assignment04

May 22, 2019

1 Functional Programming SS19

2 Assignment 04 Solutions

2.1 Exercise 1 (Semantics of Partial Functions)

Consider the following function:

```
plus :: Int -> Int -> Int
plus 0 y = y
plus x 0 = x
plus x y = x + y
```

a) *Define the semantics of* `plus` *as a function* $f_{plus} : \mathbb{Z}_\perp \times \mathbb{Z}_\perp \to \mathbb{Z}_\perp$*. Take the evaluation strategy of Haskell into account. In your definition, "+" may only be applied to arguments from* $\mathbb{Z}$*.*

We can define the semantics of `plus` by the following function:

$$f_{plus}(x, y) = \begin{cases} x + y, & x \in \mathbb{Z} \text{ and } y \in \mathbb{Z} \\ \perp & x = \perp_{\mathbb{Z}_\perp} \text{ or } y = \perp_{\mathbb{Z}_\perp} \end{cases}$$

b) *Prove or disprove that* f_{plus} *is strict.*

By definition (2.1.6 from the lecture notes), a function f is strict if and only if f is $\perp$ whenever one of its arguments is $\perp$. The strictness of f_{plus} follows directly from the definition in part (a) since the function returns $\perp$ if at least one of x or y is $\perp$. $\square$

c) *Prove that* f_{plus} *is monotonic. Do not make use of the fact that any Haskell function is computable and hence monotonic.*

The monotonicity of f_{plus} is guaranteed by lemma 2.1.8, which states that any strict function is monotonic, but we also can prove the result for f_{plus} specifically.

By definition (2.1.7 from the lecture notes), a function $f : D_1 \to D_2$ is monotonic if, for any $d, d' \in D_1$, we have that $f(d) \sqsubseteq f(d')$ whenever $d \sqsubseteq d'$. For f_{plus}, we have $D_1 = \mathbb{Z}_\perp \times \mathbb{Z}_\perp$. To show that the function is monotonic, let us consider two inputs $d = (x, y)$ and $d' = (x', y')$, where $d \sqsubseteq d'$. We now have two cases to consider:

1. $d = d'$: In this case, we either have $x = \perp, y = \perp$ or $x \in \mathbb{Z}, y \in \mathbb{Z}$. In the former case, $f_{plus}(x, y) = \perp$ due to the strictness, while in the latter case, $f_{plus}(x, y) \in \mathbb{Z}$. In both cases, $f_{plus}(x, y) \sqsubseteq f_{plus}(x', y')$ due to the reflexivity of $\sqsubseteq$.

2. $d \neq d'$: In this case, at least one of x or y must be $\perp$, but since the function is strict, $f_{plus}(x,y) = \perp$ whenever *at least one* of x, y is $\perp$. Thus, we have that either $f_{plus}(x,y) = f_{plus}(x',y') = \perp$ (if x' or y' is $\perp$) or $f_{plus}(x,y) = \perp$ and $f_{plus}(x',y') \in \mathbb{Z}$. In both cases, $f_{plus}(x,y) \sqsubseteq f_{plus}(x',y')$.

We have thus shown that $f_{plus}(x,y) \sqsubseteq f_{plus}(x',y')$ for any $(x,y) \sqsubseteq (x',y')$, which means that f_{plus} is monotonic. $\square$.

2.2 Exercise 2 (Monotonicity)

Consider the following total functions (with their usual interpretation):

a) $- : \mathbb{Z} \mapsto \mathbb{Z}_\perp$

b) $*, \max : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}_\perp$

Here, "-" denotes the function that maps 2 to -2 or -3 to 3. Moreover, "max" denotes the "maximum" function.

For all $f \in \{-, \max, *\}$, give all monotonic functions f' such that f' is an extension of f, i.e. $f'(x_1, ..., x_n) = f(x_1, ..., x_n)$ for all $x_1, ..., x_n$ different from $\perp$. The functions f' should map between the following sets:

a) $-' : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$

b) $*', \max' : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp$

a) Let us define $-$ as a function $f(x)$. If $x \in \mathbb{Z}$, then all functions of the form $x \mapsto kx - (k+1)x$, $k \in \mathbb{N}_0$ are equivalent to f. For an extension $f' : \mathbb{Z}_\perp \to \mathbb{Z}_\perp$ to be monotonic, we need to have $f'(\perp) = \perp$. Thus, any function f' that has the form below is a valid extension of f:

$$f'(x) = \begin{cases} kx - (k+1)x, & x \in \mathbb{Z}, k \in \mathbb{N}_0 \\ \perp, & x = \perp \end{cases}$$

If we allow a recursive representation of $-$, then the following function is also a valid extension of f:

$$f'(x) = \begin{cases} 0, & x = 0 \\ -1 \cdot \mathrm{sign}(x) + f'(x - \mathrm{sign}(x)), & x \in \mathbb{Z},\ |x| > 0 \\ \perp, & x = \perp \end{cases}$$

where sign is the usual sign function that returns -1 if the argument is a negative number, 1 if the argument is positive, and 0 otherwise.

b) In the case of both $*$ and max, which are functions of the form $f(x,y)$, both arguments need to be evaluated for the result to have the desired semantics. $*$ and max are thus strict functions, so any extension f' which is a monotonic function needs to map $f'(x,y) = \perp$ whenever at least one of x and y is $\perp$.

For $* = f(x,y)$, the following monotonic function $f' : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp$ is a valid extension of f:

$$f'(x,y) = \begin{cases} x \cdot y, & x \in \mathbb{N}, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

If we allow recursive functions to represent $*$, then the following two monotonic functions are also valid extensions of f:

$$f'(x,y) = \begin{cases} 0, & x \in \mathbb{N}, y = 0 \\ x + f'(x, y-1), & x, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

$$f'(x,y) = \begin{cases} 0, & x = 0, y \in \mathbb{N} \\ y + f'(x-1, y), & x, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

For $\max = f(x,y)$, there are two monotonic functions $f' : \mathbb{N}_\perp \times \mathbb{N}_\perp \to \mathbb{N}_\perp$ that can be extensions of f depending on which argument is returned when $x = y$ for $x, y \in \mathbb{N}$, namely f' can return either x or y as a result. The following are thus both valid extensions of max:

$$f'(x,y) = \begin{cases} x, & x \geq y,\ x, y \in \mathbb{N} \\ y, & x < y,\ x, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

$$f'(x,y) = \begin{cases} x, & x > y,\ x, y \in \mathbb{N} \\ y, & x \leq y,\ x, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

If we allow max to be represented as a recursive function, then the following monotonic function is also a valid extension of f:

$$f'(x,y) = \begin{cases} x, & x \in \mathbb{N}, y = 0 \\ y, & x = 0, y \in \mathbb{N} \\ 1 + f'(x-1, y-1), & x, y \in \mathbb{N} \\ \perp, & x = \perp \text{ or } y = \perp \end{cases}$$

2.3 Exercise 3 (Completeness)

Consider Theorem 2.1.13 (c) from the lecture:

Let D_1, D_2 be domains. If $\sqsubseteq_{D_2}$ is complete on D_2, then $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$.

The other direction of this theorem holds as well:

Let D_1, D_2 be arbitrary nonempty sets where D_2 is partially ordered with $\sqsubseteq_{D_2}$. If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then $\sqsubseteq_{D_2}$ is complete on D_2.

To prove this theorem, please prove the following two properties:

a) *If $\sqsubseteq_{D_1 \to D_2}$ is complete on $D_1 \to D_2$, then $\perp_{D_2}$ exists.*

Hints: *You can **not** assume that* $\bot_{D_1\to D_2}(d) = \bot_{D_2}$ *for each* $d \in D$*, as one does not know yet whether* $\bot_{D_2}$ *exists. Instead, first prove that* $\bot_{D_1\to D_2}$ *is a constant function, i.e. that* $\bot_{D_1\to D_2}(d) = \bot_{D_1\to D_2}(d')$ *for each* $d, d' \in D_1$.

Based on the definition of completeness (2.1.12 from the lecture notes), since $\sqsubseteq_{D_1\to D_2}$ is complete on $D_1 \to D_2$, there exists a smallest element that we denote by $\bot_{D_1\to D_2}$. To show that $\bot_{D_2}$ exists, it suffices to show that $\bot_{D_1\to D_2}(d) = \bot_{D_2}$ for every $d \in D_1$; however, given that we don't know if $\bot_{D_2}$ exists, we first need to show that $\bot_{D_1\to D_2}(d) = \bot_{D_1\to D_2}(d') = k$ for each $d, d' \in D_1$ and then show that $k = \bot_{D_2}$.

Let us take $d, d' \in D_1$ and assume, without loss of generality, that $\bot_{D_1\to D_2}(d) \neq \bot_{D_1\to D_2}(d')$ with $\bot_{D_1\to D_2}(d) \sqsubseteq \bot_{D_1\to D_2}(d')$. Let us now consider two chains $S = \{\bot_{D_1\to D_2}(d), f_1(d), f_2(d)... \}$ and $S' = \{\bot_{D_1\to D_2}(d'), f_1(d'), f_2(d')... \}$, where each f_i is a function $f_i : D_1 \to D_2$, such that we have that $\bot_{D_1\to D_2}(d) \sqsubseteq f_1(d) \sqsubseteq f_2(d)...$ and similarly $\bot_{D_1\to D_2}(d') \sqsubseteq f_1(d') \sqsubseteq f_2(d')....$ But if S' is a chain, then $S'' = \{\bot_{D_1\to D_2}(d), \bot_{D_1\to D_2}(d'), f_1(d'), f_2(d')...\}$ is also a chain. This however contradicts the notion that $\bot_{D_1\to D_2}$ is the smallest element for every $e \in D_1$. It thus must be the case that $\bot_{D_1\to D_2}(d) = \bot_{D_1\to D_2}(d') = k$.

Let us now take $d = \bot_{D_1}$. Since we know that $\bot_{D_1\to D_2}(\bot_{D_1}) = k$, it must be the case that either $k = \bot_{D_2}$ or $k = e$ for some element $e \in D_2$ if $\bot_{D_2}$ does not exist. Let us assume that $k = e$; then, $S = \{e, f_1(\bot_{D_1}), f_2(\bot_{D_1}), ...\}$ can be a chain only if $e \sqsubseteq f_i(\bot_{D_1})$ for every $f_i(\bot_{D_1})$. But if $e \neq \bot_{D_2}$, it can only be the case that $f_i(\bot_{D_1}) = e$. This would however exclude all strict and constant functions from consideration. This means that k cannot be equal to e, but must instead equal some smallest element, which we can denote $\bot_{D_2}$. □

b) *If* $\sqsubseteq_{D_1\to D_2}$ *is complete on* $D_1 \to D_2$*, then for all chains* S *on* D_2*, the least upper bound* $\sqcup S$ *of* S *exists in* D_2.

Hints: *You may use lemmas and theorems from the lecture before Thm. 2.1.13 for your proof.*

Let us consider a chain S on D_2 and suppose that the least upper bound $\sqcup S$ of S does not exist in D_2. Now, let us consider a chain $S' = \{f_1, f_2, ...\}$ of functions, where each f_i is a map $f_i : D_1 \to S$. Since S does not have a least upper bound, S' will also not have a least upper bound; however, this contradicts the fact that $\sqsubseteq_{D_1\to D_2}$ is complete on $D_1 \to D_2$, which means that $\sqcup S$ of S has to exist in D_2. □

2.4 Exercise 4 (Continuity)

In this exercise we will compare topological–continuity from real analysis to continuity as defined in the lecture (Scott[1] *–continuity).*

Let $a, b \in \mathbb{R}$*, then* $[a, b] := \{r \mid a \leq r \leq b\}$*. A function* $f : [a, b] \to \mathbb{R}$ *is topologically-continuous if for every converging sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $[a, b]$ *we have*

$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

A function $f : [a, b] \mapsto \mathbb{R}$ *is Scott-continuous if for any chain* $S \subseteq [a, b]$ *we have* $f(\sup S) = \sup f(S)$*, where* $f(S) = \{f(s) \mid s \in S\}$*. As usual,* $\sup S$ *denotes the least upper bound, i.e. the supremum of* S.

[1] https://en.wikipedia.org/wiki/Dana_Scott

a) *Let* $a, b \in \mathbb{R}$*. Show that the standard ordering* $\leq$ *on the real numbers is complete on the closed interval* $[a, b]$.

Hints: *You may use that $\mathbb{R}$ is complete, i.e. every set of real numbers that is bounded from above has a least upper bound.*

According to the definition of completeness (2.1.12), a reflexive partial order is complete on a set D if it has a smallest element and every chain S in D has a least upper bound. The standard ordering $\leq$ is indeed a reflexive partial order since reflexivity ($x \leq x$), transitivity ($x \leq y \wedge y \leq z \implies x \leq z$), and antisymmetry ($x \leq y \wedge y \leq x \implies x = y$) all hold.

The ordering is continuous on the interval $[a,b]$ if it has a smallest element and any subset S has a least upper bound. Since the interval is closed, the first criterion is clearly satisfied with the smallest element being a. For the second criterion, let us take an arbitrary subset $[c,d] \subseteq [a,b]$. Here, d is an upper bound since $x \leq d$ for every $x \in [c,d]$. Furthermore, d is a least upper bound since any other upper bound $y \in [a,b] \setminus [c,d]$ is greater than d (this follows from the completeness of R). The standard ordering $\leq$ is thus indeed complete. $\square$

b) *Let $a,b,c,d \in \mathbb{R}$. Show that* ***not*** *every topologically-continuous function $f : [a,b] \to [c,d]$ is Scott-continuous.*

According to definition 2.1.14 in the lecture notes, a function $f : D_1 \to D_2$ is Scott-continuous if and only if $f(\sqcup S) = \sqcup\{f(d) \mid d \in S\}$ for every chain S in d whenever $\sqsubseteq_{D_1}$ is complete on D_1 and $\sqsubseteq_{D_2}$ is complete on D_2. As shown in part (a), the ordering $\leq$ is complete on an an interval $[a,b] \in \mathbb{R}$. Given a topologically-continuous function $f : [a,b] \to [c,d]$, we thus have that $\leq$ is complete on both $D_1 = [a,b]$ and $D_2 = [c,d]$.

To show that not every topologically-continuous function f is Scott-continuous, it suffices to find a single counterexample of a topologically-continuous function that is not Scott-continuous. In this case, we can take as an example the function $f(x) = -x^2$ on the interval $[-2,2]$, for which $D_1 = [-2,2]$ and $D_2 = [-4,0]$. Let us now consider the chain $S = [-1,1] \in D_1$. In this case, we have

$$\begin{aligned} &\sup S = \sup[-1,1] = 1 \\ \implies &f(\sup S) = f(1) = -1 \end{aligned}$$

On the other hand, we have that

$$\sup\{f(d) \mid d \in S\} = \sup f(d)_{[-1,1]} = 0$$

Since $\sup S \neq \sup\{f(d) \mid d \in S\}$, $f(x) = -x^2$ is not Scott-continuous on $[-2,2]$ despite it being topologically-continuous. $\square$

c) *Let $a,b,c,d \in \mathbb{R}$. Prove that any monotonic and topologically-continuous function $f : [a,b] \to [c,d]$ is Scott-continuous. As usual, f is monotonic if $x \leq y$ implies $f(x) \leq f(y)$.*

Hints: *You can use the following lemma: Let $(x_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of real numbers which is bounded from above. Then $(x_n)_{n\in\mathbb{N}}$. converges and $\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$.*

As discussed in part (b), the first condition for Scott-continuity, namely completeness, is fulfilled by the ordering $\leq$. To show that every monotonic and topologically-continuous function $f : [a,b] \to [c,d]$ is Scott-continuous, we thus have to show that $f(\sqcup S) = \sqcup\{f(d) \mid d \in S\}$ for every chain S in D_1.

Let us consider an arbitrary interval $M = [i,j] \subseteq [a,b]$. Since $[i,j]$ is complete, we can create a monotonically increasing sequence $S = (x_n)_{n\in\mathbb{N}}$ from M, where $x_0 = i$ and $x_n < x_{n+1}$ for all

$n > 0, n \in \mathbb{N}$. As $S \subseteq [i, j]$, the sequence is bounded from above by j; we can thus apply the given lemma, based on which

$$\lim_{n\to\infty} x_n = \sup S = \sup\{x_n \mid n \in \mathbb{N}\} = j$$
$$\implies f(\sup S) = f(j)$$

Let us now construct a sequence $S' = f(x_n)_{n\in\mathbb{N}}$ from S. Since f is monotonic, we know that $f(x_n) \leq f(x_{n+1}) \leq f(j)$ for every $x_n \in S$; thus, it follows that

$$\lim_{n\to\infty} f(x_n) = \sup S' = \sup\{f(x_n) \mid n \in \mathbb{N}\} = f(j)$$

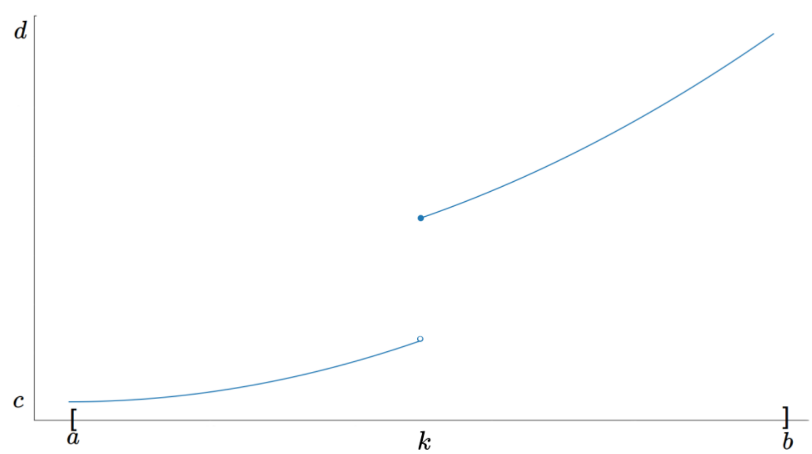
This means that $f(\sqcup S) = \sqcup\{f(d) \mid d \in S\}$ for any S, which is what we wanted to show. $\square$

d) *Let $a, b, c, d \in \mathbb{R}$. Prove or disprove that any Scott-continuous function $f : [a, b] \to [c, d]$ is topologically-continuous.*

Not every Scott-continuous function $f : [a, b] \to [c, d]$ is topologically-continuous. To show that, let us consider as an example the following function f, which is defined on the interval $[a, b]$:

$$f(x) = \begin{cases} x^2, & x \in [a, k) \\ 0.5 + x^2, & x \in [k, b] \end{cases}$$

A visualisation of the function is shown below:



discontinuous_function_example

This function is not topologically continuous on the interval $[a, b]$ since it has a discontinuity at k; however, f is Scott-continuous on $[a, b]$. For instance, let us take an interval $M = [i, k] \subseteq [a, b]$ and create a monotonically increasing sequence $S = (x_n)_{n\in\mathbb{N}}$ from M just as in part (c). For this interval, we have $\sup S = k$ and $f(\sup S) = f(k) = \sup\{f(x_n) \mid n \in \mathbb{N}\}$. This holds for any chain S in $[a, b]$.

As this example shows, Scott continuity is not a sufficient condition for topological continuity. $\square$