Physics 111 Section 1
Problem Set 07
October 3, 2016

Central Forces: Effective Potential

Help:

i Properties of a zero-curl force

4.43 In Section 4.8, I claimed that a force $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ that is central and spherically symmetric is automatically conservative. Here are two ways to prove it:

(a) Since $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ is central and spherically symmetric, it must have the form $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = f(r)\hat{r}$. Using Cartesian coordinates, show that this implies that $\nabla \times \vec{\mathbf{F}} = 0$.

(b) Even quicker, using the expression given inside the back cover for $\nabla \times \vec{\mathbf{F}}$ in spherical polars, show that $\nabla \times \vec{\mathbf{F}} = 0$.

ii Two particles joined by a massless spring

8.3 Two particles of masses m_1 and m_2 are joined by a massless spring of natural length L and force constant k. Initially, m_2 is resting on a table and I am holding m_1 vertically above m_2 at a height L. At time $t = 0$, I project m_1 vertically upward with initial velocity v_0. Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion. [*Hints*: See problem 8.2. Assume that v_0 is small enough that the two masses never collide.]

iii Effective potential for Hooke's law

8.13 Two particles whose reduced mass is μ interact via a potential energy $U = \frac{1}{2}kr^2$, where r is the distance between them.

(a) Make a sketch showing $U(r)$, the centrifugal potential energy $U_{cf}(r)$, and the effective potential energy $U_{eff}(r)$. (Treat the angular momentum l as a known, fixed constant.)

(b) Find the "equilibrium" separation r_0, the distance at which the two particles can circle each other with constant r. [*Hint*: This requires that dU_{eff}/dr be zero.]

(c) By making a Taylor expansion of $U_{eft}(r)$ about the equilibrium point r_0 and neglecting all terms in $(r - r_0)^3$ and higher, find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from separation r_0.

iv Effective potential for closed orbits. Hint: you should be able to show that

$$U''(r_0) = \frac{(n+2)l^2}{\mu {r_0}^4}$$

8.14 Consider a particle of reduced mass μ orbiting in a central force with $U = kr^n$ where $kn > 0$.

(a) Explain what the condition $kn > 0$ tells us about the force. Sketch the effective potential energy U_{eff} for the cases that $n = 2, -1$, and -3.

(b) Find the radius at which the particle (with given angular momentum l) can orbit at a fixed radius. For what values of n is this circular orbit stable? Do your sketches confirm this conclusion?

(c) For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{osc} = \tau_{orb}/\sqrt{n+2}$. Argue that if $\sqrt{n+2}$ is a rational number, these orbits are closed. Sketch them for the cases that $n = 2, -1$, and 7.