COM303: Digital Signal Processing

Lecture 11: z-transform, filter structures and design examples

Overview

- the z-transform
- filtering algorithms and filter structures
- intuitive filter design
- two more ideal filters

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- ideal filters cannot be implemented
- what is the most general, realizable LTI transformation?
  - linearity: only sums and multiplications
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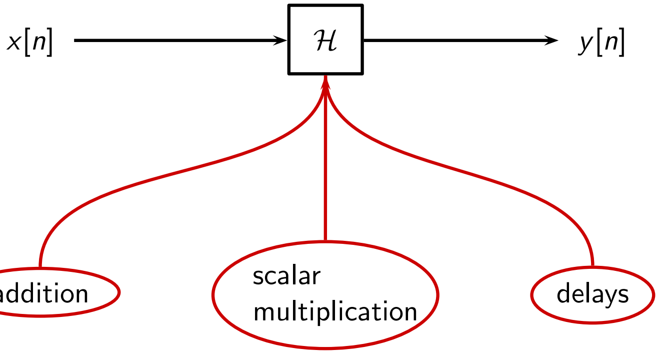
Linear, time-invariant systems



Constant-Coefficient Difference Equation

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

- uses M input and N output values
- how do we compute the frequency response?
- we need a new tool!

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The z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad z \in \mathbb{C}$$

- for us mostly a *formal operator*...
- ...but also as the extension of the DTFT to the whole complex plane:

$$X(z)\big|_{z=e^{j\omega}} = \text{DTFT}\{x[n]\}$$

- (and now the notation $X(e^{j\omega})$ should make more sense)

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Key properties

linearity:

$$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} = \alpha X(z) + \beta Y(z)$$

time shift:

$$\mathcal{Z}\{x[n - N]\} = z^{-N} X(z)$$

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Applying the z-transform to CCDE's

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

$$Y(z) \sum_{k=0}^{N-1} a_k z^{-k} = X(z) \sum_{k=0}^{M-1} b_k z^{-k}$$

IMPORTANT: this assumes $X(z)$ and $Y(z)$ exist!

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Rational Transfer Function

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}}$$

From CCDE to Frequency Response

- z-transform of CCDE:

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- convolution theorem:

$$y[n] = h[n] * x[n] \quad \Rightarrow \quad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(z)$ is the z-transform of the impulse response

Result #1: by setting $z = e^{j\omega}$ in $H(z)$ we get the frequency response!

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Leaky Integrator revisited

$$y[n] = (1-\lambda)x[n] + \lambda y[n-1]$$

$$Y(z) = (1-\lambda)X(z) + \lambda z^{-1} Y(z)$$

$$H(z) = \frac{(1-\lambda)}{1-\lambda z^{-1}}$$

$$H(e^{j\omega}) = \frac{(1-\lambda)}{1-\lambda e^{-j\omega}}$$

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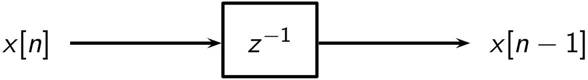
BTW, remember the delay block?



$$Y(z) = z^{-1} X(z)$$

now the notation should make more sense!

Existence and region of convergence (ROC)

$$|X(z)| < \infty \quad \text{for } z \in \text{ROC} \subseteq \mathbb{C}$$

How can we determine the ROC?

- ROC depends on the values of $x[n]$
- for rational transfer function we can use indirect methods
- we don't care about convergence in zero and infinity

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Region of convergence (ROC)

Obvious fact:

for finite-support signals, the z-transform converges everywhere (except in 0 and/or ∞)

$$X(z) = \sum_{n=-M}^{N} x[n]z^{-n} = z^{-N} P_{(M+N-1)}(z)$$

Region of convergence (ROC)

Theorem:

the z-transform is a power series, so convergence is always absolute

$$|X(z)| < \infty \Longleftrightarrow \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

Region of convergence (ROC)

Corollary #1:

the region of convergence has circular symmetry: set $z = ae^{j\theta}$:

$$\left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty \iff \sum_{n=-\infty}^{\infty} |x[n]| \, |a^{-n}| < \infty$$

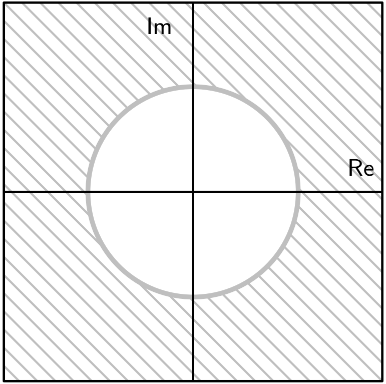
Region of convergence (ROC)

Corollary #2:

for causal signals, the ROC extends from a circle to infinity:
assume $H(z_0)$ exists and $|z_1| > |z_0|$:

$$|X(z_1)| = \left| \sum_{n=0}^{\infty} \frac{x[n]}{z_1^n} \right| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_1^n|} \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z_0^n|} \leq \infty$$

ROC shape for causal sequences



Region of convergence (ROC)

so where are the convergence problems?

in general, difficult question; but we're only interested in rational transfer functions!

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ROC for causal systems

Consider the transfer function for an LTI system:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{M-1} z^{-(M-1)}}{a_0 + a_1 z^{-1} + \ldots + a_{N-1} z^{-(N-1)}}$$

It can always be factored as:

$$H(z) = C \frac{\prod_{n=1}^{M-1} (1 - z_n z^{-1})}{\prod_{n=1}^{N-1} (1 - p_n z^{-1})}$$

ROC for causal systems

- z_n's: *zeros* of the transfer function
- p_n's: *poles* of the transfer function
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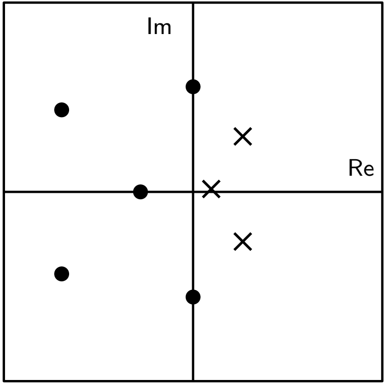

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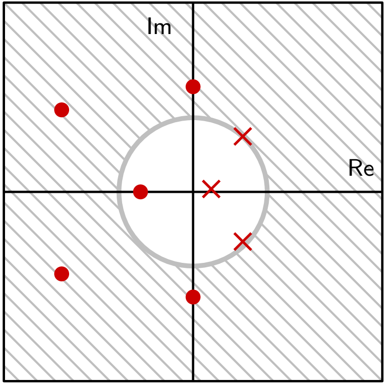

System stability criterion

Consider a filter with impulse response $h[n]$

- BIBO stability $\iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$
- $1 \in \mathrm{ROC} \iff \sum_{n=-\infty}^{\infty} |h[n]| < \infty$ because $H(z)$ converges absolutely in $z = 1$

Result #2: system is stable if and only if ROC includes the unit circle!

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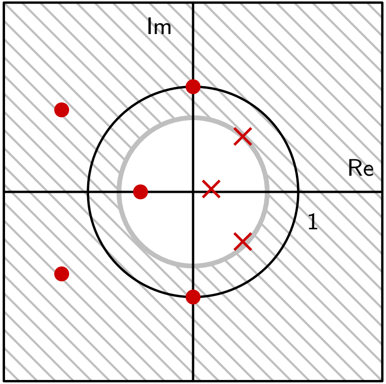
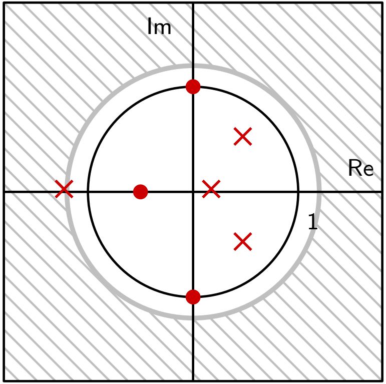
Stable system



Unstable system



Common confusion...

$$y[n] = 2y[n-1] + x[n] \qquad \text{(unstable!)}$$

apply z-transform as a formal operator:

$$H(z) = Y(z)/X(z) = \frac{1}{1 - 2z^{-1}}$$

$H(1) = -1 < \infty$, so system is stable?

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Common confusion clarified

ROC depends on $h[n]$, NOT on *formal* value of $H(z)$:

- $h[n] = 2^n \, u[n]$
- to apply the z-transform operator we *assume* to be in the ROC
- the region of convergence is $|z| > 2$ because

$$\sum_{n=0}^{\infty} a^n \, z^{-n} = \lim_{N \to \infty} \frac{1 - (a/z)^N}{1 - (a/z)} = \begin{cases} \dfrac{1}{1 - az^{-1}} & \text{if } |z| > |a| \\ \infty & \text{otherwise} \end{cases}$$

In other words...

the function $\dfrac{1}{1 - az^{-1}}$ is defined for all $z \in \mathbb{C} \setminus \{2\}$

BUT

it is the z-transform of $a^n u[n]$ *only* for $|z| > |a|$

For a generic rational transfer function

$$\sum_{k=0}^{N-1} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

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$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{\sum_{k=0}^{N-1} a_k z^{-k}} = \mathcal{Z}\{h[n]\} \qquad \leftarrow \text{this must exist!}$$

Rational Transfer Function

for which values of z does $H(z)$ exist?

- option 1: compute $h[n]$ explicitly and find ROC for the power series $\sum h[n]z^{-n}$
- option 2: derive ROC indirectly:
  - ROC is circular symmetric
  - ROC extends outwards for causal sequences
  - ROC cannot include poles

Estimating the frequency response from the pole-zero plot

The "circus tent" method:

- magnitude of z transform is like a rubber sheet over the complex plane
- zeros glue the sheet to the ground
- poles are like … poles, pushing it up
- frequency response (in magnitude) is sheet profile around the unit circle

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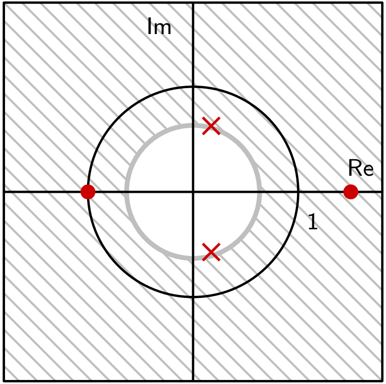

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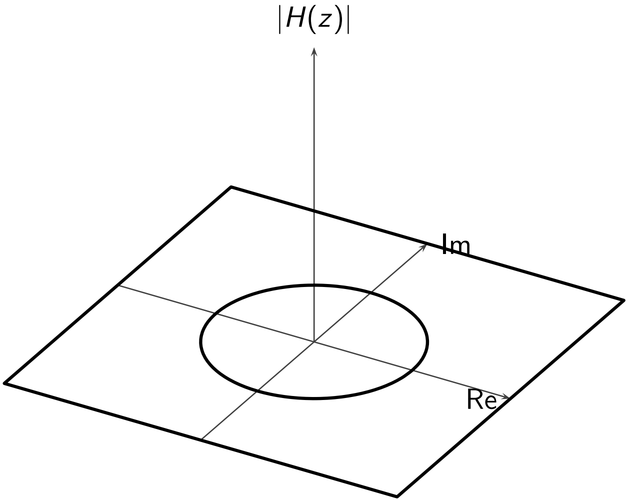

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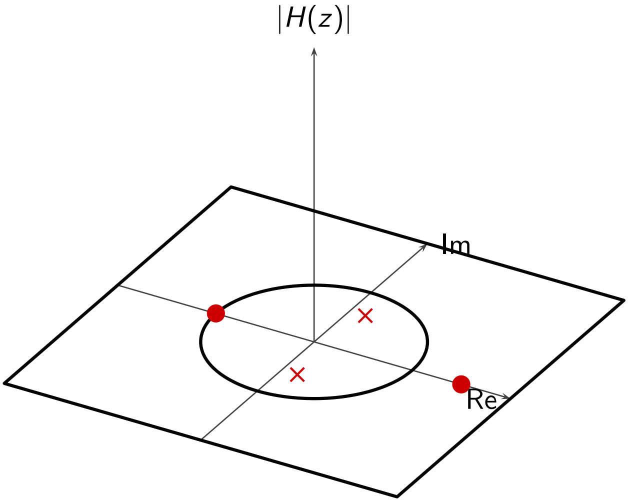

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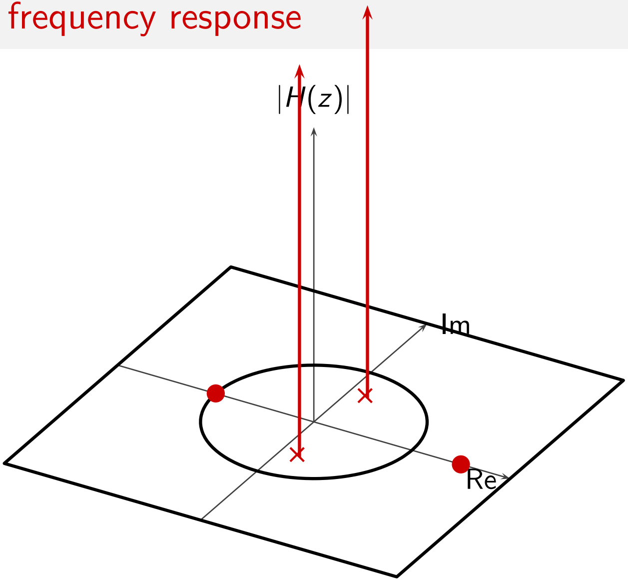

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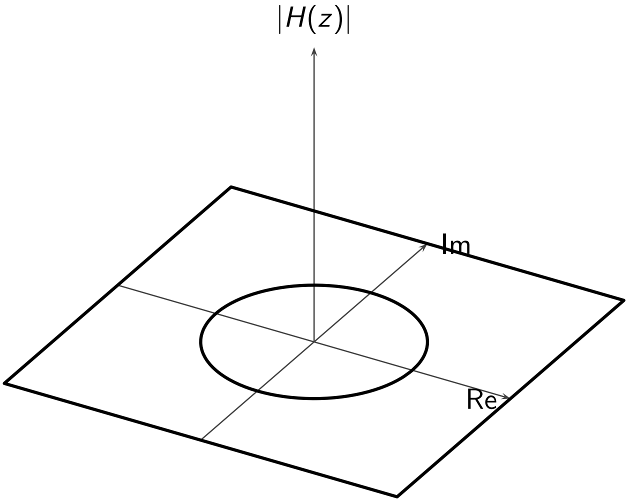

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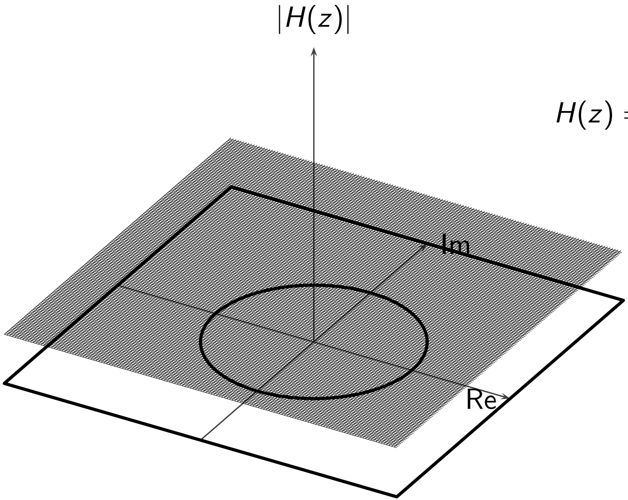
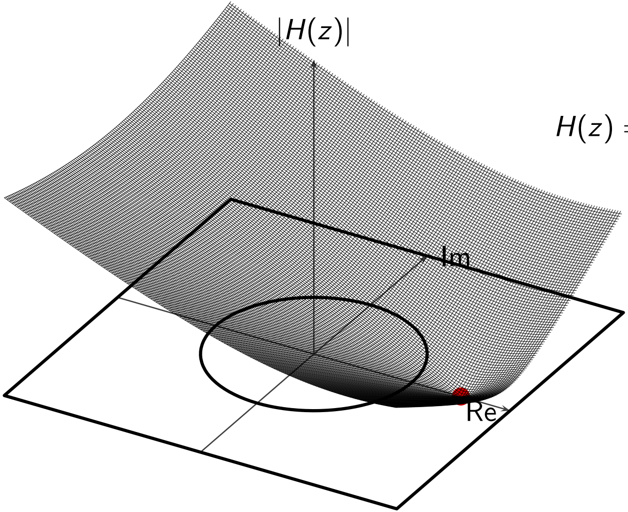


$H(z) = 1$

Estimating the frequency response



$$H(z) = (1 - z_0\, z^{-1})$$

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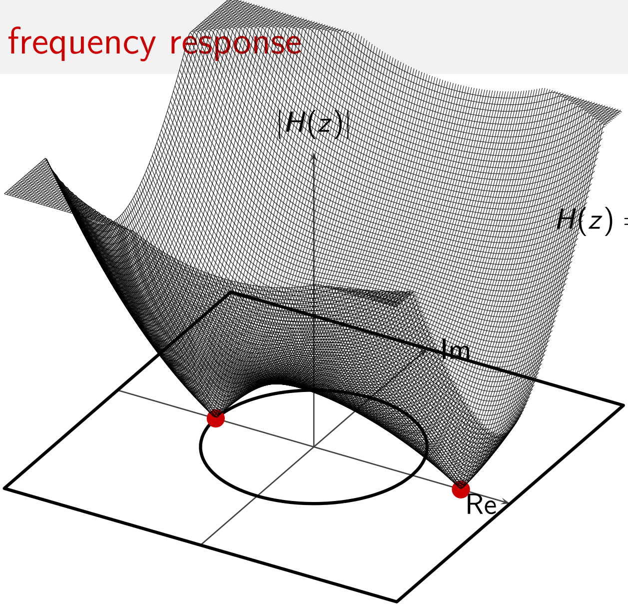

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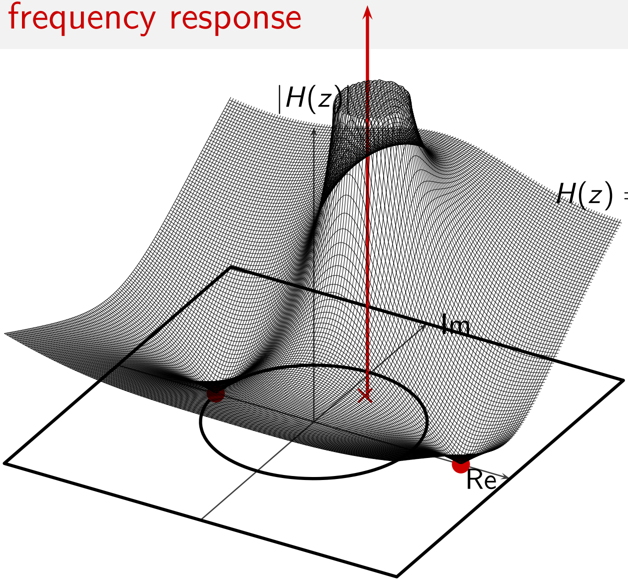

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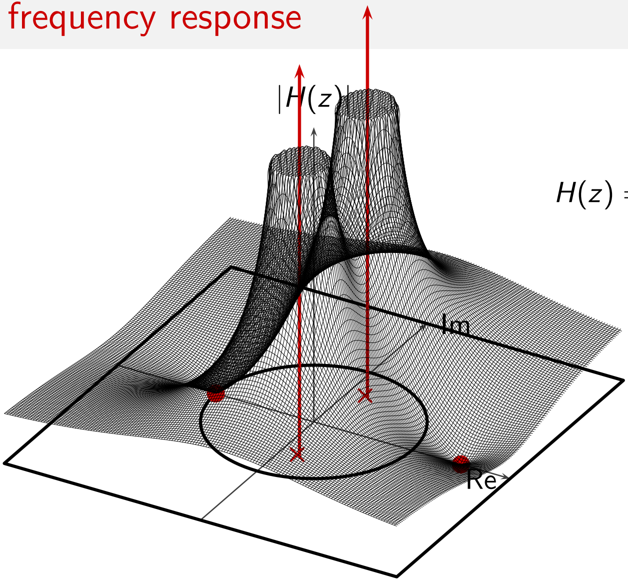

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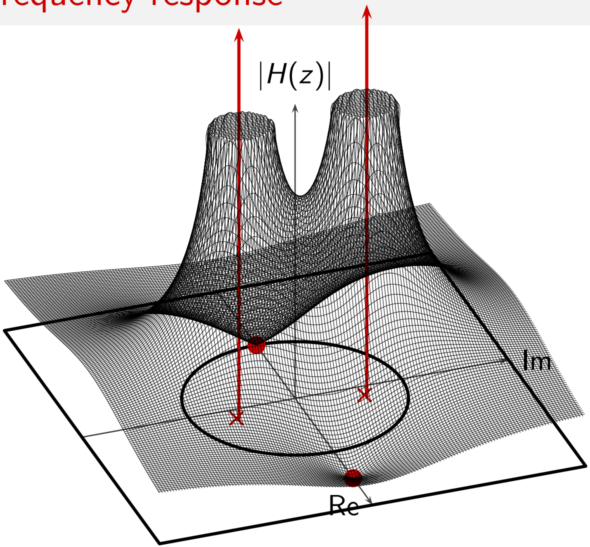

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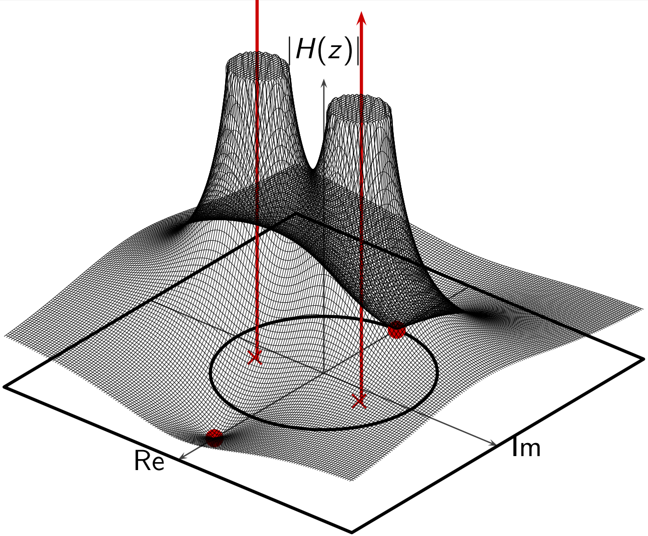

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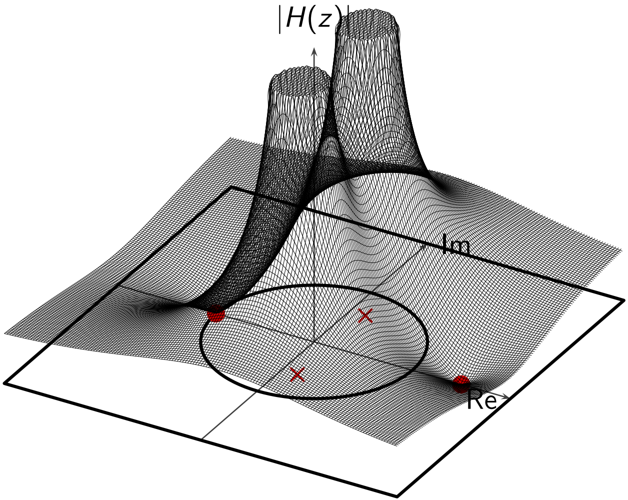

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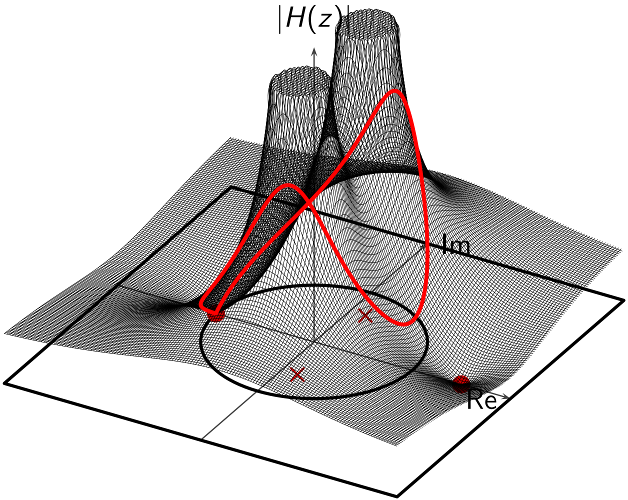

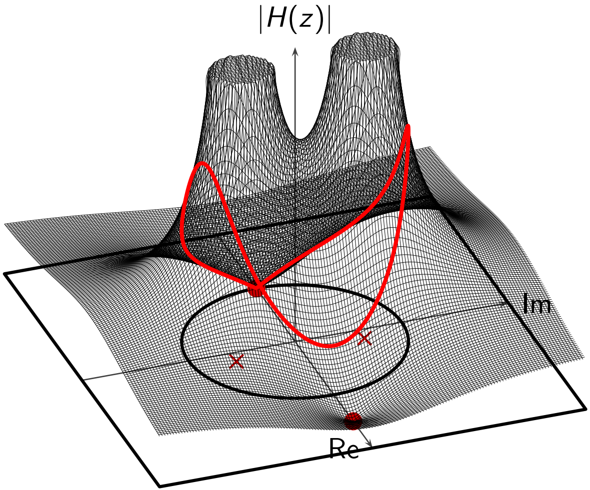
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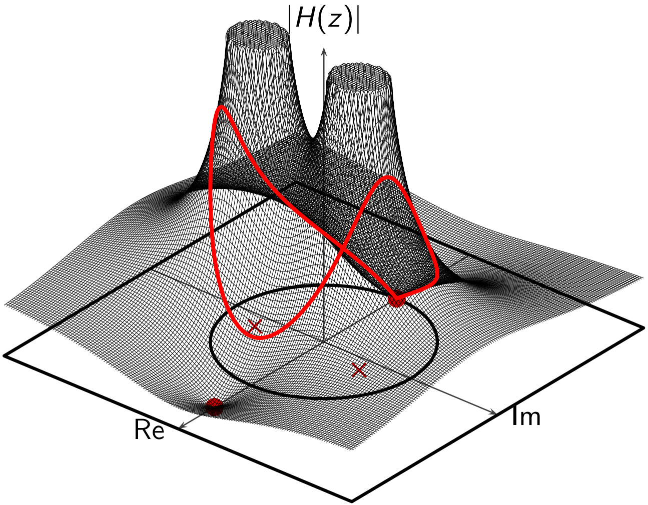
Re

Estimating the frequency response



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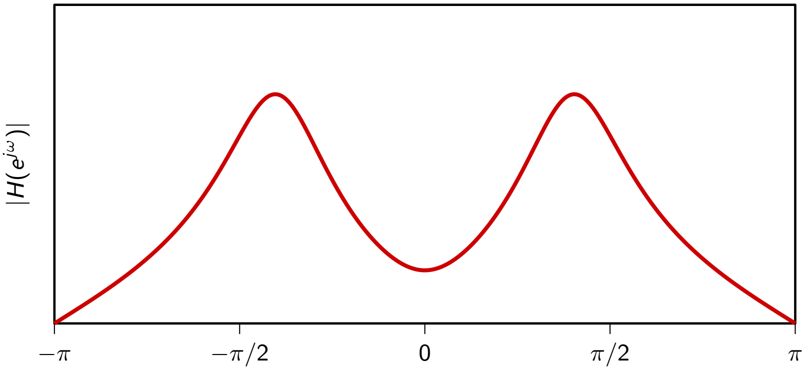


filtering algorithms and structures

Overview:

- Algorithms for CCDE's
- Block diagram
- Real-time processing

An old friend

```
class Leaky:
    def __init__(self, lmb):
        self.lmb=lmb
        self.y=0

    def compute(self, x):
        self.y = self.lmb * self.y + (1 - self.lmb) * x
        return self.y
```

Testing the code

```
>>> from leaky import Leaky
>>> li = Leaky(0.95)
>>> for v in [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]:
>>>     print li.compute(v)

[0.0, 0.0, 0.0, 0.0, 0.0500000000000000, 0.0475000000000000,
 0.0451250000000000, 0.0428687500000000, 0.0407253125000000,
 0.0386890468750000, 0.0367545945312500]
>>>
```

Key points

- we need a "memory cell" to store previous output
- we need to initialize the storage before first use
- we need 2 multiplications and one addition per output sample

Another old friend

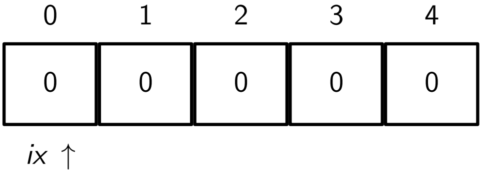
$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

Another old friend

```
class MA:
    def __init__(self, M):
        self.M = M
        self.buf = [0] * M
        self.ix = 0

    def compute(self, x):
        self.buf[self.ix] = x
        self.ix = (self.ix + 1) % self.M
        res = 0.0
        for v in self.buf:
            res += v
        return res / self.M
```

The circular buffer



The circular buffer

0	1	2	3	4
x_0	0	0	0	0
	$ix \uparrow$			

The circular buffer

0	1	2	3	4
x_0	x_1	0	0	0

$ix \uparrow$

The circular buffer

0	1	2	3	4
x_0	x_1	x_2	0	0
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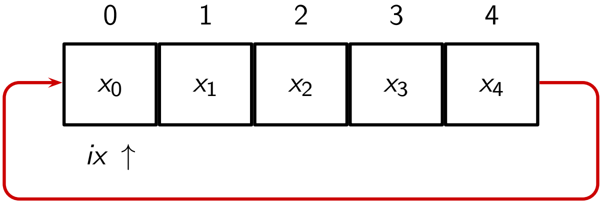

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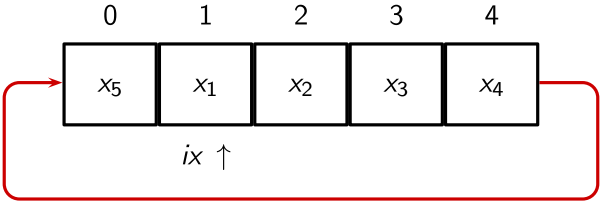

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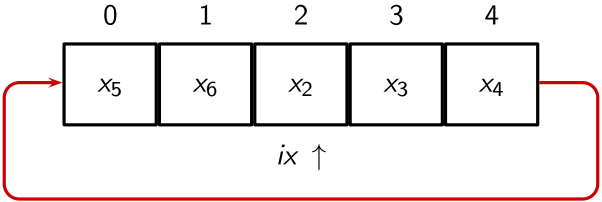

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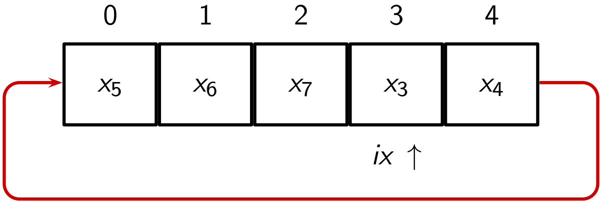
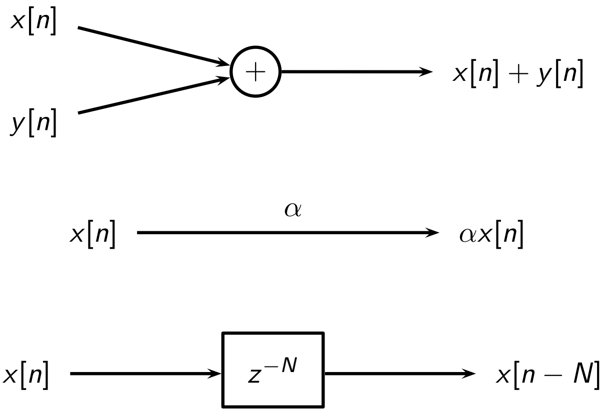


Key points

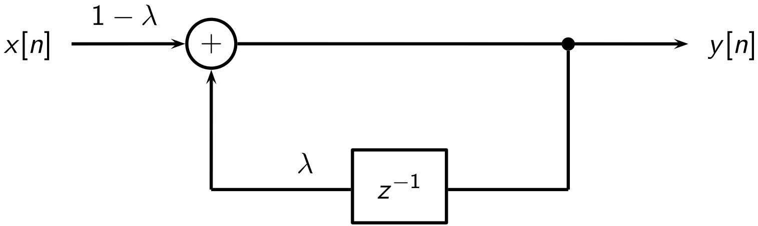
- we now need M memory cells to store previous input values
- we need to initialize the storage before first use
- we need 1 division and M additions per output sample

We can abstract from the implementation



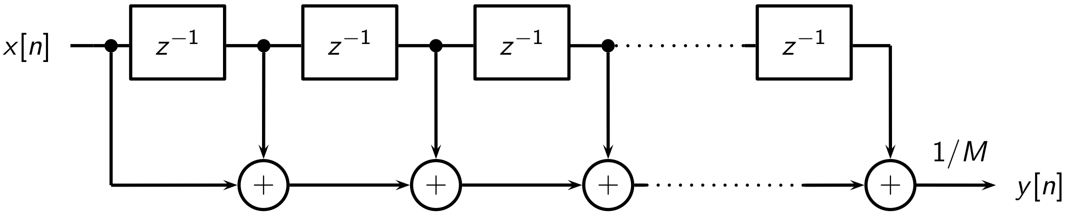
Leaky Integrator

$$y[n] = \lambda y[n-1] + (1-\lambda)x[n]$$



Moving Average

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$



The second-order section

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{B(z)}{A(z)}$$

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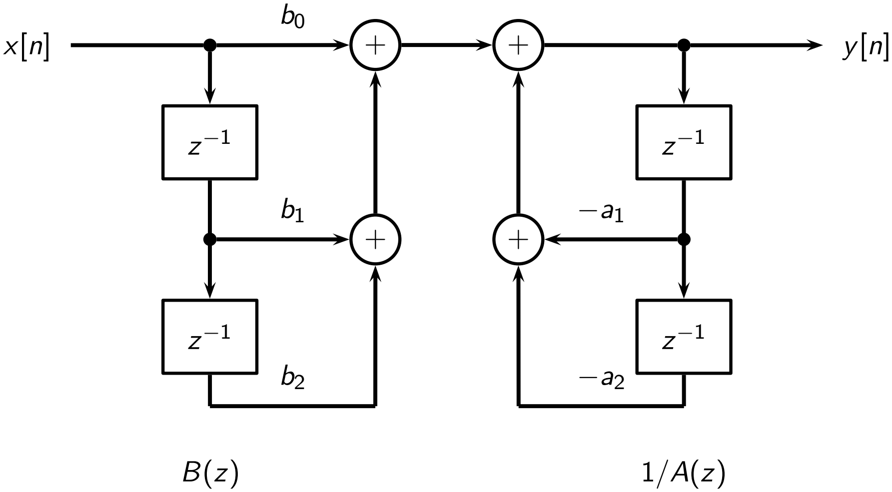
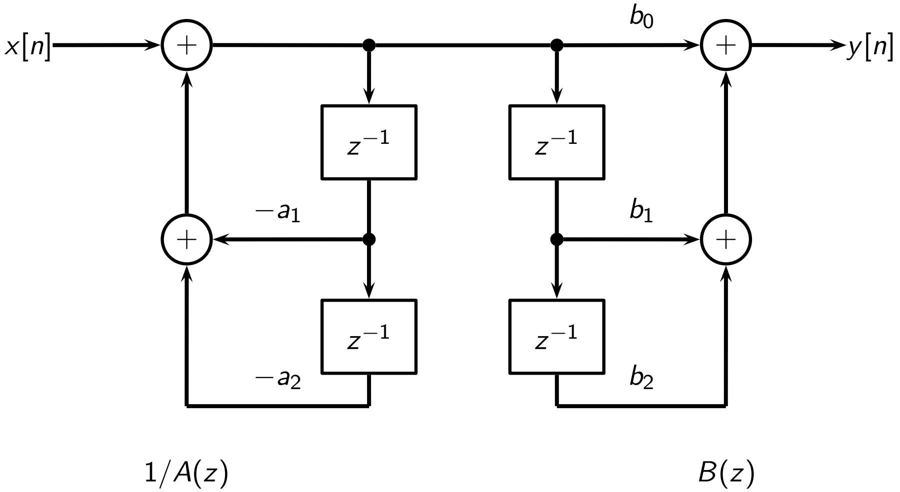
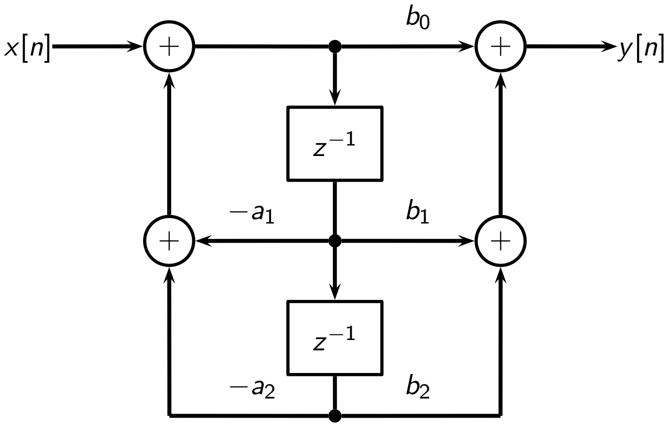
Second-order section, direct form I



Second-order section, direct form I, inverted order



Second-order section, direct form II



intuitive filter design

Overview:

- General problem
- “Intuitive” IIR design

Simple, useful filters

- many signal processing problems can be solved using simple filters
- we have seen simple lowpass filters already (Moving Average, Leaky Integrator)
- simple (low order) transfer functions allow for intuitive design and tuning

Simple lowpass

- let only low frequencies pass
- used to remove high frequency components (e.g. noise)
- useful in audio, communication, control systems
- we know a simple answer: leaky integrator

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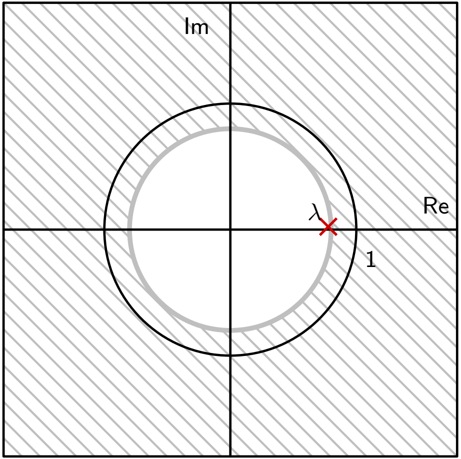
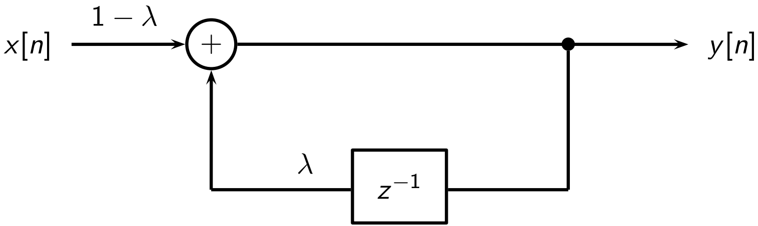
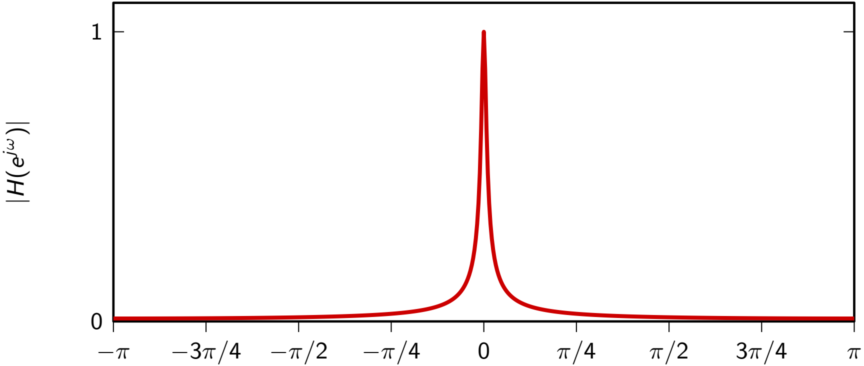


Leaky Integrator, filter structure



Leaky Integrator, $\lambda = 0.98$



Resonator

- a resonator is a narrow bandpass filter
- used to detect the presence of a sinusoid of a given frequency
- useful in communication systems and telephony (DTMF)
- idea: shift the passband of the Leaky Integrator!

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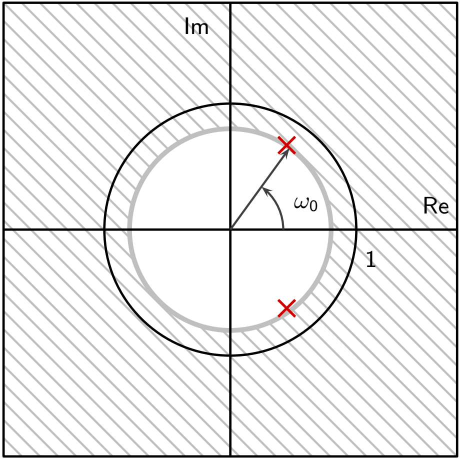
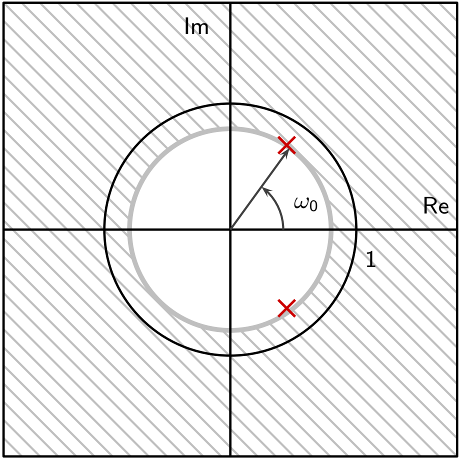


Resonator

$$H(z) = \frac{G_0}{(1 - pz^{-1})(1 - p^* z^{-1})}$$

$$p = \lambda e^{j\omega_0}$$



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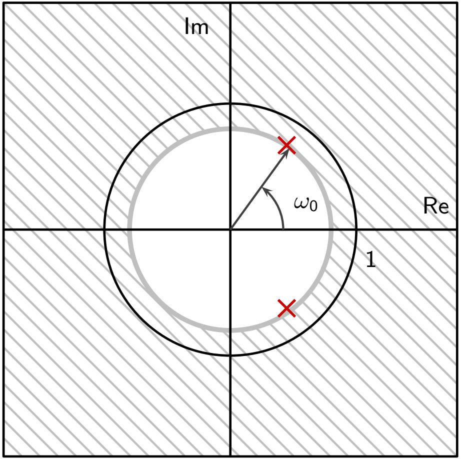

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$$= \frac{G_0}{1 - 2\Re\{p\}\, z^{-1} + |p|^2\, z^{-2}}$$

$$= \frac{G_0}{1 - 2\lambda \cos\omega_0\, z^{-1} + |\lambda|^2\, z^{-2}}$$

$$a_1 = 2\lambda \cos\omega_0$$

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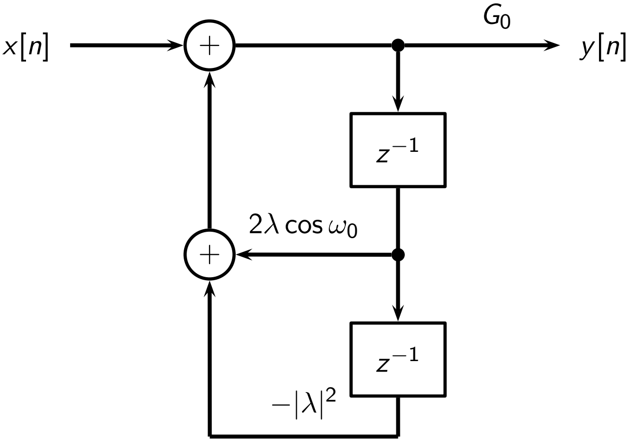
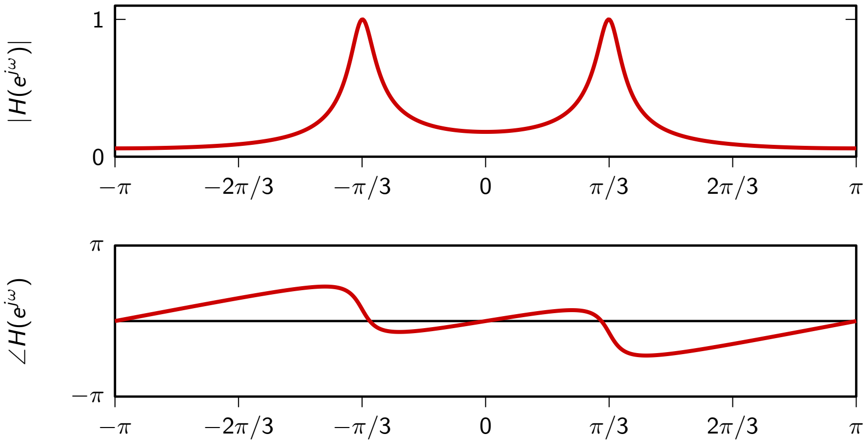
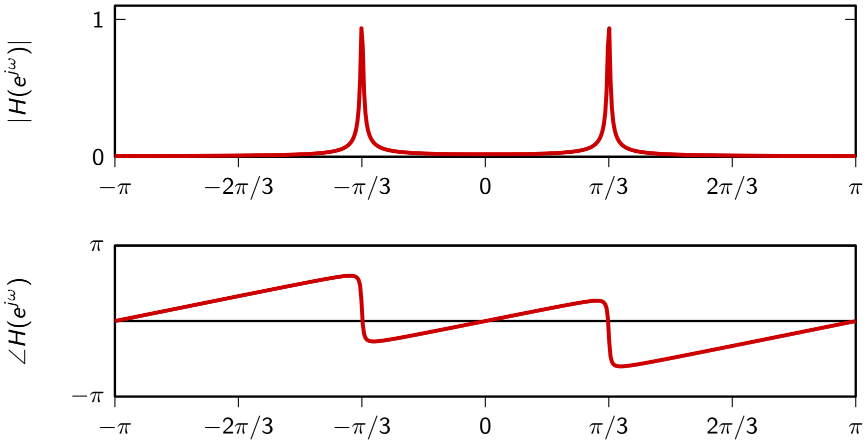
Resonator, filter structure



Resonator, $\lambda = 0.9, \omega_0 = \pi/3$



Resonator, $\lambda = 0.99, \omega_0 = \pi/3$



DC removal

- a DC-balanced signal has zero sum: $\lim_{N\to\infty} \sum_{n=-N}^{N} x[n] = 0$

  i.e. there is no Direct Current component
- its DTFT value at zero is zero
- we want to remove the DC bias from a non zero-centered signal
- we want to kill the frequency component at $\omega = 0$

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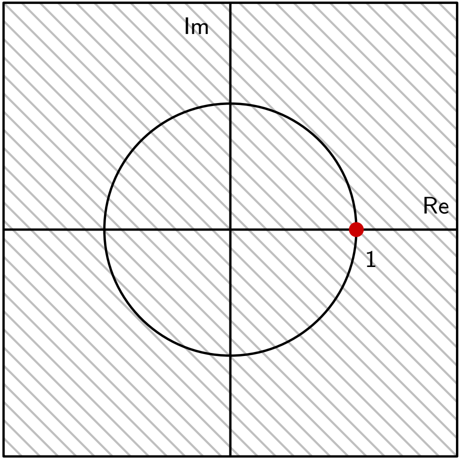

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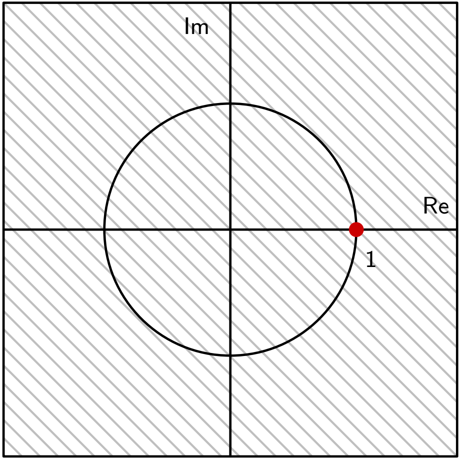

DC removal

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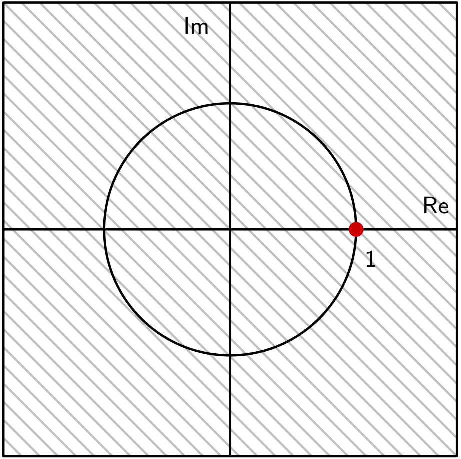
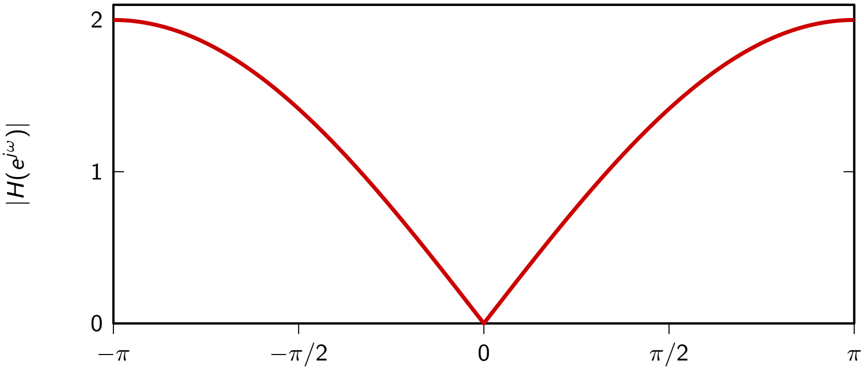
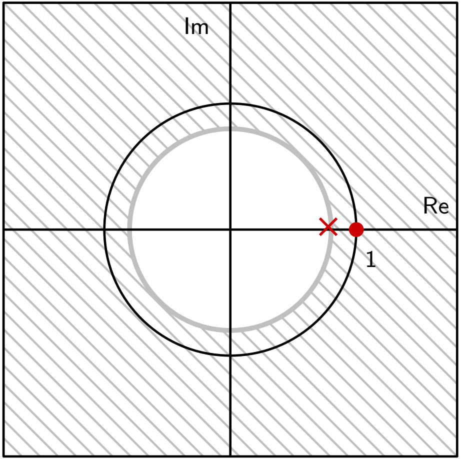


DC notch

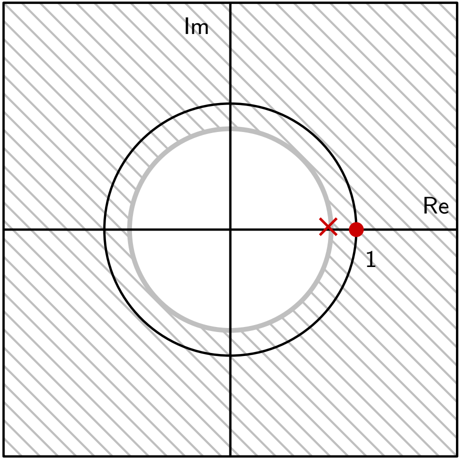


DC removal, improved



DC removal, improved

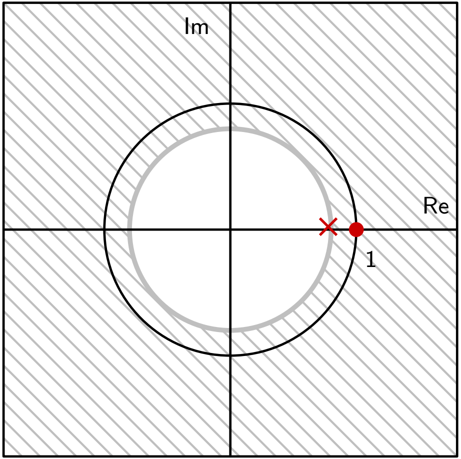
$$H(z) = \frac{1 - z^{-1}}{1 - \lambda z^{-1}}$$



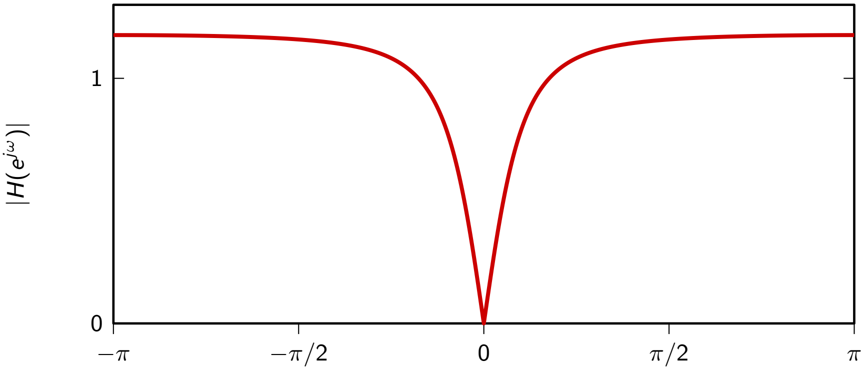
DC removal, improved

$$H(z) = \frac{1 - z^{-1}}{1 - \lambda z^{-1}}$$

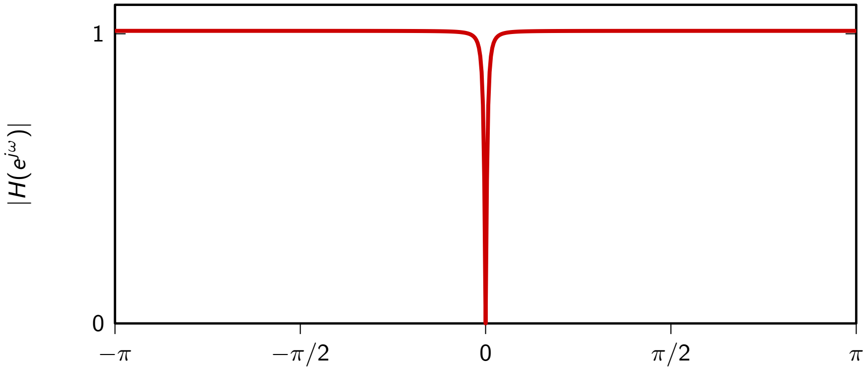
$$y[n] = \lambda y[n-1] + x[n] - x[n-1]$$



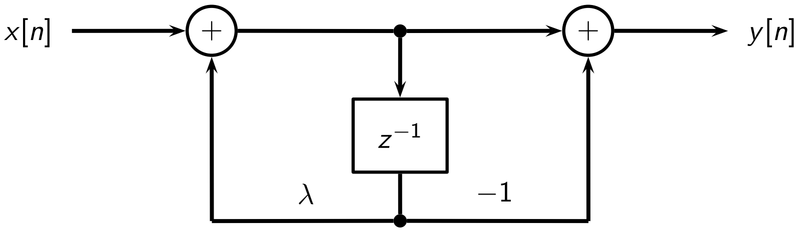
DC notch, $\lambda = 0.7$



DC notch, $\lambda = 0.98$



DC notch, filter structure



Hum removal

- similar to DC removal but we want to remove a specific nonzero frequency
- very useful for musicians: amplifiers for electric guitars pick up the hum from the electric mains (50Hz in Europe and 60Hz in North America)
- we need to tune the hum removal according to country

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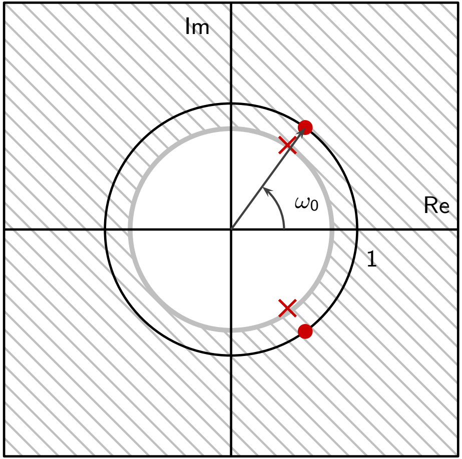
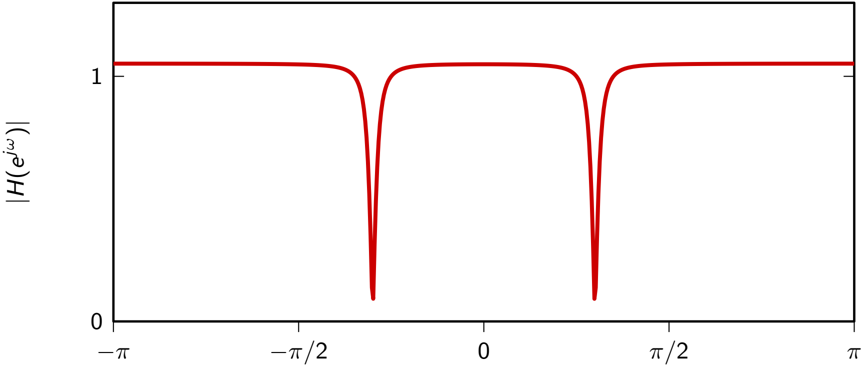


Hum removal

$$H(z) = \frac{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}{(1 - \lambda e^{j\omega_0} z^{-1})(1 - \lambda e^{-j\omega_0} z^{-1})}$$

Hum removal, $\lambda = 0.95$



Hum removal, filter structure