Physics 111 Section 1
Problem Set 19
December 5, 2016

Hamiltonian Mechanics - Hamilton's Equations (Reading: Chapter 13.1 - 13.4)

Help:

i An oscillating cart attached to a massive spring.

13.6 In discussing the oscillation of a cart on the end of a spring, we almost always ignore the mass of the spring. Set up the Hamilton $\mathcal{H}$ for a cart of mass m on a spring (force constant k) whose mass M is *not* negligible, using the extension x of the spring as the generalized coordinate. Solve Hamilton's equations and show that the mass oscillates with angular frequency $\omega = \sqrt{k/(m + M/3)}$. That is, the effect of the spring's mass is to add $M/3$ to m. (Assume that the spring's mass is distributed uniformly and that it stretches uniformly.)

ii A roller coaster: Hamiltonian versus Newton.

13.7 A roller coaster of mass m moves along a frictionless track that lies in the xy plane (x horizontally and y vertically up). The height of the track above the ground is given by $y = h(x)$.

(a) Using x as your generalized coordinate, write down the Lagrangian, the generalized momentum p, and the Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$ (as a function of x and p).

(b) Find Hamilton's equations and show that they agree with what you would get from the Newtonian approach. [*Hint*: You know from Section 4.7 that Newton's second law takes the form $F_{tang} = m\ddot{s}$, where s is the distance measured along the track. Rewrite this as an equation for $\ddot{x}$ and show that you get the same result from Hamilton's equations.]

iii The Hamiltonian and Lorentz force.

13.18 All of the examples in this chapter and all of the problems (except this one) treat forces that come from a potential energy $U(\mathbf{r})$ [or occasionally $U(\mathbf{r}, t)$]. However, the proof of Hamilton's equations given in Section 13.3 applies to any system for which Lagrange's equation hold, and this can include forces not derivable from a potential energy. An important example of such a force is the magnetic force on a charged particle.

(a) Use the Lagrangian (7.103) to show that the Hamiltonian for a charge q in an electromagnetic field is

$$\mathcal{H} = (\mathbf{p} - q\mathbf{A})^2/(2m) + qV$$

(This Hamiltonian plays an important role in the quantum mechanics of charged particles.)

(b) Show that Hamilton's equations are equivalent to the familiar Lorentz force equation $m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

iv Atwood's machine, once again, from the point of view of Hamiltonian mechanics.

13.23 Consider the modified Atwood shown in Fiqure 13.11. The two weights on the left have equal masses m and are connected by a massless spring of force constant k. The weight on the right has mass $M = 2m$, and the pulley is massless and frictionless. The coordinate x is the extension of the spring from its equilibrium length; that is, the length of the spring is $l_e + x$ where l_e is the equilibrium length (with all the weights in position and M held stationary).

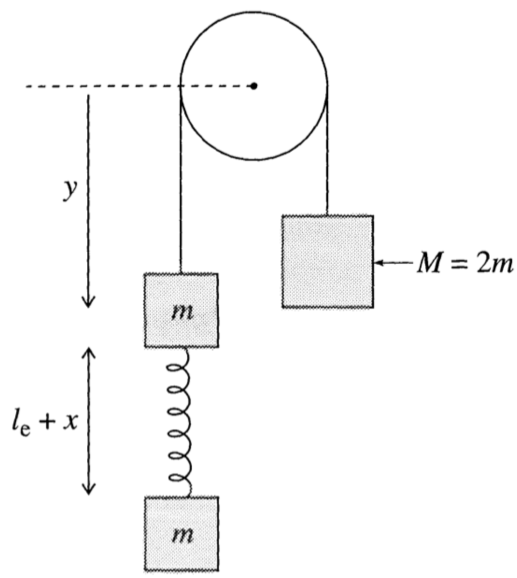

Figure 13.11 Problem 13.23

(a) Show that the total potential energy (spring plus gravitational) is just $U = \frac{1}{2}kx^2$ (plus a constant that we can take to be zero).

(b) Find the two momenta conjugate to x and y. Solve for $\dot{x}$ and $\dot{y}$, and write down the Hamiltonian. Show that the coordinate y is ignorable.

(c) Write down the four Hamilton equations and solve them for the following initial conditions: You hold the mass M fixed with the whole system in equilibrium and $y = y_0$. Still holding M fixed, you pull the lower mass m down a distance x_0, and at $t = 0$ you let go of both masses. [*Hint*: Write down the initial values of x, y and their momenta. You can solve the x equations by combining them into a second-order equation for x. Once you know $x(t)$, you can quickly write down the other three variables.] Describe the motion. In particular, find the frequency with which x oscillates.