

1 The Hodge Laplacian problem

A Hilbert complex (W^k, d^k) provides us with a natural operator which we call the *abstract Hodge Laplacian* defined as

$$L^k : D(L^k) \rightarrow W^k, u \mapsto d_*^{k+1} d^k u + d^{k-1} d_k^* u.$$

with

$$D(L^k) := \{u \in V^k \cap V_k^* \mid d^k u \in V_{k+1}^*, d_k^* u \in V^{k-1}\}$$

The name stems from the standard Hodge Laplacian on Riemannian manifolds where the differential operator is the exterior derivative (see [[empty citation](#)]). We will usually just leave out the term abstract. This section will be dealing with the operator equation

$$L^k u = f. \tag{1.0.1}$$

We will follow the corresponding section 4.4 in Arnold's book [1]. Let us start with the question of well-posedness for which the following Proposition is crucial. Also we will stick to the notation introduced in Sec. ??

1.1 Strong formulation

Proposition 1.1.1 (Kernel of the Hodge Laplacian).

$$\ker L^k = \mathfrak{H}^k \tag{1.1.1} \quad \{\text{eq:hodge_laplac}\}$$

Proof. The direction " \supseteq " is clear from the definition of harmonic forms. We will leave away the index now. Let $u \in \ker L^k$,

$$0 = \langle L^k u, u \rangle = \langle d_*^{k+1} d^k u, u \rangle + \langle d^{k-1} d_k^* u, u \rangle = \|d^k u\|^2 + \|d_k^* u\|^2$$

the norms and inner products are in the spaces W^{k-1} , W^k and W^{k+1} . Thus we see that $d_k^* u = 0$ and $d^k u = 0$ and hence $u \in \mathfrak{H}^k$. \square

This has immediate implications. Let us denote in general the orthogonal projection operator onto a subset Z as P_Z . This means that (1.1.1) can only have a solution if $P_{\mathfrak{H}} = 0$ due to the Hodge decomposition and also that solutions are only unique up to addition of harmonic forms and thus (1.1.1) is in general not well-posed. We require additional constraints. The simplest option is to require $u \perp \mathfrak{H}^k$. This leads us to the standard *strong*

formulation of the Hodge Laplacian problem with *orthogonality constraint*:
For any $f \in W^k$ find $u \in V^k \cap V_k^*$ s.t.

$$L^k u = f - P_{\mathfrak{H}} f \text{ and } u \perp \mathfrak{H}^k.$$

Because we know from $u \perp \ker L^k$ that the solution must be unique if it exists. We will later also consider different constraints on the harmonic part.

Let us investigate the Hodge-Laplacian in 3D which we are well familiar with. We have $V^0 = H^1$, $V^1 = H(\text{curl})$, $V^2 = H(\text{div})$ and $V^3 = L^2$. As before, we leave out the reference to the domain Ω which is assumed to have a compact Lipschitz boundary as in Sec.???. Then we get $V_0^* = L^2$, $V_1^* = H_0(\text{div})$, $V_2^* = H_0(\text{div})$ and $V_3^* = H_0^1$. Since $d^0 = \text{grad}$ and $d_1^* = -\text{div}$ and we set $d_0^* = 0$ to get

$$D(L^0) = \{u \in H^1 \mid \text{grad } u \in H_0(\text{div})\} \text{ and } \\ L^0 = -\text{div grad} = -\Delta.$$

Assume that our domain is path connected. Recall that then \mathfrak{H}^0 are the constant functions and thus the condition $f \perp \mathfrak{H}^0$ means that

$$\int_{\Omega} f \, dx = 0 \tag{1.1.2} \quad \{\text{eq:average_equa}\}$$

Then we see that the resulting Hodge Laplacian problem: Take $f \in L^2$ with (1.1.2) being fulfilled. Find $u \in D(L^0)$ s.t.

$$-\text{div grad } u = f \text{ and } \\ \int_{\Omega} u = 0$$

Note that $u \in D(L^0)$ requires $\text{grad } u \cdot \mathbf{n} = 0$ on $\partial\Omega$ and thus we recognize the strong formulation of the well-known Poisson problem with Neumann boundary conditions. (1.1.2) is the well-known compatibility condition for the Neumann boundary conditions.

Let us also look at situation for $k = 1$. Then

$$D(L^1) = \{\mathbf{u} \in H(\text{curl}) \cap H_0(\text{div}) \mid \text{curl } \mathbf{u} \in H_0(\text{curl}), \text{div } \mathbf{u} \in H^1\} \text{ and } \\ L^1 = \text{curl curl} - \text{grad div}.$$

This operator we already know from ???. The connection to the magnetostatic problem will be made clear later (?). The full problem then reads: Take $\mathbf{f} \in L^2(\Omega; \mathbb{R}^3)$. Find $\mathbf{u} \in D(L^1)$ s.t.

$$\text{curl curl } \mathbf{u} - \text{grad div } \mathbf{u} = \mathbf{f} - P_{\mathfrak{H}^1} \mathbf{f}. \\ u \perp \mathfrak{H}^1$$

Analogously, we obtain the same problem with changed boundary conditions for the operator L^2 and \mathfrak{H}^2 instead of \mathfrak{H}^1 . For $k = 3$ we obtain the Poisson problem, but with Dirichlet boundary conditions.

In this second part of the thesis, the situation in 2D will also become relevant. So we will

1.2 Mixed formulation

We want to deal with Hodge Laplacian problem using a finite element approach. So we would like to write this problem in variational form. The most principled way to do so is using the *mixed weak formulation*: For any $f \in W^k$ find $\sigma \in V^{k-1}$, $u \in V^k$ and $p \in \mathfrak{H}^k$ s.t.

$$\langle \sigma, \tau \rangle - \langle u, d^{k-1}\tau \rangle = 0 \quad \forall \tau \in V^{k-1}, \quad (1.2.1) \quad \{\text{eq:first_eq_mix}\}$$

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in V^k \text{ and} \quad (1.2.2) \quad \{\text{eq:second_eq_mi}\}$$

$$\langle u, q \rangle \quad \forall q \in \mathfrak{H}^k. \quad (1.2.3)$$

This formulation is of course much more complicated than the first one ??, but this formulation will turn out to be well-suited for finite element approximations. But it begs the question if the two formulations are equivalent. We will answer it in two propositions. We will treat the harmonic constraint separately.

Proposition 1.2.1. *For any $f \in W^k$, (1.2.1) and (1.2.2) hold i.i.f. $\sigma = d_k^*u$, $p = P_{\mathfrak{H}}f$ and*

$$L^k u = f - P_{\mathfrak{H}}f$$

i.e. the strong formulation is fulfilled without the additional constraint $u \perp \mathfrak{H}^k$.

Proof. We recognize that the first equation (1.2.1) is actually just the defining property of the adjoint hence $\sigma = d_k^*u$. It also implies that u is in fact in V_k^* .

Now assume additionally, that (1.2.2) holds. Then by choosing $v \in \mathfrak{H}^k$ we get from the definition of the harmonic forms and $\mathfrak{H}^k \perp \mathfrak{B}^k$ from the Hodge decomposition and thus

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in \mathfrak{H}^k$$

which is equivalent to $p = P_{\mathfrak{H}}f$. Next, we would like to show that $\langle du, dv \rangle = \langle d^*du, v \rangle$ which is fulfilled if $du \in V_{k+1}^*$. This follows from (1.2.2) since then

$$\begin{aligned} \langle du, dv \rangle &= -\langle d\sigma, v \rangle + \langle f - P_{\mathfrak{H}}f, v \rangle \\ &\leq \|f - P_{\mathfrak{H}}f - d\sigma\| \|v\| \end{aligned}$$

and so $du \in V_{k+1}^*$. Thus we have

$$\begin{aligned} & \langle d^{k-1}\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle \\ &= \langle d^{k-1}d_k^*u, v \rangle + \langle d^*du, v \rangle + \langle p, v \rangle \\ &= \langle d^{k-1}d_k^*u + d^*du + P_{\mathfrak{H}}f, v \rangle = \langle f, v \rangle \end{aligned}$$

and so since by assumption V^k is dense in W^k ,

$$d_k^*u + d^*du + P_{\mathfrak{H}}f = f.$$

For the other direction assume $u \in D(L^k)$, $d_k^*u = \sigma$ and $p = P_{\mathfrak{H}}f$ and

$$d^{k-1}d_k^*u + d^*du = f - P_{\mathfrak{H}}f. \quad (1.2.4) \quad \{\text{eq:proof_equiva}$$

The first equation holds as explained above. For the second one, we take the inner product of (1.2.4) with an arbitrary $v \in V^k$ and immediately obtain (1.2.2). \square

So we immediately obtain

Corollary 1.2.2. *The strong formulation and mixed weak formulation of the Hodge Laplacian problem with orthogonality constraint $u \perp \mathfrak{H}^k$ are equivalent.*

The reason why did not proof the equivalence directly is that we might add another harmonic constraint and would still get equivalence which we will need later. The problem with orthogonality constraint is then well-posed see [[empty citation](#)].

2 Discrete Hilbert complex

In order to approximate the Hodge Laplacian problem we want to use finite elements. We want to use them in a way that we can rebuild the structure of the Hilbert complex in our discretization. This section follows Sec. 5.2 in Arnold's book [1].

Let us assume the we have finite dimensional subspaces $V_h^k \subseteq V^k$. Then we define completely analogous to the continuous case,

$$\begin{aligned} \mathfrak{Z}_h^k &:= \{v \in V_h^k \mid dv = 0\} = \ker d \cap V_h^k \\ \mathfrak{B}_h^k &:= \{dv \mid v \in V_h^{k-1}\}. \end{aligned}$$

We can now also define the discrete harmonic forms. Now the situation is slightly different however. We will not use the continuous adjoint d_k^* to define it. Instead,

$$\mathfrak{H}_h^k := \{v \in \mathfrak{Z}_h^k \mid v \perp \mathfrak{B}_h^k\} = \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k,\perp}.$$

Notice that we have $\mathfrak{Z}_h^k \subseteq \mathfrak{Z}^k$ and $\mathfrak{B}_h^k \subseteq \mathfrak{B}^k$, but due to $\mathfrak{B}_h^{k,\perp} \supseteq \mathfrak{B}^{k,\perp}$ we have in general

$$\mathfrak{H}^k = \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k,\perp} \not\subseteq \mathfrak{Z}^k \cap \mathfrak{B}^{k,\perp} = \mathfrak{H}^k.$$

This is now already enough to define apply the mixed formulation to the discrete setting: For a given $f \in W^k$, find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$ and $p_h \in \mathfrak{H}^k$ s.t.

$$\langle \sigma_h, \tau_h \rangle - \langle u_h, d\tau_h \rangle = 0 \quad \tau_h \in V_h^{k-1}, \quad (2.0.1)$$

$$\langle d\sigma_h, v_h \rangle + \langle du_h, dv_h \rangle + \langle p_h, v_h \rangle = \langle f, v_h \rangle \quad v_h \in V_h^k, \quad (2.0.2)$$

$$\langle u_h, q_h \rangle = 0, \quad \forall q_h \in \mathfrak{H}_h^k. \quad (2.0.3) \quad \{\text{eq:third_eq_dis}\}$$

When $\mathfrak{H}_h^k \not\subseteq \mathfrak{H}^k$ this is a nonconforming method. The harmonic constraint (2.0.3) can be replaced with a different one.

There are three crucial properties that are necessary for stability and convergence of the method. The first one is the common and reasonable assumption that – as usual in finite element theory – we want that the discrete spaces V_h^k approximate the continuous ones V^j . This can be generally summarized that

$$\lim_{h \rightarrow 0} \inf_{v_h \in V_h^j} \|w - v_h\| = 0, \quad \forall w \in V^j.$$

This is usually satisfied if we use established finite elements for a given space e.g. if we take Lagrangian FE if $V = H^1$ or Raviart-Thomas if $V = H(\text{div})$ [**<empty citation>**].

The next property is more restrictive. We require that $dV_h^{k-1} \subseteq V_h^k$ and $dV_h^j \subseteq V_h^{j+1}$. This shows that the we cannot simply use arbitrary discrete subspaces independent from one another. This property has a very nice consequence. It shows that

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

is itself a Hilbert complex and we can apply the general theory from Sec. ?? directly to it. Let us do that.

Denote the restriction of d^j to V_h^j as d_h^j . Then as a linear map between finite spaces the adjoint – denoted as $d_{j,h}^* : V_h^j \rightarrow V_h^{j-1}$ – is everywhere defined. It is important to notice that in contrast to d_h the adjoint $d_{j,h}^*$ is not the restriction of the adjoint the continuous adjoint d_j^* . In general, $V_h \not\subseteq V^*$ and so the continuous adjoint might not be well-defined for a given $v_h \in V_h$.

So we obtain the Hilbert complex

$$V_h^{k-1} \xrightarrow{d^{k-1}} V_h^k \xrightarrow{d^k} V_h^{k+1}$$

and its dual complex

$$V_h^{k-1} \xleftarrow{d_k^*} V_h^k \xleftarrow{d_{k+1}^*} V_h^{k+1}$$

From the general Hilbert complex theory (Thm. ??) we thus obtain the *discrete Hodge decomposition*

$$V_h^j = \mathfrak{B}_h^j \oplus^\perp \mathfrak{H}_h^j \oplus^\perp \mathfrak{B}_{jh}^*.$$

So we achieved our goal of getting a structure like in the continuous case for our discrete approximation. Especially the question how well the discrete harmonic forms approximate the continuous one will be looked at more closely.

The third crucial assumption is the existence of *bounded cochain projections* π_h . This is a projection that is a cochain map in the sense of cochain complexes ?? i.e. the following diagram commutes: π_h are either bounded in the V or in the W -norm where W -boundedness implies V boundedness. The cochain projection will play an important role in the stability of the discrete system.

Let us now answer the question about the difference between discrete and continuous harmonic forms. In order to do that we need some way to measure the "difference" between two subspaces.

Definition 2.0.1 (Gap between subspaces). For a Banach space W with subspaces Z_1 and Z_2 . Let S_1 and S_2 be the unit spheres in Z_1 and Z_2 respectively i.e. $S_1 = \{z \in Z_1 \mid \|z\|_W = 1\}$. Then we define the gap between these subspaces as

$$\text{gap}(Z_1, Z_2) = \max\left\{\sup_{z_1 \in S_1} \text{dist}(z_1, Z_2), \sup_{z_2 \in S_2} \text{dist}(z_2, Z_1)\right\}$$

This definition is from [**kato perturbation theory**] and defines a metric on the set of closed subspaces of W (see ??Remark p.198]) If W is a Hilbert space – as it is throughout this section – and Z_1 and Z_2 are closed then the $\text{gap}(Z_1, Z_2) = \|P_{Z_1} - P_{Z_2}\|$ i.e. the difference in operator norm of the orthogonal projections onto Z_1 and Z_2 . This gives us a measure of distance between spaces which we can now apply to the question of the difference of the difference between discrete and continuous harmonic forms.

Proposition 2.0.2 (Gap between harmonic forms). *Assume that the discrete complex ?? admits a V -bounded cochain projection π_h . Then*

$$\begin{aligned} \|(I - P_{\mathfrak{H}_h^k})q\|_V &\leq \|(I - \pi_h^k)q\|_V, \forall q \in \mathfrak{H}^k \\ \|(I - P_{\mathfrak{H}^k})q_h\|_V &\leq \|(I - \pi_h^k)P_{\mathfrak{H}^k}q\|_V, \forall q \in \mathfrak{H}^k, \forall q_h \in \mathfrak{H}_h^k \end{aligned}$$

and then

$$\text{gap}(\mathfrak{H}, \mathfrak{H}_h) \leq \sup_{q \in \mathfrak{H}, \|q\|=1} \|(I - \pi_h^k)q\|_V$$

Proof. See [1, Thm. 5.2]. □

Do not forget the continuous poincare inequality

Proposition 2.0.3 (Discrete Poincare inequality). *Assume that we have a V -bounded cochain projection π_h for the discrete Hilbert complex ??. Then*

$$\|v\|_V \leq c_P \|\pi_h\|_V \|dv\|, \quad \forall v \in \mathfrak{Z}_h^{k,\perp} \cap V_h$$

with c_P being the Poincare constant from ??.

Proof. This indeed is a direct consequence of the existence of bounded cochain projections. Take $v_h \in \mathfrak{Z}_h^{k,\perp} \cap V_h$ arbitrary. Since $d(\mathfrak{Z}_h^{k,\perp} \cap V^k) = \mathfrak{B} \supseteq \mathfrak{B}_h$ we find $z \in \mathfrak{Z}_h^{k,\perp} \cap V_h$ s.t. $dz = dv$. We can apply now the continuous Poincare inequality ?? to get $\|z\|_V \leq c_P \|dz\|_V = c_P \|dv_h\|_V$. Now we can combine the different assumptions about the discrete Hilbert complex to get $v_h - \pi_h z \in V_h^k$. Now we can use the fact that π_h is a cochain map and the fact that π_h is a projection:

$$d\pi_h^k z = \pi_h^{k+1} dz = \pi_h^{k+1} dv_h = dv_h$$

For the last equality we used also the fact that we have a discrete complex i.e. $d^k V_h^k \subseteq V_h^{k+1}$. That shows that $d(v_h - \pi_h z) = 0$ i.e. $(v_h - \pi_h z) \in \mathfrak{Z}_h^k$. Because $v_h \in \mathfrak{Z}_h^{k,\perp}$ by assumption we have

$$0 = \langle v, v_h - \pi_h z \rangle = \langle v, v_h - \pi_h z \rangle + \langle dv, d(v_h - \pi_h z) \rangle = \langle v, v_h - \pi_h z \rangle_V$$

so $v_h - \pi_h z$ is V orthogonal to v_h . So

$$\|v_h\|_V^2 = \langle v_h, \pi_h^k z \rangle_V + \langle v_h, v_h - \pi_h^k z \rangle_V = \langle v_h, \pi_h^k z \rangle_V \leq \|\pi_h\|_V \|dv\| \stackrel{\text{Poincareineq.}}{\leq} c_P \|\pi_h\|_V \|dv\|_V$$

□

So we get the inf sup condition with $c_{P,h} = c_P \|\pi_h\|_V$ instead of c_P and obtain well-posedness.