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# Characteristic Classes of Topological and Generalized Manifolds

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# Characteristic Classes of Topological and Generalized Manifolds

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*To my parents*
*Antonio Carlos and Edith*
*and my siblings*
*Carol and André.*

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*Either you have a strategy, or you're part*
*of someone else's strategy.*

Alvin Toffler

# Abstract

In this work, we will first present generalized fiber bundles, a concept developed by Fadell with the aim of generalizing vector bundles, Stiefel-Whitney classes, and Wu's formula from the context of smooth manifolds to topological manifolds. After that, we will use generalized fiber bundles to obtain original results concerning the Thom, Stiefel-Whitney, Wu, and Euler classes of topological manifolds, as well as to provide a second proof of Wu's formula for topological manifolds and to establish the topological version of the Poincaré-Hopf Theorem. Finally, we will use Poincaré and Poincaré-Lefschetz dualities to construct the Stiefel-Whitney classes of generalized manifolds in a broader manner, aiming to present, for the first time in the literature, a proof of Wu's formula for such manifolds.

Keywords: characteristic classes, generalized fiber bundles, topological manifolds, generalized manifolds, Wu's formula.

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# List of Notations

1. Saying that $f : X \to Y$ is a map means the same as saying that $f$ is a continuous function between topological spaces.
2. $f : X \rightleftarrows Y : g$ denotes two maps when $f : X \to Y$ and $g : Y \to X$, not necessarily inverses of each other.
3. $1 : X \to X$ denotes the identity map on $X$.
4. $f^{-1}$ denotes the preimage of a map $f$, as well as its inverse mapping (when it exists).
5. If $f : X \to Y$ is a map, then $f(\_)$ denotes $f(x)$ for every $x \in X$.
6. If $H : X \times Y \to Z$ is a map defined on a Cartesian product, then $H(\_, y)$ denotes $H(x, y)$ for every $x \in X$. The same holds for $H(x, \_)$.
7. $p_i : X_1 \times \cdots \times X_n \to X_i$ denotes the projection on the $i$-th factor.
8. $d : X \to X \times X$ denotes the diagonal map given by $d(x) = (x, x)$.
9. Saying that $U \subset X$ is an open neighborhood of some subset $A \subset X$ means the same as saying that $U$ is an open subspace of $X$ that contains $A$.
10. Saying that $\mathcal{U}$ is an open cover of a topological space $B$ means the same as saying that $\mathcal{U} = \{U \subset B\}$, such that $U \subset B$ is an open subspace of $B$ for every $U \in \mathcal{U}$ and $\bigcup_{U \in \mathcal{U}} U = B$.
11. $X \approx Y$ denotes when two topological spaces are homeomorphic.
12. $f \sim g$ denotes when two maps are homotopic.
13. $X \sim Y$ denotes when two topological spaces have the same type of homotopy.
14. $G_1 \cong G_2$ denotes when two algebraic objects are, appropriately, isomorphic.
15. $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \ : \ \|x\| = 1\}$.
16. $D^n = \{x \in \mathbb{R}^n \ : \ \|x\| \leq 1\}$.
17. $B^n = \{x \in \mathbb{R}^n \ : \ \|x\| < 1\}$.
18. $I = [0, 1] \subset \mathbb{R}$.

19. $X^I$ denotes the topological space of paths in $X$.

20. $\Omega(X, x_0) = \{\omega \in X^I \ : \ \omega(0) = \omega(1) = x_0\}$.

21. $H_k(X, A; R)$ and $H^k(X, A; R)$ denote the $k$-th $R$-modules of singular homology and cohomology, respectively, of the pair $(X, A)$ with coefficients in a commutative ring $R$ with unity.

22. $H_k^c(X, A; R)$ and $H_c^k(X, A; R)$ denote, respectively, the $k$-th $R$-modules of singular homology and cohomology with compact support.

23. $\widetilde{H}_k(X, A; R)$ and $\widetilde{H}^k(X, A; R)$ denote, respectively, the $k$-th $R$-modules of reduced singular homology and cohomology.

24. $\check{H}^k(X, A; R)$ denotes the $k$-th $R$-module of Čech cohomology.

25. $H^k(X, A; R) = (x)$ denotes that the $k$-th $R$-module of cohomology of the pair $(X, A)$ is generated by the element $x \in H^k(X, A; R)$. The same applies for homology modules.

26. If $x \in H^k(X, A; R)$, then we denote $|x| = k$. The same applies for homology modules.

27. $<,>$: $H^k(X, A; R) \otimes H_k(X, A; R) \to R$ denotes the Kronecker product, which maps $\varphi \otimes \sigma \mapsto < \varphi, \sigma >$.

28. $\frown$: $H_k(X, A \cup B; R) \otimes H^l(X, A; R) \to H_{k-l}(X, B; R)$ denotes the cap product, which maps $\sigma \otimes \varphi \mapsto \sigma \frown \varphi$.

29. $\smile$: $H^k(X, A; R) \otimes H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$ denotes the cup product, which maps $\varphi \otimes \psi \mapsto \varphi \smile \psi$.

30. $\times : H^k(X, A; R) \otimes H^l(Y, B; R) \to H^{k+l}(X \times Y, (X \times B) \cup (A \times Y); R)$ denotes the cross product, which maps $\varphi \otimes \psi \mapsto \varphi \times \psi$.

# Chapter 1

# Introduction

*"Between the 4th and 10th of September 1935, during the International Congress of Topology held in Moscow, several works were presented that would forever change the future of Algebraic Topology, with some of these works now considered foundational research lines in this theory. Among these works, we can mention:*

- *The introduction by Witold Hurewicz to homotopy groups.*
- *The lectures by Heinz Hopf and Hassler Whitney on vector fields and sphere bundles, which initiated the study of vector bundles and, consequently, characteristic classes.*
- *The independent introductions by James Alexander and Andrei Kolmogorov to cohomology theory, as well as the cup product."*

In this work, we will contribute to the theory of characteristic classes, more specifically, characteristic classes of topological and generalized manifolds.

After this historical context on the emergence of characteristic class theory, we will begin introducing the basic concepts used for the development of this work.

In 1955, Nash introduced in [14] the concept that would become known as the field of non-singular paths of a topological manifold, which can be understood as the topological version of a non-zero vector field. Essentially, Nash showed that given a smooth manifold $M$ and fixing a point $b \in M$, the space of non-zero tangent vectors of $M$ at $b$ can also be defined from the topological viewpoint, up to a homotopy equivalence, as the set:

$$\{\omega \in M^I : \omega(t) = b \Leftrightarrow t = 0\}$$

A decade later, in 1965, Fadell defined in [7] generalized fiber bundles, a concept that not only generalized vector bundles, but also allowed the extension, through Nash's ideas in [14], of the notions of tangent and normal fiber bundles from the context of smooth manifolds to topological manifolds. Furthermore, Fadell constructed the Stiefel-Whitney classes of generalized fiber bundles in order to obtain Whitney's duality for specific topological embeddings and to prove the Wu formula for topological manifolds.

The theory developed by Fadell in [7] will serve as the foundation for the development of our entire work, which can be divided into two parts:

- The first part will consist of chapters 2, 3, and 4. These chapters can be interpreted as a modern re-reading of the results obtained by Fadell in [7], as well as a continuation of the same, since we will present additional results both on generalized fiber bundles themselves and on Thom, Stiefel-Whitney, Euler, and Wu classes of topological manifolds.
- The second part of this work will consist solely of chapter 5, in which we will construct more extensively the Stiefel-Whitney classes of generalized manifolds in order to present for the first time in the literature a proof of the Wu formula for such manifolds.

Now, we will look in more detail at how we will organize the structure of our work, pointing out our contributions and the relevance of the results that will be presented here.

In chapter 2, we will begin our work by presenting the studies conducted on generalized fiber bundles, a tool developed by Fadell in [7], which not only generalized the concepts of tangent and normal vector bundles from the context of smooth manifolds to topological manifolds, but also allowed him to define the Stiefel-Whitney classes and prove Whitney's duality and the Wu formula for the context of topological manifolds.

Concatenating definitions ??, ??, and ??, we can define a generalized fiber bundle more directly as follows:

**Definition.** *Given $E$ and $B$ topological spaces, $E_0 \subset E$ and $p : E \to B$ a onto map, we call the pair $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ an $\mathbb{R}^n-$generalized fiber bundle when:*

1. *For any maps $h : X \to E$ and $H : X \times I \to B$, such that $H(\_, 0) = p \circ h$, there exists a map $\widetilde{H} : X \times I \to E$ such that $\widetilde{H}(\_, 0) = h$ and $p \circ \widetilde{H} = H$.*
2. *If $x_0 \in X$ is such that $h(x_0) \in E_0$, then $\widetilde{H}(x_0, \_) \in E_0$.*
3. *There exists a map $s : B \to E$ such that $E_0 = E - s(B)$.*
4. *For all $b \in B$, $(p^{-1}(b), p^{-1}(b) \bigcap E_0) \sim (\mathbb{R}^n, \mathbb{R}^n - \{0\})$.*

With this definition, we can interpret a generalized fiber bundle as a fibration with the following characteristics:

- The total space is a pair of topological spaces.
- There is always at least one global section.
- The fiber behaves, up to homotopy equivalence, like a Euclidean space.

During the reading of Chapter 2, the reader will notice that the development of the chapter will not be as straightforward compared to the definition above, since our main goal will be to present the theory of generalized fiber bundles in a more detailed way and using a more modern language than the results presented by Fadell in the first half of [7].

More explicitly, we will show in Example ?? how generalized fiber bundles indeed generalize vector bundles, and in Proposition ?? how the notion of isomorphism between vector bundles remains valid when extended to the category of generalized fiber bundles. We will also show that it is possible to construct new generalized fiber bundles from others, just as it happens with vector bundles, for example: restriction bundles, product bundles, and Whitney sum bundles.

Although Chapter 2 is a preliminary chapter, we will contribute with original results concerning the pullback generalized fiber bundle, which was developed by Brown in [5] but was neither cited nor used by Fadell in [7]. These results will prove to be quite relevant when we use them in the construction of some maps regarding characteristic classes of topological manifolds in Chapters 3 and 4.

In Chapter 3, we will address the topic of characteristic classes of generalized bundles and topological manifolds, more specifically, Thom classes, Stiefel-Whitney classes, and Euler classes. Initially, we will introduce the notion of $R$-orientability of generalized fiber bundles, where $R$ is a commutative ring with unity, and their respective Thom classes, concepts originally proposed by Fadell in [7], but little explored by him, since the main topic developed in the second half of [7] was about Stiefel-Whitney classes, in which case orientability is not a concern.

Thus, we will detail a little more the definition of $R$-orientability of generalized fiber bundles and present some technical results on the behavior of Thom classes under pullback and product generalized bundles, as well as show what happens when we reverse the orientability of a generalized fiber bundle and the relation between the dimension of a $\mathbb{Z}$-orientable topological manifold and the Thom class of its tangent generalized fiber bundle. Even though these results are already known in the context of vector bundles and smooth manifolds, they can be considered original since they have not yet been described in the context of generalized fiber bundles and topological manifolds.

The second topic we will address in Chapter 3 will be about Stiefel-Whitney classes. The purpose of this topic will be to rewrite the main properties and consequences of these classes, already widespread in the literature, for the context of generalized fiber bundles and topological manifolds, following the same steps used by Milnor in ([13], Chapter 8) for vector bundles and smooth manifolds. In doing so, we will offer a broader, more modern, and detailed reinterpretation of the results proposed by Fadell in the second half of [7]. Our contributions to this topic will involve results concerning pullback generalized fiber bundles.

The third and last topic addressed in Chapter 3 will be about Euler classes. Differently from Stiefel-Whitney classes, Euler classes can only be defined for $\mathbb{Z}$-orientable generalized fiber bundles. Thus, due to the technical lemmas related to Thom classes of $\mathbb{Z}$-orientable generalized fiber bundles obtained at the beginning of Chapter 3, we will be able to conclude several consequences and applications concerning Euler classes of generalized fiber bundles and $\mathbb{Z}$-orientable topological manifolds. In this topic, except for Proposition **??**, all other results will be original, being generalizations of known results about Euler classes for vector bundles and smooth manifolds. Among these generalizations, we highlight:

**Proposition 3.7.** *Let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ be an $\mathbb{R}^n$-generalized fiber bundle that is $\mathbb{Z}$-orientable. If $(\mathcal{F}, \mathcal{F}_0)$ admits a section $s : B \to E$ such that $s(B) \subset E_0$, then*

$e(\mathcal{F}, \mathcal{F}_0) = 0$.[1]

The proposition above, in its version for vector bundles, is widely known, as it allows interpreting the Euler class of a vector bundle as an obstruction to the existence of a nowhere-vanishing section. In this work, we will present the generalized version of this interpretation, which will allow us to obtain the main application related to the Euler class in Chapter 4, the topological version of the Poincaré-Hopf theorem.

Up to this point, the reader should already have noticed the main goal of Chapters 2 and 3 of our work, which is to structure in detail and using a more up-to-date language the theory of generalized fiber bundles and their characteristic classes, while also presenting several technical contributions, aiming to generalize applications regarding characteristic classes of smooth manifolds to the context of topological manifolds, as we will see next.

The conclusion of our work regarding characteristic classes of generalized fiber bundles will be presented in Chapter 4, where we will present three major applications with original technical proofs concerning Stiefel-Whitney, Euler, and Wu classes of closed topological manifolds. Initially, we will present an alternative proof of the topological version of the famous Wu formula, which relates the Stiefel-Whitney and Wu classes of a smooth manifold through Steenrod squares.

In [7], Fadell uses generalized fiber bundles to give a first proof of Wu's formula for topological manifolds, based on the techniques used by Milnor in ([12], Chapter 9). Furthermore, the preliminary results that Fadell develops to prove Wu's formula are all in the framework of singular (co)homology $\mathbb{Z}_2$-modules. Meanwhile, the alternative proof of Wu's formula for topological manifolds that we will present in Chapter 4 will be based on different techniques also introduced by Milnor, now found in ([13], Chapter 11).

Comparing the proofs presented by us in this work and by Fadell in [7], the main differences will be found in the preliminary lemmas used in Wu's formula, as we will prove them in the framework of singular (co)homology $R$-modules with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$. Since we will use the same sequence of results employed by Milnor in [13], now using generalized fiber bundles instead of vector bundles, our main contribution will be obtaining the case $R = \mathbb{Z}$ of the following result:

**Lemma 4.1.** *Let $M^m$ be a closed, connected, $R$-orientable topological manifold with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, $b \in M$ arbitrary, $j_b : (M, M - \{b\}) \hookrightarrow (M \times M, (M \times M) - \Delta)$ the canonical inclusion, $[M]_b \in H_m(M, M - \{b\}; R)$ the local $R$-orientation class of $M$ at $b$, and $(\tau') \in H^m(M \times M, (M \times M) - \Delta; R)$ the generator uniquely defined by the Thom class of the tangent generalized fiber bundle of $M$. Then:*

$$< j_b^*(\tau'), [M]_b >= 1 \in R$$

The proof of the lemma above, in its version for smooth manifolds, can be found in ([13], Lemma 11.7, p. 123), where the Riemannian structure of the manifold and the existence of the exponential map are used, whereas our proof will be entirely algebraic, allowing generalization to the context of topological manifolds, which will be crucial for the topological applications of the Euler class.

[1] $e(\mathcal{F}, \mathcal{F}_0)$ will denote the Euler class of the generalized fiber bundle $(\mathcal{F}, \mathcal{F}_0)$.

The second application of Chapter 4 will concern Euler classes. In fact, we will present two applications on this topic, one being the relation between the Euler class and the Euler characteristic of a topological manifold, and the other being the topological version of the Poincaré-Hopf theorem. The reader will notice the importance of the $R = \mathbb{Z}$ case of Lemma 4.1 for the first application, whose statement is as follows:

**Theorem 4.2.** *If $M$ is a closed, connected, and $\mathbb{Z}$-orientable topological manifold, then*[2]*:*

$$< e(M), [M] >= \chi(M >$$

For the second map involving the Euler class, we will need to define the concept of a path field on a topological manifold, which was introduced by Nash in [14] as follows:

**Definition 4.1.** *A path field on a topological manifold $M$ is any section of its generalized fiber bundle $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$. Moreover, a nonsingular path field on $M$ is a section $s : M \to TM$ such that $s(M) \subset T_0 M$.*

As we will show in Chapter 4, generalized fiber bundles will allow us to generalize the notion of nowhere-vanishing vector fields from the smooth manifold context to the topological manifold setting, since a smooth manifold admits a nowhere-vanishing vector field if and only if it admits a nonsingular path field. With that, we will be able to prove the topological version of the Poincaré-Hopf theorem, whose statement is:

**Theorem 4.3.** *Let $M$ be a closed, connected, and $\mathbb{Z}$-orientable topological manifold. If $M$ admits a nonsingular path field, then $\chi(M) = 0$.*

This result was first presented by Brown in [5], using essentially Lefschetz numbers in his proof. In our work, we will present an alternative proof of this result using the Euler class.

As the final map in Chapter 4, we will see how some technical results about generalized bundles will allow us to prove the following:

**Theorem 4.4.** *If $i : M^m \hookrightarrow S^{m+k}$ is a locally flat embedding*[3] *between closed, connected topological manifolds with trivial normal generalized fiber bundle, then:*[4]

$$v(M) = i^*(v(S))$$

At first glance, the theorem above seems quite clear and straightforward, since if we replace the total Wu classes with the total Stiefel-Whitney classes, this result becomes an immediate consequence of Whitney duality. However, upon closer examination of the proof of Theorem 4.4 in its version for vector bundles and smooth manifolds, as given

[2] $e(M)$, $[M]$, and $\chi(M)$ will denote, respectively, the Euler class, the global orientation class, and the Euler characteristic of the manifold $M$.

[3] A locally flat embedding is a topological embedding that locally behaves like a smooth embedding, whose formal definition can be found in Definition 2.14.

[4] $v(M)$ and $v(S)$ will denote, respectively, the total Wu classes of $M$ and $S$.

by Stong in [17] and presented in more detail in [15], it becomes evident that the proof makes direct use of the existence of a tubular neighborhood for smooth embeddings.

Since we cannot guarantee the existence of a tubular neighborhood in the topological context, our main contribution was to circumvent this problem using only results about generalized fiber bundles, showing that the existence of a tubular neighborhood is not essential, but rather certain algebraic consequences of a locally flat embedding.

In the last chapter of our work, Chapter 5, we will present for the first time in the literature a proof of Wu's formula in the context of generalized manifolds, using their Poincaré and Poincaré-Lefschetz dualities. To this end, we will begin the chapter with a brief summary, based on [2], [11], and [4], about the concept of generalized manifolds. More explicitly, the constructions in this chapter will be carried out for $\mathbb{Z}_2$-homological ENR-manifolds, which are particular generalized manifolds. For convenience, we will continue to refer to these spaces simply as generalized manifolds.

In this initial summary, we will see that generalized manifolds are essentially topological spaces that behave like topological manifolds in the realm of singular (co)homology $\mathbb{Z}_2$-modules. In particular, we will be able to construct the Wu classes for such manifolds, as well as their Poincaré and Poincaré-Lefschetz dualities.

After establishing these objects, we will associate to each embedding $s : M^m \to N^{2m}$ between compact, connected generalized manifolds, such that there exists a retraction[5] $p : N \to M$, its transfer isomorphism given by the following composition of the Poincaré-Lefschetz duality of the embedding $s$ with the Poincaré duality of the manifold $M$:

$$s_! : H_k(N, N - M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}^{-1}_{N,M}} H^{2m-k}(M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}_M} H_{k-m}(M; \mathbb{Z}_2)$$

Thus, the transfer isomorphism associated to the embedding $s$ will allow us to define the Thom class also associated to the embedding $s$ as the generator:

$$(\tau_s) = H^m(N, N - M; \mathbb{Z}_2)$$

Inspired by the techniques presented by Dold in ([6], Chapter 8), we will demonstrate that the homomorphism $\phi_s : H^k(M; \mathbb{Z}_2) \to H^{k+m}(N, N - M; \mathbb{Z}_2)$ given by $\phi_s(x) = p^*(x) \smile \tau_s$ is, in fact, the dualization (via Universal Coefficients) of the transfer isomorphism $s_!$.

Having done this, we will call $\phi_s$ the Thom isomorphism associated to the embedding $s$ and define the $k$-th Stiefel-Whitney class associated to the embedding $s$ as:

$$w_k(s) = \phi_s^{-1} \circ Sq^k(\tau_s) \in H^k(M; \mathbb{Z}_2)$$

In particular, we will define the $k$-th Stiefel-Whitney class of a generalized manifold $M$ as the $k$-th Stiefel-Whitney class associated to the embedding given by the diagonal map $d : M \to M \times M$. Moreover, to ensure that this definition is indeed well-defined, we will use some results about generalized fiber bundles presented in Chapter 4 to show

[5] That is, $p \circ s = 1$.

in Theorem 5.5 that, in the context of topological manifolds, the definition of Stiefel-Whitney classes via generalized fiber bundles coincides with the definition we propose via the Stiefel-Whitney classes associated to the embedding given by the diagonal map.

Finally, motivated by the techniques presented by Bredon in ([3], Chapter 6), we will conclude Chapter 5, and consequently our work, by showing that it is possible to obtain Wu's formula for generalized manifolds using our definition of Stiefel-Whitney classes associated to the embedding given by the diagonal map of a generalized manifold.

Since Biasi, Daccach, and Saeki defined in [2] the Stiefel-Whitney classes of generalized manifolds as Wu's formula itself and presented several results in this context, we highlight the originality of Chapter 5 where we define the Stiefel-Whitney classes for generalized manifolds in an alternative way and prove Wu's formula for such manifolds.

We will conclude the introduction chapter of our work with the words of Massey, which can be found in ([10], Chapter 21), providing additional historical context for the emergence of characteristic classes:

> *"At the 1935 conference in Moscow, Hopf presented the work of one of his students, Stiefel, whose publication appeared only in the following year. In this work, Stiefel defined certain homology classes of a smooth manifold that, in modern language, are the Poincaré-dual classes of the Stiefel-Whitney classes of the tangent vector bundle. His method consisted of constructing, through a very geometric process, the cycles that represented these homology classes."*

> *"Whitney gave a lecture at the Moscow conference entitled 'Sphere spaces,' which we now call sphere bundles. These two lectures, and the subsequent papers, marked the beginning of work on the general topic of vector bundles. The most important invariants of vector bundles are generally various characteristic classes, but always cohomology classes."*

William S. Massey

# Chapter 2

# Bundles

We will begin this work by presenting the so-called generalized fiber bundles, a tool developed by Fadell in [7] with the purpose of defining the Stiefel-Whitney classes and proving Whitney's duality and Wu's formula in the context of topological manifolds.

At first, in Section ??, we will review specific concepts about vector bundles in order to fix notation and clarify to the reader how vector bundles will be naturally generalized throughout this chapter.

After that, Section ?? will serve as an intermediate step for defining generalized bundles and for presenting the results that will be shown in Section ?? in a clearer and more succinct way.

Finally, in Section ??, we will find the definition and properties involving generalized fiber bundles, almost all of which are taken from [7].

As will be explained in Observation ??, every topological manifold mentioned throughout this work will be assumed to be a manifold without boundary.

# Chapter 3

# Characteristic Classes of Topological Manifolds

# Chapter 4

# Applications in Closed Topological Manifolds

# Chapter 5

# Characteristic Classes of Generalized Manifolds

# Appendix A

# Singular (Co)homology

In order to keep this work concise, yet complete and self-explanatory, we will use this appendix as a brief review of some well-known concepts from Algebraic Topology.

When referring simultaneously to the singular homology and cohomology modules, for convenience, we will simply write singular (co)homology modules. Thus, we ask the reader to already be familiar with the concepts of these theories.

## A.1 Main results

We will use this section to state general results on singular (co)homology that will be useful for the development of this work and for a better understanding of the constructions made in the following sections of this appendix.

**Theorem A.1. (Universal Coefficients)** *Let $(X, A)$ be any pair of topological spaces. Then:*

1. ***(general case for homology)***[1] *If $H_k(X, A; \mathbb{Z})$ is a free module*[2] *for all $k \geq 0$ or $R$ is a free module, then:*

$$H_k(X, A; R) \cong H_k(X, A; \mathbb{Z}) \otimes R, \ \forall k \geq 0$$

2. ***(general case for cohomology)***[3] *If $H_k(X, A; \mathbb{Z})$ is a free module for all $k \geq 0$, then:*

$$H^k(X, A; R) \cong Hom(H_k(X, A; \mathbb{Z}); R), \ \forall k \geq 0$$

3. ***(particular case)***[4] *If $\mathbb{F}$ is a field, then the Kronecker product ensures that:*

$$H^k(X, A; \mathbb{F}) \cong Hom(H_k(X, A; \mathbb{F}); \mathbb{F}), \ \forall k \geq 0$$

*For $R = \mathbb{Z}$ in case 2 or $R = \mathbb{F}$ in case 3, the isomorphism is given by the relation $x \in H^k(X, A; R) \mapsto \overline{x}(a) =< x, a >\in R$.*

[1] Can be found in ([8], Corollary 3A.4, p. 264).
[2] A module is called free if it admits a basis.
[3] Can be found in ([8], Theorem 3.2, p. 195).
[4] Can be found in ([8], p. 198).

**Theorem A.2. (Künneth Formula)** *Let $X$ and $Y$ be any topological spaces and $R$ a finitely generated principal ideal domain. Then:*

1. ***(case for absolute cohomology)***[5] *If all the singular homology $R$-modules of $Y$ are finitely generated, then:*

$$H^k(X \times Y; R) \cong \bigoplus_{i+j=k} \left[H^i(X;R) \otimes H^j(Y;R)\right],\ \forall k \geq 0$$

2. ***(case for absolute homology)***[6]

$$H_k(X \times Y; R) \cong \bigoplus_{i+j=k} \left[H_i(X;R) \otimes H_j(Y;R)\right],\ \forall k \geq 0$$

The proofs of the general cases of the Universal Coefficients Theorem can be found in ([16], Chapter 5, Sections 2 and 5).

The proofs of the Künneth formulas, in their general versions for pairs, can be found in ([16], Chapter 5, Sections 3 and 5).

Now, let us briefly review some properties about the cap, cup, cross, and Kronecker products.

**Lemma A.1.** *Let $X$, $X'$, $Y$, and $Y'$ be arbitrary topological spaces, $f : X \to X'$ and $g : Y \to Y'$ any maps, $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the canonical projections, and $d : X \to X \times X$ the diagonal map. If $a \in H_q(X;R)$, $b \in H_r(Y;R)$, $x \in H^i(X;R)$, $x_1 \in H^{i_1}(X;R)$, $x_2 \in H^{i_2}(X;R)$, $y \in H^j(Y;R)$, $y_1 \in H^{j_1}(Y;R)$, $y_2 \in H^{j_2}(Y;R)$, $a' \in H_{q'}(X;R)$, $x' \in H^{i'}(X';R)$, $x'_1 \in H^{i'_1}(X';R)$, $x'_2 \in H^{i'_2}(X';R)$, $y' \in H^{j'}(Y';R)$, then:*

1. $1 \smile x = x = x \smile 1$
2. $0 \smile x = 0 = x \smile 0$
3. $x_1 \smile x_2 = 0 \iff x_1 = 0$ *or* $x_2 = 0$, *when* $R = \mathbb{Z}_2$
4. $a \frown 1 = a$
5. $x_1 \smile x_2 = (-1)^{|x_1|.|x_2|}(x_2 \smile x_1)$
6. $(a \frown x_1) \frown x_2 = a \frown (x_2 \smile x_1)$
7. $(x_1 \times y_1) \smile (x_2 \times y_2) = (-1)^{|y_1|.|x_2|}(x_1 \smile x_2) \times (y_1 \smile y_2)$
8. $(a \times b) \frown (x \times y) = (-1)^{|a|.(|y|-|b|)}(a \frown x) \times (b \frown y)$
9. $< x_1 \smile x_2, a >=< x_1, a \frown x_2 >$
10. $< x \times y, a \times b >= (-1)^{|x|.|y|} < x, a > . < y, b >$
11. $p_1^*(x) = x \times 1$ *and* $p_2^*(y) = 1 \times y$

[5] Can be found in ([16], Theorem 1, p. 249).
[6] Can be found in ([16], Theorem 10, p. 235).

12. $x \times y = p_1^*(x) \smile p_2^*(y)$

13. $x_1 \smile x_2 = d^*(x_1 \times x_2)$

14. $f_*(a \frown f^*(x')) = f_*(a) \frown x'$

15. $(f \times g)^*(x' \times y') = f^*(x') \times g^*(y')$

16. $f^*(x'_1 \smile x'_2) = f^*(x'_1) \smile f^*(x'_2)$

17. $< f^*(x'), a >=< x', f_*(a) >$

18. $< (f^*)^{-1}(x), a' >=< x, (f_*)^{-1}(a') >$, *if* $f^*$ *and* $f_*$ *are isomorphisms.*

The properties in the lemma above can be found, in their general versions for pairs, in ([16], Chapter 5).

Moreover, item 16 of Lemma A.1 admits a particular case when involving the inclusion map, in the following sense:

**Lemma A.2.** *Let* $(X, A)$ *and* $j : X \hookrightarrow (X, A)$ *be an arbitrary pair of topological spaces and the canonical inclusion, respectively. Then, for any* $x_1 \in H^{i_1}(X; R)$ *and* $x_2 \in H^{i_2}(X, A; R)$*, we have:*

$$j^*(x_1 \smile x_2) = x_1 \smile j^*(x_2)$$

The proof of the lemma above can also be found in ([16], Chapter 5).

That said, item 14 of Lemma A.1 admits the following particular case:

**Lemma A.3.** *Let* $(X, A)$ *be any pair of topological spaces and let* $j : X \hookrightarrow (X, A)$ *be the canonical inclusion. Then, for any* $x \in H^{i_1}(X, A; R)$ *and* $a \in H_{i_2}(X; R)$*, we have:*

$$j_*(a) \frown x = a \frown j^*(x)$$

*Proof.*

It suffices to observe that, for any $y \in H^{i_2 - i_1}(X; R)$, we have:

$$\begin{aligned} \langle y, j_*(a) \frown x \rangle &= \langle y \smile x, j^*(a) \rangle \\ &= \langle j^*(y \smile x), a \rangle \\ &= \langle y \smile j^*(x), a \rangle \\ &= \langle y, a \frown j^*(x) \rangle \end{aligned}$$

Thus, the Universal Coefficient Theorem ensures that $j_*(a) \frown x = a \frown j^*(x)$. □

Now, also due to the Universal Coefficient Theorem, the following result is a direct consequence of ([9], Theorem 4.11, p. 204):

**Proposition A.1.** *Let* $R = \mathbb{Z}$ *or* $R = \mathbb{F}$ *be any field, and let* $(X, A)$ *be any pair of topological spaces such that* $H_k(X, A; R)$ *is a finitely generated* $R$*-module for all* $k \geq 0$*. Then, for any* $k \geq 0$*,* $\alpha \in R$ *and* $a \in H_k(X, A; R)$ *with* $a \neq 0$*, there exists a unique* $x \in H^k(X, A; R)$ *such that* $x \neq 0$ *and* $\langle x, a \rangle = \alpha$*.*

Proceeding, we will construct the cohomology ring of a pair $(X, A)$.

**Definition A.1. (Cohomology Ring)** *We call $(H^*(X, A; R), +, \smile)$ the cohomology ring of the pair $(X, A)$ with coefficients in $R$, the set formed by the following formal infinite series:*

$$H^*(X, A; R) = \{x = x_0 + x_1 + x_2 + \dots \ : \ x_k \in H^k(X, A; R), \ \forall k \geq 0\}$$

*Furthermore, given $x = x_0+x_1+x_2+\dots$ and $y = y_0+y_1+y_2+\dots$ in $H^*(X, A; R)$, the operations that define this ring are given by:*

1. $x + y = z_0 + z_1 + z_2 + \dots$, *where* $z_k = x_k + y_k$ *for all* $k \geq 0$
2. $x \smile y = z_0 + z_1 + z_2 + \dots$, *where* $z_k = \sum_{i+j=k} x_i \smile y_j$ *for all* $k \geq 0$

Since the cup product is a commutative operation when $R = \mathbb{Z}_2$, then $H^*(X, A; \mathbb{Z}_2)$ will be a commutative ring with identity element $1 + 0 + 0 + \cdots \in H^*(X, A; \mathbb{Z}_2)$.

**Theorem A.3.** *The units of the ring $H^*(X, A; \mathbb{Z}_2)$ are elements of the following form:*

$$x = x_0 + x_1 + x_2 + \cdots \in H^*(X, A; \mathbb{Z}_2) \ : \ x_0 = 1$$

*Furthermore, the inverse of a unit $x = 1 + x_1 + x_2 + \dots$ is the following element:*

$$x^{-1} = 1 + x_1^{-1} + x_2^{-1} + \dots, \quad \text{where} \quad x_k^{-1} = \sum_{\substack{i+j=k \\ i \neq 0}} x_i \smile x_j^{-1}, \ \forall k \geq 1$$

The proof of the theorem above can be found in ([1], Lemma 6.1, p. 53).

At this point, let us see when a topological space has all its singular (co)homology modules finitely generated and under which conditions we can define the Euler characteristic of an arbitrary topological space.

**Proposition A.2.** *All singular homology modules of a compact ENR space*[7] *are finitely generated.*

The proof of the proposition above can be found in ([8], Corollary A.8, p. 527). As a particular consequence, all singular homology modules of a compact topological manifold are free, since every topological manifold is an ENR and every finitely generated module is free. Furthermore, due to the Universal Coefficient Theorem, every singular cohomology module of a compact topological manifold is free.

**Definition A.2. (Euler Characteristic)** *Let $X$ be a topological space such that there exists an integer $n > 0$ such that $H_k(X; \mathbb{Z}) = 0$ for $k > n$ and $H_k(X; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module for every $0 \leq k \leq n$. Thus, the Euler characteristic of $X$ is given by the following alternating sum:*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \operatorname{rank}(H_k(X; \mathbb{Z}))$$

[7] An ENR is a topological space that is a retract of an open neighborhood in some Euclidean space, that is, it can be embedded in some Euclidean space as a retract of an open neighborhood of that Euclidean space. More details about these spaces can be found in ([6], Chapter 4, Section 8).

Due to the Universal Coefficient Theorem, the Euler characteristic of a space $X$ under the conditions of the definition above can be computed using singular homology modules with coefficients in any field $\mathbb{F}$ as follows:

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim(H_k(X;\mathbb{F}))$$

To conclude this section, let us make some considerations about the infinite real projective space $\mathbb{R}P^\infty$, which will be useful for calculating the Stiefel–Whitney classes of real projective spaces using the topological version of Wu's formula.

Denoting by $\mathbb{R}P^k$ the $k$-dimensional real projective space, we can define the infinite real projective space $\mathbb{R}P^\infty$ as the direct limit of the following sequence:

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \cdots \subset \mathbb{R}P^k \subset \cdots$$

In other words, we have that $\mathbb{R}P^\infty = \bigcup_{k\geq 0} \mathbb{R}P^k$, endowed with the following topology:

"$U$ is an open subset of $\mathbb{R}P^\infty$ if and only if $U \cap \mathbb{R}P^k$ is an open subset of $\mathbb{R}P^k$ for every $k \geq 0$."

Thus, we can state the following:

**Lemma A.4.** *The canonical inclusion $i : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ gives rise to an isomorphism $i^* : H^k(\mathbb{R}P^\infty;\mathbb{Z}_2) \to H^k(\mathbb{R}P^n;\mathbb{Z}_2)$ for every $0 \leq k \leq n$.*

*Proof.* First, recall that $\mathbb{R}P^{n+1}$ is a CW-complex with one open cell in each dimension, with $\mathbb{R}P^k$ being its $k$-skeleton.

Now, consider $e_{n+1}$ as the open $(n+1)$-dimensional cell of $\mathbb{R}P^{n+1}$ and fix $x \in e_{n+1}$. Then, $\mathbb{R}P^{n+1} - \{x\}$ is a deformation retract of $\mathbb{R}P^n$.

On the other hand, considering, up to homeomorphism, $x \in D^{n+1} \subset e_{n+1}$, then $U = \mathbb{R}P^{n+1} - D^{n+1}$ will be an open subspace of $\mathbb{R}P^{n+1}$ such that:

$$\overline{U} \subset \operatorname{int}(\mathbb{R}P^{n+1} - \{x\}) = \mathbb{R}P^{n+1} - \{x\}.$$

Thus, $\mathbb{S}^n = \partial D^{n+1}$ is a deformation retract of $(\mathbb{R}P^{n+1} - \{x\}) - U = D^{n+1} - \{x\}$, and we also obtain the following excision:

$$(\mathbb{R}P^{n+1} - U, (\mathbb{R}P^{n+1} - \{x\}) - U) \hookrightarrow (\mathbb{R}P^{n+1}, \mathbb{R}P^{n+1} - \{x\}).$$

Therefore, we have the following isomorphisms:

$$\begin{aligned}
H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n;\mathbb{Z}_2) &\cong H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^{n+1} - \{x\};\mathbb{Z}_2) \\
&\cong H^k(\mathbb{R}P^{n+1} - U, (\mathbb{R}P^{n+1} - \{x\}) - U;\mathbb{Z}_2) \\
&\cong H^k(D^{n+1}, \mathbb{S}^n;\mathbb{Z}_2) \\
&\cong \begin{cases} \mathbb{Z}_2 & , \quad k = n+1 \\ 0 & , \quad k \neq n+1 \end{cases}
\end{aligned}$$

In particular, $H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for $0 \leq k \leq n$. From now on in this proof, fix $0 \leq k \leq n$.

Now, we prove by induction on $t \geq 2$ that $H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$.

To do so, consider initially the long exact cohomology sequence of the triple[8] $(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}, \mathbb{R}P^n)$:

$$\cdots \to H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}; \mathbb{Z}_2) \to H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^n; \mathbb{Z}_2) \to H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2) \to \cdots$$

Since $H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}; \mathbb{Z}_2) = 0 = H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2)$, the exactness of the sequence above ensures that $H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$.

Thus, assuming $H^k(\mathbb{R}P^{n+t_0}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for some $t_0 > 2$, we similarly obtain that $H^k(\mathbb{R}P^{n+(t_0+1)}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$, simply by using the long exact cohomology sequence of the triple $(\mathbb{R}P^{n+(t_0+1)}, \mathbb{R}P^{n+t_0}, \mathbb{R}P^n)$.

Therefore, we conclude that $H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for any $t \geq 0$. Consequently:

$$H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) = \varinjlim H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0.$$

Finally, consider the long exact cohomology sequence of the pair $(\mathbb{R}P^\infty, \mathbb{R}P^n)$:

$$\cdots \longrightarrow H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \xrightarrow{i^*} H^k(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow \cdots$$

Since $H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ and $H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H^k(\mathbb{R}P^n; \mathbb{Z}_2)$, then $i^*$ is a monomorphism between modules of the same dimension, that is, $i^*$ is an isomorphism.

□

## A.2 Slant Product

In this section, we will define a specific product between singular (co)homology modules that will be fundamental in the proof of the Wu formula for topological and homological manifolds.

This product, which we will later call the slant product, is defined for arbitrary topological spaces using singular (co)homology modules with coefficients in an arbitrary commutative unital ring, and it is also used in the proof of the Wu formula for smooth manifolds, as seen in ([13], Chapter 11).

For our context, let $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, $X$ and $Y$ be arbitrary topological spaces with $H_k(Y; R)$ finitely generated for all $k \geq 0$, and integers $i, j \geq 0$. Thus, define the following homomorphism involving $R$-modules of singular (co)homology:

$$H^i(X; R) \otimes H^j(Y; R) \otimes H_j(Y; R) \to H^i(X; R)$$

$$x \otimes y \otimes b \mapsto < y, b > x$$

Now, the Künneth formula ensures that $H^*(X \times Y; R) \cong H^*(X; R) \otimes H^*(Y; R)$, and since $H^*(X \times Y; R)$ is a ring generated by elements of the form $x \times y$, the following homomorphism is well defined:

[8]For more details about the long exact cohomology sequence of a triple, see ([8], p. 200).

$$H^{i+j}(X \times Y; R) \otimes H_j(Y; R) \to H^i(X; R)$$
$$(x \times y) \otimes b \mapsto (x \times y)/b = < y, b > x$$

Thus, we have the following:

**Definition A.3. (Slant Product)** *The slant product refers to the homomorphism $H^{i+j}(X \times Y; R) \otimes H_j(Y; R) \to H^i(X; R)$ given by $z \otimes b \mapsto z/b$.*

We will conclude this section with two particular properties of the slant product. For more details about this operation in its most general form, we suggest the reader see ([16], Chapter 6, Section 1).

**Lemma A.5.** *Let $x \times 1 \in H^i(X \times Y; R)$, $z \in H^{i+j}(X \times Y; R)$, $a \in H_{i'}(X; R)$, $b \in H_j(Y; R)$, and $p_1 : X \times Y \to X$ be the canonical projection. Then, we have the following relations:*

1. $[(x \times 1) \smile z]/b = x \smile (z/b)$
2. $(p_1)_*((a \times b) \frown z) = a \frown (z/b)$

## A.3 Steenrod Squares

The Steenrod squares are cohomological operations of great importance for the development of this work, as they are essential for defining the Stiefel-Whitney classes of vector bundles, generalized, and homological manifolds. Furthermore, the Wu formula relates, through these Steenrod squares, the Stiefel-Whitney and Wu classes of a smooth, topological, and homological variety.

Here, we will state only the basic properties of these operations. For more details on Steenrod squares, we refer the reader to ([16], Chapter 5, Section 9).

Given $(X, A)$ a pair of topological spaces and integers $m, k \geq 0$, the Steenrod squares are additive cohomological operations $Sq^k : H^m(X, A; \mathbb{Z}_2) \to H^{m+k}(X, A; \mathbb{Z}_2)$ satisfying the following properties:

1. If $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(X, A; \mathbb{Z}_2)$, then the Cartan formula holds, i.e.,

$$Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y)$$

2. If $x \in H^m(X, A; \mathbb{Z}_2)$, then:
   (a) $Sq^0(x) = x$
   (b) $Sq^m(x) = x \smile x$
   (c) $Sq^k(x) = 0$, for $k > m$

3. If $f : (X, A) \to (Y, B)$ is a map of pairs, then $Sq^k \circ f^* = f^* \circ Sq^k$, i.e., the following diagram commutes:

$$\begin{tikzcd} H^m(Y, B; \mathbb{Z}_2) \arrow[r, "f^*"] \arrow[d, "Sq^k"'] & H^m(X, A; \mathbb{Z}_2) \arrow[d, "Sq^k"'] \\ H^{m+k}(Y, B; \mathbb{Z}_2) \arrow[r, "f^*"] & H^{m+k}(X, A; \mathbb{Z}_2) \end{tikzcd}$$

Furthermore, if $f : (X, A) \to (Y, B)$ is a map that induces an isomorphism in the context of $\mathbb{Z}_2$-modules of singular cohomology, then it follows from the property above that:

$$Sq^k \circ (f^*)^{-1} = (f^*)^{-1} \circ Sq^k$$

On the other hand, given $x \in H^m(X, A; \mathbb{Z}_2)$, we can define the total square operation as follows:

$$Sq(x) = x + Sq^1(x) + Sq^2(x) + ... + Sq^m(x)$$

Thus, for any $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(X, A; \mathbb{Z}_2)$, the Cartan formula can be rewritten as:

$$Sq(x \smile y) = Sq(x) \smile Sq(y)$$

To conclude this section, given $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(Y, B; \mathbb{Z}_2)$, we can derive, using the cross product and the Cartan formula, the following relations:

$$Sq^k(x \times y) = \sum_{i+j=k} Sq^i(x) \times Sq^k(y)$$

$$Sq(x \times y) = Sq(x) \times Sq(y)$$

## A.4 $R-$Orientation Classes and e Dualities

In this section, we will define the notion of $R$-orientability of a topological manifold and state some results in this context. Afterward, we will present the most important results involving topological manifolds in the realm of (co)homology theory, known as dualities.

For a more detailed reading on orientations and the dualities mentioned here, we suggest ([8], Chapter 3, Section 3.3).

**Definition A.4. (Local Orientation)** *Consider $M^m$ a topological manifold. Then:*

1. *An $R$-local orientation of $M$ at $b \in M$ is the choice of a generator which we denote by $([M]_b) = H_m(M, M - \{b\}; R) \cong R$.*

2. *An $R$-local orientation of $M$ along a subspace $U \subset M$ is the choice of an element $[M]_U \in H_m(M, M - U; R)$ such that, for all $b \in U$, we have $((j_b^U)_*([M]_U)) = H_m(M, M - \{b\}; R)$, where $j_b^U : (M, M - U) \hookrightarrow (M, M - \{b\})$ is the canonical inclusion.*

*The elements $[M]_b$ and $[M]_U$ are called the classes of local R-orientation of $M$ at $b \in M$ and along $U$, respectively.*

**Definition A.5. (Global Orientation)** *A topological manifold $M^m$ is said to be R-orientable if there exists an open cover $\mathcal{U}$ of $M$ such that:*

1. *if $U_i, U_j \in \mathcal{U}$ and $b \in U_i \cap U_j$, then $(j_b^{U_i})_*([M]_{U_i}) = (j_b^{U_j})_*([M]_{U_j})$.*
2. *for every $U \in \mathcal{U}$ and $b \in U$, we have $[M]_b = (j_b^U)_*([M]_U)$*

After defining local and global orientations, let us consider the following:

**Proposition A.3.** *Let $M^m$ be an R-orientable topological manifold. Then:*

1. *If $M$ is connected and closed, the inclusion $j_b^M = j_b : M \hookrightarrow (M, M - \{b\})$ is such that $(j_b)_* : H_m(M; R) \to H_m(M, M - \{b\}; R)$ is an isomorphism for all $b \in M$.*
2. *For each compact $K \subset M$, there exists a unique class of R-local orientation of $M$ along $K$, $[M]_K \in H_m(M, M - K; R)$, such that $(j_b^K)_*([M]_K) = [M]_b$ for all $b \in K$.*

**Definition A.6.** *Let $M^m$ be a connected, closed, and R-orientable topological manifold. We call the global R-orientation class the generator denoted by $([M]) = H_m(M; R)$, which is such that $(j_b)_*([M]) = [M]_b$ for all $b \in M$.*

It is known in the literature that every topological manifold $M^m$ is $\mathbb{Z}_2$-orientable. Therefore, if $M$ is closed and connected, its classes of $\mathbb{Z}_2$-orientation (both local and global) are the only generators $([M]) = H_m(M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H_m(M, M - \{b\}; \mathbb{Z}_2) = ([M]_b)$ such that $(j_b)_*([M]) = [M]_b$ for all $b \in M$.

**Lemma A.6.** *If $M^m$ and $N^n$ are two closed and connected topological manifolds, then their classes of global $\mathbb{Z}_2$-orientation satisfy:*

$$[M \times N] = [M] \times [N]$$

*Proof.* First, from the Künneth formula, we have:

$$H^{m+n}(M \times N; \mathbb{Z}_2) \cong H^m(M; \mathbb{Z}_2) \otimes H^n(N; \mathbb{Z}_2)$$

On the other hand, since $H^m(M; \mathbb{Z}_2)$ and $H^n(N; \mathbb{Z}_2)$ are modules generated uniquely by the classes of global $\mathbb{Z}_2$-orientation $[M]$ and $[N]$, respectively, it follows from ([9], Corollary 5.12, p. 215) that $H^m(M; \mathbb{Z}_2) \otimes H^n(N; \mathbb{Z}_2)$ is generated uniquely by $[M] \otimes [N]$.

Since the isomorphism in the Künneth formula is given by the cross product, the class of global $\mathbb{Z}_2$-orientation of the product manifold $M \times N$ will be the product $[M] \times [N]$.

□

Before stating the dualities we will use in this work, let us examine an alternative way to visualize the cap product in the context of topological manifolds.

To do so, consider $M^m$ a topological manifold and a subspace $K \subset M$ that is compact and ENR. As seen in ([6], Chapter 8, Section 7), we can consider the cap product as a homomorphism between the following $R$-modules of (co)homology:

$$\frown: H_i(M, M-K; R) \otimes H^j(K; R) \rightarrow H_{i-j}(M, M-K; R)$$

With this, we can state the following:

**Theorem A.4. (Poincaré-Lefschetz Duality)** *Let $M^m$ be an $R$-orientable topological manifold and $K \subset M$ a compact and ENR subspace. Then, the homomorphism $\mathcal{D}_{M,K} : H^k(K; R) \rightarrow H_{m-k}(M, M-K; R)$ given by $\mathcal{D}_{M,K}(x) = [M]_K \frown x$ is an isomorphism for all $k \geq 0$.*

**Theorem A.5. (Poincaré Duality)** *If $M^m$ is a compact $R$-orientable topological manifold, then the homomorphism $\mathcal{D}_M : H^k(M; R) \rightarrow H_{m-k}(M; R)$ given by $\mathcal{D}_M(x) = [M] \frown x$ is an isomorphism for all $k \geq 0$.*

Concluding this section, let us examine some consequences of Poincaré duality.

**Theorem A.6.** *Let $M^m$ be a compact and $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a finite field. Thus, we have, for all $k \geq 0$, that the homomorphism $H^k(M; R) \rightarrow Hom(H^{m-k}(M; R); R)$ that associates $x \mapsto x'(y) =< x \smile y, [M] >$ is an isomorphism.*

*Proof.*

By directly composing the Universal Coefficients Theorem and the Poincaré duality for the topological manifold $M^m$, we obtain, for all $k \geq 0$, the following isomorphism:

$$H^k(M; R) \rightarrow Hom(H_k(M; R); R) \rightarrow Hom(H^{m-k}(M; R); R)$$

$$x \mapsto \overline{x} \mapsto \widetilde{x}$$

where $\overline{x} \in Hom(H_k(M; R); R)$ and $\widetilde{x} \in Hom(H^{m-k}(M; R); R)$ are defined, respectively, by $\overline{x}(a) =< x, a >$ and $\widetilde{x}(y) = \overline{x}([M] \frown y)$.

Thus, the isomorphism $H^k(M; R) \rightarrow Hom(H^{m-k}(M; R); R)$ is given, for all $k \geq 0$, by the following association:

$$\begin{aligned} x \mapsto \widetilde{x}(y) &= \overline{x}([M] \frown y) \\ &= < x, [M] \frown y > \\ &= < x \smile y, [M] > \\ &= x'(y) \end{aligned}$$

$\square$

**Theorem A.7. (Dual Basis)** *Consider $M^m$ a compact, $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a field. Then, for every basis $\{b_i\}_{i=1}^r$ of $H^*(M; R)$, there exists a unique corresponding basis $\{b_i^\#\}_{i=1}^r$ of $H^*(M; R)$, called the dual basis, satisfying the following identity:*

$$< b_i \smile b_j^\#, [M] > = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

*Proof.*

By the previous theorem, the correspondence $H^k(M; R) \to Hom(H^{m-k}(M; R); R)$ that maps $b \mapsto < b \smile \_, [M] >: H^{m-k}(M; R) \to R$ is an isomorphism for all $k \geq 0$.

Now, fix an arbitrary $k \geq 0$. Recall that since $H^{m-k}(M; R)$ is a finitely generated $R$-module, say by the basis $\{b_j^\#\}_{j=1}^l$, the result from ([9], Theorem 4.11, p. 204) ensures that $Hom(H^{m-k}(M; R); R)$ is also a finitely generated $R$-module by the homomorphisms $h_i : H^{m-k}(M; R) \to R$ defined, for all $i = 1, \cdots, l$, by:

$$h_i(b_j^\#) \ = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Thus, for each basic element $b \in H^k(M; R)$, there exists a unique basic element $b^\# \in H^{m-k}(M; R)$ such that $< b \smile b^\#, [M] >= 1$.

□

**Corollary A.1.** *Consider $M^m$ a compact, $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a field. Let $\{b_i\}_{i=1}^r$ be a basis of $H^*(M; R)$ and $\{b_i^\#\}_{i=1}^r$ its dual basis. Then every $x \in H^*(M; R)$ can be written as follows:*

$$x = \sum_{i=1}^{r} < x \smile b_i^\#, [M] > b_i$$

*Proof.*

Since $\{b_i\}_{i=1}^r$ is a basis of $H^*(M; R)$, for each $x \in H^*(M; R)$, there exist unique coefficients $\alpha_1, \cdots, \alpha_r \in R$ such that $x = \sum_{i=1}^{r} \alpha_i b_i$. Thus, for each $b_j^\# \in \{b_i^\#\}_{i=1}^r$, we have:

$$\begin{aligned} < x \smile b_j^\#, [M] > &= < \left( \sum_{i=1}^{r} b_i \alpha_i \right) \smile b_j^\#, [M] > \\ &= \sum_{i=1}^{r} \alpha_i < b_i \smile b_j^\#, [M] > \\ &= \alpha_j \end{aligned}$$

□

## A.5 Classes de Wu

Now, we will see how to construct the so-called Wu classes of a closed topological manifold. The construction of such classes depends solely on the Universal Coefficients Theorem and Poincaré duality.

To this end, consider a closed and connected topological manifold $M^m$, an arbitrary integer $k \geq 0$, and the Steenrod squares $Sq^k : H^{m-k}(M;\mathbb{Z}_2) \to H^m(M;\mathbb{Z}_2)$. Thus, by taking the homomorphism in $Hom(H^{m-k}(M;\mathbb{Z}_2);\mathbb{Z}_2)$ that maps $x \mapsto < Sq^k(x), [M] >$, we obtain, by Theorem A.6, that there exists a unique cohomology class $v_k(M) \in H^k(M;\mathbb{Z}_2)$ such that:

$$< v_k(M) \smile x, [M] > = < Sq^k(x), [M] >, \ \forall x \in H^{m-k}(M;\mathbb{Z}_2)$$

**Definition A.7. (Wu Class)** *Given a closed and connected topological manifold $M^m$ and an integer $k \geq 0$, we call the $k$-th Wu class of the manifold $M$ the class $v_k(M) \in H^k(M;\mathbb{Z}_2)$, uniquely characterized by the following relation:*

$$< v_k(M) \smile x, [M] > = < Sq^k(x), [M] >, \ \forall x \in H^{m-k}(M;\mathbb{Z}_2)$$

*Additionally, we call $v(M) = \sum_{k=0}^{m} v_k(M) \in H^*(M;\mathbb{Z}_2)$ the total Wu class of $M$.*

Due to the uniqueness of the Wu classes, we obtain that $v_0(M) = 1 \in H^0(M;\mathbb{Z}_2)$, since for all $x \in H^m(M;\mathbb{Z}_2)$, we have:

$$< Sq^0(x), [M] > = < x, [M] > = < 1 \smile x, [M] >$$

As an example, let us compute the total Wu class of the real projective space $\mathbb{R}P^n$:

**Example A.1.** *Given $H^1(\mathbb{R}P^n;\mathbb{Z}_2) = (a)$, then $v(\mathbb{R}P^n) = \sum_{k=0}^{n} \binom{n-k}{k} a^k$.*

*Proof.* Initially, we will show, by induction on $i \geq 0$, that $Sq^k(a^i) = \binom{i}{k} a^{i+k}$ for any $k \geq 0$. To this end, we fix that $\binom{i}{k} = 0$ when $i < k$ or $k < 0$.

Thus, for $i = 0$, we have:

$$\begin{aligned} Sq^k(a^0) &= Sq^k(1) \\ &= \begin{cases} 0 & , \ k > 0 \\ 1 & , \ k = 0 \end{cases} \\ &= \tbinom{0}{k} a^k \end{aligned}$$

Now, assuming $Sq^k(a^i) = \binom{i}{k} a^{i+k}$ for all $k \geq 0$, note that:

$$\begin{aligned} Sq^k(a^{i+1}) &= Sq^k(a^i \smile a) \\ &= \sum_{r+s=k} \left[ Sq^r(a^i) \smile Sq^s(a) \right] \\ &= \left[ Sq^k(a^i) \smile Sq^0(a) \right] + \left[ Sq^{k-1}(a^i) \smile Sq^1(a) \right] \\ &= \left[ \tbinom{i}{k} a^{i+k} \smile a \right] + \left[ \tbinom{i}{k-1} a^{i+k-1} \smile a^2 \right] \\ &= \left[ \tbinom{i}{k} + \tbinom{i}{k-1} \right] a^{i+k+1} \\ &= \tbinom{i+1}{k} a^{(i+1)+k} \end{aligned}$$

Thus, $Sq^k(a^i) = \binom{i}{k}a^{i+k}$ for any $i, k \geq 0$. Finally, let's show that $v_k(\mathbb{R}P^n) = \binom{n-k}{k}a^k$ for all $0 \leq k \leq n$. To do so, it is sufficient to verify that:

$$< \binom{n-k}{k} a^k \cup x, [\mathbb{R}P^n] =< Sq^k(x), [\mathbb{R}P^n] >, \ \forall x \in H^{n-k}(\mathbb{R}P^n; \mathbb{Z}_2)$$

Since $H^{n-k}(\mathbb{R}P^n; \mathbb{Z}_2) = (a^{n-k}) = \{0, a^{n-k}\}$, it is enough to check the above equality for $x = a^{n-k}$, as for $x = 0$ the result is immediate. Thus:

$$\begin{aligned} < \tbinom{n-k}{k}a^k \cup a^{n-k}, [\mathbb{R}P^n] >= & \ < \tbinom{n-k}{k}a^n > \\ = & \ < Sq^k(a^{n-k}), [\mathbb{R}P^n] > \end{aligned}$$

Therefore, $v(\mathbb{R}P^n) = \displaystyle\sum_{k=0}^{n} \binom{n-k}{k} a^k$.

□

**Lemma A.7.** *Given two closed and connected manifolds $M^m$ and $N^n$, we have $v(M \times N) = v(M) \times v(N)$.*

*Proof.*

Initially, denote the $k$-th Wu class of the total class $v(M) \times v(N)$ as:

$$[v(M) \times v(N)]_k = \sum_{i+j=k} [v_i(M) \times v_j(N)]$$

For $v_k(M \times N) = [v(M) \times v(N)]_k$, it is sufficient to show the following relation:

$$< [v(M) \times v(N)]_k \cup z, [M \times N] >=< Sq^k(z), [M \times N] >, \ \forall z \in H^{m+n-k}(M \times N; \mathbb{Z}_2)$$

On the other hand, the Künneth formula guarantees that:

$$H^{m+n-k}(M \times N; \mathbb{Z}_2) \cong \bigoplus_{r+s=k} \left[H^{m-r}(M; \mathbb{Z}_2) \otimes H^{n-s}(N; \mathbb{Z}_2)\right]$$

Thus, it is enough to show the previous equality for an arbitrary generator $z = x \times y \in H^{m+n-k}(M \times N; \mathbb{Z}_2)$, where $x \in H^{m-r}(M; \mathbb{Z}_2)$ and $y \in H^{n-s}(N; \mathbb{Z}_2)$, with $r + s = k$. Thus:

$$\begin{aligned} & < [v(M) \times v(N)]_k \cup z, [M \times N] >= \\ = & \ < \left(\textstyle\sum_{i+j=k} v_i(M) \times v_j(N)\right) \cup (x \times y), [M \times N] > \\ = & \ < \textstyle\sum_{i+j=k} \left[(v_i(M) \cup x) \times (v_j(N) \cup y)\right], [M \times N] > \end{aligned}$$

Note that, when considering $i + j = k = r + s$, with $i > r$ or $j > s$, we conclude that $v_i(M) \cup x \in H^{m-r+i}(M; \mathbb{Z}_2) = 0$ or $v_j(N) \cup y \in H^{n-s+j}(N; \mathbb{Z}_2) = 0$, and so:

$$\begin{aligned} & < [v(M) \times v(N)]_k \cup z, [M \times N] >= \\ = & \ < \textstyle\sum_{i+j=k}[(v_i(M) \cup x) \times (v_j(N) \cup y)], [M \times N] > \\ = & \ < (v_r(M) \cup x) \times (v_s(N) \cup y), [M \times N] > \\ = & \ < (v_r(M) \cup x) \times (v_s(N) \cup y), [M] \times [N] > \\ = & \ < v_r(M) \cup x, [M] >< v_s(N) \cup y, [N] > \\ = & \ < Sq^r(x), [M] >< Sq^s(y), [N] > \\ = & \ < Sq^r(x) \times Sq^s(y), [M] \times [N] > \\ = & \ < Sq^r(x) \times Sq^s(y), [M \times N] > \end{aligned}$$

Again, if we consider $i+j=k=r+s$, with $i>r$ or $j>s$, we obtain that $Sq^i(x) \in H^{m-r+i}(M;\mathbb{Z}_2)=0$ or $Sq^j(y) \in H^{n-s+j}(N;\mathbb{Z}_2)=0$, and consequently:

$$\begin{aligned} < [v(M)\times v(N)]_k \cup z, [M\times N] >= &\ < Sq^r(x)\times Sq^s(y), [M\times N] > \\ = &\ < \textstyle\sum_{i+j=k} Sq^i(x)\times Sq^j(y), [M\times N] > \\ = &\ < Sq^k(x\times y), [M\times N] > \\ = &\ < Sq^k(z), [M\times N] > \end{aligned}$$

Therefore, we conclude that $v(M\times N)=v(M)\times v(N)$.

$\square$

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