assignment06

June 5, 2019

1 Functional Programming SS19

2 Assignment 06 Solutions

2.0.1 Exercise 1 (Fixpoints)

Give the value of $Val[\![e]\!]_\rho$ for the following simple Haskell expressions $e \in \{e_1, e_2, e_3\}$ for the environment $\rho = \omega + \rho'$, where ω is the initial environment and ρ' is he environment with $\rho'(x) = 2, \rho'(y) = 18, \rho'(z) = 3$, and ρ' is undefined for all other variables.

Describe your computation in detail by applying rules for Val step by step. Also, for each higher-order function $f : Dom \to Dom$, where $\mathrm{lfp}\, f$ is needed in the calculation, determine what the function $f^n(\perp), n \in \mathbb{N}$, computes.

```
e1 = let isAnswer = \x -> if x < 17 then 2 else 3
     in isAnswer y

e2 = let sum = \x -> if x < 4 then x + sum (x+1)
                     else 0
     in sum x

e3 = let pow = \y -> \ z -> if z <= 0 then 1
                            else y * pow y (z-1)
     in pow 4 z
```

Hints: * You may switch between infix- and prefix-notation for Haskell operators without any intermediate steps if needed. * You may simplify $Val[\![exp_1 \quad exp_2 \quad exp_3]\!]_\rho$ to $f\ (Val[\![exp_2]\!]_\rho)\ (Val[\![exp_3]\!]_\rho)$ where $f = Val[\![exp_1]\!]_\rho$ *in* Functions *in* Dom in one step if exp_1 represents a function expecting two arguments, e.g. $+, -, *, <, <=$, etc.

- `e1 = let isAnswer = \x -> if x < 17 then 2 else 3 in isAnswer y`

$$
\begin{aligned}
Val[\![e1]\!] &= Val[\![\text{let isAnswer = \textbackslash x -> if x < 17 then 2 else 3 in isAnswer y}]\!]_\rho \\
&= Val[\![\text{isAnswer y}]\!](\rho + \{\text{isAnswer}/\ \mathrm{lfp}\ \ f\})
\end{aligned}
$$

for f *in* Functions *in* Dom where

$$f(k) = Val[\![\texttt{\textbackslash x -> if x < 17 then 2 else 3}]\!](\rho + \{\text{isAnswer}/k\})$$

Here, we have $Val[\![\text{\textbackslash x -> if x < 17 then 2 else 3}]\!] = h$ *in* Functions *in* Dom where

$$\begin{aligned} h(d) &= Val[\![\text{if x < 17 then 2 else 3}]\!](\rho + \{x/d\}) \\ &= \begin{cases} 2 & \text{if } Val[\![\text{<(x 17)}]\!](\rho + \{x/d\}) = \text{True in Constructions}_0 \text{ in Dom} \\ 3 & \text{if } Val[\![\text{<(x 17)}]\!](\rho + \{x/d\}) = \text{False in Constructions}_0 \text{ in Dom} \\ \perp & \text{otherwise} \end{cases} \end{aligned}$$

such that $Val[\![\text{<(x 17)}]\!](\rho + \{x\ /\ d\}) = g$ *in* Functions *in* Dom.
Since isAnswer is not a recursive function, we have that lfp $f = h$ and so

$$\begin{aligned} Val[\![\text{e1}]\!] &= Val[\![\text{isAnswer y}]\!](\rho + \{\text{isAnswer}\ /\ h) \\ &= Val[\![\text{if x < 17 then 2 else 3}]\!](\rho + \{x\ /\ y\}) \\ &= Val[\![\text{if x < 17 then 2 else 3}]\!](\{x\ /\ 18\}) \\ &= Val[\![\text{if 18 < 17 then 2 else 3}]\!] \\ &= 3 \end{aligned}$$

- `e2 = let sum = \x -> if x < 4 then x + sum (x+1) else 0 in sum x`

$$\begin{aligned} Val[\![e2]\!] &= Val[\![\text{let sum = \textbackslash x -> if x < 4 then x + sum (x+1) else 0 in sum x}]\!]_\rho \\ &= Val[\![\text{sum x}]\!](\rho + \{\text{sum}/\text{ lfp } f\}) \end{aligned}$$

for f *in* Functions *in* Dom where

$$f(k) = Val[\![\text{\textbackslash x -> if x < 4 then x + sum (x+1) else 0}]\!](\rho + \{\text{sum}/k\})$$

As above, we have $Val[\![\text{\textbackslash x -> if x < 4 then x + sum (x+1) else 0}]\!] = h$ *in* Functions *in* Dom where

$$\begin{aligned} h(d) &= Val[\![\text{if x < 4 then x + sum (x+1) else 0}]\!](\rho + \{\text{sum}/\text{ lfp } f,\ x/d\}) \\ &= \begin{cases} \text{x + sum(x+1)} & \text{if } Val[\![\text{<(x, 4)}]\!](\rho + \{\text{sum}/\text{ lfp } f,\ x/d\}) = \text{True in Constructions}_0 \text{ in Dom} \\ 0 & \text{if } Val[\![\text{<(x, 4)}]\!](\rho + \{\text{sum}/\text{ lfp } f,\ x/d\}) = \text{False in Constructions}_0 \text{ in Dom} \\ \perp & \text{otherwise} \end{cases} \end{aligned}$$

such that $Val[\![\text{<(x 4)}]\!](\rho + \{\text{sum}/\text{ lfp } f,\ x\ /\ d\}) = g$ *in* Functions *in* Dom.
We are now dealing with a recursive function, so let us consider $f^n(\perp), n \in \mathbb{N}$:

$$g_0 = f^0(\perp) = \perp$$

$$g_1 = f^1(\perp) = \begin{cases} x + \perp(x+1) & \text{if } x < 4 \\ 0 & \text{if } x \geq 4 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} \perp & \text{if } x < 4 \\ 0 & \text{if } x \geq 4 \\ \perp & \text{otherwise} \end{cases}$$

$$g_2 = f^2(\perp) = f(g_1) = \begin{cases} x + g_1(x+1) & \text{if } x < 4 \\ 0 & \text{if } x \geq 4 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} \perp & \text{if } x < 3 \\ 3 & \text{if } x = 3 \\ 0 & \text{if } x > 4 \\ \perp & \text{otherwise} \end{cases}$$

$$g_3 = f^3(\perp) = f(g_2) = \begin{cases} x + g_2(x+1) & \text{if } x < 4 \\ 0 & \text{if } x \geq 4 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} \perp & \text{if } x < 3 \\ 5 & \text{if } x = 2 \\ 3 & \text{if } x = 3 \\ 0 & \text{if } x > 4 \\ \perp & \text{otherwise} \end{cases}$$

This is thus the function that computes the sum of all number from x up to 3 and returns 0 otherwise; thus

$$\text{lfp } f = \begin{cases} \sum_{i=x}^{3} i & \text{if } x < 4 \\ 0 & \text{if } x \geq 4 \\ \perp & \text{otherwise} \end{cases}$$

We thus have that

$$\begin{aligned} Val[\![\text{e2}]\!] &= Val[\![\text{sum x}]\!](\rho + \{\text{sum / lfp } f) \\ &= Val[\![\text{lfp } f \text{ x}]\!](\rho) \\ &= Val[\![\text{lfp } f \text{ 2}]\!] \\ &= \sum_{i=2}^{3} i = 5 \end{aligned}$$

- `e3 = let pow = \y -> \ z -> if z <= 0 then 1 else y * pow y (z-1) in pow 4 z`

$$\begin{aligned} Val[\![e3]\!] &= Val[\![\text{let pow = \textbackslash y -> \textbackslash z -> if z <= 0 then 1 else y * pow y (z-1) in pow 4 z}]\!]_\rho \\ &= Val[\![\text{pow 4 z}]\!](\rho + \{\text{pow/ lfp } f\}) \end{aligned}$$

for f *in* Functions *in* Dom where

$$f(k) = Val[\![\text{\textbackslash y -> \textbackslash z -> if z <= 0 then 1 else y * pow y (z-1) in pow 4 z}]\!](\rho + \{\text{pow}/k\})$$

We have $Val[\![\text{\textbackslash y -> \textbackslash z -> if z <= 0 then 1 else y * pow y (z-1)}]\!] = h$ *in* Functions *in* Dom where

$$h(d_1) = Val[\![\text{\textbackslash z -> if z <= 0 then 1 else y * pow y (z-1)}]\!](\rho + \{\text{pow/ lfp } f,\ y/d_1\})$$

Since we have another lambda expression, we find that $Val[\![\text{\z -> if z <= 0 then 1 else y * pow y (z-1)}]\!] = l \ \mathit{in} \text{ Functions } \mathit{in} \text{ Dom}$ and

$$
\begin{aligned}
l(d_2) &= Val[\![\text{if z <= 0 then 1 else y * pow y (z-1)}]\!](\rho + \{\text{pow}/\,\text{lfp } f,\ y/d_1,\ z/d_2\}) \\
&= \begin{cases} 1 & \text{if } Val[\![\text{<=(z, 0)}]\!](\rho + \{\text{pow}/\,\text{lfp } f,\ y/d_1,\ z/d_2\}) = \text{True in Constructions}_0 \text{ in Dom} \\ \text{y * pow y (z-1)} & \text{if } Val[\![\text{<=(z, 0)}]\!](\rho + \{\text{pow}/\,\text{lfp } f,\ y/d_1,\ z/d_2\}) = \text{False in Constructions}_0 \text{ in Dom} \\ \perp & \text{otherwise} \end{cases}
\end{aligned}
$$

such that $Val[\![\text{<=(z 0)}]\!](\rho + \{\text{pow}/\,\text{lfp } f,\ y/d_1,\ z/d_2\}) = g \ \mathit{in} \text{ Functions } \mathit{in} \text{ Dom}$.
Let us now consider $f^n(\perp), n \in \mathbb{N}$:

$$
\begin{aligned}
g_0 &= f^0(\perp) = \perp \\
g_1 &= f^1(\perp) = \begin{cases} 1 & \text{if } z \leq 0 \\ y * \perp(z-1) & \text{if } z > 0 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } z \leq 0 \\ \perp & \text{if } z > 0 \\ \perp & \text{otherwise} \end{cases} \\
g_2 &= f^2(\perp) = f(g_1) = \begin{cases} 1 & \text{if } z \leq 0 \\ y * g_1(z-1) & \text{if } z > 0 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } z \leq 0 \\ y & \text{if } z = 1 \\ \perp & \text{if } z > 1 \\ \perp & \text{otherwise} \end{cases} \\
g_3 &= f^3(\perp) = f(g_2) = \begin{cases} 1 & \text{if } z \leq 0 \\ y * g_2(z-1) & \text{if } z > 0 \\ \perp & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } z \leq 0 \\ y & \text{if } z = 1 \\ y^2 & \text{if } z = 2 \\ \perp & \text{if } z > 2 \\ \perp & \text{otherwise} \end{cases}
\end{aligned}
$$

which is the function that computes the power z-th of a positive integer y; thus

$$
\text{lfp } f(x) = \begin{cases} y^z & \text{if } z \leq 0 \\ \perp & \text{otherwise} \end{cases}
$$

We thus have that

$$
\begin{aligned}
Val[\![\text{e3}]\!] &= Val[\![\text{pow 4 z}]\!](\rho + \{\text{pow}/\,\text{lfp } f\}) \\
&= Val[\![\text{lfp } f \text{ 4 z}]\!](\rho) \\
&= Val[\![\text{lfp } f \text{ 4 3}]\!] \\
&= 4^3 = 64
\end{aligned}
$$