lem:T_for_T_coe

**Lemma 1.5.7.** *Take* $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ *with* $c_\psi > 0$ ,$\mathbf{p} \in \mathfrak{H}^1$ *and* $\|\mathbf{p}\| = 1$. *Define* $T : X \to X$ *as*

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}, \alpha\langle\mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}).$$

*with* $\alpha < 0$ *and* $\beta > 0$. *Then* $T$ *is bounded and surjective.*

*Proof.* The boundedness is clear since all operators used in the definition are bounded w.r.t. the norms of their domains. From the Poincaré inequality we know that $\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\| \leq c_P\|\mathbf{B}_{\mathfrak{B}}\|$ and so

$$\begin{aligned}
&\|T(\sigma, \mathbf{B}, \lambda)\|_X^2 = \|\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + \|\mathbf{curl}\,\sigma + \mathbf{B} + \lambda\beta\mathbf{p}\|_{H(\mathrm{div})}^2 + \left(\alpha\langle\mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi}\right)^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^4}\|\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}}\|_{H^1}^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}_{\mathfrak{H}}\|^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq 2\|\sigma\|_{H^1}^2 + \frac{2}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + 3\|\mathbf{curl}\,\sigma\|^2 + 3\|\mathbf{B}\|_{H(\mathrm{div})}^2 + 3\lambda^2\beta^2 + 2\alpha^2\|\mathbf{B}\|_{H(\mathrm{div})}^2 + \frac{2}{c_\psi^2}\lambda^2 \\
&\leq C_T\left(\|\sigma\|_{H^1}^2 + \|\mathbf{B}\|_{H(\mathrm{div})}^2 + \lambda^2\right)
\end{aligned}$$

with

$$C_T := \max\left\{5, \frac{2}{c_P^2} + 3 + 2\alpha^2, 3\beta^2 + \frac{2}{c_\psi^2}\right\}. \tag{1.5.6}$$ {eq:bound_on_norr

So $T$ is bounded and $\|T\|_{\mathcal{L}(X,Y)} \leq \sqrt{C_T}$.

In order to prove surjectivity, we will split up $\mathbf{v} = \mathbf{v}_{\mathfrak{B}} + \mathbf{v}_{\mathfrak{H}} + \mathbf{v}_{\mathfrak{B}^*}$ using the Hodge decomposition. Take $(\tau, \mathbf{v}, \mu) \in X$ arbitrary and choose

$$\sigma = (1 + \frac{1}{c_P^2})^{-1}(\tau + \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}}) \text{ and } \mathbf{B}_{\mathfrak{B}} = \mathbf{v}_{\mathfrak{B}} - \mathbf{curl}\,\sigma.$$

So

$$\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{B}_{\mathfrak{B}} = \sigma - \frac{1}{c_P^2}(\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}} - \sigma) = (1 + \frac{1}{c_P^2})\sigma - \frac{1}{c_P^2}\mathbf{curl}^{-1}\mathbf{v}_{\mathfrak{B}} = \tau.$$

We simply choose $\mathbf{B}_{\mathfrak{B}^*} = \mathbf{v}_{\mathfrak{B}^*}$. For the harmonic part take $\kappa_v$ s.t. $\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p}$. Let us look at the system

$$\begin{pmatrix} 1 & \beta \\ \alpha & 1/c_\psi \end{pmatrix} \begin{pmatrix} \kappa_B \\ \lambda \end{pmatrix} = \begin{pmatrix} \kappa_v \\ \mu \end{pmatrix}$$

Now since $c_\psi > 0$ and $\alpha < 0$, $\beta > 0$ we get $1/c_\psi - \alpha\beta \neq 0$ and the system has a solution. Choose $\mathbf{B}_{\mathfrak{H}} = \kappa_B \mathbf{p}$. Then we see

$$\mathbf{v}_{\mathfrak{H}} = \kappa_v \mathbf{p} = \mathbf{p}(\kappa_B + \beta\lambda) = \mathbf{B}_{\mathfrak{H}} + \beta\lambda\mathbf{p}$$

and

$$\mu = \alpha\kappa_B + \frac{\lambda}{c_\psi} = \alpha\kappa_B \|\mathbf{p}\|^2 + \frac{\lambda}{c_\psi} = \alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi}.$$

By combining all that we arrive at $T(\sigma, \mathbf{B}, \lambda) = (\tau, \mathbf{v}, \mu)$. □

We assume from now on that we have always chosen $\mathbf{p}$ in a way s.t. $c_\psi$ – as defined in the previous lemma – is positive and $\mathbf{p}$ has norm one. This comes down to choosing $\mathbf{p}$ with the correct sign and normalizing it. Now we can use the T-coercivity (Prop. 1.5.2) to prove the inf-sup condition and thus well-posedness of our formulation.

**Theorem 1.5.8.** *Let $\mathbf{curl}\,\psi_0 + c_\psi \mathbf{p} = \mathbf{curl}\,\psi$ and assume we have chosen the sign of $\mathbf{p}$ s.t. $c_\psi > 0$. Then take $c_1 > 0$ s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$ (e.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$ would be a valid choice). Define $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ and $\alpha = -\frac{c_\psi}{4c_1^2 c_P^2}$. Then the bilinear form a defined at (1.5.5) satisfies the inf-sup condition i.e. (1.5.2) and (1.5.3) with $\gamma \geq \eta/\sqrt{C_T}$ with $C_T$ from (1.5.6) and*

$$\eta := \min\left\{ \frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2} \right\}.$$

*Proof.* We will use T-coercivity to prove it. Choose $(\sigma, \mathbf{B}, \lambda) \in X$ arbitrary and define $\rho := \mathbf{curl}^{-1} \mathbf{B}_{\mathfrak{B}}$. We take $T$ as in (1.5.7),

$$T(\sigma, \mathbf{B}, \lambda) = (\sigma - \frac{1}{c_P^2}\rho, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}, \alpha\langle \mathbf{p}, \mathbf{B}\rangle + \frac{\lambda}{c_\psi})$$

Then $T$ is surjective due to Lemma 1.5.7. Note

$$\langle \mathbf{B}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2 \langle \frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}, \mathbf{p}\rangle^2 = \|\mathbf{B}_{\mathfrak{H}}\|^2$$

where we used in the last equality that $\frac{\mathbf{B}_{\mathfrak{H}}}{\|\mathbf{B}_{\mathfrak{H}}\|}$ is either $+\mathbf{p}$ or $-\mathbf{p}$ because $\mathfrak{H}^1$ is assumed to be one-dimensional. We split up $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_0 + c_\psi \mathbf{p}$ to get

$$
\begin{aligned}
& a(\sigma, \mathbf{B}, \lambda; T(\sigma, \mathbf{B}, \lambda)) \\
&= \langle \sigma, \sigma - \frac{1}{c_P^2}\rho\rangle - \langle \mathbf{B}, \mathbf{curl}\,\sigma - \frac{1}{c_P^2}\,\mathbf{curl}\,\rho\rangle + \langle \mathbf{curl}\,\sigma, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\,\mathrm{div}\,\mathbf{p}\rangle \\
&\quad + \langle \mathrm{div}\,\mathbf{B}, \mathrm{div}\,\mathbf{curl}\,\sigma + \mathrm{div}\,\mathbf{B} + \beta\lambda\,\mathrm{div}\,\mathbf{p}\rangle \\
&\quad + \langle \lambda\mathbf{p}, \mathbf{curl}\,\sigma + \mathbf{B} + \beta\lambda\mathbf{p}\rangle - (\alpha\langle \mathbf{B}, \mathbf{p}\rangle + \frac{\lambda}{c_\psi})\langle \mathbf{B}, \mathbf{curl}\,\psi\rangle \\
&= \|\sigma\|^2 - \frac{1}{c_P^2}\langle \sigma, \rho\rangle + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\beta - \alpha c_\psi \|\mathbf{B}_{\mathfrak{H}}\|^2 \\
&\quad - \alpha\langle \mathbf{p}, \mathbf{B}_{\mathfrak{H}}\rangle\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle - \frac{\lambda}{c_\psi}\langle \mathbf{B}_{\mathfrak{B}}, \mathbf{curl}\,\psi_0\rangle
\end{aligned}
$$

Due to the Poincaré inequality

$$
\|\rho\| \le \|\rho\|_{H^1} \overset{\text{Poincaré}}{\le} c_P \|\mathbf{curl}\,\rho\| = c_P \|\mathbf{B}_{\mathfrak{B}}\|.
$$

Using $\epsilon$-Young combined with Cauchy-Schwarz inequality several times we obtain the lower bound.

$$
\begin{aligned}
& \|\sigma\|^2 - \left(\frac{1}{2}\|\sigma\|^2 + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2}{2c_P^2}\right) + \frac{1}{c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 \\
&\quad + \lambda^2\beta - \alpha c_\psi\|\mathbf{B}_{\mathfrak{H}}\|^2 - \left(\frac{\epsilon_1 \alpha^2 \|\mathbf{B}_{\mathfrak{H}}\|^2}{2} + \frac{\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2\epsilon_1}\right) - \left(\frac{\lambda^2}{2\epsilon_2 c_\psi^2} + \frac{\epsilon_2\|\mathbf{B}_{\mathfrak{B}}\|^2\|\mathbf{curl}\,\psi_0\|^2}{2}\right)
\end{aligned}
$$

Choose $\epsilon_1 = 4c_1^2 c_P^2$ to get

$$
\begin{aligned}
& \frac{1}{2}\|\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{1}{2\epsilon_2 c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(-\alpha c_\psi - \frac{4c_1^2 c_P^2 \alpha^2}{2}\right) - \|\mathbf{B}_{\mathfrak{B}}\|^2 \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2 c_P^2} - \|\mathbf{B}_{\mathfrak{B}}\|^2 \frac{\epsilon_2\|\mathbf{curl}\,\psi_0\|^2}{2}
\end{aligned}
$$

Now choose $\epsilon_2 = \frac{1}{4c_1^2 c_P^2}$, plug in the definition of $\alpha$ and use $\|\mathbf{curl}\,\psi_0\| \le c_1$ to get the next lower bound

$$
\begin{aligned}
& \frac{1}{2}\|\sigma\|^2 + \|\mathbf{B}_{\mathfrak{B}}\|^2\left(\frac{1}{2c_P^2} - \frac{1}{8c_P^2} - \frac{\|\mathbf{curl}\,\psi_0\|^2}{8c_1^2 c_P^2}\right) + \|\mathbf{curl}\,\sigma\|^2 + \|\mathrm{div}\,\mathbf{B}\|^2 + \lambda^2\left(\beta - \frac{4c_1^2 c_P^2}{2c_\psi^2}\right) \\
&\quad + \|\mathbf{B}_{\mathfrak{H}}\|^2\left(\frac{c_\psi^2}{4c_1^2 c_P^2} - \frac{c_1^2 c_P^2 c_\psi^2}{8c_1^4 c_P^4}\right)
\end{aligned}
$$

and finally by using the Poincaré inequality $\|\mathbf{B}_{\mathfrak{B}^*}\| \leq c_P \|\operatorname{div} \mathbf{B}\|$ and $\beta = \frac{3c_1^2 c_P^2}{c_\psi^2}$ we obtain the next bound

$$\frac{1}{2}\|\sigma\|^2 + \frac{1}{4c_P^2}\|\mathbf{B}_{\mathfrak{B}}\|^2 + \|\mathbf{curl}\,\sigma\|^2 + \frac{1}{2c_P^2}\|\mathbf{B}_{\mathfrak{B}^*}\|^2 + \frac{1}{2}\|\operatorname{div} \mathbf{B}\|^2 + \frac{c_1^2 c_P^2}{c_\psi^2}\lambda^2 + \frac{c_\psi^2}{8c_1^2 c_P^2}\|\mathbf{B}_{\mathfrak{H}}\|^2.$$
$$\geq \eta \|(\sigma, \mathbf{B}, \lambda)\|_X^2$$

where we chose

$$\eta := \min\left\{\frac{1}{2}, \frac{1}{4c_P^2}, \frac{c_1^2 c_P^2}{c_\psi^2}, \frac{c_\psi^2}{8c_1^2 c_P^2}\right\}$$

to obtain the $T$-coercivity. We can then choose $\gamma$ from (1.5.2) as $\eta/\|T\|_{\mathcal{L}(X,X)}$ and then use $C_T$ from (1.5.6) to get a lower bound

$$\gamma \geq \frac{\eta}{\sqrt{C_T}}.$$

$\square$

**Corollary 1.5.9** (Well-posedness)**.** *The variational formulation of the magnetostatic problem* (Problem 1.4.1) *is well-posed. For a solution* $(\sigma, \mathbf{B}, \lambda) \in X$ *we have the stability estimate*

$$\|\mathbf{B}\| = \|(\sigma, \mathbf{B}, \lambda)\|_X \leq \frac{\|J\| + |C_1|}{\gamma}.$$

*Proof.* Recall that when $(\sigma, \mathbf{B}, \lambda)$ is a solution then $\sigma = 0$ and $\lambda = 0$ and $\operatorname{div} \mathbf{B} = 0$ which implies the first equality. The statement follows immediately from the previous theorem and Thm. 1.5.1 and the fact that

$$|\ell(\tau, \mathbf{v}, \mu)| = |-\langle J, \tau\rangle - C_1\mu| \leq (\|J\| + |C_1|)\|(\tau, \mathbf{v}, \mu)\|_X$$

and thus $\|\ell\|_{X'} \leq \|J\| + |C_1|$. $\square$

**Remark 1.5.10.** Note that $1/c_\psi$ terms arise in the stability constant $1/\gamma$. This is not surprising since the term $\langle \mathbf{curl}\,\psi, \mathbf{B}\rangle$ will not enforce the harmonic part if the harmonic part of $\mathbf{curl}\,\psi$ would be zero because $\mathbf{B}_{\mathfrak{H}}$ will disappear from the formulation. So we expect stability issues if the harmonic part is too small. This also forces us to choose $\psi$ with this in mind to obtain a stable system.

# 2 Discretized magnetostatic problem

In order to approximate solutions of the 2D magnetostatic problem we want to use finite elements. A very typical question that arises for any discretization of a model is what notions of the continuous model are represented in the discretized one. In our case, a fundamental structure of our problem is the Hilbert complex we introduced in Section 1.1. We would like to represent it in our discretization, leading to the discrete Hilbert complex. This section follows Sec. 5.2 in Arnold's book [1]. We start with reviewing the general theory behind the discretization of general Hilbert complexes before applying this theory to our problem. Once we discretized our problem, we will utilize the inf-sup condition proven in Section 1.1 to prove well-posedness of the discrete formulation and derive a quasi-optimal error estimate.

## 2.1 Discrete Hilbert complex

{sec:discrete_hi

Let us at first stick to the general situation. We assume that we have a Hilbert complex $(W^k, d^k)$ with its corresponding domain complex $(V^k, d^k)$ and dual complex $(V_k^*, d_k^*)$ for $k \in \mathbb{Z}$. For this chapter we will only need a short subsequence

$$V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1}$$

for some fixed $k$ and $j \in \{k-1, k, k+1\}$ will always be the index to refer to all three spaces. We will sometimes leave out the indices if the meaning is clear from the context.

Let us assume that we have finite dimensional subspaces $V_h^j \subseteq V^j$. As usual in numerical analysis, $h > 0$ stands loosely for the fineness of our discretization, e.g. the grid size or the maximal diameter of the elements in the mesh. Then we define completely analogous to the continuous case,

$$\begin{aligned} \mathfrak{Z}_h^j &:= \{v_h \in V_h^j \mid d^j v_h = 0\} = \ker d^j \cap V_h^j \\ \mathfrak{B}_h^j &:= \{d^j v_h \mid v_h \in V_h^{j-1}\}. \end{aligned}$$

We can now also define the discrete harmonic forms. Now the situation is slightly different however. We will not use the adjoint $d_j^*$ to define it. Instead,

$$\mathfrak{H}_h^j := \{v \in \mathfrak{Z}_h^j \mid v \perp \mathfrak{B}_h^j\} = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp}.$$

Notice that we have $\mathfrak{Z}_h^j \subseteq \mathfrak{Z}^j$ and $\mathfrak{B}_h^j \subseteq \mathfrak{B}^j$, but due to $\mathfrak{B}_h^{j,\perp} \supseteq \mathfrak{B}^{j,\perp}$ we have in general

$$\mathfrak{H}_h^j = \mathfrak{Z}_h^j \cap \mathfrak{B}_h^{j,\perp} \not\subseteq \mathfrak{Z}^j \cap \mathfrak{B}^{j,\perp} = \mathfrak{H}^j.$$

We will later investigate the difference between the space of the discrete and continuous harmonic forms more closely.

There are three crucial assumptions that are necessary for stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want that the discrete spaces $V_h^j$ approximate the continuous ones $V^j$.

{ass:convergence

**Assumption 2.1.1.** For the discrete spaces, we require $V_h^j \subseteq V^j$ and

$$\lim_{h\to 0} \inf_{v_h \in V_h^j} \|w - v_h\|_V = 0, \quad \forall w \in V^j.$$

This is usually satisfied for a reasonable choice of finite element space. The next property is more restrictive and is a compatibility condition between the spaces.

**Assumption 2.1.2.** For $j = k-1, k$,

$$dV_h^j \subseteq V_h^{j+1}.$$

This means that we cannot simply use arbitrary discrete subspaces independent from one another. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. **??** directly to it. Let us do that.

Denote the restriction of $d^j$ to $V_h^j$ as $d_h^j$. Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \to V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to $d_h^j$ the adjoint $d_{j,h}^*$ is not the restriction of the adjoint $d_j^*$. In general, $V_h^j \not\subseteq V_j^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h^j$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1} \tag{2.1.1}$$ {eq:discrete_hil

and its dual complex

$$V_h^{k-1} \xleftarrow{d_{k,h}^*} V_h^k \xleftarrow{d_{k+1,h}^*} V_h^{k+1}$$

From the general Hilbert complex theory (Thm. **??**) we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \overset{\perp}{\oplus} \mathfrak{H}_h^j \overset{\perp}{\oplus} \mathfrak{B}_{jh}^*.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. We will investigate the question how well the discrete harmonic forms approximate the continuous ones more thoroughly later.

**Assumption 2.1.3.** There exist *bounded cochain projections* $\Pi_h^j : V^j \to V_h^j$. This is a projection that is a cochain map in the sense of cochain complexes **??** i.e. the following diagram commutes:

$$\begin{tikzcd}
V^{k-1} \arrow[r, "d^{k-1}"] \arrow[d, "\Pi_h^{k-1}"] & V^k \arrow[r, "d^k"] \arrow[d, "\Pi_h^k"] & V^{k+1} \arrow[d, "\Pi_h^{k+1}"] \\
V_h^{k-1} \arrow[r, "d^{k-1}"] & V_h^k \arrow[r, "d^k"] & V_h^{k+1}
\end{tikzcd}$$

$\Pi_h^j$ are either bounded in the $V$- or in the $W$-norm.

The cochain projection will play an important role in the stablity of the discrete system. Thanks to the commuting property, if $\Pi_h^j$ are $W$-bounded then they are $V$-bounded. In this section, we will make the weaker assumption of $V$-boundedness.

The fact that $\Pi_h$ is a $V$-bounded projection immediately allows a quasi optimal estimate. For any $v \in V^j$, we can take $w_h \in V_h^j$ arbitrary s.t.

$$\begin{aligned}
\|v - \Pi_h^j v\|_V &= \|v - w_h + \Pi_h^j(w_h - v)\|_V && (2.1.2)\\
&\leq \|I - \Pi_h^j\|_{\mathcal{L}(V,V)} \|v - w_h\|_V. && (2.1.3)
\end{aligned}$$

{eq:bound_projec

From now on we will denote the operator norm $\|\cdot\|_{\mathcal{L}(V,V)}$ by slight abuse of notation as $\|\cdot\|_V$. Since $w_h$ was arbitrary we can take the infimum over $w_h \in V_h^k$ and obtain a quasi optimal estimate.

Let us now answer the question about the difference between discrete and continous harmonic forms. In order to do that, we need some way to measure the "difference" between two subspaces.

For a general metric space $(X, d)$, we will use the standard notation $d(x, M) := \inf_{m \in M} d(x, m)$ for $x \in X$ being in a metric space and $M \subseteq X$. If we are dealing with a normed space then we take the metric induced by the norm.

**Definition 2.1.4** (Gap between subspaces)**.** For a Banach space $W$ with subspaces $Z_1$ and $Z_2$ let $S_1$ and $S_2$ be the unit spheres in $Z_1$ and $Z_2$ respectively i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$ and analogous for $S_2$. Then we define the gap between these subspaces as

$$\operatorname{gap}(Z_1, Z_2) = \max\{\sup_{z_1 \in S_1} d(z_1, Z_2), \sup_{z_2 \in S_2} d(z_2, Z_1)\}$$

This definition is from [7, Ch.4 §2.1] and defines a metric on the set of closed subspaces of $W$. If $W$ is a Hilbert space – as it is throughout this section – and $Z_1$ and $Z_2$ are closed then the $\operatorname{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|_{\mathcal{L}(W,W)}$ i.e. the difference in operator norm of the orthogonal projections onto $Z_1$ and $Z_2$. This gives us a measure of distance between spaces which we can now apply to the question about the difference between discrete and continuous harmonic forms.

**Proposition 2.1.5** (Gap between harmonic forms)**.** *Assume that the discrete complex (2.1.1) admits a $V$-bounded cochain projection $\Pi_h$. Then*

$$\|(I - P_{\mathfrak{H}_h^k})q\|_V \leq \|(I - \Pi_h^k)q\|_V, \forall q \in \mathfrak{H}^k \tag{2.1.4}$$ {eq:difference_i

$$\|(I - P_{\mathfrak{H}^k})q_h\|_V \leq \|(I - \Pi_h^k)P_{\mathfrak{H}^k}q_h\|_V, \quad \forall q_h \in \mathfrak{H}_h^k \tag{2.1.5}$$

*and then*

$$\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q \in \mathfrak{H}, \|q\|=1} \|(I - \Pi_h^k)q\|_V$$

*Proof.* See [1, Thm. 5.2]. □

Do not forget the continuous poincare inequality The following proposition clarifies how close a discrete harmonic form can be chosen. But in order to prove it, we will need a small lemma.

{lem:distance_un

**Lemma 2.1.6.** *Let $W$ be a Banach space, $Z \subseteq W$ a closed subspace. Denote $S_Z := \{z \in Z \mid \|z\|_W = 1\}$. Then for any $w \in W$ with $\|w\|_W = 1$ we get*

$$d(w, S_Z) \leq 2\, d(w, Z)$$

*Proof.* See [7, Ch.4 §2, (2.13)]. □

{prop:choice_of_

**Proposition 2.1.7.** *Take $p \in \mathfrak{H}^k$ with $\|p\| = 1$. Then we can choose $p_h \in \mathfrak{H}_h^k$ with $\|p_h\| = 1$ s.t.*

$$\|p - p_h\| \leq 2\,\|I - \Pi_h^k\|_V \inf_{v_h \in V_h} \|p - v_h\|_V.$$

*Proof.* Notice that since $\mathfrak{H}^k \subseteq \mathfrak{Z}^k$ we always have $\|q\|_V = \|q\|$ for all $q \in \mathfrak{H}^k$. The same is true for $\mathfrak{H}_h^k$. Denote $S_h := \{q_h \in \mathfrak{H}_h^k \mid \|q_h\| = 1\}$. Since $S_h$ is closed we can find $p_h \in S_h$ s.t.

$$\|p_h - p\| = \inf_{q_h \in S_h} \|q_h - p\|$$

The right hand side can be estimated using (2.1.4) and then the quasi optimal bound for the projection derived at (2.1.3).

$$\inf_{q_h \in S_h} \|q_h - p\|_V = \inf_{q_h \in S_h} \|q_h - p\| \overset{\text{Lem. 2.1.6}}{\leq} 2 \inf_{q_h \in \mathfrak{H}_h} \|q_h - p\| = 2\, \|P_{\mathfrak{H}_h} p - p\| \overset{(2.1.4)}{\leq} 2\, \|\Pi_h^k p - p\|_V$$
$$\leq 2\, \|I - \Pi_h^k\|_{\mathcal{L}(V,V)} \inf_{v_h \in V_h} \|p - v_h\|_V$$

which gives us the estimate. □

If we have the standard situation that we have for $v \in V^j$

$$\|v - \Pi_h^j v\|_V \leq C\|v\|_V h^s \tag{2.1.6}$$ {eq:standard_est

for some generic constant $C > 0$ independent of $v$ and the mesh size $h$ and some $s > 0$ then we can improve the estimate to

$$\|p_h - p\| \leq Ch^s$$

by applying it in the last estimate of the proof.

Also if we assume $\|\Pi_h\| \leq c_\Pi$ for $h$ small enough and $c_\Pi > 0$ independent of $h$ then we Assumption 2.1.1 implies

$$p_h \xrightarrow{V} p \text{ as } h \to 0.$$

{thm:dim_of_disc

**Theorem 2.1.8** (Dimension of $\mathfrak{H}_h^k$)**.** *Assume that we have a finite-dimensional subcomplex with a $V$-bounded cochain projection. Assume further, that*

$$\|q - \Pi_h^k q\| < \|q\|, \quad \forall q \in \mathfrak{H}^k \setminus \{0\}. \tag{2.1.7}$$ {eq:assumption_s

*Then $\mathfrak{H}^k$ and $\mathfrak{H}_h^k$ are isomorphic. In particular,* $\dim \mathfrak{H}^k = \dim \mathfrak{H}_h^k$.

*Proof.* See [1, Thm 5.1] and the explanation after the proof. □

If we assume a standard error estimate for the projection as (2.1.6) then (2.1.7) is fulfilled if $Ch^s < 1$ which will be true for $h$ small enough.

{prop:gap_image_

**Proposition 2.1.9.** *Assuming again a finite-dimensional subcomplex with a $V$-bounded cochain projection. Then we can bound the gap between the image spaces*

$$\operatorname{gap}(\mathfrak{B}_h^j, \mathfrak{B}^j) \leq \|I - \Pi_h^j\|_V \sup_{\substack{z \in \mathfrak{B}^j \\ \|z\|=1}} \inf_{v_h \in V_h^j} \|z - v_h\|_V.$$

*Proof.* At first note, that for any $z \in \mathfrak{B}$, $\|z\|_V = \|z\|$. Since $\mathfrak{B}_h \subseteq \mathfrak{B}$, $d(z_h, \mathfrak{B}) = 0$ for any $z_h \in \mathfrak{B}_h$.

Take $z \in \mathfrak{B}^j$ arbitrary i.e. there exists $w \in V^{j-1}$ s.t. $dw = z$. Since $\Pi_h$ is a cochain projection, we thus have

$$\Pi_h z = \Pi_h dw = d\Pi_h w \in \mathfrak{B}_h^j$$

so $\Pi_h$ maps $\mathfrak{B}^j$ into $\mathfrak{B}_h^j$. Putting all that together,

$$d(z, \mathfrak{B}_h^j) = \inf_{z_h \in \mathfrak{B}_h} \|z - z_h\| \leq \|z - \Pi_h z\| = \|z - \Pi_h z\|_V \overset{(2.1.3)}{\leq} \|I - \Pi_h\|_V \inf_{v_h \in V_h} \|z - v_h\|_V$$

and then the claim follows from the definition of the gap. □

**Proposition 2.1.10** (Discrete Poincare inequality)**.** *Assume that we have a $V$-bounded cochain projection $\Pi_h$ for the discrete Hilbert complex. Then*

$$\|v_h\|_V \leq c_{P,h} \|dv_h\|, \quad \forall v_h \in \mathfrak{Z}_h^{k\perp} \cap V_h$$

*with $c_{P,h} := c_P \|\Pi_h\|_V$ and $c_P$ being the Poincare constant from* **??**.

*Proof.* This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}^{k,\perp} \cap V^k) = \mathfrak{B}^{k+1} \supseteq \mathfrak{B}_h^{k+1}$ we find $z \in \mathfrak{Z}^{k\perp} \cap V_h$ s.t. $dz = dv_h$. We can apply now the continuous Poincare inequality **??** to get $\|z\|_V \leq c_P \|dz\|_V = c_P \|dv_h\|_V$. Now we can use the fact that $\Pi_h$ is a cochain map and the fact that $\Pi_h$ is a projection:

$$d\Pi_h^k z = \Pi_h^{k+1} dz = \Pi_h^{k+1} dv_h = dv_h$$

For the last equality we used also the fact that we have a discrete complex i.e. $d^k V_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \Pi_h z) = 0$ i.e. $(v_h - \Pi_h z) \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v_h, v_h - \Pi_h z \rangle = \langle v_h, v_h - \Pi_h z \rangle + \langle dv_h, d(v_h - \Pi_h z) \rangle = \langle v_h, v_h - \Pi_h z \rangle_V$$

so $v_h - \Pi_h z$ is $V$ orthogonal to $v_h$. So

$$\begin{aligned}\|v_h\|_V^2 &= \langle v_h, \Pi_h^k z \rangle_V + \langle v_h, v_h - \Pi_h^k z \rangle_V = \langle v_h, \Pi_h^k z \rangle_V \leq \|v_h\|_V \|\Pi_h\|_V \|z\| \\ &\overset{\text{Poincare ineq.}}{\leq} \|v_h\|_V c_P \|\Pi_h\|_V \|dv_h\|_V\end{aligned}$$

□

Notice that if we assume that $\lim_{h \to 0} \|\Pi_h\|_V = 1$ (which is true if e.g. the standard estimate (2.1.6) is fulfilled) then $c_{P,h} \to c_P$ for $h \to 0$.

In conclusion, we obtain a discrete version of the Hilbert complex where the harmonic forms are accurately represented if $h$ is small enough.

## 2.2 Discretized magnetostatic problem

Let us apply the theory of discrete Hilbert complexes to the 2D Hilbert complex (1.1.2). We assume that we have finite dimensional subspaces $V_h^0 \subseteq H_0^1$, $V_h^1 \subseteq H_0(\mathrm{div})$ and $V_h^2 \subseteq L^2$ that approximate the full spaces in the sense of Assumption 2.1.1 and

$$V_h^0 \xrightarrow{\mathbf{curl}} V_h^1 \xrightarrow{\mathrm{div}} V_h^2$$

and the dual complex

$$V_h^0 \xleftarrow{\widetilde{\mathrm{curl}}_h} V_h^1 \xleftarrow{-\widetilde{\mathrm{grad}}_h} V_h^2$$

where $\widetilde{\mathrm{curl}}_h$ is the adjoint of $\mathbf{curl}_h$ and can thus be seen as a weak approximation of curl and the same for $\widetilde{\mathrm{grad}}_h$. Analogous to the continuous case we assume that $\dim \mathfrak{H}_h^1 = 1$ which is not unreasonable thanks to Thm. 2.1.8 for $h > 0$ small enough.

For our domain, we assume from now on that $\Omega$ is suitable for discretization in the sense that the functions in the discrete spaces and the continuous ones are both defined on it. What that means exactly depends on the chosen discretization and we will explain it later when we go into more detail about the actual implementation (see Assumption 3.2.2).

The discretized version of the strong formulation of the magnetostatic problem (Problem 1.2.1) then states: Find $\mathbf{B}_h \in V_h^1$ s.t.

$$\widetilde{\mathrm{curl}}_h \mathbf{B}_h = J \text{ and } \operatorname{div} \mathbf{B}_h = 0$$

plus the additional curve integral constraint. If $J \notin V_h^0$ then this clearly does not have a solution. Note that the divergence is enforced strongly while the curl is only enforced weakly.

As explained in Sec. 1.4, we will substitute the curve integral constraint from Problem 1.2.1 with (1.4.1). This gives us the following discrete formulation. Choose $\mathbf{p}_h \in \mathfrak{H}_h^1$ s.t. $\|\mathbf{p}_h\| = 1$.

{prob:magnetosta

**Problem 2.2.1.** Find $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned}
\langle \sigma_h, \tau_h \rangle - \langle \mathbf{B}_h, \mathbf{curl}\, \tau_h \rangle &= -\langle J, \tau_h \rangle \quad \forall \tau_h \in V_h^0, \\
\langle \mathbf{curl}\, \sigma_h, \mathbf{v}_h \rangle + \langle \operatorname{div} \mathbf{B}_h, \operatorname{div} \mathbf{v}_h \rangle + \langle \lambda \mathbf{p}_h, \mathbf{v}_h \rangle &= 0 \quad \forall \mathbf{v}_h \in V_h^1, \\
\mu \langle \mathbf{curl}\, \psi, \mathbf{B}_h \rangle &= \mu C_1 \quad \forall \mu \in \mathbb{R}.
\end{aligned}$$

Here we assume for simplicity that $\mathbf{curl}\,\psi \in V_h^1$. Since we can choose $\psi$ and we assume to be on a discretizable domain this is not unreasonable. In practice, $\mathbf{p}_h$ is computed numerically before assembling the system.

We define $X_h := V_h^0 \times V_h^1 \times \mathbb{R}$. Note that this trial and test space is indeed conforming i.e. $X_h \subseteq X$, but we choose the discrete bilinear form $a_h : X_h \times X_h \to \mathbb{R}$

$$\begin{aligned}&a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu)\\&= \langle \sigma_h, \tau_h\rangle - \langle \mathbf{B}_h, \mathbf{curl}\,\tau_h\rangle + \langle \mathbf{curl}\,\sigma_h, \mathbf{v}_h\rangle + \langle \operatorname{div}\mathbf{B}_h, \operatorname{div}\mathbf{v}_h\rangle + \langle \lambda\mathbf{p}_h, \mathbf{v}_h\rangle - \mu\langle \mathbf{curl}\,\psi, \mathbf{B}_h\rangle.\end{aligned}$$

with $\mathbf{p}_h \in \mathfrak{H}_h^1$ so the resulting bilinear forms are different since we have $\mathbf{p}_h$ instead of $\mathbf{p}$. So we can write the discrete problem in standard form: Find $\sigma_h, \mathbf{B}_h, \lambda \in X_h$ s.t.

$$a_h(\sigma_h, \mathbf{B}_h, \lambda; \tau_h, \mathbf{v}_h, \mu) = \ell(\tau_h, \mathbf{v}_h, \mu) \quad \forall (\tau_h, \mathbf{v}_h, \mu) \in X_h.$$

For simplicity, we assume for the theoretical considerations that we can compute all inner products exactly and that $C_1$ is given exactly as well. That also means that the right hand side $\ell$ is the same for the continuous and discrete problem.

Because we have transfered the continuous structures to the discrete case we can apply the same arguments as in Sec. 1.5.

**Theorem 2.2.2** (Well-posedness of the discrete problem)**.** *For the following assumptions we always tacitly require that $h > 0$ is small enough. We assume that the Hilbert complex admits uniformly $V$-bounded cochain projections $\Pi_h$ i.e. there exists $c_\Pi > 0$ independent of $h$ s.t. $\|\Pi_h\|_{\mathcal{L}(V,V)} \le c_\Pi$. We assume that $\mathbf{curl}\,\psi \in V_h^1$. Then we find $\psi_{0,h} \in V_h^0$ and $c_{\psi,h} > 0$ s.t. $\mathbf{curl}\,\psi = \mathbf{curl}\,\psi_{0,h} + c_{\psi,h}\mathbf{p}_h$. We also assume that (2.1.7) holds and thus $\dim \mathfrak{H}_h^1 = \dim \mathfrak{H}^1 = 1$ and we choose $\mathbf{p}_h$ according to Prop. 2.1.7. Then the discrete variational problem (Problem 2.2.1 ) is well-posed i.e. there exists a unique solution and for a solution $(\sigma_h, \mathbf{B}_h, \lambda) \in X_h$ we have the stability estimate*

$$\|\mathbf{B}_h\|_{H(\mathrm{div})} \le \frac{\|J\| + |C_1|}{\gamma_h}.$$

*where $\gamma_h$ has the same expression as $\gamma$ except $c_{P,h}$ instead of $c_P$ and $c_{\psi,h}$ instead of $c_\psi$. Furthermore if $c_{P,h} \to c_P$ then $\gamma_h \to \gamma$ for $h \to 0$.*

*Proof.* By following exactly the same arguments as in Sec. 1.5, we can prove the well-posedness through the BNB-theorem. However, we have to argue why $c_{\psi,h} > 0$ if $c_\psi > 0$. Notice since $\|\mathbf{p}_h\| = \|\mathbf{p}\| = 1$ and $\dim \mathfrak{H}^1 = \dim \mathfrak{H}_h^1 = 1$ we have

$$c_\psi = \langle \mathbf{p}, \mathbf{curl}\,\psi\rangle \text{ and } c_{\psi,h} = \langle \mathbf{p}_h, \mathbf{curl}\,\psi\rangle.$$

So

$$|c_{\psi,h} - c_\psi| = |\langle \mathbf{p} - \mathbf{p}_h, \mathbf{curl}\,\psi\rangle| \leq \|\mathbf{curl}\,\psi\|\|\mathbf{p} - \mathbf{p}_h\|.$$

and so because we chose $\mathbf{p}_h$ as described in Prop. 2.1.7 and assume $\Pi_h$ being uniformly bounded we have $\|\mathbf{p} - \mathbf{p}_h\| \to 0$ as $h \to 0$ and thus we obtain $c_{\psi,h} \to c_\psi$ for $h \to 0$ and hence we can assume $c_{\psi,h} > 0$ for $h$ small enough.

The next question is why we can choose $c_1$ for the discrete case just as in the continuous one. Remember that $c_1 > 0$ was chosen s.t. $\|\mathbf{curl}\,\psi_0\| \leq c_1$. Choose now w.l.o.g. $c_1 = \|\mathbf{curl}\,\psi_0\| + 1$. Then it would be clear, that we can choose the same $c_1$ if $\|\mathbf{curl}\,\psi_{0,h}\| \to \|\mathbf{curl}\,\psi_0\|$. This is indeed true:

$$\begin{aligned}&\big|\|\mathbf{curl}\,\psi_{0,h}\| - \|\mathbf{curl}\,\psi_0\|\big| \leq \|\mathbf{curl}\,\psi_{0,h} - \mathbf{curl}\,\psi_0\| = \|P_{\mathfrak{B}^1_h}\,\mathbf{curl}\,\psi - P_{\mathfrak{B}^1}\,\mathbf{curl}\,\psi\| \\ &\leq \|P_{\mathfrak{B}^1_h} - P_{\mathfrak{B}^1}\|\|\mathbf{curl}\,\psi\| = \operatorname{gap}(\mathfrak{B}^1_h, \mathfrak{B}^1)\|\mathbf{curl}\,\psi\| \to 0\end{aligned}$$

where we use Prop. 2.1.9 combined with the uniform boundedness of $\Pi_h$ to obtain convergence.

If $c_{P,h} \to c_P$ then $\gamma_h \to \gamma$ is clear because $\gamma$ depends continuously on $c_\psi$ and $c_P$. □

Finally, we want to use the inf-sup condition to derive an a-priori error estimate. In order to do so, we have to consider the fact that the bilinear forms are different in the discrete and continuous case. So we will use the following lemma.

{lem:a_priori_es

**Lemma 2.2.3.** *Let $x \in X$ be a solution of a general variational problem of the form (1.5.1) and $x_h \in X_h$ be a solution of the discretized version i.e for $X_h \subseteq X$ and $Y_h \subseteq Y$ finite-dimensional subspaces*

$$a_h(x_h, y_h) = \ell(y_h) \quad \forall y_h \in Y_h.$$

*Assume that an inf-sup condition holds for the discrete problem with constant $\gamma_h$. Define $\delta_h(x) \in Y'$ as*

$$\langle \delta(x), y\rangle_{Y' \times Y} := a(x,y) - a_h(x,y).$$

*Then*

$$\|x - x_h\|_X \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in X_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

*Proof.* Take $z_h \in X_h$ arbitrary. Then with the triangular inequality

$$\|x - x_h\|_X \leq \|x - z_h\|_X + \|x_h - z_h\|_X. \tag{2.2.1}$$ {eq:a_priori:tri

We now have to bound the last term on the right hand side. Assume w.l.o.g. that $x_h - z_h$ is not zero. Then from the inf-sup condition we can find $y_h \in Y_h \setminus \{0\}$ s.t.

$$\begin{aligned}
&\gamma_h \|x_h - z_h\|_X \|y_h\|_Y \leq a_h(x_h - z_h, y_h) \\
&= a_h(x - z_h, y_h) + a_h(x_h, y_h) - a(x, y_h) + a(x, y_h) - a_h(x, y_h) \\
&= a_h(x - z_h, y_h) + \langle \delta_h(x), y_h \rangle_{Y' \times Y} \\
&\leq \|a_h\| \|x - z_h\|_X \|y_h\|_Y + \|\delta_h(x)\|_{Y'} \|y_h\|_Y
\end{aligned}$$

In the third step, we used the fact that $x$ and $x_h$ are solutions and the discrete problem has the same right hand side as the continous one. So we can bound $\|x_h - z_h\|_X$ by

$$\|x_h - z_h\|_X \leq \frac{\|a_h\|}{\gamma_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}$$

and plugging this in (2.2.1) and taking the infimum over $z_h \in X_h$ we get

$$\|x - x_h\| \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{z_h \in V_h} \|x - z_h\|_X + \frac{\|\delta_h(x)\|_{Y'}}{\gamma_h}.$$

□

We now have to apply this lemma to the magnetostatic formulation.

**Theorem 2.2.4** (Quasi optimal a-priori estimate)**.** *Let $(\sigma, \mathbf{B}, \lambda) \in X$ be the exact solution of Problem 1.4.1 and $(\sigma_h, \mathbf{B}_h, \lambda_h) \in X_h$ the solution of the discrete Problem 2.2.1. Then*

$$\|\mathbf{B} - \mathbf{B}_h\| \leq \left(1 + \frac{\|a_h\|}{\gamma_h}\right) \inf_{\mathbf{z}_h \in V_h^1} \|\mathbf{B} - \mathbf{z}_h\|_{H(\mathrm{div})}$$

*Proof.* At first recall that if $(\sigma, \mathbf{B}, \lambda)$ is solution then $\sigma = 0$, $\lambda = 0$ and $\operatorname{div} \mathbf{B} = 0$. So $\|(\sigma, \mathbf{B}, \lambda)\|_X = \|\mathbf{B}\|$ and analogous for $(\sigma_h, \mathbf{B}_h, \lambda_h)$. Also recognize then for any $y = (\tau, \mathbf{v}, \mu) \in X$

$$\langle \delta_h(x), y \rangle = \lambda \langle \mathbf{p}, \mathbf{v} \rangle - \lambda \langle \mathbf{p}_h, \mathbf{v} \rangle = 0.$$

Thus the estimate follows immediately from Lemma 2.2.3. □

# 3 Implementation on a single patch domain

We start with very short introduction on Splines and their tensor product space which will be our choice of basis on the reference domain. Then we choose our degrees of freedom and basis dual to them. We will work at first on a reference domain and then transfer it to the physical domain using pushforwards. In the end, we will explain how the homogeneous boundary conditions are enforced and state the final system to be solved.

## 3.1 Splines

For the finite element spaces, we will use the pushforwards of tensor product splines defined on a rectangular reference domain $\hat{\Omega}$. This section is a recollection of [4, Sec. 4.2] since we use the same method as presented in this paper. We start with a very brief introduction of B-Splines and the spaces of tensor products of these.

We choose a knot sequence $\boldsymbol{\xi} = (\xi_i)_{i=0}^{n+p}$ with $\xi_0 \leq \xi_1 \leq ...\xi_{n+p}$. B-Splines $B_i^p$ of degree $p \geq 0$ for $i = 0, ..., n-1$ are defined recursively as

$$B_i^p(x) := \frac{x - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(x) + \frac{\xi_{i+p+1} - x}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(x)$$

and

$$B_i^0(x) := \begin{cases} 1, & \text{if } x \in [\xi_i, \xi_{i+1}), \\ 0, & \text{else} \end{cases}$$

We choose two types of sequences, periodic and non-periodic ones. On an interval $[a, b]$, let $a = x_0 < x_1 < ... < x_N = b$ be our grid. We will stick to the equidistant case i.e. $h$ will be our grid size $x_{i+1} - x_i$. For the non-periodic case, we choose an *open* knot sequence by $\xi_0 = ... = \xi_p = x_0$, $\xi_{p+k} = x_k$ for $k = 0, 1, ..., N$ and $\xi_n = \xi_{n+1} = ... = \xi_{n+p} = x_N$ i.e.

$$\boldsymbol{\xi} = (\underbrace{x_0, x_0, ..., x_0}_{p+1 \text{ times}}, x_1, x_2, ..., x_{N-1}, \underbrace{x_N, x_N, ..., x_N}_{p+1 \text{ times}})$$

and for the periodic case $\xi_0 = x_0 - ph$, $\xi_1 = x_0 - (p-1)h, ..., \xi_p = x_0$, $\xi_{p+k} = x_k$ and $\xi_{n+k} = x_N + kh$ for $k = 0, ..., p$,

$$\boldsymbol{\xi} = (x_0 - ph, x_0 - (p-1)h, ..., x_0 - h, x_0, x_1, x_2, ..., x_{N-1}, x_N, x_N + h, x_N + 2h, ..., x_N + ph).$$

In the periodic case, the splines at the beginning and end of the interval are extended periodically. In both cases, we have $n + p + 1 = N + 2p + 1$ knots in our knot sequence $\boldsymbol{\xi}$ and hence obtain $n = N + p$ splines. We then define the spline space $\mathbb{S}^p(\boldsymbol{\xi}) := \operatorname{span}\{B_i^p\}_{i=0}^{n-1}$ which has dimension $N + p$.

Note that all the knot multiplicities in the interior are one and thus our spline space has maximal regularity which implies that it is equal to the piecewise polynomial space

$$\mathbb{S}^p(\boldsymbol{\xi}) = \{v \in C^{p-1} \mid v|_{[x_j, x_{j+1})} \in \mathbb{P}_p([x_j, x_{j+1})) \text{ for } j \in [N-1]\}.$$

where $\mathbb{P}_p([x_j, x_{j+1}))$ are the polynomials of degree $p$ on $[x_j, x_{j+1})$ and we use the standard notation $[K] = \{0, 1, ..., K\}$ for $k \in \mathbb{N}$.

This construction can now be generalized to 2D by using a tensor product approach. Denote $a_1 = x_0 < x_1 < x_2 < ... < x_N = b_1$ the grid in $x$-direction and $a_2 = y_0 < y_1 < y_2 < ... < y_N = b_2$ in $y$-direction and assume that we have the same number of grid points in both dimensions for simplicity. Using the same definition of knot sequence as above gives us the knot sequence $\boldsymbol{\xi}$ in $x$-direction and $\boldsymbol{\eta}$ in $y$-direction. The resulting splines are are denoted as $B^{q_1}_{i,\xi}$, $i = 0, ..., n_1 = N + q_1$, and $B^{q_2}_{j,\eta}$, $j = 0, ..., N + q_2$ and the spaces $\mathbb{S}^{q_1}(\boldsymbol{\xi})$ and $\mathbb{S}^{q_2}(\boldsymbol{\eta})$ respectively.

We use the notation with $\mathbf{q} \in \{p-1, p\}^2$, $n_1 = N + q_1$, $n_2 = N + q_2$ and we define for $\mathbf{i} \in [n_1 - 1] \times [n_2 - 1]$ the *tensor product spline*

$$B^{\mathbf{q}}_{\mathbf{i}}(x, y) = B^{q_1}_{i_1,\xi}(x) B^{q_2}_{i_2,\eta}(y)$$

and $\mathbb{S}^{\mathbf{q}}(\xi, \eta)$ the span of these tensor product splines. We will from now on leave out the reference to the knot sequences and assume them to be fixed. The spline spaces used in the tensor product can also be periodic or be periodic in one direction and non-periodic in the other which will the case in our implementation later.

Then we obtain the following discrete Hilbert complex on our reference domain $\hat{\Omega}$

$$\mathbb{S}^{p,p} \xrightarrow{\mathbf{curl}} \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \xrightarrow{\text{div}} \mathbb{S}^{p-1,p-1}$$

and we denote

$$\hat{V}^0_h = \mathbb{S}^{p,p}, \ \hat{V}^1_h = \begin{pmatrix} \mathbb{S}^{p,p-1} \\ \mathbb{S}^{p-1,p} \end{pmatrix} \text{ and } \hat{V}^2_h = \mathbb{S}^{p-1,p-1}. \tag{3.1.1}$$

{eq:discrete_spl

It is well-known that if we have a function $\mathbf{v}$ that is piecewise smooth then $\mathbf{v} \in H(\text{div})$ i.i.f. the normal trace across element interfaces agrees. Analogously a piecewise smooth $\tau \in H^1$ i.i.f. the values agree on the interfaces. We will always assume $p \geq 2$ and thus know that all our tensor splines are at least continuous globally and so $\hat{V}^0_h \subseteq H^1(\hat{\Omega})$ and $\hat{V}^1_h \subseteq H(\text{div}, \hat{\Omega})$ as desired.

## 3.2 Basis and degrees of freedom

Let us now investigate the degrees of freedom and the corresponding basis functions. At first, we will define these only on the reference domain $\hat{\Omega}$. Then we will use pullbacks and pushforwards to transfer them from the reference domain to the physical domain $\Omega$. We will also use this opportunity to specify more precisely, what domains we consider such that this approach works.

We now fix $p$ and get $n = N + p$. Here it is crucial to remember that we assumed to have the same number of grid points in both dimensions. We will use geometric degrees of freedom i.e. each degree of freedom can be associated with some geometrical element of our grid. We define Greville points by

$$\zeta_i^x := \frac{\xi_{i+1} + ... + \xi_{i+p}}{p}$$

i.e. the knot averages for $i = 0, ..., n-1$ and analogous $\zeta_j^y$ in $y$-direction using the knot sequence $\boldsymbol{\eta}$. Then the spline interpolation at these points is well-defined (see [9, Sec. 3.3.1]). Note that in the periodic case some Greville points lie outside of the grid, but we assume the function that is interpolated to be periodic as well, so the values at the Greville points are well-defined.

This gives us the following geometric elements nodes, edges and cells

$$\begin{aligned} \hat{\mathtt{n}}_{\mathbf{i}} &:= (\zeta_{i_1}^x, \zeta_{i_2}^y), \quad \mathbf{i} \in \mathcal{M}^0 \\ \hat{\mathtt{e}}_{d,\mathbf{i}} &:= [\hat{\mathtt{n}}_{\mathbf{i}}, \hat{\mathtt{n}}_{\mathbf{i}+\mathbf{e}_d}], \quad (d, \mathbf{i}) \in \mathcal{M}^1 \\ \hat{\mathtt{c}}_{\mathbf{i}} := [\hat{\mathtt{e}}_{1,\mathbf{i}}, \hat{\mathtt{e}}_{1,\mathbf{i}+\mathbf{e}_1}] &= [\zeta_{i_1}^x, \zeta_{i_1+1}^x] \times [\zeta_{i_2}^y, \zeta_{i_2+1}^y], \quad \mathbf{i} \in \mathcal{M}^2 \end{aligned}$$

with $[\cdot]$ being the convex hull. As before, $\mathbf{e}_d$ for $d = 1, 2$ is the standard basis vector of $\mathbb{R}^2$. The set of multiindices are defined as

$$\begin{aligned} \mathcal{M}^0 &:= [n-1]^2 \\ \mathcal{M}^1 &:= \{(d, \mathbf{i}) \mid d \in \{1, 2\}, \mathbf{i} \in [0, n-1]^2, i_d < n-1\} \\ \mathcal{M}^2 &:= [n-2]^2 \end{aligned}$$

Now that we have defined the geometric elements we define the corresponding degrees of freedom. We define $\mathbf{e}_d^\perp$ as $\mathbf{R}_{\pi/2}\mathbf{e}_d$ i.e. the rotation by $\pi/2$ in counter clockwise direction which leads to $\mathbf{e}_1^\perp = \mathbf{e}_2$ and $\mathbf{e}_2^\perp = -\mathbf{e}_1$. The degrees of freedom are then

$$\begin{aligned} \hat{\sigma}_{\mathbf{i}}^0(v) &:= v(\hat{\mathtt{n}}_{\mathbf{i}}), \quad \mathbf{i} \in \mathcal{M}^0 \\ \hat{\sigma}_{\mathbf{d},\mathbf{i}}^1(\mathbf{v}) &:= \int_{\hat{\mathtt{e}}_{d,\mathbf{i}}} \mathbf{v} \cdot \mathbf{e}_d^\perp, \quad (d, \mathbf{i}) \in \mathcal{M}^1 \\ \hat{\sigma}_{\mathbf{i}}^2(v) &:= \int_{\hat{\mathtt{c}}_{\mathbf{i}}} v, \quad \mathbf{i} \in \mathcal{M}^2 \end{aligned}$$

These degrees of freedom are unisolvent on $\hat{V}_h^\ell$ i.e. with $N_\ell = |\mathcal{M}^\ell|$ and some ordering $\mu_0, \mu_1, ..., \mu_{N_\ell}$ of the indices of $\mathcal{M}^\ell$, denoting $\sigma_{\mu_i}^\ell = \sigma_i^\ell$, we define

$$\hat{\boldsymbol{\sigma}}^\ell := (\hat{\sigma}_0^\ell, \hat{\sigma}_1^\ell, ..., \hat{\sigma}_{N_\ell}^\ell)^\top : \hat{V}_h^\ell \to \mathbb{R}^{N_\ell}$$

which is bijective. And we can thus define our basis functions $\hat{\Lambda}^\ell_\mu$, $\mu \in \mathcal{M}^\ell$, (denoted in bold if vector valued) as the basis which is dual to the degrees of freedom in the sense

$$\hat{\sigma}^\ell_\mu(\hat{\Lambda}^\ell_\nu) = \delta_{\mu,\nu} \quad \forall \mu, \nu \in \mathcal{M}^\ell.$$

**Remark 3.2.1.** For implementational purposes, these basis functions dual to the degrees of freedom are not necessarily the best option. For the computation of mass matrices etc. it is more convenient to use the B-splines directly due to their local support and fast computation. We will not go too deeply into the details of implementation however. More details about the use of B-splines and the connection with the basis $\Lambda^\ell_\mu$ can be found in [4, Sec. 4.8]

The question is now on what function spaces these degrees of freedom are defined. We note first that the standard choice with $\hat{V}^0 = H^1(\hat{\Omega})$, $\hat{V}^1 = H(\mathrm{div}, \hat{\Omega})$ and $\hat{V}^2 = L^2(\hat{\Omega})$ can not work because e.g. the evaluation at point values is generally not well-defined for $H^1$-functions in 2D. Thus, we need to choose function spaces with higher regularity or integrability.

Let us define the spaces

$$\begin{aligned} W^1_{1,2}(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_1 \partial_2 v \in L^1(\hat{\Omega})\} \\ W^1_d(\hat{\Omega}) &:= \{v \in L^1(\hat{\Omega}) \mid \partial_d v \in L^1(\hat{\Omega})\} \text{ for } d \in \{1, 2\} \end{aligned}$$

and then denote $\hat{U}^0_{L^1} := W^1_{1,2}(\hat{\Omega})$, $\hat{U}^1_{L^1} := W^1_1(\hat{\Omega}) \times W^1_2(\hat{\Omega})$ and $\hat{U}^2_{L^1} := L^1(\hat{\Omega})$. Then the degrees of freedom $\sigma^\ell_\mu$ are well-defined on $\hat{U}^\ell := \hat{U}^\ell_{L^1} \cap \hat{V}^\ell$.

We thus obtained the degrees of freedom and basis functions on the reference domain $\hat{\Omega}$. The idea to continue is now to define the basis functions on the physical domain $\Omega$ by a pushforward of the basis functions from the reference domain. We will now clarify what types of domain we will consider for the discretization. Then we will define the pushforward as the inverse of the pullback and apply it to the basis functions and degrees of freedom to transfer them to the physical domain.

{ass:discretizab

**Assumption 3.2.2.** There exists a rectangular reference domain $\hat{\Omega} = [a_1, b_1] \times [0, T]$, $a_1 < b_1$, $T > 0$ and an orientation preserving diffeomorphism $\mathbf{F}$ defined on $\hat{\Omega}$, which is $T$-periodic in the second variable, such that $\Omega = F(\hat{\Omega})$.

We have the tensor grid on the reference domain

$$\{a_1 = x_0 < x_1 < ... < x_N = b_1\} \times \{0 = y_0 < y_1 < ... < y_N = T\}.$$

For the curve $\Gamma$, we assume that there exists $\hat{\Gamma} \subseteq \hat{\Omega}$ that goes along the edges of the grid such that $\Gamma = \mathbf{F}(\hat{\Gamma})$.

This is usually referred as the *single patch case* since we only use one mapping from the reference domain. We see that this assumption is made so that the notions on the physical domain are well-represented on the reference domain. Notice that under this assumption the preimage of $\Omega_\Gamma$ corresponds to a subgrid on the reference domain.

We already introduced the pullback in Sec.**??** in the 3D case. Then we define the pullbacks in 2D

$$\begin{aligned}
&\mathcal{P}^0_{\mathbf{F}} : v \mapsto \hat{v} := v \circ \mathbf{F} \\
&\mathcal{P}^1_{\mathbf{F}} : \mathbf{v} \mapsto \hat{\mathbf{v}} := (\det D\mathbf{F})\, D\mathbf{F}^{-1}(\mathbf{v} \circ \mathbf{F}) \\
&\mathcal{P}^2_{\mathbf{F}} : v \mapsto \hat{v} := (\det D\mathbf{F})(v \circ \mathbf{F})
\end{aligned}$$

which map functions on the physical domain $\Omega$ to functions on the reference domain $\hat{\Omega}$. Then we have the commuting properties

$$\begin{aligned}
\widehat{\mathbf{curl}}\mathcal{P}^0_{\mathbf{F}} v &= \mathcal{P}^1_{\mathbf{F}}\, \mathbf{curl}\, v \\
\widehat{\operatorname{div}}\mathcal{P}^1_{\mathbf{F}} \mathbf{v} &= \mathcal{P}^1_{\mathbf{F}} \operatorname{div} \mathbf{v}
\end{aligned}$$

when the **curl** and divergence are well-defined.

**Remark 3.2.3.** Just as in Sec.**??** these pullbacks can be derived from the pullbacks of differential forms. Keep in mind however, that in 2D there are two different ways to identify a 1-form with a vector field (cf. **??**). This leads to different pullbacks. We have chosen it here s.t. we have the commuting property with the differential operators in our Hilbert complex (1.1.2).

We now define the pushforwards as $\mathcal{F}^\ell := (\mathcal{P}^\ell_{\mathbf{F}})^{-1}$ which then read

$$\begin{aligned}
&\mathcal{F}^0 : \hat{v} \mapsto v := \hat{v} \circ \mathbf{F}^{-1} \\
&\mathcal{F}^1 : \hat{\mathbf{v}} \mapsto \mathbf{v} := (\det D\mathbf{F}^{-1})\, D\mathbf{F}(\hat{\mathbf{v}} \circ \mathbf{F}^{-1}) \\
&\mathcal{F}^2 : \hat{v} \mapsto v := (\det D\mathbf{F}^{-1})(\hat{v} \circ \mathbf{F}^{-1}).
\end{aligned}$$

By applying these pushforwards we get the basis functions on the physical domain

$$\Lambda^\ell_\mu := \mathcal{F}^\ell \hat{\Lambda}^\ell_\mu$$

and then

$$\bar{V}^\ell_h := \operatorname{span}\{\Lambda^\ell_\mu \mid \mu \in \mathcal{M}^\ell\}$$

are our discrete spaces without boundary conditions.

Using the geometric degrees of freedom on the reference domain, we can now construct the corresponding degrees of freedom on the physical domain as

$$\sigma_\mu^\ell := \hat{\sigma}_\mu^\ell \circ \mathcal{P}_F^\ell$$

Then we have by construction that $\sigma_\mu^\ell(\Lambda_\nu^\ell) = \delta_{\mu,\nu}$ for all $\mu, \nu \in \mathcal{M}^\ell$.

The degrees of freedom then also correspond to geometric objects in the physical domain. The reference grid gets mapped to the physical domain which gives us the geometric objects there. We define the mapped nodes in the physical domain as

$$\mathtt{n}_{\mathbf{i}} := \mathbf{F}(\hat{\mathtt{n}}_{\mathbf{i}}),$$

and analogous edges $\mathtt{e}_{d,\mathbf{i}}$, $(d, \mathbf{i}) \in \mathcal{M}^1$, and cells $\mathtt{c}_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{M}^2$. $\sigma^0$ corresponds to point values at the nodes, $\sigma^1$ to the fluxes through the edges and $\sigma^2$ to the integral over the mapped cells.

**Remark 3.2.4.** If we would take the now obvious choice of $v \mapsto \sum_{\mu\in\mathcal{M}^0} \sigma_\mu^0(v)\Lambda_\mu^0 \in \bar{V}_h^0$ and analogous for the other function spaces, the result would be a cochain projection, but it would not be bounded and thus insufficient for the theory introduced in Sec. 2.1.

However, there are $L^2$-stable quasi-interpolants which also commute with the differential operators (see [2, Sec. 4]) and provide us with the stable cochain projections that we need for the theory.

## 3.3 Building the discrete system

{sec:assembling_

Now that we have specified the basis and degrees of freedom we will talk about how we will build the system to solve the magnetostatic problem on $\Omega$. In particular, we will explain how we will enforce homogeneous boundary conditions since this is the way the magnetostatic problem was posed. To achive that, we define the boundary projections and reformulate the discrete problem with them. When this is done, we will write the equations in matrix form to see what system is solved in practice.

Recall that $\mathbf{F}$, the parametrization of the physical domain, is assumed to be periodic in the second variable and so the "north" and "south" edge of the rectangle $\hat{\Omega}$ do not correspond to a boundary of the physical domain $\Omega$. This motivates the (slight abuse of) notation

$$\partial\hat{\Omega} := \{a, b\} \times [0, T].$$

Let us define the sets of boundary indices $\mathcal{B}^\ell$ which correspond to the multiindices of the geometric elements on the boundary i.e.

$$\mathcal{B}^0 := \{\mathbf{i} \in \mathcal{M}^0 \mid \hat{\mathtt{n}}_{\mathbf{i}} \in \partial\hat{\Omega}\}$$
$$\mathcal{B}^1 := \{(d, \mathbf{i}) \in \mathcal{M}^1 \mid \hat{\mathtt{e}}_{d,\mathbf{i}} \subseteq \partial\hat{\Omega}\}$$

Since $V_h^2 \subseteq L^2$ we do not require any boundary conditions there. Then we define the spaces

$$V_h^\ell := \{v_h \in \bar{V}_h^\ell \mid \sigma_\mu^\ell(v_h) = 0 \quad \forall \mu \in \mathcal{B}^\ell\} = \operatorname{span}\{\Lambda_\mu^\ell \mid \mu \in \mathcal{M}^\ell \setminus \mathcal{B}^\ell\}$$

For these spaces it holds then

$$V_h^0 = \bar{V}_h^0 \cap H_0^1(\Omega)$$
$$V_h^1 = \bar{V}_h^1 \cap H_0(\operatorname{div}, \Omega)$$

and so we see that the homogeneous boundary conditions simply correspond to the boundary degrees of freedom being zero. This means for $V_h^0$ we require the nodal values on the boundary to vanish and for $V_h^1$ zero flux through the boundary edges.

Then we define the projections $P_h^\ell : \bar{V}_h^\ell \to \bar{V}_h^\ell$ which set the boundary degrees of freedom to zero and are thus a projection onto $V_h^\ell$. The matrix representation $\mathbb{P}^\ell$ is then simply $(\mathbb{P}^\ell)_{\mu,\nu} = 1$ i.i.f. $\mu = \nu$ and $\mu$ does not correspond to a geometric element on the boundary. They are easily constructed by taking the identity matrix and setting the diagonal entries to zero that belong to boundary degrees of freedom. These matrices are obviously symmetric.

We now reformulate the discrete system using these projections. We apply the boundary projections to all functions involved and then add the boundary penalization terms. With boundary penalities the discrete system is: Find $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ s.t.

$$\langle (I - P_h^0)\sigma_h, (I - P_h^0)\tau_h \rangle + \langle P_h^0\sigma_h, P_h^0\tau_h \rangle - \langle P_h^1\mathbf{B}_h, \mathbf{curl}\, P_h^0\tau_h \rangle = -\langle J, P_h^0\tau_h \rangle \quad \forall \tau_h \in \bar{V}_h^0, \tag{3.3.1}$$
{eq:system_with_

$$\begin{aligned} &\langle \mathbf{curl}\, P_h^0\sigma_h, P_h^1\mathbf{v}_h \rangle + \langle (I - P_h^1)\mathbf{B}_h, (I - P_h^1)\mathbf{v}_h \rangle \\ &\qquad + \langle \operatorname{div} P_h^1\mathbf{B}_h, \operatorname{div} P_h^1\mathbf{v}_h \rangle + \langle \lambda\mathbf{p}_h, P_h^1\mathbf{v}_h \rangle = 0 \quad \forall \mathbf{v}_h \in \bar{V}_h^1, \end{aligned} \tag{3.3.2}$$
{eq:system_with_

$$\langle \mathbf{curl}\, \psi, P_h^1\mathbf{B}_h \rangle = C_1. \tag{3.3.3}$$
{eq:system_with_

Since we apply the projection everywhere it is then easy to show that $\sigma_h \in \bar{V}_h^0$, $\mathbf{B}_h \in \bar{V}_h^1$ and $\lambda \in \mathbb{R}$ solve (3.3.1)-(3.3.3) i.i.f. $\sigma_h \in V_h^0$, $\mathbf{B}_h \in V_h^1$ and

$(\sigma_h, \mathbf{B}_h, \lambda)$ solves the system with homogeneous discrete spaces (Prob. 2.2.1). So the two formulations are equivalent.

We will from now on often use implicitly some flattening of the multi-indices in $\mathcal{M}^\ell$. Define $\mathbb{M}^\ell$ as the mass matrix of $\bar{V}_h^\ell$ i.e. $\mathbb{M}_{ij}^\ell = \langle \Lambda_i^\ell, \Lambda_j^\ell \rangle$. We define the matrix $\mathbb{D}$ as the matrix representation of the divergence applied to $\bar{V}_h^1$ i.e. $\operatorname{div}|_{\bar{V}_h^1} : \bar{V}_h^1 \to \bar{V}_h^2$. Analogously $\mathbb{C}$ is the matrix representation of **curl**. Then we have, as mentioned before, the matrix representation of the boundary projections $\mathbb{P}^\ell$ and $\mathbb{I}^\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ is the identity matrix. We denote the vector of coefficients of a function with an underline e.g. $\underline{\sigma} \in \mathbb{R}^{N_0}$ is the vector of coefficients of $\sigma$ in the basis $\Lambda_\mu^0$, $\mu \in \mathcal{M}^0$. $\underline{\mathbf{B}}, \underline{\mathbf{p}} \in \mathbb{R}^{N_1}$ are the coefficients of $\mathbf{B}_h$ and $\mathbf{p}_h$ in the basis $\boldsymbol{\Lambda}_\mu^1$, $\mu \in \mathcal{M}^1$. So rewriting the equations (3.3.1)-(3.3.3) in matrix-vector form gives us

$$\begin{aligned}
\underline{\tau}^\top(\mathbb{I}^0 - \mathbb{P}^0)\mathbb{M}^0(\mathbb{I}^0 - \mathbb{P}^0)\underline{\sigma} + \underline{\tau}^\top\mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \underline{\tau}^\top\mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \underline{\tau}^\top\underline{\tilde{J}} \quad \forall \underline{\tau} \in \mathbb{R}^{N_0} \\
\underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + \underline{\mathbf{v}}^\top(\mathbb{I}^1 - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1 - \mathbb{P}^1)\underline{\mathbf{B}} \\
+\underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \underline{\mathbf{v}}^\top\mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0 \quad \forall \underline{\mathbf{v}} \in \mathbb{R}^{N_1} \\
\underline{\psi}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1
\end{aligned}$$

where $\underline{\tilde{J}} = (\langle J, \Lambda_i^0 \rangle)_{i=1}^{N_0}$ which gives us the final system to be solved

$$\begin{aligned}
(\mathbb{I}^0 - \mathbb{P}^0)^\top\mathbb{M}^0(\mathbb{I}^0 - \mathbb{P}^0)\underline{\sigma} + \mathbb{P}^0\mathbb{M}^0\mathbb{P}^0\underline{\sigma} + \mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = \underline{\tilde{J}} \\
\mathbb{P}^1\mathbb{M}^1\mathbb{C}\mathbb{P}^0\underline{\sigma} + (\mathbb{I} - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I} - \mathbb{P}^1)\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1\underline{\mathbf{B}} + \mathbb{P}^1\mathbb{M}^1\underline{\mathbf{p}} = 0 \\
\underline{\psi}^\top\mathbb{C}^\top\mathbb{M}^1\mathbb{P}^1\underline{\mathbf{B}} = C_1
\end{aligned}$$

Now we will explain how the discrete harmonic form is computed. In order to do that we will characterize harmonic forms in a different way.

**Proposition 3.3.1.** *Define the penalized discrete Hodge Laplacian operator* $\bar{L}_h^1 : \bar{V}_h^1 \to \bar{V}_h^1$ *as*

$$\bar{L}_h^1 = -\widetilde{\mathrm{grad}}_h \operatorname{div} P_h^1 + \widetilde{\mathrm{curl}}_h \,\mathbf{curl}\, P_h^0 + (I - P_h^1)^*(I - P_h^1)$$

*where* $\widetilde{\mathrm{grad}}_h$ *and* $\widetilde{\mathrm{curl}}_h$ *are the adjoints of* $\operatorname{div} P_h^1$ *and* $\mathbf{curl}\, P_h^0$ *respectively. Then*

$$\ker \bar{L}_h^1 = \mathfrak{H}_h^1.$$

Note that the spaces in the discrete Hilbert complex (2.1.1) have homogeneous boundary conditions and thus this is also true for the spaces $\mathfrak{H}_h^j$ and $\mathfrak{B}_h^j$ and $\mathfrak{B}_{jh}^*$ by definition.

*Proof.* See [3, Thm. 3.2] □

Notice that for any linear operator $\phi : V \to W$ between finite dimensional inner product spaces $V$ and $W$ with matrix representation $A$, the matrix representation of the adjoint $\phi^*$ is $G_V^{-1} A^\top G_W$ where $G_V$ and $G_W$ are the Gramian matrices of the chosen bases in $V$ and $W$ respectively. So the penalized discrete Hodge Laplacian has then the matrix representation

$$(\mathbb{M}^1)^{-1}\mathbb{P}^1\mathbb{D}^\top\mathbb{M}^1\mathbb{D}\mathbb{P}^1 + \mathbb{C}\mathbb{P}^0(\mathbb{M}^0)^{-1}\mathbb{P}^0\mathbb{C}^\top\mathbb{M}^1 + (\mathbb{M}^1)^{-1}(\mathbb{I}^1 - \mathbb{P}^1)\mathbb{M}^1(\mathbb{I}^1 - \mathbb{P}^1)$$

and we compute the coefficients $\underline{\mathbf{p}}$ of the discrete harmonic form by computing an element of the kernel of this matrix. One can also multiply it with $\mathbb{M}^1$ from the left which does not change the kernel, but avoids having to compute $(\mathbb{M}^1)^{-1}$.

**Remark 3.3.2.** We acknowledge the fact that computing the inverse of the mass matrix $\mathbb{M}^0$ is a computational bottleneck of this implementation. The inverse of the mass matrix is in general dense and is thus problematic memory-wise. One way to avoid this would be to use multiple patches instead and then use a CONGA approach described in detail in [4]. Then the mass matrices are block diagonal and inverting them much less costly.

# 4 Numerical examples

We will consider two numerical examples, the simulation of the induced magnetic field from a wire, where the exact solution is given by the Biot-Savart law, and a manufactured solution. The Biot-Savart solution will be approximated on a standard annulus and a "distorted annulus" and the manufactured solution will be approximated on a standard annulus with two different curve integrals given.

## 4.1 Magnetic field induced by wire

As a first simple numerical example, we consider a standard example from magnetostatics which is the magnetic field induced by an infinitely long, straight wire with radius zero. The *Biot-Savart law* can be used to compute it. Let the wire be equal to the $z$-axis and $I$ be the electrical current flowing through it. $\boldsymbol{\ell}(s) = s\mathbf{e}_3$, $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{\ell' \times (x - \ell(s))}{|x - \ell(s)|^3}\, ds = \frac{\mu_0 I}{4\pi |x|^2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

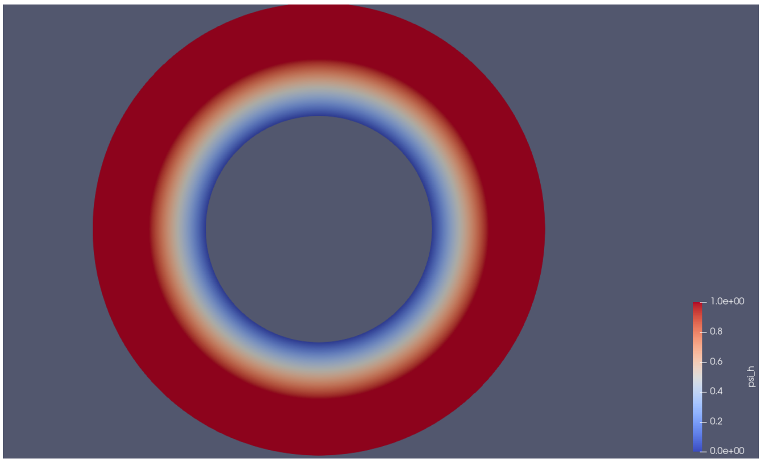



Figure 4.1: A possible choice of $\psi$ on an annulus. $\Gamma$ is here in the middle of the annulus. $\psi = 0$ at the inner boundary and $\psi = 1$ at $\Gamma$ and between $\Gamma$ and the outer boundary {fig:psi_annulus

for convenience we pick now $I = \frac{2\pi}{\mu_0}$ to get

$$\mathbf{B}(x) = \frac{2}{|x|^2} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

We will only focus on the first two components since our approximation is in 2D. We choose as our domain of computation $\Omega$ the annulus with inner radius 1 and outer radius 2. We choose as our curve $\Gamma$ the parametrization of the circle with radius 1.5 in anticlockwise direction which gives us the curve integral

$$C_0 = \int_\Gamma \mathbf{B} \cdot d\ell = 4\pi$$

$J = 0$ on our domain and hence $C_1 = C_0$. The reference domain $\hat{\Omega} = [0, 1] \times [0, 2\pi]$ and the mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} (\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

Then we choose $\psi$ as a simple interpolation from the inner boundary to the curve $\Gamma$ i.e. in logical coordinates

$$\psi(\hat{x}_1, \hat{x}_2) = \begin{cases} 2\hat{x}_1 & \text{for } \hat{x}_1 \leq 0.5 \\ 1 & \text{else.} \end{cases} \tag{4.1.1}$$ {eq:linear_inter

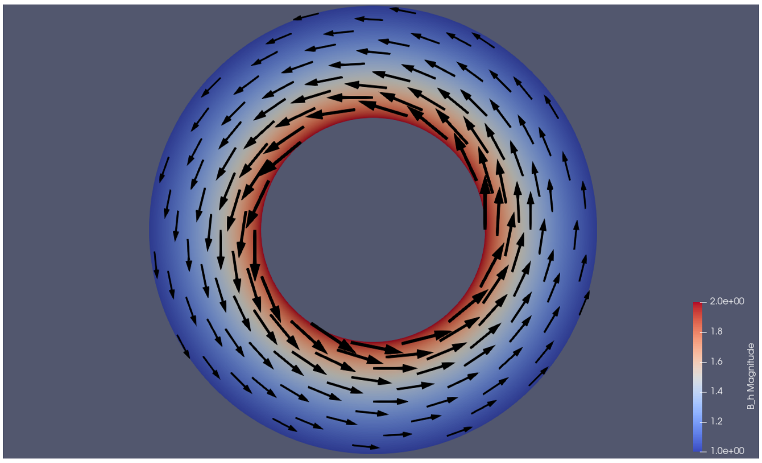



Figure 4.2: Solution of magnetostatic problem induced by a wire through the origin in $z$-direction on an annulus for 64 grid cells on the reference domain in both dimensions and spline degree $p = 2$ {fig:biot_savart

This fulfills all the criteria we had for $\psi$ for this given $\Gamma$ (see Fig. 4.1).

For the implementation we are using the PSYDAC library (`https://github.com/pyccel/psydac`) which is an open source Python 3 library for isogeometric analysis (see [6] for more details). See Fig. 4.2 for the solution where we chose 64 cells for our grid on the reference domain in both dimensions and $p = 2$. Note that $p = 2$ refers to the choice of our spaces from (3.1.1) and hence the spline space for approximating the magnetic field will be $\mathbb{S}_{2,1} \times \mathbb{S}_{1,2}$ so the lower spline degree will be one. The pushforward spline space achieves then an approximation error of $h^{p+1}$ [8, Ch.4, (4.48)] if the solution is at least $H^{p+1}$-regular. Thus the expected rate of convergence is two in our case which can be observed (see Fig. 4.3).

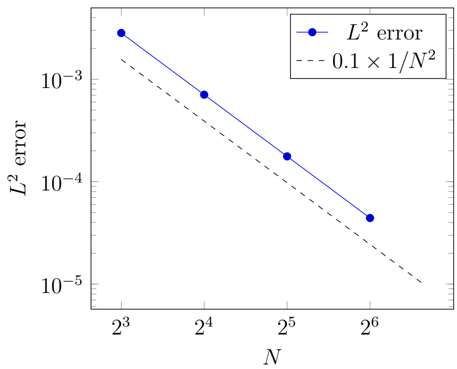
Maybe Different curves for Biot Savart ?

Now we will change the domain where the problem is posed, but we will stick to the Biot-Savart setting. This means we still assume the wire goes through the origin in $z$-direction, which results in the same solution, but we will approximate on a "distorted annulus" using the same reference domain, but the different mapping

$$\mathbf{F}(\hat{x}) = \begin{pmatrix} (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\cos(\hat{x}_2) \\ (\cos^2(\hat{x}_2) + 1)(\hat{x}_1 + 1)\sin(\hat{x}_2) \end{pmatrix}.$$

which results in a kind of "distorted annulus"(see **??**).

Note that now for this domain we do not have the homogeneous boundary condition $\mathbf{B} \cdot \mathbf{n} = 0$ anymore. Thus we have to deal with the boundary conditions by using a standard lifting approach which means we take an



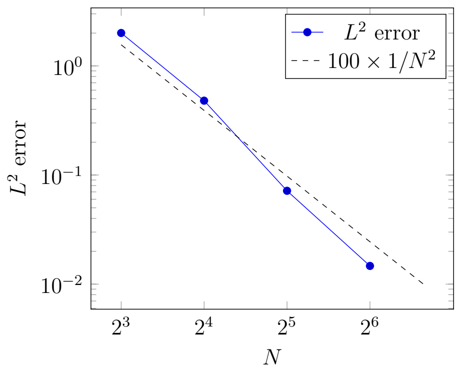




Figure 4.3: Convergence analysis for the approximation of the Biot-Savart solution with $p = 2$ on the annulus domain (left) and the distorted annulus domain (right)

{fig:convergence

interpolation of the boundary conditions $\mathbf{B}_{h,g}$ which we compute before and then split $\mathbf{B}_h = \mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ with $\mathbf{B}_{h,0} \in V_h^1$. Then we substitute $P_h^1 \mathbf{B}_h$ in (3.3.1)-(3.3.3) with $\mathbf{B}_{h,0} + \mathbf{B}_{h,g}$ and put the terms with $\mathbf{B}_{h,g}$ on the right hand side. This leads to the same left hand side, but as right hand side we get $-\langle J, \tau_h \rangle + \langle \mathbf{B}_{h,g}, \mathbf{curl}\, \tau_h \rangle$ in the first equation, $-\langle \operatorname{div} \mathbf{B}_{h,g}, \operatorname{div} \mathbf{v}_h \rangle$ in the second and $C_1 - \langle \mathbf{curl}\, \psi, \mathbf{B}_{h,g} \rangle$ in the third. In practice, since we know the exact solution $\mathbf{B}$ we simply take $\mathbf{B}_{h,g} := (I - P_h^1)\mathbf{B}$. The remaining steps of assembling the system are completely analogous to Sec. 3.3.

As $\psi$ we use the exact same definition as in (4.1.1) which results in a different curve due to the different domain parametrization (see Fig. 4.4). $J$ is of course still zero and the curve integral does not change either. We obtain again second order convergence for $p = 2$ (see Fig.4.3), but the error is several orders of magnitude larger compared to the normal annulus.

## 4.2 Manufactured solution

As an example with a non-vanishing $J$, we will use a manufactured solution. Let $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$. It is easy to see that $\operatorname{div} \mathbf{B} = 0$ and $\mathbf{B} \cdot \mathbf{n} = 0$ for the annulus. This results in

$$J(x) = 4|x|^2 - 12|x| + 8.$$

We will pose this problem on the annulus domain from before, but we will use two different choices for $\Gamma$. $\Gamma_1$ will be the same $\Gamma$ as before, but $\Gamma_2$ will be the outer boundary of $\Omega$ i.e. it is equal to $\partial\Omega_{out}$. Since $J$ is not zero this

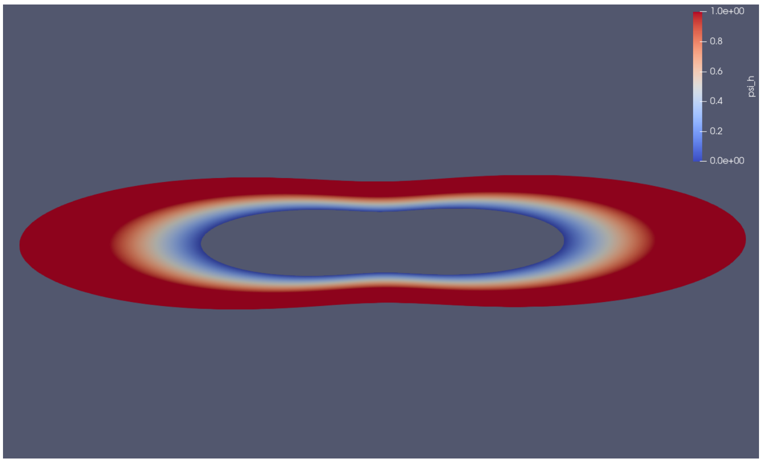



Figure 4.4: $\psi$ on the distorted annulus domain. $\Gamma$ is again in the middle between the interior and exterior boundary. {fig:psi_distort

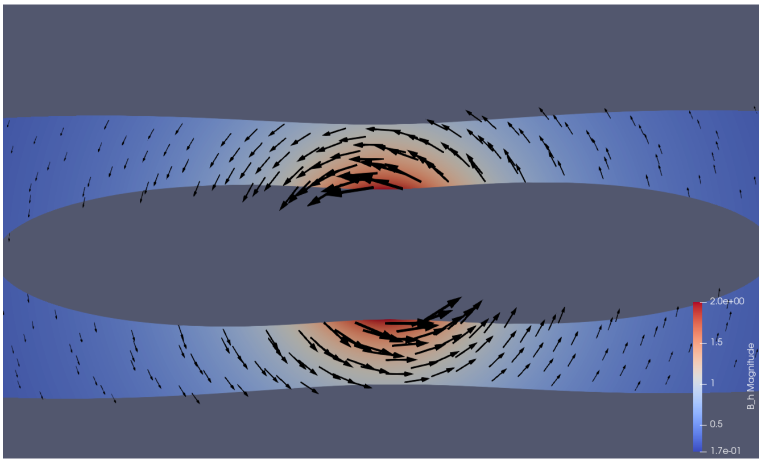



Figure 4.5: Solution of the Biot-Savart law on the distorted annulus with 64 cells in both dimensions and $p = 2$. The wire is in the centre pointing towards the point of view. {fig:distorted_a

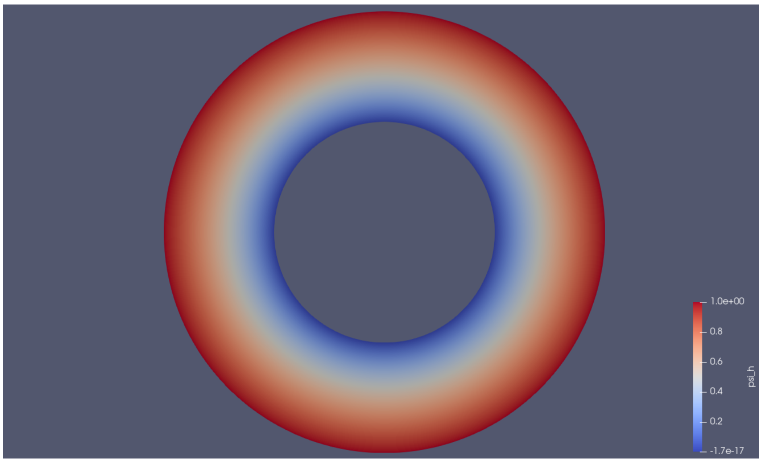



Figure 4.6: $\psi$ as the solution of a Poisson problem with $\psi = 0$ on the interior and $\psi = 1$ on the exterior boundary. $\Gamma$ is here equal to $\partial\Omega_{out}$. {fig:psi_poisson

leads to different curve integrals

$$\int_{\Gamma_1} \mathbf{B} \cdot d\ell = \frac{9\pi}{8} \quad \text{and} \quad \int_{\Gamma_2} \mathbf{B} \cdot d\ell = 0.$$

For $\Gamma_1$ we will choose $\psi$ as before and for $\Gamma_2$ we take $\psi$ as the solution of the Laplace problem with boundary condition $\psi = 0$ on $\partial\Omega_{in}$ and $\psi = 1$ on $\partial\Omega_{out}$ (see Fig. 4.6). We observe the same convergence rate (Fig. 4.7).

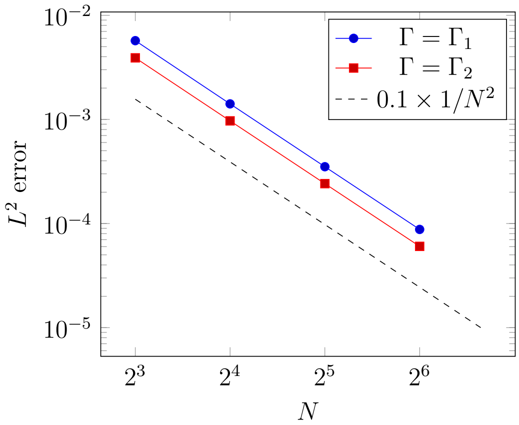



Figure 4.7: Convergence analysis for the manufactured solution $\mathbf{B}(x) = (|x|^2 - 2)(-x_2, x_1)^\top$ with different choices of $\psi$ corresponding to different curves $\Gamma_1$ (as before) and $\Gamma_2$ (equal to the exterior boundary). {fig:convergence