**FEDERAL UNIVERSITY OF SÃO CARLOS**
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# Characteristic Classes of
# Topological and Generalized Manifolds

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# Characteristic Classes of
# Topological and Generalized Manifolds

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*To my parents*
*Antonio Carlos and Edith*
*and my siblings*
*Carol and André.*

# Acknowledgements

*In memory of my father, Antonio Carlos Barbosa. Unfortunately, you couldn't be physically present on the day of my defense, but you certainly were and will always be present in the hearts of our entire family. Thank you for everything!*

*Either you have a strategy,*
*or you're part of someone else's strategy.*

Alvin Toffler

# Abstract

In this work, we will first present generalized bundles, a concept developed by Fadell with the aim of generalizing vector bundles, Stiefel-Whitney classes, and Wu's formula from the context of smooth manifolds to topological manifolds. After that, we will use generalized bundles to obtain original results concerning the Thom, Stiefel-Whitney, Wu, and Euler classes of topological manifolds, as well as to provide a second proof of Wu's formula for topological manifolds and to establish the topological version of the Poincaré-Hopf Theorem. Finally, we will use Poincaré and Poincaré-Lefschetz dualities to construct the Stiefel-Whitney classes of generalized manifolds in a broader manner, aiming to present, for the first time in the literature, a proof of Wu's formula for such manifolds.

Keywords: characteristic classes, generalized bundles, topological manifolds, generalized manifolds, Wu's formula.

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# List of Notations

1. Saying that $f : X \to Y$ is a map means the same as saying that $f$ is a continuous function between topological spaces.

2. $f : X \rightleftarrows Y : g$ denotes two maps when $f : X \to Y$ and $g : Y \to X$, not necessarily inverses of each other.

3. $1 : X \to X$ denotes the identity map on $X$.

4. $f^{-1}$ denotes the preimage of a map $f$, as well as its inverse mapping (when it exists).

5. If $f : X \to Y$ is a map, then $f(\_)$ denotes $f(x)$ for every $x \in X$.

6. If $H : X \times Y \to Z$ is a map defined on a Cartesian product, then $H(\_, y)$ denotes $H(x, y)$ for every $x \in X$. The same holds for $H(x, \_)$.

7. $p_i : X_1 \times \cdots \times X_n \to X_i$ denotes the projection on the $i$-th factor.

8. $d : X \to X \times X$ denotes the diagonal map given by $d(x) = (x, x)$.

9. Saying that $U \subset X$ is an open neighborhood of some subset $A \subset X$ means the same as saying that $U$ is an open subspace of $X$ that contains $A$.

10. Saying that $\mathcal{U}$ is an open cover of a topological space $B$ means the same as saying that $\mathcal{U} = \{U \subset B\}$, such that $U \subset B$ is an open subspace of $B$ for every $U \in \mathcal{U}$ and $\bigcup_{U \in \mathcal{U}} U = B$.

11. $X \approx Y$ denotes when two topological spaces are homeomorphic.

12. $f \sim g$ denotes when two maps are homotopic.

13. $X \sim Y$ denotes when two topological spaces have the same type of homotopy.

14. $G_1 \cong G_2$ denotes when two algebraic objects are, appropriately, isomorphic.

15. $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \; : \; \|x\| = 1\}$.

16. $D^n = \{x \in \mathbb{R}^n \; : \; \|x\| \leq 1\}$.

17. $B^n = \{x \in \mathbb{R}^n \; : \; \|x\| < 1\}$.

18. $I = [0, 1] \subset \mathbb{R}$.

19. $X^I$ denotes the topological space of paths in $X$.

20. $\Omega(X, x_0) = \{\omega \in X^I \ : \ \omega(0) = \omega(1) = x_0\}$.

21. $H_k(X, A; R)$ and $H^k(X, A; R)$ denote the $k$-th $R$-modules of singular homology and cohomology, respectively, of the pair $(X, A)$ with coefficients in a commutative ring $R$ with unity.

22. $H_k^c(X, A; R)$ and $H_c^k(X, A; R)$ denote, respectively, the $k$-th $R$-modules of singular homology and cohomology with compact support.

23. $\widetilde{H}_k(X, A; R)$ and $\widetilde{H}^k(X, A; R)$ denote, respectively, the $k$-th $R$-modules of reduced singular homology and cohomology.

24. $\check{H}^k(X, A; R)$ denotes the $k$-th $R$-module of Čech cohomology.

25. $H^k(X, A; R) = (x)$ denotes that the $k$-th $R$-module of cohomology of the pair $(X, A)$ is generated by the element $x \in H^k(X, A; R)$. The same applies for homology modules.

26. If $x \in H^k(X, A; R)$, then we denote $|x| = k$. The same applies for homology modules.

27. $<,>: H^k(X, A; R) \otimes H_k(X, A; R) \rightarrow R$ denotes the Kronecker product, which maps $\varphi \otimes \sigma \mapsto < \varphi, \sigma >$.

28. $\frown: H_k(X, A \cup B; R) \otimes H^l(X, A; R) \rightarrow H_{k-l}(X, B; R)$ denotes the cap product, which maps $\sigma \otimes \varphi \mapsto \sigma \frown \varphi$.

29. $\smile: H^k(X, A; R) \otimes H^l(X, B; R) \rightarrow H^{k+l}(X, A \cup B; R)$ denotes the cup product, which maps $\varphi \otimes \psi \mapsto \varphi \smile \psi$.

30. $\times : H^k(X, A; R) \otimes H^l(Y, B; R) \rightarrow H^{k+l}(X \times Y, (X \times B) \cup (A \times Y); R)$ denotes the cross product, which maps $\varphi \otimes \psi \mapsto \varphi \times \psi$.

# Chapter 1

# Introduction

*"Between the 4th and 10th of September 1935, during the International Congress of Topology held in Moscow, several works were presented that would forever change the future of Algebraic Topology, with some of these works now considered foundational research lines in this theory. Among these works, we can mention:*

- *The introduction by Witold Hurewicz to homotopy groups.*

- *The lectures by Heinz Hopf and Hassler Whitney on vector fields and sphere bundles, which initiated the study of vector bundles and, consequently, characteristic classes.*

- *The independent introductions by James Alexander and Andrei Kolmogorov to cohomology theory, as well as the cup product."*

In this work, we will contribute to the theory of characteristic classes, more specifically, characteristic classes of topological and generalized manifolds.

After this historical context on the emergence of characteristic class theory, we will begin introducing the basic concepts used for the development of this work.

In 1955, Nash introduced in [42] the concept that would become known as the field of non-singular paths of a topological manifold, which can be understood as the topological version of a non-zero vector field. Essentially, Nash showed that given a smooth manifold $M$ and fixing a point $b \in M$, the space of non-zero tangent vectors of $M$ at $b$ can also be defined from the topological viewpoint, up to a homotopy equivalence, as the set:

$$\{\omega \in M^I : \omega(t) = b \Leftrightarrow t = 0\}$$

A decade later, in 1965, Fadell defined in [20] generalized bundles, a concept that not only generalized vector bundles, but also allowed the extension, through Nash's ideas in [42], of the notions of tangent and normal fiber bundles from the context of smooth manifolds to topological manifolds. Furthermore, Fadell constructed the Stiefel-Whitney classes of generalized fiber bundles in order to obtain Whitney's duality for specific topological embeddings and to prove the Wu formula for topological manifolds.

The theory developed by Fadell in [20] will serve as the foundation for the development of our entire work, which can be divided into two parts:

- The first part will consist of chapters 2, 3, and 4. These chapters can be interpreted as a modern re-reading of the results obtained by Fadell in [20], as well as a continuation of the same, since we will present additional results both on generalized bundles themselves and on Thom, Stiefel-Whitney, Euler, and Wu classes of topological manifolds.

- The second part of this work will consist solely of chapter 5, in which we will construct more extensively the Stiefel-Whitney classes of generalized manifolds in order to present for the first time in the literature a proof of the Wu formula for such manifolds.

Now, we will look in more detail at how we will organize the structure of our work, pointing out our contributions and the relevance of the results that will be presented here.

In chapter 2, we will begin our work by presenting the studies conducted on generalized fiber bundles, a tool developed by Fadell in [20], which not only generalized the concepts of tangent and normal vector bundles from the context of smooth manifolds to topological manifolds, but also allowed him to define the Stiefel-Whitney classes and prove Whitney's duality and the Wu formula for the context of topological manifolds.

Concatenating definitions 2.5, 2.7, and 2.8, we can define a generalized bundle more directly as follows:

**Definition.** *Given $E$ and $B$ topological spaces, $E_0 \subset E$ and $p : E \to B$ a onto map, we call the pair $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ an $\mathbb{R}^n-$generalized bundle when:*

1. *For any maps $h : X \to E$ and $H : X \times I \to B$, such that $H(\_, 0) = p \circ h$, there exists a map $\widetilde{H} : X \times I \to E$ such that $\widetilde{H}(\_, 0) = h$ and $p \circ \widetilde{H} = H$.*

2. *If $x_0 \in X$ is such that $h(x_0) \in E_0$, then $\widetilde{H}(x_0, \_) \in E_0$.*

3. *There exists a map $s : B \to E$ such that $E_0 = E - s(B)$.*

4. *For all $b \in B$, $(p^{-1}(b), p^{-1}(b) \bigcap E_0) \sim (\mathbb{R}^n, \mathbb{R}^n - \{0\})$.*

With this definition, we can interpret a generalized bundle as a fibration with the following characteristics:

- The total space is a pair of topological spaces.

- There is always at least one global section.

- The fiber behaves, up to homotopy equivalence, like a Euclidean space.

During the reading of Chapter 2, the reader will notice that the development of the chapter will not be as straightforward compared to the definition above, since our main goal will be to present the theory of generalized bundles in a more detailed way and using a more modern language than the results presented by Fadell in the first half of [20].

More explicitly, we will show in Example 2.5 how generalized bundles indeed generalize vector bundles, and in Proposition 2.1 how the notion of isomorphism between vector bundles remains valid when extended to the category of generalized bundles. We will also show that it is possible to construct new generalized bundles from others, just as it happens with vector bundles, for example: restriction bundles, product bundles, and Whitney sum bundles.

Although Chapter 2 is a preliminary chapter, we will contribute with original results concerning the pullback generalized bundle, which was developed by Brown in [9] but was neither cited nor used by Fadell in [20]. These results will prove to be quite relevant when we use them in the construction of some maps regarding characteristic classes of topological manifolds in Chapters 3 and 4.

In Chapter 3, we will address the topic of characteristic classes of generalized bundles and topological manifolds, more specifically, Thom classes, Stiefel-Whitney classes, and Euler classes. Initially, we will introduce the notion of $R-$orientability of generalized bundles, where $R$ is a commutative ring with unity, and their respective Thom classes, concepts originally proposed by Fadell in [20], but little explored by him, since the main topic developed in the second half of [20] was about Stiefel-Whitney classes, in which case orientability is not a concern.

Thus, we will detail a little more the definition of $R-$orientability of generalized bundles and present some technical results on the behavior of Thom classes under pullback and product generalized bundles, as well as show what happens when we reverse the orientability of a generalized bundle and the relation between the dimension of a $\mathbb{Z}-$orientable topological manifold and the Thom class of its tangent generalized bundle. Even though these results are already known in the context of vector bundles and smooth manifolds, they can be considered original since they have not yet been described in the context of generalized bundles and topological manifolds.

The second topic we will address in Chapter 3 will be about Stiefel-Whitney classes. The purpose of this topic will be to rewrite the main properties and consequences of these classes, already widespread in the literature, for the context of generalized bundles and topological manifolds, following the same steps used by Milnor in ([41], Chapter 8) for vector bundles and smooth manifolds. In doing so, we will offer a broader, more modern, and detailed reinterpretation of the results proposed by Fadell in the second half of [20]. Our contributions to this topic will involve results concerning pullback generalized bundles.

The third and last topic addressed in Chapter 3 will be about Euler classes. Differently from Stiefel-Whitney classes, Euler classes can only be defined for $\mathbb{Z}-$orientable generalized bundles. Thus, due to the technical lemmas related to Thom classes of $\mathbb{Z}-$orientable generalized bundles obtained at the beginning of Chapter 3, we will be able to conclude several consequences and applications concerning Euler classes of generalized bundles and $\mathbb{Z}-$orientable topological manifolds. In this topic, except for Proposition 3.4, all other results will be original, being generalizations of known results about Euler classes for vector bundles and smooth manifolds. Among these generalizations, we highlight:

**Proposition 3.7.** *Let* $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ *be an* $\mathbb{R}^n-$*generalized bundle that is* $\mathbb{Z}$*-orientable. If* $(\mathcal{F}, \mathcal{F}_0)$ *admits a section* $s : B \to E$ *such that* $s(B) \subset E_0$*, then*

$e(\mathcal{F}, \mathcal{F}_0) = 0.$[1]

The proposition above, in its version for vector bundles, is widely known, as it allows interpreting the Euler class of a vector bundle as an obstruction to the existence of a nowhere-vanishing section. In this work, we will present the generalized version of this interpretation, which will allow us to obtain the main application related to the Euler class in Chapter 4, the topological version of the Poincaré-Hopf theorem.

Up to this point, the reader should already have noticed the main goal of Chapters 2 and 3 of our work, which is to structure in detail and using a more up-to-date language the theory of generalized bundles and their characteristic classes, while also presenting several technical contributions, aiming to generalize applications regarding characteristic classes of smooth manifolds to the context of topological manifolds, as we will see next.

The conclusion of our work regarding characteristic classes of generalized bundles will be presented in Chapter 4, where we will present three major applications with original technical proofs concerning Stiefel-Whitney, Euler, and Wu classes of closed topological manifolds. Initially, we will present an alternative proof of the topological version of the famous Wu formula, which relates the Stiefel-Whitney and Wu classes of a smooth manifold through Steenrod squares.

In [20], Fadell uses generalized bundles to give a first proof of Wu's formula for topological manifolds, based on the techniques used by Milnor in ([40], Chapter 9). Furthermore, the preliminary results that Fadell develops to prove Wu's formula are all in the framework of singular (co)homology $\mathbb{Z}_2-$modules. Meanwhile, the alternative proof of Wu's formula for topological manifolds that we will present in Chapter 4 will be based on different techniques also introduced by Milnor, now found in ([41], Chapter 11).

Comparing the proofs presented by us in this work and by Fadell in [20], the main differences will be found in the preliminary lemmas used in Wu's formula, as we will prove them in the framework of singular (co)homology $R-$modules with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$. Since we will use the same sequence of results employed by Milnor in [41], now using generalized bundles instead of vector bundles, our main contribution will be obtaining the case $R = \mathbb{Z}$ of the following result:

**Lemma 4.1.** *Let $M^m$ be a closed, connected, $R$-orientable topological manifold with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, $b \in M$ arbitrary, $j_b : (M, M - \{b\}) \hookrightarrow (M \times M, (M \times M) - \Delta)$ the canonical inclusion, $[M]_b \in H_m(M, M - \{b\}; R)$ the local $R$-orientation class of $M$ at $b$, and $(\tau') \in H^m(M \times M, (M \times M) - \Delta; R)$ the generator uniquely defined by the Thom class of the tangent generalized bundle of $M$. Then:*

$$< j_b^*(\tau'), [M]_b >= 1 \in R$$

The proof of the lemma above, in its version for smooth manifolds, can be found in ([41], Lemma 11.7, p. 123), where the Riemannian structure of the manifold and the existence of the exponential map are used, whereas our proof will be entirely algebraic, allowing generalization to the context of topological manifolds, which will be crucial for the topological applications of the Euler class.

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[1]$e(\mathcal{F}, \mathcal{F}_0)$ will denote the Euler class of the generalized bundle $(\mathcal{F}, \mathcal{F}_0)$.

The second application of Chapter 4 will concern Euler classes. In fact, we will present two applications on this topic, one being the relation between the Euler class and the Euler characteristic of a topological manifold, and the other being the topological version of the Poincaré-Hopf theorem. The reader will notice the importance of the $R = \mathbb{Z}$ case of Lemma 4.1 for the first application, whose statement is as follows:

**Theorem 4.2.** *If $M$ is a closed, connected, and $\mathbb{Z}$-orientable topological manifold, then[2]:*

$$< e(M), [M] >= \chi(M >$$

For the second map involving the Euler class, we will need to define the concept of a path field on a topological manifold, which was introduced by Nash in [42] as follows:

**Definition 4.1.** *A path field on a topological manifold $M$ is any section of its generalized bundle $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$. Moreover, a nonsingular path field on $M$ is a section $s : M \to TM$ such that $s(M) \subset T_0 M$.*

As we will show in Chapter 4, generalized bundles will allow us to generalize the notion of nowhere-vanishing vector fields from the smooth manifold context to the topological manifold setting, since a smooth manifold admits a nowhere-vanishing vector field if and only if it admits a nonsingular path field. With that, we will be able to prove the topological version of the Poincaré-Hopf theorem, whose statement is:

**Theorem 4.3.** *Let $M$ be a closed, connected, and $\mathbb{Z}$-orientable topological manifold. If $M$ admits a nonsingular path field, then $\chi(M) = 0$.*

This result was first presented by Brown in [9], using essentially Lefschetz numbers in his proof. In our work, we will present an alternative proof of this result using the Euler class.

As the final map in Chapter 4, we will see how some technical results about generalized bundles will allow us to prove the following:

**Theorem 4.4.** *If $i : M^m \hookrightarrow S^{m+k}$ is a locally flat embedding[3] between closed, connected topological manifolds with trivial normal generalized bundle, then:[4]*

$$v(M) = i^*(v(S))$$

At first glance, the theorem above seems quite clear and straightforward, since if we replace the total Wu classes with the total Stiefel-Whitney classes, this result becomes an immediate consequence of Whitney duality. However, upon closer examination of the proof of Theorem 4.4 in its version for vector bundles and smooth manifolds, as given

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[2]$e(M)$, $[M]$, and $\chi(M)$ will denote, respectively, the Euler class, the global orientation class, and the Euler characteristic of the manifold $M$.

[3]A locally flat embedding is a topological embedding that locally behaves like a smooth embedding, whose formal definition can be found in Definition 2.14.

[4]$v(M)$ and $v(S)$ will denote, respectively, the total Wu classes of $M$ and $S$.

by Stong in [46] and presented in more detail in [43], it becomes evident that the proof makes direct use of the existence of a tubular neighborhood for smooth embeddings.

Since we cannot guarantee the existence of a tubular neighborhood in the topological context, our main contribution was to circumvent this problem using only results about generalized bundles, showing that the existence of a tubular neighborhood is not essential, but rather certain algebraic consequences of a locally flat embedding.

In the last chapter of our work, Chapter 5, we will present for the first time in the literature a proof of Wu's formula in the context of generalized manifolds, using their Poincaré and Poincaré-Lefschetz dualities. To this end, we will begin the chapter with a brief summary, based on [4], [38], and [8], about the concept of generalized manifolds. More explicitly, the constructions in this chapter will be carried out for $\mathbb{Z}_2-$homological ENR-manifolds, which are particular generalized manifolds. For convenience, we will continue to refer to these spaces simply as generalized manifolds.

In this initial summary, we will see that generalized manifolds are essentially topological spaces that behave like topological manifolds in the realm of singular (co)homology $\mathbb{Z}_2-$modules. In particular, we will be able to construct the Wu classes for such manifolds, as well as their Poincaré and Poincaré-Lefschetz dualities.

After establishing these objects, we will associate to each embedding $s : M^m \to N^{2m}$ between compact, connected generalized manifolds, such that there exists a retraction[5] $p : N \to M$, its transfer isomorphism given by the following composition of the Poincaré-Lefschetz duality of the embedding $s$ with the Poincaré duality of the manifold $M$:

$$s_! : H_k(N, N - M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}_{N,M}^{-1}} H^{2m-k}(M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}_M} H_{k-m}(M; \mathbb{Z}_2)$$

Thus, the transfer isomorphism associated to the embedding $s$ will allow us to define the Thom class also associated to the embedding $s$ as the generator:

$$(\tau_s) = H^m(N, N - M; \mathbb{Z}_2)$$

Inspired by the techniques presented by Dold in ([17], Chapter 8), we will demonstrate that the homomorphism $\phi_s : H^k(M; \mathbb{Z}_2) \to H^{k+m}(N, N - M; \mathbb{Z}_2)$ given by $\phi_s(x) = p^*(x) \smile \tau_s$ is, in fact, the dualization (via Universal Coefficients) of the transfer isomorphism $s_!$.

Having done this, we will call $\phi_s$ the Thom isomorphism associated to the embedding $s$ and define the $k$-th Stiefel-Whitney class associated to the embedding $s$ as:

$$w_k(s) = \phi_s^{-1} \circ Sq^k(\tau_s) \in H^k(M; \mathbb{Z}_2)$$

In particular, we will define the $k$-th Stiefel-Whitney class of a generalized manifold $M$ as the $k$-th Stiefel-Whitney class associated to the embedding given by the diagonal map $d : M \to M \times M$. Moreover, to ensure that this definition is indeed well-defined, we

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[5]That is, $p \circ s = 1$.

will use some results about generalized bundles presented in Chapter 4 to show in Theorem 5.5 that, in the context of topological manifolds, the definition of Stiefel-Whitney classes via generalized bundles coincides with the definition we propose via the Stiefel-Whitney classes associated to the embedding given by the diagonal map.

Finally, motivated by the techniques presented by Bredon in ([7], Chapter 6), we will conclude Chapter 5, and consequently our work, by showing that it is possible to obtain Wu's formula for generalized manifolds using our definition of Stiefel-Whitney classes associated to the embedding given by the diagonal map of a generalized manifold.

Since Biasi, Daccach, and Saeki defined in [4] the Stiefel-Whitney classes of generalized manifolds as Wu's formula itself and presented several results in this context, we highlight the originality of Chapter 5 where we define the Stiefel-Whitney classes for generalized manifolds in an alternative way and prove Wu's formula for such manifolds.

We will conclude the introduction chapter of our work with the words of Massey, which can be found in ([32], Chapter 21), providing additional historical context for the emergence of characteristic classes:

> *"At the 1935 conference in Moscow, Hopf presented the work of one of his students, Stiefel, whose publication appeared only in the following year. In this work, Stiefel defined certain homology classes of a smooth manifold that, in modern language, are the Poincaré-dual classes of the Stiefel-Whitney classes of the tangent vector bundle. His method consisted of constructing, through a very geometric process, the cycles that represented these homology classes."*

> *"Whitney gave a lecture at the Moscow conference entitled 'Sphere spaces,' which we now call sphere bundles. These two lectures, and the subsequent papers, marked the beginning of work on the general topic of vector bundles. The most important invariants of vector bundles are generally various characteristic classes, but always cohomology classes."*

William S. Massey

# Chapter 2

# Bundles

We will begin this work by presenting the so-called generalized bundles, a tool developed by Fadell in [20] with the purpose of defining the Stiefel-Whitney classes and proving Whitney's duality and Wu's formula in the context of topological manifolds.

At first, in Section 2.1, we will review specific concepts about vector bundles in order to fix notation and clarify to the reader how vector bundles will be naturally generalized throughout this chapter.

After that, Section 2.2 will serve as an intermediate step for defining generalized bundles and for presenting the results that will be shown in Section 2.3 in a clearer and more succinct way.

Finally, in Section 2.3, we will find the definition and properties involving generalized bundles, almost all of which are taken from [20]. *As will be explained in Observation 2.1, every topological manifold mentioned throughout this work will be assumed to be a manifold without boundary.*

## 2.1 Vector Bundle

For a more specific and detailed reading about the theory of vector bundles, aimed at defining characteristic classes, we suggest [2], [24], [31], and [41].

**Definition 2.1. (Vector Bundle)** *Considering $E$ and $B$ topological spaces and a onto map $p : E \to B$, we call the triple $\xi = (E, p, B)$ an $\mathbb{R}^n-$vector bundle if:*

- *For every $b \in B$, the set $p^{-1}(b)$ admits a structure of a real $n$-dimensional vector space.*

- *We require that $\xi$ is locally trivial, that is: there exists an open cover $\bigcup_{\alpha} U_\alpha = B$ such that, for each open set $U_\alpha$, there exists $h_\alpha : U_\alpha \times \mathbb{R}^n \to p^{-1}(U_\alpha)$ a homeomorphism making the following diagram commute:*

$$U_\alpha \times \mathbb{R}^n \xrightarrow{\quad h_\alpha \quad} p^{-1}(U_\alpha)$$

$$p_1 \searrow \quad \swarrow p$$

$$U_\alpha$$

- *Moreover, $h_\alpha|_{\{b\}\times\mathbb{R}^n} : \{b\} \times \mathbb{R}^n \to p^{-1}(b)$ is a vector space isomorphism for every $b \in U_\alpha$ and each $\alpha$.*

In this context, we call $\xi$ a vector bundle over a base space $B$, with total space $E$ and fiber $F = p^{-1}(b)$ over $b \in B$. Furthermore, we say that a map $s : B \to E$ is a section of $\xi$ if $p \circ s = 1$.
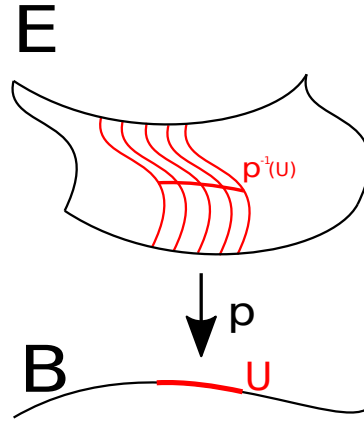


Figure 2.1: Illustration of the local triviality of a vector bundle.

Note that every vector bundle $\xi = (E, p, B)$ admits a section $s : B \to E$, called the zero section, defined by $s(b) = 0 \in p^{-1}(b)$.

**Definition 2.2.** *We say that two $\mathbb{R}^n-$vector bundles $\xi = (E, p, B)$ and $\xi' = (E', q, B)$ are isomorphic, denoted by $\xi \cong \xi'$, if there exists a homeomorphism $h : E \to E'$ such that $q \circ h = p$ and $h|_{p^{-1}(b)} : p^{-1}(b) \to q^{-1}(b)$ is a vector space isomorphism for every $b \in B$.*

Now, let us recall how to specifically define the tangent and normal vector bundles of a smooth manifold and of a smooth embedding, respectively.

Consider $M^m$ a smooth $m$-dimensional manifold and define $TM = \bigcup_{b \in M} \{b\} \times T_b M$, recalling that $T_b M$ is the tangent space of $M$ at $b$.

Thus, $\tau(M) = (TM, p_1, M)$ defines an $\mathbb{R}^m$-vector bundle, where the fiber over $b \in M$ is $p_1^{-1}(b) = \{b\} \times T_b M$.

**Definition 2.3.** *$\tau(M)$ is called the tangent vector bundle of the smooth manifold $M$.*

Recall that an immersion $i : N^n \to M^{n+k}$ is a map between smooth manifolds such that its differential $d_b i : T_b N \to T_{i(b)} M$ is an injective linear transformation for every $b \in N$.

In this context, we can ensure that $d_b i(T_b N)$ is a linear subspace of $T_{i(b)} M$. Thus, the set $E(i) = \{(b,v) \in N \times T_{i(b)} M \; : \; v \in [d_b i(T_b N)]^\perp \subset T_{i(b)} M\}$ is well defined.

Therefore, $\nu(i) = (E(i), p_1, N)$ is an $\mathbb{R}^k$-vector bundle, where the fiber over $b \in N$ is given by $p_1^{-1}(b) = \{b\} \times [d_b i(T_b N)]^\perp$.

**Definition 2.4.** *$\nu(i)$ is called the normal vector bundle of the immersion $i : N \to M$.*

Note that the normal vector bundle can also be defined for a smooth embedding, since every smooth embedding is both a topological embedding and an immersion.

To conclude the topics about vector bundles, let us present the following relation between the tangent and normal vector bundles of a smooth embedding between smooth manifolds, whose proof can be found in ([41], Corollaries 3.4 and 3.5, pp. 30–31).

**Theorem 2.1.** *If $i : N^n \to M^{n+k}$ is a smooth embedding between smooth manifolds, then:*

$$\tau(N) \oplus \nu(i) \cong i^*(\tau(M))$$

## 2.2 Fibration and Pair Fibration

In this section, we will define the concept of pair fibration, also developed by Fadell in [20], which can be considered as a fibration whose total space is a pair of topological spaces.

The definition of pair fibration will be necessary when we define generalized bundles in the next section. We will use the properties developed in this section as an intermediate step in the proofs of results involving generalized bundles.

Except for Example 2.2, all the results presented throughout this section can be found, with few details, in [20] and [9].

**Definition 2.5.** *We say that a map $p : E \to B$ satisfies the homotopy lifting property (HLP) over a topological space $X$ if, for any maps $h : X \to E$ and $H : X \times I \to B$ such that $H(\_, 0) = p \circ h$, there exists a continuous map $\widetilde{H} : X \times I \to E$ such that $\widetilde{H}(\_, 0) = h$ and $p \circ \widetilde{H} = H$. In this case, we have the following commutative*

*diagram:*

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ \ h\ \ } & E \\
\downarrow & \nearrow{\widetilde{H}} & \downarrow{p} \\
X \times I & \xrightarrow{\ \ H\ \ } & B
\end{array}
$$

**Definition 2.6.** *A onto map $p : E \to B$ is said to be a fibration if $p$ satisfies the HLP over any topological space $X$.*

*In this context, we call the triple $\mathcal{F} = (E, p, B)$ a fibration over a base space $B$, with total space $E$ and fiber $F = p^{-1}(b)$ over $b \in B$. We say that a map $s : B \to E$ is a section of $\mathcal{F}$ if $p \circ s = 1$.*

**Definition 2.7. (Pair Fibration)** *Let $p : E \to B$ be a onto map and $E_0 \subset E$. We call $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ a pair fibration if, for any topological space $X$ and any maps $h : X \to E$ and $H : X \times I \to B$ such that $H(\_, 0) = p \circ h$, there exists a continuous map $\widetilde{H} : X \times I \to E$ such that $\widetilde{H}(\_, 0) = h$, $p \circ \widetilde{H} = H$, and if $x_0 \in X$ is such that $h(x_0) \in E_0$, then $\widetilde{H}(x_0, \_) \in E_0$.*

*In this context, we say that $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration over a base space $B$, with total space $(E, E_0)$ and fiber $(F, F_0) = (p^{-1}(b), p^{-1}(b) \cap E_0)$ over $b \in B$.*

*We also say that a map $s : B \to E$ is a section of $(\mathcal{F}, \mathcal{F}_0)$ if $p \circ s = 1$.*

Note that Definition 2.7 guarantees that $\mathcal{F} = (E, p, B)$ and $\mathcal{F}_0 = (E_0, p_0 = p_{|E_0}, B)$ are fibrations.

Historically, the concept of fibration coincides with the concept of "fiber space" given by Hurewicz in ([28], Section 1, p. 956), as also shown in ([28], Section 2, p. 957). Thus, the concept of pair fibration coincides with the so-called "fibered pair" given by Fadell in ([20], Definition 2.3, p. 489), as shown in ([9], Lemma 1.4, p. 183).

As the simplest example of a pair fibration, we have the expected one:

**Example 2.1.** *Considering $B$ an arbitrary topological space, we obtain a pair fibration $(\varepsilon_B^n, \varepsilon_B^{n,0}) = (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\}), p_1, B)$ with fiber homeomorphic to $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$.*

*Proof.*

Initially, consider $X$ any topological space and maps $h : X \to B \times \mathbb{R}^n$ and $H : X \times I \to B$ such that the following commutative diagram is obtained:

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ \ h\ \ } & B \times \mathbb{R}^n \\
\downarrow & & \downarrow{p_1} \\
X \times I & \xrightarrow{\ \ H\ \ } & B
\end{array}
$$

Since $h = (h_1, h_2)$, define $\widetilde{H} : X \times I \to B \times \mathbb{R}^n$ by $\widetilde{H}(x,t) = (H(x,t), h_2(x))$. Clearly, $\widetilde{H}$ is continuous, $p_1 \circ \widetilde{H} = H$, and $\widetilde{H}(\_,0) = h$, since $H(\_,0) = h_1$. Furthermore, if $x_0 \in X$ is such that $h(x_0) \in B \times (\mathbb{R}^n - \{0\})$, then $h_2(x_0) \in \mathbb{R}^n - \{0\}$, and thus, $\widetilde{H}(x_0, \_) \in B \times (\mathbb{R}^n - \{0\})$.

Finally, given $(F, F_0)$ the fiber of $(\mathcal{F}, \mathcal{F}_0)$ over any $b \in B$, we have:

$$
\begin{aligned}
F &= p_1^{-1}(b) \\
&= \{b\} \times \mathbb{R}^n \\
&\approx \mathbb{R}^n
\end{aligned}
$$

$$
\begin{aligned}
F_0 &= p_1^{-1}(b) \cap [B \times (\mathbb{R}^n - \{0\})] \\
&= [\{b\} \times \mathbb{R}^n] \cap [B \times (\mathbb{R}^n - \{0\})] \\
&= \{b\} \times (\mathbb{R}^n - \{0\}) \\
&\approx \mathbb{R}^n - \{0\}
\end{aligned}
$$

Thus, $(\varepsilon_B^n, \varepsilon_B^{n,0})$ is indeed a pair fibration.

$\square$

Note that Example 2.1 also remains valid if we replace $\mathbb{R}^n$ with any topological space $F$ and $\mathbb{R}^n - \{0\}$ with any subspace $F_0 \subset F$. However, in this case, the fiber would be homeomorphic to the pair $(F, F_0)$.

Next, we will present an alternative way of constructing pair fibrations, whose proof can be found in ([1], Theorem 2.5, p.241).

**Theorem 2.2.** *Let $p : E \to B$ be a onto map with $B$ paracompact, and pairs $(F, F_0)$ and $(E, E_0)$. If there exists an open covering $\mathcal{U}$ of $B$ such that, for each $U \in \mathcal{U}$, there exists a homeomorphism $h_U : (U \times F, U \times F_0) \to (p^{-1}(U), p^{-1}(U) \cap E_0)$ such that $p \circ h_U = p_1$, then $(E, E_0, p, B)$ is a pair fibration.*

Theorem 2.2 can be considered the version of the Hurewicz Uniformization Theorem[1] for fibrations of pairs.

Thus, we can obtain the following:

**Example 2.2.** *Every $\mathbb{R}^n$-vector bundle, over a paracompact base, can be associated with a pair fibration, whose fiber is homeomorphic to the pair $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$.*

*Proof.*

First, consider $\xi = (E, p, B)$ an $\mathbb{R}^n$-vector bundle over a paracompact $B$. By Definition 2.1, we know that there exists an open cover $\mathcal{U}$ of $B$ such that, for every $U \in \mathcal{U}$, there exists a homeomorphism $h_U : U \times \mathbb{R}^n \to p^{-1}(U)$ satisfying $p \circ h_U = p_1$.

Denoting by $s : B \to E$ the zero section of $\xi$ and $E_0 = E - s(B)$, it is clear that we can consider $h_U : (U \times \mathbb{R}^n, U \times (\mathbb{R}^n - \{0\})) \to (p^{-1}(U), p^{-1}(U) \cap E_0)$ as a homeomorphism of pairs.

---

[1]For more details on the Hurewicz Uniformization Theorem, see [28].

Therefore, it follows from Theorem 2.2 that $(\xi, \xi_0) = (E, E_0, p, B)$ is a pair fibration.

$\square$

Now, as the final part of this section, let us see how to construct some pair fibrations from others.

**Lemma 2.1.** *Let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ be a pair fibration and $U \subset B$ any subset. Then, $(\mathcal{F}, \mathcal{F}_0)_{|U} = (p^{-1}(U), p^{-1}(U) \cap E_0, p_{|p^{-1}(U)}, U)$ will be a pair fibration with fiber equal to the fiber of $(\mathcal{F}, \mathcal{F}_0)$.*

*Proof.*
First, denote $q = p_{|p^{-1}(U)}$ and consider $X$ an arbitrary topological space and maps $h : X \to p^{-1}(U)$ and $H : X \times I \to U$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ h\ } & p^{-1}(U) \\
\cap \ \downarrow & & \downarrow q \\
X \times I & \xrightarrow{\ H\ } & U
\end{array}
$$

Since $Im(H) \subset U \subset B$ and $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, there exists a map $\widetilde{H} : X \times I \to E$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(\_, 0) = h$. On the other hand, $Im(p \circ \widetilde{H}) = Im(H) \subset U$, and thus $Im(\widetilde{H}) \subset p^{-1}(U)$.

Moreover, if $x_0 \in X$ is such that $h(x_0) \in E_0$, then $\widetilde{H}(x_0, \_) \in E_0$. That is, if $h(x_0) \in p^{-1}(U) \cap E_0$, then $\widetilde{H}(x_0, \_) \in p^{-1}(U) \cap E_0$.

Finally, letting $(F', F_0')$ and $(F, F_0)$ be the fibers of $(\mathcal{F}, \mathcal{F}_0)_{|U}$ and $(\mathcal{F}, \mathcal{F}_0)$, respectively, over the same $b \in U$, we have:

$$
\begin{aligned}
F' &= q^{-1}(b) \\
&= p^{-1}(b) \cap p^{-1}(U) \\
&= p^{-1}(b) \\
&= F
\end{aligned}
$$

$$
\begin{aligned}
F_0' &= q^{-1}(b) \bigcap [p^{-1}(U) \cap E_0] \\
&= p^{-1}(b) \bigcap [p^{-1}(U) \cap E_0] \\
&= p^{-1}(b) \cap E_0 \\
&= F_0
\end{aligned}
$$

Therefore, $(\mathcal{F}, \mathcal{F}_0)_{|U}$ is a pair fibration.

$\square$

**Lemma 2.2.** *Let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ be a pair fibration and $f : B' \to B$ a map. Then, $f^*(\mathcal{F}, \mathcal{F}_0) = (f^*E, f^*E_0, p_1, B')$ is also a pair fibration with fiber homeomorphic to the fiber of $(\mathcal{F}, \mathcal{F}_0)$, where:*

1. $f^*E = \{(b', e) \in B' \times E \ : \ f(b') = p(e)\}$

2. $f^*E_0 = \{(b', e) \in f^*E \ : \ e \in E_0\}$

*Proof.*

At first, let $X$ be any topological space and $h : X \to f^*E$ and $H : X \times I \to B'$ maps such that the following diagram commutes:

$$
\begin{array}{ccccc}
X \times \{0\} & \xrightarrow{\ h\ } & f^*E & \xrightarrow{\ p_2\ } & E \\
\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\
X \times I & \xrightarrow{\ H\ } & B' & \xrightarrow{\ f\ } & B
\end{array}
$$

Denoting $g = p_2 \circ h : X \to E$ and $G = f \circ H : X \times I \to B$, since the diagram above commutes and $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, there exists $\widetilde{G} : X \times I \to E$ such that $p \circ \widetilde{G} = G$, $\widetilde{G}(\_, 0) = g$, and if $x_0 \in X$ is such that $g(x_0) \in E_0$, then $\widetilde{G}(x_0, \_) \in E_0$.

On the other hand, it is clear that $\widetilde{H} = (H, \widetilde{G}) : X \times I \to f^*E$ is well-defined, since $f \circ H = G = p \circ \widetilde{G}$. Moreover, $p_1 \circ \widetilde{H} = H$ and $\widetilde{H}(\_, 0) = h$, since $H(\_, 0) = p_1 \circ h$ and $\widetilde{G}(\_, 0) = p_2 \circ h$.

Thus, if $x_0 \in X$ is such that $h(x_0) \in f^*E_0$, then $g(x_0) = p_2 \circ h(x_0) \in E_0$, and consequently, $\widetilde{G}(x_0, \_) \in E_0$. Therefore, $\widetilde{H}(x_0, \_) \in f^*E_0$.

Finally, if $(F', F_0')$ and $(F, F_0)$ are fibers of $f^*(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}, \mathcal{F}_0)$ over $b_0' \in B'$ and $f(b_0') \in B$, respectively, then:

$$
\begin{aligned}
F' &= p_1^{-1}(b_0') \\
&= \{(b', e) \in f^*E \ : \ b' = p_1(b', e) = b_0'\} \\
&= \{(b_0', e) \in B' \times E \ : \ f(b_0') = p(e)\} \\
&= \{(b_0', e) \in B' \times E \ : \ e \in p^{-1}(f(b_0'))\} \\
&= \{b_0'\} \times p^{-1}(f(b_0')) \\
&= \{b_0'\} \times F \\
&\approx F
\end{aligned}
$$

$$
\begin{aligned}
F_0' &= p_1^{-1}(b_0') \cap f^*E_0 \\
&= (\{b_0'\} \times F) \cap f^*E_0 \\
&= \{b_0'\} \times (F \cap E_0) \\
&= \{b_0'\} \times F_0 \\
&\approx F_0
\end{aligned}
$$

Therefore, $f^*(\mathcal{F}, \mathcal{F}_0)$ is indeed a pair fibration.                  $\square$

**Lemma 2.3.** *Let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $(\mathcal{F}', \mathcal{F}'_0) = (E', E_0', q, B')$ be two pair fibrations. Then, $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) = (E'', E_0'', r, B'')$ will be a pair fibration with fiber equal to the product of the fibers of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$, where:*

1.  $E'' = E \times E'$

2.  $E_0'' = (E \times E_0') \cup (E_0 \times E')$

3.  $r = p \times q$

*Proof.*

At first, let $X$ be any topological space and $h : X \to E''$ and $H : X \times I \to B''$ maps such that the following diagram commutes:

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ h\ } & E'' \\
\Big\downarrow & & \Big\downarrow r \\
X \times I & \xrightarrow{\ H\ } & B''
\end{array}
$$

Since $h = (h_1, h_2)$, $H = (H_1, H_2)$, and $H(\_, 0) = r \circ h$, we have that $H_1(\_, 0) = p \circ h_1$ and $H_2(\_, 0) = q \circ h_2$. As $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ are pair fibrations, there exist maps $\widetilde{H_1} : X \times I \to E$ and $\widetilde{H_2} : X \times I \to E'$ such that $p \circ \widetilde{H_1} = H_1$, $q \circ \widetilde{H_2} = H_2$, $\widetilde{H_1}(\_, 0) = h_1$, $\widetilde{H_2}(\_, 0) = h_2$, and if $x_0 \in X$ is such that $h_1(x_0) \in E_0$, then $\widetilde{H_1}(x_0, \_) \in E_0$. Moreover, if $x_0 \in X$ is such that $h_2(x_0) \in E_0'$, then $\widetilde{H_2}(x_0, \_) \in E_0'$.

Thus, defining $\widetilde{H} = (\widetilde{H_1}, \widetilde{H_2}) : X \times I \to E''$, it is clear that $\widetilde{H}(\_, 0) = h$ and $r \circ \widetilde{H} = H$. Furthermore, if $x_0 \in X$ is such that $h(x_0) \in E_0''$, then $h_1(x_0) \in E_0$ or $h_2(x_0) \in E_0'$. Hence, $\widetilde{H_1}(x_0, \_) \in E_0$ or $\widetilde{H_2}(x_0, \_) \in E_0'$, that is, $\widetilde{H}(x_0, \_) \in E_0''$.

Finally, consider, respectively, $(F, F_0)$, $(F', F_0')$, and $(F'', F_0'')$ the fibers of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ over $b \in B$, $b' \in B'$, and $(b, b') \in B''$. Then:

$$
\begin{aligned}
F'' &= r^{-1}(b, b') \\
&= p^{-1}(b) \times q^{-1}(b') \\
&= F \times F'
\end{aligned}
$$

$$
\begin{aligned}
F_0'' &= r^{-1}(b, b') \cap E_0'' \\
&= (F \times F') \bigcap [(E \times E_0') \cup (E_0 \times E')] \\
&= [(F \times F') \cap (E \times E_0')] \bigcup [(F \times F') \cap (E_0 \times E')] \\
&= [F \times (F' \cap E_0')] \bigcup [(F \cap E_0) \times F'] \\
&= (F \times F_0') \cup (F_0 \times F')
\end{aligned}
$$

Therefore, $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ will be a pair fibration.

$\square$

## 2.3   Generalized Bundle

In this section, we will present the tool developed by Fadell in [20] that allows us to generalize tangent and normal vector bundles of smooth manifolds to topological manifolds. This tool is called the generalized bundle.

Except for examples 2.4 and 2.5, lemmas 2.5 and 2.7, and propositions 2.1, 2.4, 2.5, and 2.8, all other results were taken from [20]. However, we will provide more detailed proofs of these results here.

**Definition 2.8. (Generalized Bundle)** *A pair fibration* $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ *is said to be an* $\mathbb{R}^n$-*generalized bundle if:*

1. *There exists a section* $s : B \to E$ *of* $(\mathcal{F}, \mathcal{F}_0)$ *such that* $E_0$ *is realized, i.e.,* $E_0 = E - s(B)$.

2. *Every fiber* $(F, F_0)$ *of* $(\mathcal{F}, \mathcal{F}_0)$ *satisfies* $(F, F_0) \sim (\mathbb{R}^n, \mathbb{R}^n - \{0\})$.

**Example 2.3.** *The pair fibration* $(\varepsilon_B^n, \varepsilon_B^{n,0})$ *is an* $\mathbb{R}^n-$*generalized bundle.*

*Proof.*

Since Example 2.1 already shows that $(\varepsilon_B^n, \varepsilon_B^{n,0})$ is a pair fibration with a fiber homeomorphic to the pair $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, it is enough to define a section of $(\varepsilon_B^n, \varepsilon_B^{n,0})$ that realizes $B \times (\mathbb{R}^n - \{0\})$.

Note that $s : B \to B \times \mathbb{R}^n$ defined by $s(b) = (b, 0)$ is the required section.

Therefore, $(\varepsilon_B^n, \varepsilon_B^{n,0})$ is an $\mathbb{R}^n-$generalized bundle.

□

**Lemma 2.4.** *If* $(\mathcal{F}, \mathcal{F}_0)$ *is an* $\mathbb{R}^n-$*generalized bundle over* $B$, *then* $(\mathcal{F}, \mathcal{F}_0)_{|U}$ *will also be an* $\mathbb{R}^n-$*generalized bundle for any* $U \subset B$.

*Proof.*

From Lemma 2.1, we already know that if $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, then for any arbitrary $U \subset B$, $(\mathcal{F}, \mathcal{F}_0)_{|U}$ will also be a pair fibration with fiber equal to the fiber of $(\mathcal{F}, \mathcal{F}_0)$.

Thus, let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $(\mathcal{F}, \mathcal{F}_0)_{|U} = (p^{-1}(U), p^{-1}(U) \cap E_0, p_{|p^{-1}(U)}, U)$. It is enough to find a section of $(\mathcal{F}, \mathcal{F}_0)_{|U}$ that realizes $p^{-1}(U) \cap E_0$.

Since $(\mathcal{F}, \mathcal{F}_0)$ is an $\mathbb{R}^n-$generalized bundle, there exists a section $s : B \to E$ of $(\mathcal{F}, \mathcal{F}_0)$ that realizes $E_0$.

Now, define $s' = s_{|U} : U \to p^{-1}(U)$. Clearly, $Im(s') \subset p^{-1}(U)$ because $p \circ s(U) = U$. Furthermore, we have:

$$
\begin{aligned}
p^{-1}(U) \cap E_0 &= p^{-1}(U) \bigcap [E - s(B)] \\
&= [p^{-1}(U) \cap E] - [p^{-1}(U) \cap s(B)] \\
&= p^{-1}(U) - s(U) \\
&= p^{-1}(U) - s'(U)
\end{aligned}
$$

Therefore, $(\mathcal{F}, \mathcal{F}_0)_{|U}$ will also be an $\mathbb{R}^n-$generalized bundle for all $U \subset B$.

□

Following the notation of Lemma 2.4, if $(\mathcal{F}, \mathcal{F}_0)$ is an $\mathbb{R}^n-$generalized bundle, we call $(\mathcal{F}, \mathcal{F}_0)_{|U}$ the restriction of the $\mathbb{R}^n-$generalized bundle of $(\mathcal{F}, \mathcal{F}_0)$ to $U \subset B$.

**Definition 2.9.** *Let* $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ *and* $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B)$ *be* $\mathbb{R}^n$-*generalized bundles over the same base. We call* $\phi : (\mathcal{F}, \mathcal{F}_0) \to (\mathcal{F}', \mathcal{F}'_0)$ *a fiber map if* $\phi : (E, E_0) \to (E', E'_0)$ *is a map that preserves the fibers, i.e.,* $q \circ \phi = p$.

**Definition 2.10.** *We say that two* $\mathbb{R}^n-$*generalized bundles over the same base,* $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ *and* $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B)$, *are homotopically isomorphic if there exist fiber maps* $\phi : (\mathcal{F}, \mathcal{F}_0) \rightleftarrows (\mathcal{F}', \mathcal{F}'_0) : \psi$ *and* $H : (E, E_0) \times I \to (E, E_0)$ *and* $G : (E', E'_0) \times I \to (E', E'_0)$ *homotopies such that:*

1. $H(\_, 0) = \psi \circ \phi$      4. $G(\_, 0) = \phi \circ \psi$
2. $H(\_, 1) = 1$      5. $G(\_, 1) = 1$
3. $p \circ H(\_, t) = p, \forall t \in I$      6. $q \circ G(\_, t) = q, \forall t \in I$

*The homotopy isomorphism, defined above, will be denoted by* $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}', \mathcal{F}'_0)$.

The notion of homotopy isomorphism, as given in the definition above, coincides with the concept of "fiber homotopy equivalence" given by Fadell in ([20], Definition 2.4, p. 489).

**Definition 2.11.** *We call a* $\mathbb{R}^n-$*generalized bundle* $(\mathcal{F}, \mathcal{F}_0)$, *over* $B$, *locally trivial if there exists an open cover* $\mathcal{U}$ *of* $B$ *such that* $(\mathcal{F}, \mathcal{F}_0)_{|U} \sim_f (\varepsilon_U^n, \varepsilon_U^{n,0})$, *for every* $U \in \mathcal{U}$. *In particular,* $(\mathcal{F}, \mathcal{F}_0)$ *will be called a trivial* $\mathbb{R}^n-$*generalized bundle if* $(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_B^n, \varepsilon_B^{n,0})$.

**Example 2.4.** *Every* $\mathbb{R}^n-$*generalized bundle over a point is trivial.*

*Proof.*

Let us denote by $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, \{*\})$ a $\mathbb{R}^n-$generalized bundle whose base space consists of only one point.

Thus, it is clear that $(E, E_0) = (p^{-1}(*), p^{-1}(*) \cap E_0) \sim (\mathbb{R}^n, \mathbb{R}^n - \{0\})$, that is, there exist fiber maps $\phi : (E, E_0) \rightleftarrows (\mathbb{R}^n, \mathbb{R}^n - \{0\}) : \psi$ and homotopies $H : (E, E_0) \times I \to (E, E_0)$ and $G : (\mathbb{R}^n, \mathbb{R}^n - \{0\}) \times I \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ such that $H(\_, 0) = \psi \circ \phi$, $H(\_, 1) = 1$, $G(\_, 0) = \phi \circ \psi$ and $G(\_, 1) = 1$.

Thus, the following fiber maps and homotopies are well-defined:

1. $\overline{\phi} : (E, E_0) \rightleftarrows (\{*\} \times \mathbb{R}^n, \{*\} \times \mathbb{R}^n - \{0\}) : \overline{\psi}$
   $\overline{\phi}(e) = (*, \phi(e))$
   $\overline{\psi}(*, x) = \psi(x)$

2. $\overline{H} : (E, E_0) \times I \to (E, E_0)$
   $\overline{H}(e, t) = H(e, t)$

3. $\overline{G} : (\{*\} \times \mathbb{R}^n, \{*\} \times \mathbb{R}^n - \{0\}) \times I \to (\{*\} \times \mathbb{R}^n, \{*\} \times \mathbb{R}^n - \{0\})$
   $\overline{G}((*, x), t) = (*, G(x, t))$

It is clear that $\overline{H}(\_, 1) = 1$, $\overline{G}(\_, 1) = 1$, $p \circ \overline{H}(\_, t) = p$ and $p_1 \circ \overline{G}(\_, t) = p_1$ for any $t \in I$. Also:

$$\forall e \in E, \; \begin{aligned} \overline{H}(e, 0) &= H(e, 0) \\ &= \psi \circ \phi(e) \\ &= \overline{\psi}(*, \phi(e)) \\ &= \overline{\psi} \circ \overline{\phi}(e) \end{aligned}$$

$$\forall x \in \mathbb{R}^n, \; \begin{aligned} \overline{G}((*, x), 0) &= (*, G(x, 0)) \\ &= (*, \phi \circ \psi(x)) \\ &= \overline{\phi}(\psi(x)) \\ &= \overline{\phi} \circ \overline{\psi}(*, x) \end{aligned}$$

Therefore, $(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon^n_{\{*\}}, \varepsilon^{n,0}_{\{*\}})$.

$\square$

Regarding specific conditions, the homotopy isomorphism can be guaranteed more simply, as follows:

**Lemma 2.5.** *Let $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ be two $\mathbb{R}^n-$generalized bundles, over the same base. If $\phi : (\mathcal{F}, \mathcal{F}_0) \to (\mathcal{F}', \mathcal{F}'_0)$ is a fiber map such that $\phi$ is a homeomorphism, then $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}', \mathcal{F}'_0)$.*

*Proof.*
Initially, since $\phi$ is a homeomorphism and $q \circ \phi = p$, it follows that $p \circ \phi^{-1} = q$, that is, $\phi^{-1} : (\mathcal{F}, \mathcal{F}_0) \to (\mathcal{F}', \mathcal{F}'_0)$ is also a fiber map.

Thus, it is evident that $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}, \mathcal{F}_0)$, since it is sufficient to define trivial homotopies for the conditions of homotopy isomorphism to be satisfied.

$\square$

In this way, using Lemma 2.5 and the construction from Example 2.2, we obtain the following:

**Example 2.5.** *If $\xi$ is an $\mathbb{R}^n-$vector bundle over a paracompact base, then its associated pair fibration $(\xi, \xi_0)$ will be a locally trivial $\mathbb{R}^n-$generalized bundle.*

Therefore, we can consider that the notion of generalized bundles generalizes the concept of vector bundles. Thus, it is expected that the isomorphism structure between vector bundles is preserved when passing to homotopy isomorphism, as in the following:

**Proposition 2.1.** *Let $\xi$ and $\eta$ be two isomorphic $\mathbb{R}^n-$vector bundles over the same paracompact base. Then, their respective associated $\mathbb{R}^n-$generalized bundles $(\xi, \xi_0)$ and $(\eta, \eta_0)$ are homotopy isomorphic.*

*Proof.*

First, let $\xi = (E, p, B) \cong \eta = (E', q, B)$. Thus, by definition 2.2, there exists a homeomorphism $h : E \to E'$ such that $q \circ h = p$ and $h_{|F} : F \to F'$ is a vector isomorphism, for any fibers $F$ and $F'$ of $\xi$ and $\eta$, respectively, over the same $b \in B$.

Let $s : B \to E$ and $s' : B \to E'$ be the zero sections of $\xi$ and $\eta$, respectively, and consider $E_0 = E - s(B)$ and $E_0' = E' - s'(B)$. If we show that $h(E_0) \subset E_0'$, then $h : (E, E_0) \to (E', E_0')$ will be a homeomorphism and a fiber map, and thus, Lemma 2.5 will guarantee that $(\xi, \xi_0) \sim_f (\eta, \eta_0)$.

To this end, note that:

$$\begin{aligned} e \in E_0 \implies & \; e \in F = p^{-1}(p(e)) \cong \mathbb{R}^n \text{ and } e \neq 0 \\ \implies & \; h(e) \in F' = q^{-1}(p(e)) \cong \mathbb{R}^n \text{ and } h(e) \neq 0 \\ \implies & \; h(e) \in E_0' \end{aligned}$$

Therefore, we conclude that $(\xi, \xi_0) \sim_f (\eta, \eta_0)$.

$\square$

**Lemma 2.6.** *If $(\mathcal{F}, \mathcal{F}_0)$ is a $\mathbb{R}^n-$generalized bundle over $B$ and $f : B' \to B$ is any map, then $f^*(\mathcal{F}, \mathcal{F}_0)$ will also be a $\mathbb{R}^n-$generalized bundle. In particular, if $(\mathcal{F}, \mathcal{F}_0)$ is trivial, then $f^*(\mathcal{F}, \mathcal{F}_0)$ will also be trivial.*

*Proof.*

Due to lemma 2.2, we already know that if $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, then $f^*(\mathcal{F}, \mathcal{F}_0)$ will be a pair fibration with fiber homeomorphic to the fiber of $(\mathcal{F}, \mathcal{F}_0)$. Thus, denoting $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $f^*(\mathcal{F}, \mathcal{F}_0) = (f^*E, f^*E_0, p_1, B')$, it is enough to show that there exists a section of $f^*(\mathcal{F}, \mathcal{F}_0)$ that realizes $f^*E_0$.

To this end, since $(\mathcal{F}, \mathcal{F}_0)$ is a $\mathbb{R}^n-$generalized bundle, there exists a section $s : B \to E$ of $(\mathcal{F}, \mathcal{F}_0)$ that realizes $E_0$. In this way, the section $s' : B' \to f^*E$ of $f^*(\mathcal{F}, \mathcal{F}_0)$ is well-defined, given by $s'(b') = (b', s(f(b')))$. Finally:

$$\begin{aligned} f^*E_0 & = \; [B' \times E_0] \bigcap f^*E \\ & = \; [B' \times (E - s(B))] \bigcap f^*E \\ & = \; [(B' \times E) - (B' \times s(B))] \bigcap f^*E \\ & = \; [(B' \times E) \cap f^*E] - [(B' \times s(B)) \cap f^*E] \\ & = \; f^*E - \{(b', s(b)) \in B' \times s(B) \; : \; f(b') = p(s(b)) = b\} \\ & = \; f^*E - [B' \times s(f(B'))] \\ & = \; f^*E - s'(B') \end{aligned}$$

Therefore, $f^*(\mathcal{F}, \mathcal{F}_0)$ will be a $\mathbb{R}^n-$generalized bundle.

Now, we need to show that if $(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_B^n, \varepsilon_B^{n,0})$, then $f^*(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_{B'}^n, \varepsilon_{B'}^{n,0})$. So, suppose there exist the following fiber maps and homotopies satisfying the relations below:

1. $\phi : (E, E_0) \rightleftarrows (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\})) : \psi$

2. $H : (E, E_0) \times I \to (E, E_0)$

3. $G : (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\})) \times I \to (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\}))$

4. $\phi$, $\psi$, $H$, and $G$ satisfy:

$$
\begin{array}{ll}
H(\_,0) = \psi \circ \phi & G(\_,0) = \phi \circ \psi \\
H(\_,1) = 1 & G(\_,1) = 1 \\
p \circ H(\_,t) = p, \forall t \in I & p_1 \circ G(\_,t) = p_1, \forall t \in I
\end{array}
$$

In this way, the following fiber maps and homotopies are well-defined:

1. $\overline{\phi} : (f^*E, f^*E_0) \rightleftarrows (B' \times \mathbb{R}^n, B' \times (\mathbb{R}^n - \{0\})) : \overline{\psi}$
   $\overline{\phi}(b', e) = (b', p_2 \circ \phi(e))$
   $\overline{\psi}(b', x) = (b', \psi(f(b'), x))$

2. $\overline{H} : (f^*E, f^*E_0) \times I \to (f^*E, f^*E_0)$
   $\overline{H}((b', e), t) = (b', H(e, t))$

3. $\overline{G} : (B' \times \mathbb{R}^n, B' \times (\mathbb{R}^n - \{0\})) \times I \to (B' \times \mathbb{R}^n, B' \times (\mathbb{R}^n - \{0\}))$
   $\overline{G}((b', x), t) = (b', p_2 \circ G((f(b'), x), t))$

Let's show that $\overline{\phi}$, $\overline{\psi}$, $\overline{H}$, and $\overline{G}$ satisfy the conditions of definition 2.10. Indeed:

1. It is clear that $p_1 \circ \overline{H}((\_, \_), t) = p_1$ and $p_1 \circ \overline{G}((\_, \_), t) = p_1$ for all $t \in I$

2. It is also clear that $\overline{H}((\_, \_), 1) = 1$ and $\overline{G}((\_, \_), 1) = 1$

3. $\forall (b', e) \in f^*E$,

$$
\begin{aligned}
\overline{H}((b', e), 0) &= (b', H(e, 0)) \\
&= (b', \psi \circ \phi(e)) \\
&= (b', \psi(p_1 \circ \phi(e), p_2 \circ \phi(e))) \\
&= (b', \psi(p(e), p_2 \circ \phi(e))) \\
&= (b', \psi(f(b'), p_2 \circ \phi(e))) \\
&= \overline{\psi}(b', p_2 \circ \phi(e)) \\
&= \overline{\psi} \circ \overline{\phi}(b', e)
\end{aligned}
$$

4. $\forall (b', x) \in B' \times \mathbb{R}^n$,

$$
\begin{aligned}
\overline{G}((b', x), 0) &= (b', p_2 \circ G((f(b'), x), 0)) \\
&= (b', p_2 \circ \phi \circ \psi(f(b'), x)) \\
&= \overline{\phi}(b', \psi(f(b'), x)) \\
&= \overline{\phi} \circ \overline{\psi}(b', x)
\end{aligned}
$$

Therefore, $f^*(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_{B'}^n, \varepsilon_{B'}^{n,0})$.

$\square$

Following the notation of Lemma 2.6, given $(\mathcal{F}, \mathcal{F}_0)$ a $\mathbb{R}^n-$generalized bundle, we call $f^*(\mathcal{F}, \mathcal{F}_0)$ the $\mathbb{R}^n-$generalized bundle pullback of $(\mathcal{F}, \mathcal{F}_0)$ by the map $f$.

Now, let's look at some examples involving generalized fibration pullbacks.

**Example 2.6.** *Let $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ be two $\mathbb{R}^n-$generalized bundles, over the same base $B$, such that $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}', \mathcal{F}'_0)$. Then, let $1 : B \to B$ be the identity map, and we have $(\mathcal{F}, \mathcal{F}_0) \sim_f 1^*(\mathcal{F}', \mathcal{F}'_0)$.*

*Proof.*

Initially, let us denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B)$. Since $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}', \mathcal{F}'_0)$, there exist the following fiber maps and homotopies satisfying the following relations:

1. $\phi : (E, E_0) \rightleftarrows (E', E'_0) : \psi$

2. $H : (E, E_0) \times I \to (E, E_0)$

3. $G : (E', E'_0) \times I \to (E', E'_0)$

4. $\phi$, $\psi$, $H$, and $G$ satisfy the following relations:
$$H(\_, 0) = \psi \circ \phi \qquad G(\_, 0) = \phi \circ \psi$$
$$H(\_, 1) = 1 \qquad G(\_, 1) = 1$$
$$p \circ H(\_, t) = p, \ \forall t \in I \qquad q \circ G(\_, t) = q, \ \forall t \in I$$

Recall that the $\mathbb{R}^n-$generalized bundle pullback $1^*(\mathcal{F}', \mathcal{F}'_0) = (1^*E', 1^*E'_0, p_1, B)$ is such that:
$$1^*E' = \{(b, e') \in B \times E' : b = q(e')\}$$
$$1^*E'_0 = \{(b, e') \in 1^*E' : e' \in E'_0\}$$

So, the following fiber maps and homotopies are well-defined:

1. $\overline{\phi} : (E, E_0) \rightleftarrows (1^*E', 1^*E'_0) : \overline{\psi}$
   $\overline{\phi}(e) = (p(e), \phi(e))$
   $\overline{\psi}(b, e') = \psi(e')$

2. $\overline{H} : (E, E_0) \times I \to (E, E_0)$
   $\overline{H}(e, t) = H(e, t)$

3. $\overline{G} : (1^*E', 1^*E'_0) \times I \to (1^*E', 1^*E'_0)$
   $\overline{G}((b, e'), t) = (b, G(e', t))$

Thus, it is clear that $\overline{H}(e, 1) = e$, for all $e \in E$, and $p \circ \overline{H}(\_, t) = p$, for all $t \in I$, since $H$ has these properties. Similarly, $\overline{G}((b, e'), 1) = (b, e')$, for all $(b, e') \in 1^*E'$, and $p_1 \circ \overline{G}(\_, t) = p_1$, for all $t \in I$. Additionally, we have that:

$$\forall e \in E, \ \overline{H}(e,0) = \ H(e,0)$$
$$= \ \psi \circ \phi(e)$$
$$= \ \overline{\psi}(p(e),\phi(e))$$
$$= \ \overline{\psi} \circ \overline{\phi}(e)$$

$$\forall (b,e') \in 1^*E', \ \overline{G}((b,e'),0) = \ (b,G(e',0))$$
$$= \ (b,\phi \circ \psi(e'))$$
$$= \ (q(e'),\phi \circ \psi(e'))$$
$$= \ (p \circ \psi(e'),\phi \circ \psi(e'))$$
$$= \ \overline{\phi}(\psi(e'))$$
$$= \ \overline{\phi} \circ \overline{\psi}(b,e')$$

Therefore, $(\mathcal{F},\mathcal{F}_0) \sim_f 1^*(\mathcal{F}',\mathcal{F}'_0)$.

$\square$

**Example 2.7.** *Let $B$ be a topological space, $b_0 \in B$ any point, and $c : B \to \{b_0\}$ the constant map. Then, $(\varepsilon_B^n, \varepsilon_B^{n,0}) \sim_f c^*(\varepsilon_{\{b_0\}}^n, \varepsilon_{\{b_0\}}^{n,0})$.*

*Proof.*
    Recall that $c^*(\varepsilon_{\{b_0\}}^n, \varepsilon_{\{b_0\}}^{n,0}) = (c^*(\{b_0\} \times \mathbb{R}^n), c^*[\{b_0\} \times (\mathbb{R}^n - \{0\})], p_1, B)$ is the $\mathbb{R}^n$-generalized bundle pullback such that:

$$c^*(\{b_0\} \times \mathbb{R}^n) = \ \{(b,(b_0,x)) \in B \times (\{b_0\} \times \mathbb{R}^n) : b_0 = c(b) = p_1(b_0,x) = b_0\}$$
$$= \ B \times \{b_0\} \times \mathbb{R}^n$$

$$c^*[\{b_0\} \times (\mathbb{R}^n - \{0\})] = B \times \{b_0\} \times (\mathbb{R}^n - \{0\})$$

    Thus, $\phi : (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\})) \rightleftarrows (B \times \{b_0\} \times \mathbb{R}^n, B \times \{b_0\} \times (\mathbb{R}^n - \{0\})) : \psi$, given by $\phi(b,x) = (b,b_0,x)$ and $\psi(b,b_0,x) = (b,x)$, respectively, are well-defined fiber maps, with $\phi$ being a homeomorphism with inverse $\psi$.
    Therefore, by lemma 2.5, we conclude that $(\varepsilon_B^n, \varepsilon_B^{n,0}) \sim_f c^*(\varepsilon_{\{b_0\}}^n, \varepsilon_{\{b_0\}}^{n,0})$. $\square$

    The next result is a more general version of lemma 2.5, now involving generalized bundles over different bases.

**Lemma 2.7.** *Assume $(\mathcal{F},\mathcal{F}_0) = (E, E_0, p, B)$ is an $\mathbb{R}^n$-generalized bundle. Given homeomorphisms $h : B' \to B$ and $H : (E', E'_0) \to (E, E_0)$, the structure $(\mathcal{F}',\mathcal{F}'_0) = (E', E'_0, h^{-1} \circ p \circ H, B')$ also forms an $\mathbb{R}^n$-generalized bundle. We also have to $(\mathcal{F}',\mathcal{F}'_0) \sim_f h^*(\mathcal{F},\mathcal{F}_0)$.*

*Proof.*
    Denote $q = h^{-1} \circ p \circ H$. We begin by proving that $(\mathcal{F}',\mathcal{F}'_0)$ is a pair fibration. To this end, let $X$ be any topological space, and let $f : X \to E'$ and $F : X \times I \to B'$ be maps such that $F(\_,0) = q \circ f$.

By defining the maps $g = H \circ f : X \to E$ and $G = h \circ F : X \times I \to B$, it is clear that:

$$
\begin{aligned}
G(\_,0) =& \ h \circ F(\_,0) \\
=& \ h \circ q \circ f \\
=& \ p \circ H \circ f \\
=& \ p \circ g
\end{aligned}
$$

Thus, we obtain the following commutative diagram:



Since $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, we know there exists a map $\widetilde{G} : X \times I \to E$ such that $\widetilde{G}(\_,0) = g$, $p \circ \widetilde{G} = G$, and if $g(x) \in E_0$, then $\widetilde{G}(x, \_) \in E_0$. Thus, defining the map $\widetilde{F} = H^{-1} \circ \widetilde{G} : X \times I \to E'$, we have:

$$
\begin{aligned}
\widetilde{F}(\_,0) =& \ H^{-1} \circ \widetilde{G}(\_,0) \\
=& \ H^{-1} \circ g \\
=& \ H^{-1} \circ H \circ f \\
=& \ f
\end{aligned}
$$

$$
\begin{aligned}
q \circ \widetilde{F} =& \ h^{-1} \circ p \circ H \circ H^{-1} \circ \widetilde{G} \\
=& \ h^{-1} \circ p \circ \widetilde{G} \\
=& \ h^{-1} \circ G \\
=& \ h^{-1} \circ h \circ F \\
=& \ F
\end{aligned}
$$

$$
\begin{aligned}
f(x) \in E_0' \implies& \ g(x) = H \circ f(x) \in E_0 \\
\implies& \ \widetilde{G}(x, \_) \in E_0 \\
\implies& \ \widetilde{F}(x, \_) = H^{-1} \circ \widetilde{G}(x, \_) \in E_0'
\end{aligned}
$$

Thus, we conclude that $(\mathcal{F}', \mathcal{F}'_0)$ is indeed a pair fibration.

Now, we prove that $(\mathcal{F}', \mathcal{F}'_0)$ is an $\mathbb{R}^n-$generalized bundle. Since there exists a section $s : B \to E$ of $(\mathcal{F}, \mathcal{F}_0)$ that realizes $E_0$, we define $s' : H^{-1} \circ s \circ h : B' \to E'$. We have:

$$
\begin{aligned}
q \circ s' &= h^{-1} \circ p \circ H \circ H^{-1} \circ s \circ h \\
&= h^{-1} \circ p \circ s \circ h \\
&= h^{-1} \circ h \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E' - s'(B') &= H^{-1}(E) - H^{-1}(s(h(B'))) \\
&= H^{-1}(E - s(B)) \\
&= H^{-1}(E_0) \\
&= E'_0
\end{aligned}
$$

Thus, $s'$ is a section of $(\mathcal{F}', \mathcal{F}'_0)$ that realizes $E'_0$. Moreover, since every fiber of $(\mathcal{F}, \mathcal{F}_0)$ has the same type of homotopy as $(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, for every $b' \in B'$, we have:

$$
\begin{aligned}
q^{-1}(b') &= H^{-1}(p^{-1}(h(b'))) \\
&\approx p^{-1}(h(b')) \\
&\sim \mathbb{R}^n
\end{aligned}
$$

$$
\begin{aligned}
q^{-1}(b') \cap E'_0 &= H^{-1}(p^{-1}(h(b'))) \cap H^{-1}(E_0) \\
&= H^{-1}(p^{-1}(h(b')) \cap E_0) \\
&\approx p^{-1}(h(b')) \cap E_0 \\
&\sim \mathbb{R}^n - \{0\}
\end{aligned}
$$

Thus, $(\mathcal{F}', \mathcal{F}'_0)$ will be an $\mathbb{R}^n-$generalized bundle.

Finally, we show that $(\mathcal{F}', \mathcal{F}'_0) \sim_f h^*(\mathcal{F}, \mathcal{F}_0)$. To this end, recall that $h^*(\mathcal{F}, \mathcal{F}_0) = (h^*E, h^*E_0, p_1, B')$, where:

$$
h^*E = \{(b', e) \in B' \times E : h(b') = p(e)\}
$$

$$
h^*E_0 = \{(b', e) \in h^*E : e \in E_0\}
$$

Considering $\phi : (E', E'_0) \rightleftarrows (h^*E, h^*E_0) : \psi$ given by $\phi(e') = (q(e'), H(e'))$ and $\psi(b', e) = H^{-1}(e)$, we conclude that $\phi$ and $\psi$ are well-defined pair fibrations, with $\phi$ being a homeomorphism with inverse $\psi$.

Therefore, by Lemma 2.5, we have $(\mathcal{F}', \mathcal{F}'_0) \sim_f h^*(\mathcal{F}, \mathcal{F}_0)$.

$\square$

**Lemma 2.8.** *Assuming that $(\mathcal{F}, \mathcal{F}_0)$ is an $\mathbb{R}^n-$generalized bundle and $(\mathcal{F}', \mathcal{F}'_0)$ is an $\mathbb{R}^m-$generalized bundle, then $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ will be an $\mathbb{R}^{n+m}-$generalized bundle. In particular, if $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ are trivial, then $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ will also be trivial.*

*Proof.*

First, let us denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B')$, and consider $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) = (E'', E''_0, r, B'')$, where:

1. $E'' = E \times E'$

2. $E_0'' = (E \times E_0') \cup (E_0 \times E')$

3. $r = p \times q$

4. $B'' = B \times B'$

Since Lemma 2.3 states that if $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ are pair fibrations, then $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ will be a pair fibration with fiber equal to the product of the fibers of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$. Therefore, it remains to show that there is a section of $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ that realizes $E_0''$.

To do this, since $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ are generalized bundles, there exist sections $s : B \to E$ and $s' : B' \to E'$ of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ that realize $E_0$ and $E_0'$, respectively. Therefore, we have that $s'' : B'' \to E''$, defined by $s''(b, b') = (s(b), s'(b'))$, is a section of $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ such that:

$$
\begin{aligned}
E_0'' &= (E \times E_0') \bigcup (E_0 \times E') \\
&= [E \times (E' - s'(B'))] \bigcup [(E - s(B)) \times E'] \\
&= [(E \times E') - (E \times s'(B'))] \bigcup [(E \times E') - (s(B) \times E')] \\
&= (E \times E') - [(E \times s'(B')) \bigcap (s(B) \times E')] \\
&= E'' - (s(B) \times s'(B')) \\
&= E'' - s''(B'')
\end{aligned}
$$

Thus, $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ will be an $\mathbb{R}^{n+m}$-generalized bundle.

Now, we must show that if $(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_B^n, \varepsilon_B^{n,0})$ and $(\mathcal{F}', \mathcal{F}'_0) \sim_f (\varepsilon_{B'}^m, \varepsilon_{B'}^{m,0})$, then $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) \sim_f (\varepsilon_{B''}^{n+m}, \varepsilon_{B''}^{n+m,0})$. To do so, suppose that there exist the following fiber maps and homotopies satisfying the relations below:

1. $\phi : (E, E_0) \rightleftarrows (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\})) : \psi$

2. $H : (E, E_0) \times I \to (E, E_0)$

3. $G : (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\})) \times I \to (B \times \mathbb{R}^n, B \times (\mathbb{R}^n - \{0\}))$

4. $\phi$, $\psi$, $H$, and $G$ are such that:

$$
\begin{array}{ll}
H(\_, 0) = \psi \circ \phi & G(\_, 0) = \phi \circ \psi \\
H(\_, 1) = 1 & G(\_, 1) = 1 \\
p \circ H(\_, t) = p, \forall t \in I & p_1 \circ G(\_, t) = p_1, \forall t \in I
\end{array}
$$

Also, suppose that there exist the following fiber maps and homotopies satisfying the relations below:

1. $\phi' : (E', E_0') \rightleftarrows (B' \times \mathbb{R}^m, B' \times (\mathbb{R}^m - \{0\})) : \psi'$

2. $H' : (E', E_0') \times I \to (E', E_0')$

3. $G' : (B' \times \mathbb{R}^m, B' \times (\mathbb{R}^m - \{0\})) \times I \to (B' \times \mathbb{R}^m, B' \times (\mathbb{R}^m - \{0\}))$

4. $\phi'$, $\psi'$, $H'$, and $G'$ are such that:

$$H'(\_,0) = \psi' \circ \phi' \qquad\qquad G'(\_,0) = \phi' \circ \psi'$$
$$H'(\_,1) = 1 \qquad\qquad\qquad G'(\_,1) = 1$$
$$q \circ H'(\_,t) = q, \forall t \in I \qquad p_1 \circ G'(\_,t) = p_1, \forall t \in I$$

With this, the following fiber maps and homotopies are well-defined:

1. $\phi'' : (E'', E_0'') \rightleftarrows (B'' \times \mathbb{R}^{n+m}, B'' \times (\mathbb{R}^{n+m} - \{0\})) : \psi''$
   $\phi''(e,e') = ((p(e), q(e')), (p_2 \circ \phi(e), p_2 \circ \phi'(e')))$
   $\psi''((b,b'), (x,x')) = (\psi(b,x), \psi'(b',x'))$

2. $H'' : (E'', E_0'') \times I \to (E'', E_0'')$
   $H''((e,e'),t) = (H(e,t), H'(e',t))$

3. $G'' : (B'' \times \mathbb{R}^{n+m}, B'' \times (\mathbb{R}^{n+m} - \{0\})) \times I \to (B'' \times \mathbb{R}^{n+m}, B'' \times (\mathbb{R}^{n+m} - \{0\}))$
   $G''(((b,b'),(x,x')),t) = ((b,b'),(p_2 \circ G((b,x),t), p_2 \circ G'((b',x'),t)))$

Let's show that $\phi''$, $\psi''$, $H''$, and $G''$ satisfy the conditions of definition 2.10. Indeed:

1. It is clear that $r \circ H''((\_,\_),t) = r$ and $p_1 \circ G''((\_,\_),t) = p_1$.

2. It is clear that $H''(\_,1) = \mathrm{Id}_{(E'',E_0'')}$ and $G''(\_,1) = \mathrm{Id}_{(B'' \times \mathbb{R}^{n+m}, B'' \times (\mathbb{R}^{n+m} - \{0\}))}$.

3. $\forall (e,e') \in E''$,

$$
\begin{aligned}
H''((e,e'),0) &= (H(e,0), H'(e',0)) \\
&= (\psi \circ \phi(e), \psi' \circ \phi'(e')) \\
&= (\psi(p_1 \circ \phi(e), p_2 \circ \phi(e)), \psi'(p_1 \circ \phi'(e'), p_2 \circ \phi'(e'))) \\
&= (\psi(p(e), p_2 \circ \phi(e)), \psi'(q(e'), p_2 \circ \phi'(e'))) \\
&= \psi''((p(e), q(e')), (p_2 \circ \phi(e), p_2 \circ \phi'(e'))) \\
&= \psi'' \circ \phi''(e,e')
\end{aligned}
$$

4. $\forall ((b,b'),(x,x')) \in B'' \times \mathbb{R}^{n+m}$,

$$
\begin{aligned}
G''(((b,b'),(x,x')),0) &= ((b,b'),(p_2 \circ G((b,x),0), p_2 \circ G'((b',x'),0))) \\
&= ((b,b'),(p_2 \circ \phi \circ \psi(b,x), p_2 \circ \phi' \circ \psi'(b',x'))) \\
&= \phi''(\psi(b,x), \psi'(b',x')) \\
&= \phi'' \circ \psi''((b,b'),(x,x'))
\end{aligned}
$$

Thus, $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) \sim_f (\varepsilon_{B''}^{n+m}, \varepsilon_{B''}^{n+m,0})$.                                    $\square$

Following the notations of Lemma 2.8, given $(\mathcal{F}, \mathcal{F}_0)$ as an $\mathbb{R}^n$-generalized bundle and $(\mathcal{F}', \mathcal{F}'_0)$ as an $\mathbb{R}^m$-generalized bundle, we define $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ as the $\mathbb{R}^{n+m}$-generalized bundle product of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$.

**Definition 2.12. (Whitney Sum)** *Consider* $(\mathcal{F}, \mathcal{F}_0)$ *be an* $\mathbb{R}^n$*-generalized bundle and* $(\mathcal{F}', \mathcal{F}'_0)$ *be an* $\mathbb{R}^m$*-generalized bundle, both defined over the same base* $B$*. Let* $d : B \to B \times B$ *denote the diagonal map. The Whitney sum of* $(\mathcal{F}, \mathcal{F}_0)$ *and* $(\mathcal{F}', \mathcal{F}'_0)$ *is defined as the following* $\mathbb{R}^{n+m}$*-generalized bundle:*

$$(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0) = d^*[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)]$$

## 2.3.1   Generalized Tangent Bundle of a Topological Manifold

In this subsection, we will show that the concept of generalized bundle allows us to extend the notion of tangent vector bundle of smooth manifolds to topological manifolds.

To this end, let $M^m$ be a topological manifold and define:

- $T_0 M = \{\omega \in M^I \ : \ \omega(t) = \omega(0) \Leftrightarrow t = 0\}$

- $TM = T_0 M \cup \{\omega \in M^I \ : \ \omega(t) = \omega(0), \ \forall t \in I\}$

- $p : TM \to M$, given by $p(\omega) = \omega(0)$

Thus, we obtain the following:

**Proposition 2.2.** *Let* $M^m$ *be a topological manifold. Then, the pair* $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$ *is a locally trivial* $\mathbb{R}^m$*-generalized bundle.*

The proof of the proposition above can be found in ([20], Proposition 3.8, p. 493).

**Observation 2.1.** *It is worth noting that the construction carried out by Fadell in [20] to prove Proposition 2.2 is only valid for topological manifolds without boundary.*

*Therefore, all topological manifolds mentioned throughout this work will be assumed to have no boundary.*

**Definition 2.13.** *Given a topological manifold* $M^m$*, we call the pair* $(\tau M, \tau_0 M)$ *the* $\mathbb{R}^m$*-generalized tangent bundle of* $M$*.*

On the other hand, let $M^m$ be a smooth manifold and $\xi$ its $\mathbb{R}^m$-tangent vector bundle, as in Definition 2.3, and consider its associated $\mathbb{R}^m$-generalized bundle $(\xi, \xi_0)$. Then, we obtain the following relationship between the tangent vector bundle and the generalized tangent bundle of a smooth manifold:

**Proposition 2.3.** $(\xi, \xi_0) \sim_f (\tau M, \tau_0 M)$

*Sketch of the proof:*

The proof essentially consists in showing that the fibers of these generalized bundles, over the same point, are homotopically equivalent.

Due to the importance of this result, we provide here only a brief intuition of how this is done. The full proof can be found in ([20], Proposition 3.17, p. 495) and [42].

Fixing $b \in M$, we know that the fiber of $\xi$ over $b$ consists of all tangent vectors to $M$ at $b$. Moreover, nonzero tangent vectors can be seen as the derivatives at zero of geodesics $\gamma :] - \delta, \delta[ \to M$ such that $\gamma(0) = b$.

Considering $M$ as a smooth manifold without boundary, the nonzero tangent vectors of $T_b M$ can be identified with derivatives at zero of geodesics $\gamma : [0,1] \to M$ that are arc-length reparametrized and have unit length, satisfying $\gamma(0) = b$.

Now, due to Whitney's embedding theorem[2], we know that $M$ can be embedded in a Euclidean space. Thus, we have enough ambient space to ensure the homotopy equivalence between $T_b M$ and the following space:

$$E = \{\gamma \in M^I : \gamma \text{ is a geodesic reparametrized by arc length with}$$
$$\text{length equal to 1 and } \gamma(0) = b\} \bigcup \{\gamma \in M^I \text{ constant at } b\}$$

On the other hand, Nash states in [42] that the space $E$ defined above is homotopy equivalent to the fiber of $(\tau M, \tau_0 M)$ over $b$, that is, the following set:

$$\{\omega \in M^I : \omega(t) = b \Leftrightarrow t = 0\} \bigcup \{\omega \in M^I \text{ constant at } b\}$$

Thus, we intuitively conclude that the fibers of $(\xi, \xi_0)$ and $(\tau M, \tau_0 M)$ over the same point are homotopy equivalent.
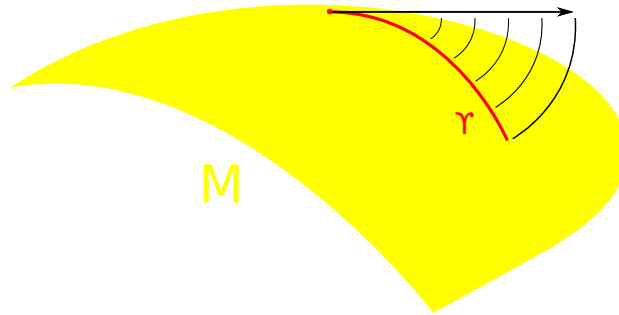
$\square$



Figure 2.2: Homotopy equivalence between a geodesic $\gamma \in M^I$ and its tangent vector at $\gamma(0)$.

Thus, Proposition 2.3 allows us to state that the notion of tangent bundle can be extended from the concept of smooth manifolds to topological manifolds.

---

[2]For more details on Whitney's embedding theorem, see ([35], Theorem 6.15, p. 134).

**Proposition 2.4.** *If $h : M \to N$ is a homeomorphism between topological manifolds, then $(\tau M, \tau_0 M) \sim_f h^*(\tau N, \tau_0 N)$.*

*Proof.*

Since $h$ is a homeomorphism, the map $H : (TM, T_0 M) \to (TN, T_0 N)$ given by $H(\omega) = h \circ \omega$ is one ass well, with inverse $H^{-1}(\omega) = h^{-1} \circ \omega$.

If we denote by $p : TM \to M$ and $q : TN \to N$ the projections of $(\tau M, \tau_0 M)$ and $(\tau N, \tau_0 N)$, respectively, then the map $h^{-1} \circ q \circ H : TM \to M$ satisfies $h^{-1} \circ q \circ H(\omega) = p(\omega)$.

Thus, Lemma 2.7 guarantees that $(\tau M, \tau_0 M) \sim_f h^*(\tau N, \tau_0 N)$.

$\square$

Let us now see that the notion of generalized tangent bundle is natural with respect to the Cartesian product of topological manifolds, in the following sense:

**Proposition 2.5.** *Let $M$ and $S$ be two arbitrary topological manifolds. Then:*

$$(\tau(M \times S), \tau_0(M \times S)) \sim_f (\tau M, \tau_0 M) \times (\tau S, \tau_0 S)$$

*Proof.*

Initially, consider $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$, $(\tau S, \tau_0 S) = (TS, T_0 S, q, S)$, and $(\tau(M \times S), \tau_0(M \times S)) = (T(M \times S), T_0(M \times S), r, M \times S)$. Denote the product $(\tau M, \tau_0 M) \times (\tau S, \tau_0 S) = (E, E_0, p \times q, M \times S)$, where:

$$E = (TM) \times (TS)$$

$$E_0 = [(TM) \times (T_0 S)] \bigcup [(T_0 M) \times (TS)]$$

Consequently, the map $\phi : (T(M \times S), T_0(M \times S)) \to (E, E_0)$ defined by $\phi(\omega) = (p_1 \circ \omega, p_2 \circ \omega)$ is well defined, because if $\omega \in T_0(M \times S)$, then $\omega(t) \neq \omega(0)$ for all $0 < t \leq 1$. That is, $p_i \circ \omega(t) \neq p_i \circ \omega(0)$ for all $0 < t \leq 1$ and $i = 1$ or $2$, and thus, $p_i \circ \omega \in E_0$ for $i = 1$ or $2$.

On the other hand, observe that $\phi$ is a homeomorphism whose inverse map $\phi^{-1} : (E, E_0) \to (T(M \times S), T_0(M \times S))$ is naturally given by $\phi^{-1}(\omega_1, \omega_2) = (\omega_1, \omega_2)$. Moreover, it is clear that $(p \times q) \circ \phi = r$.

Thus, it follows from Lemma 2.5 that:

$$(\tau(M \times S), \tau_0(M \times S)) \sim_f (\tau M, \tau_0 M) \times (\tau S, \tau_0 S)$$

$\square$

## 2.3.2  Generalized Normal Bundle of a Local-Flat Embedding

In this subsection, we will show that the concept of generalized bundle also allows the generalization of the notion of normal vector bundle from smooth manifolds to topological manifolds.

Before moving forward, we recall[3] that if $M^m$ and $S^{m+k}$ are smooth manifolds and if $i : M \hookrightarrow S$ is a smooth embedding, then $i(M)$ constitutes a smooth submanifold of $S$ such that, for every $b \in M$, there exists an open neighborhood $U \subset S$ of $i(b)$ with $(U, U \cap i(M)) \approx (\mathbb{R}^{m+k}, \mathbb{R}^m)$.

This feature leads us to the following:

**Definition 2.14.** *A topological embedding $i : M^m \hookrightarrow S^{m+k}$, between topological manifolds, is said to be locally-flat, or simply local-flat, if for every $b \in M$, there exists an open neighborhood $U \subset S$ of $i(b)$ such that $(U, U \cap i(M)) \approx (\mathbb{R}^{m+k}, \mathbb{R}^m)$.*

In the notation of the definition above, since $M \approx i(M)$, we can rewrite $M^m \subset S^{m+k}$ as a local-flat embedding such that $(U, U \cap M) \approx (\mathbb{R}^{m+k}, \mathbb{R}^m)$. The notation used to describe a local-flat embedding will depend on the problem at hand.

For the next result, consider:

- $M^m \subset S^{m+k}$ a local-flat embedding

- $N_0 = \{\omega \in S^I \ : \ \omega(t) \in M \Leftrightarrow t = 0\}$

- $N = N_0 \bigcup \{\omega \in S^I \ : \ \omega(t) = \omega(0) \in M, \ \forall t \in I\}$

- $q : N \to M$ given by $q(\omega) = \omega(0)$

Thus:

**Proposition 2.6.** *Let $M^m \subset S^{m+k}$ be a local-flat embedding. Then, the pair $(\mathcal{N}, \mathcal{N}_0) = (N, N_0, q, M)$ is a locally trivial $\mathbb{R}^k-$generalized bundle.*

The proof of the proposition above can be found in ([20], Proposition 4.1, p. 496).

**Definition 2.15.** *We call $(\mathcal{N}, \mathcal{N}_0)$ the $\mathbb{R}^k-$generalized normal bundle of the local-flat embedding $M^m \subset S^{m+k}$.*

On the other hand, consider $M^m \subset \mathbb{R}^{m+k}$ a smooth embedding of a smooth manifold into Euclidean space, and let $\eta$ be the normal $\mathbb{R}^k$-vector bundle of this embedding, as in Definition 2.4. Thus, we have the following relation between the normal vector bundle and the generalized normal bundle of the local-flat embedding $M \subset \mathbb{R}^{m+k}$:

---

[3] For more details, see ([35], Proposition 5.16, p. 106).

**Proposition 2.7.** $(\eta, \eta_0) \sim_f (\mathcal{N}, \mathcal{N}_0)$

The proof of the proposition above can be found in ([20], Corollary 4.9, p. 498).

In ([20], Theorem 4.11, p. 498), Fadell shows that Theorem 2.1 is valid in the context of local-flat embeddings, as follows:

**Theorem 2.3.** *If $M^m \subset S^{m+k}$ is a local-flat embedding with $\mathbb{R}^k-$ generalized normal bundle $(\mathcal{N}, \mathcal{N}_0)$, then:*

$$(\tau M, \tau_0 M) \oplus (\mathcal{N}, \mathcal{N}_0) \sim_f (\tau S, \tau_0 S)_{|M}$$

Furthermore, we can obtain the following:

**Proposition 2.8.** *If $i : M^m \hookrightarrow S^{m+k}$ is a local-flat embedding, then:*

$$(\tau S, \tau_0 S)_{|M} \sim_f i^*(\tau S, \tau_0 S)$$

*Proof.*

First, let us fix the following notations:

1. $(\tau S, \tau_0 S) = (T, T_0, p, S)$, where:

   - $T_0 = \{\omega \in S^I \ : \ \omega(t) = \omega(0) \Leftrightarrow t = 0\}$
   - $T = T_0 \bigcup \{\omega \in S^I \ : \ \omega(t) = \omega(0), \ \forall t \in I\}$
   - $p : T \to S$ given by $p(\omega) = \omega(0)$

2. $(\tau S, \tau_0 S)_{|M} = (p^{-1}(M), p^{-1}(M) \cap T_0, q, M)$, where:

   - $p^{-1}(M) = \{\omega \in T \ : \ \omega(0) \in M\}$
   - $p^{-1}(M) \cap T_0 = \{\omega \in T_0 \ : \ \omega(0) \in M\}$
   - $q = p_{|p^{-1}(M)} : p^{-1}(M) \to M$ given by $q(\omega) = \omega(0)$

3. $i^*(\tau S, \tau_0 S) = (i^*T, i^*T_0, p_1, M)$, where:

   - $i^*T = \{(b, \omega) \in M \times T \ : \ i(b) = \omega(0)\}$
   - $i^*T_0 = \{(b, \omega) \in i^*T \ : \ \omega \in T_0\}$
   - $p_1 : i^*T \to M$ given by $p_1(b, \omega) = b$

Thus, the fiber map $\phi : (p^{-1}(M), p^{-1}(M) \cap T_0) \to (i^*T, i^*T_0)$ given by $\phi(\omega) = (\omega(0), \omega)$ is well-defined.

On the other hand, note that the map $\psi : (i^*T, i^*T_0) \to (p^{-1}(M), p^{-1}(M) \cap T_0)$ given by $\psi(b, \omega) = \omega$ is also well-defined.

Furthermore, it is clear that $\phi$ is a homeomorphism with inverse $\psi$. Thus, Lemma 2.5 guarantees that $(\tau S, \tau_0 S)_{|M} \sim_f i^*(\tau S, \tau_0 S)$.

$\square$

With this, we conclude in this chapter the study of generalized bundles, a concept developed by Fadell in [20] to generalize the notions of tangent and normal vector bundles from the context of smooth manifolds to topological manifolds, where we presented here only the intuitive proof, based on Nash's ideas in [42], of how the generalization of the tangent vector bundle occurs.

In order to not only present a modern reinterpretation of the first half of the results presented by Fadell in [20], but also to complement [20], we have carefully shown in detail how the generalized bundles indeed generalize vector bundles, as well as how the notion of vector bundle isomorphism is preserved when extended to the category of generalized bundles.

It is worth noting that we also developed in this chapter the concept of pullback generalized bundle, as well as some consequences of such a bundle, a concept that was not mentioned by Fadell in [20].

In any case, the results about generalized bundles carefully developed in this chapter suggest that we view this concept as a theory in itself and not just as a tool to construct characteristic classes, as we will present in the following chapter.

# Chapter 3

# Characteristic Classes of Topological Manifolds

In this chapter, we will construct the Thom classes, Stiefel-Whitney classes, and Euler classes of generalized bundles, and present some consequences of such objects. In particular, we will examine the behavior of these classes for the generalized tangent bundles of topological manifolds.

To that end, in Section 3.1, we will introduce the concept of orientability of generalized bundles, which was originally proposed by Fadell in [20], in order to guarantee the existence of the Thom class and Thom isomorphism for such bundles. We will also discuss how the Thom class behaves in specific generalized bundles.

In Section 3.2, we will define the Stiefel-Whitney classes of generalized bundles in a manner identical to the definition of the Stiefel-Whitney classes of vector bundles presented in [41]. Furthermore, we will see how the notion of pullback generalized bundle introduced in the previous chapter will be relevant to deduce some consequences concerning the Stiefel-Whitney classes, since Fadell did not address this concept in [20].

Concluding the chapter, in Section 3.3, we will define the Euler class of generalized bundles, a topic that was scarcely addressed by Fadell in [20]. In this section, we will present several well-known results about Euler classes of vector bundles and smooth manifolds, but in their versions for generalized bundles and topological manifolds.

As explained in Remark 2.1, we emphasize that every topological manifold mentioned in this chapter will be a manifold without boundary.

## 3.1   Orientability and Thom Class

Before constructing the Stiefel–Whitney and Euler classes of generalized bundles, we need to introduce the concept of orientability of these bundles. This concept will be fundamental to ensure the existence of the Thom class and the Thom isomorphism, as will be seen later.

The concept of orientability of generalized bundles was proposed by Fadell in [20]. However, Fadell did not delve deeply into this matter, since the main topic developed in [20] concerned the Stiefel–Whitney classes, for which there is no need to address orientability.

In this section, we will explore in more detail the notion of orientability of generalized bundles and present some technical results about the Thom class—more specifically, the behavior of the Thom class in pullback and product generalized bundles, what happens when we reverse the orientation of a generalized bundle, and the relationship between the dimension of a topological manifold and the Thom class of its tangent generalized bundle.

With the exception of Lemmas 3.1, 3.2, 3.3, and 3.4, the results presented in this section were taken from [20].

Thus, let us first consider $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ to be a $\mathbb{R}^n-$ generalized bundle, and the following set:

$$\Omega_p = \{(e, \omega) \in E \times B^I \ : \ p(e) = \omega(0)\}$$

In particular, since $(\mathcal{F}, \mathcal{F}_0)$ is a pair fibration, then by defining the maps $h : \Omega_p \to E$ and $H : \Omega_p \times I \to B$ by $h(e, \omega) = e$ and $H((e, \omega), t) = \omega(t)$, respectively, there exists a map $\widetilde{H} : \Omega_p \times I \to E$ such that the following diagram commutes:



Moreover, if $(e, \omega) \in \Omega_p$ is such that $h(e, \omega) \in E_0$, then $\widetilde{H}((e, \omega), \_) \in E_0$. In this way, we can define the map $\lambda : \Omega_p \to E^I$ by $[\lambda(e, \omega)](t) = \widetilde{H}((e, \omega), t)$.

Thus, fixing the fiber $(F, F_0)$ of $(\mathcal{F}, \mathcal{F}_0)$ over $b_0 \in B$, the following map is well-defined:

$$\Omega(B, b_0) \times (F, F_0) \to (F, F_0)$$
$$(\omega, e) \longmapsto \omega \cdot e = [\lambda(e, \omega)](1)$$

Note that, fixing $\omega \in \Omega(B, b_0)$, it is evident that the map $(F, F_0) \to (F, F_0)$, which sends $e \mapsto \omega \cdot e$, induces an action of $\Omega(B, b_0)$ on $H_n(F, F_0; R)$ for any commutative unital ring $R$.

With this, we have the following:

**Definition 3.1. (Orientability)** *A $\mathbb{R}^n-$ generalized bundle $(\mathcal{F}, \mathcal{F}_0)$ over $B$ is said to be $R-$orientable if, for every $b_0 \in B$, the action of $\Omega(B, b_0)$ on $H_n(F, F_0; R)$ defined above is trivial, where $(F, F_0)$ denotes the fiber of $(\mathcal{F}, \mathcal{F}_0)$ over $b_0 \in B$.*

Naturally, the orientation of a topological manifold is directly related to the orientation of its tangent generalized bundle, as we state in the next result, whose proof can be found in ([20], Proposition 3.16, p. 495).

**Proposition 3.1.** *A topological manifold $M$ is $R-$orientable if and only if its generalized tangent bundle $(\tau M, \tau_0 M)$ is $R$-orientable.*

**Observation 3.1.** *Every generalized bundle is $\mathbb{Z}_2-$orientable, as guaranteed in ([1], Corollary 2.8, p. 243).*

With the aim of defining the Stiefel–Whitney classes of any generalized bundle and the Euler class of a $\mathbb{Z}-$orientable generalized bundle, we need to ensure the existence of the Thom class and Thom isomorphism, just as it is done for vector bundles, as in ([41], Chapters 8, 9, and 10).

**Theorem 3.1.** *Let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ be an $R-$orientable $\mathbb{R}^n-$ generalized bundle. Then, there exists a unique class $\tau \in H^n(E, E_0; R)$ such that the homomorphism $\phi : H^k(B; R) \to H^{n+k}(E, E_0; R)$, given by $\phi(x) = p^*(x) \smile \tau$, is an isomorphism for every integer $k \geq 0$. Moreover, letting $i : (F, F_0) \hookrightarrow (E, E_0)$ be the inclusion of a fiber $(F, F_0)$ of $(\mathcal{F}, \mathcal{F}_0)$ into its total space, and $(u) = H^n(F, F_0; R) \cong R$, the class $\tau$ is uniquely determined by $i^*(\tau) = u$.*

The proof of the theorem above can be found in ([20], Theorem 5.2, p. 502).

**Observation 3.2.** *In the course of the proof of Theorem 3.1, it is shown that the induced maps $i^* : H^n(E, E_0; R) \to H^n(F, F_0; R)$ and $p^* : H^k(B; R) \to H^k(E; R)$ are isomorphisms for all $k \geq 0$.*

**Definition 3.2. (Thom Class and Thom Isomorphism)** *Given an $R-$orientable $\mathbb{R}^n-$ generalized bundle $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, the generator $(\tau) = H^n(E, E_0; R) \cong R$ is called the Thom class, and the isomorphism $\phi : H^k(B; R) \to H^{n+k}(E, E_0; R)$ from Theorem 3.1 is referred to as the Thom isomorphism of $(\mathcal{F}, \mathcal{F}_0)$.*

Now, let us examine some consequences involving orientability and the Thom classes of generalized bundles. Since we will be working with Stiefel–Whitney and Euler classes throughout this chapter, the final lemmas of this section will be stated for $R = \mathbb{Z}_2$ and $R = \mathbb{Z}$.

**Lemma 3.1.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $R-$orientable $\mathbb{R}^n-$ generalized bundle over a base $B$, let $f : B' \to B$ be any map, and let $\tau$ be the Thom class of $(\mathcal{F}, \mathcal{F}_0)$. Then:*

*1. $f^*(\mathcal{F}, \mathcal{F}_0)$ is an $R-$orientable $\mathbb{R}^n-$ generalized bundle.*

*2. If $\tau^*$ is the Thom class of $f^*(\mathcal{F}, \mathcal{F}_0)$, then $\tau^* = 1 \times \tau$.*

*Proof.*

First, denoting $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, we already know from Lemma 2.6 that $f^*(\mathcal{F}, \mathcal{F}_0) = (f^*E, f^*E_0, p_1, B')$ is a generalized $\mathbb{R}^n-$bundle, where:

$$f^*E = \{(b', e) \in B' \times E \ : \ f(b') = p(e)\}$$

$$f^*E_0 = \{(b', e) \in f^*E \ : \ e \in E_0\}$$

Now, fixing $b'_0 \in B'$ and $f(b'_0) \in B$, let $(F^*, F_0^*)$ and $(F, F_0)$ be the fibers of $f^*(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}, \mathcal{F}_0)$ over $b'_0 \in B'$ and $f(b'_0) \in B$, respectively. Due to Lemma 2.2, we also know that $(F^*, F_0^*) = \{b'_0\} \times (F, F_0)$.

Thus, the action defined in Definition 3.1 of $\Omega(B', b'_0)$ on $H_n(F^*, F_0^*; R)$ reduces to the action of $\Omega(B, f(b'_0))$ on $H_n(F, F_0; R)$.

Since $(\mathcal{F}, \mathcal{F}_0)$ is an $R-$orientable generalized bundle, the action of $\Omega(B, f(b'_0))$ on $H_n(F, F_0; R)$ is trivial. Therefore, the action of $\Omega(B', b'_0)$ on $H_n(F^*, F_0^*; R)$ is also trivial, and consequently, $f^*(\mathcal{F}, \mathcal{F}_0)$ is an $R-$orientable generalized bundle.

Finally, denoting the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$ and $f^*(\mathcal{F}, \mathcal{F}_0)$ respectively by the generators $(\tau) = H^n(E, E_0; R)$ and $(\tau^*) = H^n(f^*E, f^*E_0; R)$, we will prove that $\tau^* = 1 \times \tau$.

To do so, consider $i : (F, F_0) \hookrightarrow (E, E_0)$ and $j : (F^*, F_0^*) \hookrightarrow (f^*E, f^*E_0)$ the canonical inclusions. Since $(F^*, F_0^*) = \{b'_0\} \times (F, F_0)$, it follows that $j = 1 \times i$.

Now, fixing the generators $(u) = H^n(F, F_0; R)$ and $(1 \times u) = H^n(F^*, F_0^*; R)$[1], recall that the Thom classes $\tau$ and $\tau^*$ are uniquely determined so that $i^*(\tau) = u$ and $j^*(\tau^*) = 1 \times u$.

However, note that:

$$
\begin{aligned}
j^*(1 \times \tau) &= (1 \times i)^*(1 \times \tau) \\
&= 1 \times i^*(\tau) \\
&= 1 \times u
\end{aligned}
$$

Therefore, by uniqueness, we conclude that $\tau^* = 1 \times \tau$.

$\square$

**Lemma 3.2.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $\mathbb{R}^n-$generalized bundle, $(\mathcal{F}', \mathcal{F}'_0)$ an $\mathbb{R}^m-$generalized bundle, both $R-$orientable. Denoting by $\tau$ and $\tau'$ the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$, respectively, then:*

1. *$(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ is an $\mathbb{R}^{n+m}-$generalized $R-$orientable bundle.*

2. *If $\tau''$ is the Thom class of $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$, then $\tau'' = \tau \times \tau'$.*

*Proof.*

Recalling Lemma 2.8, we know that $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ is an $\mathbb{R}^{n+m}-$generalized bundle. Furthermore, we denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B')$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) = (E'', E''_0, r, B'')$, where:

---

[1]The Künneth formula allows us to assert that $1 \times u$ is a generator of $H^n(F^*, F_0^*; R)$.

$$E'' = E \times E'$$
$$E_0'' = (E \times E_0') \cup (E_0 \times E')$$
$$r = p \times q$$
$$B'' = B \times B'$$

Let $b_0 \in B$, $b_0' \in B'$, and $(b_0, b_0') \in B''$. Consider $(F, F_0)$, $(F', F_0')$, and $(F'', F_0'')$ as the fibers of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ over $b_0 \in B$, $b_0' \in B'$, and $(b_0, b_0') \in B''$, respectively.

Since $(F'', F_0'') = (F, F_0) \times (F', F_0')$, the Künneth formula guarantees us that:

$$H_{n+m}(F'', F_0''; R) \cong H_n(F, F_0; R) \otimes H_m(F', F_0'; R)$$

Moreover, we also have the following homeomorphism:

$$\Omega(B'', (b_0, b_0')) \approx \Omega(B, b_0) \times \Omega(B', b_0')$$

Therefore, the action given in Definition 3.1 of $\Omega(B'', (b_0, b_0'))$ on $H_{n+m}(F'', F_0''; R)$ reduces to the actions of $\Omega(B, b_0)$ and $\Omega(B', b_0')$ on $H_n(F, F_0; R)$ and $H_m(F', F_0'; R)$, respectively.

Since $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ are $R$−orientable generalized bundles, the actions of $\Omega(B, b_0)$ and $\Omega(B', b_0')$ on $H_n(F, F_0; R)$ and $H_m(F', F_0'; R)$, respectively, are trivial. Thus, the action of $\Omega(B'', (b_0, b_0'))$ on $H_{n+m}(F'', F_0''; R)$ is trivial and, consequently, $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ is an $R$−orientable generalized bundle.

Finally, let $(\tau) = H^n(E, E_0; R)$, $(\tau') = H^m(E', E_0'; R)$, and $(\tau'') = H^{n+m}(E'', E_0''; R)$ be the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$, respectively. We now prove that $\tau'' = \tau \times \tau'$.

Consider the canonical inclusions $i : (F, F_0) \hookrightarrow (E, E_0)$, $i' : (F', F_0') \hookrightarrow (E', E_0')$, and $i'' : (F'', F_0'') \hookrightarrow (E'', E_0'')$. Since $(F'', F_0'') = (F, F_0) \times (F', F_0')$ and $(E'', E_0'') = (E, E_0) \times (E', E_0')$, then $i'' = i \times i'$.

Fix the generators $(u) = H^n(F, F_0; R)$, $(u') = H^m(F', F_0'; R)$, and $(u \times u') = H^{n+m}(F'', F_0''; R)$[2] and recall that the Thom classes $\tau$, $\tau'$, and $\tau''$ are uniquely characterized by the equalities $i^*(\tau) = u$, $i'^*(\tau') = u'$, and $i''^*(\tau'') = u \times u'$, respectively.

However, note that:

$$
\begin{aligned}
i''^*(\tau \times \tau') &= (i \times i')^*(\tau \times \tau') \\
&= i^*(\tau) \times i'^*(\tau') \\
&= u \times u'
\end{aligned}
$$

Hence, by uniqueness, we conclude that $\tau'' = \tau \times \tau'$.

$\square$

**Lemma 3.3.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be a $\mathbb{R}^n$−generalized bundle that is $\mathbb{Z}$−orientable, and let $\tau$ be its Thom class. Denoting by $-(\mathcal{F}, \mathcal{F}_0)$ the same $\mathbb{R}^n$−generalized bundle with the orientation reversed, and by $\tau'$ its Thom class, then $\tau' = -\tau$.*

---

[2]The Künneth formula guarantees that $u \times u'$ is a generator of $H^{n+m}(F'', F_0''; R)$.

*Proof.*

Let us denote by $(E, E_0)$ the total space of $(\mathcal{F}, \mathcal{F}_0)$ and by $(F, F_0)$ an arbitrary fiber of $(\mathcal{F}, \mathcal{F}_0)$.

The result follows directly from Theorem 3.1 and Definition 3.2, since the Thom class of $(\mathcal{F}, \mathcal{F}_0)$ is the generator $(\tau) = H^n(E, E_0; \mathbb{Z}) \cong \mathbb{Z}$, which is directly related to the generator $(u) = H^n(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$, and consequently, the Thom class of $-(\mathcal{F}, \mathcal{F}_0)$ is the generator $(-\tau) = H^n(E, E_0; \mathbb{Z}) \cong \mathbb{Z}$, directly related to the generator $(-u) = H^n(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$.

$\square$

**Lemma 3.4.** *Let $M^m$ be a closed topological manifold that is $\mathbb{Z}-$orientable and of odd dimension, and let $(\tau M, \tau_0 M)$ be its $\mathbb{R}^m-$generalized tangent bundle. Then the Thom class $\tau$ of $(\tau M, \tau_0 M)$ satisfies $\tau \smile \tau = 0$.*

*Proof.*

First, since $M^m$ is a $\mathbb{Z}-$orientable topological manifold, Proposition 3.1 ensures that the $\mathbb{R}^m-$generalized tangent bundle $(\tau M, \tau_0 M)$ of $M$ is also $\mathbb{Z}-$orientable.

On the other hand, denoting by $(TM, T_0M)$ the total space of $(\tau M, \tau_0 M)$ and recalling that $M$ is compact, the Thom isomorphism for $(\tau M, \tau_0 M)$ gives:

$$H^{2m}(TM, T_0M; \mathbb{Z}) \cong H^m(M; \mathbb{Z}) \cong \mathbb{Z}$$

Therefore, denoting the generator $(\tau) = H^m(TM, T_0M; \mathbb{Z})$ as the Thom class of $(\tau M, \tau_0 M)$, and since item 5 of Lemma A.1 states that $\tau \smile \tau = (-1)^{m^2}(\tau \smile \tau)$ and $m$ is odd, it follows that $\tau \smile \tau = -(\tau \smile \tau)$, i.e., $2(\tau \smile \tau) = 0$.

Since $2(\tau \smile \tau) \in H^{2m}(TM, T_0M; \mathbb{Z}) \cong \mathbb{Z}$, we conclude that $\tau \smile \tau = 0$.

$\square$

## 3.2 Stiefel-Whitney Classes

The construction of the Stiefel–Whitney classes for generalized bundles will follow, due to Theorem 3.1, in exactly the same way as the construction given in ([41], Chapter 8) for vector bundles. Note that, since every generalized bundle is $\mathbb{Z}_2-$orientable and the Stiefel–Whitney classes are defined in the context of singular cohomology $\mathbb{Z}_2-$modules, we do not need to impose any orientability condition on the generalized bundles throughout this section.

With the exception of Propositions 3.2 and 3.3, the results in this section were taken from [20] and [21]. Furthermore, it is worth noting that Theorems 3.3 and 3.6, and Corollary 3.3, can be found in ([21], Lemma 2.11, p. 39), ([21], Theorem 5.2, p. 52), and ([20], Theorem 6.11, p. 504), respectively. In these references, Fadell presents more technical proofs without using pullback generalized bundles. In contrast, the proofs provided here highlight the importance of the pullback generalized bundle.

Thus, let $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ be an $\mathbb{R}^n-$generalized bundle and $\phi$ its Thom isomorphism. Then, for every $k \geq 0$, the following composition makes sense:[3]

$$H^n(E, E_0; \mathbb{Z}_2) \xrightarrow{\;\;Sq^k\;\;} H^{n+k}(E, E_0; \mathbb{Z}_2) \xrightarrow{\;\;\phi^{-1}\;\;} H^k(B; \mathbb{Z}_2)$$

**Definition 3.3. (Stiefel–Whitney Classes)** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $\mathbb{R}^n-$generalized bundle over a base $B$, and let $\phi$ and $\tau$ denote its Thom isomorphism and Thom class, respectively. For each $k \geq 0$, the $k$th Stiefel–Whitney class of $(\mathcal{F}, \mathcal{F}_0)$ is defined as:*

$$w_k(\mathcal{F}, \mathcal{F}_0) = \phi^{-1} \circ Sq^k(u) \in H^k(B; \mathbb{Z}_2)$$

*We also define the element $W(\mathcal{F}, \mathcal{F}_0) = \displaystyle\sum_{k \geq 0} w_k(\mathcal{F}, \mathcal{F}_0) \in H^*(B; \mathbb{Z}_2)$ as the total Stiefel–Whitney class of $(\mathcal{F}, \mathcal{F}_0)$.*

Now, let us examine some consequences of the Stiefel–Whitney classes.

**Theorem 3.2.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be a $\mathbb{R}^n-$generalized bundle. Then $w_0(\mathcal{F}, \mathcal{F}_0) = 1$ and $w_k(\mathcal{F}, \mathcal{F}_0) = 0$ for $k > n$. In other words, $W(\mathcal{F}, \mathcal{F}_0) = 1 + \displaystyle\sum_{k=1}^{n} w_k(\mathcal{F}, \mathcal{F}_0)$.*

*Proof.*

Consider $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, $\phi$ its Thom isomorphism, and $\tau \in H^n(E, E_0; \mathbb{Z}_2)$ its Thom class. Since $\phi(1) = \tau$ and $Sq^0 = 1$, we have:

$$\begin{aligned} w_0(\mathcal{F}, \mathcal{F}_0) &= \phi^{-1} \circ Sq^0(\tau) \\ &= \phi^{-1}(\tau) \\ &= 1 \end{aligned}$$

On the other hand, for $k > n$, we have $Sq^k = 0$, and thus $w_k(\mathcal{F}, \mathcal{F}_0) = 0$. Therefore, the total Stiefel-Whitney class of $(\mathcal{F}, \mathcal{F}_0)$ is $W(\mathcal{F}, \mathcal{F}_0) = 1 + \displaystyle\sum_{k=1}^{n} w_k(\mathcal{F}, \mathcal{F}_0)$.

$\square$

Note that, given a $\mathbb{R}^n-$generalized bundle $(\mathcal{F}, \mathcal{F}_0)$ over $B$, since Theorem 3.2 guarantees that $w_0(\mathcal{F}, \mathcal{F}_0) = 1$, the total Stiefel-Whitney class is an invertible element[4] in the ring $H^*(B; \mathbb{Z}_2)$, whose inverse element will be denoted by $W^{-1}(\mathcal{F}, \mathcal{F}_0)$.

---

[3] $Sq^k$ denotes the Steenrod square operation, some of whose properties can be found in Section A.3.

[4] See Theorem A.3.

**Theorem 3.3.** *Let $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ be two $\mathbb{R}^n-$generalized bundles over $B$ and $B'$, respectively. If $f : B \to B'$ is a map such that $(\mathcal{F}, \mathcal{F}_0) \sim_f f^*(\mathcal{F}', \mathcal{F}'_0)$, then $W(\mathcal{F}, \mathcal{F}_0) = f^*(W(\mathcal{F}', \mathcal{F}'_0))$.*

*Proof.*

Initially, consider $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B')$, and recall that $f^*(\mathcal{F}', \mathcal{F}'_0) = (f^*E', f^*E'_0, p_1, B)$ is the $\mathbb{R}^n-$ generalizedbundle such that:

$$f^*E' = \{(b, e') \in B \times E' \ : \ f(b) = q(e')\}$$

$$f^*E'_0 = \{(b, e') \in f^*E' \ : \ e' \in E'_0\}$$

Since $(\mathcal{F}, \mathcal{F}_0) \sim_f f^*(\mathcal{F}', \mathcal{F}'_0)$, Definition 2.10 ensures that there exists a fibered map $g : (E, E_0) \to (f^*E', f^*E'_0)$ such that $g$ is, in particular, a homotopy equivalence.

Thus, considering the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$ and $f^*(\mathcal{F}', \mathcal{F}'_0)$ as the generators $(\tau) = H^n(E, E_0; \mathbb{Z}_2)$ and $(\tau^*) = H^n(f^*E', f^*E'_0; \mathbb{Z}_2)$, respectively, the fact that $g$ is a homotopy equivalence allows us to assert that:

$$g^*(\tau^*) = \tau$$

Conversely, by regarding the Thom class of $(\mathcal{F}', \mathcal{F}'_0)$ as the generator $(\tau') = H^n(E', E'_0; \mathbb{Z}_2)$, the canonical projection $p_2 : (f^*E', f^*E'_0) \to (E', E'_0)$ satisfies, guarantees by Lemma 3.1, that :

$$p_2^*(\tau') = \tau^* = 1 \times \tau'$$

Therefore, we have:

$$g^* \circ p_2^*(\tau') = \tau$$

Furthermore, by definition, we have the following commutative diagram:



Now, denoting the Thom isomorphisms of $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ respectively by $\phi : H^k(B; \mathbb{Z}_2) \to H^{k+n}(E, E_0; \mathbb{Z}_2)$ and $\phi' : H^k(B'; \mathbb{Z}_2) \to H^{k+n}(E', E'_0; \mathbb{Z}_2)$, we can also assert that $\phi \circ f^* = g^* \circ p_2^* \circ \phi'$. Indeed:

$$\forall x \in H^*(B'; \mathbb{Z}_2),\ g^* \circ p_2^* \circ \phi'(x) \begin{aligned} &= g^* \circ p_2^*(q^*(x) \smile \tau') \\ &= [g^* \circ p_2^* \circ q^*(x)] \smile [g^* \circ p_2^*(\tau')] \\ &= [p^* \circ f^*(x)] \smile \tau \\ &= \phi \circ f^*(x) \end{aligned}$$

With this, we have:

$$\forall k \geq 0, f^*(w_k(\mathcal{F}', \mathcal{F'}_0)) \begin{aligned} &= f^* \circ \phi'^{-1} \circ Sq^k(\tau') \\ &= \phi^{-1} \circ g^* \circ p_2^* \circ Sq^k(\tau') \\ &= \phi^{-1} \circ Sq^k \circ g^* \circ p_2^*(\tau') \\ &= \phi^{-1} \circ Sq^k(\tau) \\ &= w_k(\mathcal{F}, \mathcal{F}_0) \end{aligned}$$

Therefore, $f^*(W(\mathcal{F}', \mathcal{F'}_0)) = W(\mathcal{F}, \mathcal{F}_0)$.

$\square$

**Corollary 3.1.** *Let* $(\mathcal{F}, \mathcal{F}_0)$ *and* $(\mathcal{F}', \mathcal{F'}_0)$ *be two* $\mathbb{R}^n$*-generalized bundles over the same base. If* $(\mathcal{F}, \mathcal{F}_0) \sim_f (\mathcal{F}', \mathcal{F'}_0)$*, then* $W(\mathcal{F}, \mathcal{F}_0) = W(\mathcal{F}', \mathcal{F'}_0)$.

*Proof.*

Let $B$ be the base of both $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F'}_0)$, and let $1 : B \to B$ be the identity map. From Example 2.6, we know that $(\mathcal{F}, \mathcal{F}_0) \sim_f 1^*(\mathcal{F}', \mathcal{F'}_0)$.

Thus, it follows from Theorem 3.3 that:

$$W(\mathcal{F}, \mathcal{F}_0) = 1^*(W(\mathcal{F}', \mathcal{F'}_0)) = W(\mathcal{F}', \mathcal{F'}_0)$$

$\square$

**Corollary 3.2.** *If* $(\mathcal{F}, \mathcal{F}_0)$ *is a trivial* $\mathbb{R}^n$*-generalized bundle, then* $W(\mathcal{F}, \mathcal{F}_0) = 1$.

*Proof.*

Initially, it follows from Theorem 3.2 that $w_0(\mathcal{F}, \mathcal{F}_0) = 1$.

Now, let $B$ be the base of $(\mathcal{F}, \mathcal{F}_0)$. Since $(\mathcal{F}, \mathcal{F}_0)$ is a trivial $\mathbb{R}^n$-generalized bundle, we have $(\mathcal{F}, \mathcal{F}_0) \sim_f (\varepsilon_B^n, \varepsilon_B^{n,0})$. Moreover, given any $b \in B$, let $c : B \to \{b\}$ be the constant map. From Example 2.7, it follows that $(\varepsilon_B^n, \varepsilon_B^{n,0}) \sim_f c^*(\varepsilon_{\{b\}}^n, \varepsilon_{\{b\}}^{n,0})$.

Thus, combining Corollary 3.1 with Theorem 3.3, we obtain:

$$W(\mathcal{F}, \mathcal{F}_0) = c^*(W(\varepsilon_{\{b\}}^n, \varepsilon_{\{b\}}^{n,0}))$$

But since $w_k(\varepsilon_{\{b\}}^n, \varepsilon_{\{b\}}^{n,0}) \in H^k(\{b\}; \mathbb{Z}_2) = 0$ for $k > 0$, it follows that $w_k(\mathcal{F}, \mathcal{F}_0) = 0$ for $k > 0$.

Therefore, $W(\mathcal{F}, \mathcal{F}_0) = 1$.

$\square$

**Theorem 3.4.** *Let $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ be an $\mathbb{R}^n-$generalized bundle and an $\mathbb{R}^m-$generalized bundle, respectively. Then:*

$$W[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)] = W(\mathcal{F}, \mathcal{F}_0) \times W(\mathcal{F}', \mathcal{F}'_0)$$

*Proof.*

Initially, let us denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B')$ and recall that $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) = (E'', E''_0, r, B'')$ is the $\mathbb{R}^{n+m}-$generalized bundle such that:

$E'' = E \times E'$
$E''_0 = (E \times E'_0) \bigcup (E_0 \times E')$
$r = p \times q$
$B'' = B \times B'$

Considering the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$ as the generators $(\tau) \in H^n(E, E_0; \mathbb{Z}_2)$, $(\tau') \in H^m(E', E'_0; \mathbb{Z}_2)$, and $(\tau'') \in H^{n+m}(E'', E''_0; \mathbb{Z}_2)$, respectively, we know from Lemma 3.2 that $\tau'' = \tau \times \tau'$.

Now, let $\phi : H^k(B; \mathbb{Z}_2) \to H^{k+n}(E, E_0; \mathbb{Z}_2)$, $\phi' : H^k(B'; \mathbb{Z}_2) \to H^{k+m}(E', E'_0; \mathbb{Z}_2)$ and $\phi'' : H^k(B''; \mathbb{Z}_2) \to H^{k+n+m}(E'', E''_0; \mathbb{Z}_2)$ be the Thom isomorphisms of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$, and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$, respectively. We obtain that $\phi'' = \phi \times \phi'$. In fact:

$$
\begin{aligned}
\forall x \times x' \in H^*(B''; \mathbb{Z}_2), \; \phi''(x \times x') &= r^*(x \times x') \smile \tau'' \\
&= [p^*(x) \times q^*(x')] \smile (\tau \times \tau') \\
&= [p^*(x) \smile \tau] \times [q^*(x') \smile \tau'] \\
&= \phi(x) \times \phi'(x') \\
&= (\phi \times \phi')(x \times x')
\end{aligned}
$$

With this, we conclude that:
$$
\begin{aligned}
\forall k \geq 0, \; w_k[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)] &= \phi''^{-1} \circ Sq^k(\tau'') \\
&= \phi''^{-1} \circ Sq^k(\tau \times \tau') \\
&= \phi''^{-1} \left[ \sum_{a+b=k} \left( Sq^a(\tau) \times Sq^b(\tau') \right) \right] \\
&= \sum_{a+b=k} \left[ \phi''^{-1} \left( Sq^a(\tau) \times Sq^b(\tau') \right) \right] \\
&= \sum_{a+b=k} \left[ \left( \phi^{-1} \circ Sq^a(\tau) \right) \times \left( \phi'^{-1} \circ Sq^b(\tau') \right) \right] \\
&= \sum_{a+b=k} \left[ w_a(\mathcal{F}, \mathcal{F}_0) \times w_b(\mathcal{F}', \mathcal{F}'_0) \right]
\end{aligned}
$$

Therefore, $W[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)] = W(\mathcal{F}, \mathcal{F}_0) \times W(\mathcal{F}', \mathcal{F}'_0)$.

$\square$

**Corollary 3.3. (Whitney product)** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $\mathbb{R}^n$-generalized bundle and $(\mathcal{F}', \mathcal{F}'_0)$ an $\mathbb{R}^m$-generalized bundle, both over the same base. Then:*

$$W[(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0)] = W(\mathcal{F}, \mathcal{F}_0) \smile W(\mathcal{F}', \mathcal{F}'_0)$$

*Proof.*

Let $B$ be the base of both $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$, and let $d : B \to B \times B$ be the diagonal map. Then, we know that $(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0) = d^*[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)]$.

Theorems 3.3 and 3.4 then ensure that:

$$
\begin{aligned}
W[(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0)] &= d^*[W[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)]] \\
&= d^*[W(\mathcal{F}, \mathcal{F}_0) \times W(\mathcal{F}', \mathcal{F}'_0)] \\
&= W(\mathcal{F}, \mathcal{F}_0) \smile W(\mathcal{F}', \mathcal{F}'_0)
\end{aligned}
$$

$\square$

As a direct combination of the Whitney product with Corollary 3.2, we obtain the following:

**Corollary 3.4.** *Let $(\mathcal{F}, \mathcal{F}_0)$ and $(\mathcal{F}', \mathcal{F}'_0)$ be an $\mathbb{R}^n$-generalized bundle and an $\mathbb{R}^m$-generalized bundle, respectively, both over the same base. Then:*

1. *if $(\mathcal{F}, \mathcal{F}_0)$ is trivial, then $W[(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0)] = W(\mathcal{F}', \mathcal{F}'_0)$*

2. *if $(\mathcal{F}, \mathcal{F}_0) \oplus (\mathcal{F}', \mathcal{F}'_0)$ is trivial, then $W(\mathcal{F}, \mathcal{F}_0) = W^{-1}(\mathcal{F}', \mathcal{F}'_0)$*

Now, let us examine how the Stiefel–Whitney classes behave in the context of topological manifolds. For this purpose, we need the following:

**Definition 3.4.** *Given a topological manifold $M$, we denote its total Stiefel–Whitney class by $W(M) = W(\tau M, \tau_0 M)$.*

**Theorem 3.5.** *If $M^m$ is a smooth manifold, then the construction of $W(M)$ (developed in this section) coincides with the classical notion of the Stiefel–Whitney classes as in ([41], Chapter 8).*

*Proof.*

Since $M$ is smooth, we can combine Proposition 2.3 with Corollary 3.1, since denoting by $\xi$ the tangent $\mathbb{R}^m$-vector bundle of $M$, and by $(\xi, \xi_0)$ its associated $\mathbb{R}^m$-generalized bundle, the construction of $W(\xi, \xi_0)$ coincides with the classical construction of $W(\xi)$ as in ([41], Chapter 8).

$\square$

**Proposition 3.2.** *Let $M$ and $S$ be two topological manifolds. Then:*

$$W(M \times S) = W(M) \times W(S)$$

*Proof.*

It follows directly from the combination of Proposition 2.5, Corollary 3.1, and Theorem 3.4.

$\square$

As a direct combination of Lemma 2.7 with Theorem 3.3, we obtain the following:

**Proposition 3.3.** *If $h : M \to S$ is a homeomorphism between two topological manifolds, then $W(M) = h^*(W(S))$.*

Proposition 3.3 can be extended to homotopy equivalence; however, since its proof requires more advanced tools, we postpone it to the next chapter.

Let us also consider the behavior of Stiefel–Whitney classes under local-flat embeddings.

**Theorem 3.6. (Whitney Duality)** *If $i : M^m \hookrightarrow S^{m+k}$ is a local-flat embedding with $\mathbb{R}^k-$generalized normal bundle $(\mathcal{N}, \mathcal{N}_0)$, then:*

$$W(M) \smile W(\mathcal{N}, \mathcal{N}_0) = i^*(W(S))$$

*Proof.*

Due to Theorem 2.3 and Proposition 2.8, we have:

$$(\tau M, \tau_0 M) \oplus (\mathcal{N}, \mathcal{N}_0) \sim_f (\tau S, \tau_0 S)_{|M}$$

$$(\tau S, \tau_0 S)_{|M} \sim_f i^*(\tau S, \tau_0 S)$$

Combining the Whitney product with Corollary 3.1 and Theorem 3.3, we conclude that $W(M) \smile W(\mathcal{N}, \mathcal{N}_0) = i^*(W(S))$.

$\square$

**Corollary 3.5.** *If $M^m \subset \mathbb{R}^{m+k}$ is a local-flat embedding with $\mathbb{R}^k-$generalized normal bundle $(\mathcal{N}, \mathcal{N}_0)$, then $W(M) = W^{-1}(\mathcal{N}, \mathcal{N}_0)$.*

*Proof.* It follows directly from Theorem 3.6, since $\mathbb{R}^{m+k}$ is a contractible topological space, and consequently, $W(\mathbb{R}^{n+k}) = 1$.

$\square$

## 3.3   Euler Class

We conclude this chapter by defining the Euler class of a generalized bundle, for which we now need to impose the condition of $\mathbb{Z}-$orientability.

We will show how the Euler class relates to the Stiefel–Whitney classes, and how this condition induces some results analogous to those in the context of Stiefel–Whitney classes, but now for the Euler class. Furthermore, we will see under which conditions the Euler class vanishes.

In this section, only Proposition 3.4 was taken from [20].

**Definition 3.5. (Euler Class)** *Let $(\mathcal{F}, \mathcal{F}_0)$ be a $\mathbb{Z}-$orientable $\mathbb{R}^n-$generalized bundle over a base $B$, and let $\phi$ and $\tau$ denote its Thom isomorphism and Thom class, respectively. The Euler class of $(\mathcal{F}, \mathcal{F}_0)$ is defined as the following class:*

$$e(\mathcal{F}, \mathcal{F}_0) = \phi^{-1}(\tau \smile \tau) \in H^n(B; \mathbb{Z})$$

*In particular, we denote the Euler class of a topological manifold $M$ by:*

$$e(M) = e(\tau M, \tau_0 M)$$

Recalling Observation 3.2, the next result provides an alternative characterization of the Euler class, whose proof can be found in ([20], Proposition 7.11, p. 510).

**Proposition 3.4.** *Let us consider $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ an $\mathbb{R}^n-$generalized bundle that is $\mathbb{Z}-$orientable, $\tau$ its Thom class, and $i_E : E \hookrightarrow (E, E_0)$ the canonical inclusion. Then:*

$$e(\mathcal{F}, \mathcal{F}_0) = (p^*)^{-1} \circ i_E^*(\tau)$$

Now, we show how the Stiefel–Whitney and Euler classes are related.

**Theorem 3.7.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $\mathbb{R}^n-$generalized bundle that is $\mathbb{Z}-$orientable. Then, the n-th Stiefel–Whitney class of $(\mathcal{F}, \mathcal{F}_0)$ is the mod 2 reduction of its Euler class. In other words, the canonical projection $\rho_2 : \mathbb{Z} \to \mathbb{Z}_2$ is such that:*

$$(\rho_2)_n(e(\mathcal{F}, \mathcal{F}_0)) = w_n(\mathcal{F}, \mathcal{F}_0)$$

*Proof.*
     First, let us see how to construct a mod 2 reduction in the setting of singular cohomology modules. To do this, consider the following short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}_2 \longrightarrow 0$$

If $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, we deduce from ([44], Theorem 11, p. 239) that there are homomorphisms[5] $(\times 2)_k : H^k(B; \mathbb{Z}) \to H^k(B; \mathbb{Z})$, $(\rho_2)_k : H^k(B; \mathbb{Z}) \to H^k(B; \mathbb{Z}_2)$, and $\beta^k : H^k(B; \mathbb{Z}_2) \to H^{k+1}(B; \mathbb{Z})$ such that the following is a long exact sequence:

$$\cdots \longrightarrow H^k(B; \mathbb{Z}) \xrightarrow{(\times 2)_k} H^k(B; \mathbb{Z}) \xrightarrow{(\rho_2)_k} H^k(B; \mathbb{Z}_2) \xrightarrow{\beta^k} H^{k+1}(B; \mathbb{Z}) \longrightarrow \cdots$$

Similarly, the same homomorphisms above are also defined for the pair $(E, E_0)$.
     Since $(\mathcal{F}, \mathcal{F}_0)$ is an $\mathbb{Z}-$orientable $\mathbb{R}^n-$generalized bundle, we may define its Thom class $\tau \in H^n(E, E_0; \mathbb{Z})$ and Thom isomorphism $\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$.

---

[5]The connecting homomorphism $\beta^k$ is known as the Bockstein cohomology homomorphism.

On the other hand, since $(\mathcal{F}, \mathcal{F}_0)$ is also an $\mathbb{Z}_2-$orientable $\mathbb{R}^n-$generalized bundle, we can also consider its Thom class and Thom isomorphism $\tau_2 \in H^n(E, E_0; \mathbb{Z}_2)$ and $\phi_2 : H^k(B; \mathbb{Z}_2) \to H^{k+n}(E, E_0; \mathbb{Z}_2)$, respectively.

Due to the naturality of the mod 2 reduction homomorphism, we obtain that the following diagram commutes:

$$
\begin{array}{ccc}
H^k(B; \mathbb{Z}) & \xrightarrow{\ \ (\rho_2)_k\ \ } & H^k(B; \mathbb{Z}_2) \\[2em]
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi_2} \\[2em]
H^{k+n}(E, E_0; \mathbb{Z}) & \xrightarrow{\ \ (\overline{\rho_2})_{k+n}\ \ } & H^{k+n}(E, E_0; \mathbb{Z}_2)
\end{array}
$$

Just to clarify the notation above, $\rho_2$ and $\overline{\rho_2}$ denote the mod 2 reductions for $B$ and $(E, E_0)$, respectively.

In particular, if we consider the diagram above for $k = 0$, we obtain:

$$
\begin{aligned}
(\overline{\rho_2})_n(\tau) &= (\overline{\rho_2})_n \circ \phi(1) \\
&= \phi_2 \circ (\rho_2)_0(1) \\
&= \phi_2(1) \\
&= \tau_2
\end{aligned}
$$

Hence, we conclude that:

$$
\begin{aligned}
(\rho_2)_n(e(\mathcal{F}, \mathcal{F}_0)) &= (\rho_2)_n \circ \phi^{-1}(\tau \smile \tau) \\
&= \phi_2^{-1} \circ (\overline{\rho_2})_{2n}(\tau \smile \tau) \\
&= \phi_2^{-1}((\overline{\rho_2})_n(\tau) \smile (\overline{\rho_2})_n(\tau)) \\
&= \phi_2^{-1}(\tau_2 \smile \tau_2) \\
&= \phi_2^{-1} \circ Sq^n(\tau_2) \\
&= w_n(\mathcal{F}, \mathcal{F}_0)
\end{aligned}
$$

$\square$

**Theorem 3.8.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be an $\mathbb{Z}-orientable$ $\mathbb{R}^n-generalized$ bundle over a base $B$, and let $f : B' \to B$ be any map. Then:*

$$
e\left(f^*(\mathcal{F}, \mathcal{F}_0)\right) = f^*\left(e(\mathcal{F}, \mathcal{F}_0)\right)
$$

*Proof.*

First, denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ and $f^*(\mathcal{F}, \mathcal{F}_0) = (f^*E, f^*E_0, p_1, B')$, where:

$$
f^*E = \{(b', e) \in B' \times E \ : \ f(b') = p(e)\}
$$

$$
f^*E_0 = \{(b', e) \in f^*E \ : \ e \in E_0\}
$$

Now, consider the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$ and $f^*(\mathcal{F}, \mathcal{F}_0)$ as being, respectively, the generators $(\tau) \in H^n(E, E_0; \mathbb{Z})$ and $(\tau^*) \in H^n(f^*E, f^*E_0; \mathbb{Z})$.

Let $p_2 : (f^*E, f^*E_0) \to (E, E_0)$ be the canonical projection; then Lemma 3.1 ensures that:

$$p_2^*(\tau) = \tau^* = 1 \times \tau$$

On the other hand, considering the Thom isomorphisms of $(\mathcal{F}, \mathcal{F}_0)$ and $f^*(\mathcal{F}, \mathcal{F}_0)$ as $\phi : H^k(B; \mathbb{Z}) \to H^{n+k}(E, E_0; \mathbb{Z})$ and $\psi : H^k(B'; \mathbb{Z}) \to H^{k+n}(f^*E, f^*E_0; \mathbb{Z})$, respectively, we obtain, via computations analogous to those in Theorem 3.3, that:

$$p_2^* \circ \phi = \psi \circ f^*$$

Therefore, we conclude that:

$$
\begin{aligned}
e\left(f^*(\mathcal{F}, \mathcal{F}_0)\right) &= \psi^{-1}(\tau^* \smile \tau^*) \\
&= \psi^{-1}(p_2^*(\tau) \smile p_2^*(\tau)) \\
&= \psi^{-1} \circ p_2^*(\tau \smile \tau) \\
&= f^* \circ \phi^{-1}(\tau \smile \tau) \\
&= f^*\left(e(\mathcal{F}, \mathcal{F}_0)\right)
\end{aligned}
$$

$\square$

**Theorem 3.9.** *Let $(\mathcal{F}, \mathcal{F}_0)$ be a $\mathbb{R}^n-$generalized bundle and $(\mathcal{F}', \mathcal{F}'_0)$ a $\mathbb{R}^m-$generalized bundle, both $\mathbb{Z}-$orientable. Then:*

$$e[(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)] = e(\mathcal{F}, \mathcal{F}_0) \times e(\mathcal{F}', \mathcal{F}'_0)$$

*Proof.*
First, let us denote $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$, $(\mathcal{F}', \mathcal{F}'_0) = (E', E'_0, q, B')$ and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0) = (E'', E''_0, r, B'')$, where:

$E'' = E \times E'$
$E''_0 = (E \times E'_0) \cup (E_0 \times E')$
$r = p \times q$
$B'' = B \times B'$

Considering the generators $(\tau) = H^n(E, E_0; \mathbb{Z})$, $(\tau') = H^m(E', E'_0; \mathbb{Z})$ and $(\tau'') = H^{n+m}(E'', E''_0; \mathbb{Z})$ as the Thom classes of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$ and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$, respectively, we know from Lemma 3.2 that $\tau'' = \tau \times \tau'$.

Now, letting $\phi : H^n(B; \mathbb{Z}) \to H^{2n}(E, E_0; \mathbb{Z})$, $\phi' : H^m(B'; \mathbb{Z}) \to H^{2m}(E', E'_0; \mathbb{Z})$ and $\phi'' : H^{n+m}(B''; \mathbb{Z}) \to H^{2(n+m)}(E'', E''_0; \mathbb{Z})$ be the Thom isomorphisms of $(\mathcal{F}, \mathcal{F}_0)$, $(\mathcal{F}', \mathcal{F}'_0)$ and $(\mathcal{F}, \mathcal{F}_0) \times (\mathcal{F}', \mathcal{F}'_0)$, respectively, we obtain by arguments analogous to Theorem 3.4 that:

$$\phi'' = (-1)^{nm}(\phi \times \phi')$$

Thus, we conclude that:

$$
\begin{aligned}
e\left[(\mathcal{F},\mathcal{F}_0) \times (\mathcal{F}',\mathcal{F}'_0)\right] &= \phi''^{-1}(\tau'' \smile \tau'') \\
&= (-1)^{nm}(\phi^{-1} \times \phi'^{-1})\left(\tau'' \smile \tau''\right) \\
&= (-1)^{nm}(\phi^{-1} \times \phi'^{-1})\left((\tau \times \tau') \smile (\tau \times \tau')\right) \\
&= (-1)^{nm}(\phi^{-1} \times \phi'^{-1})\left((-1)^{nm}(\tau \smile \tau) \times (\tau' \smile \tau')\right) \\
&= (-1)^{2nm}(\phi^{-1} \times \phi'^{-1})\left((\tau \smile \tau) \times (\tau' \smile \tau')\right) \\
&= \phi^{-1}(\tau \smile \tau) \times \phi'^{-1}(\tau' \smile \tau') \\
&= e(\mathcal{F},\mathcal{F}_0) \times e(\mathcal{F}',\mathcal{F}'_0)
\end{aligned}
$$

$\square$

**Corollary 3.6. (Whitney Product)** *Let $(\mathcal{F},\mathcal{F}_0)$ be a $\mathbb{R}^n-$generalized bundle and $(\mathcal{F}',\mathcal{F}'_0)$ a $\mathbb{R}^m-$generalized bundle, both $R-$orientable and over the same base. Then:*

$$
e[(\mathcal{F},\mathcal{F}_0) \oplus (\mathcal{F}',\mathcal{F}'_0)] = e(\mathcal{F},\mathcal{F}_0) \smile e(\mathcal{F}',\mathcal{F}'_0)
$$

*Proof.*

The proof is analogous to Corollary 3.3, by applying Theorems 3.8 and 3.9.

$\square$

**Proposition 3.5.** *Let $(\mathcal{F},\mathcal{F}_0)$ be a $\mathbb{R}^n-$generalized bundle that is $\mathbb{Z}-$orientable. Denoting by $-(\mathcal{F},\mathcal{F}_0)$ the same generalized bundle but with reversed orientation, we have:*

$$
e\left(-(\mathcal{F},\mathcal{F}_0)\right) = -e(\mathcal{F},\mathcal{F}_0)
$$

*Proof.*

Let us denote $(\mathcal{F},\mathcal{F}_0) = (E, E_0, p, B)$. Thus, considering the generators $(\tau) = H^n(E, E_0; \mathbb{Z})$ and $(\tau') = H^n(E, E_0; \mathbb{Z})$ as the Thom classes of $(\mathcal{F},\mathcal{F}_0)$ and $-(\mathcal{F},\mathcal{F}_0)$, respectively, then Lemma 3.3 ensures that $\tau' = -\tau$.

On the other hand, denoting the Thom isomorphisms of $(\mathcal{F},\mathcal{F}_0)$ and $-(\mathcal{F},\mathcal{F}_0)$ by $\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$ and $\phi' : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$, respectively, we obtain:

$$
\begin{aligned}
\forall x \in H^k(B; \mathbb{Z}), \ \phi'(x) &= p^*(x) \smile \tau' \\
&= p^*(x) \smile (-\tau) \\
&= -\left(p^*(x) \smile \tau\right) \\
&= -\phi(x)
\end{aligned}
$$

Therefore, we conclude that:

$$
\begin{aligned}
e\left(-(\mathcal{F},\mathcal{F}_0)\right) &= \phi'^{-1}(\tau' \smile \tau') \\
&= \phi'^{-1}((-\tau) \smile (-\tau)) \\
&= \phi'^{-1}(\tau \smile \tau) \\
&= -\phi^{-1}(\tau \smile \tau) \\
&= -e(\mathcal{F},\mathcal{F}_0)
\end{aligned}
$$

$\square$

Finally, let us see under which conditions the Euler class vanishes.

**Proposition 3.6.** *If $M$ is a closed $\mathbb{Z}-orientable$ topological manifold of odd dimension, then $e(M) = 0$.*

*Proof.*

Let $m = \dim(M)$ and denote $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$ the $\mathbb{R}^m-$generalized tangent bundle of $M$, and consider the generator $(\tau) = H^m(TM, T_0 M; \mathbb{Z})$ as the Thom class of $(\tau M, \tau_0 M)$.

We already know from Lemma 3.4 that $\tau \smile \tau = 0$.

Thus, denoting by $\phi : H^k(M; \mathbb{Z}) \to H^{k+m}(TM, T_0 M; \mathbb{Z})$ the Thom isomorphism of $(\tau M, \tau_0 M)$, we obtain that $e(M) = \phi^{-1}(\tau \smile \tau) = 0$.

$\square$

**Proposition 3.7.** *Let us consider $(\mathcal{F}, \mathcal{F}_0) = (E, E_0, p, B)$ an $\mathbb{R}^n-generalized$ bundle that is $\mathbb{Z}-orientable$. If $(\mathcal{F}, \mathcal{F}_0)$ admits a section $s : B \to E$ such that $s(B) \subset E_0$, then $e(\mathcal{F}, \mathcal{F}_0) = 0$.*

*Proof.*

We already know from Observation 3.2 that if $(\mathcal{F}, \mathcal{F}_0)$ is a $\mathbb{Z}-orientable$ generalized bundle, then $p^* : H^n(B; \mathbb{Z}) \to H^n(E; \mathbb{Z})$ is an isomorphism. Since $s$ is a section of $p$, that is, $p \circ s = 1$, it follows that $s^* = (p^*)^{-1}$.

On the one hand, from Proposition 3.4 we have that $e(\mathcal{F}, \mathcal{F}_0) = s^* \circ i_E^*(\tau)$, where $i_E : E \hookrightarrow (E, E_0)$ is the canonical inclusion and $\tau \in H^n(E, E_0; \mathbb{Z})$ is the Thom class of $(\mathcal{F}, \mathcal{F}_0)$.

On the other hand, denoting by $j : (E_0, E_0) \hookrightarrow (E, E_0)$ the canonical inclusion, we have the following commutative diagram:

$$
\begin{array}{ccccc}
B & \xrightarrow{\ \ s\ \ } & E_0 & \xrightarrow{\ (i_E)_{|E_0}\ } & (E_0, E_0) \\
\downarrow{\scriptstyle s} & & & & \downarrow{\scriptstyle j} \\
E & \xrightarrow{\quad\quad i_E \quad\quad} & & & (E, E_0)
\end{array}
$$

However, $j^* : H^n(E, E_0; \mathbb{Z}) \to H^n(E_0, E_0; \mathbb{Z})$ is the zero homomorphism, since $H^n(E_0, E_0; \mathbb{Z}) = 0$. Thus, $s^* \circ i_E^* = s^* \circ (i_E)_{|E_0}^* \circ j^* = 0$.

Hence, $e(\mathcal{F}, \mathcal{F}_0) = s^* \circ i_E^*(\tau) = 0$.

$\square$

Thus, we conclude this chapter by presenting several results, taken from [20] as well as original, about the Thom, Stiefel–Whitney, and Euler classes of generalized bundles and, consequently, of topological manifolds.

On the one hand, when we compare this chapter with the second half of the results presented by Fadell in [20], concerning the Stiefel–Whitney and Euler classes, we conclude that this chapter is a reinterpretation of Fadell's work in a more modern language and with more detailed proofs of a large part of the results in [20].

On the other hand, we also conclude that this chapter is a continuation of [20], as we present results related to the pullback generalized bundle, a concept not even mentioned in [20], and we also present results involving the orientability of generalized bundles, which were barely addressed in [20]. In particular, Section 3.3 on the Euler class consists almost entirely of original results.

In any case, the way we developed our contributions to the theory of characteristic classes of topological manifolds in this chapter reinforces the importance of all the properties of generalized bundles developed in the previous chapter.

# Chapter 4

# Applications in Closed Topological Manifolds

In this chapter, we present some applications concerning the Stiefel–Whitney, Euler, and Wu classes of closed topological manifolds.

To begin with, in Section 4.1, we provide an alternative proof of the topological version of Wu's formula, which was first presented by Fadell in [20]. We also explore some consequences of this result.

Using the preliminary lemmas of Wu's formula, we show in Section 4.2 how the Euler class and the Euler characteristic of a topological manifold are related, and how this allowed us to generalize one of the consequences of the Poincaré–Hopf theorem from the smooth manifold setting to the topological one.

To conclude the chapter, in Section 4.3, we show how the theory of generalized bundles enabled us to extend, once again from the smooth to the topological context, one of the results presented by de Stong in [46], involving Wu classes, whose proof relies heavily on the existence of tubular neighborhoods for smooth embeddings.

## 4.1 Topological Version of Wu's Formula

In ([40], Chapter 9) and ([41], Chapter 11), Milnor presents two proofs of the Wu formula for smooth manifolds, which are very similar to each other, differing only in some technical details.

In [20], Fadell uses generalized bundles to provide a first proof of the Wu formula for topological manifolds, based on [40]. It is worth noting that the preliminary results developed by Fadell in [20] to prove the Wu formula are all within the framework of singular (co)homology with $\mathbb{Z}_2-$modules.

In this section, we will also use generalized bundles to provide a second proof of the Wu formula for topological manifolds, this time based on [41]. However, the preliminary results developed here for the Wu formula will be within the framework of singular (co)homology with $R-$modules, where $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, since the versions of these results for $R = \mathbb{Z}$ will be extremely valuable for establishing, in the next section, the applications concerning the Euler class.

To this end, throughout this section we will fix the following objects:

- $M^m$ a closed, connected, and $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$.

- Given a basis $\{b_i\}_{i=1}^r$ for $H^*(M; R)$, let $\{b_i^\#\}_{i=1}^r$ be its dual basis.

- $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$ the $\mathbb{R}^m-$generalized tangent bundle of $M$.

- $(\tau) = H^m(TM, T_0 M; R) \cong R$ the Thom class of $(\tau M, \tau_0 M)$.

- $\phi : H^k(M; R) \to H^{m+k}(TM, T_0 M; R)$ the Thom isomorphism of $(\tau M, \tau_0 M)$ defined by $\phi(x) = p^*(x) \smile \tau$.

Let $d : M \to M \times M$ be the diagonal map and $\Delta = d(M)$. It follows from ([20], Proposition 6.14, p. 505) that the map $\psi : (TM, T_0 M) \to (M \times M, M \times M - \Delta)$ defined by $\psi(\omega) = (\omega(0), \omega(1))$ induces an isomorphism in the context of singular cohomology with $R-$modules. Thus, we may denote $(\tau') = H^m(M \times M, M \times M - \Delta; R) \cong R$ such that $\psi^*(\tau') = \tau$.

Before stating the first technical result for the proof of the Wu formula, let us consider an arbitrary point $b \in M$ and $j_b : (M, M - \{b\}) \hookrightarrow (M \times M, M \times M - \Delta)$ the canonical inclusion given by $j_b(x) = (b, x)$. Then the induced map $j_b^*$ relates the class $\tau'$ to the local $R-$orientation class[1] of $M$ at $b \in M$ as follows:

**Lemma 4.1.** *For any $b \in M$, we have:*

$$< j_b^*(\tau'), [M]_b >= 1 \in R$$

*Proof.*
First, fix $b \in M$ and let $(F, F_0)$ be the fiber of $(\tau M, \tau_0 M)$ over $b \in M$.
Consider the canonical inclusion $f_b : (F, F_0) \hookrightarrow (TM, T_0 M)$ and define the map $\psi_0 : (F, F_0) \to (M, M - \{b\})$ by $\psi_0(\omega) = \omega(1)$. Then the following diagram commutes:

$$
\begin{array}{ccc}
(F, F_0) & \xrightarrow{\psi_0} & (M, M - \{b\}) \\
\downarrow{f_b} & & \downarrow{j_b} \\
(TM, T_0 M) & \xrightarrow{\psi} & (M \times M, M \times M - \Delta)
\end{array}
$$

On the other hand, recalling that the local $R-$orientation class of $M$ at $b \in M$ is the generator $([M]_b) = H_m(M; R) \cong R$, Proposition A.1 ensures that there exists a unique element $x_b \in H^m(M; R)$ such that $x_b \neq 0$ and $< x_b, [M]_b >= 1$.

Now, since $(F, F_0) \sim (\mathbb{R}^m, \mathbb{R}^m - \{0\})$ and, according to ([20], Proposition 3.12, p. 494), $\psi_0$ induces an isomorphism in singular (co)homology with $R-$modules, we may choose a generator $([F]_0) = H_m(F, F_0; R)$ such that $(\psi_0)_*([F]_0) = [M]_b$.

---

[1]For more details on orientation classes, see Section A.4.

Furthermore, since Proposition A.1 again ensures that there exists a unique element $x_F \in H^m(F, F_0; R)$ such that $x_F \neq 0$ and $< x_F, [F]_0 >= 1$, it follows that $\psi_0^*(x_b) = x_F$, because:

$$
\begin{aligned}
< \psi_0^*(x_b), [F]_0 > &= \; < x_b, (\psi_0)_*([F]_0) > \\
&= \; < x_b, [M]_b > \\
&= \; 1
\end{aligned}
$$

Finally, since the Thom class $\tau$ of $(\tau M, \tau_0 M)$ is uniquely determined such that $f_b^*(\tau) = x_F$, we conclude that:

$$
\begin{aligned}
< j_b^*(\tau'), [M]_b > &= \; < j_b^* \circ (\psi^*)^{-1}(\tau), [M]_b > \\
&= \; < (\psi_0^*)^{-1} \circ f_b^*(\tau), [M]_b > \\
&= \; < (\psi_0^*)^{-1}(x_F), [M]_b > \\
&= \; < x_b, [M]_b > \\
&= \; 1
\end{aligned}
$$

$\square$

Note that the version of Lemma 4.1 for $R = \mathbb{Z}_2$ follows directly from the commutativity of the diagram presented in its proof, since $j_b^*$ is an isomorphism and the only nonzero elements of $H^m(M \times M, M \times M - \Delta; \mathbb{Z}_2)$ and $H_m(M, M - \{b\}; \mathbb{Z}_2)$ are, respectively, $\tau'$ and $[M]_b$.

With this, we want to highlight that our contribution to the proof of Lemma 4.1 concerns the case $R = \mathbb{Z}$, which is essential for the versions of Lemma 4.2 and Proposition 4.1 for $R = \mathbb{Z}$, and consequently, for the maps that will be presented in Section 4.2 on the Euler class.

We also emphasize that the proof of Lemma 4.1, in its version for smooth manifolds, can be found in ([41], Lemma 11.7, p. 123). That proof uses the Riemannian structure of the manifold as well as the existence of the exponential map, whereas our proof was obtained entirely by algebraic means, allowing us to generalize to the context of topological manifolds.

Now, returning to the preliminary results regarding the proof of the topological version of Wu's formula, let us consider $j : M \times M \hookrightarrow (M \times M, M \times M - \Delta)$ the canonical inclusion and define $U = j^*(\tau') \in H^m(M \times M; R)$. Then, the slant product relates $U$ and the global $R-$orientation class of $M$ as follows:

**Lemma 4.2.** $U/[M] = 1 \in H^0(M; R) \cong R$

*Proof.*

Let $b \in M$ be arbitrary and denote by $k_b : \{b\} \hookrightarrow M$ the canonical inclusion.

The following diagram is commutative:

$$
\begin{array}{ccc}
H^m(M \times M; R) & \xrightarrow{\ /[M]\ } & H^0(M; R) \\
\downarrow{\scriptstyle k_b^* \times 1} & & \downarrow{\scriptstyle k_b^*} \\
H^m(\{b\} \times M; R) & \xrightarrow{\ /[M]\ } & H^0(\{b\}; R)
\end{array}
$$

Indeed, for any $x \otimes y \in H^0(M; R) \otimes H^m(M; R)$, we have:

$$
\begin{aligned}
k_b^*\left((x \times y)/[M]\right) &= k_b^*\left(< y, [M] > x\right) \\
&= < y, [M] > k_b^*(x) \\
&= \left(k_b^*(x) \times y\right)/[M] \\
&= \left((k_b^* \times 1)(x \times y)\right)/[M]
\end{aligned}
$$

Now, recall that the global and local orientation classes of $M$ are related by the inclusion $\rho_b : M \hookrightarrow (M, M - \{b\})$ via $(\rho_b)_*([M]) = [M]_b$. Also, denoting by $i_b : M \hookrightarrow M \times M$ the inclusion defined by $i_b(x) = (b, x)$, it is clear that the following diagrams commute:

$$
\begin{array}{ccc}
M & \xhookrightarrow{\ \rho_b\ } & (M, M - \{b\}) \\
\downarrow{\scriptstyle i_b} & & \downarrow{\scriptstyle j_b} \\
M \times M & \xhookrightarrow{\ j\ } & (M \times M, M \times M - \Delta)
\end{array}
$$

$$
\begin{array}{ccc}
\{b\} \times M & \xhookrightarrow{\ 1 \times i_b\ } & \{b\} \times (M \times M) \\
{\scriptstyle k_b \times 1}\downarrow & \swarrow{\scriptstyle p_2} & \\
M \times M & &
\end{array}
$$

Thus, we obtain:

$$
\begin{aligned}
k_b^*(U/[M]) &= \left((k_b^* \times 1)(U)\right)/[M] \\
&= \left((1 \times i_b^*) \circ p_2^*(U)\right)/[M] \\
&= \left((1 \times i_b^*)(1 \times U)\right)/[M] \\
&= \left(1 \times i_b^*(U)\right)/[M] \\
&= <i_b^*(U), [M]>1 \\
&= <i_b^* \circ j^*(\tau'), [M]>1 \\
&= <\rho_b^* \circ j_b^*(\tau'), [M]>1 \\
&= <j_b^*(\tau'), (\rho_b)_*([M])>1 \\
&= <j_b^*(\tau'), [M]_b>1 \\
&= 1
\end{aligned}
$$

Hence, since $k_b^*(U/[M]) = 1 \in H^0(\{b\}; R) \cong R$ for any $b \in M$, we conclude that $U/[M] = 1 \in H^0(M; R) \cong R$.

$\square$

**Lemma 4.3.** $(x \times 1) \smile U = (1 \times x) \smile U, \ \forall x \in H^*(M; R)$

*Proof.*

First, recall that the canonical projections $p_1, p_2 : M \times M \to M$ satisfy $(p_1)_{|\Delta} = (p_2)_{|\Delta}$. Moreover, since $M$ is a topological manifold, it is an ENR.

Therefore, it follows from ([17], Proposition 8.6, p. 81) that there exists an open neighborhood $\Delta_\epsilon \subset M \times M$ of $\Delta$ such that $(p_1)_{|\Delta_\epsilon} \sim (p_2)_{|\Delta_\epsilon}$. In other words, if $e' : \Delta_\epsilon \hookrightarrow M \times M$ denotes the canonical inclusion, then $e'^* \circ p_1^* = e'^* \circ p_2^*$.

On the other hand, consider the inclusion $e : (\Delta_\epsilon, \Delta_\epsilon - \Delta) \hookrightarrow (M \times M, M \times M - \Delta)$ as the pair version of $e'$. It is clear that $e$ is an excision, and therefore its induced map in the context of singular cohomology with coefficients in $R-$modules is an isomorphism.

Thus, for every $x \in H^*(M; R)$, we have:

$$
\begin{aligned}
e'^* \circ p_1^*(x) = e'^* \circ p_2^*(x) \implies & \ e'^* \circ p_1^*(x) \smile e^*(\tau') = e'^* \circ p_2^*(x) \smile e^*(\tau') \\
\implies & \ e^*(p_1^*(x) \smile \tau') = e^*(p_2^*(x) \smile \tau') \\
\implies & \ p_1^*(x) \smile \tau' = p_2^*(x) \smile \tau' \\
\implies & \ j^*(p_1^*(x) \smile \tau') = j^*(p_2^*(x) \smile \tau') \\
\implies & \ p_1^*(x) \smile j^*(\tau') = p_2^*(x) \smile j^*(\tau') \\
\implies & \ p_1^*(x) \smile U = p_2^*(x) \smile U
\end{aligned}
$$

Hence, we conclude that $(x \times 1) \smile U = (1 \times x) \smile U$ for all $x \in H^*(M; R)$.

$\square$

**Proposition 4.1.** $U = \sum_{i=1}^{r} (-1)^{|b_i|} \left( b_i \times b_i^{\#} \right)$

*Proof.*

Recall that $U \in H^m(M \times M; R) \cong \bigoplus_{i+j=m} H^i(M; R) \otimes H^j(M; R)$, so there exist $c_1, \cdots, c_r \in H^*(M; R)$ such that:

- $|b_i| + |c_i| = m$

- $U = \sum_{i=1}^{r} (b_i \times c_i)$

Note that, for each $b_j \in \{b_i\}_{i=1}^{r}$, we have:

$$
\begin{aligned}
\left[(b_j \times 1) \smile U\right]/[M] &= b_j \smile (U/[M]) \\
&= b_j \smile 1 \\
&= b_j
\end{aligned}
$$

On the other hand, the previous lemma ensures that:

$$
\begin{aligned}
\left[(b_j \times 1) \smile U\right]/[M] &= \left[(1 \times b_j) \smile U\right]/[M] \\
&= \left[(1 \times b_j) \smile \left(\sum_{i=1}^{r} (b_i \times c_i)\right)\right]/[M] \\
&= \sum_{i=1}^{r} \left[((1 \times b_j) \smile (b_i \times c_i))/[M]\right] \\
&= \sum_{i=1}^{r} (-1)^{|b_j| \cdot |b_i|} \left[((1 \smile b_i) \times (b_j \smile c_i))/[M]\right] \\
&= \sum_{i=1}^{r} (-1)^{|b_j| \cdot |b_i|} \langle b_j \smile c_i, [M]\rangle b_i
\end{aligned}
$$

Thus, since $b_j = \sum_{i=1}^{r} (-1)^{|b_j| \cdot |b_i|} b_i \langle b_j \smile c_i, [M]\rangle$, we conclude that:

$$
(-1)^{|b_j| \cdot |b_i|} \langle b_j \smile c_i, [M]\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
$$

Finally, note that for $i = j$, we have $(-1)^{|b_i| \cdot |b_i|} = (-1)^{|b_i|}$, while for $i \neq j$, $\langle b_j \smile c_i, [M]\rangle = 0$ regardless of the sign $(-1)^{|b_j| \cdot |b_i|}$. Therefore, we may rewrite the above equality as:

$$
\langle b_j \smile (-1)^{|b_i|} c_i, [M]\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
$$

By the uniqueness of the dual basis, as in Theorem A.7, it follows that $b_i^{\#} = (-1)^{|b_i|}c_i$, that is, $c_i = (-1)^{|b_i|}b_i^{\#}$. Hence,

$$U = \sum_{i=1}^{r}(-1)^{|b_i|}\left(b_i \times b_i^{\#}\right).$$

$\square$

**Lemma 4.4.** *For any $k \geq 0$, we have:*

$$w_k(M) = Sq^k(U)/[M]$$

*Proof.*

Recalling that $w_k(M) = \phi^{-1} \circ Sq^k(\tau)$, then:

$$Sq^k(\tau) = \phi(w_k(M)) = p^*(w_k(M)) \smile \tau$$

Defining the homomorphisms $\phi' : H^k(M;\mathbb{Z}_2) \to H^{k+m}(M \times M, M \times M - \Delta;\mathbb{Z}_2)$ and $\phi'' : H^k(M;\mathbb{Z}_2) \to H^{k+m}(M \times M;\mathbb{Z}_2)$, respectively, by $\phi'(x) = p_1^*(x) \smile \tau'$ and $\phi''(x) = p_1^*(x) \smile U$, we have that the following diagram commutes:



Indeed, for any $x \in H^k(M;\mathbb{Z}_2)$, we have:

$$
\begin{aligned}
\psi^* \circ \phi'(x) &= \psi^*(p_1^*(x) \smile \tau') \\
&= p^*(x) \smile \psi^*(\tau') \\
&= p^*(x) \smile \tau \\
&= \phi(x)
\end{aligned}
$$

$$
\begin{aligned}
j^* \circ \phi'(x) &= j^*(p_1^*(x) \smile \tau') \\
&= p_1^*(x) \smile j^*(\tau') \\
&= p_1^*(x) \smile U \\
&= \phi''(x)
\end{aligned}
$$

Note that since $\phi$ and $\psi^*$ are isomorphisms such that $\phi = \psi^* \circ \phi'$, then $\phi'$ is also an isomorphism. Thus:

$$
\begin{aligned}
\phi'^{-1} \circ Sq^k(\tau') &= \phi'^{-1} \circ Sq^k \circ (\psi^*)^{-1}(\tau) \\
&= \phi'^{-1} \circ (\psi^*)^{-1} \circ Sq^k(\tau) \\
&= \phi^{-1} \circ Sq^k(\tau) \\
&= w_k(M)
\end{aligned}
$$

Hence, we get that $Sq^k(\tau') = \phi'(w_k(M))$ and:

$$
\begin{aligned}
Sq^k(U) &= Sq^k \circ j^*(\tau') \\
&= j^* \circ Sq^k(\tau') \\
&= j^* \circ \phi'(w_k(M)) \\
&= \phi''(w_k(M)) \\
&= p_1^*(w_k(M)) \smile U \\
&= (w_k(M) \times 1) \smile U
\end{aligned}
$$

Therefore, we conclude that:

$$
\begin{aligned}
Sq^k(U)/[M] &= [(w_k(M) \times 1) \smile U]/[M] \\
&= w_k(M) \smile (U/[M]) \\
&= w_k(M) \smile 1 \\
&= w_k(M)
\end{aligned}
$$

$\square$

Finally, we can state and prove the main result of this section, the Wu formula:

**Theorem 4.1. (Wu's Formula)** *For any $k \geq 0$, we have:*

$$
w_k(M) = \sum_{i+j=k} Sq^i(v_j(M))
$$

*Proof.*
By Corollary A.1 and the very definition of the Wu classes, we have:

$$
\begin{aligned}
v_j(M) &= \sum_{l=1}^{r} \langle v_j(M) \smile b_l^{\#}, [M] \rangle b_l \\
&= \sum_{l=1}^{r} \langle Sq^j(b_l^{\#}), [M] \rangle b_l
\end{aligned}
$$

Therefore, we conclude that:

$$
\begin{aligned}
\sum_{i+j=k} Sq^i(v_j(M)) &= \sum_{i+j=k}\sum_{l=1}^{r}\langle Sq^j(b_l^\#),[M]\rangle Sq^i(b_l)\\
&= \sum_{i+j=k}\sum_{l=1}^{r}\left[\left(Sq^i(b_l)\times Sq^j(b_l^\#)\right)/[M]\right]\\
&= \sum_{l=1}^{r}\left[\left(\sum_{i+j=k}\left(Sq^i(b_l)\times Sq^j(b_l^\#)\right)\right)/[M]\right]\\
&= \sum_{l=1}^{r}\left[Sq^k(b_l\times b_l^\#)/[M]\right]\\
&= Sq^k\left(\sum_{l=1}^{r}(b_l\times b_l^\#)\right)/[M]\\
&= Sq^k(U)/[M]\\
&= w_k(M)
\end{aligned}
$$

$\square$

Now, let us look at some consequences of Wu's formula.

**Corollary 4.1.** *If $f : M^m \to S^m$ is a map between closed, connected topological manifolds, which induces an isomorphism on the singular (co)homology groups with $\mathbb{Z}_2-$coefficients, then $f^*(W(S)) = W(M)$. In particular, the same holds if $f$ is a homotopy equivalence.*

*Proof.*

It is enough to show that, under these conditions, the Wu classes of $M$ and $S$ satisfy $f^*(v(S)) = v(M)$, since then Wu's formula ensures that:

$$
\begin{aligned}
f^*(W(S)) &= f^*(Sq(v(S)))\\
&= Sq(f^*(v(S)))\\
&= Sq(v(M))\\
&= W(M)
\end{aligned}
$$

Due to the uniqueness of the Wu classes, it suffices to show that for all $0 \le i \le m$:

$$
\langle f^*(v_i(S)) \smile x, [M]\rangle = \langle Sq^i(x),[M]\rangle, \quad \forall x \in H^{m-i}(M;\mathbb{Z}_2)
$$

First, since $f_*$ and $f^*$ are isomorphisms, we have $f_*([M]) = [S]$ and, for each $0 \le i \le m$ and any $x \in H^{m-i}(M;\mathbb{Z}_2)$, there exists a unique $y \in H^{m-i}(S;\mathbb{Z}_2)$ such that $f^*(y) = x$. Thus:

$$
\begin{aligned}
\langle f^*(v_i(S)) \smile x, [M]\rangle &= \langle f^*(v_i(S)) \smile f^*(y), [M]\rangle\\
&= \langle f^*(v_i(S) \smile y),[M]\rangle\\
&= \langle v_i(S) \smile y, f_*([M])\rangle\\
&= \langle v_i(S) \smile y, [S]\rangle\\
&= \langle Sq^i(y),[S]\rangle\\
&= \langle Sq^i(y), f_*([M])\rangle\\
&= \langle f^*(Sq^i(y)),[M]\rangle\\
&= \langle Sq^i(f^*(y)),[M]\rangle\\
&= \langle Sq^i(x),[M]\rangle
\end{aligned}
$$

Hence, $f^*(v(S)) = v(M)$ and, consequently, $f^*(W(S)) = W(M)$.

$\square$

**Corollary 4.2.** *Let $M^m$ be a closed and connected topological manifold such that $w_1(M) = \cdots = w_r(M) = 0$ for some integer $0 \leq r \leq m$. Then, the Wu classes of $M$ satisfy $v_1(M) = \cdots = v_r(M) = 0$ and $v_{r+1}(M) = w_{r+1}(M)$.*

*Proof.*

Using directly Wu's formula and the properties of Steenrod squares, we obtain recursively:

$$
\begin{aligned}
0 &= w_1(M) \\
&= Sq^0(v_1(M)) + Sq^1(v_0(M)) \\
&= Sq^0(v_1(M)) \\
&= v_1(M)
\end{aligned}
$$

$$
\begin{aligned}
0 &= w_2(M) \\
&= Sq^0(v_2(M)) + Sq^1(v_1(M)) + Sq^2(v_0(M)) \\
&= Sq^0(v_2(M)) \\
&= v_2(M)
\end{aligned}
$$

$$
\vdots
$$

$$
\begin{aligned}
0 &= w_r(M) \\
&= \sum_{i=0}^{r} Sq^i(v_{r-i}(M)) \\
&= Sq^0(v_r(M)) + \sum_{i=1}^{r} Sq^i(v_{r-i}(M)) \\
&= Sq^0(v_r(M)) \\
&= v_r(M)
\end{aligned}
$$

$$
\begin{aligned}
w_{r+1}(M) &= \sum_{i=0}^{r+1} Sq^i(v_{r+1-i}(M)) \\
&= Sq^0(v_{r+1}(M)) + \sum_{i=1}^{r+1} Sq^i(v_{r+1-i}(M)) \\
&= Sq^0(v_{r+1}(M)) \\
&= v_{r+1}(M)
\end{aligned}
$$

$\square$

**Corollary 4.3.** *Let $M^m$ be a closed and connected topological manifold. Then, for every integer $0 < i < m$ such that $2i > m$, we have $v_i(M) = 0$.*

*Proof.*

Recall that the Wu class $v_i(M) \in H^i(M; \mathbb{Z}_2)$ is uniquely determined by the following relation:

$$
< v_i(M) \smile x, [M] > = < Sq^i(x), [M] >, \ \forall x \in H^{m-i}(M; \mathbb{Z}_2)
$$

Now, since $2i > m$, that is, $i > m - i$, then for any $x \in H^{m-i}(M; \mathbb{Z}_2)$, we have $Sq^i(x) = 0$ and, consequently, $< Sq^i(x), [M] >= 0$. Thus, $< v_i(M) \smile x, [M] >= 0$, which implies $v_i(M) \smile x = 0$.

If $H^{m-i}(M; \mathbb{Z}_2) = 0$, then by Poincaré duality and the Universal Coefficient Theorem, we have $H^i(M; \mathbb{Z}_2) = 0$. Hence, $v_i(M) = 0$.

On the other hand, if $H^{m-i}(M; \mathbb{Z}_2) \neq 0$, then $v_i(M) \smile x = 0$, in particular for $x \neq 0$, and thus $v_i(M) = 0$.

In any case, we obtain $v_i(M) = 0$ whenever $2i > m$.

$\square$

**Corollary 4.4.** *Let $M^m$ be a closed and connected topological manifold with $m = 2r$ or $m = 2r + 1$ for some integer $r \geq 0$. If $w_1(M) = \cdots = w_r(M) = 0$, then $W(M) = 1$.*

*Proof.*

It follows from Corollary 4.2 that $v_1(M) = \cdots = v_r(M) = 0$.

On the other hand, if $i \geq r + 1$, then $2i \geq 2r + 2 > m$, and thus, from Corollary 4.3, we have that $v_i(M) = 0$ for $i \geq r + 1$.

Hence, $v(M) = 1$ and, by the Wu formula, we conclude that $W(M) = 1$.

$\square$

**Corollary 4.5.** *Let $M^m$ be a closed and connected topological manifold such that $W(M) \neq 1$. If $i > 0$ is the smallest integer for which $w_i(M) \neq 0$, then $i$ must be a power of two.*

*Proof.*

Suppose $i > 0$ is the smallest integer such that $w_i(M) \neq 0$, then:

$$w_1(M) = \cdots = w_{i-1}(M) = 0$$

By Corollary 4.2, it follows that $v_1(M) = \cdots = v_{i-1}(M) = 0$ and $v_i(M) = w_i(M) \neq 0$.

Since $< Sq^k(x), [M] >=< v_k(M) \smile x, [M] >= 0$ for every $k = 1, \cdots, i - 1$ and $x \in H^{m-k}(M; \mathbb{Z}_2)$, we have that $Sq^k$ is the zero homomorphism for all $k = 1, \cdots, i - 1$.

Now, if $H^{m-i}(M; \mathbb{Z}_2) = 0$, then Poincaré duality and the Universal Coefficient Theorem ensure that $H^i(M; \mathbb{Z}_2) = 0$, and consequently, $w_i(M) = 0$, which contradicts the hypothesis. Therefore, since there exists $x \neq 0$ in $H^{m-i}(M; \mathbb{Z}_2)$, we get $v_i(M) \smile x \neq 0$. Thus, $< Sq^i(x), [M] >=< v_i(M) \smile x, [M] >\neq 0$, so $Sq^i(x) \neq 0$, i.e., $Sq^i$ is not the zero homomorphism.

As a result, it is not possible to decompose $Sq^i$ as a sum of compositions of $Sq^k$ with $1 \leq k \leq i - 1$.

From this, by ([7], Theorem 15.8, p. 407)[2], we conclude that $i$ must be a power of two.

$\square$

As a final application of the Wu formula, let us see how this formula allows us to compute the total Stiefel–Whitney class of real projective spaces without relying on results about generalized bundles, as in Theorem 3.5, and without relying on their smooth structure, as done in ([41], Theorem 4.5, p. 45).

In other words, the topological version of the Wu formula allows us to compute the total Stiefel–Whitney class of $\mathbb{R}P^n$ depending only on its cellular structure and no longer on its differential structure.

**Corollary 4.6.** *Let* $(a) = H^1(\mathbb{R}P^n; \mathbb{Z}_2)$, *then* $W(\mathbb{R}P^n) = (1+a)^{n+1}$.

*Proof.*

Initially, recall that the canonical inclusion $i : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ induces, by Lemma A.4, an isomorphism $i^* : H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$. Thus, denoting by $(b) = H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$, we have $i^*(b) = a$.

Fixing now the Newton binomial $\binom{i}{k} = 0$ when $i < k$ or $k < 0$, define for every integer $m \geq 0$ the following element:

$$\widetilde{v}(m) = \sum_{k=0}^{m} \binom{m-k}{k} b^k \in H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$$

Note that $\widetilde{v}(0) = 1 = \widetilde{v}(1)$. On the other hand, for $m > 1$, we have:

$$
\begin{aligned}
\widetilde{v}(m+1) &= \sum_{k=0}^{m+1} \binom{m+1-k}{k} b^k \\
&= \sum_{k=0}^{m+1} \left[ \binom{m-k}{k} + \binom{m-k}{k-1} \right] b^k \\
&= \sum_{k=0}^{m+1} \binom{m-k}{k} b^k + \sum_{k=0}^{m+1} \binom{m-k}{k-1} b^k \\
&= \sum_{k=0}^{m} \binom{m-k}{k} b^k + \sum_{k=1}^{m} \binom{m-k}{k-1} b^k \\
&= \widetilde{v}(m) + \sum_{k=0}^{m-1} \binom{m-1-k}{k} b^{k+1} \\
&= \widetilde{v}(m) + b \smile \widetilde{v}(m-1)
\end{aligned}
$$

On the other hand, define, for every integer $m \geq 0$, the element:

$$\beta(m) = Sq(\widetilde{v}(m)) \in H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$$

Again, note that $\beta(0) = 1 = \beta(1)$. Also, for $m > 1$, we have:

$$
\begin{aligned}
\beta(m+1) &= Sq(\widetilde{v}(m+1)) \\
&= Sq(\widetilde{v}(m) + b \smile \widetilde{v}(m-1)) \\
&= Sq(\widetilde{v}(m)) + [Sq(b) \smile Sq(\widetilde{v}(m-1))] \\
&= \beta(m) + [(b + b^2) \smile \beta(m-1)]
\end{aligned}
$$

---

[2] *"If an integer $i > 0$ is not a power of two, then $Sq^i$ decomposes as a sum of compositions of $Sq^k$, with $0 < k < i$".*

Now, we prove by induction on $m \geq 0$ that $\beta(m) = (1+b)^{m+1} - b^{m+1}$. First, it is clear that $\beta(0) = 1 = (1+b) - b$. Assuming valid $\beta(k) = (1+b)^{k+1} - b^{k+1}$ for every $0 \leq k \leq m$, note that:

$$
\begin{aligned}
\beta(m+1) &= \beta(m) + [(b+b^2) \smile \beta(m-1)] \\
&= [(1+b)^{m+1} - b^{m+1}] + [(b+b^2) \smile [(1+b)^m - b^m]] \\
&= (1+b)^{m+1} - b^{m+1} + [(b+b^2) \smile (1+b)^m] - [(b+b^2) \smile b^m] \\
&= (1+b)^{m+1} - b^{m+1} + [b \smile (1+b)^{m+1}] - b^{m+1} - b^{m+2} \\
&= [(1+b)^{m+1} \smile (1+b)] - b^{m+2} \\
&= (1+b)^{m+2} - b^{m+2}
\end{aligned}
$$

Remembering that Example A.1 guarantees that $v(\mathbb{R}P^n) = \sum_{k=0}^{n} \binom{n-k}{k} a^k$, we have:

$$
\begin{aligned}
Sq(v(\mathbb{R}P^n)) &= Sq\left(\sum_{k=0}^{n} \binom{n-k}{k} a^k\right) \\
&= Sq\left(\sum_{k=0}^{n} \binom{n-k}{k} (i^*(b))^k\right) \\
&= i^*\left[Sq\left(\sum_{k=0}^{n} \binom{n-k}{k} b^k\right)\right] \\
&= i^*(\beta(n)) \\
&= i^*[(1+b)^{n+1} - b^{n+1}] \\
&= (1+i^*(b))^{n+1} - (i^*(b))^{n+1} \\
&= (1+a)^{n+1}
\end{aligned}
$$

Therefore, Wu's formula guarantees that $W(\mathbb{R}P^n) = (1+a)^{n+1}$.

$\square$

# 4.2 Topological Version of Poincaré-Hopf Theorem

In this section, we will present some applications of the Euler class of a topological manifold, whose proofs were only made possible due to the preliminary results used in the derivation of Wu's formula also being developed in the context of singular cohomology $\mathbb{Z}-$modules.

First, we will examine the relationship between the Euler class and the Euler characteristic of a topological manifold. Then, we will show how to generalize the notion of vector field from the smooth manifold setting to the topological one, and how to combine these results to state the topological version of the Poincaré-Hopf theorem.

Let us begin with the following relationship between the global $\mathbb{Z}-$orientation class, the Euler class and the Euler characteristic of a topological manifold:

**Theorem 4.2.** *If $M$ is a closed, connected, and $\mathbb{Z}-$orientable topological manifold, then:*

$$< e(M), [M] > = \chi(M)$$

*Proof.*

Let $dim(M) = m$, and $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$ denote the $\mathbb{R}^m$-generalized tangent bundle of $M$, with $(\tau) = H^m(TM, T_0 M; \mathbb{Z})$ representing the Thom class of $(\tau M, \tau_0 M)$.

Let $d : M \to M \times M$ be the diagonal map, and let $i_{TM} : TM \hookrightarrow (TM, T_0M)$ and $j : M \times M \hookrightarrow (M \times M, M \times M - \Delta)$ be the canonical inclusions, where $\Delta = d(M)$.

Now, let $s : M \to TM$ be the section of $(\tau M, \tau_0 M)$ that assigns to each $b \in M$ the constant path at $b$ in $M$. Proposition 3.4 ensures that $e(M) = s^* \circ i_{TM}^*(\tau)$.

On the other hand, if we define the map $\psi : (TM, T_0 M) \to (M \times M, M \times M - \Delta)$ by $\psi(\omega) = (\omega(0), \omega(1))$, it is clear that the following diagram commutes:

$$
\begin{array}{ccc}
(TM, T_0 M) & \xrightarrow{\ \ \psi\ \ } & (M \times M, M \times M - \Delta) \\
\big\uparrow{\scriptstyle i_{TM}} & & \big\uparrow{\scriptstyle j} \\
TM \xleftarrow{\ s\ } M & \xrightarrow{\ d\ } & M \times M
\end{array}
$$

Since we already know that $\psi$ induces an isomorphism in the context of singular cohomology $\mathbb{Z}$-modules, defining $\tau' = (\psi^*)^{-1}(\tau) \in H^m(M \times M, M \times M - \Delta; \mathbb{Z})$ and $U = j^*(\tau') \in H^m(M \times M; \mathbb{Z})$, we obtain:

$$
\begin{aligned}
e(M) & = s^* \circ i_{TM}^*(\tau) \\
& = s^* \circ i_{TM}^* \circ \psi^*(\tau') \\
& = d^* \circ j^*(\tau') \\
& = d^*(U)
\end{aligned}
$$

Finally, let $\{b_i\}_{i=1}^r$ be a basis for $H^*(M; \mathbb{Z})$ and $\{b_i^{\#}\}_{i=1}^r$ its dual basis. Since $e(M) = d^*(U)$, Proposition 4.1 ensures that:

$$
\begin{aligned}
< e(M), [M] > & = \ < d^*(U), [M] > \\
& = \ < d^* \left( \sum_{i=1}^r (-1)^{|b_i|} (b_i \times b_i^{\#}) \right), [M] > \\
& = \ \sum_{i=1}^r (-1)^{|b_i|} < d^*(b_i \times b_i^{\#}), [M] > \\
& = \ \sum_{i=1}^r (-1)^{|b_i|} < b_i \smile b_i^{\#}, [M] > \\
& = \ \sum_{i=1}^r (-1)^{|b_i|} \\
& = \ \sum_{k=0}^m (-1)^k \operatorname{rank}(H_k(M; \mathbb{Z})) \\
& = \ \chi(M)
\end{aligned}
$$

$\square$

Naturally, due to Theorem 3.7, if we drop the $\mathbb{Z}-$orientability condition in the corollary above, we can rewrite it as follows:

**Corollary 4.7.** *If $M$ is a closed and connected topological manifold, then:*

$$< w_n(M), [M] > \equiv \chi(M) \ (mod \ 2)$$

The next result is a direct combination of Proposition 3.6 with Theorem 4.2.

**Corollary 4.8.** *If $M$ is a closed, connected, and $\mathbb{Z}-$orientable topological manifold of odd dimension, then $\chi(M) = 0$.*

The next definition generalizes the concept of vector field from smooth manifolds to the context of topological manifolds, aiming to state the topological version of one of the consequences of the Poincaré-Hopf Theorem for smooth manifolds.

**Definition 4.1. (Path Field)** *A* path field *on a topological manifold $M$ is any section of its tangent $\mathbb{R}^n$-generalized bundle $(\tau M, \tau_0 M) = (TM, T_0M, p, M)$. Furthermore, a* non-singular path field *on $M$ is a section $s : M \to TM$ of $(\tau M, \tau_0 M)$ such that $s(M) \subset T_0 M$.*

From the definition of path field and Proposition 2.3, it is clear that a smooth manifold admits a nowhere vanishing vector field if and only if it admits a non-singular path field.

That said, let us recall the statement of one of the consequences of the Poincaré-Hopf Theorem:

> **(Poincaré-Hopf Corollary)** *"A closed, connected, and $\mathbb{Z}-$orientable smooth manifold $M$ admits a nowhere vanishing vector field if and only if $\chi(M) = 0$."*

In [9], Brown presented for the first time the topological version of the above result, replacing the notion of nowhere vanishing vector field with non-singular path field, stated as follows:

> *"A closed, connected, and $\mathbb{Z}-$orientable topological manifold $M$ admits a non-singular path field if and only if $\chi(M) = 0$."*

Later, in [10], Brown and Fadell showed that this result can be extended to topological manifolds with boundary and non-orientable ones as follows:

> *"A compact, connected, topological manifold $M$, with or without boundary and not necessarily $\mathbb{Z}-$orientable, admits a non-singular path field if and only if $\chi(M) = 0$."*

In both [9] and [10], techniques involving generalized bundles and the Lefschetz number were used, but the notion of Euler class was not employed in the proofs.

The Euler class allows for a more concise proof of the result presented in [9], though only in one direction, as follows:

**Theorem 4.3. (Poincaré-Hopf)** *Let $M$ be a closed, connected, and $\mathbb{Z}-$orientable topological manifold. If $M$ admits a non-singular path field, then $\chi(M) = 0$.*

*Proof.*

Since $M$ is a compact, $\mathbb{Z}-$orientable topological manifold that admits a non-singular path field, i.e., there exists a section $s : M \to TM$ of the $\mathbb{R}^n-$generalized tangent bundle $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$ such that $s(M) \subset T_0 M$, it follows from Proposition 3.7 that $e(M) = 0$.

Therefore, Corollary 4.2 ensures that $\chi(M) = 0$.

$\square$

## 4.3   A Tubular Neighborhood Problem

We conclude this chapter by presenting the only map that does not involve Stiefel-Whitney classes or Euler classes, but only Wu classes and some technical results about generalized bundles.

To better understand the significance of this result, let us make a few observations. First, recall that by combining Theorem 3.6 with Corollary 3.2, we obtain:

> *"If $i : M^m \hookrightarrow S^{m+k}$ is a local-flat embedding with trivial generalized normal bundle, then $W(M) = i^*(W(S))$"*

Our goal is to prove the same result for the total Wu class, that is:

> *"If $i : M^m \hookrightarrow S^{m+k}$ is a local-flat embedding with trivial generalized normal bundle, then $v(M) = i^*(v(S))$"*

In ([46], Lemma 7, p. 274), Stong proves the same result in the smooth setting, as follows:

> *"If $i : M^m \hookrightarrow S^{m+k}$ is a smooth embedding with trivial vector normal bundle, then $v(M) = i^*(v(S))$"*

A more detailed proof of the result above can be found in ([43], Lemma 7, p. 33). In that proof, the direct use of the existence of a tubular neighborhood for smooth embeddings becomes evident.

To illustrate how such a generalization was made and to motivate the proof of the final result of this chapter, let $M^m \subset S^{m+k}$ be a smooth embedding with trivial vector normal bundle and $V$ the tubular neighborhood of $M$ in $S$, as in the following figure:
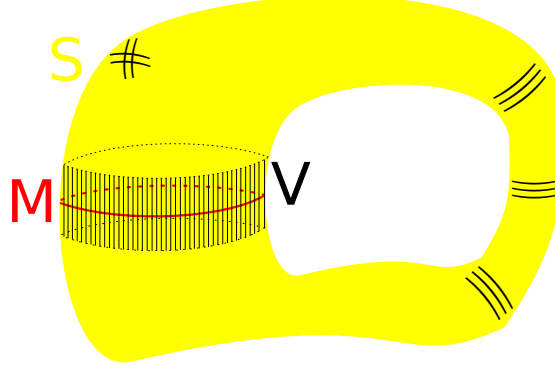


Figure 4.1: Tubular neighborhood of a smooth embedding.

With that, the proof by Stong in ([46], Lemma 7, p. 274) essentially consists of finding a commutative diagram under the following conditions:

$$
\begin{array}{ccc}
H^*(S;\mathbb{Z}_2) \otimes H^*(S;\mathbb{Z}_2) & \longrightarrow & H^*(S;\mathbb{Z}_2) \\
\uparrow & & \uparrow \\
H^*(S;\mathbb{Z}_2) \otimes H^*(S,S-V;\mathbb{Z}_2) & \longrightarrow & H^*(S,S-V;\mathbb{Z}_2) \\
\downarrow & & \downarrow \\
H^*(V;\mathbb{Z}_2) \otimes H^*(V,\partial V;\mathbb{Z}_2) & & \\
\downarrow & & \downarrow \\
H^*(V,\partial V;\mathbb{Z}_2) & \longrightarrow & H^*(S;\mathbb{Z}_2)
\end{array}
$$

Since there is no tubular neighborhood in the topological setting, we circumvent this issue using only results about generalized bundles, in order to construct a similar commutative diagram and thus show that the existence of a tubular neighborhood is not essential to this result, but rather certain algebraic consequences of a local-flat embedding.

Finally, we have the following:

**Theorem 4.4.** *If $i : M^m \hookrightarrow S^{m+k}$ is a local-flat embedding between closed and connected topological manifolds, with trivial $\mathbb{R}^k-$generalized normal bundle, then $v(M) = i^*(v(S))$.*

*Proof.*

First, let $(\mathcal{N}, \mathcal{N}_0) = (N, N_0, q, M)$ be the $\mathbb{R}^k$−generalized normal bundle of the local-flat embedding $i$, and let $f : (N, N_0) \rightleftarrows (M \times \mathbb{R}^k, M \times (\mathbb{R}^k - \{0\})) : g$ be the homotopy equivalences given by the triviality of $(\mathcal{N}, \mathcal{N}_0)$. Also recall that $f$ and $g$ are fiber maps, that is, $p_1 \circ f = q$ and $q \circ g = p_1$.

Let $c : S \hookrightarrow (S, S - M)$ be the canonical inclusion and $\xi : (N, N_0) \to (S, S - M)$ the map given by $\xi(\omega) = \omega(1)$, which, according to ([20], Theorem 7.5, p. 509), induces an isomorphism in the context of singular (co)homology with coefficients in $\mathbb{Z}_2$.

Thus, for any integer $t \geq 0$, the homomorphisms are well defined:

$$B = f_* \circ (\xi_*)^{-1} \circ c_* : H_t(S; \mathbb{Z}_2) \to H_t(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2)$$

$$A = c^* \circ (\xi^*)^{-1} \circ f^* : H^t(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2) \to H^t(S; \mathbb{Z}_2)$$

Note that the homomorphism $A$ is the cohomological dual of the homomorphism $B$.

Furthermore, we also know that the Künneth formula ensures that:

$$H^{t+k}(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2) \cong H^t(M; \mathbb{Z}_2) \otimes H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$$

Then, by letting $(\varphi) = H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$, we define the homomorphism $\bar{i} : H^t(M; \mathbb{Z}_2) \to H^{t+k}(S; \mathbb{Z}_2)$ by $\bar{i}(x) = A(x \times \varphi)$.

With the above notations fixed, let us begin the proof.

If we denote $(\sigma) = H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$, then $B([S]) = [M] \times \sigma$. Indeed, observe that:

$$B([S]) \in H_{m+k}(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2) \cong H_m(M; \mathbb{Z}_2) \otimes H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

Since $H_m(M; \mathbb{Z}_2)$ and $H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$ are generated by $[M]$ and $\sigma$, respectively, then $H_m(M; \mathbb{Z}_2) \otimes H_k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$ is generated by $[M] \otimes \sigma$. That is, the only nonzero class of $H_{m+k}(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2)$ is $[M] \times \sigma$.

Therefore, in order to ensure $B([S]) = [M] \times \sigma$, it is enough to prove that $B([S]) \neq 0$. On the other hand, since $B = f_* \circ (\xi_*)^{-1} \circ c_*$ and both $f_*$ and $\xi_*$ are isomorphisms, we have $B([S]) \neq 0$ if and only if $c_*([S]) \neq 0$.

Now, for an arbitrary point $b \in M \subset S$, the inclusion $j_b : S \hookrightarrow (S, S - \{b\})$ can be decomposed as the following inclusions:

$$S \xhookrightarrow{\quad c \quad} (S, S - M) \xhookrightarrow{\quad\quad} (S, S - \{b\})$$

Since $(j_b)_*$ is an isomorphism[3], $c_*$ is a monomorphism and hence $c_*([S]) \neq 0$ since $[S] \neq 0$.

---

[3]See Proposition A.3.

Therefore, $B([S]) = [M] \times \sigma$.

Let us now show that, by defining $e = \xi \circ g : M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to (S, S - M)$, the following diagram commutes for all $t \geq 0$:

$$
\begin{array}{ccc}
H^{m-t}(S; \mathbb{Z}_2) \otimes H^{t+k}(S; \mathbb{Z}_2) & \xrightarrow{\smile} & H^{m+k}(S; \mathbb{Z}_2) \\
\uparrow{\scriptstyle 1 \times c^*} & & \uparrow{\scriptstyle c^*} \\
H^{m-t}(S; \mathbb{Z}_2) \otimes H^{t+k}(S, S - M; \mathbb{Z}_2) & \xrightarrow{\smile} & H^{m+k}(S, S - M; \mathbb{Z}_2) \\
\downarrow{\scriptstyle i^* \times e^*} & & \\
H^{m-t}(M \times \mathbb{R}^k; \mathbb{Z}_2) \otimes H^{t+k}(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2) & & \Big\downarrow{\scriptstyle c^*} \\
\downarrow{\scriptstyle \smile} & {\scriptstyle e^*} & \\
H^{m+k}(M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}); \mathbb{Z}_2) & \xleftarrow{\ A\ } & H^{m+k}(S; \mathbb{Z}_2)
\end{array}
$$

Let us prove this commutativity step by step. Initially, due to the very definitions of the homomorphisms $A$ and $e^*$, it is clear that $A \circ e^* = c^*$.

On the other hand, Lemma A.2 ensures that the inclusion $c : S \hookrightarrow (S, S - M)$ satisfies the property that $c^*(x \smile y) = x \smile c^*(y)$ for any $x \in H^{m-t}(S; \mathbb{Z}_2)$ and $y \in H^{t+k}(S, S - M; \mathbb{Z}_2)$.

Now, we can assume, without loss of generality, that the embedding $i : M \hookrightarrow S$ induces $i^* : H^t(S; \mathbb{Z}_2) \to H^t(M \times \mathbb{R}^k; \mathbb{Z}_2)$, since $\mathbb{R}^k$ is a contractible topological space. Moreover, we can take this contraction as $M \to M \times \mathbb{R}^k$ that maps $b \mapsto (b, 0)$.

Thus, in order to restrict the image of $e : M \times (\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to (S, S - M)$ to $S$, it is necessary to restrict the domain to $M \times \{0\}$. In this way, the restriction $e_{|M \times \{0\}} : M \times \{0\} \to S$ reduces to the embedding $i$, since for a fixed $b \in M$, $g(b, 0) \in N - N_0$ is the constant path in $S$ equal to $b \in M$, and thus:

$$
\begin{aligned}
e(b, 0) &= \xi \circ g(b, 0) \\
&= \\
&= b \\
&= i(b)
\end{aligned}
$$

Hence, we obtain for any $x \in H^{m-t}(S; \mathbb{Z}_2)$ and $y \in H^{t+k}(S, S - M; \mathbb{Z}_2)$ that $e^*(x \smile y) = i^*(x) \smile e^*(y)$.

With this, the diagram shown above is indeed commutative.

Moreover, recalling that $\bar{i} : H^t(M; \mathbb{Z}_2) \to H^{t+k}(S; \mathbb{Z}_2)$ is the homomorphism defined by $\bar{i}(x) = A(x \times \varphi)$, where $\varphi \in H^k(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$, we can obtain for any $x \in H^t(M; \mathbb{Z}_2)$ and $y \in H^{m-t}(S; \mathbb{Z}_2)$ that:

$$
\begin{aligned}
A(i^*(y) \smile (x \times \varphi)) &= y \smile c^*((e^*)^{-1}(x \times \varphi)) \\
&= y \smile A(x \times \varphi) \\
&= y \smile \bar{i}(x)
\end{aligned}
$$

Therefore, we have for any $x \in H^t(M; \mathbb{Z}_2)$ and $y \in H^{m-t}(S; \mathbb{Z}_2)$ that:

$$
\begin{aligned}
< \bar{i}(x) \smile y, [S] > &= \ < A(i^*(y) \smile (x \times \varphi)), [S] > \\
&= \ < i^*(y) \smile (x \times \varphi), B([S]) > \\
&= \ < i^*(y), ([M] \times \sigma) \frown (x \times \varphi) > \\
&= \ < i^*(y), ([M] \frown x) \times (\sigma \frown \varphi) > \\
&= \ < i^*(y), [M] \frown x > \\
&= \ < x \smile i^*(y), [M] >
\end{aligned}
$$

Finally, we are ready to prove that $i^*(v(S)) = v(M)$. To do so, due to the uniqueness of the Wu classes, it suffices to show that for all $0 \le t \le n$:

$$
< i^*(v_t(S)) \smile x, [M] > = < Sq^t(x), [M] >, \ \forall x \in H^{m-t}(M; \mathbb{Z}_2)
$$

But first, note that $Sq^t(\varphi) = 0$ when $t > 0$, since $H^{k+t}(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2) = 0$ for $t > 0$ and $Sq^t(\varphi) \in H^{k+t}(\mathbb{R}^k, \mathbb{R}^k - \{0\}; \mathbb{Z}_2)$. With this, we obtain that $Sq^t(x \times \varphi) = Sq^t(x) \times \varphi$, for any $x \in H^*(M; \mathbb{Z}_2)$.

Thus, we have for any $0 \le t \le m$ and $x \in H^{m-t}(M; \mathbb{Z}_2)$ that:

$$
\begin{aligned}
< i^*(v_t(S)) \smile x, [M] > &= \ < \bar{i}(x) \smile v_t(S), [S] > \\
&= \ < Sq^t(\bar{i}(x)), [S] > \\
&= \ < Sq^t(A(x \times \varphi)), [S] > \\
&= \ < A(Sq^t(x) \times \varphi), [S] > \\
&= \ < \bar{i}(Sq^t(x)), [S] > \\
&= \ < \bar{i}(Sq^t(x)) \smile 1, [S] > \\
&= \ < i^*(1) \smile Sq^t(x), [M] > \\
&= \ < Sq^t(x), [M] >
\end{aligned}
$$

Therefore, $i^*(v(S)) = v(M)$.

$\square$

Thus, we conclude this chapter with several contributions in the context of characteristic classes of topological manifolds. More specifically, we presented maps related to the Stiefel–Whitney, Euler, and Wu classes of closed topological manifolds.

Among these contributions, we highlight:

- A second proof of the topological version of the Wu formula, with some of the preliminary lemmas developed both for $\mathbb{Z}_2-$modules and for $\mathbb{Z}-$modules of singular (co)homology.

- How the preliminary lemmas of the Wu formula allowed us to relate the Euler class and the Euler characteristic of a topological manifold and, consequently, ensure the vanishing of the Euler characteristic of an odd-dimensional topological manifold.

- An alternative proof of one direction of the topological version of the Poincaré–Hopf theorem, which shows the relationship between the existence of a nonsingular path field on a topological manifold and the vanishing of its Euler characteristic, using the Euler class in its proof.

- How the theory of generalized bundles was essential to relate the Wu classes of topological manifolds via a local-flat embedding with a trivial generalized normal bundle.

Furthermore, with all the results presented so far, we also conclude the importance of generalized bundles not only as a tool for the theory of characteristic classes of topological manifolds, but as a theory in its own right.

# Chapter 5

# Characteristic Classes of Generalized Manifolds

In this chapter, we present for the first time in the literature a proof of Wu's formula in the context of generalized manifolds.

We begin in Section 5.1 with a brief overview, based on [4], [38], and [8], of the concept of generalized manifolds, introducing the definition of such an object as well as some of its properties.

In Section 5.2, we algebraically construct the Thom class and Thom isomorphism associated to an embedding between generalized manifolds with a specific codimension, following the results presented by Dold in ([17], Chapter 8).

Then, in Section 5.3, we construct the Stiefel–Whitney classes of embeddings between generalized manifolds with a specific codimension, the Stiefel–Whitney classes of generalized manifolds, and also point out the similarities between these constructions and the topological context presented in Chapter 3.

Finally, to conclude the chapter, we present in Section 5.4 the proof of Wu's formula for generalized manifolds using only their dualities—that is, without requiring the existence of a tangent bundle for such manifolds. The techniques used are based on the results presented by Bredon in ([7], Chapter 6).

## 5.1   Generalized Manifolds

This section aims to present the basic results on generalized manifolds, as well as the formal definition of such objects.

For a more detailed discussion on generalized manifolds, see [4], [8], [12], [11], [38], and [50].

We begin the final chapter of this work with the following definitions:

**Definition 5.1. (Cohomological Dimension)** *Let $M$ be a locally compact Hausdorff topological space. We say that $M$ has finite cohomological $\mathbb{Z}_2-$dimension, denoted by $dim_{\mathbb{Z}_2} M < \infty$, if there exists an integer $m \geq 0$ such that $H_c^{m+1}(U; \mathbb{Z}_2) = 0$ for every open neighborhood $U \subset M$.*

Definition 5.1 is, in fact, a consequence concerning the cohomological dimension of a space, whose proof can be found in ([8], Theorem 16.14, p. 115).

**Definition 5.2. (Homology Manifold)** *A locally compact Hausdorff topological space M is called an m−dimensional $\mathbb{Z}_2$−homology manifold if:*

- $dim_{\mathbb{Z}_2} M < \infty;$

- $\forall b \in M, \ H_k(M, M - \{b\}; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & , \ if \ k = m \\ 0 & , \ if \ k \neq m \end{cases}$

**Definition 5.3. (Hereditarily Paracompact)** *A Hausdorff topological space M is called hereditarily paracompact if every open subset of M is paracompact.*

In particular, every metric space is hereditarily paracompact[1].

**Definition 5.4. (Locally Homologically Connected)** *A topological space M is said to be locally homologically connected, denoted by $HLC^\infty_{\mathbb{Z}_2}$, if for every $b \in M$ and every open neighborhood $U \subset M$ of b, there exists an open neighborhood $V \subset U$ of b such that the homomorphism $\widetilde{H}_k(V; \mathbb{Z}_2) \to \widetilde{H}_k(U; \mathbb{Z}_2)$ induced by the inclusion $V \hookrightarrow U$ is trivial for all integers $k \geq 0$.*

**Definition 5.5. (Generalized Manifold)** *A locally compact Hausdorff topological space M is called an m−dimensional $\mathbb{Z}_2$−generalized manifold if:*

1. *M is an m-dimensional $\mathbb{Z}_2$−homology manifold;*

2. *M is hereditarily paracompact;*

3. *M is $HLC^\infty_{\mathbb{Z}_2}$.*

Note that the above definitions can be stated for an arbitrary coefficient group, not necessarily $\mathbb{Z}_2$, but since the purpose of this chapter is to work with Stiefel–Whitney and Wu classes, we will restrict ourselves to $\mathbb{Z}_2$−generalized manifolds.

On the other hand, although the definition of a generalized manifold is somewhat complex, these manifolds are well-behaved from an algebraic perspective, as we shall see.

First, the dimension of a generalized manifold coincides with its cohomological dimension, as shown in the proposition below, whose proof can be found in ([8], Remark 9.6, p. 331).

---

[1]For more details on hereditarily paracompact spaces, see ([8], Chapter 1, Section 6).

**Proposition 5.1.** *If $M^m$ is a $\mathbb{Z}_2-$ generalized manifold, then $dim_{\mathbb{Z}_2}M = m$.*

The singular cohomology modules of generalized manifolds are well-behaved in the following sense:

**Proposition 5.2.** *If $M^m$ is a compact $\mathbb{Z}_2-$ generalized manifold, then $H^k(M; \mathbb{Z}_2)$ is a finitely generated module for every integer $0 \leq k \leq m$.*

The proof of the proposition above can be found in ([8], Corollary 17.7, p. 129).

As a matter of curiosity, generalized manifolds differ from topological manifolds starting in dimension three, as we can see in the theorem below, whose proof is found in ([8], Theorem 16.32, p. 388).

**Theorem 5.1.** *Let $M^m$ be a second-countable homological manifold[2] and suppose $m \leq 2$. Then $M$ is a topological manifold.*

Before proceeding, we would like to highlight that there are several studies concerning the existence of generalized manifolds that are not topological manifolds. In fact, there are specific conditions that determine whether a generalized manifold has the same homotopy type as a topological manifold or not. However, since we will not pursue this direction, we suggest [12], [11], and [13] for readers interested in this topic.

Continuing our summary on generalized manifolds, the following result ensures the existence of Poincaré duality in the context of these manifolds, as presented in ([8], Corollary 10.2, p. 338), with a statement identical to that of smooth and topological manifolds:

**Theorem 5.2.** *Consider $M^m$ a compact and connected generalized manifold, and let $([M]) = H^m(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Then, for every $k \geq 0$, the duality obtained by homomorphism $\mathcal{D}_M : H^k(M; \mathbb{Z}_2) \to H_{m-k}(M; \mathbb{Z}_2)$, defined by $\mathcal{D}_M(x) = [M] \frown x$, is indeed an isomorphism.*

More generally, Halverson proved in ([22], Theorem 5.2, p. 242) that the Poincaré-Hopf duality is also valid for generalized manifolds, with the following statement:

**Theorem 5.3.** *Let $N^n$ be a compact generalized manifold, let $M \subset N$ be a closed subspace, and let $([N]_M) = H_n(N, N - M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ be its generator. Then, for every $k \geq 0$, the homomorphism $\mathcal{D}_{N,M} : \check{H}^k(M; \mathbb{Z}_2) \to H_{n-k}(N, N - M; \mathbb{Z}_2)$, defined by $\mathcal{D}_{N,M}(x) = [N]_M \frown x$, is indeed an isomorphism.*

---

[2]A topological space is said to be second-countable if it has a countable base.

After these initial considerations about generalized manifolds, we may assume that these objects are, roughly speaking, topological spaces that behave like topological manifolds in the context of singular (co)homology with $\mathbb{Z}_2-$module coefficients.

Moreover, due to some techniques we will use throughout this chapter, we will now work with a particular, yet broad, class of generalized manifolds: homological ENR-manifolds[3]. To briefly illustrate this class of manifolds that we will work with in this chapter, let us consider an example of a homological ENR-manifold that is not homeomorphic to a topological manifold.

**Example 5.1.** *Let $M^m$ be a topological manifold that is not homeomorphic to the sphere $\mathbb{S}^m$, but that is a $(\mathbb{Z}_2, m)-$homology sphere, i.e., $H_k(M; \mathbb{Z}_2) \cong H_k(\mathbb{S}^m; \mathbb{Z}_2)$ for every integer $k \geq 0$. Then, the suspension $S(M)$ of $M$ is a homological ENR-manifold of dimension $m + 1$.*

*Indeed, since $M$ is an ENR space, its suspension $S(M)$ is also an ENR. On the other hand, we know that $H_k(S(M), S(M) - \{b\}; \mathbb{Z}_2) \cong H_k(\mathbb{R}^{m+1}, \mathbb{R}^{m+1} - \{0\}; \mathbb{Z}_2)$ for any $b \in S(M)$ and any integer $k \geq 0$, that is, $S(M)$ is a homological manifold.*

*However, $S(M)$ is not a topological manifold, because for any open neighborhood $V \subset S(M)$ of the top vertex $v_+ \in S(M)$ of the suspension, we also know that $V \approx C_+(M)$, where $C_+(M)$ is the upper cone of the suspension $S(M)$. But $C_+(M)$ is not homeomorphic to the disk $D^{m+1}$, since its boundary $\partial(C_+(M)) = M$ is not homeomorphic to the sphere $\mathbb{S}^m$.*

The example above is taken from ([38], Example 2.2.4, p. 23).

## 5.2   Thom's Class and Isomorphism

In ([17], Chapter 8), Dold constructed the Thom class and Thom isomorphism associated with an embedding between topological manifolds using Poincaré and Poincaré-Lefschetz dualities. After examining these constructions more closely, we realized that Dold used only algebraic properties of topological manifolds, which motivated us to attempt the same approach in the context of generalized manifolds.

With some adaptations of the results presented by Dold, we will see in this section how to algebraically construct the Thom class and Thom isomorphism associated with a topological embedding, of specific codimension, between ENR homology manifolds using only the Poincaré and Poincaré-Lefschetz dualities of these manifolds.

To that end, we will fix the following assumptions throughout this section:

- $s : M^m \to N^{2m}$ is a topological embedding between compact, connected ENR homology manifolds;

- since we can assume, without loss of generality, that $M \subset N$, we will suppose that $M$ is a retract of $N$, that is, we assume there exists a map $p : N^{2m} \to M^m$ such that $p \circ s = 1$;

---

[3]The ENR condition is sufficient to ensure that a homological manifold is a generalized manifold.

- $\mathcal{D}_M : H^k(M; \mathbb{Z}_2) \rightarrow H_{m-k}(M; \mathbb{Z}_2)$ is the Poincaré duality of $M$, defined by $\mathcal{D}_M(x) = [M] \frown x$;

- $\mathcal{D}_N : H^k(N; \mathbb{Z}_2) \rightarrow H_{2m-k}(N; \mathbb{Z}_2)$ is the Poincaré duality of $N$, defined by $\mathcal{D}_N(x) = [N] \frown x$;

- $\mathcal{D}_{N,M} : H^k(M; \mathbb{Z}_2) \rightarrow H_{2m-k}(N, N - M; \mathbb{Z}_2)$ is the Poincaré-Lefschetz duality of $M \subset N$, defined by $\mathcal{D}_{N,M}(x) = [N]_M \frown x$.

Before proceeding, recall that since every homology manifold is $\mathbb{Z}_2-$orientable, there is no obstacle to considering the dualities mentioned above. Also recall that the global orientation classes of $M$ and $N$ are, respectively, the generators $([M]) = H_m(M; \mathbb{Z}_2)$ and $([N]) = H_{2m}(N; \mathbb{Z}_2)$, and the local orientation class of $N$ along $M$ is the generator $([N]_M) = H_{2m}(N, N - M; \mathbb{Z}_2)$.

Moreover, by fixing the canonical inclusion $j : N \hookrightarrow (N, N - M)$, we also know that the dualities $\mathcal{D}_N$ and $\mathcal{D}_{N,M}$ are related by the following commutative diagram:

$$
\begin{array}{ccc}
H^k(N; \mathbb{Z}_2) & \xrightarrow{\ s^*\ } & H^k(M; \mathbb{Z}_2) \\
\Big\downarrow{\scriptstyle \mathcal{D}_N} & {\scriptstyle [N]_M \frown} \searrow & \Big\downarrow{\scriptstyle \mathcal{D}_{N,M}} \\
H_{2m-k}(N; \mathbb{Z}_2) & \xrightarrow{\ j_*\ } & H_{2m-k}(N, N - M; \mathbb{Z}_2)
\end{array}
$$

Since $\mathcal{D}_N(1) = [N]$ and the diagram above commutes, we conclude the following relation:

$$j_*([N]) = [N]_M$$

Furthermore, the diagram above also ensures the following:

**Lemma 5.1.** *Given any $a \in H_k(N, N - M; \mathbb{Z}_2)$, there exists $x \in H^{2m-k}(N; \mathbb{Z}_2)$ such that $a = [N]_M \frown x$*

*Proof.*

This follows directly from the relation $\mathcal{D}_{N,M} \circ s^*(\_) = [N]_M \frown (\_)$ and from the facts that $\mathcal{D}_{N,M}$ is an isomorphism and $s^*$ is surjective, since $p \circ s = 1$.

$\square$

Now that we have fixed all necessary notations for the development of the rest of this chapter, let us construct the Thom class associated with the embedding $s : M^m \rightarrow N^{2m}$. To do so, we begin by defining the transfer isomorphism:

**Definition 5.6. (Transfer Isomorphism)** *To the embedding* $s : M^m \to N^{2m}$ *corresponds a natural homomorphism* $s_! : H_k(N, N - M; \mathbb{Z}_2) \to H_{k-m}(M; \mathbb{Z}_2)$, *called the transfer isomorphism, obtained as a composition of following canonical maps:*

$$s_! : H_k(N, N - M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}_{N,M}^{-1}} H^{2m-k}(M; \mathbb{Z}_2) \xrightarrow{\mathcal{D}_M} H_{k-m}(M; \mathbb{Z}_2)$$

Thanks to the transfer isomorphism, we have:

$$H_k(N, N - M; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & , \ k = m \text{ and } k = 2m \\ 0 & , \ k < m \text{ or } k > 2m \end{cases}$$

Hence, the Universal Coefficient Theorem guarantees that:

$$H^k(N, N - M; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & , \ k = m \text{ and } k = 2m \\ 0 & , \ k < m \text{ or } k > 2m \end{cases}$$

**Definition 5.7. (Thom class)** *We call the Thom class associated with the embedding* $s : M^m \to N^{2m}$ *the generator* $(\tau_s) = H^m(N, N - M; \mathbb{Z}_2)$

Let us also note that the transfer isomorphism $s_!$ and the embedding $s$ are related in the following way:

**Lemma 5.2.** *For any* $x \in H^k(N; \mathbb{Z}_2)$, *we have:*

$$s_!([N]_M \frown x) = [M] \frown s^*(x)$$

*Proof.*

Given any $x \in H^k(N; \mathbb{Z}_2)$ and recalling that $\mathcal{D}_{N,M} \circ s^*(x) = [N]_M \frown x$, we conclude:

$$\begin{aligned} s_!([N]_M \frown x) &= s_! \circ \mathcal{D}_{N,M} \circ s^*(x) \\ &= \mathcal{D}_M \circ \mathcal{D}_{N,M}^{-1} \circ \mathcal{D}_{N,M} \circ s^*(x) \\ &= \mathcal{D}_M \circ s^*(x) \\ &= [M] \frown s^*(x) \end{aligned}$$

$\square$

Now, the following two lemmas will provide a representative in $H^m(N; \mathbb{Z}_2)$ for the Thom class $\tau_s \in H^m(N, N - M; \mathbb{Z}_2)$ associated with the embedding $s : M^m \to N^{2m}$.

**Lemma 5.3.** *There exists a unique nonzero class* $U_s \in H^m(N; \mathbb{Z}_2)$ *such that:*

$$s_*([M]) = [N] \frown U_s$$

*Proof.*

Initially, since $[M] \neq 0$ in $H_m(M; \mathbb{Z}_2)$ and $s_*$ is injective (given that $p \circ s = 1$), it follows that $s_*([M]) \neq 0$ in $H_m(N; \mathbb{Z}_2)$.

Since $\mathcal{D}_N : H^m(N; \mathbb{Z}_2) \to H_m(N; \mathbb{Z}_2)$ is an isomorphism, there exists a unique nonzero class $U_s \in H^m(N; \mathbb{Z}_2)$ such that $\mathcal{D}_N(U_s) = s_*([M])$, that is, $[N] \frown U_s = s_*([M])$.

$\square$

**Proposition 5.3.** *The canonical inclusion $j : N \hookrightarrow (N, N - M)$ satisfies:*

$$j^*(\tau_s) = U_s$$

*Proof.*

Initially, since $U_s \neq 0$ in $H^m(N; \mathbb{Z}_2)$ and $\tau_s \neq 0$ in $H^m(N, N - M; \mathbb{Z}_2)$, we have $\tau_s \smile \tau_s \neq 0$ and $U_s \smile \tau_s \neq 0$, both in $H^{2m}(N, N - M; \mathbb{Z}_2)$. Thus, since $H^{2m}(N, N - M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, we get:

$$(\tau_s \smile \tau_s) = H^{2m}(N, N - M; \mathbb{Z}_2) = (U_s \smile \tau_s)$$

Also, since $([N]_M) = H_{2m}(N, N - M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, then:

$$< \tau_s \smile \tau_s, [N]_M > = \ 1 \ = <U_s \smile \tau_s, [N]_M >$$

Now, let us prove that $j^*(\tau_s) \neq 0$. To that end, observe:

$$
\begin{aligned}
< j^*(\tau_s), [N]_M \frown \tau_s > &= \ < j^*(\tau_s) \smile \tau_s, [N]_M > \\
&= \ < j^*(\tau_s) \smile \tau_s, j_*([N]) > \\
&= \ < j^*(j^*(\tau_s) \smile \tau_s), [N] > \\
&= \ < j^*(\tau_s) \smile j^*(\tau_s), [N] > \\
&= \ < j^*(\tau_s \smile \tau_s), [N] > \\
&= \ < \tau_s \smile \tau_s, j_*([N]) > \\
&= \ < \tau_s \smile \tau_s, [N]_M > \\
&= \ 1
\end{aligned}
$$

On the other hand, if $j^*(\tau_s) = 0$, we would have $< j^*(\tau_s), [N]_M \frown \tau_s >= 0$, which is a contradiction. Therefore, $j^*(\tau_s) \neq 0$.

Thus, the Proposition A.1 ensures that there exists a unique class $x \in H^m(N; \mathbb{Z}_2)$ with $x \neq 0$ such that $< x, \mathcal{D}_N(j^*(\tau_s)) >= 1$.

If we prove that $< U_s, \mathcal{D}_N(j^*(\tau_s)) >= 1 =< j^*(\tau_s), \mathcal{D}_N(j^*(\tau_s)) >$, we conclude by uniqueness that $j^*(\tau_s) = U_s$. Let us do that:

$$
\begin{aligned}
< U_s, \mathcal{D}_N(j^*(\tau_s)) > &= \ < U_s, [N] \frown j^*(\tau_s) > \\
&= \ < U_s \smile j^*(\tau_s), [N] > \\
&= \ < j^*(U_s \smile \tau_s), [N] > \\
&= \ < U_s \smile \tau_s, j_*([N]) > \\
&= \ < U_s \smile \tau_s, [N]_M > \\
&= \ 1
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
< j^*(\tau_s), \mathcal{D}_N(j^*(\tau_s)) > &= \ < j^*(\tau_s), [N] \frown j^*(\tau_s) > \\
&= \ < j^*(\tau_s) \smile j^*(\tau_s), [N] > \\
&= \ < j^*(\tau_s \smile \tau_s), [N] > \\
&= \ < \tau_s \smile \tau_s, j_*([N]) > \\
&= \ < \tau_s \smile \tau_s, [N]_M > \\
&= \ 1
\end{aligned}
$$

Therefore, $j^*(\tau_s) = U_s$.

$\square$

Thus, we only need to prove a few more technical results about the Thom class so that we can finally define the Thom isomorphism.

**Lemma 5.4.** $s_*([M]) = [N]_M \frown \tau_s$

*Proof.* It suffices to combine Lemma 5.3, Proposition 5.3, and Lemma A.3 as follows:

$$
\begin{aligned}
s_*([M]) &= \ [N] \frown U_s \\
&= \ [N] \frown j^*(\tau_s) \\
&= \ j_*([N]) \frown \tau_s \\
&= \ [N]_M \frown \tau_s
\end{aligned}
$$

$\square$

**Proposition 5.4.** *For any* $a \in H_k(N, N - M; \mathbb{Z}_2)$, *we have:*

$$
s_* \circ s_!(a) = a \frown \tau_s
$$

*Proof.*

First, we know from Lemma 5.1 that for any $a \in H_k(N, N - M; \mathbb{Z}_2)$, there exists $x \in H^{2m-k}(N; \mathbb{Z}_2)$ such that $a = [N]_M \frown x$.

Thus, Lemmas 5.2 and 5.4 ensure that:

$$
\begin{aligned}
s_* \circ s_!(a) &= \ s_* \circ s_!([N]_M \frown x) \\
&= \ s_*([M] \frown s^*(x)) \\
&= \ s_*([M]) \frown x \\
&= \ ([N]_M \frown \tau_s) \frown x \\
&= \ [N]_M \frown (\tau_s \cup x) \\
&= \ [N]_M \frown (x \cup \tau_s) \\
&= \ ([N]_M \frown x) \frown \tau_s \\
&= \ a \frown \tau_s
\end{aligned}
$$

$\square$

**Theorem 5.4.** *The transfer isomorphism* $s_! : H_k(N, N - M; \mathbb{Z}_2) \to H_{k-m}(M; \mathbb{Z}_2)$ *can be decomposed as follows:*

$$s_! : H_k(N, N - M; \mathbb{Z}_2) \xrightarrow{\;\frown \tau_s\;} H_{k-m}(N; \mathbb{Z}_2) \xrightarrow{\;p_*\;} H_{k-m}(M; \mathbb{Z}_2)$$

*Proof.* Given any $a \in H_k(N, N - M; \mathbb{Z}_2)$ and recalling that $p \circ s = 1$, the previous proposition ensures that:

$$
\begin{aligned}
p_*(a \frown \tau_s) &= p_* \circ s_* \circ s_!(a) \\
&= s_!(a)
\end{aligned}
$$

$\square$

**Theorem 5.5. (Thom isomorphism)** *For any integer $k \geq 0$, the following composition is an isomorphism:*

$$\phi_s : H^k(M; \mathbb{Z}_2) \xrightarrow{\;p^*\;} H^k(N; \mathbb{Z}_2) \xrightarrow{\;\smile \tau_s\;} H^{k+m}(N, N - M; \mathbb{Z}_2)$$

*Proof.*
First, note that for any $a \in H_{k+m}(N, N - M; \mathbb{Z}_2)$ and $x \in H^k(M; \mathbb{Z}_2)$, we have:

$$
\begin{aligned}
< \phi_s(x), a > &= < p^*(x) \smile \tau_s, a > \\
&= < p^*(x), a \frown \tau_s > \\
&= < x, p_*(a \frown \tau_s) > \\
&= < x, s_!(a) >
\end{aligned}
$$

Thus, $\phi_s$ is the cohomological dual of $s_!$, and since $s_!$ is an isomorphism, the Universal Coefficient Theorem ensures that $\phi_s$ is an isomorphism.

$\square$

**Definition 5.8.** *The isomorphism $\phi_s : H^k(M; \mathbb{Z}_2) \to H^{k+m}(N, N - M; \mathbb{Z}_2)$ described in Theorem 5.5 is referred to as the Thom isomorphism associated with the embedding $s : M^m \to N^{2m}$.*

## 5.3   Stiefel-Whitney Classes

We will construct in this section, in a fully algebraic way, the Stiefel–Whitney classes of embeddings between homological manifolds and the Stiefel–Whitney classes of homological manifolds. Moreover, using technical results obtained by Fadell in [20], we will show some similarities between these constructions and the Stiefel–Whitney classes of $\mathbb{R}^n-$generalized normal bundles of local-flat embeddings and the Stiefel–Whitney classes of topological manifolds presented in Chapter 3.

Initially, since we can associate to each topological embedding between homological manifolds, under specific conditions, its Thom class and Thom isomorphism, it is possible

to define the Stiefel–Whitney classes of this embedding in a way analogous to what was done for vector bundles in ([41], Chapter 4) and for generalized bundles in Chapter 3, as follows:

**Definition 5.9. (Stiefel–Whitney classes)** *Let $s : M^m \to N^{2m}$ be a topological embedding between compact, connected ENR-homological manifolds such that $M$ is a retract of $N$. Denoting by $\tau_s$ and $\phi_s$ the Thom class and Thom isomorphism associated to the embedding $s$, respectively, we define the k-th Stiefel–Whitney class of the embedding $s$ as:*

$$w_k(s) = \phi_s^{-1} \circ Sq^k(\tau_s) \in H^k(M; \mathbb{Z}_2)$$

*Furthermore, we call $W(s) = \displaystyle\sum_{k \geq 0} w_k(s) \in H^*(M; \mathbb{Z}_2)$ the total Stiefel–Whitney class of $s$.*

Due to the properties of Steenrod squares, the following result is analogous to Theorem 3.2.

**Proposition 5.5.** *If $s : M^m \to N^{2m}$ is a topological embedding between compact, connected ENR-homological manifolds such that $M$ is a retract of $N$, then $w_0(s) = 1$ and $w_k(s) = 0$ for $k > m$. In other words, $W(s) = 1 + \displaystyle\sum_{k=1}^{m} w_k(s)$.*

Now, since every local-flat embedding is a topological embedding and every topological manifold is an ENR-homological manifold, we obtain the following:

**Theorem 5.6.** *Let $s : M^m \to S^{2m}$ be a local-flat embedding between closed, connected topological manifolds such that $M$ is a retract of $S$, and let $(\mathcal{N}, \mathcal{N}_0)$ be its $\mathbb{R}^m-$generalized normal bundle. If $W(\mathcal{N}, \mathcal{N}_0)$ and $W(s)$ denote, respectively, the total Stiefel–Whitney classes defined in Definitions 3.4 and 5.9, then $W(\mathcal{N}, \mathcal{N}_0) = W(s)$.*

*Proof.*
    Initially, recall that the normal $\mathbb{R}^m-$generalized bundle of the local-flat embedding $s : M \to S$ is the pair $(\mathcal{N}, \mathcal{N}_0) = (N, N_0, q, M)^4$, where:

- $N_0 = \{\omega \in S^I \ : \ \omega(t) \in M \Leftrightarrow t = 0\}$

- $N = N_0 \cup \{\omega \in S^I \ : \ \omega(t) = \omega(0) \in M, \ \forall t \in I\}$

- $q : N \to M$ is given by $q(\omega) = \omega(0)$

---

[4] In this chapter, the letter $N$ will not denote a manifold except in the proof of Theorem 5.6.

Since $s : M \to S$ is a local-flat embedding, denote by $\tau \in H^m(N, N_0; \mathbb{Z}_2)$ and $\phi : H^k(M; \mathbb{Z}_2) \to H^{k+m}(N, N_0; \mathbb{Z}_2)$ the Thom class and Thom isomorphism, respectively, of $(\mathcal{N}, \mathcal{N}_0)$, where $\phi(x) = q^*(x) \smile \tau$.

On the other hand, considering $s : M \to S$ as a topological embedding between compact, connected ENR homological manifolds such that $M$ is a retract of $S$, let $p : S \to M$ be such a retraction and denote by $\tau_s \in H^m(S, S - M; \mathbb{Z}_2)$ the Thom class and $\phi_s : H^k(M; \mathbb{Z}_2) \to H^{k+m}(S, S - M; \mathbb{Z}_2)$ the Thom isomorphism, both associated to the embedding $s$, where $\phi_s(x) = p^*(x) \smile \tau_s$.

Now, defining $\xi : (N, N_0) \to (S, S - M)$ by $\xi(\omega) = \omega(1)$, then ([20], Theorem 7.5, p. 509) guarantees that $\xi$ induces an isomorphism in the context of $\mathbb{Z}_2-$modules of singular cohomology such that $\xi^*(\tau_s) = \tau$.

Moreover, since $N - N_0 = \{\omega \in S^I \ : \ \omega(t) = \omega(0) \in M, \ \forall t \in I\}$, we can consider the restriction $\xi_{|N-N_0} : (N - N_0) \to (S - (S - M))$ defined by:

$$
\begin{aligned}
\xi_{|N-N_0}(\omega) &= \omega(1) \\
&= \omega(0) \\
&= q(\omega)
\end{aligned}
$$

Thus, we have that $\xi^* \circ \phi_s = \phi$. Indeed:

$$
\begin{aligned}
\forall k \geq 0, \ \forall x \in H^k(M; \mathbb{Z}_2), \ \xi^* \circ \phi_s(x) &= \xi^*(p^*(x) \smile \tau_s) \\
&= q^*(x) \smile \xi^*(\tau_s) \\
&= q^*(x) \smile \tau \\
&= \phi(x)
\end{aligned}
$$

Hence, the following diagram is commutative for any integer $k \geq 0$:



Therefore, for all $k \geq 0$, we have:

$$
\begin{aligned}
w_k(\mathcal{N}, \mathcal{N}_0) &= \phi^{-1} \circ Sq^k(\tau) \\
&= \phi^{-1} \circ Sq^k \circ \xi^*(\tau_s) \\
&= \phi^{-1} \circ \xi^* \circ Sq^k(\tau_s) \\
&= \phi_s^{-1} \circ Sq^k(\tau_s) \\
&= w_k(s)
\end{aligned}
$$

$\square$

Note that, even though there is no notion of tangent bundle for homology manifolds, it is possible to define the Stiefel-Whitney classes of such manifolds as follows:

**Definition 5.10. (Stiefel-Whitney Classes)** *Let $M^m$ be a compact connected homology ENR manifold. Since the diagonal map $d : M \to M \times M$ is a topological embedding and the canonical projection $p_1 : M \times M \to M$ is a retraction of $M \times M$ onto $M$, we can then define the total Stiefel-Whitney class of $M$ by $W(M) = W(d)$.*

For the construction of the Stiefel-Whitney classes of a homology manifold to be indeed well-defined, we must pay attention to the following point: since every topological manifold is a homology manifold, the construction of the Stiefel-Whitney classes of a topological manifold must be the same regardless of whether we use the construction given by Fadell for topological manifolds as in Definition 3.4 or our construction for generalized manifolds as in Definition 5.10. In other words:

**Theorem 5.7.** *Consider $M^m$ a closed connected topological manifold, $(\tau M, \tau_0 M)$ its $\mathbb{R}^m-$generalized tangent bundle and $d : M \to M \times M$ the diagonal. Denoting by $W(\tau M, \tau_0 M)$ and $W(d)$ the total Stiefel-Whitney classes of $M$ given by Definitions 3.4 and 5.10, respectively, then $W(\tau M, \tau_0 M) = W(d)$.*

*Proof.*

Recall that $(\tau M, \tau_0 M) = (TM, T_0 M, p, M)$, where $p : TM \to M$ is defined by $p(\omega) = \omega(0)$, and that the canonical projection $p_1 : M \times M \to M$ is a retraction of $M \times M$ onto $M$, i.e. $p_1 \circ d = 1$, so fix some notations.

Since $M^m$ is a topological manifold with generalized tangent bundle $(\tau M, \tau_0 M)$, denote by $\tau \in H^m(TM, T_0 M; \mathbb{Z}_2)$ and $\phi : H^k(M; \mathbb{Z}_2) \to H^{k+m}(TM, T_0 M; \mathbb{Z}_2)$ the Thom class and Thom isomorphism, respectively, of $(\tau M, \tau_0 M)$, where $\phi(x) = p^*(x) \smile \tau$.

On the other hand, considering $M^m$ as a compact connected homology ENR manifold, denote $\Delta = d(M)$, $\tau_d \in H^m(M \times M, M \times M - \Delta; \mathbb{Z}_2)$ the Thom class and $\phi_d : H^k(M; \mathbb{Z}_2) \to H^{k+m}(M \times M, M \times M - \Delta; \mathbb{Z}_2)$ the Thom isomorphism, both associated to the embedding $d$, where $\phi_d(x) = p_1^*(x) \smile \tau_d$.

Now, due to part of the proof of Lemma 4.4, we know that defining the map $\psi : (TM, T_0 M) \to (M \times M, M \times M - \Delta)$ by $\psi(\omega) = (\omega(0), \omega(1))$, then $\psi$ induces an isomorphism in the category of $\mathbb{Z}_2-$modules of singular cohomology such that $\psi^*(\tau_d) = \tau$ and $\psi^* \circ \phi_d = \phi$.

Thus, the following diagram is commutative for any integer $k \geq 0$:

$$
\begin{array}{ccc}
H^m(M \times M, M \times M - \Delta; \mathbb{Z}_2) & \xrightarrow{\psi^*} & H^m(TM, T_0 M; \mathbb{Z}_2) \\
\Big\downarrow{\scriptstyle Sq^k} & & \Big\downarrow{\scriptstyle Sq^k} \\
H^{k+m}(M \times M, M \times M - \Delta; \mathbb{Z}_2) & \xrightarrow{\psi^*} & H^{k+m}(TM, T_0 M; \mathbb{Z}_2) \\
& \searrow{\scriptstyle \phi_d^{-1}} \qquad \swarrow{\scriptstyle \phi^{-1}} & \\
& H^k(M; \mathbb{Z}_2) &
\end{array}
$$

Therefore, for all $k \geq 0$, we have:

$$
\begin{aligned}
w_k(\tau M, \tau_0 M) &= \phi^{-1} \circ Sq^k(\tau) \\
&= \phi^{-1} \circ Sq^k \circ \psi^*(\tau_d) \\
&= \phi^{-1} \circ \psi^* \circ Sq^k(\tau_d) \\
&= \phi_d^{-1} \circ Sq^k(\tau_d) \\
&= w_k(d)
\end{aligned}
$$

$\square$

## 5.4 Wu's Formula for Generalized Manifolds

In ([7], Chapter 6), Bredon presented an alternative proof of Wu's formula for closed, connected smooth manifolds using the diagonal map, Poincaré and Poincaré-Lefschetz dualities, and some properties arising from the existence of the tangent vector bundle. Noting that Bredon used algebraic properties related to smooth manifolds and tangent vector bundles, all combined with the diagonal map, we were motivated to attempt the same for the context of homological manifolds by using the algebraic construction of the Thom class and Thom isomorphism for these manifolds obtained in section 5.2.

After some adaptations of the results presented by Bredon, we will prove in this section Wu's formula for homological manifolds without necessarily requiring the existence of a tangent bundle for such manifolds, but only using their dualities.

To this end, throughout this section, we fix $M^m$ as a compact, connected ENR homological manifold, $d : M \to M \times M$ the diagonal map, and the canonical projection $p_1 : M \times M \to M$ as a retraction of $M \times M$ onto $M$, i.e., $p_1 \circ d = 1$.

Also fixing an arbitrary basis $\{b_i\}_{i=1}^r$ of $H^*(M; \mathbb{Z}_2)$ and its dual basis $\{b_i^{\#}\}_{i=1}^r$, we can write the class $U_d \in H^m(M \times M; \mathbb{Z}_2)$ obtained in Lemma 5.3 as follows:

**Lemma 5.5.** $U_d = \displaystyle\sum_{i=1}^{r} (b_i \times b_i^{\#})$

*Proof.* As $\{b_i\}_{i=1}^r$ and $\{b_i^{\#}\}_{i=1}^r$ are bases for $H^*(M; \mathbb{Z}_2)$, then $\{b_i \times b_j^{\#}\}_{i,j=1}^r$ forms a basis for $H^*(M \times M; \mathbb{Z}_2)$. Thus, since $U_d \in H^*(M \times M; \mathbb{Z}_2)$, for each $i, j = 1, \ldots, r$ there exists a unique $\alpha_{i,j} \in \mathbb{Z}_2$ such that:

$$
U_d = \sum_{i,j=1}^{r} \alpha_{i,j} (b_i \times b_j^{\#})
$$

Before proceeding with the proof, note that due to the uniqueness of the dual basis as in Theorem A.7, the set $\{b_i^{\#} \times b_j\}_{i,j=1}^r$ is the dual basis of $\{b_i \times b_j^{\#}\}_{i,j=1}^r$ in $H^*(M \times M; \mathbb{Z}_2)$, since:

$$
\begin{aligned}
< (b_i \times b_j^{\#}) \smile (b_{i'}^{\#} \times b_{j'}), [M \times M] > &= \; < (b_i \smile b_{i'}^{\#}) \times (b_j^{\#} \smile b_{j'}), [M \times M] > \\
&= \; < b_i \smile b_{i'}^{\#}, [M] >< b_j^{\#} \smile b_{j'}, [M] > \\
&= \; \begin{cases} 1, & i' = i \text{ and } j' = j \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Thus, $(b_i \times b_j^{\#})^{\#} = b_i^{\#} \times b_j$ for any $i, j = 1, \dots, r$.
Now, by Corollary A.1, we also have:

$$
U_d = \sum_{i,j=1}^{r} (b_i \times b_j^{\#}) < U_d \smile (b_i \times b_j^{\#})^{\#}, [M \times M] >
$$

Therefore, Lemma 5.3 ensures that for any $i, j = 1, \dots, r$:

$$
\begin{aligned}
\alpha_{i,j} &= \; < U_d \smile (b_i \times b_j^{\#})^{\#}, [M \times M] > \\
&= \; < b_i^{\#} \times b_j, [M \times M] \frown U_d > \\
&= \; < b_i^{\#} \times b_j, d_*([M]) > \\
&= \; < d^*(b_i^{\#} \times b_j), [M] > \\
&= \; < b_i^{\#} \smile b_j, [M] > \\
&= \; \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
\end{aligned}
$$

Hence, $U_d = \sum_{i=1}^{r} (b_i \times b_i^{\#})$.

$\square$

**Lemma 5.6.** $U_d/[M] = 1 \in H^0(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$

*Proof.*

Initially, by fixing the basic element $(b_1) = H^0(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, we have $b_1 = 1$. Thus, it is evident that $< b_1^{\#}, [M] >= 1$. In other words, $(b_1^{\#}) = H^m(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Still, since $b_i \notin H^0(M; \mathbb{Z}_2)$ for $i \neq 1$, it is clear that $b_i^{\#} \notin H^m(M; \mathbb{Z}_2)$ when $i \neq 1$. Therefore, from the previous lemma we obtain:

$$
\begin{aligned}
U_d/[M] &= \; \left( \sum_{i=1}^{r} (b_i \times b_i^{\#}) \right) /[M] \\
&= \; \sum_{i=1}^{r} \left[ \left( b_i \times b_i^{\#} \right) /[M] \right] \\
&= \; \sum_{i=1}^{r} b_i < b_i^{\#}, [M] > \\
&= \; b_1 < b_1^{\#}, [M] > \\
&= \; b_1 \\
&= \; 1
\end{aligned}
$$

$\square$

Now, combining Lemmas 5.3 and 5.4 for the embedding $d : M \rightarrow M \times M$ and denoting $\Delta = d(M)$, we obtain the following relation:

$$[M \times M] \frown U_d = [M \times M]_\Delta \frown \tau_d$$

This induces the following:

**Lemma 5.7.** *For every integer $k \geq 0$, we have the following relation:*

$$[M \times M] \frown Sq^k(U_d) = [M \times M]_\Delta \frown Sq^k(\tau_d)$$

*Proof.*

We simply observe that for any $k \geq 0$, Proposition 5.3 and Lemma A.3 ensure that:

$$
\begin{aligned}
[M \times M] \frown Sq^k(U_d) &= [M \times M] \frown Sq^k(j^*(\tau_d)) \\
&= [M \times M] \frown j^*(Sq^k(\tau_d)) \\
&= j^*([M \times M]) \frown Sq^k(\tau_d) \\
&= [M \times M]_\Delta \frown Sq^k(\tau_d)
\end{aligned}
$$

$\square$

Thus, denoting by $\phi_d : H^k(M; \mathbb{Z}_2) \rightarrow H^{k+m}(M \times M, M \times M - \Delta; \mathbb{Z}_2)$ the Thom isomorphism associated to the embedding $d$, which is defined by $\phi_d(x) = p_1^*(x) \smile \tau_d$, and recalling that $w_k(M) = \phi_d^{-1} \circ Sq^k(\tau_d)$, we obtain a first relation between the Stiefel-Whitney classes of $M$ and the class $U_d$ as follows:

**Lemma 5.8.** *For every integer $k \geq 0$, we have the following relation:*

$$d_*([M] \frown w_k(M)) = [M \times M] \frown Sq^k(U_d)$$

*Proof.*

Using Lemma 5.4 and the previous lemma, and recalling that $p_1 \circ d = 1$, we obtain for all $k \geq 0$:

$$
\begin{aligned}
d_*([M] \frown w_k(M)) &= d_*([M] \frown d^* \circ p_1^*(w_k(M))) \\
&= d_*([M]) \frown p_1^*(w_k(M)) \\
&= ([M \times M]_\Delta \frown \tau_d) \frown p_1^*(w_k(M)) \\
&= [M \times M]_\Delta \frown (\tau_d \smile p_1^*(w_k(M))) \\
&= [M \times M]_\Delta \frown \phi_d(w_k(M)) \\
&= [M \times M]_\Delta \frown Sq^k(\tau_d) \\
&= [M \times M] \frown Sq^k(U_d)
\end{aligned}
$$

$\square$

This leads to the following lemma, which shows a more elegant relation between the Stiefel–Whitney classes of $M$ and the class $U_d$.

**Lemma 5.9.** *For any integer $k \geq 0$, we have:*

$$w_k(M) = Sq^k(U_d)/[M]$$

*Proof.*

Recalling that $p_1 \circ d = 1$ and using the previous lemma together with item 2 of Lemma A.5, we obtain for all $k \geq 0$:

$$
\begin{aligned}
\mathcal{D}_M(w_k(M)) &= (p_1)_* \circ d_* \circ \mathcal{D}_M(w_k(M)) \\
&= (p_1)_* \left( d_* \left( [M] \frown w_k(M) \right) \right) \\
&= (p_1)_* \left( [M \times M] \frown Sq^k(U_d) \right) \\
&= (p_1)_* \left( ([M] \times [M]) \frown Sq^k(U_d) \right) \\
&= [M] \frown \left( Sq^k(U_d)/[M] \right) \\
&= \mathcal{D}_M(Sq^k(U_d)/[M])
\end{aligned}
$$

Therefore, since the Poincaré duality $\mathcal{D}_M$ is an isomorphism, we conclude that $w_k(M) = Sq^k(U_d)/[M]$.

$\square$

Therefore, we finally obtain the:

**Theorem 5.8. (Wu Formula)** *Let $M^m$ be a compact and connected homological ENR manifold. Then:*

$$W(M) = Sq(v(M))$$

*Proof.*

We must prove that, for every integer $k \geq 0$, we have $w_k(M) = \displaystyle\sum_{i+j=k} Sq^i(v_j(M))$.

Due to Corollary A.1 and the very definition of Wu classes, we have:

$$
\begin{aligned}
v_j(M) &= \sum_{l=1}^{r} < v_j(M) \smile b_l^{\#}, [M] > b_l \\
&= \sum_{l=1}^{r} < Sq^j(b_l^{\#}), [M] > b_l
\end{aligned}
$$

Thus, by the previous lemma and Lemma 5.5, we conclude:

$$
\begin{aligned}
w_k(M) &= Sq^k(U_d)/[M] \\
&= Sq^k \left( \sum_{l=1}^{r} (b_l \times b_l^{\#}) \right) /[M] \\
&= \sum_{l=1}^{r} \left[ Sq^k(b_l \times b_l^{\#})/[M] \right] \\
&= \sum_{l=1}^{r} \left[ \left( \sum_{i+j=k} \left( Sq^i(b_l) \times Sq^j(b_l^{\#}) \right) \right) /[M] \right] \\
&= \sum_{i+j=k} \sum_{l=1}^{r} \left[ \left( Sq^i(b_l) \times Sq^j(b_l^{\#}) \right) /[M] \right] \\
&= \sum_{i+j=k} \sum_{l=1}^{r} < Sq^j(b_l^{\#}), [M] > Sq^i(b_l) \\
&= \sum_{i+j=k} Sq^i \left( \sum_{l=1}^{r} < Sq^j(b_l^{\#}), [M] > b_l \right) \\
&= \sum_{i+j=k} Sq^i(v_j(M))
\end{aligned}
$$

$\square$

Thus, we conclude this chapter with entirely original contributions to the theory of characteristic classes of generalized manifolds, which, roughly speaking, are topological spaces with no geometric condition that share the same homological and cohomological behavior as topological manifolds. In particular, we showed that the fact that Poincaré and Poincaré–Lefschetz dualities also hold in the context of generalized manifolds was fundamental to the development of this chapter.

Even though there is no notion of a normal bundle for topological embeddings between generalized manifolds, we algebraically constructed the Thom class and Thom isomorphism associated with such embeddings. Hence, we defined the Stiefel–Whitney classes of these embeddings analogously to both the classical definition and the one presented in Chapter 3, using Steenrod squares, the Thom isomorphism, and the Thom class.

Moreover, since every local-flat embedding between topological manifolds can be viewed as a topological embedding between generalized manifolds, we showed that the definition of the Stiefel–Whitney classes of the normal generalized bundle of a local-flat embedding coincides with the definition of these classes via the construction in the context of generalized manifolds.

On the other hand, even though there is no notion of a tangent bundle for generalized manifolds, we constructed the Thom class, Thom isomorphism, and the Stiefel–Whitney classes of a generalized manifold by considering the particular case of the embedding given by the diagonal map of that manifold. We also showed that the definition of the Stiefel–Whitney classes of the tangent generalized bundle of a topological manifold coincides with the definition of these classes when viewing the manifold as a generalized manifold.

Since Biasi, Daccach, and Saeki defined in [4] the Stiefel–Whitney classes of generalized manifolds using the Wu formula itself and presented several results in this context, we emphasize the originality of Chapter 5, where we defined the Stiefel–Whitney classes for generalized manifolds via an alternative approach and proved the Wu formula for such manifolds.

Thus, we conclude this work with several contributions to the theory of characteristic classes, both for topological and for generalized manifolds.

# Appendix A

# Singular (Co)homology

In order to keep this work concise, yet complete and self-explanatory, we will use this appendix as a brief review of some well-known concepts from Algebraic Topology.

When referring simultaneously to the singular homology and cohomology modules, for convenience, we will simply write singular (co)homology modules. Thus, we ask the reader to already be familiar with the concepts of these theories.

## A.1  Main results

We will use this section to state general results on singular (co)homology that will be useful for the development of this work and for a better understanding of the constructions made in the following sections of this appendix.

**Theorem A.1. (Universal Coefficients)** *Let $(X, A)$ be any pair of topological spaces. Then:*

1. *(**general case for homology**)[1] If $H_k(X, A; \mathbb{Z})$ is a free module[2] for all $k \geq 0$ or $R$ is a free module, then:*

$$H_k(X, A; R) \cong H_k(X, A; \mathbb{Z}) \otimes R, \ \forall k \geq 0$$

2. *(**general case for cohomology**)[3] If $H_k(X, A; \mathbb{Z})$ is a free module for all $k \geq 0$, then:*

$$H^k(X, A; R) \cong Hom(H_k(X, A; \mathbb{Z}); R), \ \forall k \geq 0$$

3. *(**particular case**)[4] If $\mathbb{F}$ is a field, then the Kronecker product ensures that:*

$$H^k(X, A; \mathbb{F}) \cong Hom(H_k(X, A; \mathbb{F}); \mathbb{F}), \ \forall k \geq 0$$

---

[1] Can be found in ([23], Corollary 3A.4, p. 264).
[2] A module is called free if it admits a basis.
[3] Can be found in ([23], Theorem 3.2, p. 195).
[4] Can be found in ([23], p. 198).

*For $R = \mathbb{Z}$ in case 2 or $R = \mathbb{F}$ in case 3, the isomorphism is given by the relation $x \in H^k(X, A; R) \mapsto \overline{x}(a) = <x, a> \in R$.*

**Theorem A.2. (Künneth Formula)** *Let $X$ and $Y$ be any topological spaces and $R$ a finitely generated principal ideal domain. Then:*

1. *(**case for absolute cohomology**)[5] If all the singular homology $R$-modules of $Y$ are finitely generated, then:*

$$H^k(X \times Y; R) \cong \bigoplus_{i+j=k} \left[ H^i(X; R) \otimes H^j(Y; R) \right], \ \forall k \geq 0$$

2. *(**case for absolute homology**)[6]*

$$H_k(X \times Y; R) \cong \bigoplus_{i+j=k} \left[ H_i(X; R) \otimes H_j(Y; R) \right], \ \forall k \geq 0$$

The proofs of the general cases of the Universal Coefficients Theorem can be found in ([44], Chapter 5, Sections 2 and 5).

The proofs of the Künneth formulas, in their general versions for pairs, can be found in ([44], Chapter 5, Sections 3 and 5).

Now, let us briefly review some properties about the cap, cup, cross, and Kronecker products.

**Lemma A.1.** *Let $X$, $X'$, $Y$, and $Y'$ be arbitrary topological spaces, $f : X \to X'$ and $g : Y \to Y'$ any maps, $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ the canonical projections, and $d : X \to X \times X$ the diagonal map. If $a \in H_q(X; R)$, $b \in H_r(Y; R)$, $x \in H^i(X; R)$, $x_1 \in H^{i_1}(X; R)$, $x_2 \in H^{i_2}(X; R)$, $y \in H^j(Y; R)$, $y_1 \in H^{j_1}(Y; R)$, $y_2 \in H^{j_2}(Y; R)$, $a' \in H_{q'}(X; R)$, $x' \in H^{i'}(X'; R)$, $x'_1 \in H^{i'_1}(X'; R)$, $x'_2 \in H^{i'_2}(X'; R)$, $y' \in H^{j'}(Y'; R)$, then:*

1. $1 \smile x = x = x \smile 1$

2. $0 \smile x = 0 = x \smile 0$

3. $x_1 \smile x_2 = 0 \iff x_1 = 0 \ or \ x_2 = 0, \ when \ R = \mathbb{Z}_2$

4. $a \frown 1 = a$

5. $x_1 \smile x_2 = (-1)^{|x_1|.|x_2|}(x_2 \smile x_1)$

6. $(a \frown x_1) \frown x_2 = a \frown (x_2 \smile x_1)$

7. $(x_1 \times y_1) \smile (x_2 \times y_2) = (-1)^{|y_1|.|x_2|}(x_1 \smile x_2) \times (y_1 \smile y_2)$

---

[5] Can be found in ([44], Theorem 1, p. 249).

[6] Can be found in ([44], Theorem 10, p. 235).

8. $(a \times b) \frown (x \times y) = (-1)^{|a| \cdot (|y| - |b|)} (a \frown x) \times (b \frown y)$

9. $< x_1 \smile x_2, a >=< x_1, a \frown x_2 >$

10. $< x \times y, a \times b >= (-1)^{|x| \cdot |y|} < x, a > . < y, b >$

11. $p_1^*(x) = x \times 1$ and $p_2^*(y) = 1 \times y$

12. $x \times y = p_1^*(x) \smile p_2^*(y)$

13. $x_1 \smile x_2 = d^*(x_1 \times x_2)$

14. $f_*(a \frown f^*(x')) = f_*(a) \frown x'$

15. $(f \times g)^*(x' \times y') = f^*(x') \times g^*(y')$

16. $f^*(x_1' \smile x_2') = f^*(x_1') \smile f^*(x_2')$

17. $< f^*(x'), a >=< x', f_*(a) >$

18. $< (f^*)^{-1}(x), a' >=< x, (f_*)^{-1}(a') >$, if $f^*$ and $f_*$ are isomorphisms.

The properties in the lemma above can be found, in their general versions for pairs, in ([44], Chapter 5).

Moreover, item 16 of Lemma A.1 admits a particular case when involving the inclusion map, in the following sense:

**Lemma A.2.** *Let $(X, A)$ and $j : X \hookrightarrow (X, A)$ be an arbitrary pair of topological spaces and the canonical inclusion, respectively. Then, for any $x_1 \in H^{i_1}(X; R)$ and $x_2 \in H^{i_2}(X, A; R)$, we have:*

$$j^*(x_1 \smile x_2) = x_1 \smile j^*(x_2)$$

The proof of the lemma above can also be found in ([44], Chapter 5).

That said, item 14 of Lemma A.1 admits the following particular case:

**Lemma A.3.** *Let $(X, A)$ be any pair of topological spaces and let $j : X \hookrightarrow (X, A)$ be the canonical inclusion. Then, for any $x \in H^{i_1}(X, A; R)$ and $a \in H_{i_2}(X; R)$, we have:*

$$j_*(a) \frown x = a \frown j^*(x)$$

*Proof.*

It suffices to observe that, for any $y \in H^{i_2 - i_1}(X; R)$, we have:

$$
\begin{aligned}
\langle y, j_*(a) \frown x \rangle &= \langle y \smile x, j^*(a) \rangle \\
&= \langle j^*(y \smile x), a \rangle \\
&= \langle y \smile j^*(x), a \rangle \\
&= \langle y, a \frown j^*(x) \rangle
\end{aligned}
$$

Thus, the Universal Coefficient Theorem ensures that $j_*(a) \frown x = a \frown j^*(x)$.

$\square$

Now, also due to the Universal Coefficient Theorem, the following result is a direct consequence of ([27], Theorem 4.11, p. 204):

**Proposition A.1.** *Let $R = \mathbb{Z}$ or $R = \mathbb{F}$ be any field, and let $(X, A)$ be any pair of topological spaces such that $H_k(X, A; R)$ is a finitely generated $R$-module for all $k \geq 0$. Then, for any $k \geq 0$, $\alpha \in R$ and $a \in H_k(X, A; R)$ with $a \neq 0$, there exists a unique $x \in H^k(X, A; R)$ such that $x \neq 0$ and $\langle x, a \rangle = \alpha$.*

Proceeding, we will construct the cohomology ring of a pair $(X, A)$.

**Definition A.1. (Cohomology Ring)** *We call $(H^*(X, A; R), +, \smile)$ the cohomology ring of the pair $(X, A)$ with coefficients in $R$, the set formed by the following formal infinite series:*

$$H^*(X, A; R) = \{x = x_0 + x_1 + x_2 + \ldots \ : \ x_k \in H^k(X, A; R), \ \forall k \geq 0\}$$

*Furthermore, given $x = x_0 + x_1 + x_2 + \ldots$ and $y = y_0 + y_1 + y_2 + \ldots$ in $H^*(X, A; R)$, the operations that define this ring are given by:*

1. *$x + y = z_0 + z_1 + z_2 + \ldots$, where $z_k = x_k + y_k$ for all $k \geq 0$*

2. *$x \smile y = z_0 + z_1 + z_2 + \ldots$, where $z_k = \displaystyle\sum_{i+j=k} x_i \smile y_j$ for all $k \geq 0$*

Since the cup product is a commutative operation when $R = \mathbb{Z}_2$, then $H^*(X, A; \mathbb{Z}_2)$ will be a commutative ring with identity element $1 + 0 + 0 + \cdots \in H^*(X, A; \mathbb{Z}_2)$.

**Theorem A.3.** *The units of the ring $H^*(X, A; \mathbb{Z}_2)$ are elements of the following form:*

$$x = x_0 + x_1 + x_2 + \cdots \in H^*(X, A; \mathbb{Z}_2) \ : \ x_0 = 1$$

*Furthermore, the inverse of a unit $x = 1 + x_1 + x_2 + \ldots$ is the following element:*

$$x^{-1} = 1 + x_1^{-1} + x_2^{-1} + \ldots, \quad \text{where} \quad x_k^{-1} = \sum_{\substack{i+j=k \\ i \neq 0}} x_i \smile x_j^{-1}, \ \forall k \geq 1$$

The proof of the theorem above can be found in ([2], Lemma 6.1, p. 53).

At this point, let us see when a topological space has all its singular (co)homology modules finitely generated and under which conditions we can define the Euler characteristic of an arbitrary topological space.

**Proposition A.2.** *All singular homology modules of a compact ENR space[7] are finitely generated.*

---

[7]An ENR is a topological space that is a retract of an open neighborhood in some Euclidean space, that is, it can be embedded in some Euclidean space as a retract of an open neighborhood of that Euclidean space. More details about these spaces can be found in ([17], Chapter 4, Section 8).

The proof of the proposition above can be found in ([23], Corollary A.8, p. 527). As a particular consequence, all singular homology modules of a compact topological manifold are free, since every topological manifold is an ENR and every finitely generated module is free. Furthermore, due to the Universal Coefficient Theorem, every singular cohomology module of a compact topological manifold is free.

**Definition A.2. (Euler Characteristic)** *Let $X$ be a topological space such that there exists an integer $n > 0$ such that $H_k(X; \mathbb{Z}) = 0$ for $k > n$ and $H_k(X; \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module for every $0 \leq k \leq n$. Thus, the Euler characteristic of $X$ is given by the following alternating sum:*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \operatorname{rank}(H_k(X; \mathbb{Z}))$$

Due to the Universal Coefficient Theorem, the Euler characteristic of a space $X$ under the conditions of the definition above can be computed using singular homology modules with coefficients in any field $\mathbb{F}$ as follows:

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim(H_k(X; \mathbb{F}))$$

To conclude this section, let us make some considerations about the infinite real projective space $\mathbb{R}P^\infty$, which will be useful for calculating the Stiefel–Whitney classes of real projective spaces using the topological version of Wu's formula.

Denoting by $\mathbb{R}P^k$ the $k$-dimensional real projective space, we can define the infinite real projective space $\mathbb{R}P^\infty$ as the direct limit of the following sequence:

$$\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \cdots \subset \mathbb{R}P^k \subset \cdots$$

In other words, we have that $\mathbb{R}P^\infty = \bigcup_{k \geq 0} \mathbb{R}P^k$, endowed with the following topology:

*"U is an open subset of $\mathbb{R}P^\infty$ if and only if $U \cap \mathbb{R}P^k$ is an open subset of $\mathbb{R}P^k$ for every $k \geq 0$."*

Thus, we can state the following:

**Lemma A.4.** *The canonical inclusion $i : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$ gives rise to an isomorphism $i^* : H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^k(\mathbb{R}P^n; \mathbb{Z}_2)$ for every $0 \leq k \leq n$.*

*Proof.* First, recall that $\mathbb{R}P^{n+1}$ is a CW-complex with one open cell in each dimension, with $\mathbb{R}P^k$ being its $k$-skeleton.

Now, consider $e_{n+1}$ as the open $(n+1)$-dimensional cell of $\mathbb{R}P^{n+1}$ and fix $x \in e_{n+1}$. Then, $\mathbb{R}P^{n+1} - \{x\}$ is a deformation retract of $\mathbb{R}P^n$.

On the other hand, considering, up to homeomorphism, $x \in D^{n+1} \subset e_{n+1}$, then $U = \mathbb{R}P^{n+1} - D^{n+1}$ will be an open subspace of $\mathbb{R}P^{n+1}$ such that:

$$\overline{U} \subset \text{int}(\mathbb{R}P^{n+1} - \{x\}) = \mathbb{R}P^{n+1} - \{x\}.$$

Thus, $\mathbb{S}^n = \partial D^{n+1}$ is a deformation retract of $(\mathbb{R}P^{n+1} - \{x\}) - U = D^{n+1} - \{x\}$, and we also obtain the following excision:

$$(\mathbb{R}P^{n+1} - U, (\mathbb{R}P^{n+1} - \{x\}) - U) \hookrightarrow (\mathbb{R}P^{n+1}, \mathbb{R}P^{n+1} - \{x\}).$$

Therefore, we have the following isomorphisms:

$$
\begin{aligned}
H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2) &\cong H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^{n+1} - \{x\}; \mathbb{Z}_2) \\
&\cong H^k(\mathbb{R}P^{n+1} - U, (\mathbb{R}P^{n+1} - \{x\}) - U; \mathbb{Z}_2) \\
&\cong H^k(D^{n+1}, \mathbb{S}^n; \mathbb{Z}_2) \\
&\cong \begin{cases} \mathbb{Z}_2 &, \quad k = n+1 \\ 0 &, \quad k \neq n+1 \end{cases}
\end{aligned}
$$

In particular, $H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for $0 \le k \le n$. From now on in this proof, fix $0 \le k \le n$.

Now, we prove by induction on $t \ge 2$ that $H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$.

To do so, consider initially the long exact cohomology sequence of the triple[8] $(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}, \mathbb{R}P^n)$:

$$\cdots \to H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}; \mathbb{Z}_2) \to H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^n; \mathbb{Z}_2) \to H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2) \to \cdots$$

Since $H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^{n+1}; \mathbb{Z}_2) = 0 = H^k(\mathbb{R}P^{n+1}, \mathbb{R}P^n; \mathbb{Z}_2)$, the exactness of the sequence above ensures that $H^k(\mathbb{R}P^{n+2}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$.

Thus, assuming $H^k(\mathbb{R}P^{n+t_0}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for some $t_0 > 2$, we similarly obtain that $H^k(\mathbb{R}P^{n+(t_0+1)}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$, simply by using the long exact cohomology sequence of the triple $(\mathbb{R}P^{n+(t_0+1)}, \mathbb{R}P^{n+t_0}, \mathbb{R}P^n)$.

Therefore, we conclude that $H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ for any $t \ge 0$. Consequently:
$$H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) = \varinjlim H^k(\mathbb{R}P^{n+t}, \mathbb{R}P^n; \mathbb{Z}_2) = 0.$$

Finally, consider the long exact cohomology sequence of the pair $(\mathbb{R}P^\infty, \mathbb{R}P^n)$:

$$\cdots \longrightarrow H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \overset{i^*}{\longrightarrow} H^k(\mathbb{R}P^n; \mathbb{Z}_2) \longrightarrow \cdots$$

Since $H^k(\mathbb{R}P^\infty, \mathbb{R}P^n; \mathbb{Z}_2) = 0$ and $H^k(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H^k(\mathbb{R}P^n; \mathbb{Z}_2)$, then $i^*$ is a monomorphism between modules of the same dimension, that is, $i^*$ is an isomorphism.

$\square$

---

[8]For more details about the long exact cohomology sequence of a triple, see ([23], p. 200).

## A.2   Slant Product

In this section, we will define a specific product between singular (co)homology modules that will be fundamental in the proof of the Wu formula for topological and homological manifolds.

This product, which we will later call the slant product, is defined for arbitrary topological spaces using singular (co)homology modules with coefficients in an arbitrary commutative unital ring, and it is also used in the proof of the Wu formula for smooth manifolds, as seen in ([41], Chapter 11).

For our context, let $R = \mathbb{Z}$ or $R = \mathbb{Z}_2$, $X$ and $Y$ be arbitrary topological spaces with $H_k(Y; R)$ finitely generated for all $k \geq 0$, and integers $i, j \geq 0$. Thus, define the following homomorphism involving $R$-modules of singular (co)homology:

$$H^i(X; R) \otimes H^j(Y; R) \otimes H_j(Y; R) \to H^i(X; R)$$

$$x \otimes y \otimes b \mapsto <y, b> x$$

Now, the Künneth formula ensures that $H^*(X \times Y; R) \cong H^*(X; R) \otimes H^*(Y; R)$, and since $H^*(X \times Y; R)$ is a ring generated by elements of the form $x \times y$, the following homomorphism is well defined:

$$H^{i+j}(X \times Y; R) \otimes H_j(Y; R) \to H^i(X; R)$$

$$(x \times y) \otimes b \mapsto (x \times y)/b = <y, b> x$$

Thus, we have the following:

**Definition A.3. (Slant Product)** *The slant product refers to the homomorphism* $H^{i+j}(X \times Y; R) \otimes H_j(Y; R) \to H^i(X; R)$ *given by* $z \otimes b \mapsto z/b$.

We will conclude this section with two particular properties of the slant product. For more details about this operation in its most general form, we suggest the reader see ([44], Chapter 6, Section 1).

**Lemma A.5.** *Let* $x \times 1 \in H^i(X \times Y; R)$, $z \in H^{i+j}(X \times Y; R)$, $a \in H_{i'}(X; R)$, $b \in H_j(Y; R)$, *and* $p_1 : X \times Y \to X$ *be the canonical projection. Then, we have the following relations:*

*1.* $[(x \times 1) \smile z]/b = x \smile (z/b)$

*2.* $(p_1)_*((a \times b) \frown z) = a \frown (z/b)$

# A.3   Steenrod Squares

The Steenrod squares are cohomological operations of great importance for the development of this work, as they are essential for defining the Stiefel-Whitney classes of vector bundles, generalized, and homological manifolds. Furthermore, the Wu formula relates, through these Steenrod squares, the Stiefel-Whitney and Wu classes of a smooth, topological, and homological variety.

Here, we will state only the basic properties of these operations. For more details on Steenrod squares, we refer the reader to ([44], Chapter 5, Section 9).

Given $(X, A)$ a pair of topological spaces and integers $m, k \geq 0$, the Steenrod squares are additive cohomological operations $Sq^k : H^m(X, A; \mathbb{Z}_2) \to H^{m+k}(X, A; \mathbb{Z}_2)$ satisfying the following properties:

1. If $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(X, A; \mathbb{Z}_2)$, then the Cartan formula holds, i.e.,

$$Sq^k(x \smile y) = \sum_{i+j=k} Sq^i(x) \smile Sq^j(y)$$

2. If $x \in H^m(X, A; \mathbb{Z}_2)$, then:

   (a) $Sq^0(x) = x$

   (b) $Sq^m(x) = x \smile x$

   (c) $Sq^k(x) = 0$, for $k > m$

3. If $f : (X, A) \to (Y, B)$ is a map of pairs, then $Sq^k \circ f^* = f^* \circ Sq^k$, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
H^m(Y, B; \mathbb{Z}_2) & \xrightarrow{\;\;f^*\;\;} & H^m(X, A; \mathbb{Z}_2) \\
\downarrow{\scriptstyle Sq^k} & & \downarrow{\scriptstyle Sq^k} \\
H^{m+k}(Y, B; \mathbb{Z}_2) & \xrightarrow{\;\;f^*\;\;} & H^{m+k}(X, A; \mathbb{Z}_2)
\end{array}
$$

Furthermore, if $f : (X, A) \to (Y, B)$ is a map that induces an isomorphism in the context of $\mathbb{Z}_2$-modules of singular cohomology, then it follows from the property above that:

$$Sq^k \circ (f^*)^{-1} = (f^*)^{-1} \circ Sq^k$$

On the other hand, given $x \in H^m(X, A; \mathbb{Z}_2)$, we can define the total square operation as follows:

$$Sq(x) = x + Sq^1(x) + Sq^2(x) + ... + Sq^m(x)$$

Thus, for any $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(X, A; \mathbb{Z}_2)$, the Cartan formula can be rewritten as:

$$Sq(x \smile y) = Sq(x) \smile Sq(y)$$

To conclude this section, given $x \in H^m(X, A; \mathbb{Z}_2)$ and $y \in H^n(Y, B; \mathbb{Z}_2)$, we can derive, using the cross product and the Cartan formula, the following relations:

$$Sq^k(x \times y) = \sum_{i+j=k} Sq^i(x) \times Sq^k(y)$$

$$Sq(x \times y) = Sq(x) \times Sq(y)$$

# A.4  $R-$Orientation Classes and e Dualities

In this section, we will define the notion of $R$-orientability of a topological manifold and state some results in this context. Afterward, we will present the most important results involving topological manifolds in the realm of (co)homology theory, known as dualities.

For a more detailed reading on orientations and the dualities mentioned here, we suggest ([23], Chapter 3, Section 3.3).

**Definition A.4. (Local Orientation)** *Consider $M^m$ a topological manifold. Then:*

1. *An $R$-local orientation of $M$ at $b \in M$ is the choice of a generator which we denote by $([M]_b) = H_m(M, M - \{b\}; R) \cong R$.*

2. *An $R$-local orientation of $M$ along a subspace $U \subset M$ is the choice of an element $[M]_U \in H_m(M, M-U; R)$ such that, for all $b \in U$, we have $((j_b^U)_*([M]_U)) = H_m(M, M - \{b\}; R)$, where $j_b^U : (M, M - U) \hookrightarrow (M, M - \{b\})$ is the canonical inclusion.*

*The elements $[M]_b$ and $[M]_U$ are called the classes of local $R$-orientation of $M$ at $b \in M$ and along $U$, respectively.*

**Definition A.5. (Global Orientation)** *A topological manifold $M^m$ is said to be $R$-orientable if there exists an open cover $\mathcal{U}$ of $M$ such that:*

1. *if $U_i, U_j \in \mathcal{U}$ and $b \in U_i \cap U_j$, then $(j_b^{U_i})_*([M]_{U_i}) = (j_b^{U_j})_*([M]_{U_j})$.*

2. *for every $U \in \mathcal{U}$ and $b \in U$, we have $[M]_b = (j_b^U)_*([M]_U)$*

After defining local and global orientations, let us consider the following:

**Proposition A.3.** *Let $M^m$ be an $R$-orientable topological manifold. Then:*

1. *If $M$ is connected and closed, the inclusion $j_b^M = j_b : M \hookrightarrow (M, M - \{b\})$ is such that $(j_b)_* : H_m(M; R) \to H_m(M, M - \{b\}; R)$ is an isomorphism for all $b \in M$.*

2. *For each compact $K \subset M$, there exists a unique class of R-local orientation of $M$ along $K$, $[M]_K \in H_m(M, M - K; R)$, such that $(j_b^K)_*([M]_K) = [M]_b$ for all $b \in K$.*

**Definition A.6.** *Let $M^m$ be a connected, closed, and R-orientable topological manifold. We call the global R-orientation class the generator denoted by $([M]) = H_m(M; R)$, which is such that $(j_b)_*([M]) = [M]_b$ for all $b \in M$.*

It is known in the literature that every topological manifold $M^m$ is $\mathbb{Z}_2$-orientable. Therefore, if $M$ is closed and connected, its classes of $\mathbb{Z}_2$-orientation (both local and global) are the only generators $([M]) = H_m(M; \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong H_m(M, M - \{b\}; \mathbb{Z}_2) = ([M]_b)$ such that $(j_b)_*([M]) = [M]_b$ for all $b \in M$.

**Lemma A.6.** *If $M^m$ and $N^n$ are two closed and connected topological manifolds, then their classes of global $\mathbb{Z}_2$-orientation satisfy:*

$$[M \times N] = [M] \times [N]$$

*Proof.* First, from the Künneth formula, we have:

$$H^{m+n}(M \times N; \mathbb{Z}_2) \cong H^m(M; \mathbb{Z}_2) \otimes H^n(N; \mathbb{Z}_2)$$

On the other hand, since $H^m(M; \mathbb{Z}_2)$ and $H^n(N; \mathbb{Z}_2)$ are modules generated uniquely by the classes of global $\mathbb{Z}_2$-orientation $[M]$ and $[N]$, respectively, it follows from ([27], Corollary 5.12, p. 215) that $H^m(M; \mathbb{Z}_2) \otimes H^n(N; \mathbb{Z}_2)$ is generated uniquely by $[M] \otimes [N]$.

Since the isomorphism in the Künneth formula is given by the cross product, the class of global $\mathbb{Z}_2$-orientation of the product manifold $M \times N$ will be the product $[M] \times [N]$.

$\square$

Before stating the dualities we will use in this work, let us examine an alternative way to visualize the cap product in the context of topological manifolds.

To do so, consider $M^m$ a topological manifold and a subspace $K \subset M$ that is compact and ENR. As seen in ([17], Chapter 8, Section 7), we can consider the cap product as a homomorphism between the following $R$-modules of (co)homology:

$$\frown: H_i(M, M - K; R) \otimes H^j(K; R) \to H_{i-j}(M, M - K; R)$$

With this, we can state the following:

**Theorem A.4. (Poincaré-Lefschetz Duality)** *Let $M^m$ be an R-orientable topological manifold and $K \subset M$ a compact and ENR subspace. Then, the homomorphism $\mathcal{D}_{M,K} : H^k(K; R) \to H_{m-k}(M, M - K; R)$ given by $\mathcal{D}_{M,K}(x) = [M]_K \frown x$ is an isomorphism for all $k \geq 0$.*

**Theorem A.5. (Poincaré Duality)** *If $M^m$ is a compact $R$-orientable topological manifold, then the homomorphism $\mathcal{D}_M : H^k(M;R) \to H_{m-k}(M;R)$ given by $\mathcal{D}_M(x) = [M] \frown x$ is an isomorphism for all $k \geq 0$.*

Concluding this section, let us examine some consequences of Poincaré duality.

**Theorem A.6.** *Let $M^m$ be a compact and $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a finite field. Thus, we have, for all $k \geq 0$, that the homomorphism $H^k(M;R) \to Hom(H^{m-k}(M;R);R)$ that associates $x \mapsto x'(y) = <x \smile y, [M]>$ is an isomorphism.*

*Proof.*

By directly composing the Universal Coefficients Theorem and the Poincaré duality for the topological manifold $M^m$, we obtain, for all $k \geq 0$, the following isomorphism:

$$H^k(M;R) \to Hom(H_k(M;R);R) \to Hom(H^{m-k}(M;R);R)$$

$$x \mapsto \overline{x} \mapsto \widetilde{x}$$

where $\overline{x} \in Hom(H_k(M;R);R)$ and $\widetilde{x} \in Hom(H^{m-k}(M;R);R)$ are defined, respectively, by $\overline{x}(a) = <x, a>$ and $\widetilde{x}(y) = \overline{x}([M] \frown y)$.

Thus, the isomorphism $H^k(M;R) \to Hom(H^{m-k}(M;R);R)$ is given, for all $k \geq 0$, by the following association:

$$
\begin{aligned}
x \mapsto \widetilde{x}(y) &= \overline{x}([M] \frown y) \\[2mm]
&= <x, [M] \frown y> \\[2mm]
&= <x \smile y, [M]> \\[2mm]
&= x'(y)
\end{aligned}
$$

$\square$

**Theorem A.7. (Dual Basis)** *Consider $M^m$ a compact, $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a field. Then, for every basis $\{b_i\}_{i=1}^r$ of $H^*(M;R)$, there exists a unique corresponding basis $\{b_i^{\#}\}_{i=1}^r$ of $H^*(M;R)$, called the dual basis, satisfying the following identity:*

$$<b_i \smile b_j^{\#}, [M]> = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

*Proof.*

By the previous theorem, the correspondence $H^k(M;R) \to Hom(H^{m-k}(M;R);R)$ that maps $b \mapsto <b \smile \_, [M]> : H^{m-k}(M;R) \to R$ is an isomorphism for all $k \geq 0$.

Now, fix an arbitrary $k \geq 0$. Recall that since $H^{m-k}(M; R)$ is a finitely generated $R$-module, say by the basis $\{b_j^\#\}_{j=1}^l$, the result from ([27], Theorem 4.11, p. 204) ensures that $Hom(H^{m-k}(M; R); R)$ is also a finitely generated $R$-module by the homomorphisms $h_i : H^{m-k}(M; R) \to R$ defined, for all $i = 1, \cdots, l$, by:

$$h_i(b_j^\#) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Thus, for each basic element $b \in H^k(M; R)$, there exists a unique basic element $b^\# \in H^{m-k}(M; R)$ such that $< b \smile b^\#, [M] >= 1$.

$\square$

**Corollary A.1.** *Consider $M^m$ a compact, $R$-orientable topological manifold, with $R = \mathbb{Z}$ or $R = \mathbb{F}$ a field. Let $\{b_i\}_{i=1}^r$ be a basis of $H^*(M; R)$ and $\{b_i^\#\}_{i=1}^r$ its dual basis. Then every $x \in H^*(M; R)$ can be written as follows:*

$$x = \sum_{i=1}^r < x \smile b_i^\#, [M] > b_i$$

*Proof.*
Since $\{b_i\}_{i=1}^r$ is a basis of $H^*(M; R)$, for each $x \in H^*(M; R)$, there exist unique coefficients $\alpha_1, \cdots, \alpha_r \in R$ such that $x = \sum_{i=1}^r \alpha_i b_i$. Thus, for each $b_j^\# \in \{b_i^\#\}_{i=1}^r$, we have:

$$
\begin{aligned}
< x \smile b_j^\#, [M] > &= \left< \left( \sum_{i=1}^r b_i \alpha_i \right) \smile b_j^\#, [M] \right> \\
&= \sum_{i=1}^r \alpha_i < b_i \smile b_j^\#, [M] > \\
&= \alpha_j
\end{aligned}
$$

$\square$

## A.5 Wu Classes

Now, we will see how to construct the so-called Wu classes of a closed topological manifold. The construction of such classes depends solely on the Universal Coefficients Theorem and Poincaré duality.

To this end, consider a closed and connected topological manifold $M^m$, an arbitrary integer $k \geq 0$, and the Steenrod squares $Sq^k : H^{m-k}(M; \mathbb{Z}_2) \to H^m(M; \mathbb{Z}_2)$. Thus, by taking the homomorphism in $Hom(H^{m-k}(M; \mathbb{Z}_2); \mathbb{Z}_2)$ that maps $x \mapsto < Sq^k(x), [M] >$, we obtain, by Theorem A.6, that there exists a unique cohomology class $v_k(M) \in H^k(M; \mathbb{Z}_2)$ such that:

$$< v_k(M) \smile x, [M] >=< Sq^k(x), [M] >, \forall x \in H^{m-k}(M; \mathbb{Z}_2)$$

**Definition A.7. (Wu Class)** *Given a closed and connected topological manifold $M^m$ and an integer $k \geq 0$, we call the $k$-th Wu class of the manifold $M$ the class $v_k(M) \in H^k(M; \mathbb{Z}_2)$, uniquely characterized by the following relation:*

$$< v_k(M) \smile x, [M] > = < Sq^k(x), [M] >, \ \forall x \in H^{m-k}(M; \mathbb{Z}_2)$$

*Additionally, we call $v(M) = \sum_{k=0}^{m} v_k(M) \in H^*(M; \mathbb{Z}_2)$ the total Wu class of $M$.*

Due to the uniqueness of the Wu classes, we obtain that $v_0(M) = 1 \in H^0(M; \mathbb{Z}_2)$, since for all $x \in H^m(M; \mathbb{Z}_2)$, we have:

$$< Sq^0(x), [M] > = < x, [M] > = < 1 \smile x, [M] >$$

As an example, let us compute the total Wu class of the real projective space $\mathbb{R}P^n$:

**Example A.1.** *Given $H^1(\mathbb{R}P^n; \mathbb{Z}_2) = (a)$, then $v(\mathbb{R}P^n) = \sum_{k=0}^{n} \binom{n-k}{k} a^k$.*

*Proof.*
Initially, we will show, by induction on $i \geq 0$, that $Sq^k(a^i) = \binom{i}{k} a^{i+k}$ for any $k \geq 0$. To this end, we fix that $\binom{i}{k} = 0$ when $i < k$ or $k < 0$.
Thus, for $i = 0$, we have:

$$Sq^k(a^0) = Sq^k(1)$$

$$= \begin{cases} 0 & , \ k > 0 \\ 1 & , \ k = 0 \end{cases}$$

$$= \binom{0}{k} a^k$$

Now, assuming $Sq^k(a^i) = \binom{i}{k} a^{i+k}$ for all $k \geq 0$, note that:

$$Sq^k(a^{i+1}) = Sq^k(a^i \smile a)$$

$$= \sum_{r+s=k} \left[ Sq^r(a^i) \smile Sq^s(a) \right]$$

$$= \left[ Sq^k(a^i) \smile Sq^0(a) \right] + \left[ Sq^{k-1}(a^i) \smile Sq^1(a) \right]$$

$$= \left[ \binom{i}{k} a^{i+k} \smile a \right] + \left[ \binom{i}{k-1} a^{i+k-1} \smile a^2 \right]$$

$$= \left[ \binom{i}{k} + \binom{i}{k-1} \right] a^{i+k+1}$$

$$= \binom{i+1}{k} a^{(i+1)+k}$$

Thus, $Sq^k(a^i) = \binom{i}{k}a^{i+k}$ for any $i, k \geq 0$. Finally, let's show that $v_k(\mathbb{R}P^n) = \binom{n-k}{k}a^k$ for all $0 \leq k \leq n$. To do so, it is sufficient to verify that:

$$< \binom{n-k}{k}a^k \smile x, [\mathbb{R}P^n] = < Sq^k(x), [\mathbb{R}P^n] >, \ \forall x \in H^{n-k}(\mathbb{R}P^n; \mathbb{Z}_2)$$

Since $H^{n-k}(\mathbb{R}P^n; \mathbb{Z}_2) = (a^{n-k}) = \{0, a^{n-k}\}$, it is enough to check the above equality for $x = a^{n-k}$, as for $x = 0$ the result is immediate. Thus:

$$< \binom{n-k}{k}a^k \smile a^{n-k}, [\mathbb{R}P^n] >= \ < \binom{n-k}{k}a^n >$$

$$= \ < Sq^k(a^{n-k}), [\mathbb{R}P^n] >$$

Therefore, $v(\mathbb{R}P^n) = \sum_{k=0}^{n} \binom{n-k}{k}a^k$.

$\square$

**Lemma A.7.** *Given two closed and connected manifolds $M^m$ and $N^n$, we have $v(M \times N) = v(M) \times v(N)$.*

*Proof.*
Initially, denote the $k$-th Wu class of the total class $v(M) \times v(N)$ as:

$$[v(M) \times v(N)]_k = \sum_{i+j=k} [v_i(M) \times v_j(N)]$$

For $v_k(M \times N) = [v(M) \times v(N)]_k$, it is sufficient to show the following relation:

$$< [v(M) \times v(N)]_k \smile z, [M \times N] >=< Sq^k(z), [M \times N] >, \ \forall z \in H^{m+n-k}(M \times N; \mathbb{Z}_2)$$

On the other hand, the Künneth formula guarantees that:

$$H^{m+n-k}(M \times N; \mathbb{Z}_2) \cong \bigoplus_{r+s=k} \left[ H^{m-r}(M; \mathbb{Z}_2) \otimes H^{n-s}(N; \mathbb{Z}_2) \right]$$

Thus, it is enough to show the previous equality for an arbitrary generator $z = x \times y \in H^{m+n-k}(M \times N; \mathbb{Z}_2)$, where $x \in H^{m-r}(M; \mathbb{Z}_2)$ and $y \in H^{n-s}(N; \mathbb{Z}_2)$, with $r + s = k$. Thus:

$$< [v(M) \times v(N)]_k \smile z, [M \times N] > =$$

$$= \ < \left( \sum_{i+j=k} v_i(M) \times v_j(N) \right) \smile (x \times y), [M \times N] >$$

$$= \ < \sum_{i+j=k} [(v_i(M) \smile x) \times (v_j(N) \smile y)], [M \times N] >$$

Note that, when considering $i + j = k = r + s$, with $i > r$ or $j > s$, we conclude that $v_i(M) \smile x \in H^{m-r+i}(M; \mathbb{Z}_2) = 0$ or $v_j(N) \smile y \in H^{n-s+j}(N; \mathbb{Z}_2) = 0$, and so:

$$< [v(M) \times v(N)]_k \smile z, [M \times N] > =$$

$$= \; < \sum_{i+j=k} [(v_i(M) \smile x) \times (v_j(N) \smile y)], [M \times N] >$$

$$= \; < (v_r(M) \smile x) \times (v_s(N) \smile y), [M \times N] >$$

$$= \; < (v_r(M) \smile x) \times (v_s(N) \smile y), [M] \times [N] >$$

$$= \; < v_r(M) \smile x, [M] >< v_s(N) \smile y, [N] >$$

$$= \; < Sq^r(x), [M] >< Sq^s(y), [N] >$$

$$= \; < Sq^r(x) \times Sq^s(y), [M] \times [N] >$$

$$= \; < Sq^r(x) \times Sq^s(y), [M \times N] >$$

Again, if we consider $i + j = k = r + s$, with $i > r$ or $j > s$, we obtain that $Sq^i(x) \in H^{m-r+i}(M; \mathbb{Z}_2) = 0$ or $Sq^j(y) \in H^{n-s+j}(N; \mathbb{Z}_2) = 0$, and consequently:

$$< [v(M) \times v(N)]_k \smile z, [M \times N] > = \; < Sq^r(x) \times Sq^s(y), [M \times N] >$$

$$= \; < \sum_{i+j=k} Sq^i(x) \times Sq^j(y), [M \times N] >$$

$$= \; < Sq^k(x \times y), [M \times N] >$$

$$= \; < Sq^k(z), [M \times N] >$$

Therefore, we conclude that $v(M \times N) = v(M) \times v(N)$.

$\square$

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