

EE-559 – Deep learning

2.1. Loss and risk

François Fleuret

<https://fleuret.org/ee559/>

Mon Feb 18 13:33:33 UTC 2019

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There are multiple types of inference that we can roughly split into three categories:

- Classification (e.g. object recognition, cancer detection, speech processing),
- regression (e.g. customer satisfaction, stock prediction, epidemiology), and
- density estimation (e.g. outlier detection, data visualization, sampling/synthesis).

The standard formalization considers a measure of probability

$$\mu_{X,Y}$$

over the observation/value of interest, and i.i.d. training samples

$$(x_n, y_n), \quad n = 1, \dots, N.$$

Intuitively, for classification it can often be interpreted as

$$\mu_{X,Y}(x,y) = \mu_{X|Y=y}(x) P(Y=y)$$

that is, draw Y first, and given its value, generate X .

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Here

$$\mu_{X|Y=y}$$

stands for the population of the observable signal for class y (e.g. “sound of an /ē/”, “image of a cat”).

For regression, one would interpret the joint law more naturally as

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which would be: first, generate X , and given its value, generate Y .

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In the simple cases

$$Y = f(X) + \epsilon$$

where f is the deterministic dependency between x and y , and ϵ is a random noise, independent of X .

With such a model, we can more precisely define the three types of inferences we introduced before:

Classification,

- (X, Y) random variables on $\mathcal{Z} = \mathbb{R}^D \times \{1, \dots, C\}$,
- we want to estimate $\text{argmax}_y P(Y = y | X = x)$.

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Density estimation,

- X random variable on $\mathcal{Z} = \mathbb{R}^D$,
- we want to estimate μ_X .

The boundaries between these categories are fuzzy:

- Regression allows to do classification through class scores.
- Density models allow to do classification thanks to Bayes' law.

etc.

We call **generative** classification methods with an explicit data model, and **discriminative** the ones bypassing such a modeling .

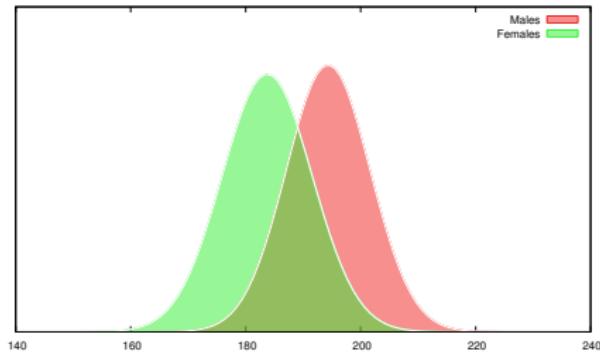
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Example: Can we predict a Brazilian basketball player's gender G from his/her height H ?

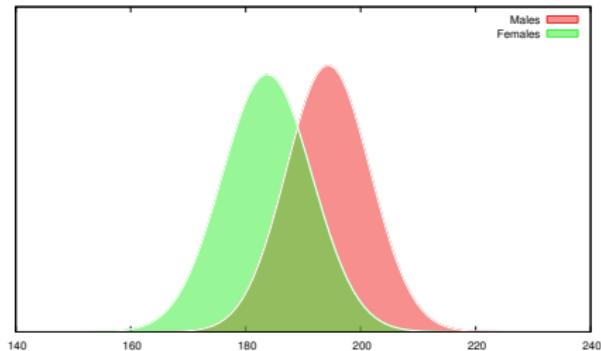
Females: 190 182 188 184 196 173 180 193 179 186 185 169

Males: 192 190 183 199 200 190 195 184 190 203 205 201

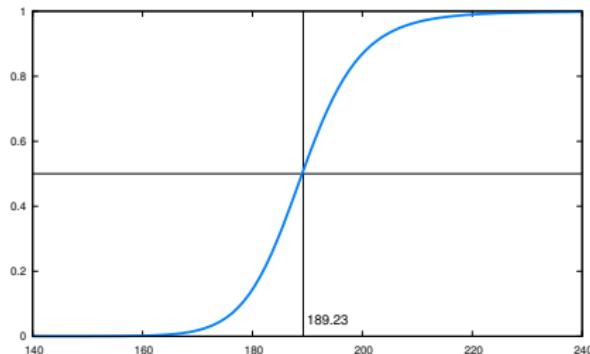
In the **generative** approach, we model $\mu_{H|G=g}(h)$



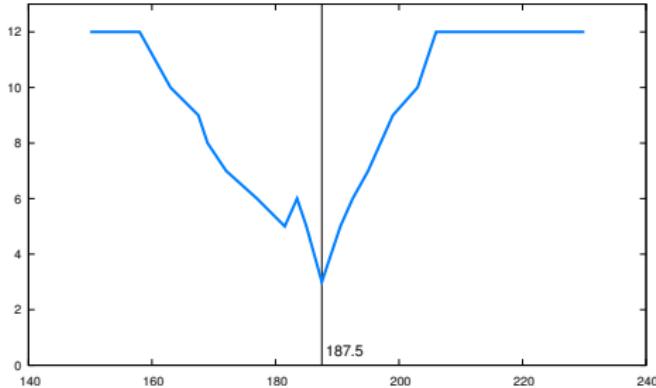
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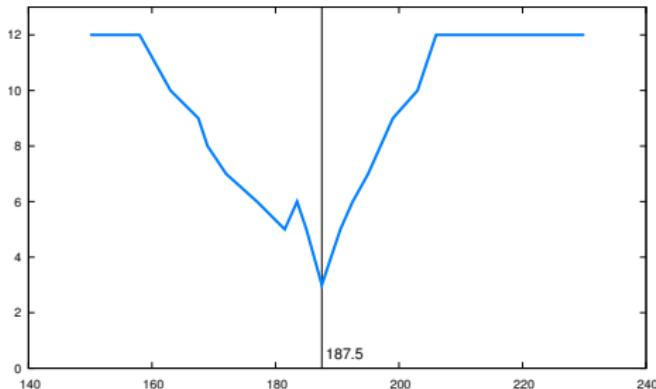
and use Bayes's law $P(G = g | H = h) = \frac{\mu_{H|G=g}(h)P(G=g)}{\mu_H(h)}$



In the **discriminative** approach we directly pick the threshold that works the best on the data:



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Note that it is harder to design a confidence indicator.

Risk, empirical risk

Learning consists of finding in a set \mathcal{F} of functionals a “good” f^* (or its parameters’ values) usually defined through a loss

$$\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}$$

such that $\ell(f, z)$ increases with how wrong f is on z .

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- for classification:

$$\ell(f, (x, y)) = \mathbf{1}_{\{f(x) \neq y\}},$$

- for regression:

$$\ell(f, (x, y)) = (f(x) - y)^2,$$

- for density estimation:

$$\ell(q, z) = -\log q(z).$$

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The loss may include additional terms related to f itself.

We are looking for an f with a small **expected risk**

$$R(f) = \mathbb{E}_Z (\ell(f, Z)),$$

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Although this quantity is unknown, if we have i.i.d. training samples

$$\mathcal{D} = \{Z_1, \dots, Z_N\},$$

we can compute an estimate, the **empirical risk**:

$$\hat{R}(f; \mathcal{D}) = \hat{\mathbb{E}}_{\mathcal{D}} (\ell(f, Z)) = \frac{1}{N} \sum_{n=1}^N \ell(f, Z_n).$$

We have

$$\mathbb{E}_{Z_1, \dots, Z_N} (\hat{R}(f; \mathcal{D})) = \mathbb{E}_{Z_1, \dots, Z_N} \left(\frac{1}{N} \sum_{n=1}^N \ell(f, Z_n) \right)$$

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The empirical risk is an **unbiased estimator** of the expected risk.

Finally, given \mathcal{D} , \mathcal{F} , and ℓ , “learning” aims at computing

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- Can we bound $R(f)$ with $\hat{R}(f; \mathcal{D})$?

Yes if f is not chosen using \mathcal{D} . Since the Z_n are independent, we just need to take into account the variance of $\hat{R}(f; \mathcal{D})$.

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For instance if $|\mathcal{F}| = 1$, we can!

Note that in practice, we call “loss” both the functional

$$\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}$$

and the empirical risk minimized during training

$$\mathcal{L}(f) = \frac{1}{N} \sum_{n=1}^N \ell(f, z_n).$$

The end

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2.2. Over and under fitting

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What about interviewing ten candidates and picking the best? What about interviewing one thousand?

With

$$Q_k^n \sim \mathcal{B}(0.5), \quad n = 1, \dots, 1000, \quad k = 1, \dots, 10,$$

independent standing for “candidate n answered question k correctly”, we have

$$\forall n, \quad P(\forall k, Q_k^n = 1) = \frac{1}{1024}$$

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So there is 62% chance that among 1,000 candidates answering completely at random, one will score perfectly.

Selecting a candidate based on a statistical estimator biases the said estimator for that candidate. And you need a greater number of “competence checks” if you have a larger pool of candidates.

Over and under-fitting, capacity. K -nearest-neighbors

A simple classification procedure is the “ K -nearest neighbors.”

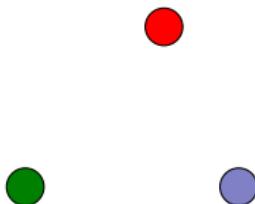
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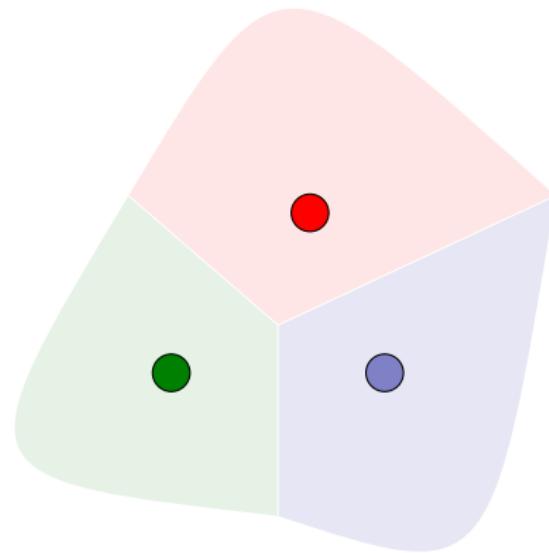
$$(x_n, y_n) \in \mathbb{R}^D \times \{1, \dots, C\}, \quad n = 1, \dots, N$$

to predict the y associated to a new x , take the y_n of the closest x_n :

$$\begin{aligned} n^*(x) &= \operatorname{argmin}_n \|x_n - x\| \\ f^*(x) &= y_{n^*(x)}. \end{aligned}$$

This recipe corresponds to $K = 1$, and makes the empirical training error zero.





$$K = 1$$

Under mild assumptions of regularities of $\mu_{X,Y}$, for $N \rightarrow \infty$ the asymptotic error rate of the 1-NN is less than twice the (optimal!) Bayes' Error rate.

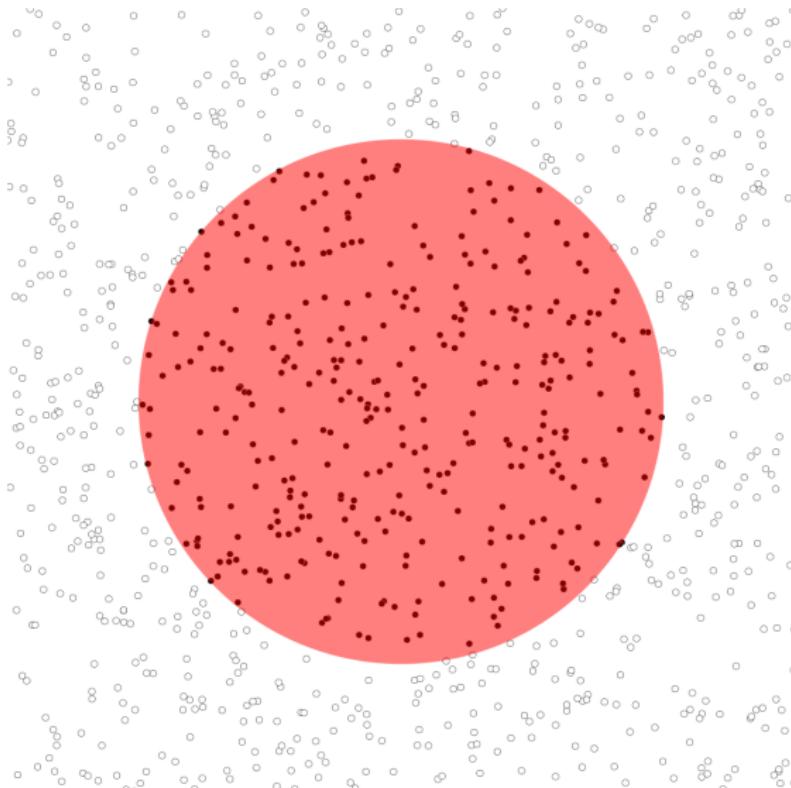
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It can be made more stable by looking at the $K > 1$ closest training points, and taking the majority vote.

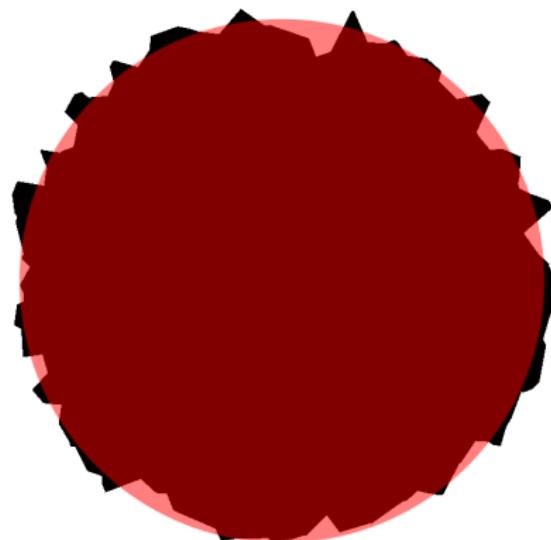
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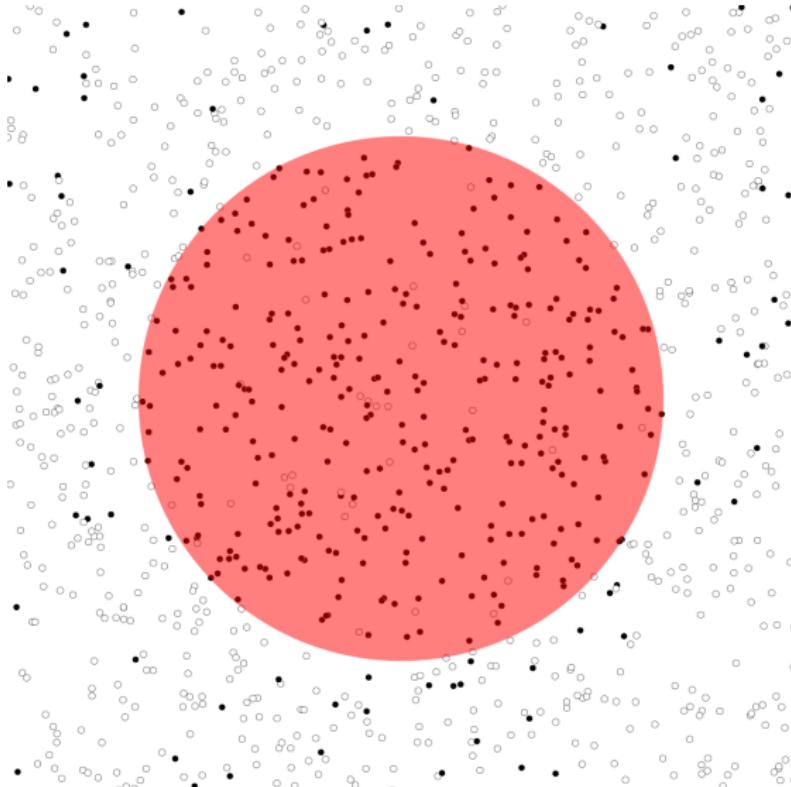
If we let also $K \rightarrow \infty$ “not too fast”, the error rate is the (optimal!) Bayes' Error rate.



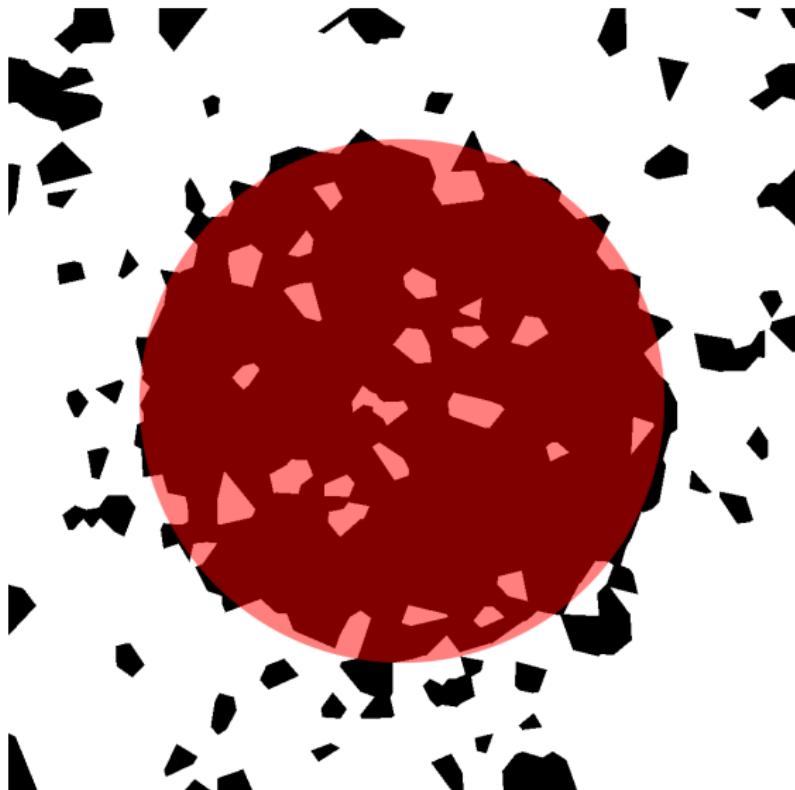
Training set



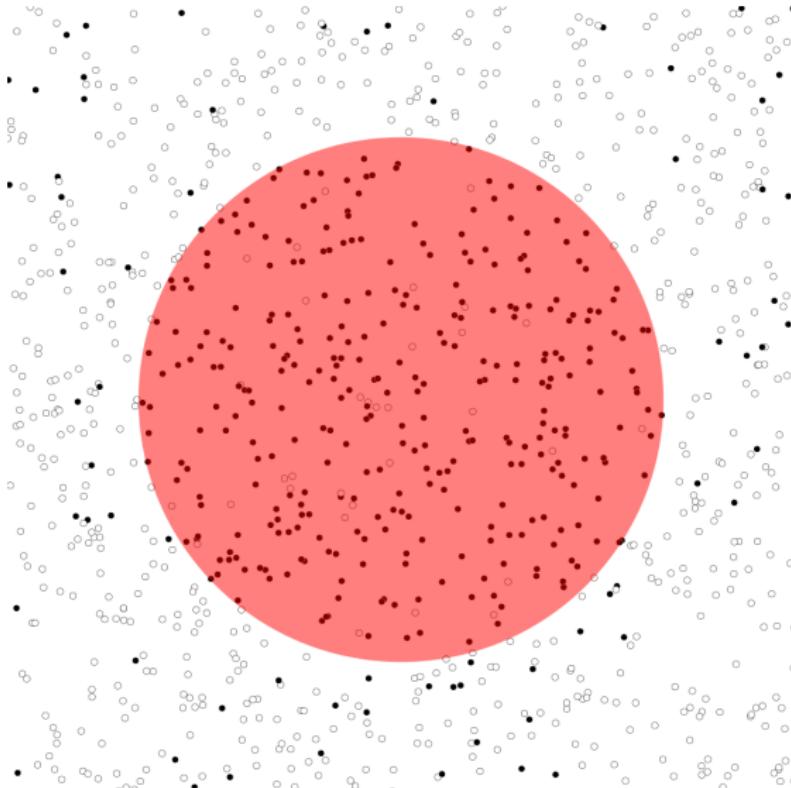
Prediction (K=1)



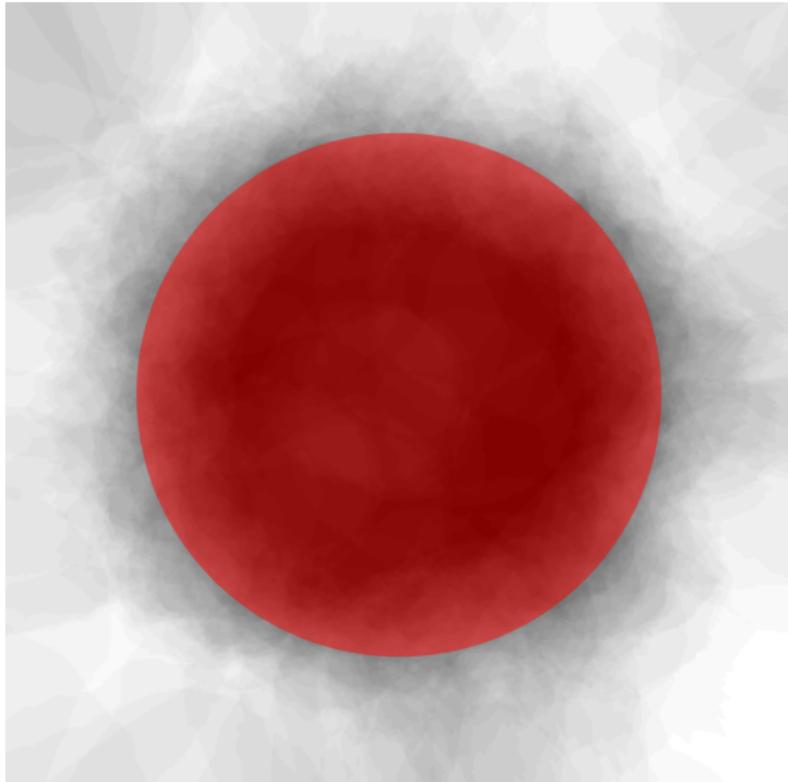
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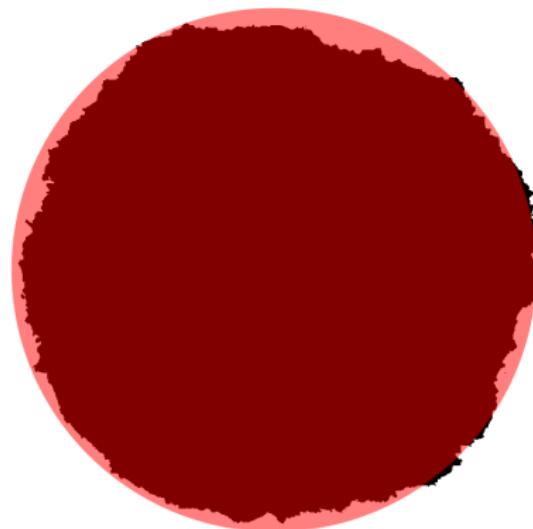
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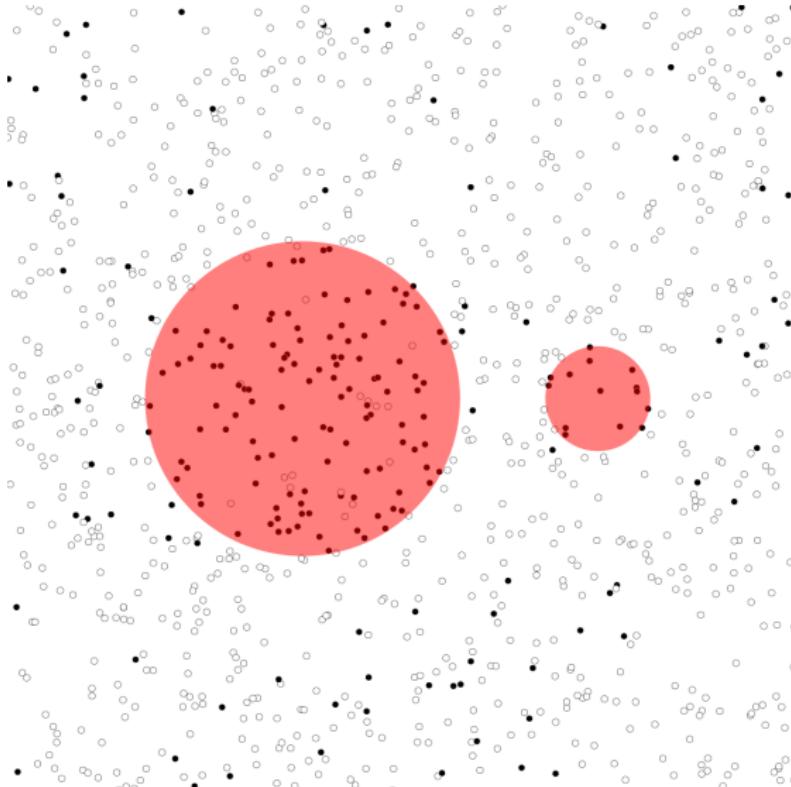
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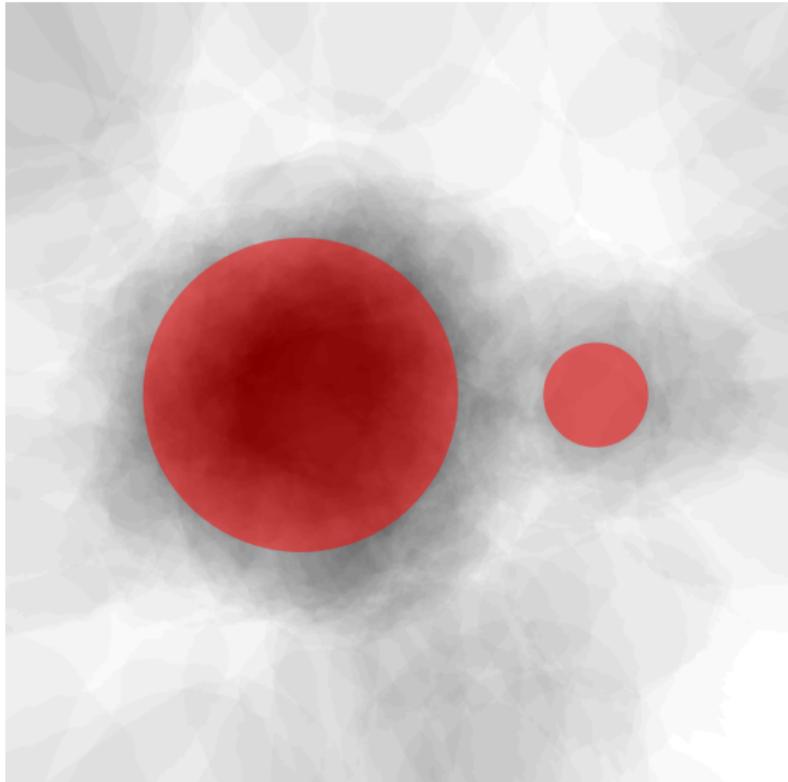
Votes (K=51)



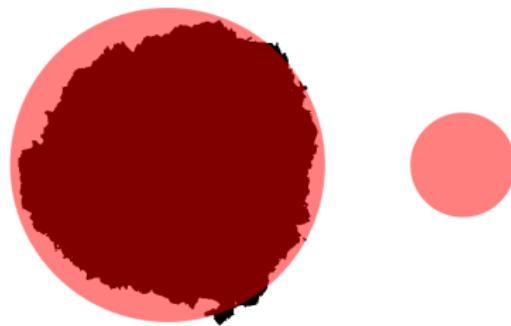
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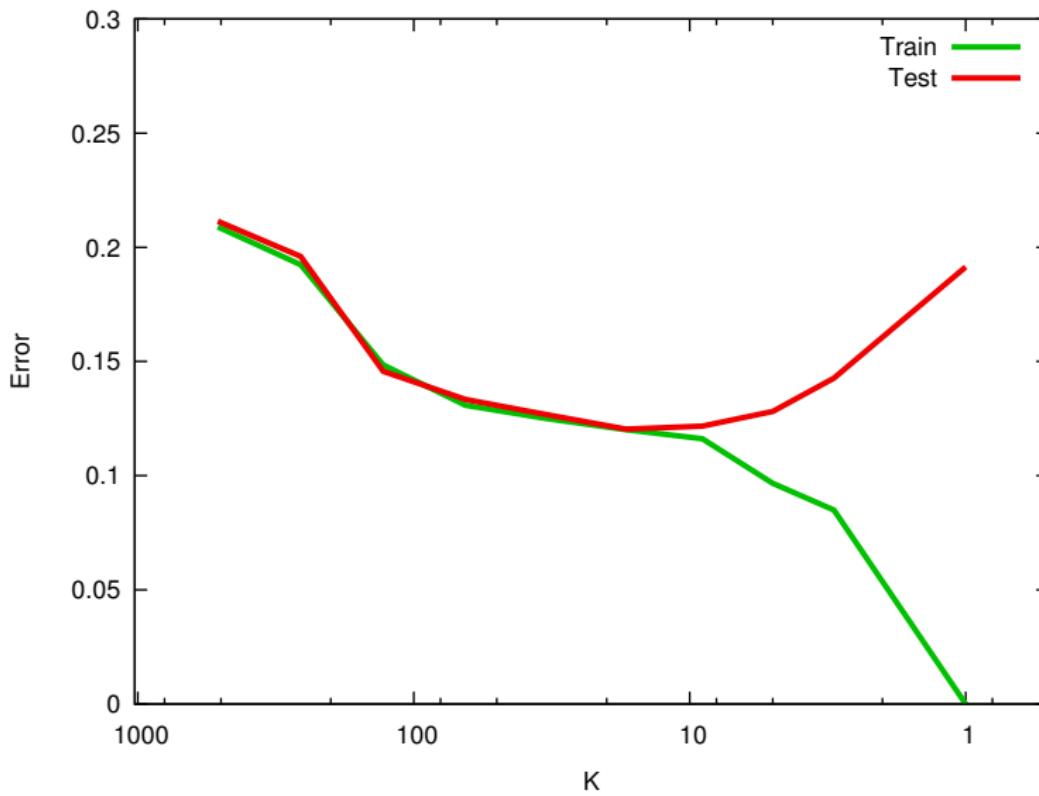
Training set

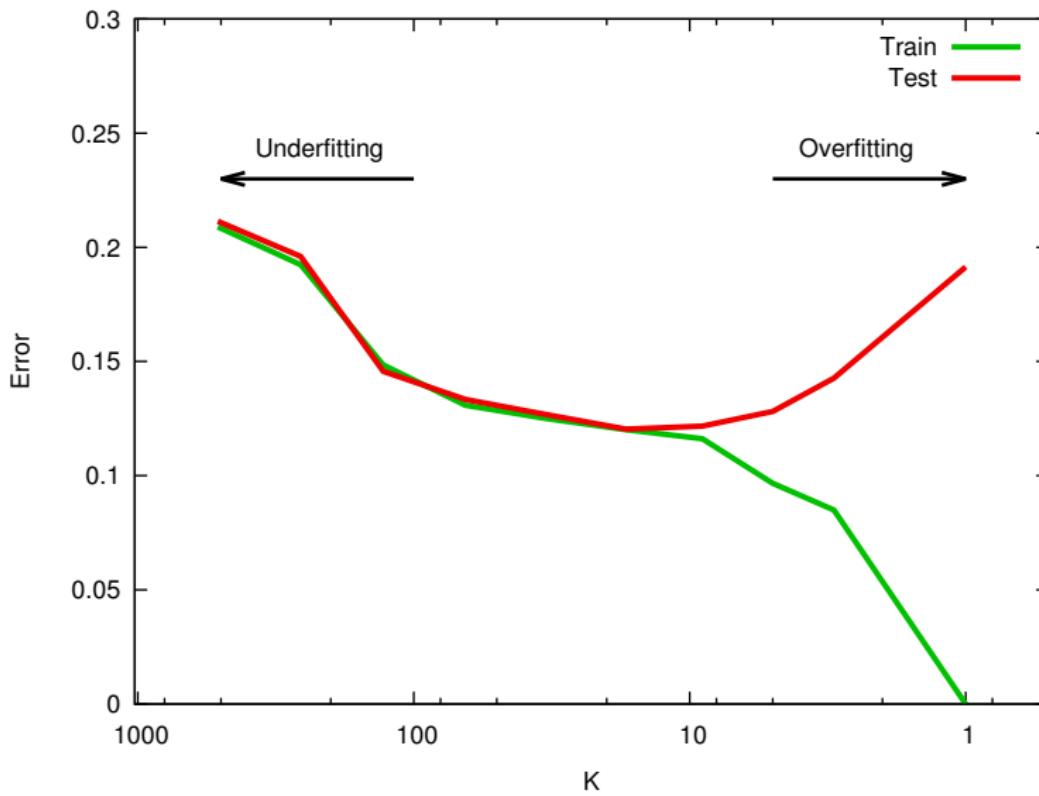


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Over and under-fitting, capacity, polynomials

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Hence, minimizing this loss is a standard quadratic problem, for which we have efficient algorithms.

$$\underset{\alpha}{\operatorname{argmin}} \left\| \begin{pmatrix} x_1^0 & \dots & x_1^D \\ \vdots & & \vdots \\ x_N^0 & \dots & x_N^D \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_D \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \right\|^2$$

```

def fit_polynomial(D, x, y):
    X = torch.empty(x.size(0), D + 1)
    for d in range(D + 1):
        X[:, d] = x.pow(d)

    # gels expects a matrix for target
    Y = y.view(-1, 1)

    # LAPACK's GEneralized Least-Square
    alpha, _ = torch.gels(Y, X)

    return alpha[:D+1, 0]

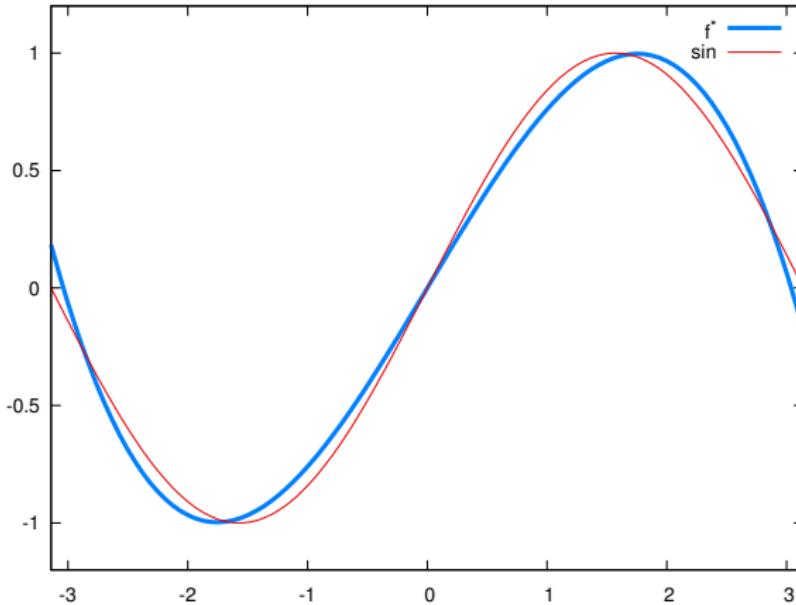
```

```
D, N = 4, 100
x = torch.linspace(-math.pi, math.pi, N)
y = x.sin()
alpha = fit_polynomial(D, x, y)

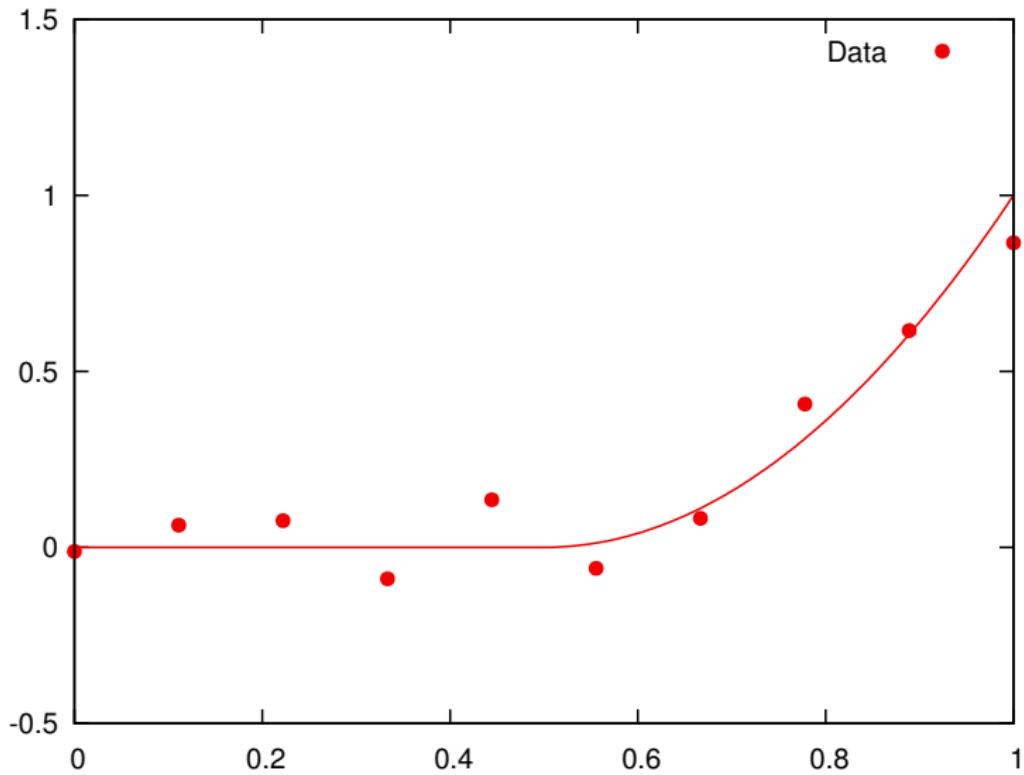
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yhat = X.mv(alpha)

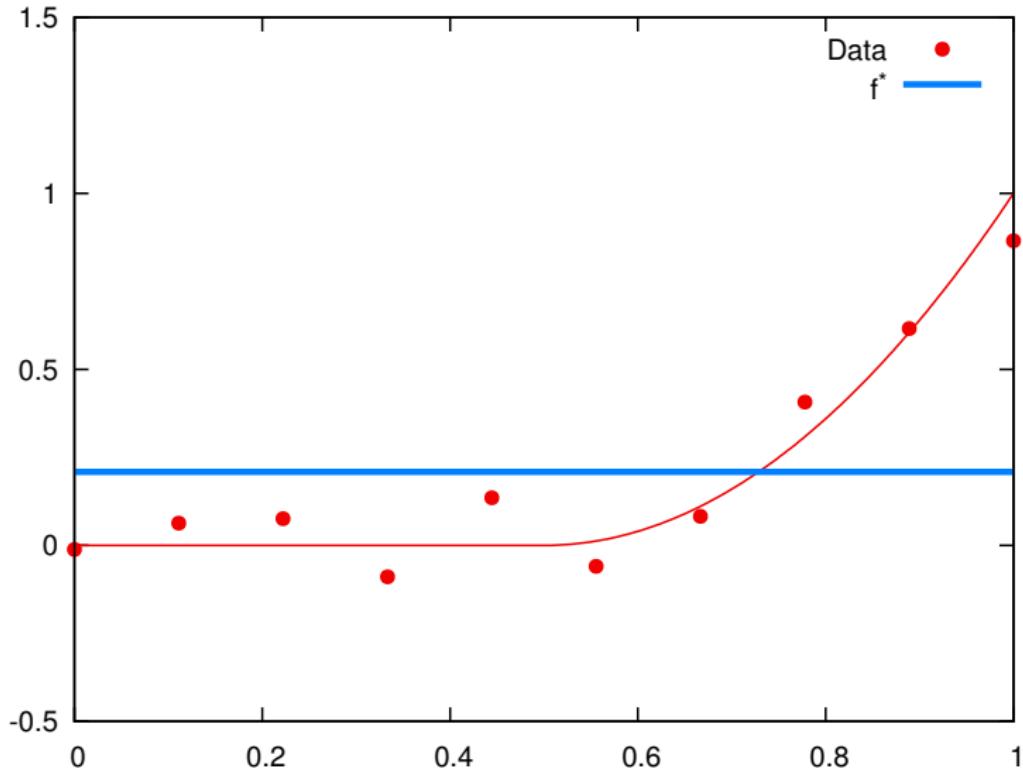
for k in range(N):
    print(x[k].item(), y[k].item(), yhat[k].item())
```



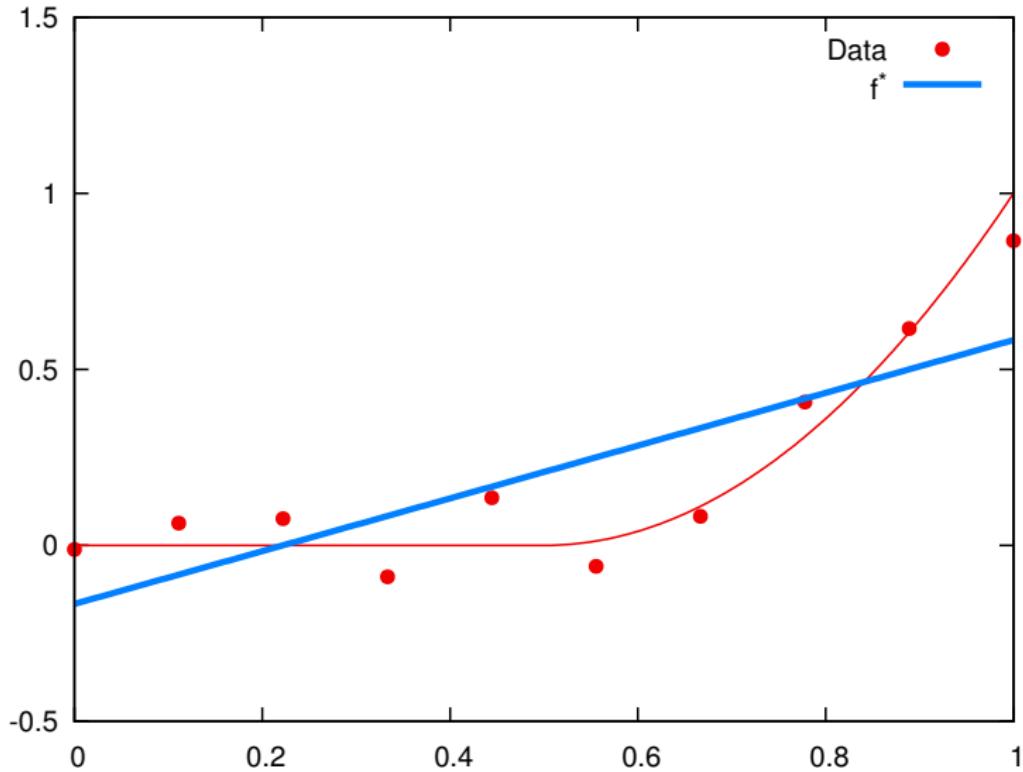
We can use that model to illustrate how the prediction changes when we increase the degree or the regularization.



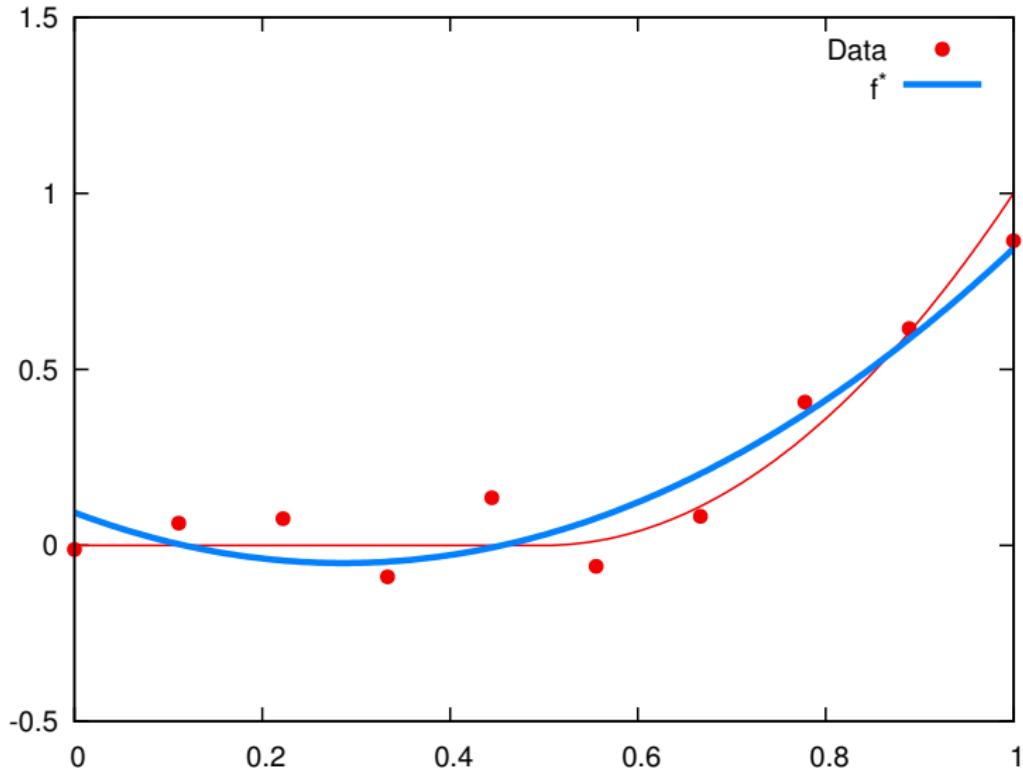
Degree D=0



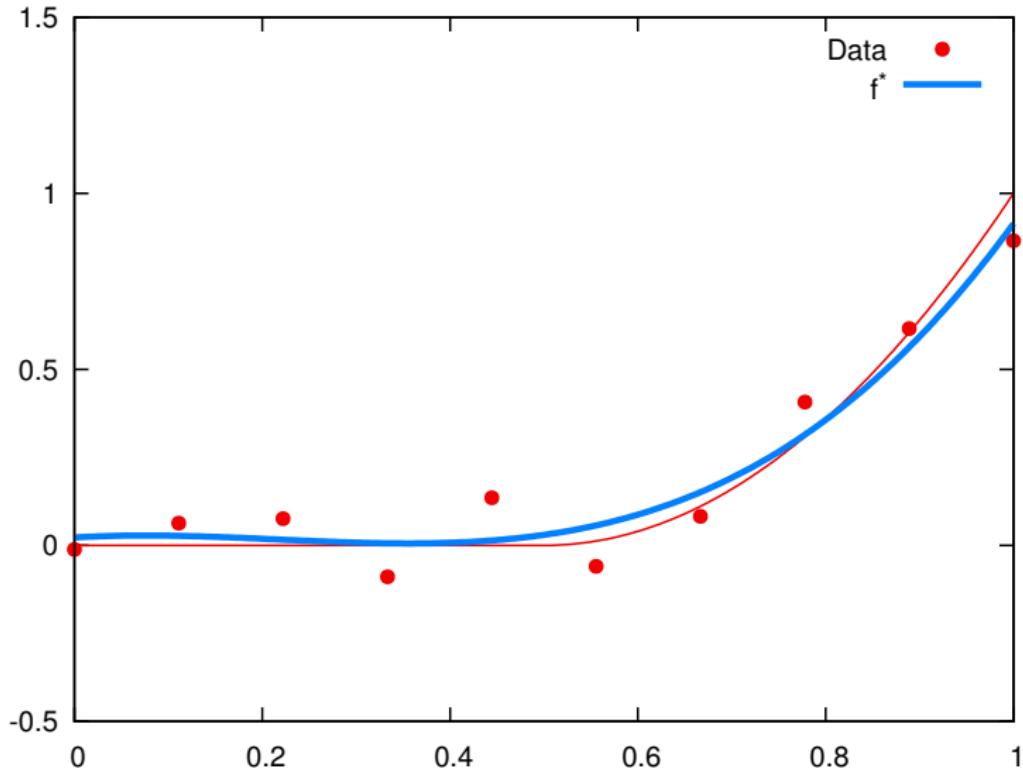
Degree D=1



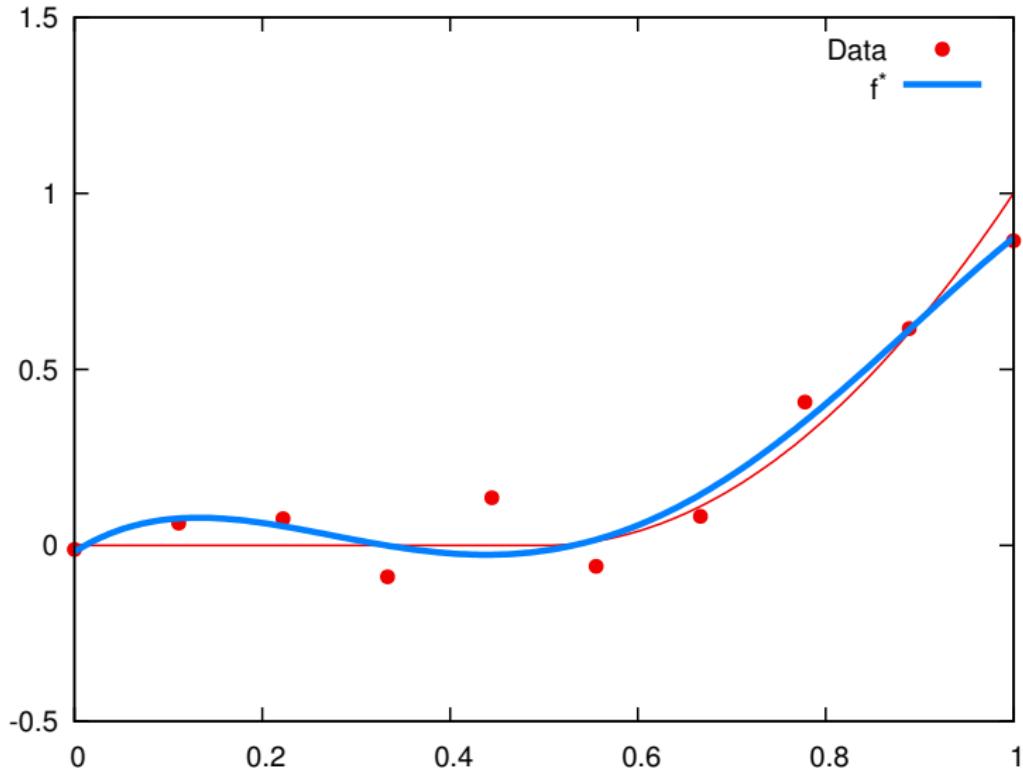
Degree D=2



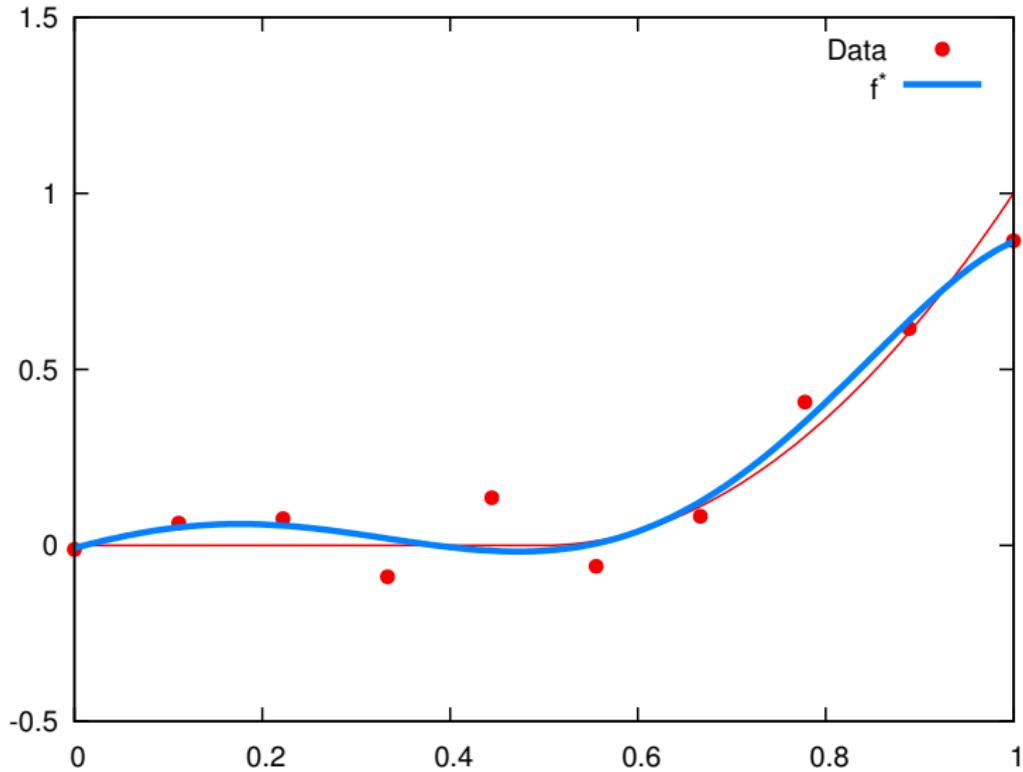
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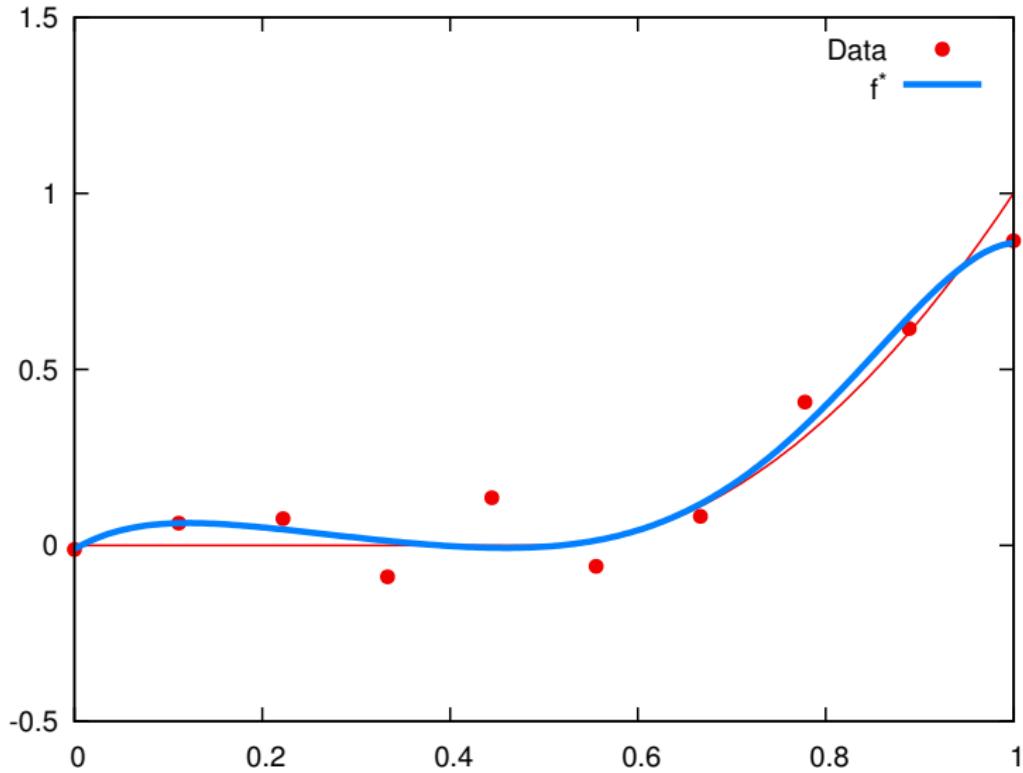
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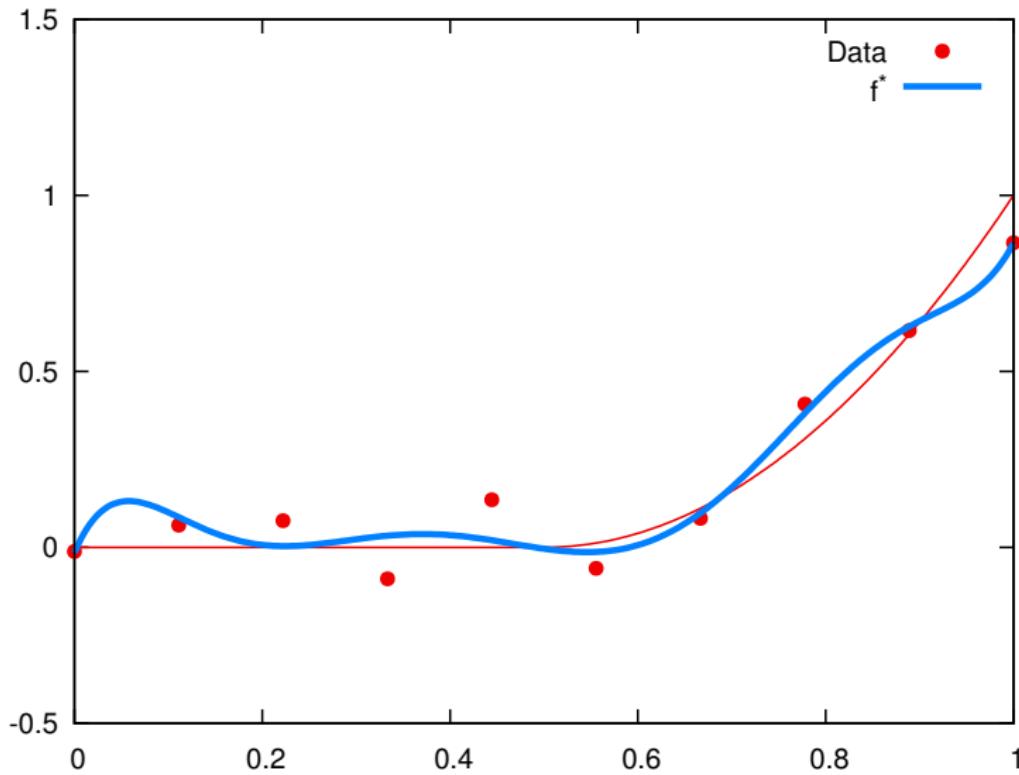
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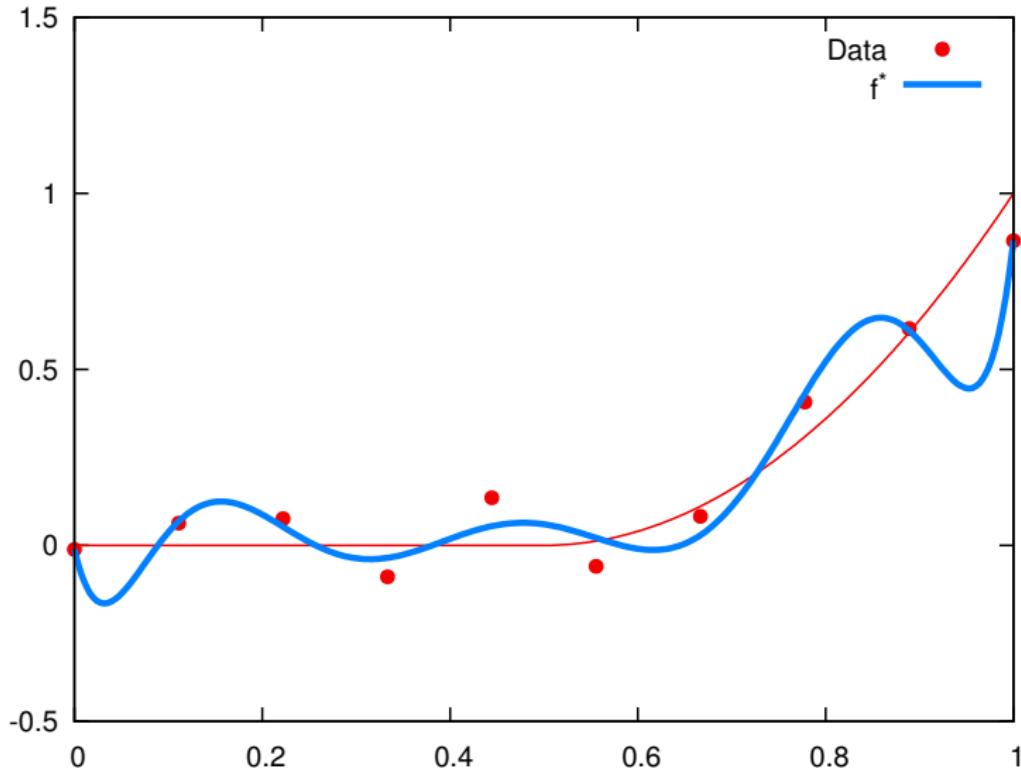
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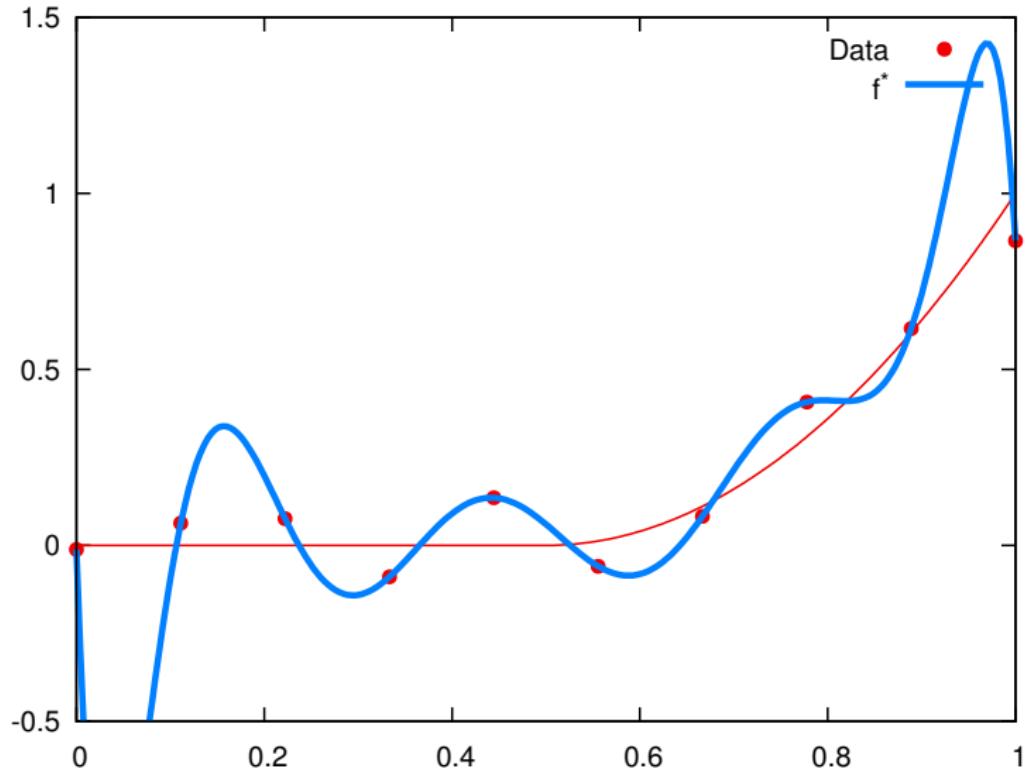
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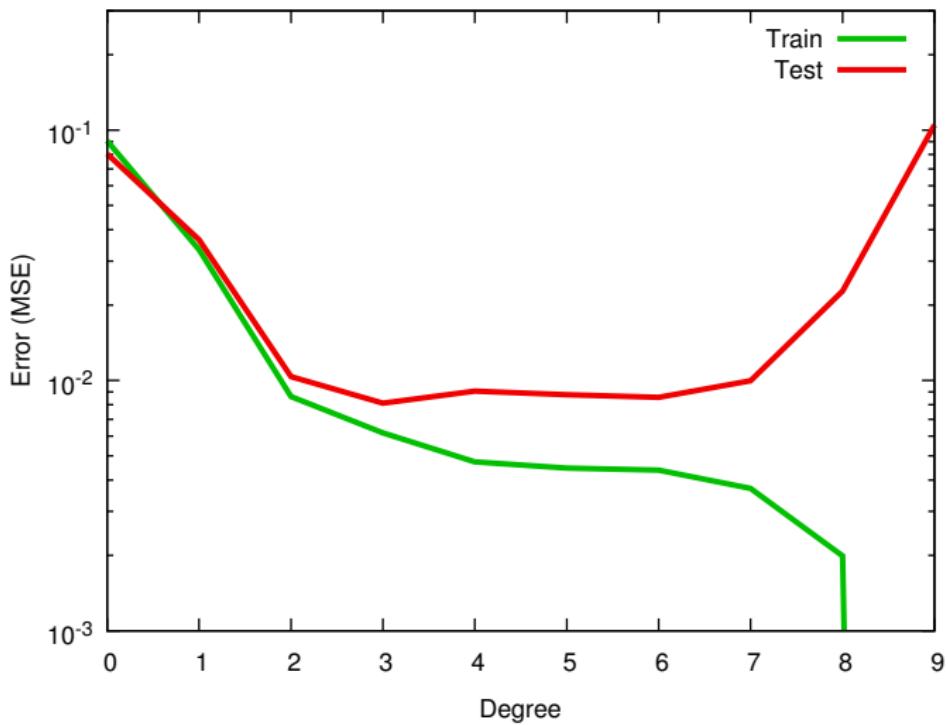


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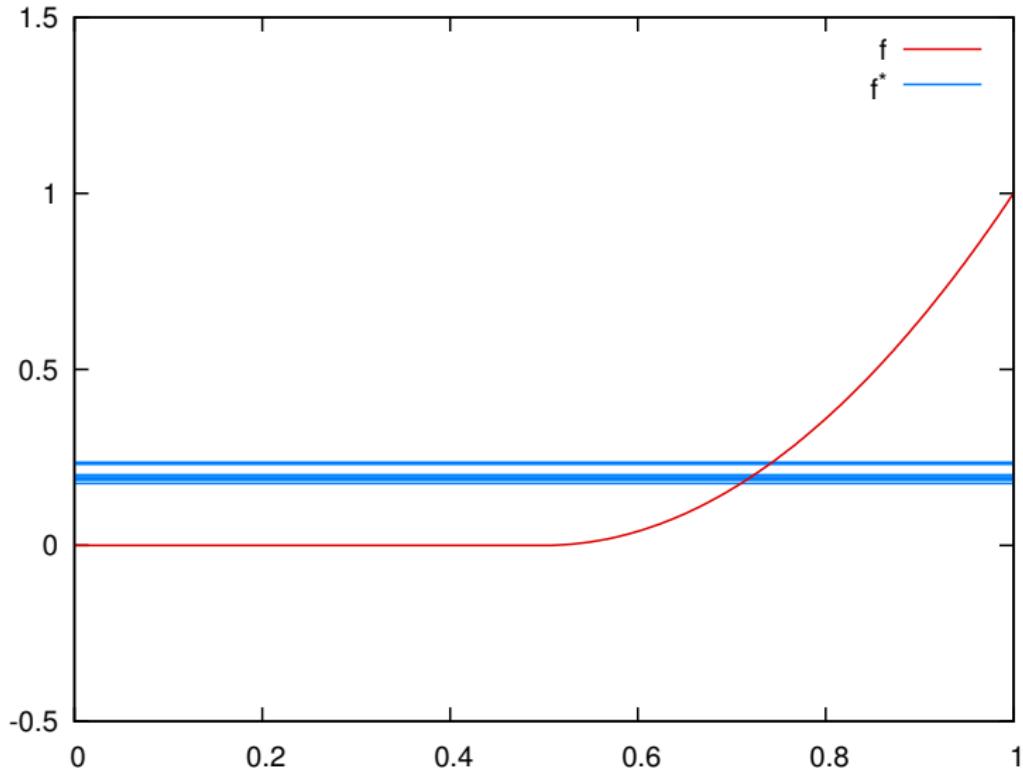
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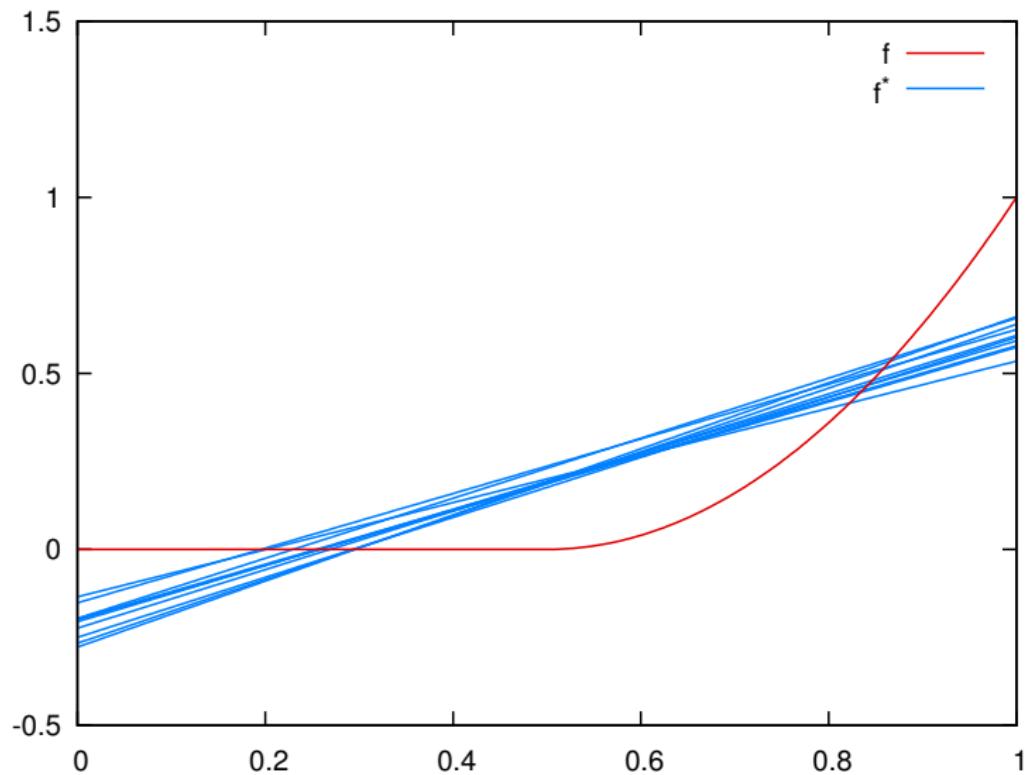


We can visualize the influence of the noise by generating multiple training sets $\mathcal{D}_1, \dots, \mathcal{D}_M$ with different noise, and training one model on each.

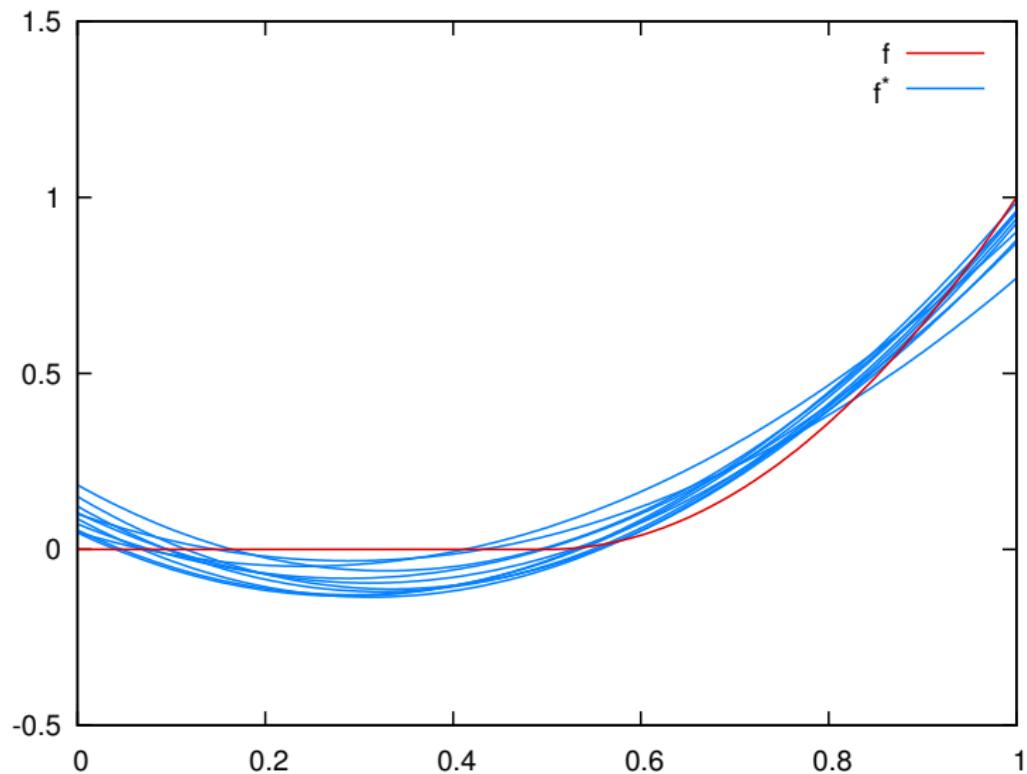
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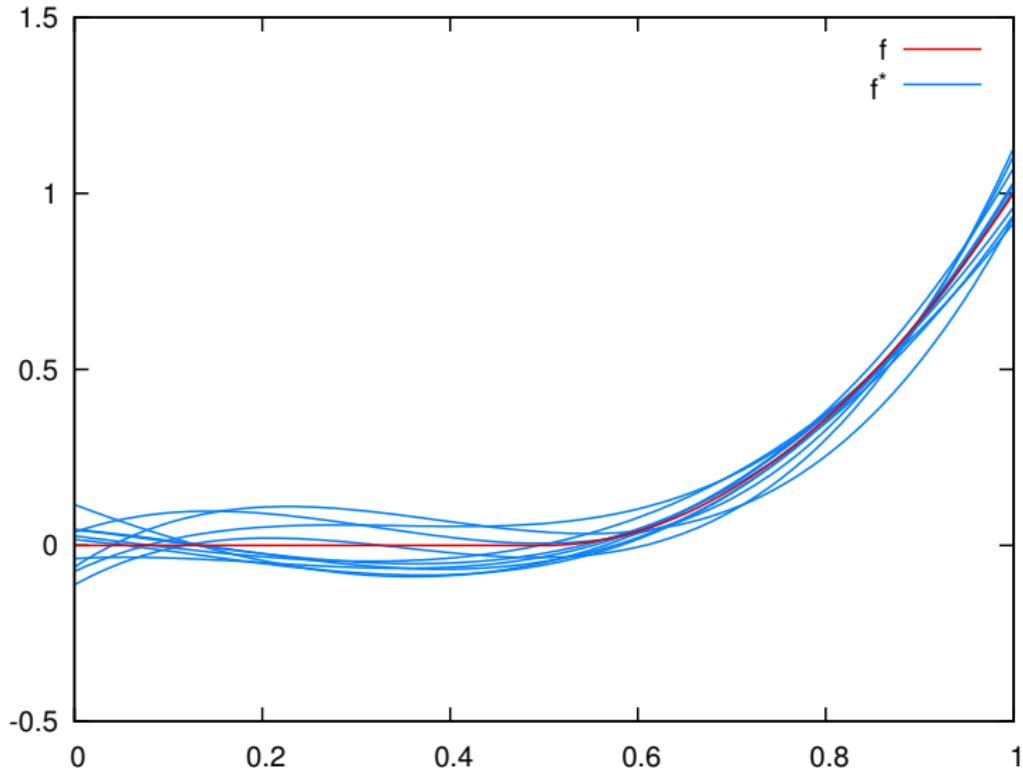
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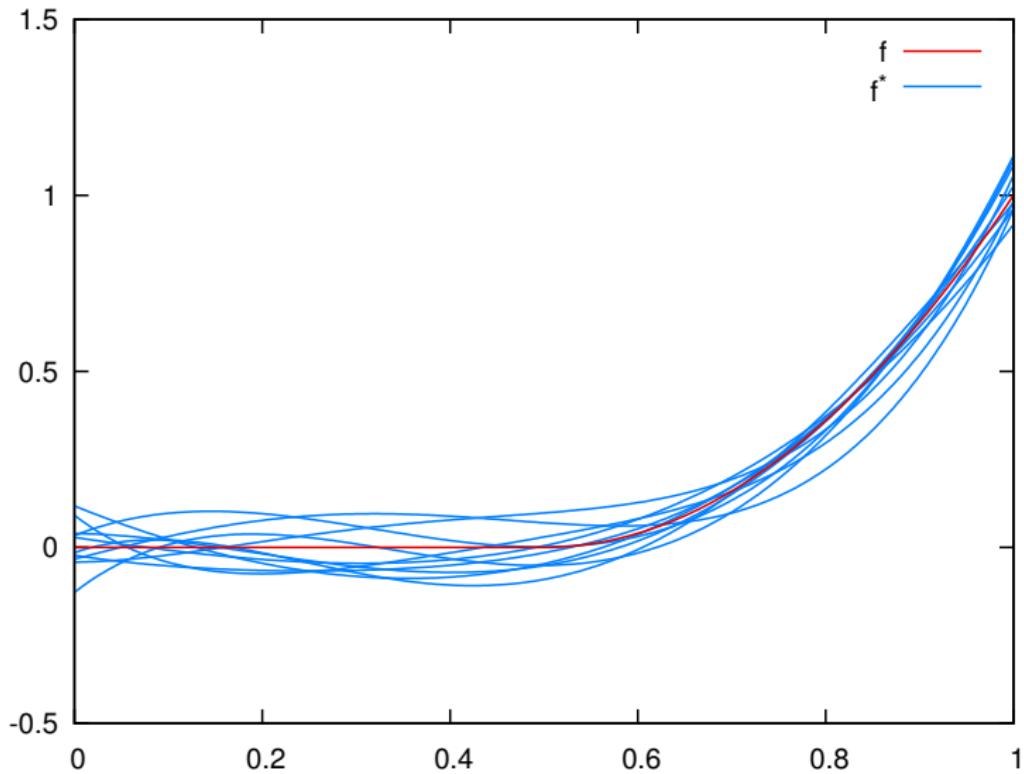
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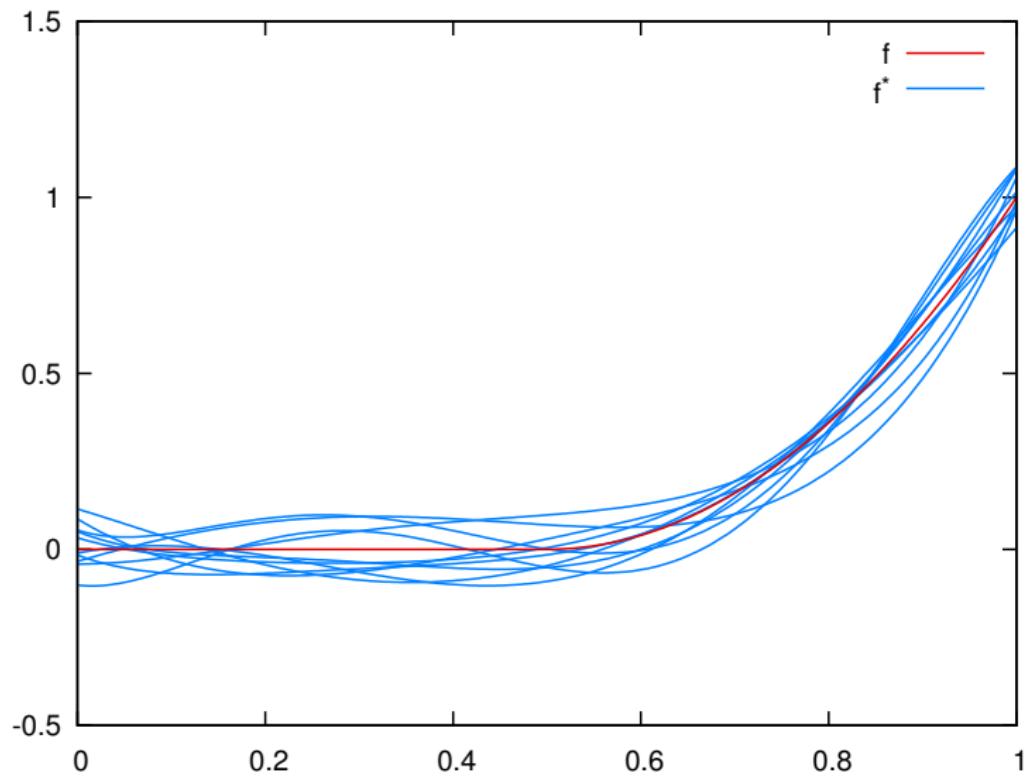
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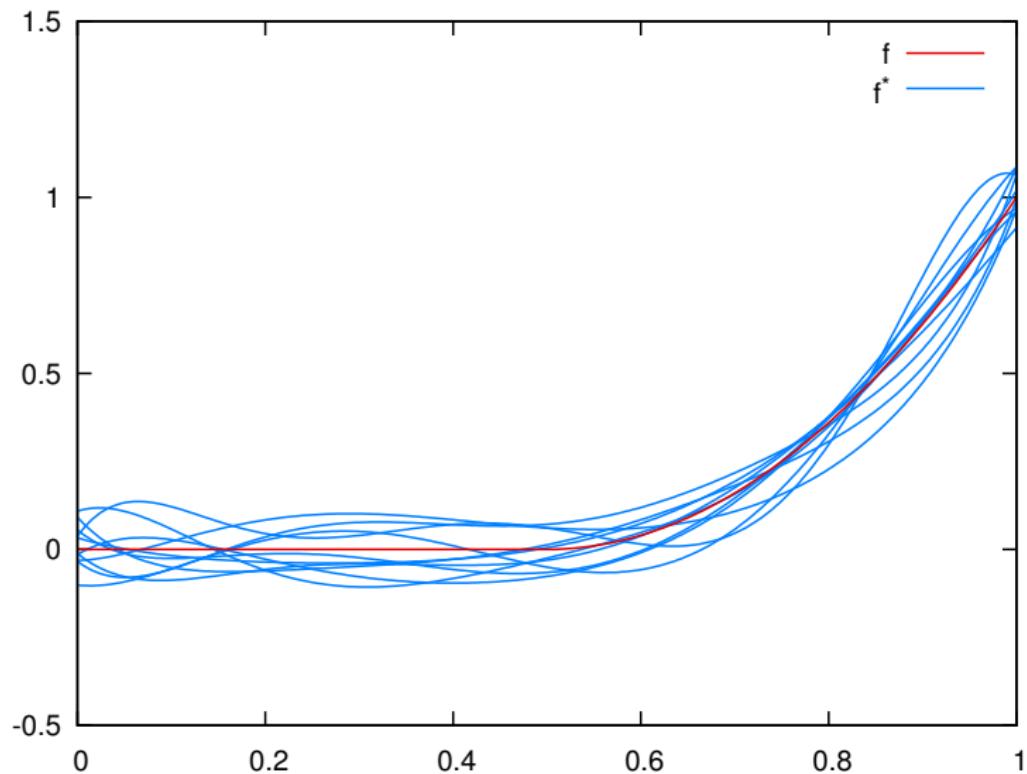
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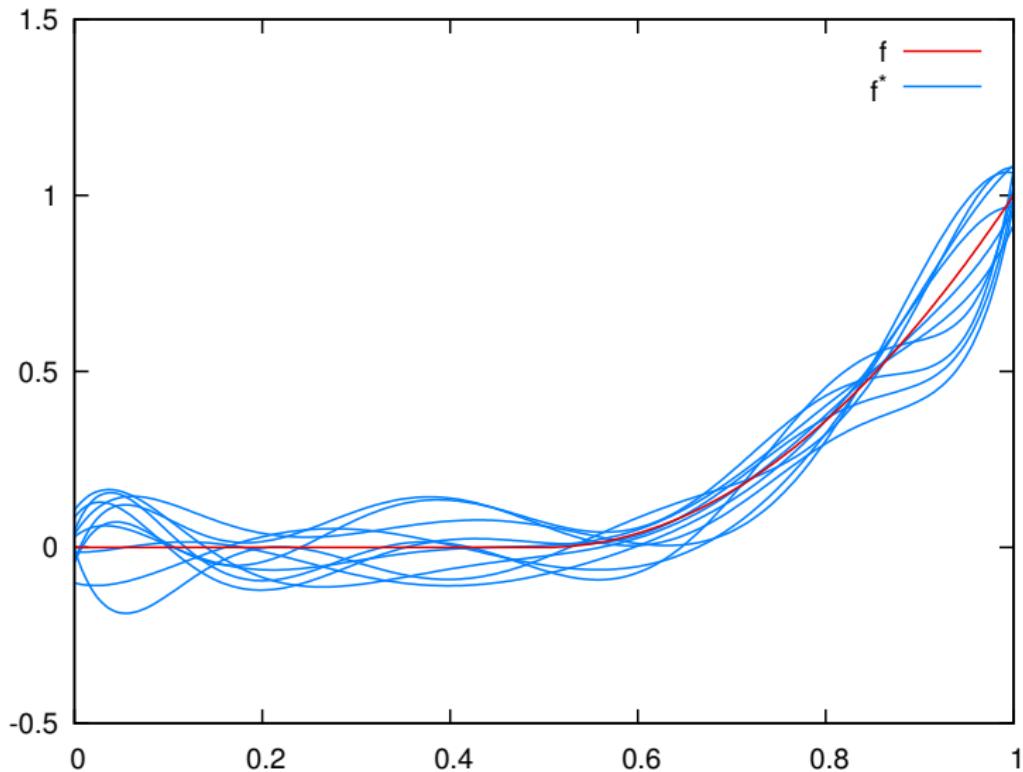
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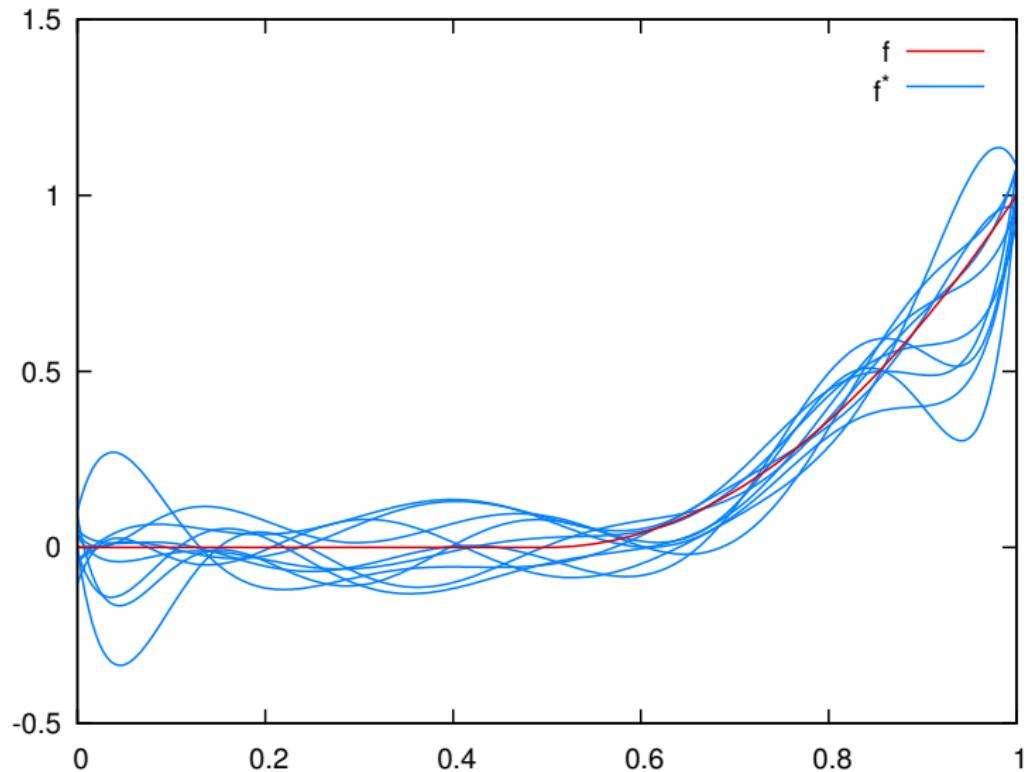
Degree D=6



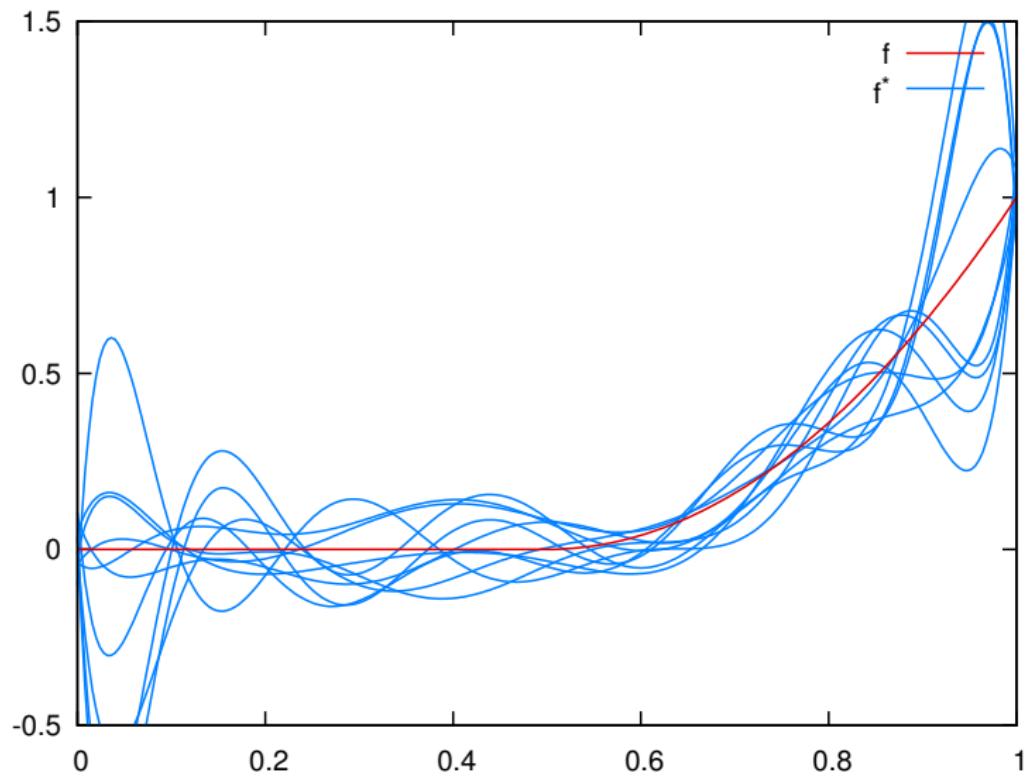
Degree D=7



Degree D=8



Degree D=9



We can reformulate this control of the degree with a penalty

$$\mathcal{L}(\alpha) = \sum_n (f(x_n; \alpha) - y_n)^2 + \sum_d I_d(\alpha_d)$$

where

$$I_d(\alpha) = \begin{cases} 0 & \text{if } d \leq D \text{ or } \alpha = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Such a penalty kills any term of degree $> D$.

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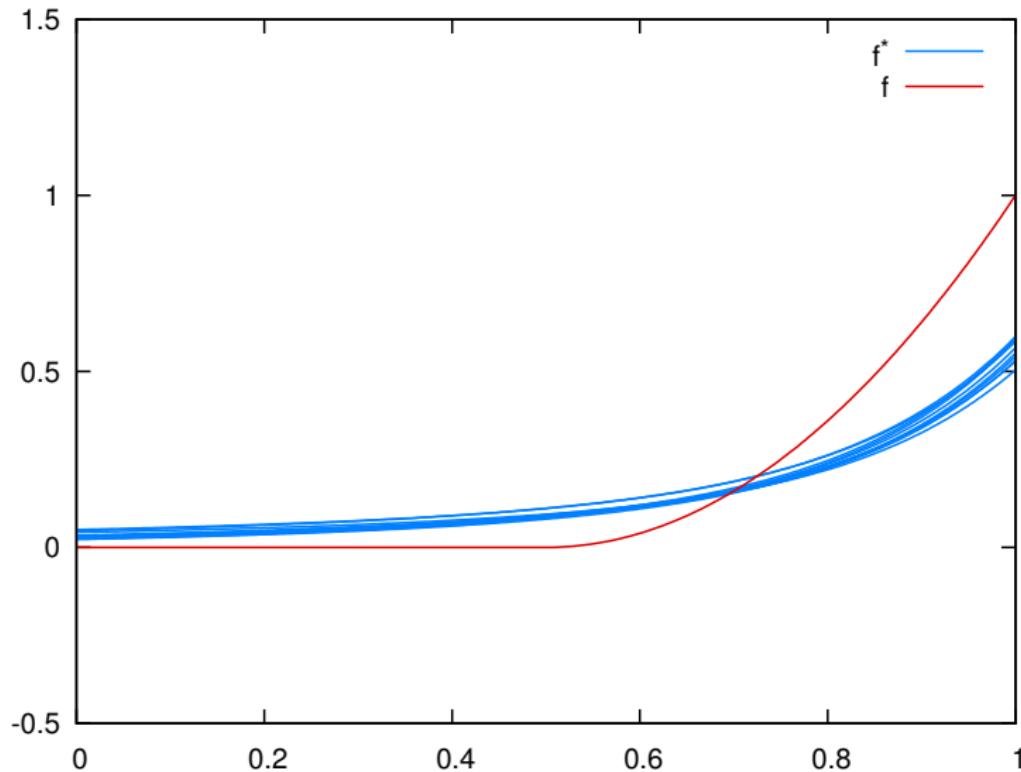
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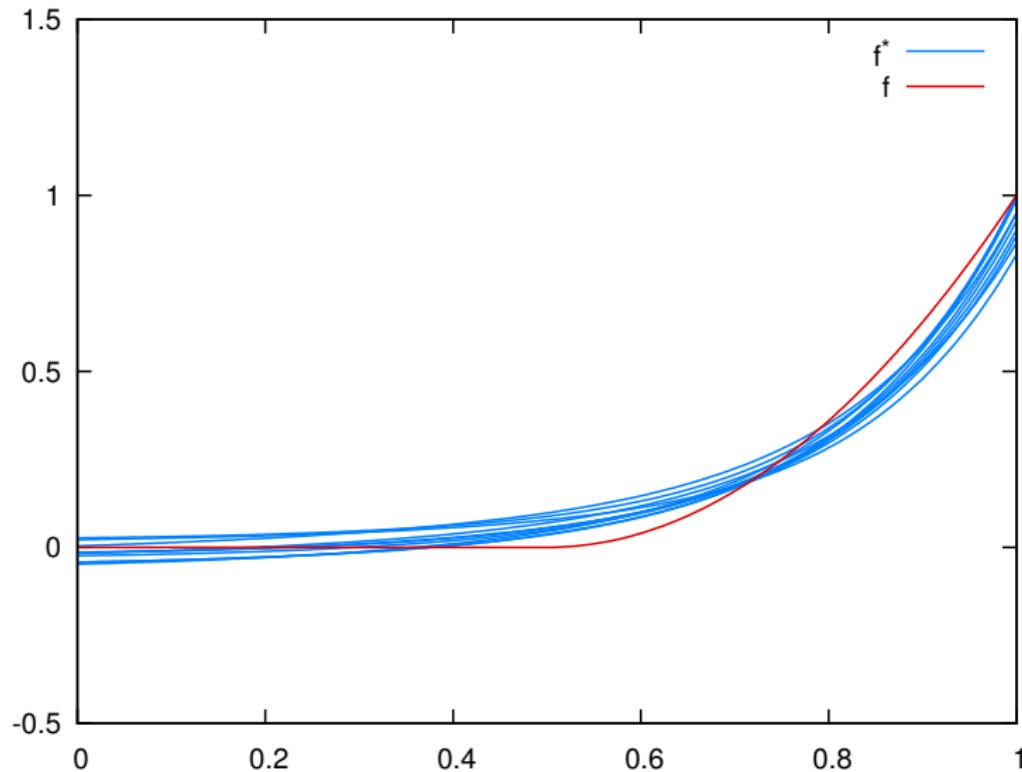
This motivates the use of more subtle variants. For instance, to keep all this quadratic

$$\mathcal{L}(\alpha) = \sum_n (f(x_n; \alpha) - y_n)^2 + \rho \sum_d \alpha_d^2.$$

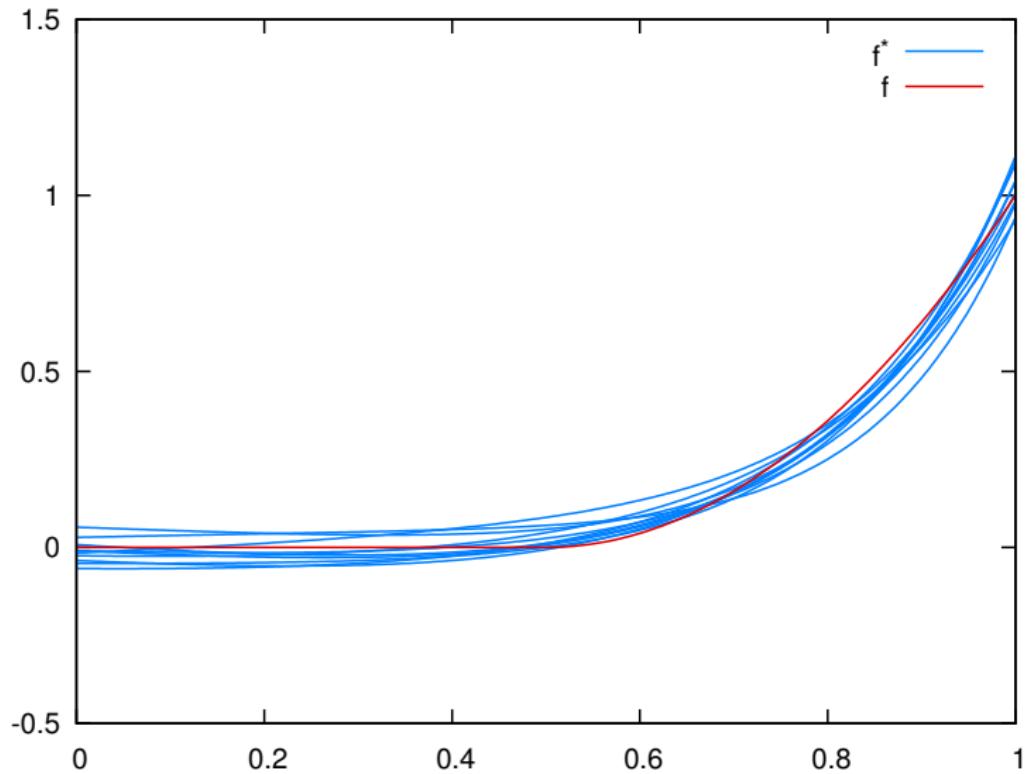
$D=9, \rho=1e1$



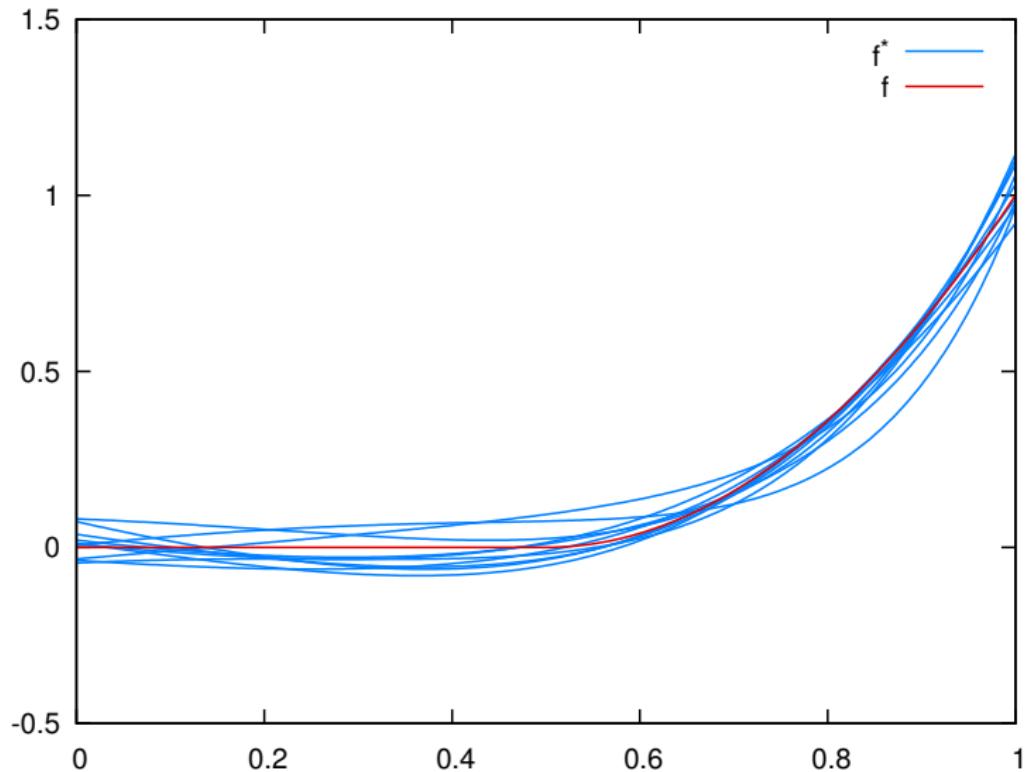
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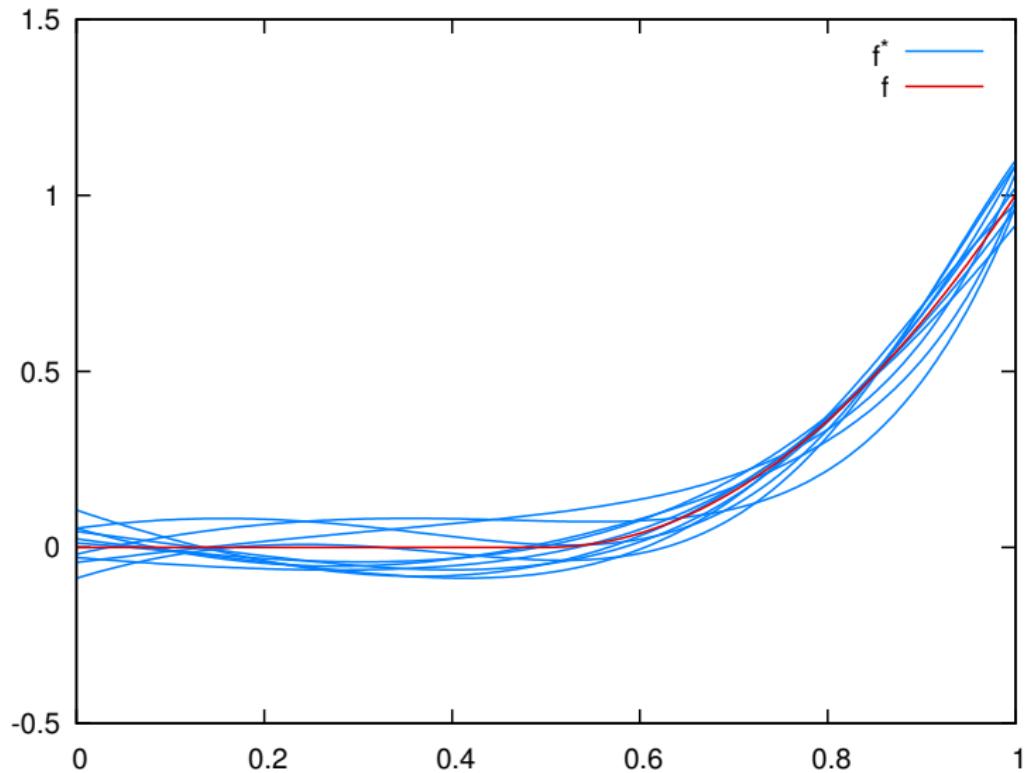
$D=9, \rho=1e-1$



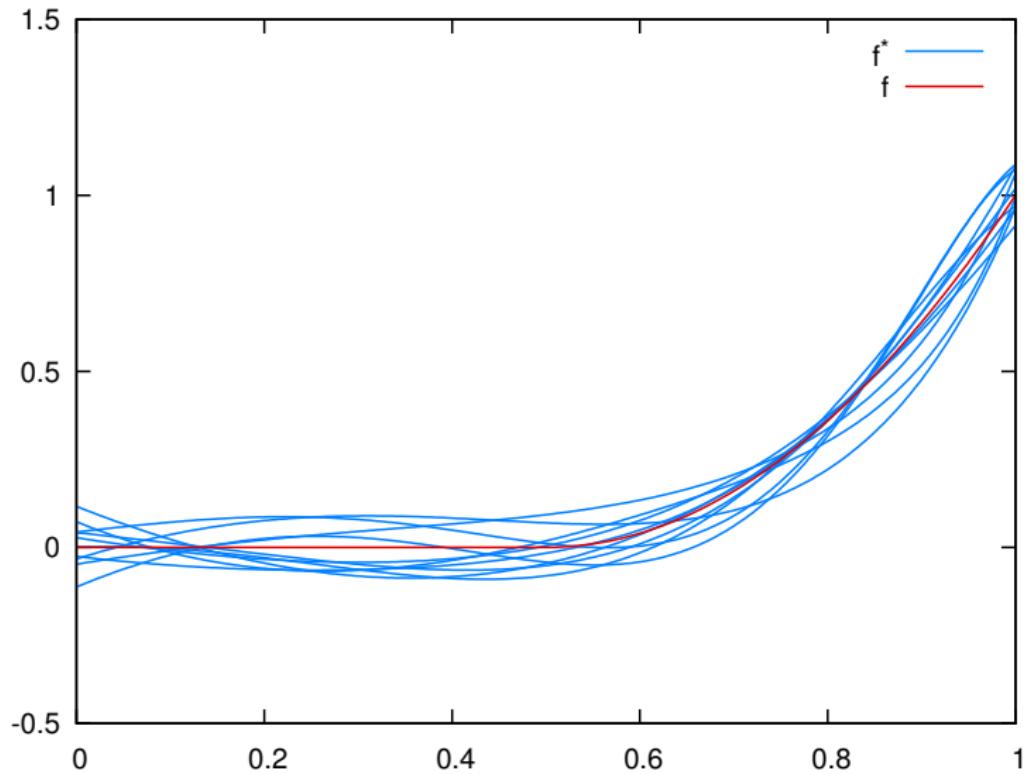
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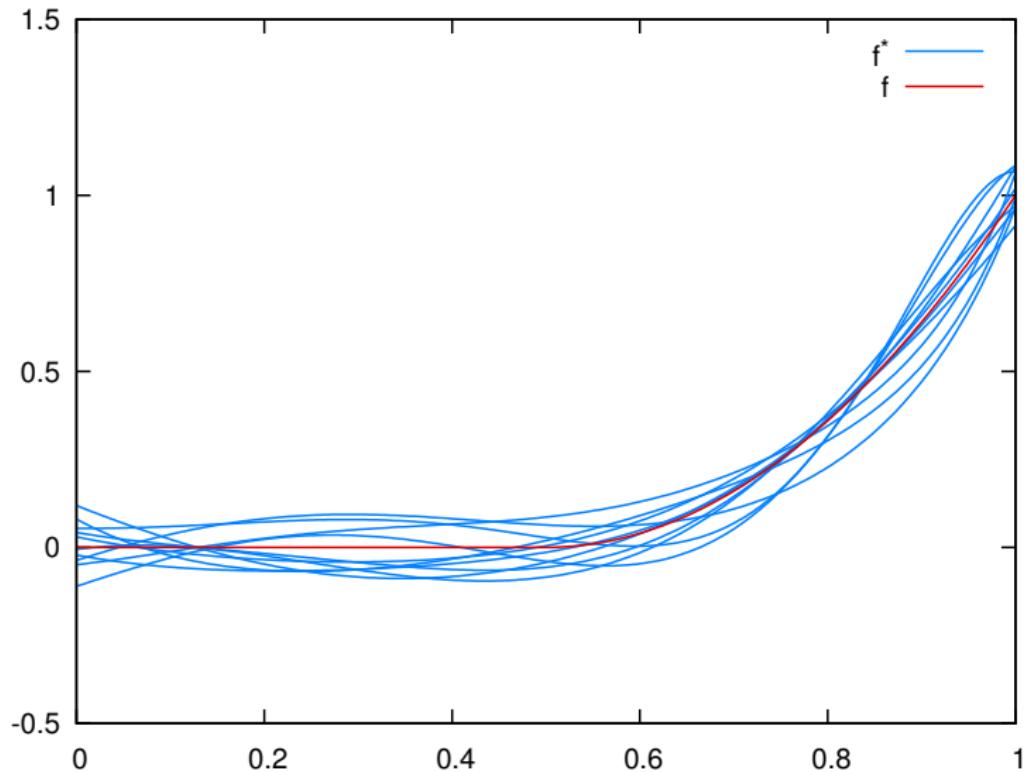
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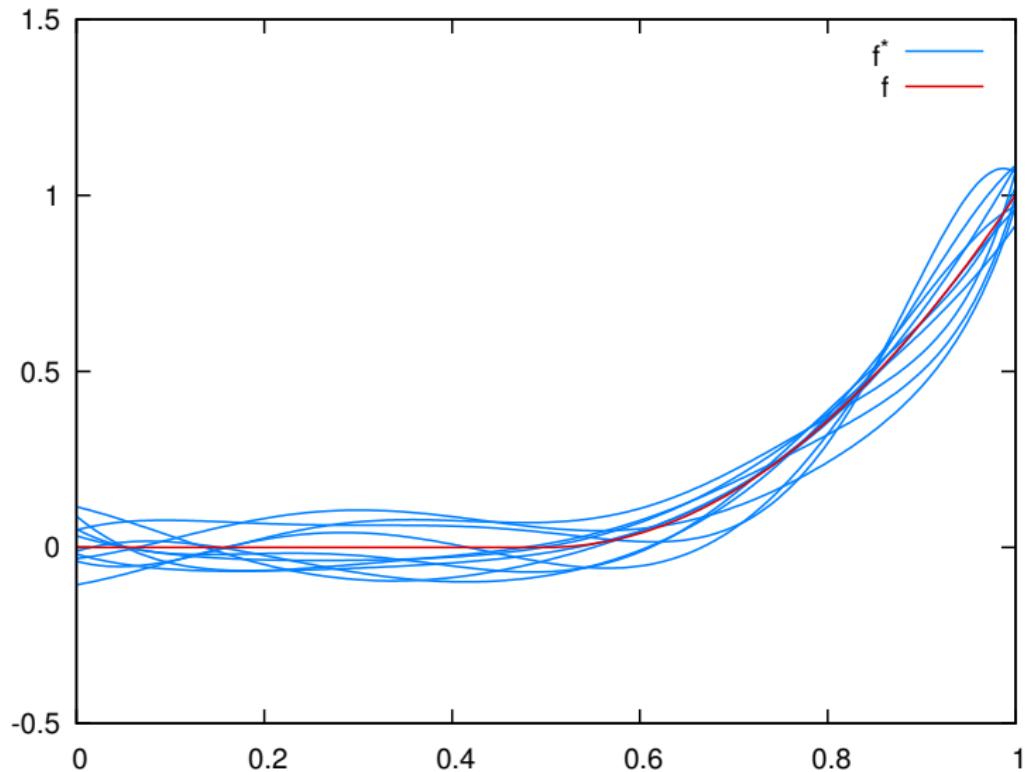
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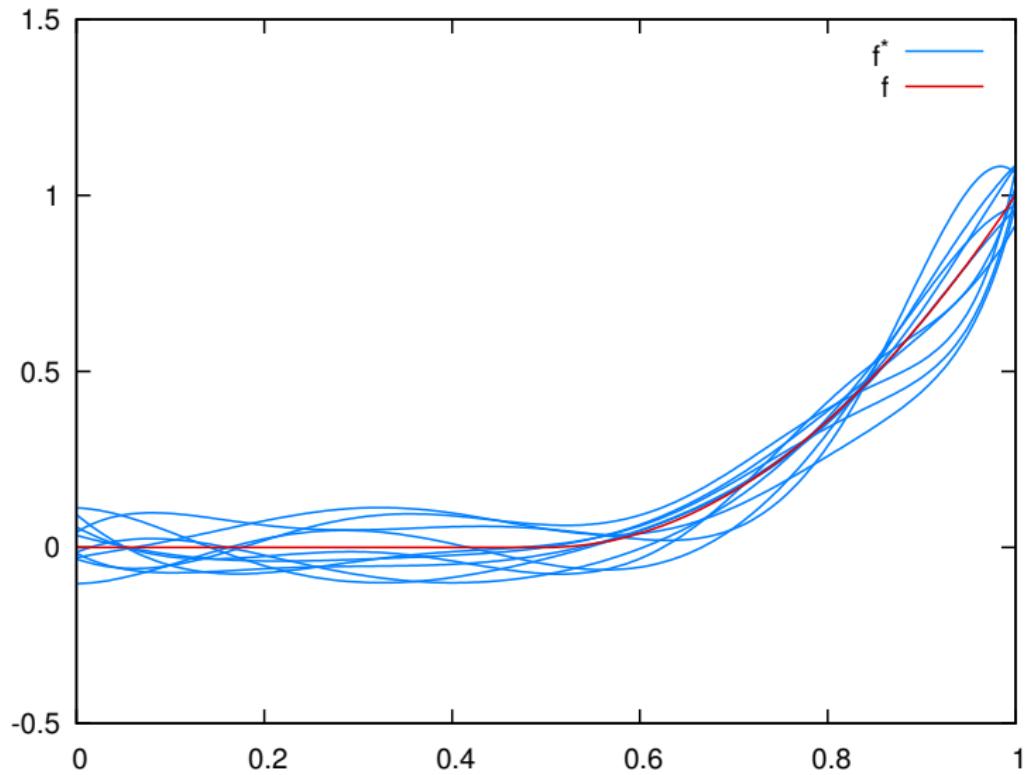
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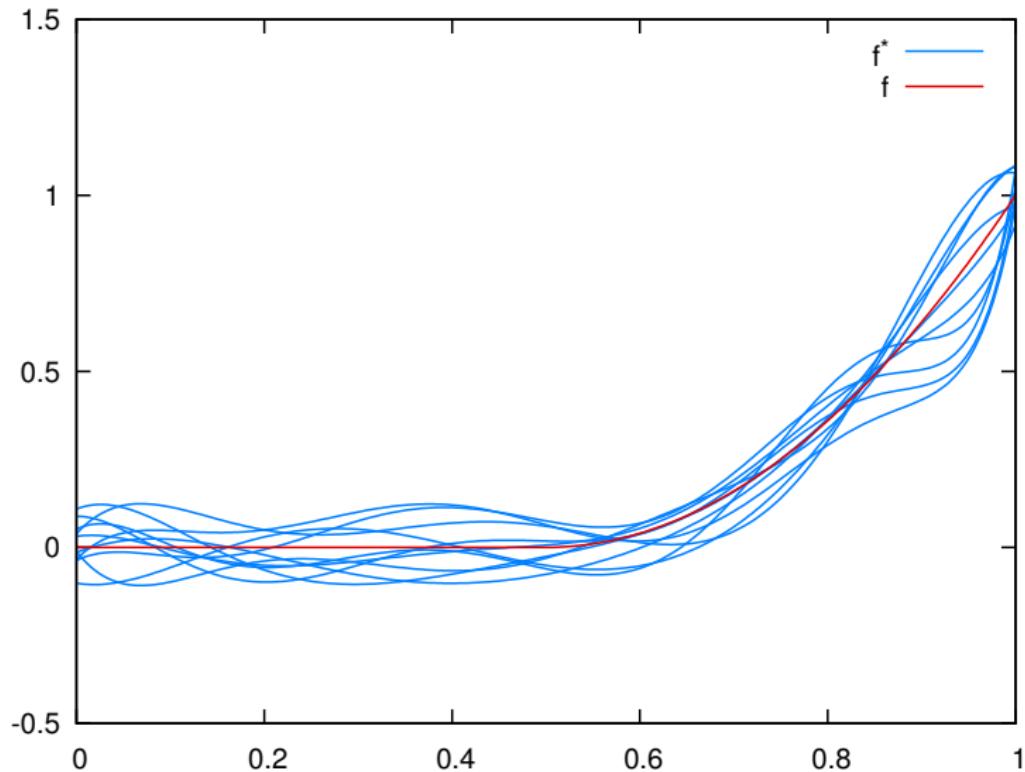
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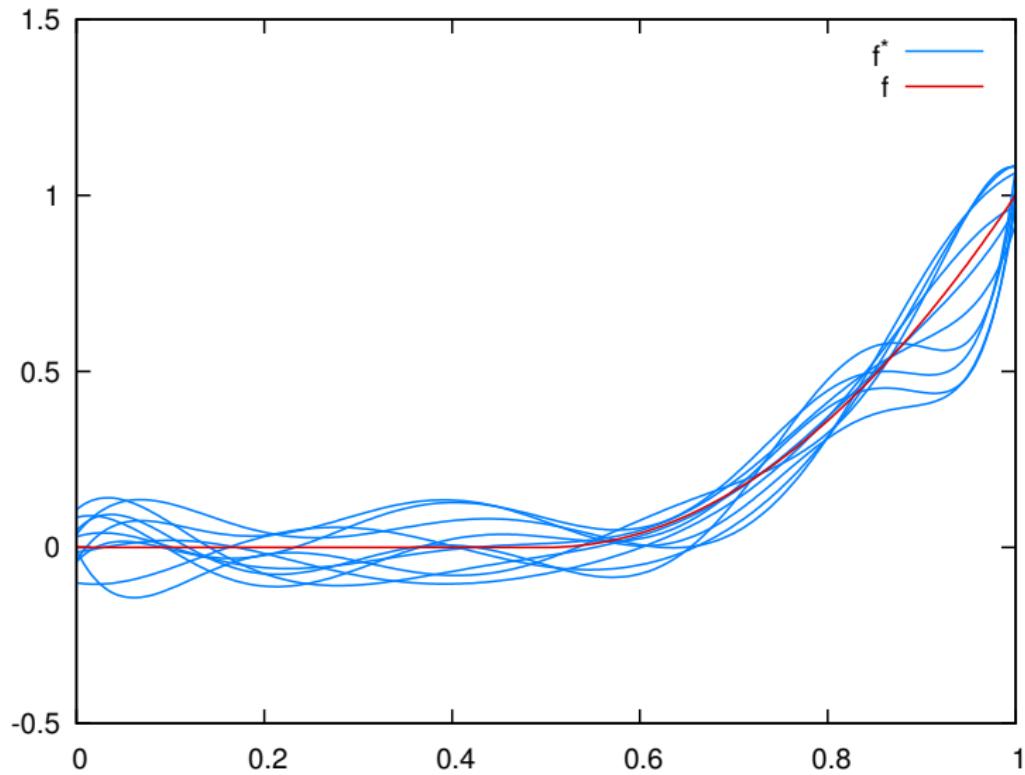
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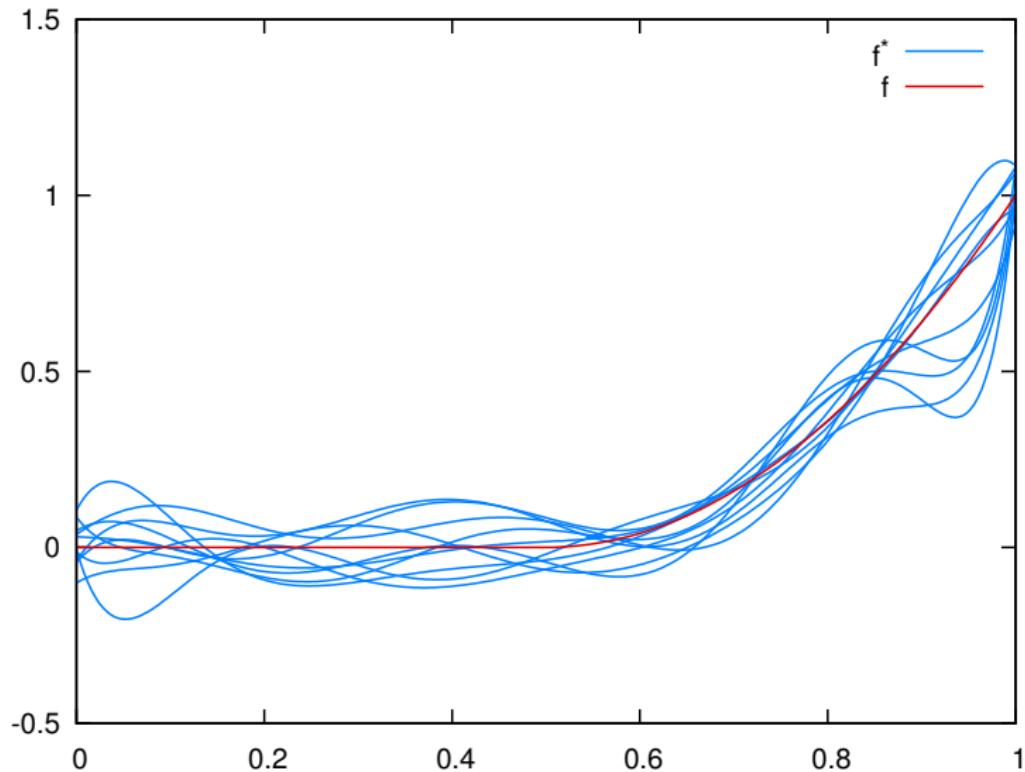
$D=9, \rho=1e-8$



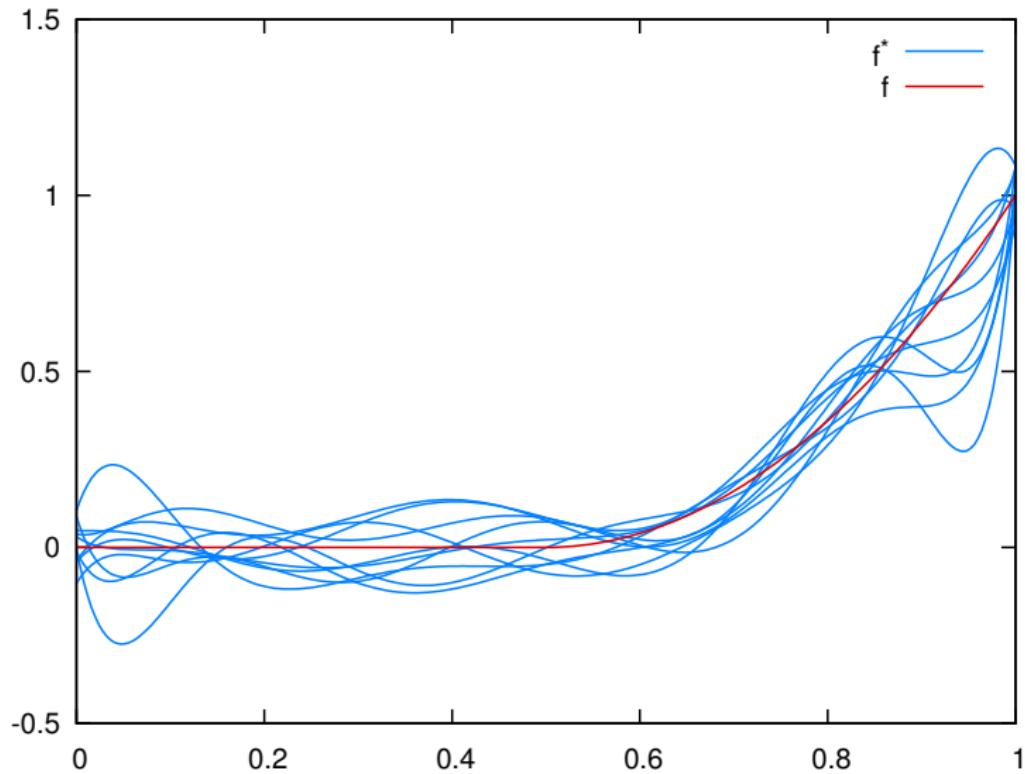
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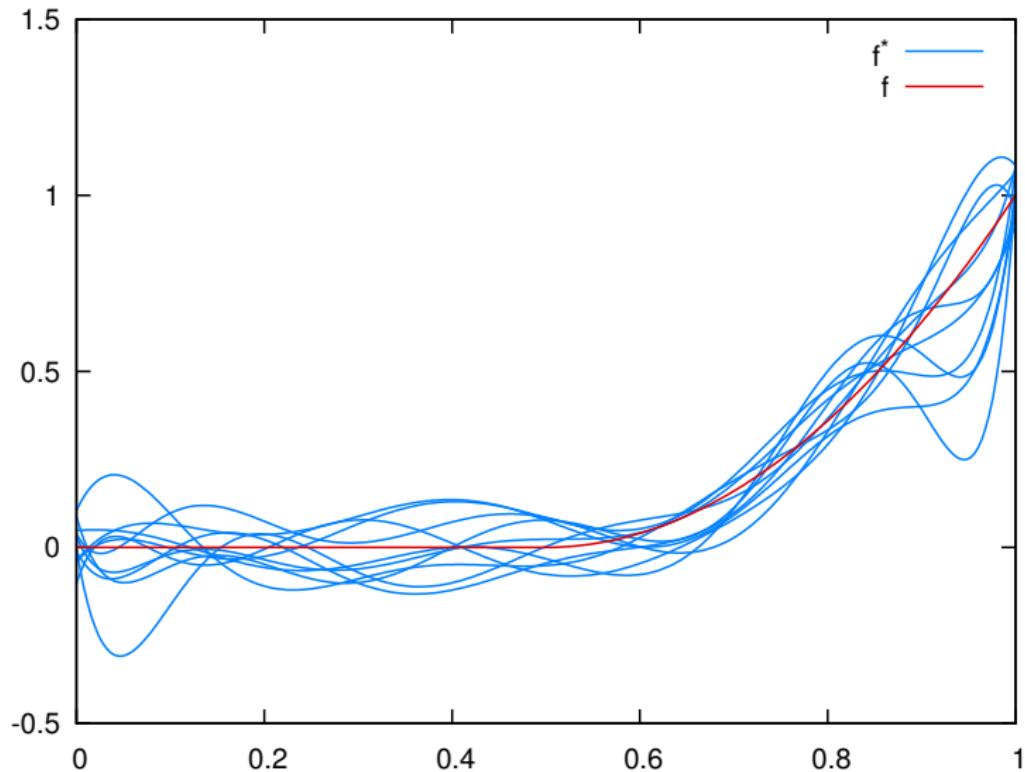
$D=9, \rho=1e-10$



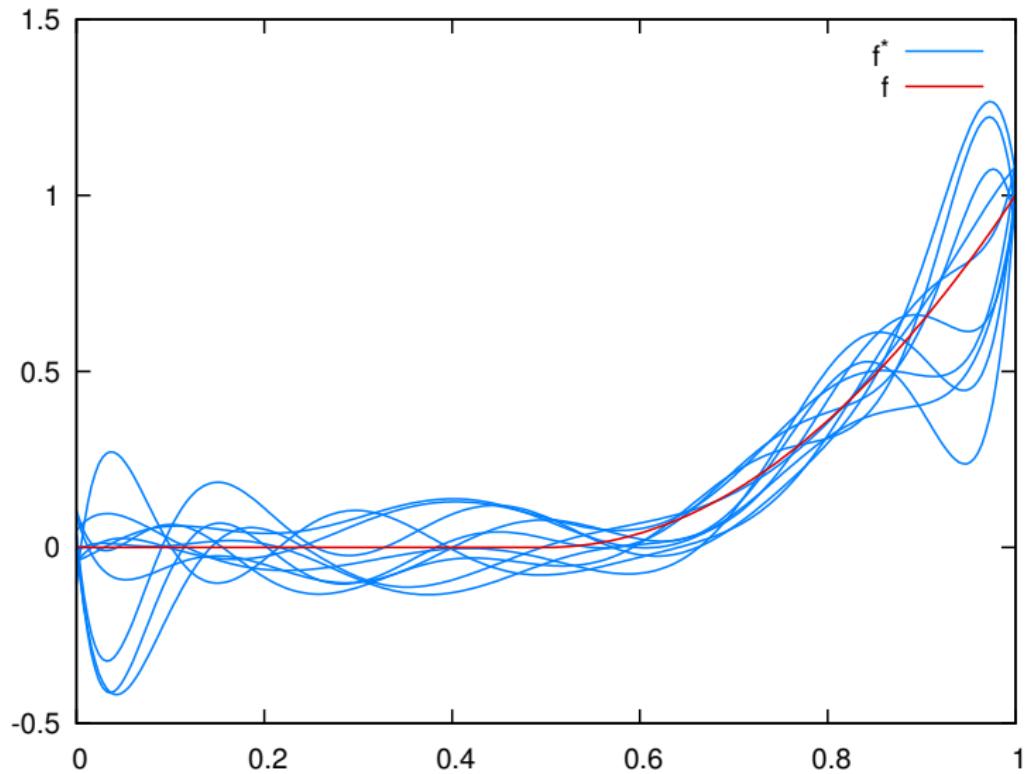
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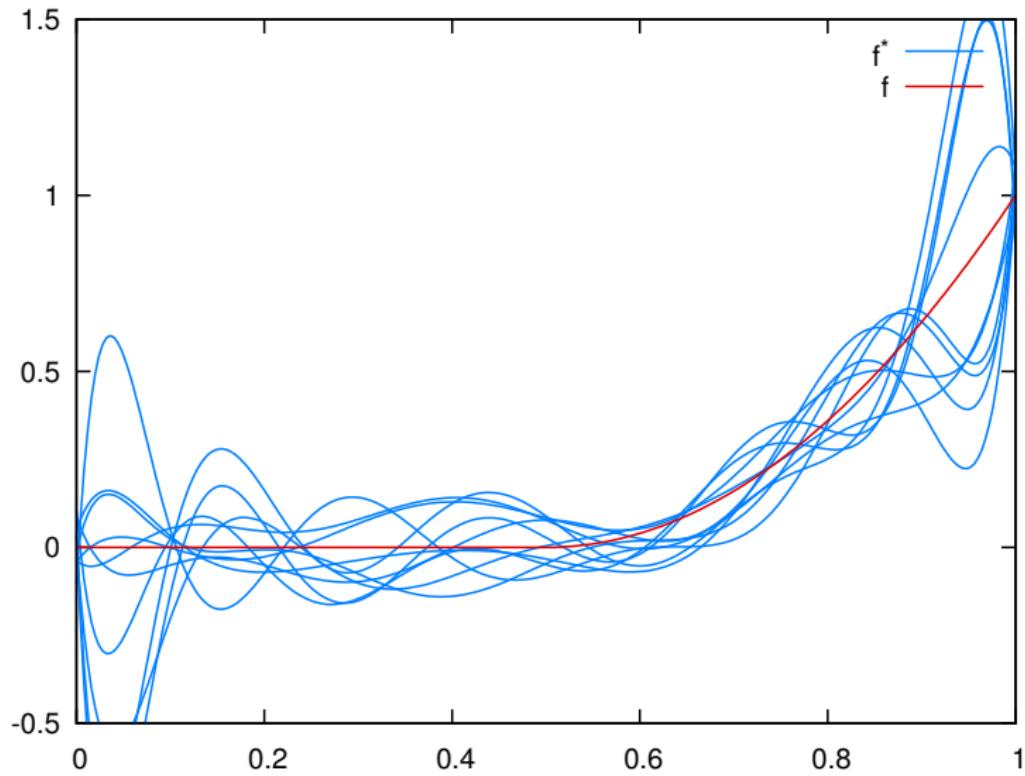
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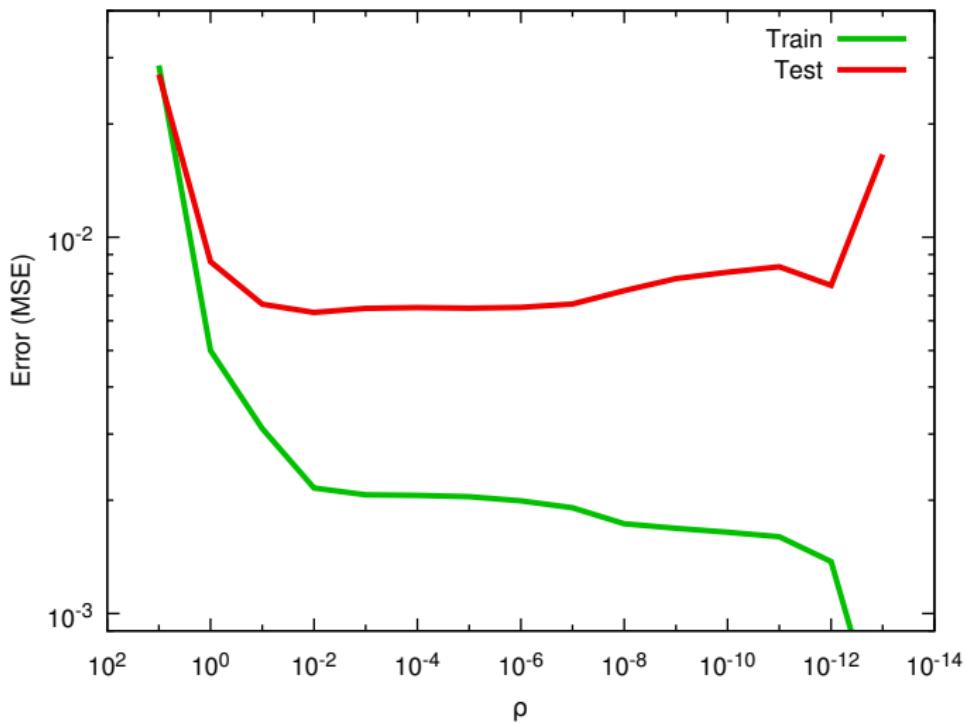


$D=9, \rho=1e-13$



$D=9, \rho=0.0$





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A mathematically precise notion is the Vapnik–Chervonenkis dimension of a set of functions, which, in the Binary classification case, is the cardinality of the largest set that can be labeled arbitrarily (Vapnik, 1995).

It is a very powerful concept, but is poorly adapted to neural networks. We will not say more about it in this course.

Although the capacity is hard to define precisely, it is quite clear in practice how to modulate it for a given class of models.

In particular one can control over-fitting either by

- Reducing the space \mathcal{F} (less functionals, constrained or degraded optimization), or
- Making the choice of f^* less dependent on data (penalty on coefficients, margin maximization, ensemble methods).

The end

References

V. N. Vapnik. *The Nature of Statistical Learning Theory*. Springer-Verlag, New York, 1995.

EE-559 – Deep learning

2.3. Bias-variance dilemma

François Fleuret

<https://fleuret.org/ee559/>

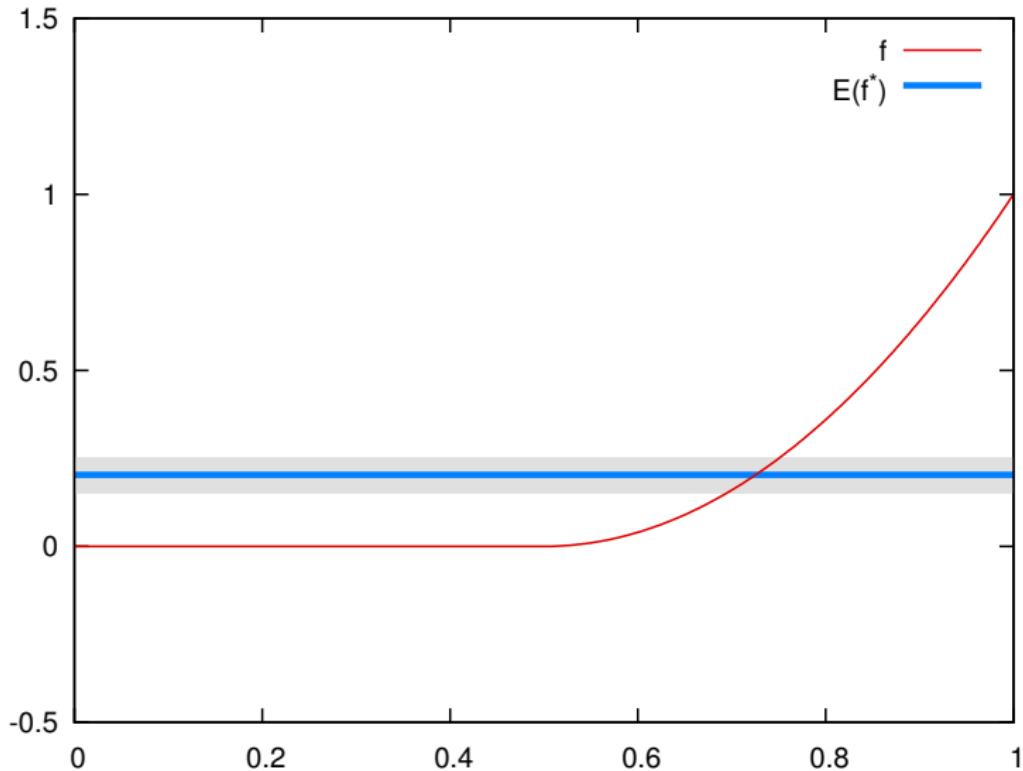
Mon Feb 18 13:33:39 UTC 2019

We can visualize over-fitting for our polynomial regression by generating multiple training sets $\mathcal{D}_1, \dots, \mathcal{D}_M$, training as many models f_1, \dots, f_M , and computing empirically the mean and standard deviation of the prediction at every point.

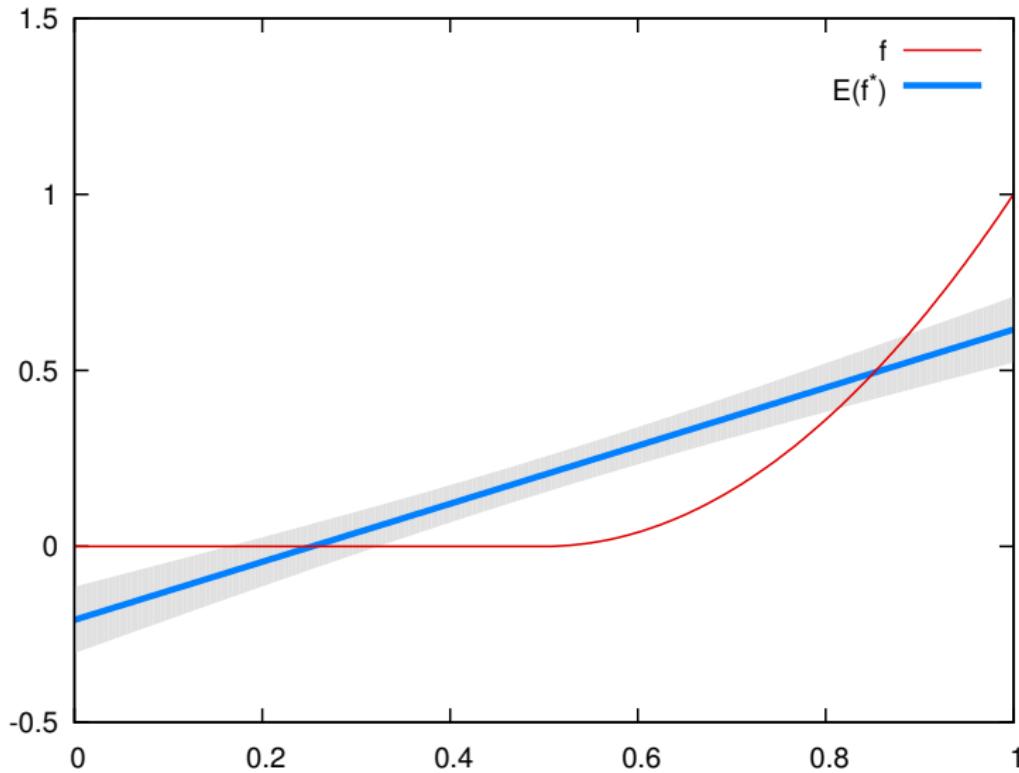
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When the capacity increases or regularization decreases, the mean of the predicted value gets right on target, but the prediction varies more across runs.

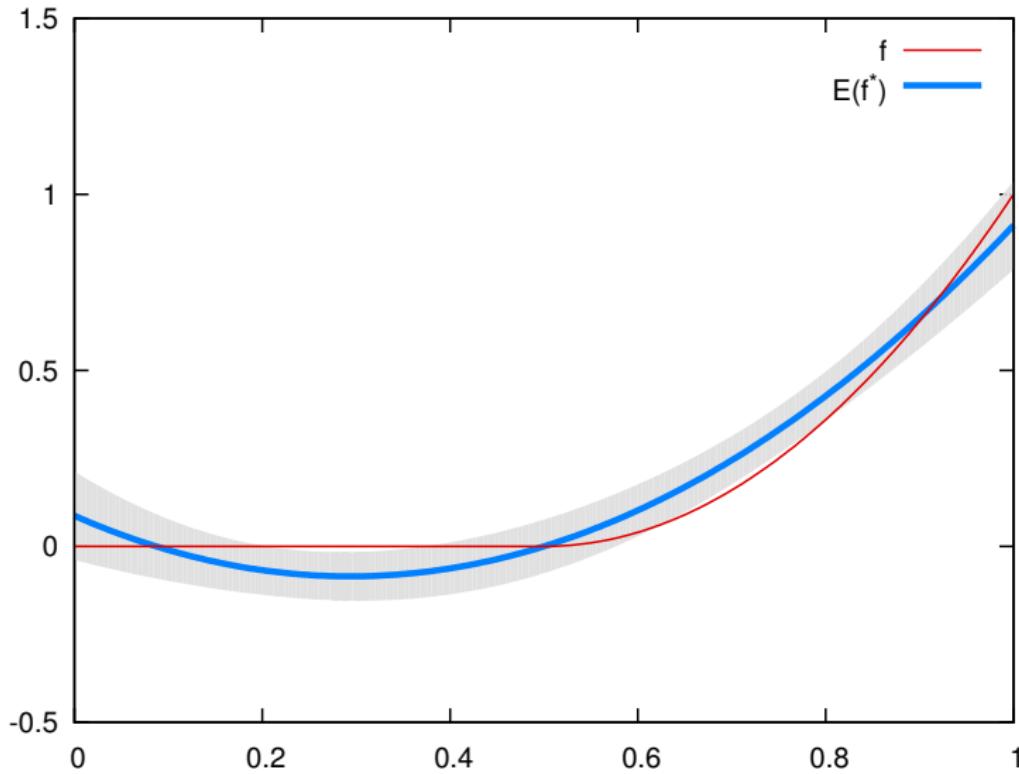
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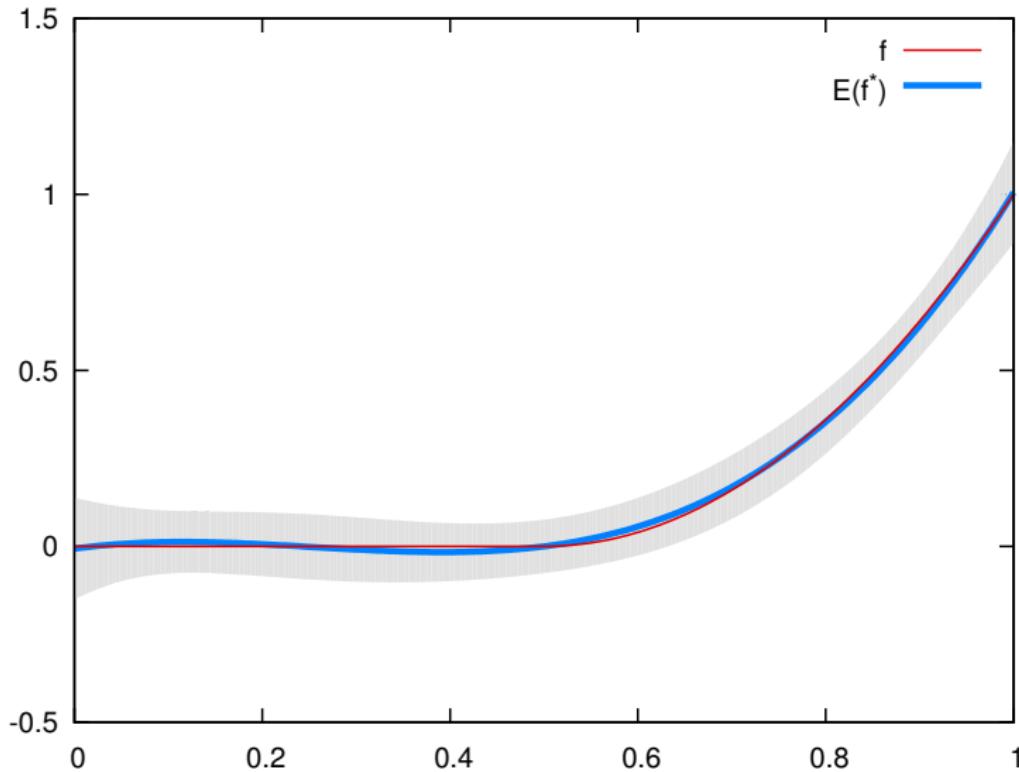
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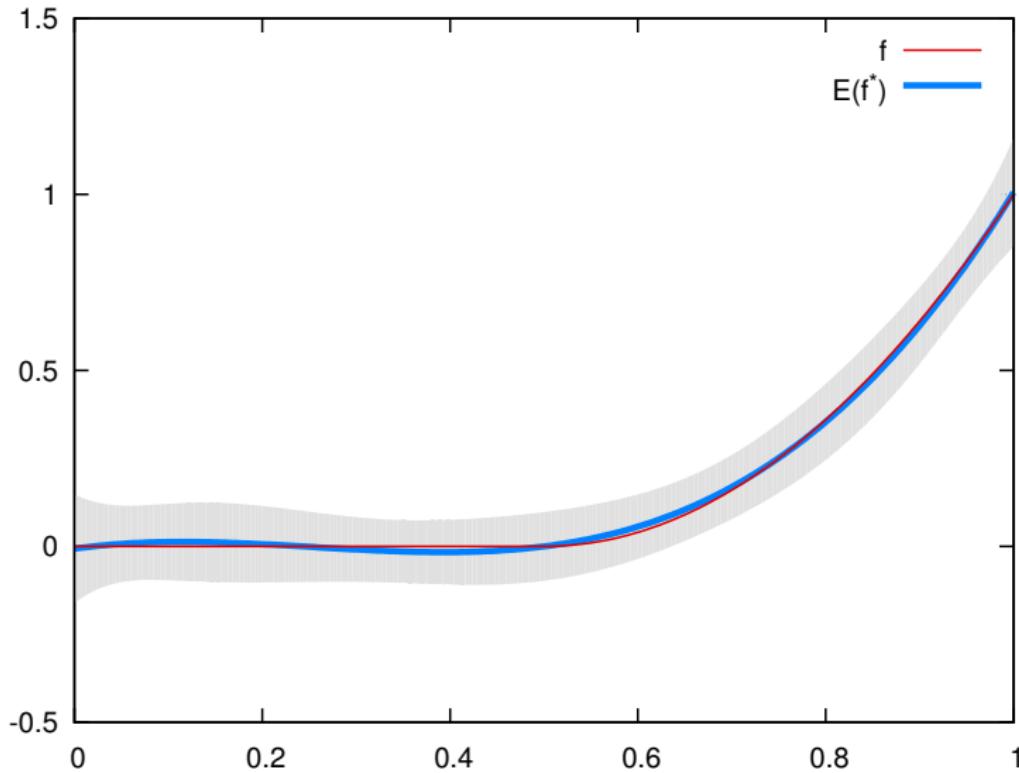
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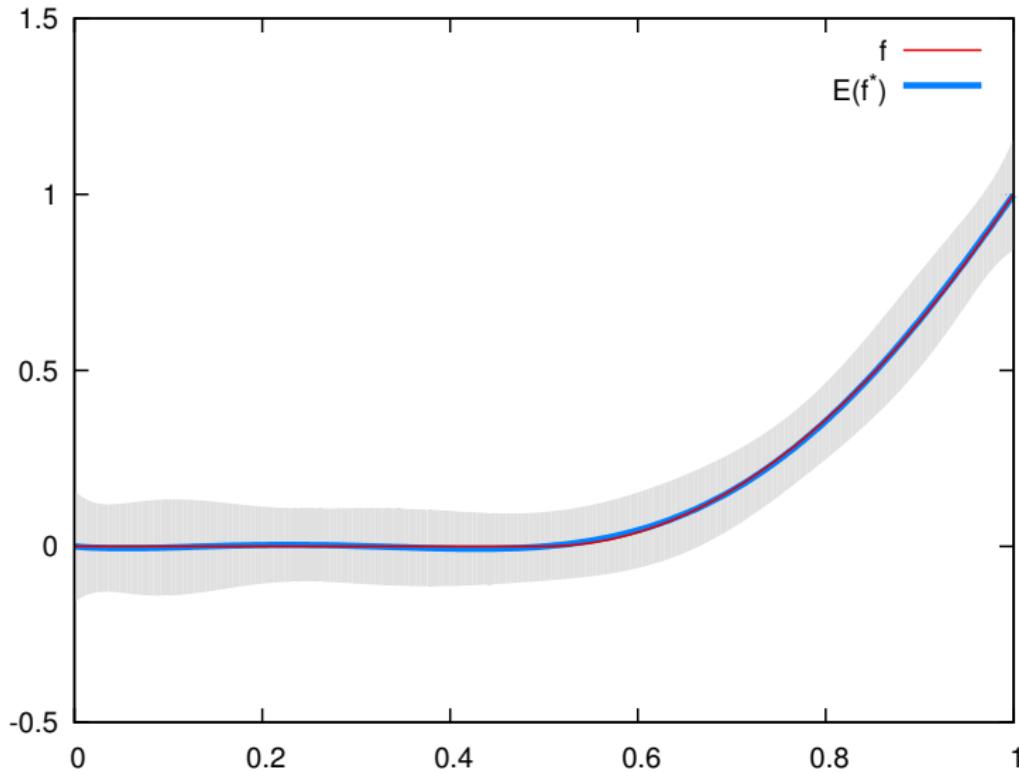
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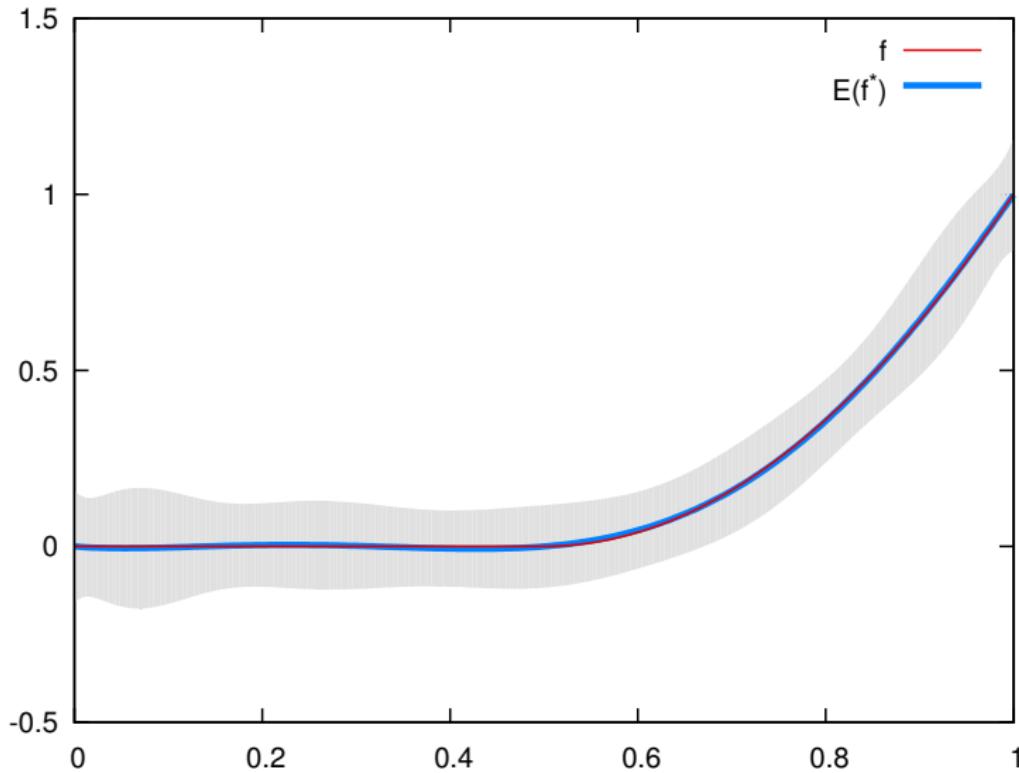
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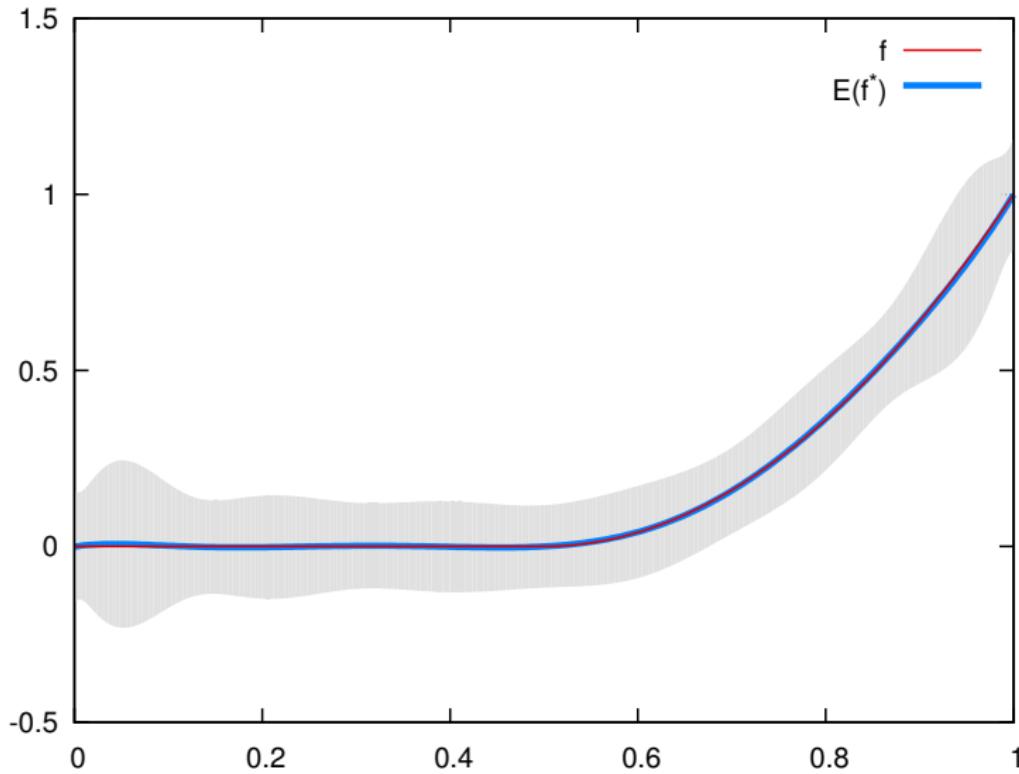
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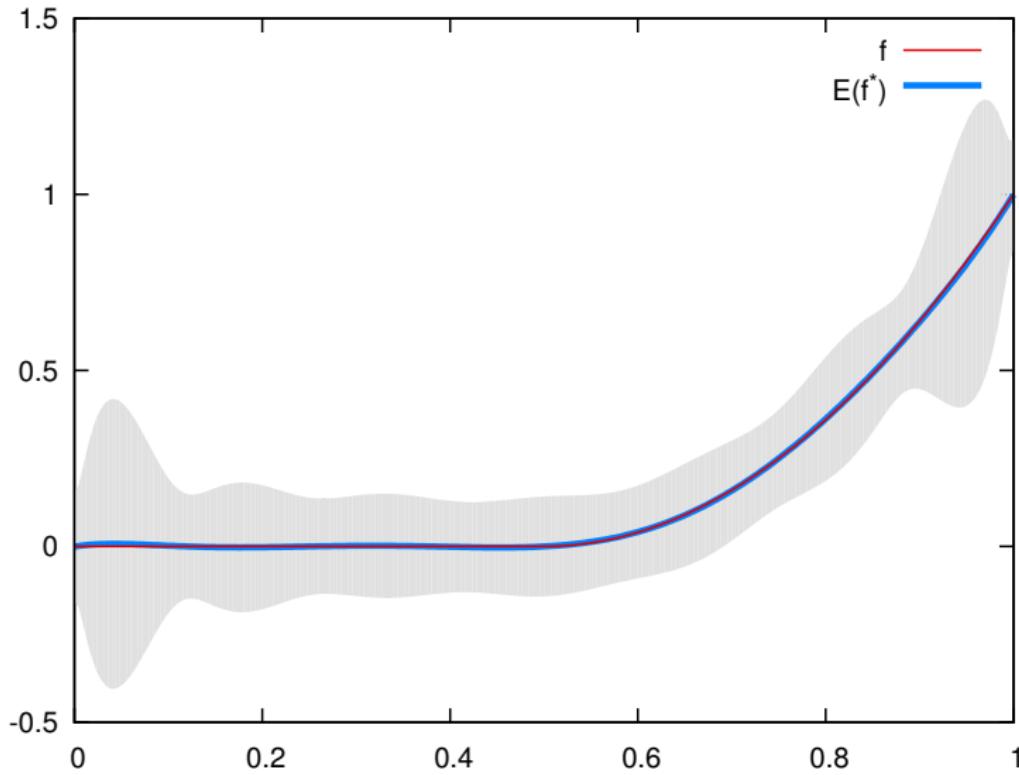
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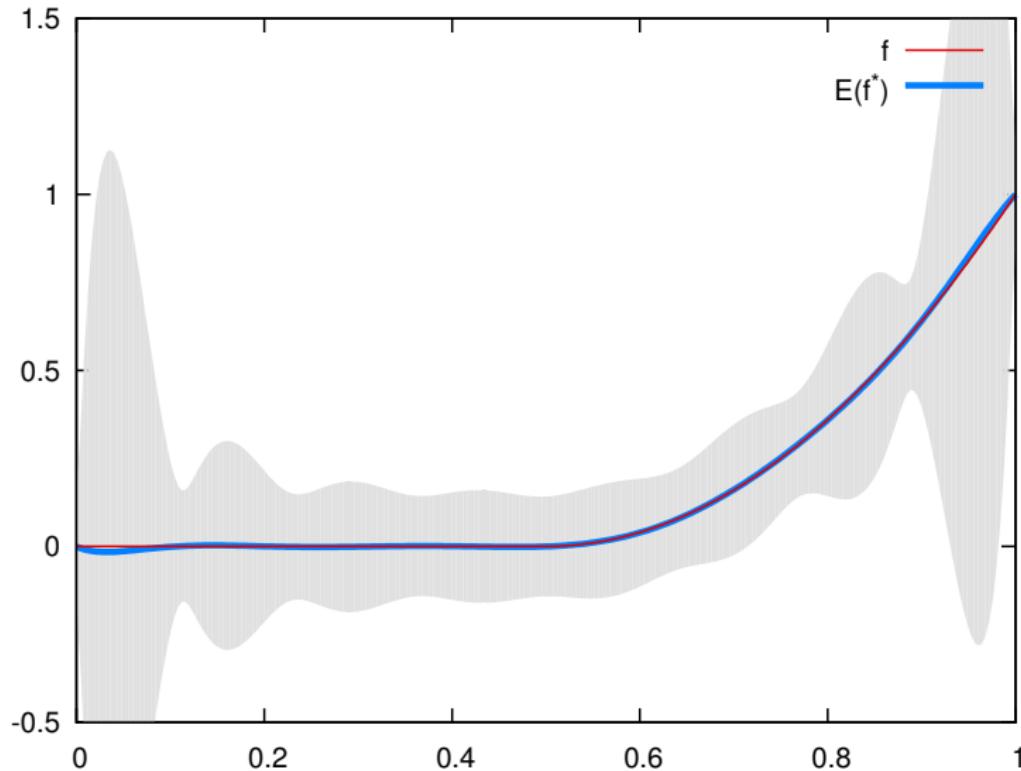
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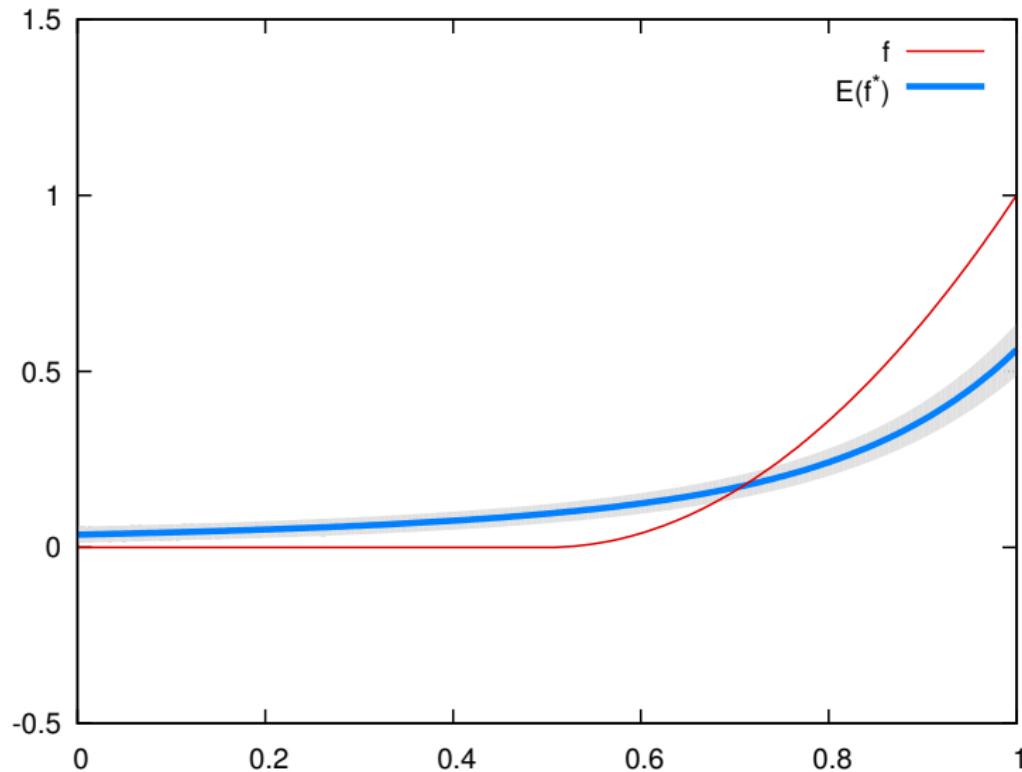
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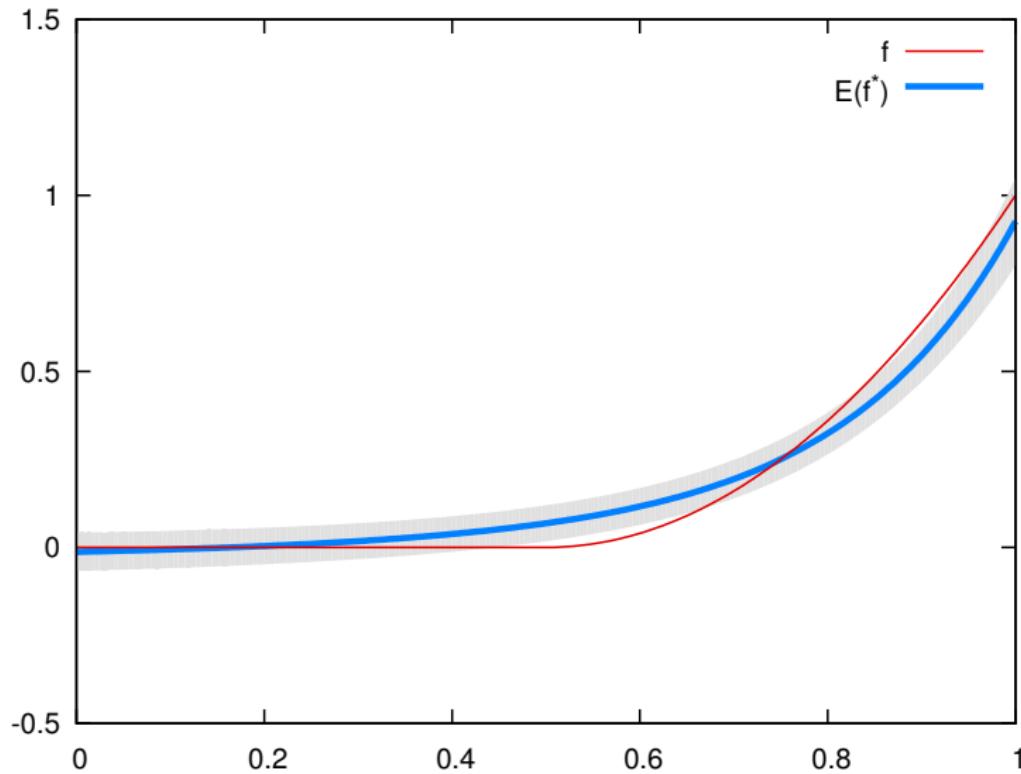
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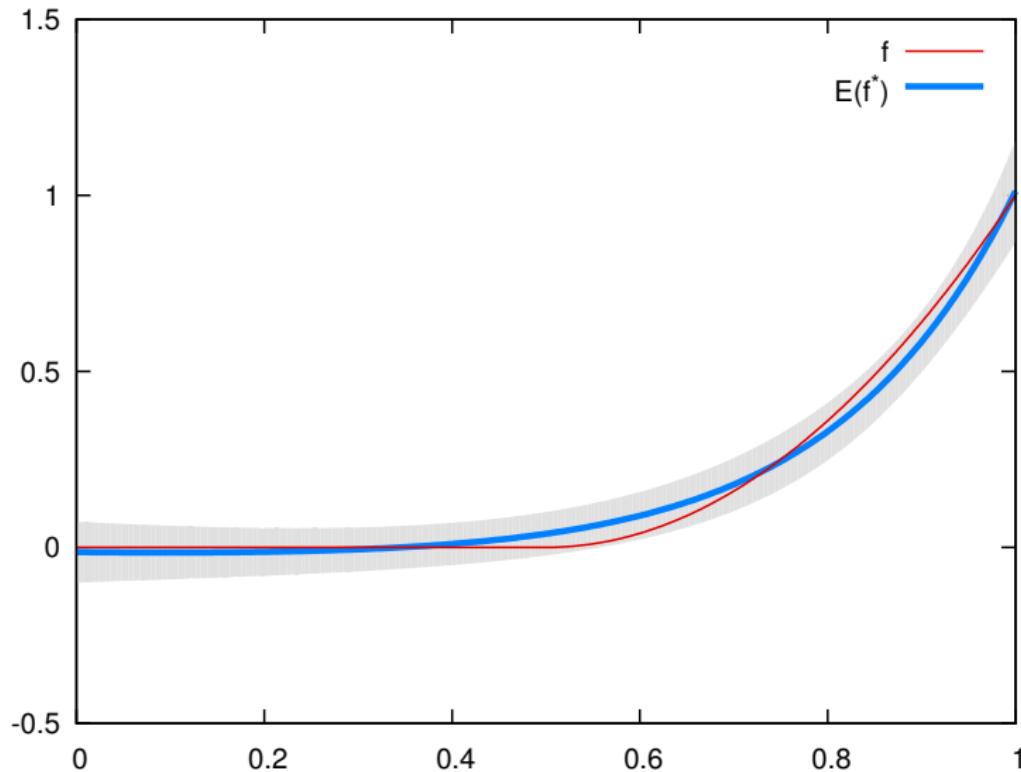
$D=9, \rho=1e1$



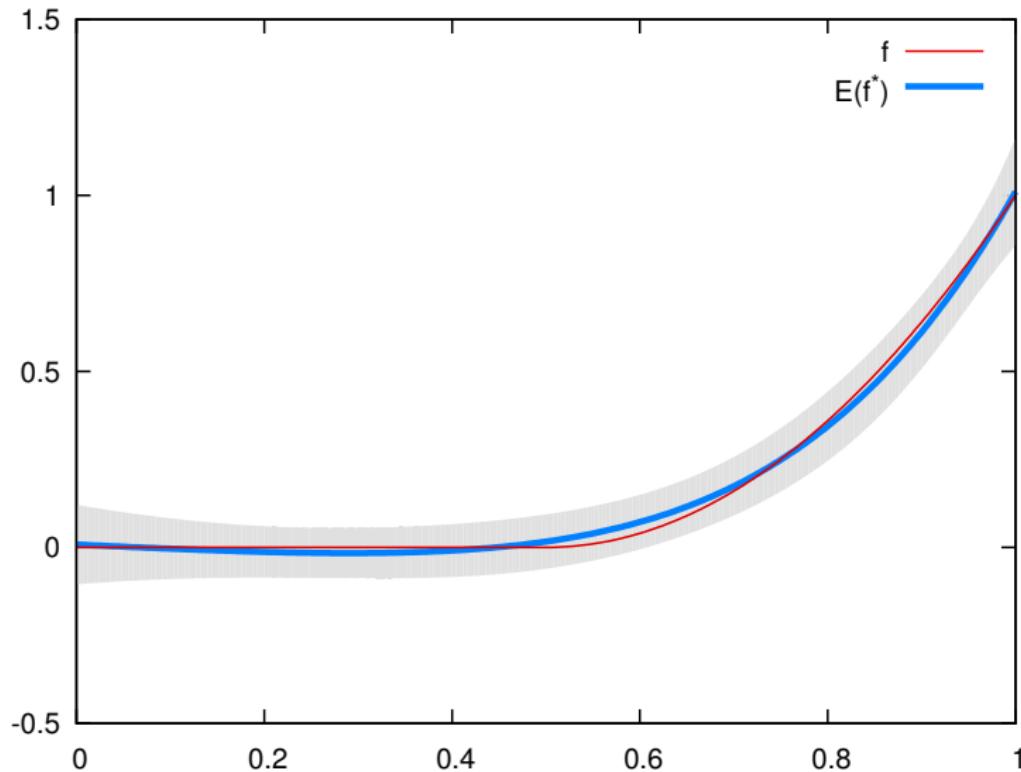
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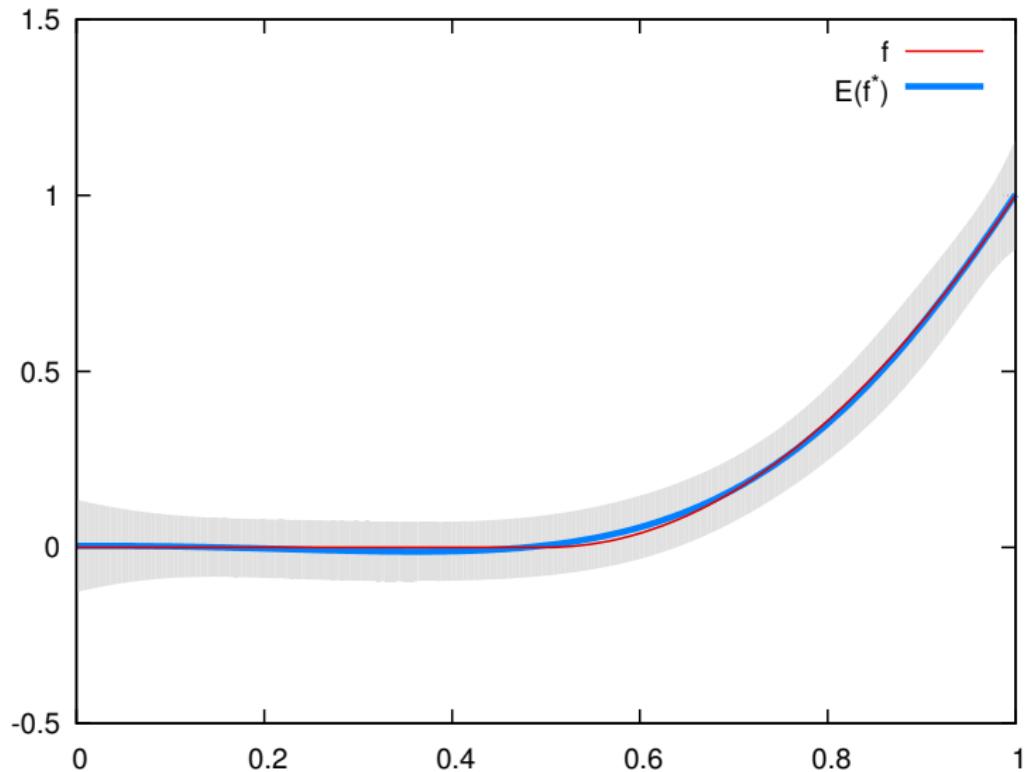
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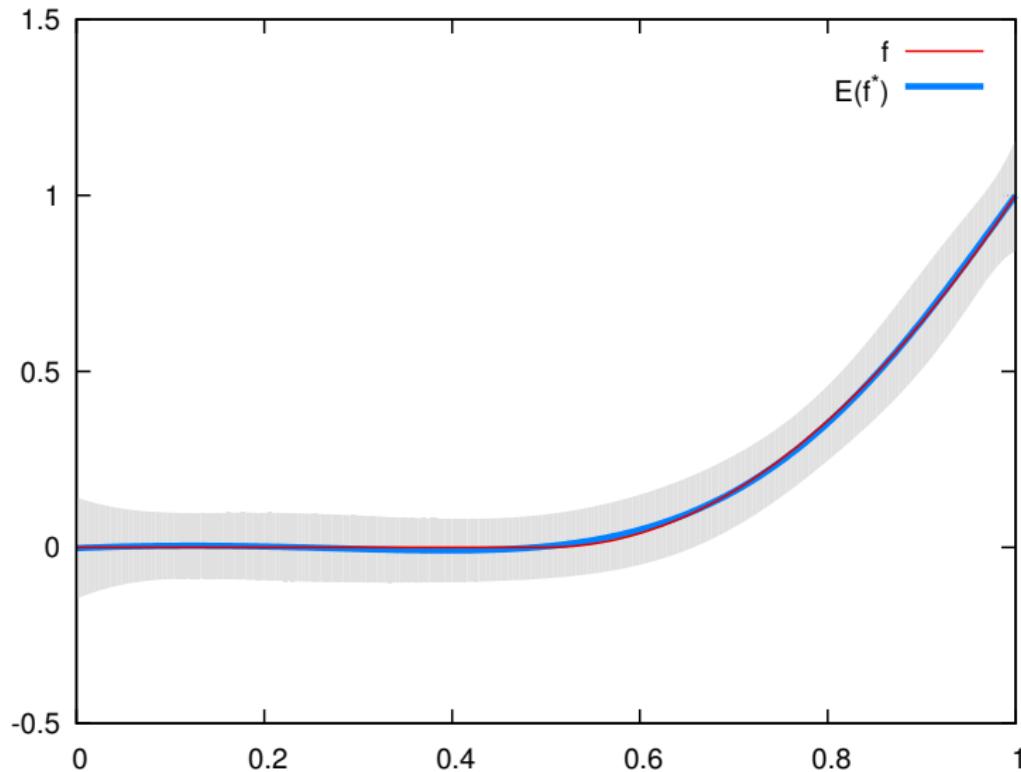
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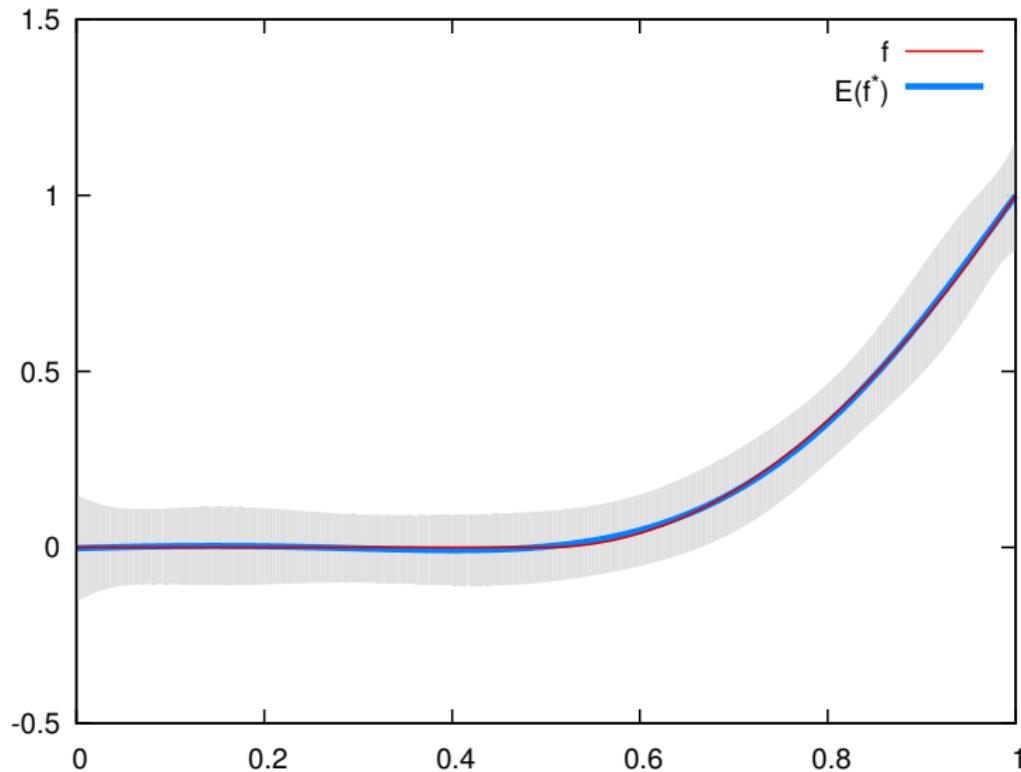
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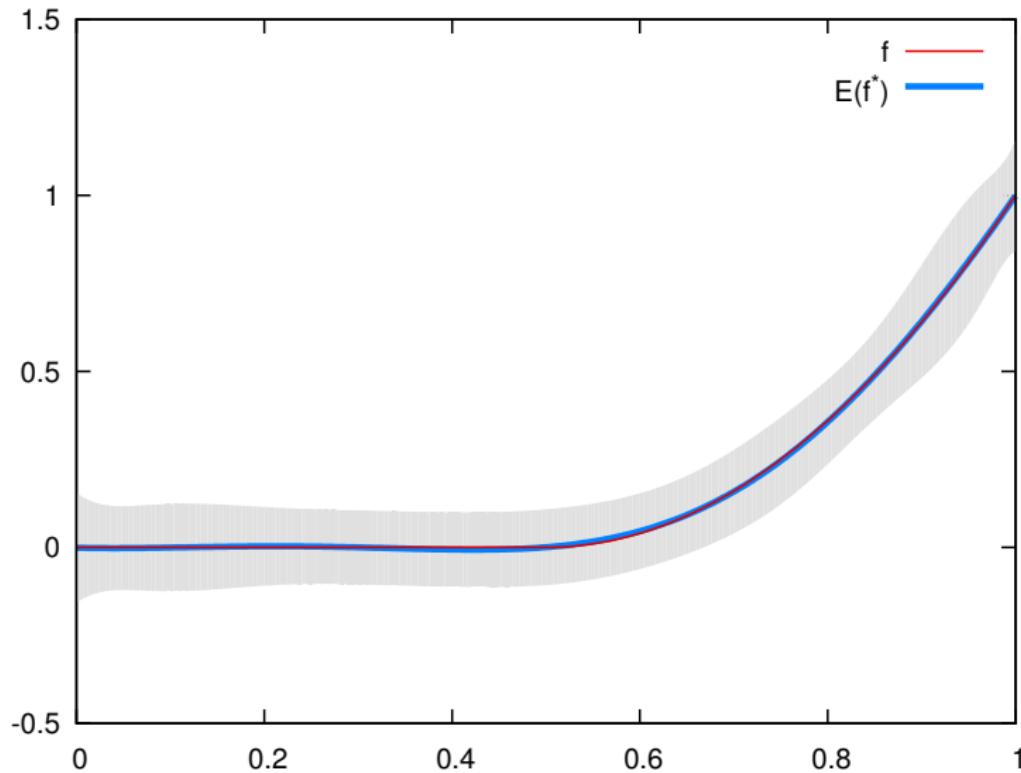
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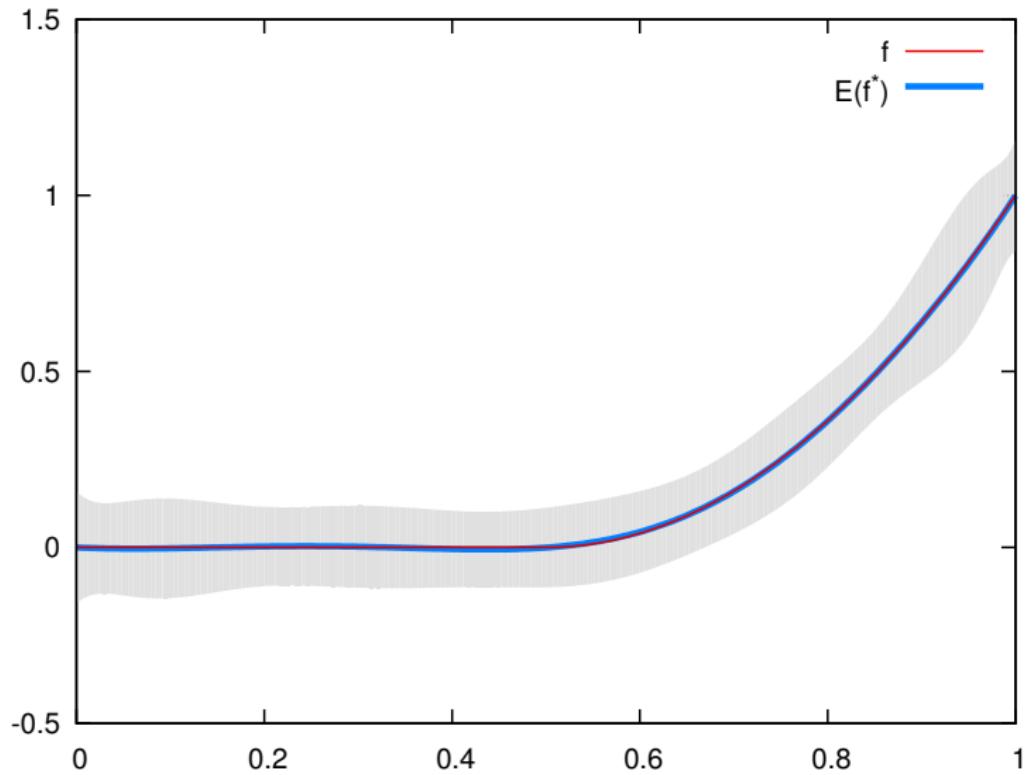
$D=9, \rho=1e-5$



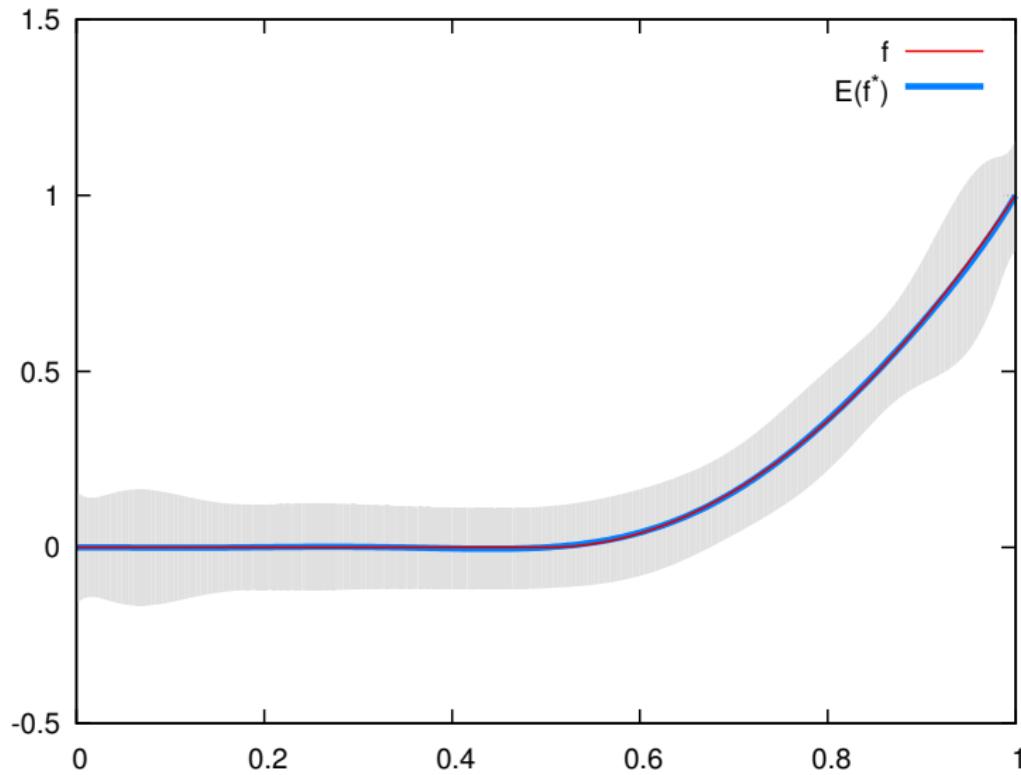
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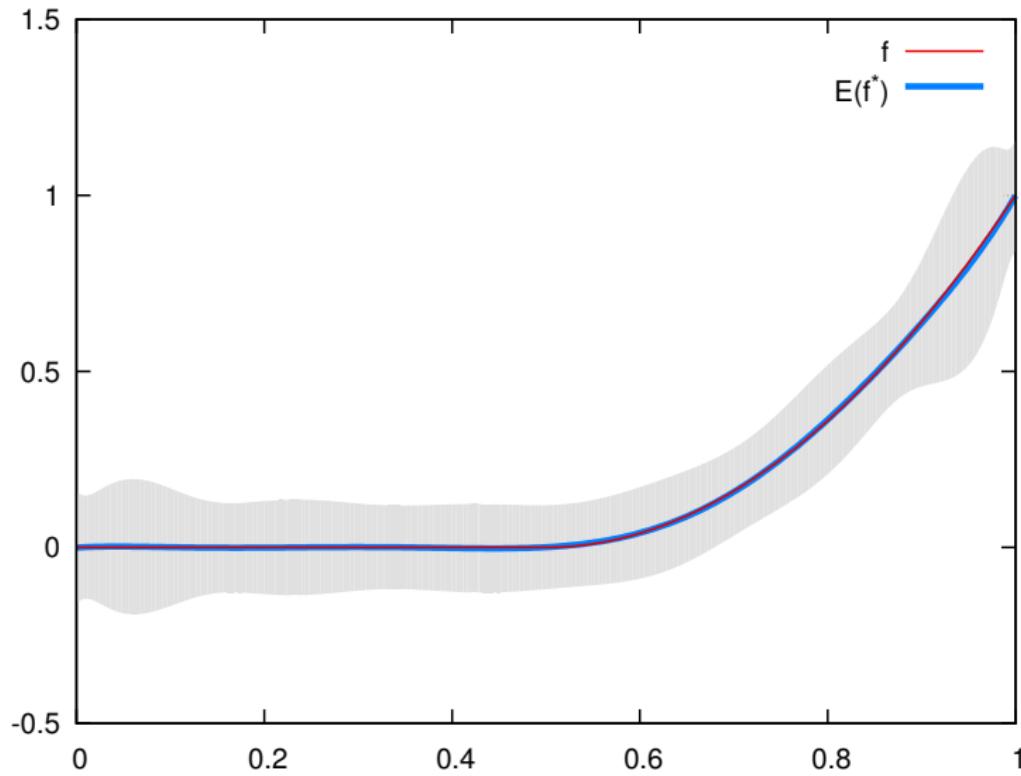
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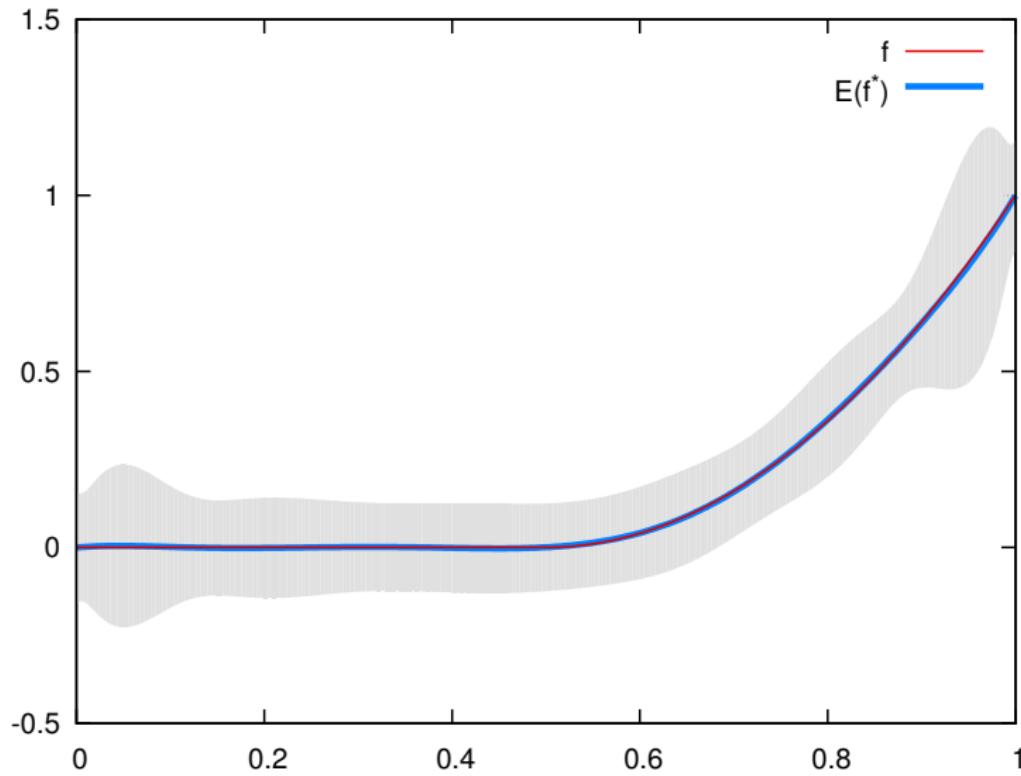
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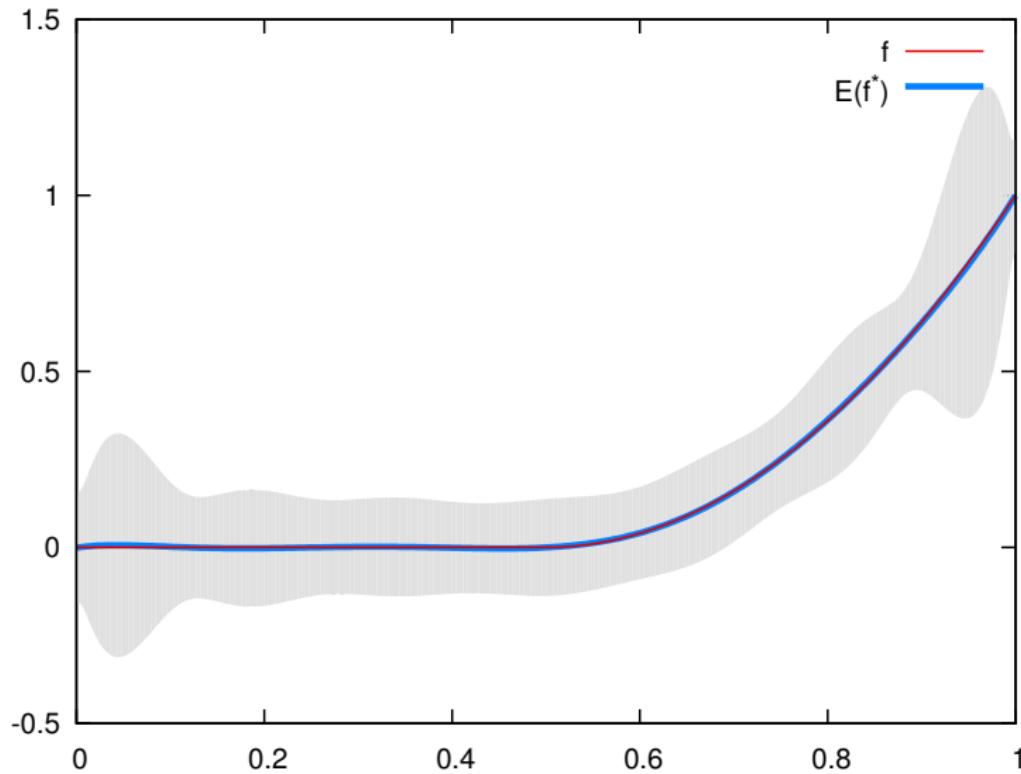
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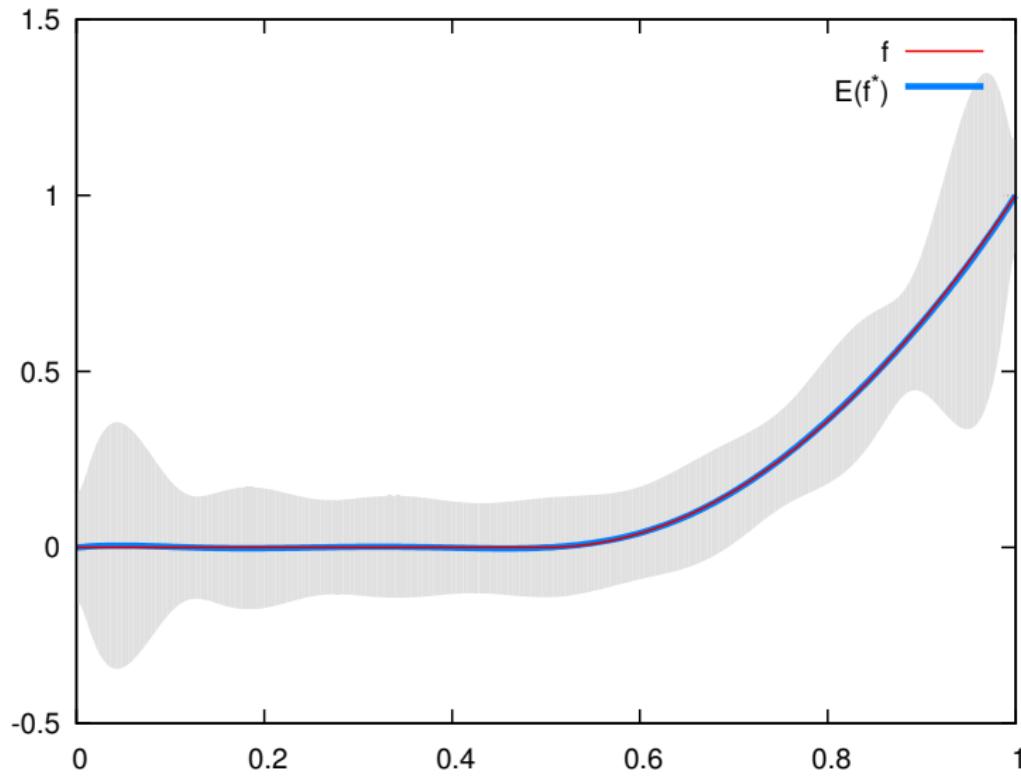
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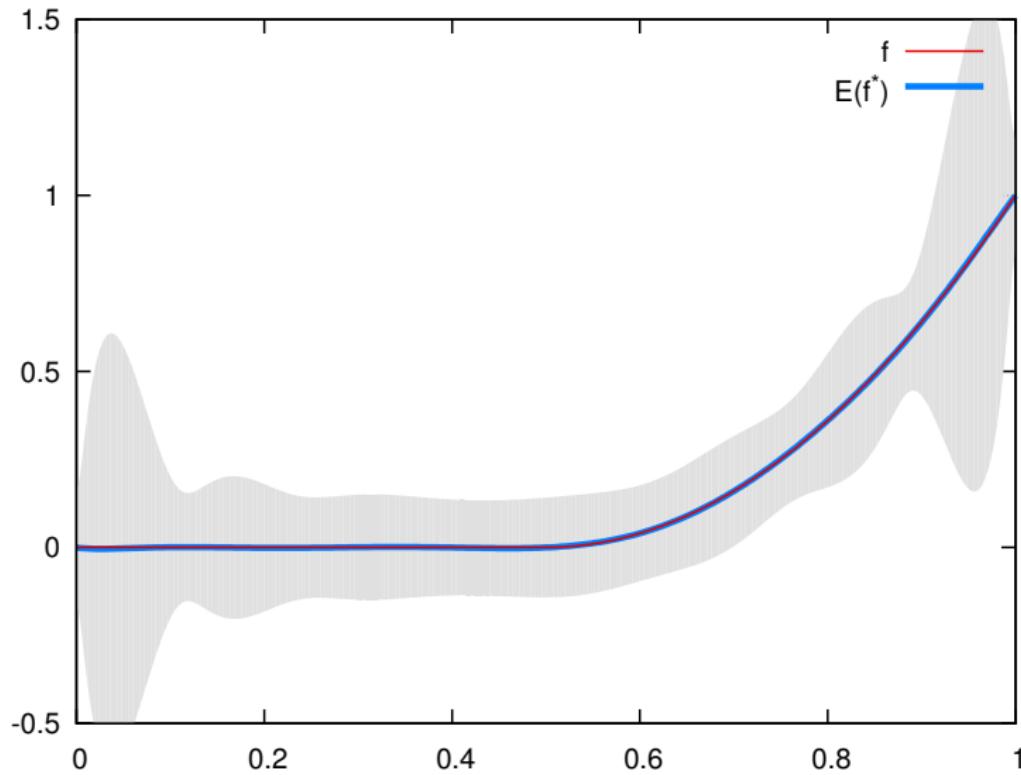
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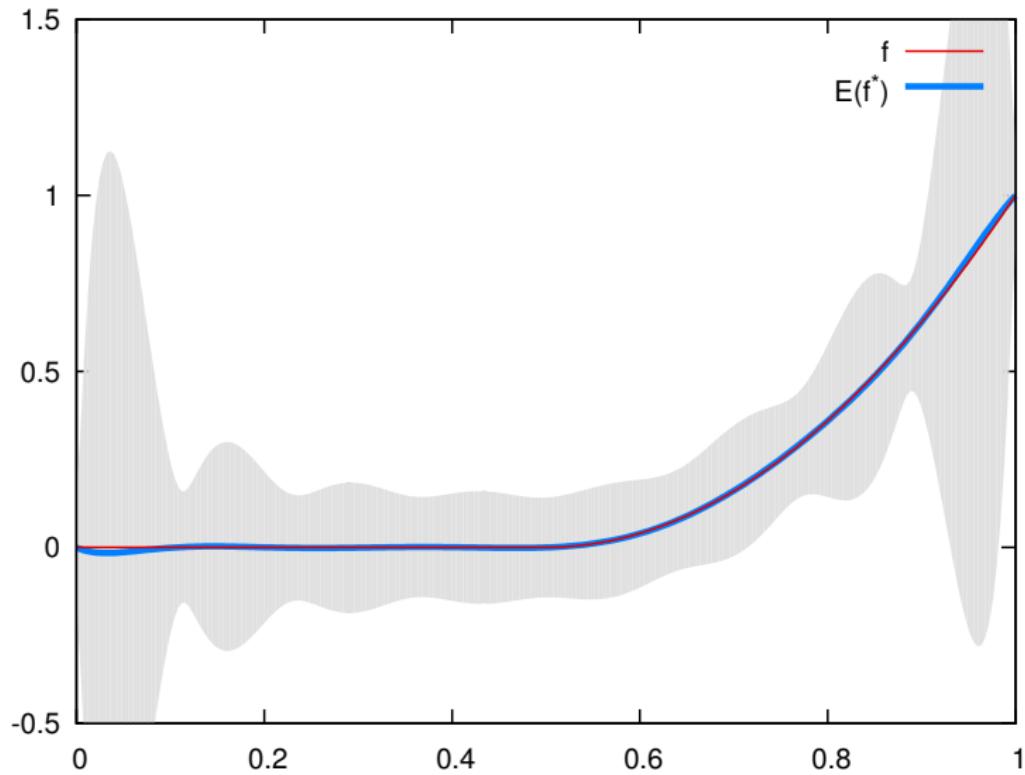
$D=9, \rho=1e-12$



$D=9, \rho=1e-13$



$D=9, \rho=0.0$



We can formalize these observations as follows:

Let x be fixed, y the “true” value associated to it, f^* the predictor we learned from the data-set \mathcal{D} , and $Y = f^*(x)$ be the value we predict at x .

If we consider that the training set \mathcal{D} is a random quantity, then f^* is random, and consequently Y is.

We have

$$\mathbb{E}_{\mathcal{D}} ((Y - y)^2)$$

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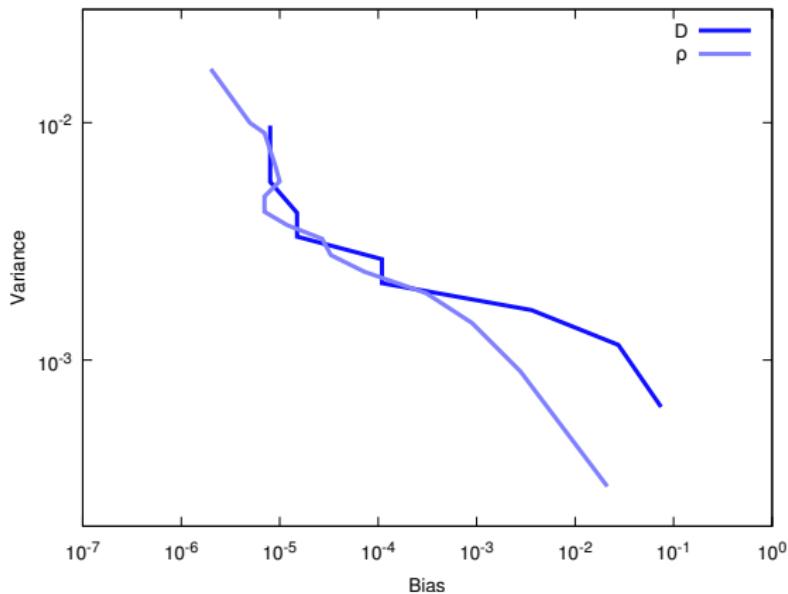
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This is the **bias-variance decomposition**:

- the bias term quantifies how much the model fits the data on average,
- the variance term quantifies how much the model changes across data-sets.

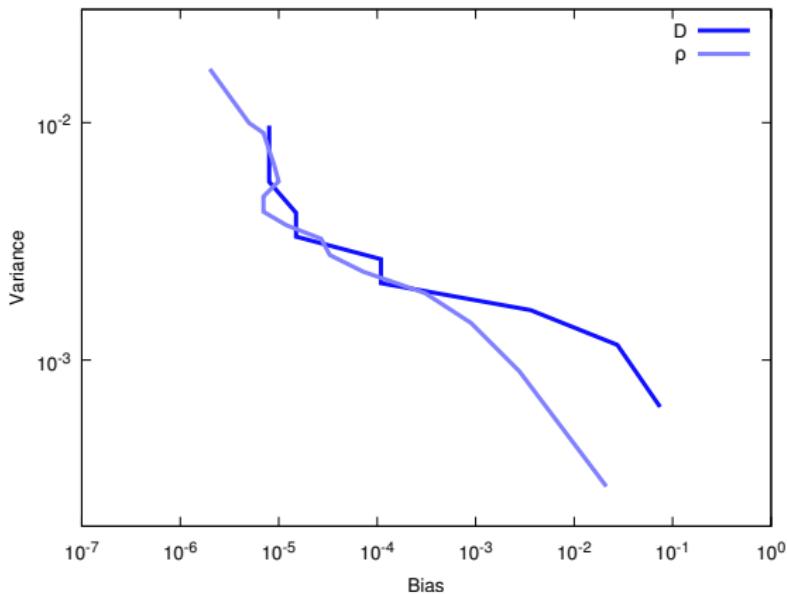
(Geman and Bienenstock, 1992)

From this comes the **bias variance tradeoff**:



Reducing the capacity makes f^* fit the data less on average, which increases the bias term.

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Reducing the capacity makes f^* fit the data less on average, which increases the bias term. Increasing the capacity makes f^* vary a lot with the training data, which increases the variance term.

Is all this probabilistic?

Conceptually model-fitting and regularization can be interpreted as Bayesian inference.

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By looking at the data \mathcal{D} , we can estimate a posterior distribution for the said parameters,

$$\mu_A(\alpha | \mathcal{D} = \mathbf{d}) \propto \mu_{\mathcal{D}}(\mathbf{d} | A = \alpha) \mu_A(\alpha),$$

and from that their most likely values.

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and from that their most likely values.

So instead of a penalty term, we define a prior distribution, which is usually more intellectually satisfying.

For instance, consider a polynomial model with Gaussian prior, that is

$$\forall n, \quad Y_n = \sum_{d=0}^D A_d X_n^d + \Delta_n,$$

where

$$\forall d, \quad A_d \sim \mathcal{N}(0, \xi), \quad \forall n, \quad X_n \sim \mu_X, \quad \Delta_n \sim \mathcal{N}(0, \sigma)$$

all independent.

For instance, consider a polynomial model with Gaussian prior, that is

$$\forall n, \quad Y_n = \sum_{d=0}^D A_d X_n^d + \Delta_n,$$

where

$$\forall d, \quad A_d \sim \mathcal{N}(0, \xi), \quad \forall n, \quad X_n \sim \mu_X, \quad \Delta_n \sim \mathcal{N}(0, \sigma)$$

all independent.

For clarity, let $A = (A_0, \dots, A_D)$ and $\alpha = (\alpha_0, \dots, \alpha_D)$.

Remember that $\mathcal{D} = \{(X_1, Y_1), \dots, (X_N, Y_N)\}$ is the (random) training set and $\mathbf{d} = \{(x_1, y_1), \dots, (x_N, y_N)\}$ is a realization.

$$\log \mu_A(\alpha \mid \mathcal{D} = \mathbf{d})$$

$$\begin{aligned} & \log \mu_A(\alpha \mid \mathcal{D} = \mathbf{d}) \\ &= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid A = \alpha) \mu_A(\alpha)}{\mu_{\mathcal{D}}(\mathbf{d})} \end{aligned}$$

$$\begin{aligned}\log \mu_A(\alpha \mid \mathcal{D} = \mathbf{d}) \\ &= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid A = \alpha) \mu_A(\alpha)}{\mu_{\mathcal{D}}(\mathbf{d})} \\ &= \log \mu_{\mathcal{D}}(\mathbf{d} \mid A = \alpha) + \log \mu_A(\alpha) - \log Z\end{aligned}$$

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&= \log \prod_n \mu(x_n, y_n \mid A = \alpha) + \log \mu_A(\alpha) - \log Z \\
&= \log \prod_n \mu(y_n \mid X_n = x_n, A = \alpha) \underbrace{\mu(x_n \mid A = \alpha)}_{= \mu(x_n)} + \log \mu_A(\alpha) - \log Z
\end{aligned}$$

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&= \log \prod_n \mu(y_n \mid X_n = x_n, A = \alpha) + \log \mu_A(\alpha) - \log Z'
\end{aligned}$$

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&= \log \prod_n \mu(y_n \mid X_n = x_n, A = \alpha) + \log \mu_A(\alpha) - \log Z' \\
&= - \underbrace{\frac{1}{2\sigma^2} \sum_n \left(y_n - \sum_d \alpha_d x_n^d \right)^2}_{\text{Gaussian noise on } Y} - \underbrace{\frac{1}{2\xi^2} \sum_d \alpha_d^2}_{\text{Gaussian prior on } A} - \log Z''.
\end{aligned}$$

Taking $\rho = \sigma^2/\xi^2$ gives the penalty term of the previous slides.

$$\begin{aligned}
& \log \mu_A(\alpha \mid \mathcal{D} = \mathbf{d}) \\
&= \log \frac{\mu_{\mathcal{D}}(\mathbf{d} \mid A = \alpha) \mu_A(\alpha)}{\mu_{\mathcal{D}}(\mathbf{d})} \\
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\end{aligned}$$

Taking $\rho = \sigma^2/\xi^2$ gives the penalty term of the previous slides.

Regularization seen through that prism is intuitive: The stronger the prior, the more evidence you need to deviate from it.

The end

References

- S. Geman and E. Bienenstock. Neural networks and the bias/variance dilemma. *Neural Computation*, 4:1–58, 1992.

EE-559 – Deep learning

2.4. Proper evaluation protocols

François Fleuret

<https://fleuret.org/ee559/>

Mon Feb 18 13:33:44 UTC 2019

Learning algorithms, in particular deep-learning ones, require the tuning of many meta-parameters.

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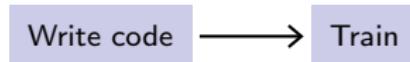
These parameters have a strong impact on the performance, resulting in a “meta” over-fitting through experiments.

We must be extra careful with performance estimation.

Running 100 times the same experiment on MNIST, with randomized weights, we get:

Worst	Median	Best
1.3%	1.0%	0.82%

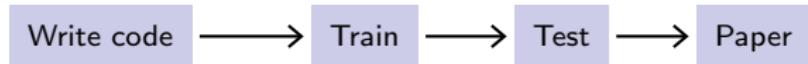
The ideal development cycle is



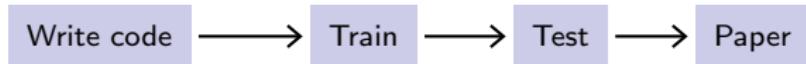
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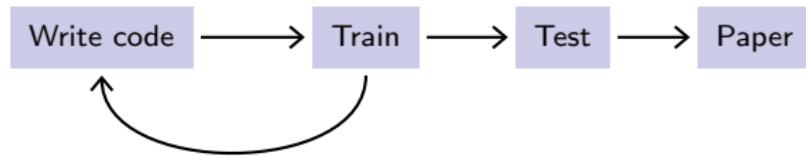
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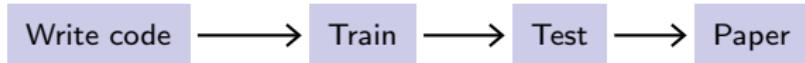
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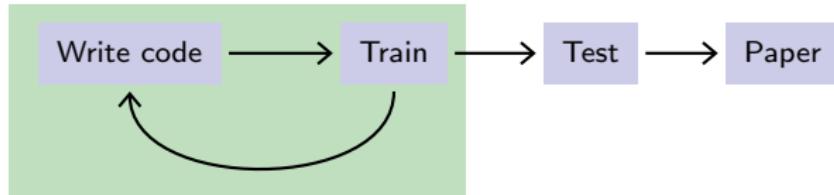
or in practice something like



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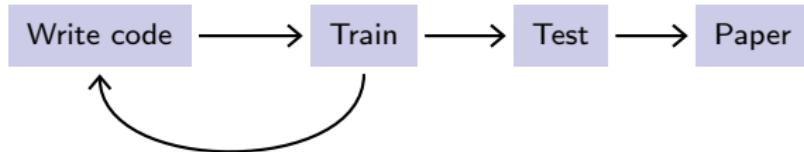


or in practice something like

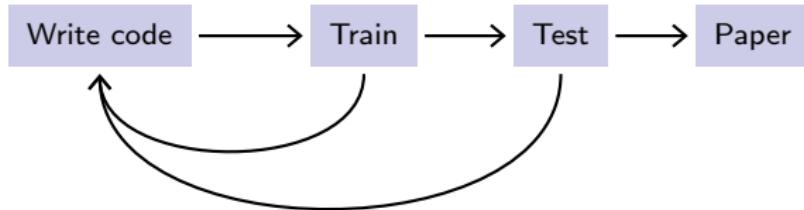


There may be over-fitting, but it does not bias the final performance evaluation.

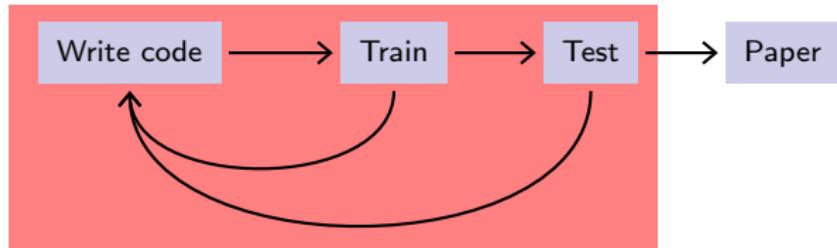
Unfortunately, it often looks like



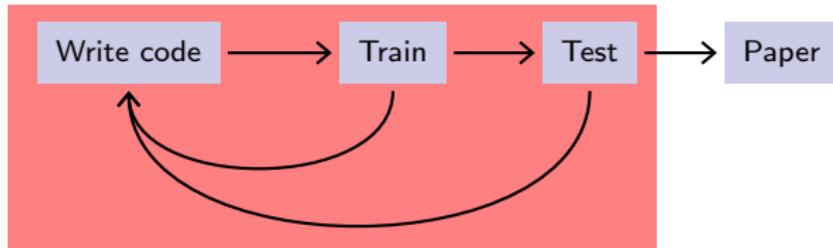
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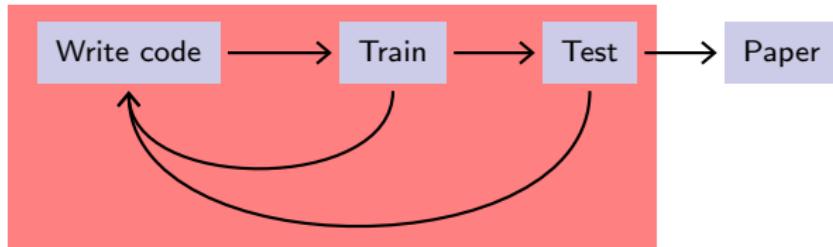


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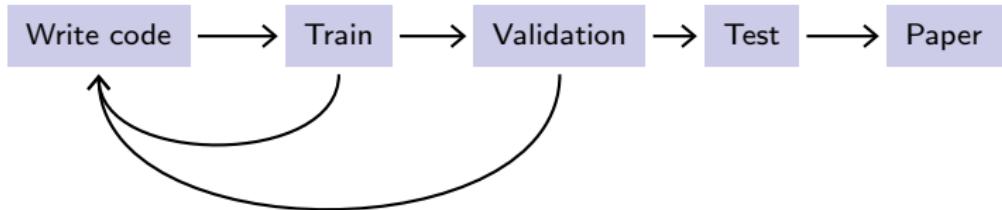


This should be avoided at all costs. The standard strategy is to have a separate validation set for the tuning.

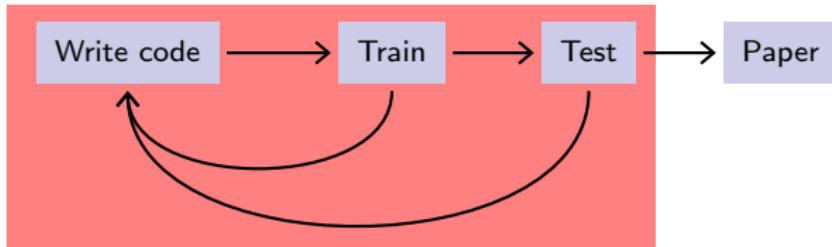
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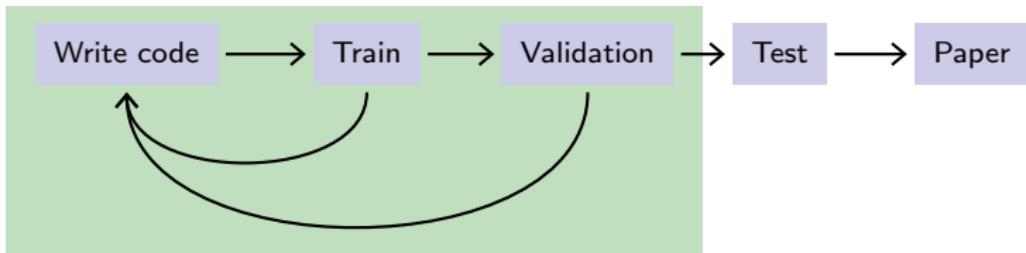
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When data is scarce, one can use cross-validation: average through multiple random splits of the data in a train and a validation sets.

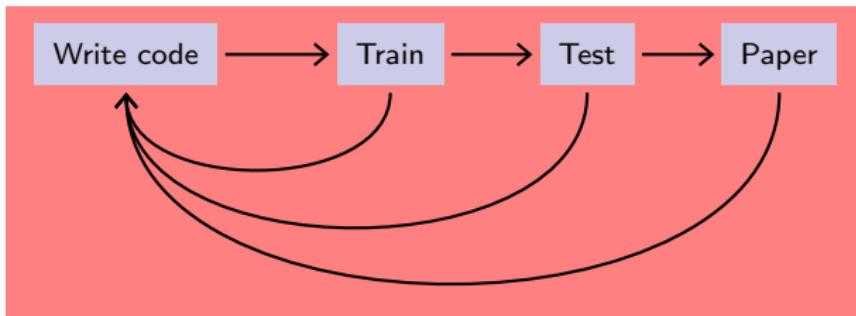
When data is scarce, one can use cross-validation: average through multiple random splits of the data in a train and a validation sets.

There is no unbiased estimator of the variance of cross-validation valid under all distributions (Bengio and Grandvalet, 2004).

Some data-sets (MNIST!) have been used by thousands of researchers, over millions of experiments, in hundreds of papers.

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The global overall process looks more like



“Cheating” in machine learning, from bad to “are you kidding?”:

- “Early evaluation stopping”,
- meta-parameter (over-)tuning,
- data-set selection,
- algorithm data-set specific clauses,
- seed selection.

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Top-tier conferences are demanding regarding experiments, and are biased against “complicated” pipelines.

The community pushes toward accessible implementations, reference data-sets, leader boards, and constant upgrades of benchmarks.

The end

References

- Y. Bengio and Y. Grandvalet. No unbiased estimator of the variance of k-fold cross-validation. *Journal of Machine Learning Research (JMLR)*, 5:1089–1105, 2004.

EE-559 – Deep learning

2.5. Basic clustering and embeddings

François Fleuret

<https://fleuret.org/ee559/>

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Deep learning models combine embeddings and dimension reduction operations.

They parametrize and re-parametrize multiple times the input signal into representations that get more and more invariant and noise free.

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To get an intuition of how this is possible, we consider here two standard algorithms:

- K -means, and
- Principal Component Analysis (PCA).

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- K -means, and
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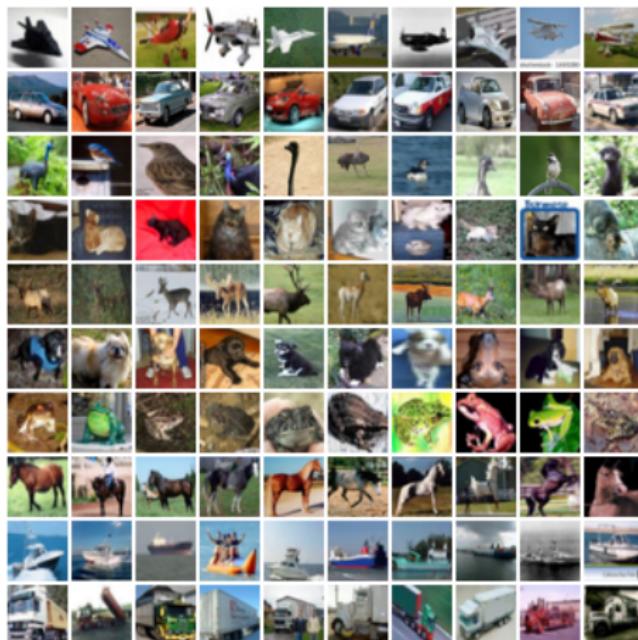
We will illustrate these methods on our two favorite data-sets.

MNIST data-set

1 1 8 3 6 1 0 3 1 0 0 1 1 2 7 3 0 4 6 5
2 6 4 7 1 8 9 9 3 0 7 1 0 2 0 3 5 4 6 5
8 6 3 7 5 8 0 9 1 0 3 1 2 2 3 3 6 4 7 5
0 6 2 7 9 8 5 9 2 1 1 4 4 5 6 4 1 2 5 3
9 3 9 0 5 9 6 5 7 4 1 3 4 0 4 8 0 4 3 6
8 7 6 0 9 7 5 7 2 1 1 6 8 9 4 1 5 2 2 9
0 3 9 6 7 2 0 3 5 4 3 4 5 8 9 5 4 7 4 2
1 3 4 8 9 1 9 2 8 7 9 1 8 7 4 1 3 1 1 0
2 3 9 4 9 2 1 6 8 4 1 7 4 4 9 2 8 7 2 4
4 2 1 9 7 2 8 7 6 9 2 3 8 1 6 5 1 1 0
4 0 9 1 1 2 4 3 2 7 3 8 6 9 0 5 6 0 7 6
2 6 4 5 8 3 1 5 1 9 2 7 4 4 4 8 1 5 8 9
5 6 7 9 9 3 7 0 9 0 6 6 2 3 9 0 7 5 4 8
0 9 4 1 2 8 7 1 2 6 1 0 3 0 1 1 8 2 0 3
9 4 0 5 0 6 1 7 7 8 1 9 2 0 5 1 2 2 7 3
5 4 9 7 1 8 3 9 6 0 3 1 1 2 6 3 5 7 6 8
3 9 5 8 5 7 4 1 1 3 1 7 5 5 5 2 5 8 7 0
9 7 7 5 0 9 0 0 8 9 2 4 8 1 6 1 6 5 1 8
3 4 0 5 5 8 3 6 2 3 9 2 1 1 5 2 1 3 2 8
7 3 7 2 4 6 9 7 2 4 2 8 1 1 3 8 4 0 6 5

28 × 28 grayscale images, 60k train samples, 10k test samples.

CIFAR10 data-set



32 × 32 color images, 50k train samples, 10k test samples.

(Krizhevsky, 2009, chap. 3)

Given

$$x_n \in \mathbb{R}^D, \quad n = 1, \dots, N,$$

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Given a point, the index of its closest centroid is a good coding.

Formally, [Lloyd's algorithm for] K -means (approximately) solves

$$\operatorname{argmin}_{c_1, \dots, c_K \in \mathbb{R}^D} \sum_n \min_k \|x_n - c_k\|^2.$$

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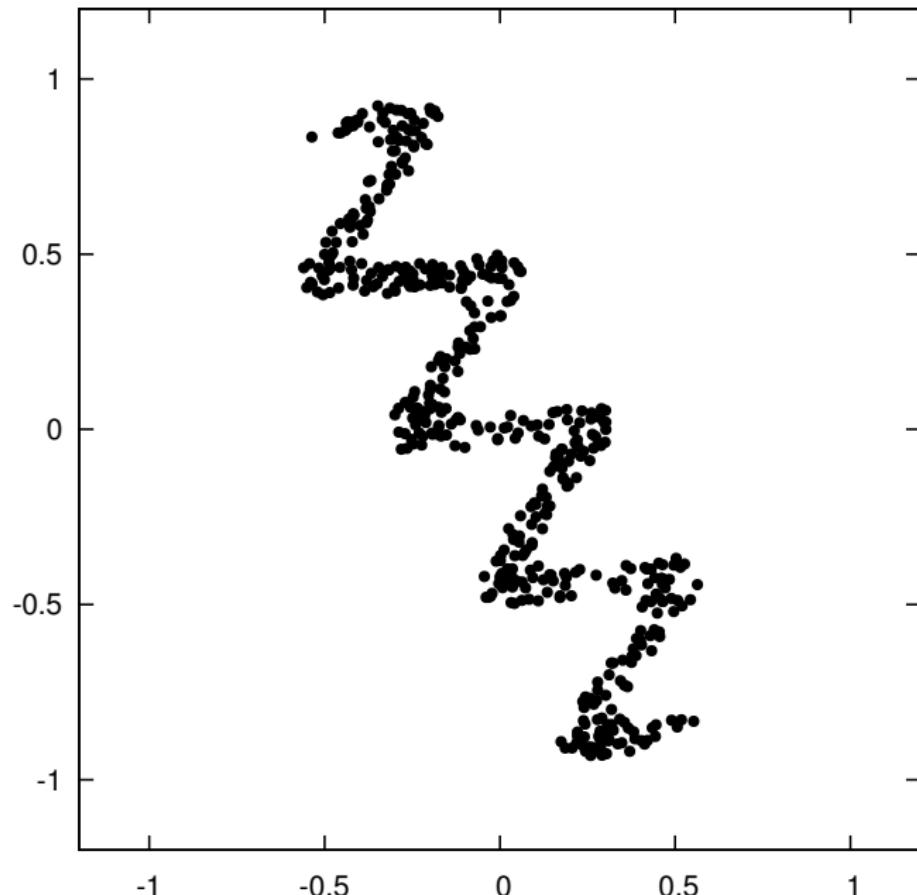
$$\underset{c_1, \dots, c_K \in \mathbb{R}^D}{\operatorname{argmin}} \sum_n \min_k \|x_n - c_k\|^2.$$

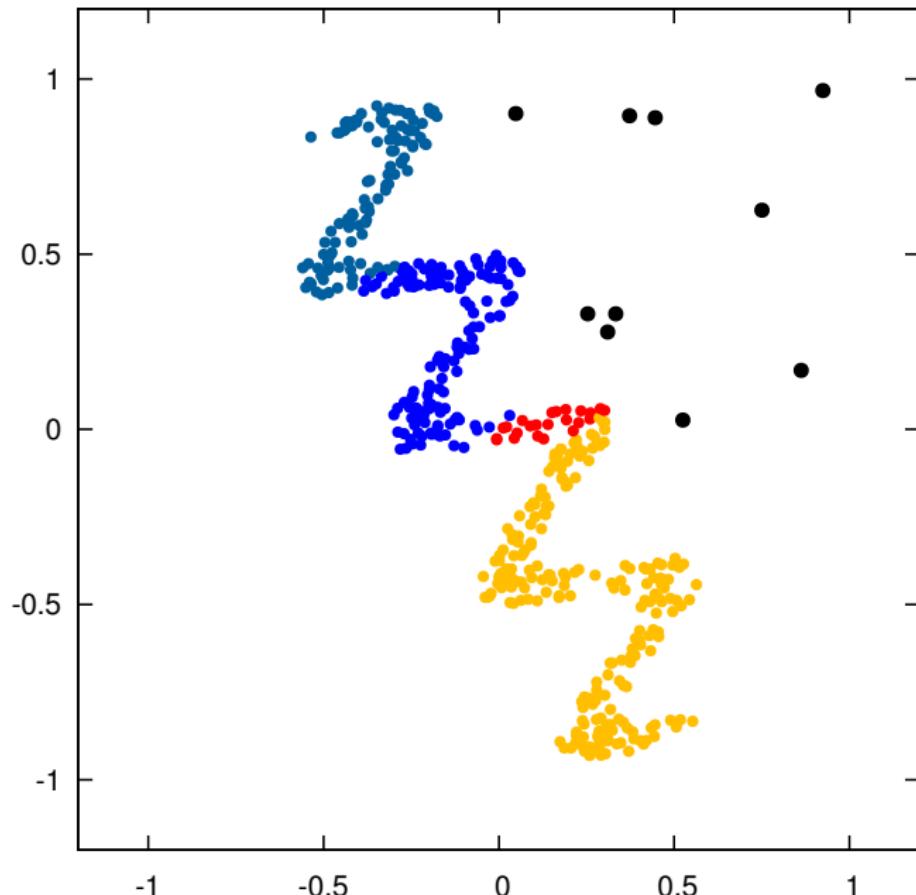
This is achieved with a random initialization of c_1^0, \dots, c_K^0 followed by repeating until convergence:

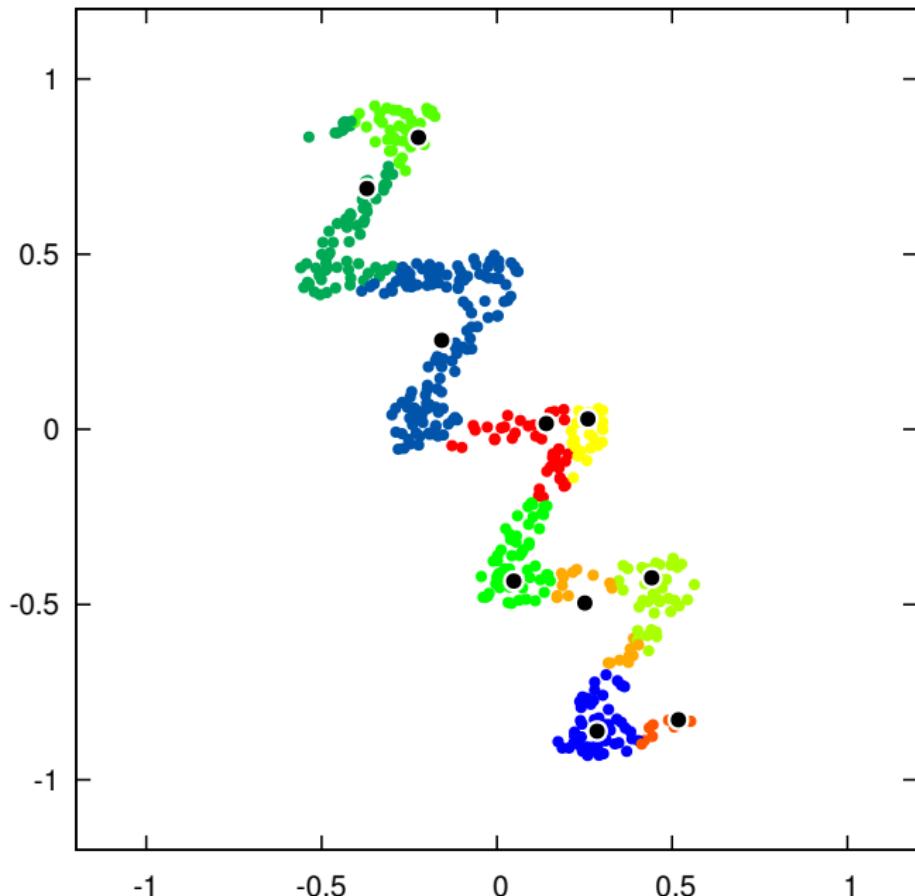
$$\forall n, k_n^t = \underset{k}{\operatorname{argmin}} \|x_n - c_k^t\| \quad (1)$$

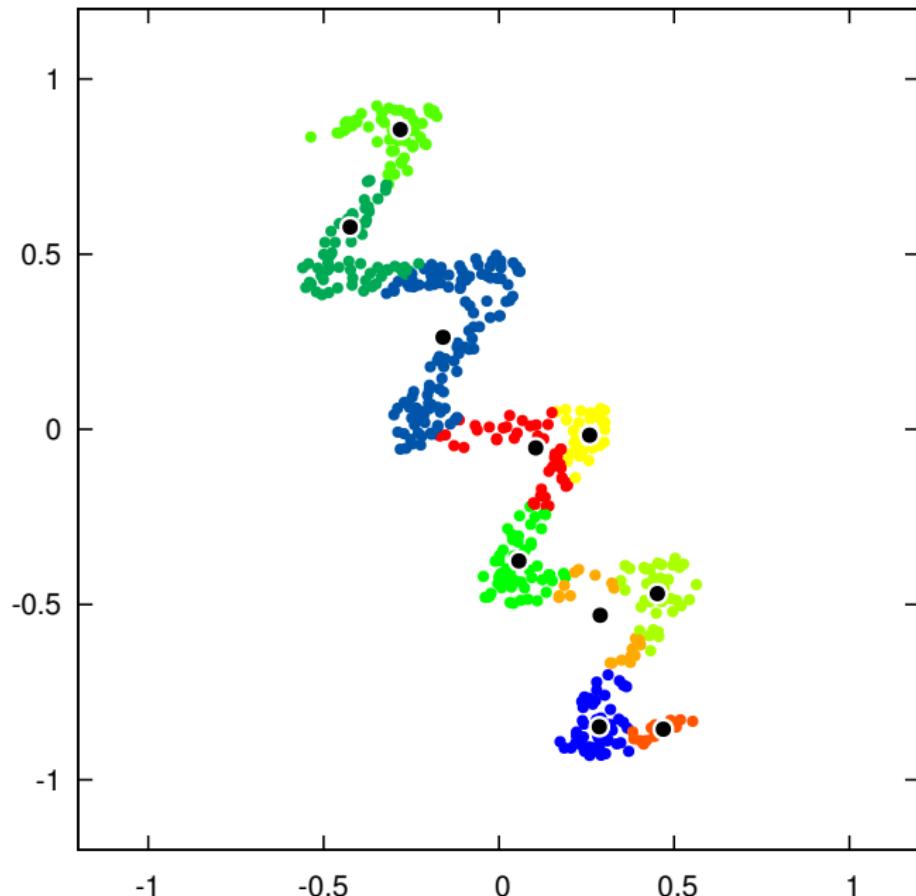
$$\forall k, c_k^{t+1} = \frac{1}{|n : k_n^t = k|} \sum_{n : k_n^t = k} x_n \quad (2)$$

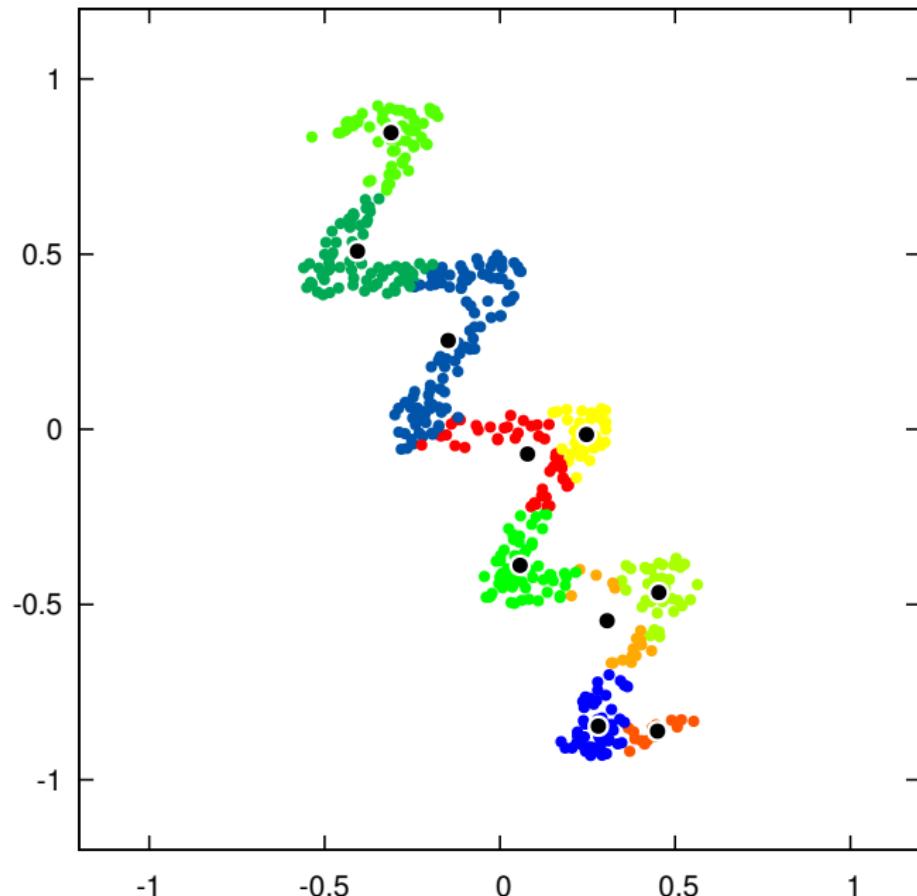
At every iteration, (1) each sample is associated to its closest centroid's cluster, and (2) each centroid is updated to the average of its cluster.

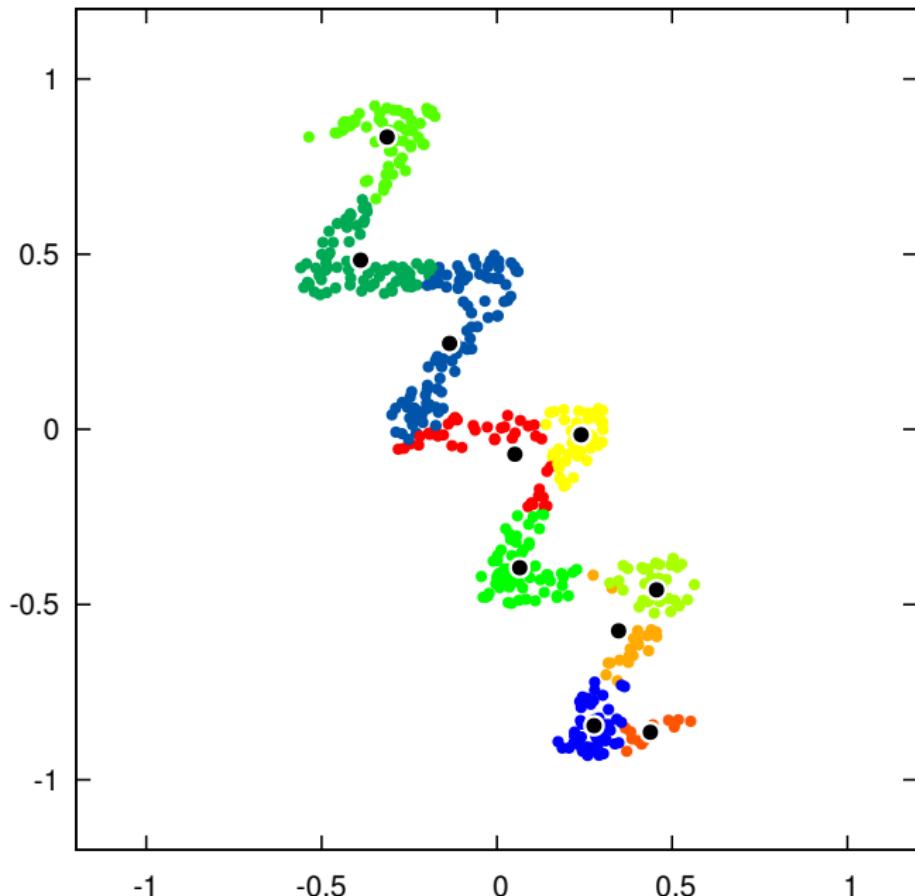


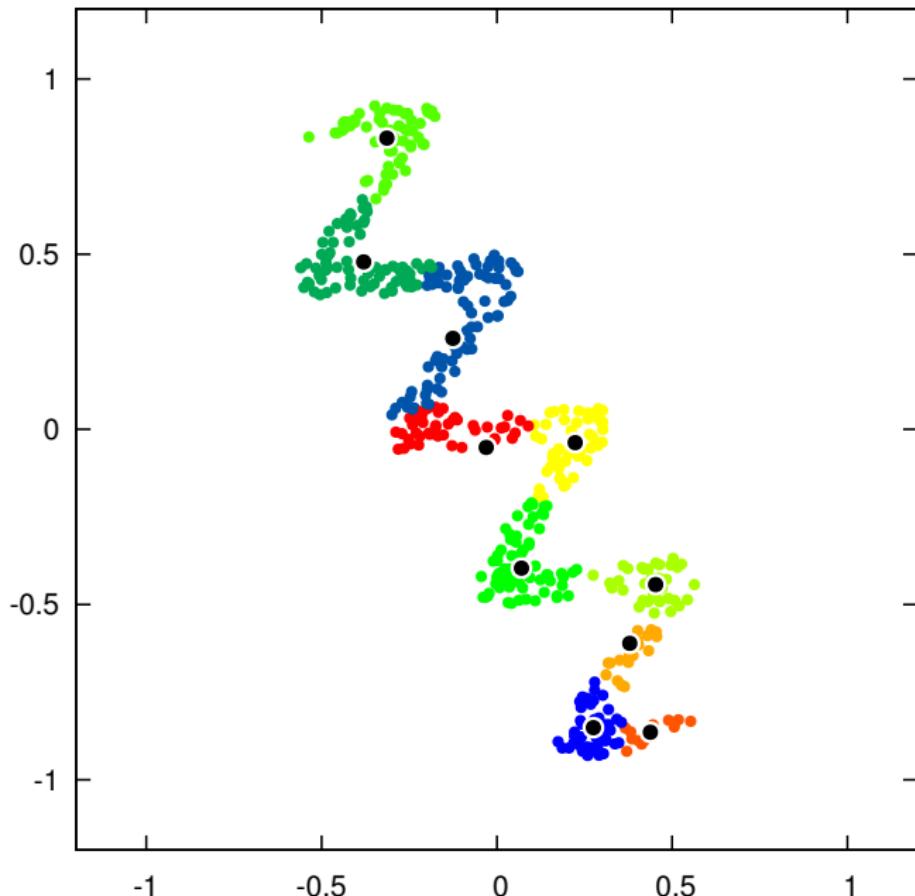


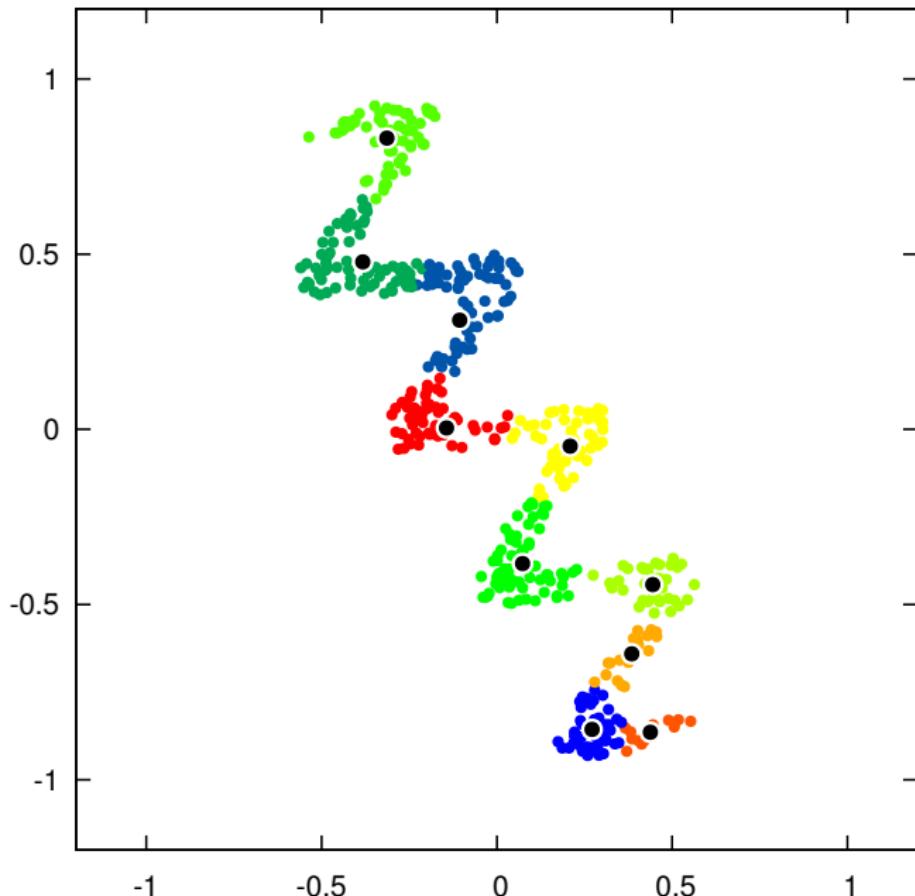


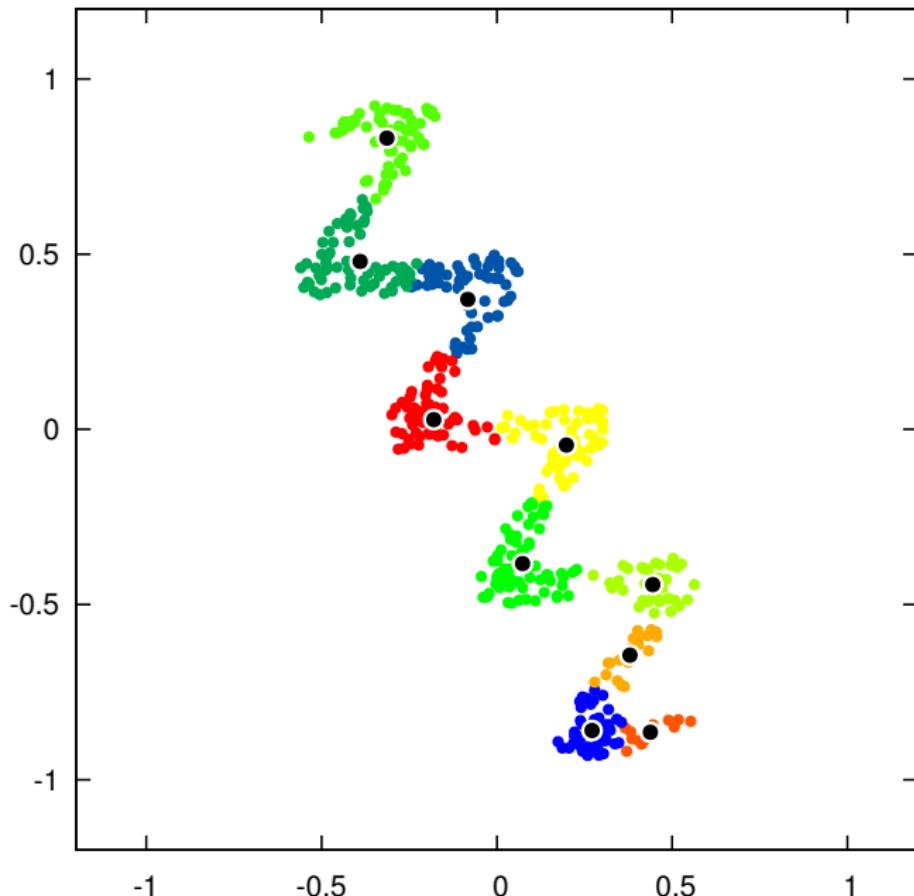


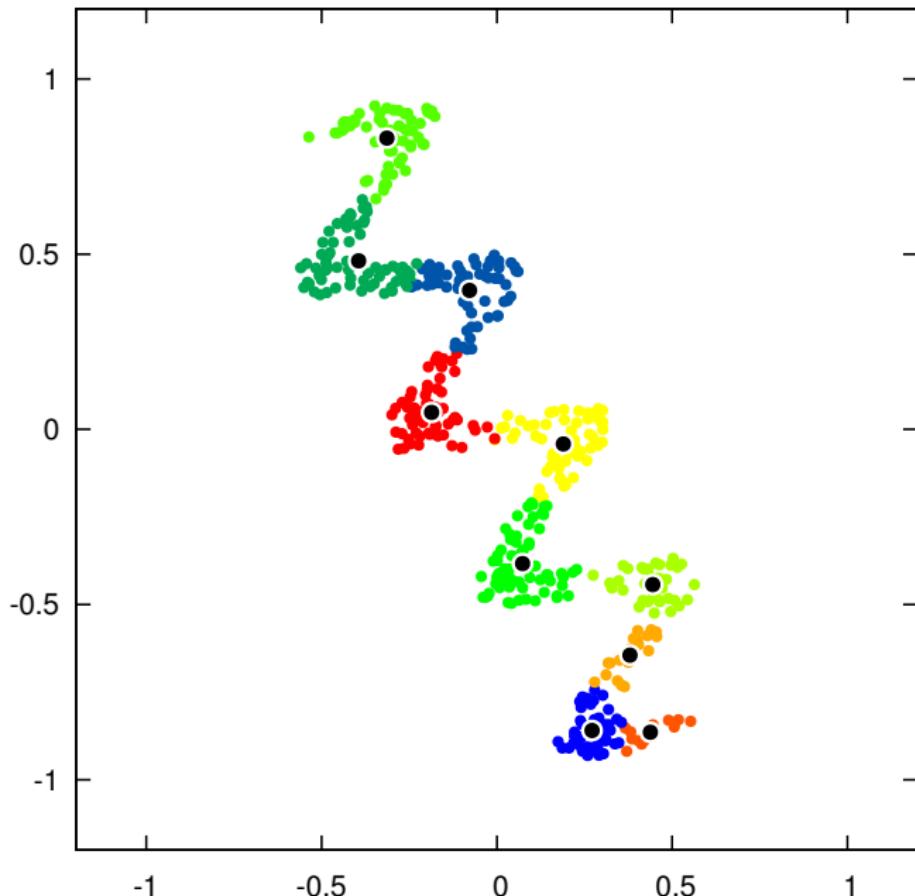


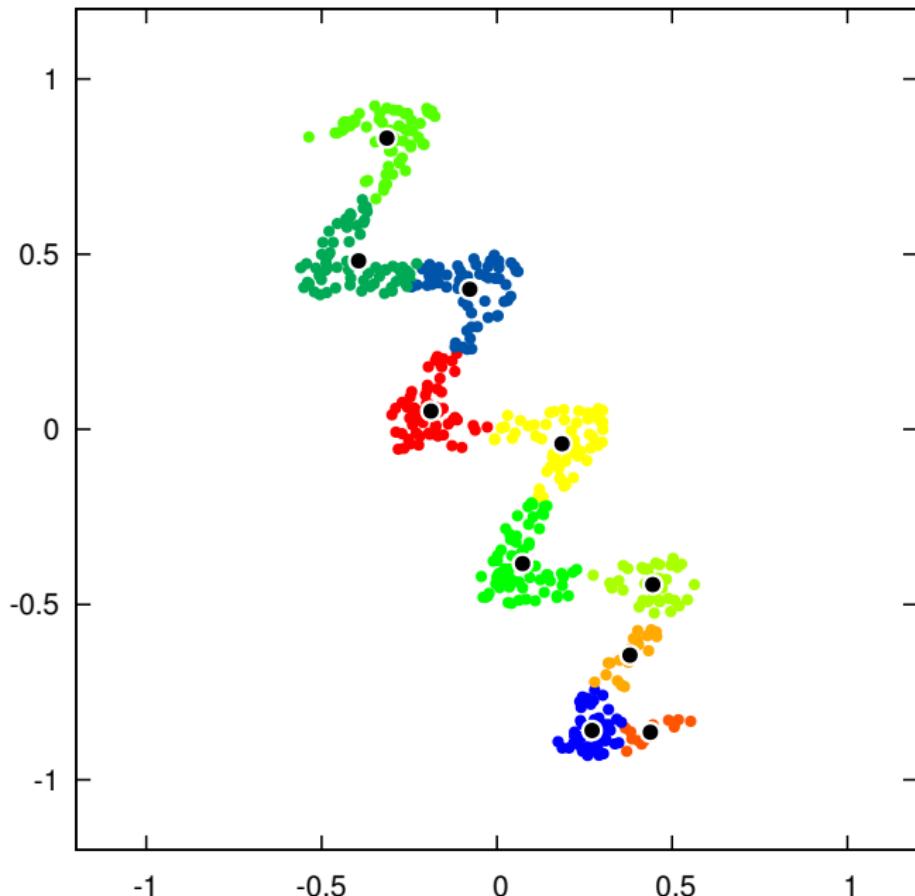


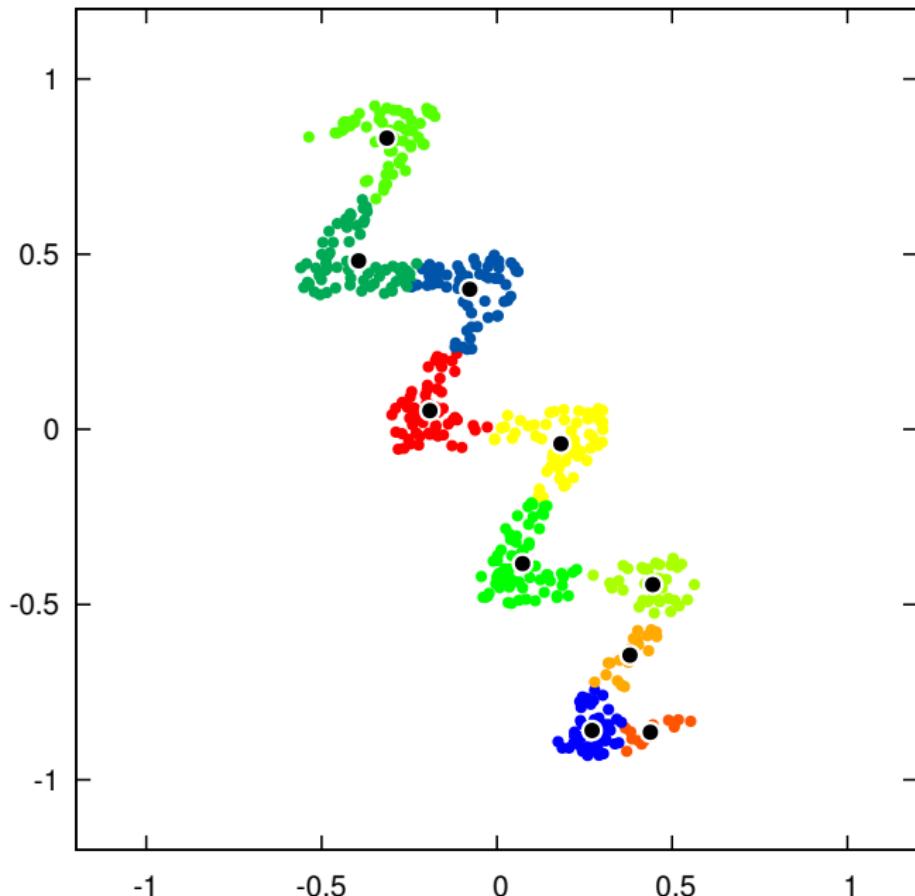












We can apply that algorithm to images from MNIST ($1 \times 28 \times 28$) or CIFAR ($3 \times 32 \times 32$) by considering them as vectors from \mathbb{R}^{784} and \mathbb{R}^{3072} respectively.

Centroids can similarly be visualized as images, and clustering can be done per-class, or for all the classes mixed.

0
1
2
3
4
5
6
7
8
9
0

$K = 1$

00
11
22
33
44
55
66
77
88
99
00

$K = 2$

0000
1111
2222
3333
4444
5555
6666
7777
8888
9999
0000

$K = 4$

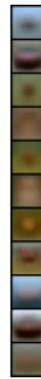
0000000000
1111111111
2222222222
3333333333
4444444444
5555555555
6666666666
7777777777
8888888888
9999999999
0000000000

$K = 8$

0000000000000000
1111111111111111
2222222222222222
3333333333333333
4444444444444444
5555555555555555
6666666666666666
7777777777777777
8888888888888888
9999999999999999
0521019305986792

$K = 16$

$$K = 32$$



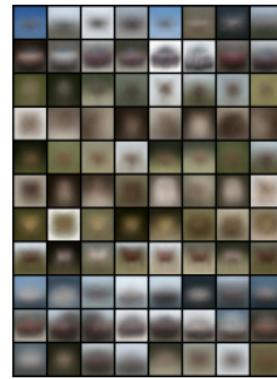
$K = 1$



$K = 2$



$K = 4$



$K = 8$



$K = 16$



$K = 32$

The Principal Component Analysis (PCA) aims also at extracting an information in a L^2 sense. Instead of clusters, it looks for an “affine subspace”, i.e. a point and a basis, that spans the data.

Given data-points

$$x_n \in \mathbb{R}^D, n = 1, \dots, N$$

(A) compute the average and center the data

$$\begin{aligned}\bar{x} &= \frac{1}{N} \sum_n x_n \\ \forall n, x_n^{(0)} &= x_n - \bar{x}\end{aligned}$$

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and then for $t = 1, \dots, D$,

(B) pick the direction and project the data

$$\begin{aligned}v_t &= \underset{\|v\|=1}{\operatorname{argmax}} \sum_n \left(v \cdot x_n^{(t-1)} \right)^2 \\ \forall n, \quad x_n^{(t)} &= x_n^{(t-1)} - \left(v_t \cdot x_n^{(t-1)} \right) v_t.\end{aligned}$$

Although this is a simple way to envision PCA, standard implementations rely on an eigendecomposition. With

$$X = \begin{pmatrix} - & x_1 & - \\ & \vdots & \\ - & x_N & - \end{pmatrix}$$

we have

$$\sum_n (v \cdot x_n)^2$$

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$$\sum_n (v \cdot x_n)^2 = \left\| \begin{pmatrix} v \cdot x_1 \\ \vdots \\ v \cdot x_N \end{pmatrix} \right\|_2^2$$

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we have

$$\begin{aligned}\sum_n (v \cdot x_n)^2 &= \left\| \begin{pmatrix} v \cdot x_1 \\ \vdots \\ v \cdot x_N \end{pmatrix} \right\|_2^2 \\ &= \|vX^T\|_2^2 \\ &= (vX^T)(vX^T)^T \\ &= v(X^TX)v^T.\end{aligned}$$

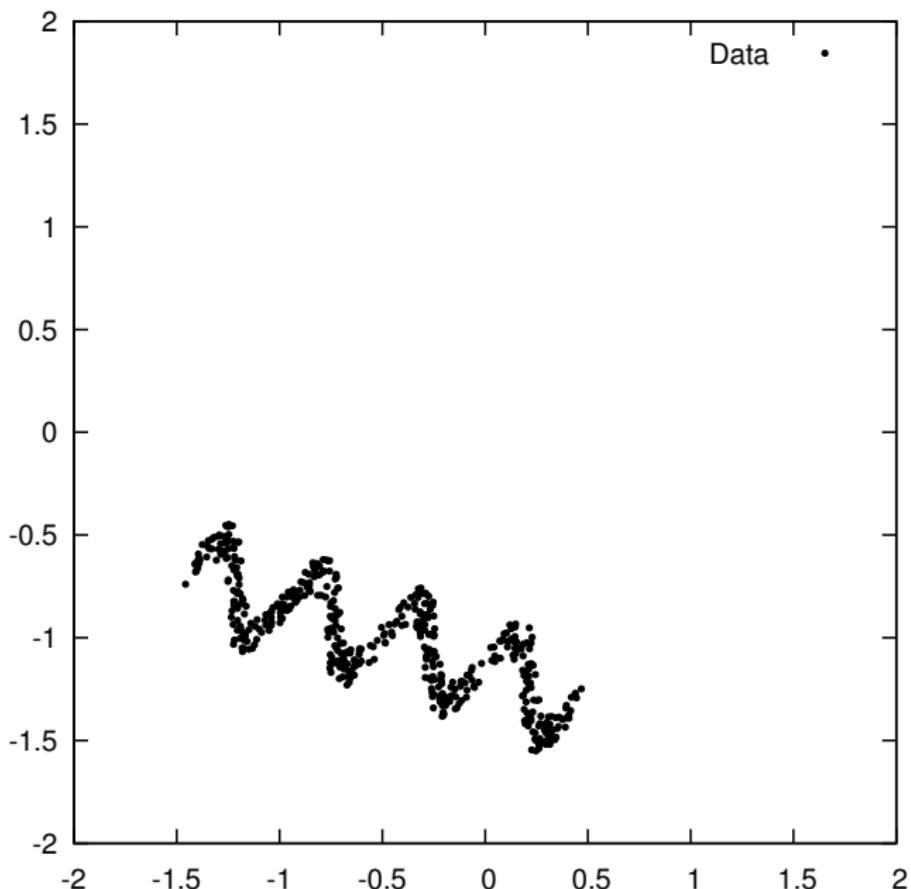
Although this is a simple way to envision PCA, standard implementations rely on an eigendecomposition. With

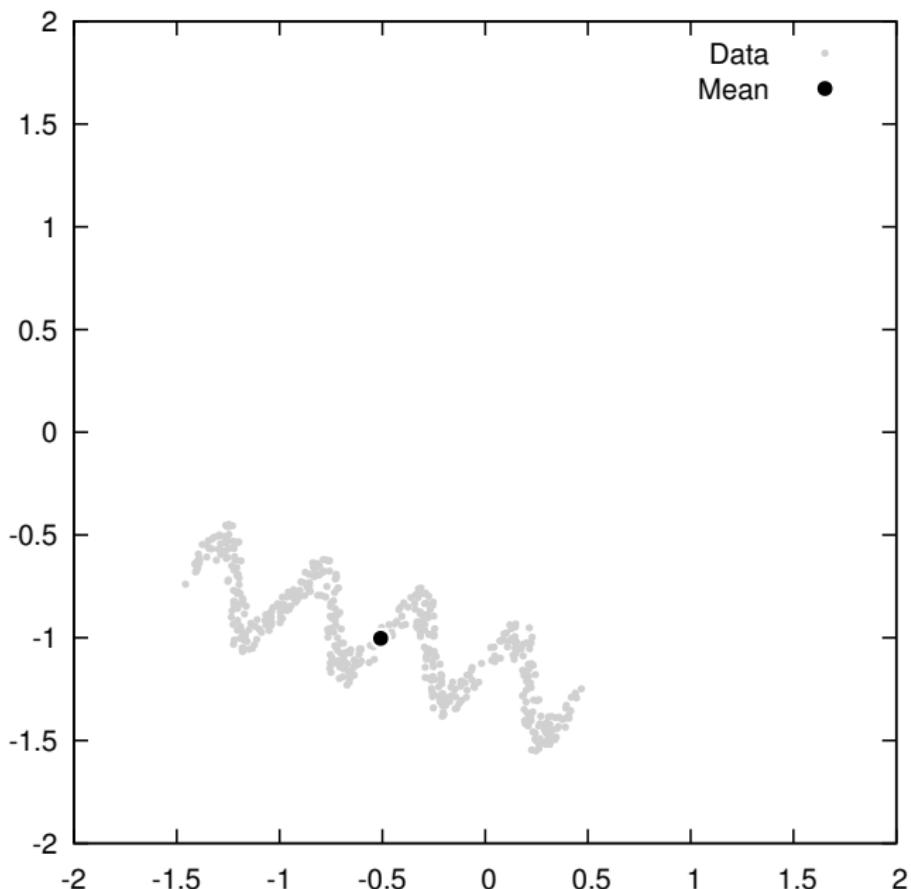
$$X = \begin{pmatrix} - & x_1 & - \\ & \vdots & \\ - & x_N & - \end{pmatrix}$$

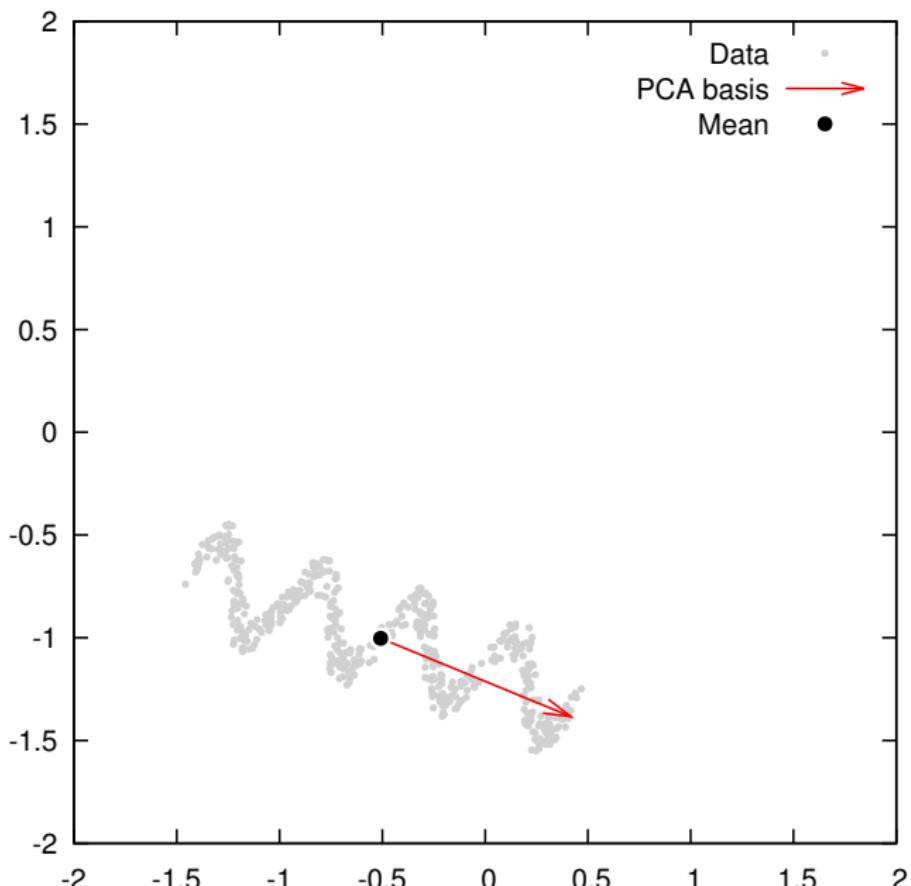
we have

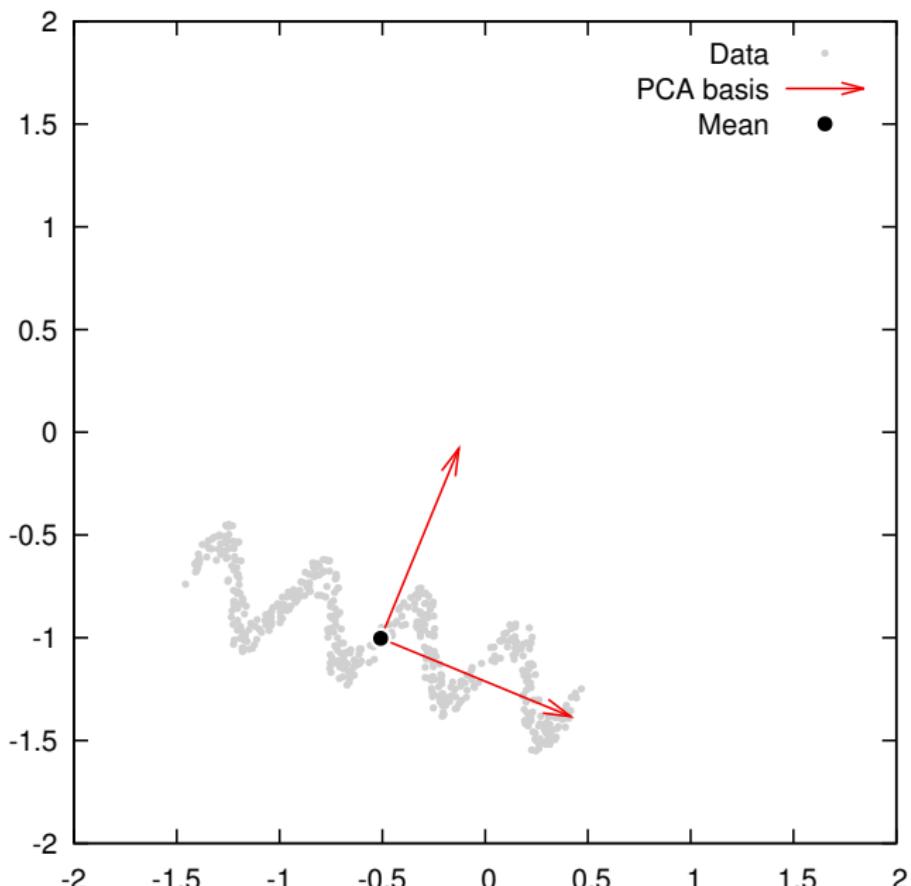
$$\begin{aligned}\sum_n (v \cdot x_n)^2 &= \left\| \begin{pmatrix} v \cdot x_1 \\ \vdots \\ v \cdot x_N \end{pmatrix} \right\|_2^2 \\ &= \|vX^T\|_2^2 \\ &= (vX^T)(vX^T)^T \\ &= v(X^TX)v^T.\end{aligned}$$

From this we can derive that v_1, v_2, \dots, v_D are the eigenvectors of X^TX ranked according to [the absolute values of] their eigenvalues.







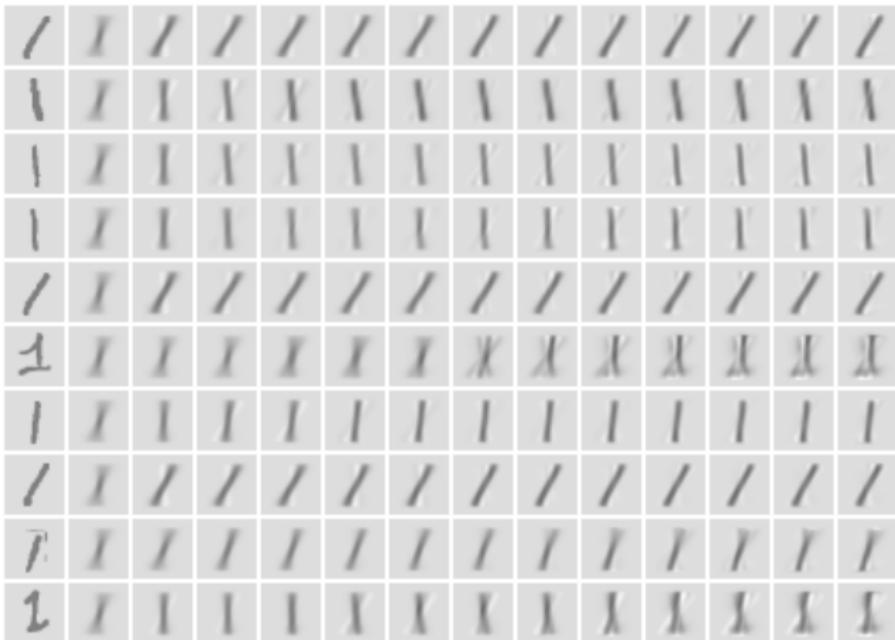


As for K -means, we can apply that algorithm to images from MNIST or CIFAR by considering them as vectors.

For any sample x and any T , we can compute a reconstruction using T vectors from the PCA basis, *i.e.*

$$\bar{x} + \sum_{t=1}^T (v_t \cdot x) v_t.$$

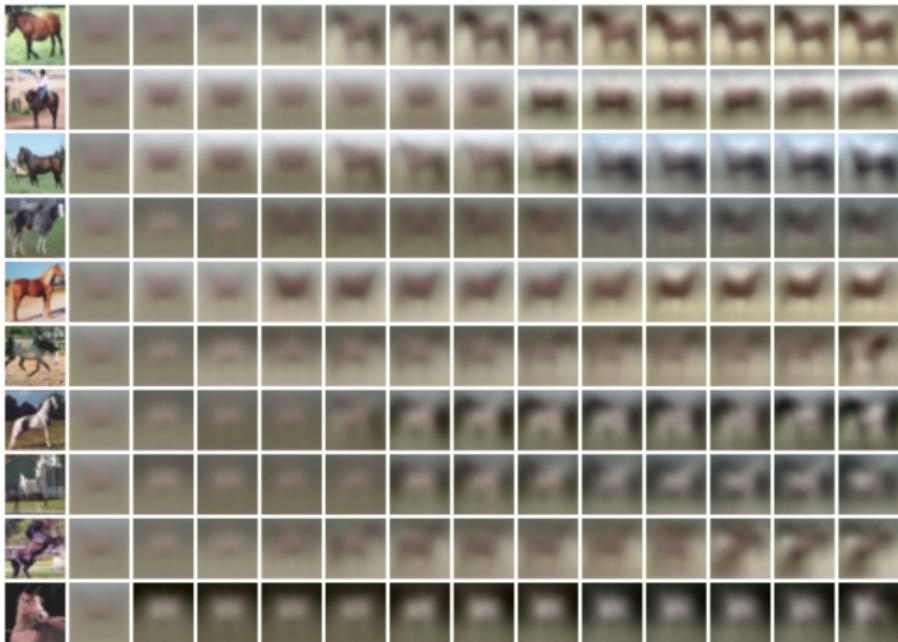
\bar{x} v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12}

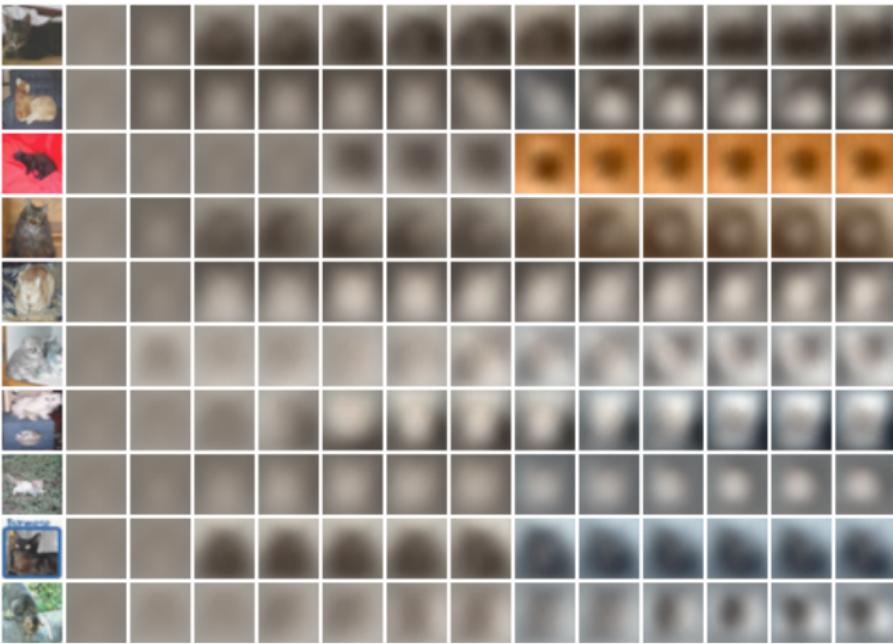


\bar{x} v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12}



\bar{x} v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_{10} v_{11} v_{12}





These results show that even crude embeddings capture something meaningful. Changes in pixel intensity as expected, but also deformations in the “indexing” space (*i.e.* the image plan).

However, translations and deformations damage the representation badly, and “composition” (*e.g.* object on background) is not handled at all.

These strengths and shortcomings provide an intuitive motivation for “deep neural networks”, and the rest of this course.

We would like

- to use many encoding “of these sorts” for small local structures with limited variability,
- have different “channels” for different components,
- process at multiple scales.

Computationally, we would like to deal with large signals and large training sets, so we need to avoid super-linear cost in one or the other.

The end

References

- A. Krizhevsky. Learning multiple layers of features from tiny images. Master's thesis, Department of Computer Science, University of Toronto, 2009.