**FEDERAL UNIVERSITY OF SÃO CARLOS**
CENTER OF EXACT SCIENCES AND TECHNOLOGY
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# Characteristic Classes of Topological and Generalized Manifolds

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*To my parents*
*Antonio Carlos and Edith*
*and my siblings*
*Carol and André.*

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*Either you have a strategy, or you're part of someone else's strategy.*

Alvin Toffler

# Abstract

In this work, we will initially present generalized bundles, a concept developed by Fadell with the objective of generalizing vector bundles, Stiefel-Whitney classes and Wu's formula from the context of smooth manifolds to topological manifolds. After that, we will use the generalized bundles to obtain original results of Thom, Stiefel-Whitney, Wu and Euler classes of topological manifolds, as well as present a second proof of Wu's formula for topological manifolds and the topological version of the Poincaré-Hopf theorem. Finally, we will use the Poincaré and Poincaré-Lefschetz dualities to more comprehensively construct the Stiefel-Whitney classes of generalized manifolds in order to present, for the first time in the literature, a proof of the Wu's formula for such manifolds.

Keywords: characteristic classes, generalized bundles, topological manifolds, generalized manifolds, Wu's formula.

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# List of notations

1. Saying that $f : X \to Y$ is a map means the same as $f$ being a continuous function between topological spaces.

2. $f : X \rightleftarrows Y : g$ denotes two maps when $f : X \to Y$ and $g : Y \to X$, not necessarily inverses of each other.

3. $1 : X \to X$ denotes the identity map on $X$.

4. $f^{-1}$ denotes the preimage of a map $f$, as well as its inverse map (when it exists).

5. If $f : X \to Y$ is a map, then $f(\_)$ denotes $f(x)$ for every $x \in X$.

6. If $H : X \times Y \to Z$ is a map defined on a Cartesian product, then $H(\_, y)$ denotes $H(x, y)$ for every $x \in X$. The same applies for $H(x, \_)$.

7. $p_i : X_1 \times \cdots \times X_n \to X_i$ denota a projeção no $i-$ésimo fator.

8. $d : X \to X \times X$ denota a aplicação diagonal dada por $d(x) = (x, x)$.

9. Dizer que $U \subset X$ é uma vizinhança aberta de algum subconjunto $A \subset X$ significa o mesmo que $U$ ser um subespaço aberto de $X$ que contem A.

10. Dizer que $\mathcal{U}$ é uma cobertura aberta de um espaço topológico $B$ significa o mesmo que dizer que $\mathcal{U} = \{U \subset B\}$, de modo que $U \subset B$ é um subespaço aberto de $B$ para todo $U \in \mathcal{U}$ e $\bigcup\limits_{U \in \mathcal{U}} U = B$.

11. $X \approx Y$ denota quando dois espaços topológicos são homeomorfos.

12. $f \sim g$ denota quando duas funções são homotópicas.

13. $X \sim Y$ denota quando dois espaços topológicos tem o mesmo tipo de homotopia.

14. $G_1 \cong G_2$ denota quando dois objetos algébricos são, apropriadamente, isomorfos.

15. $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ : \ \|x\| = 1\}$ .

16. $D^n = \{x \in \mathbb{R}^n \ : \ \|x\| \leq 1\}$.

17. $B^n = \{x \in \mathbb{R}^n \ : \ \|x\| < 1\}$.

18. $I = [0, 1] \subset \mathbb{R}$.

19. $X^I$ denota o espaço topológico dos caminhos em $X$.

20. $\Omega(X, x_0) = \{\omega \in X^I \ : \ \omega(0) = \omega(1) = x_0\}$

21. $H_k(X, A; R)$ e $H^k(X, A; R)$ denotam os $k-$ésimos $R-$módulos de homologia e cohomologia singular, respectivamente, do par $(X, A)$ com coeficientes em um anel $R$ comutativo e com unidade.

22. $H_k^c(X, A; R)$ e $H_c^k(X, A; R)$ denotam, respectivamente, os $k-$ésimos $R-$módulos de homologia e cohomologia singular com suporte compacto.

23. $\widetilde{H}_k(X, A; R)$ e $\widetilde{H}^k(X, A; R)$ denotam, respectivamente, os $k-$ésimos $R-$módulos de homologia e cohomologia singular reduzidos.

24. $\check{H}^k(X, A; R)$ denota o $k-$ésimo $R-$módulo de cohomologia de Čech.

25. $H^k(X, A; R) = (x)$ denota que o $k-$ésimo $R-$módulo de cohomologia do par $(X, A)$ é gerado pelo elemento $x \in H^k(X, A; R)$. O mesmo para módulos de homologia.

26. se $x \in H^k(X, A; R)$, então denotamos $|x| = k$. O mesmo para módulos de homologia.

27. $<,>$: $H^k(X, A; R) \otimes H_k(X, A; R) \to R$, que associa $\varphi \otimes \sigma \mapsto < \varphi, \sigma >$, denota o produto de Kronecker.

28. $\frown$: $H_k(X, A \cup B; R) \otimes H^l(X, A; R) \to H_{k-l}(X, B; R)$, que associa $\sigma \otimes \varphi \mapsto \sigma \frown \varphi$, denota o produto cap.

29. $\smile$: $H^k(X, A; R) \otimes H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$, que associa $\varphi \otimes \psi \mapsto \varphi \smile \psi$, denota o produto cup.

30. $\times$ : $H^k(X, A; R) \otimes H^l(Y, B; R) \to H^{k+l}(X \times Y, (X \times B) \cup (A \times Y); R)$, que associa $\varphi \otimes \psi \mapsto \varphi \times \psi$, denota o produto cross.

# Chapter 1

# Introduction