Physics 111 Section 1
Problem Set 03
September 12, 2016

The Lagrangian, Constrained Systems (Reading: Chapter 7.2–7.5)

Help:

i Modify Lagrange's equations for a 1D system with a non-conservative contraint force such as friction. In this case, which forces are included in the definition of $U(x)$? This is a straightforward problem, don't over think it.

7.12 Lagrange's equations in the form discussed in this chapter hold only if the forces (at least the nonconstraint forces) are derivable from a potential energy. To get an idea how they can be modified to include forces like friction, consider the following: A single particle in one dimension is subject to various conservative forces (net conservative force$= F = -\partial U/\partial x$) and a nonconservative force (let's call it F_{fric}). Define the Lagrangian as $\mathcal{L} = T - U$ and show that the appropriate modification is

$$\frac{\partial \mathcal{L}}{\partial x} + F_{\text{fric}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

■

ii Prove that Lagrange's equations hold for two particles moving under unspecified constraints.

7.13 In Section 7.4 [Equations (7.41 through (7.51)], I proved Lagrange's equations for a single particle constrained to move on a two-dimensional surface. Go through the same steps to prove Lagrange's equations for a system consisting of two particles subject to various unspecified constraints.

[*Hint*: The net force on particle 1 is the sum of the total constraint force $\mathbf{F}_1^{\text{cstr}}$ and the total nonconstraint force $\mathbf{F}_1$, and likewise for particle 2. The constraint forces come in many guises (the normal force of a surface, the tension force of a string tied between the particles, etc.), but it is always true that the net work done by all constraint forces in any displacement consistent with the constraints is zero—this is the defining property of constraint forces. Meanwhile, we take for granted that the nonconstraint forces are derivable from a potential energy $U(\mathbf{r}_1, \mathbf{r}_2, t)$; that is $\mathbf{F}_1 = -\nabla_1 U$ and likewise for particle 2. Write down the difference δS between the action integral for the right path given by $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ and any nearby wrong path given by $\mathbf{r}_1(t) + \boldsymbol{\epsilon}_1(t)$ and $\mathbf{r}_2(t) + \boldsymbol{\epsilon}_2(t)$. Paralleling the steps of Section 7.4, you can show that δS is given by an integral analogous to (7.49), and this is zero by the defining property of constraint forces.]

■

iii An oscillating bead on a spinning hoop. Getting the correct expression for the velocity of the bead is tricky; keep in mind that while ω is constant, $\phi = \phi(t)$.

7.35 Figure 7.16 is a bird's-eye view of a smooth horizontal wire hoop that is forced to rotate at a fixed angular velocity ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center with the diameter AB. Find the Lagrangian for this system using ϕ as your generalised coordinate. (Read the hint in Problem 7.29.) Use the Lagrange equation of motion to show that the bead oscillates about the point B exactly like a simple pendulum. What is the frequency of these oscillations if their amplitude is small?

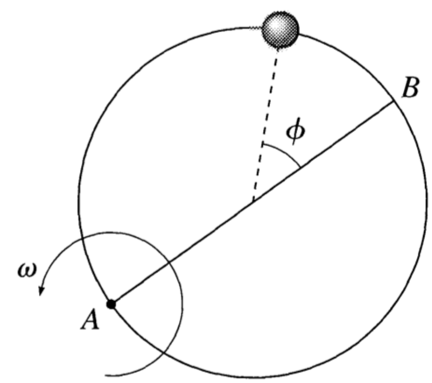

Figure 7.16 Problem 7.35

■

iv A central force potential. In part (c) focus on the θ equation and the type of motion it implies, and in part (d) do the same thing but for your ϕ equation.

7.39

(a) Write down the Lagrangian for a particle moving in three dimensions under the influence of a conservative force with potential energy $U(r)$, using spherical polar coordinates (r, θ, ϕ).

(b) Write down the three Lagrange equations and explain their significance in terms of radial acceleration, angular momentum, and so forth. (The θ equation is the tricky one, since you will find it implies that the ϕ component of $\boldsymbol{\ell}$ varies with time, which seems to contradict conservation of angular momentum. Remember, however that ℓ_ϕ is the component of $\boldsymbol{\ell}$ in a *variable* direction.)

(c) Suppose that initially the motion is in the equatorial plane (that is, $\theta_0 = \pi/2$ and $\dot{\theta}_0 = 0$). Describe the subsequent motion.

(d) Suppose instead that the initial motion is along a line of longitude (that is, $\dot{\phi}_0 = 0$). Describe the subsequent motion.

■