Physics 111 Section 1
Problem Set 07
October 3, 2016

Central Forces: Effective Potential (Chapter 8.1 - 8.4)

**Help:**

**i** Properties of a zero-curl force

**4.43** In Section 4.8, I claimed that a force $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ that is central and spherically symmetric is automatically conservative. Here are two ways to prove it:

(a) Since $\vec{\mathbf{F}}(\vec{\mathbf{r}})$ is central and spherically symmetric, it must have the form $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = f(r)\hat{r}$. Using Cartesian coordinates, show that this implies that $\nabla \times \vec{\mathbf{F}} = 0$.

(b) Even quicker, using the expression given inside the back cover for $\nabla \times \vec{\mathbf{F}}$ in spherical polars, show that $\nabla \times \vec{\mathbf{F}} = 0$.

**ii** Two particles joined by a massless spring

**8.3** Two particles of masses $m_1$ and $m_2$ are joined by a massless spring of natural length $L$ and force constant $k$. Initially, $m_2$ is resting on a table and I am holding $m_1$ vertically above $m_2$ at a height $L$. At time $t = 0$, I project $m_1$ vertically upward with initial velocity $v_0$. Find the positions of the two masses at any subsequent time $t$ (before either mass returns to the table) and describe the motion. [*Hints*: See problem 8.2. Assume that $v_0$ is small enough that the two masses never collide.]

**iii** Effective potential for Hooke's law

**8.13** Two particles whose reduced mass is $\mu$ interact via a potential energy $U = \frac{1}{2}kr^2$, where $r$ is the distance between them.

(a) Make a sketch showing $U(r)$, the centrifugal potential energy $U_{cf}(r)$, and the effective potential energy $U_{eff}(r)$. (Treat the angular momentum $l$ as a known, fixed constant.)

(b) Find the "equilibrium" separation $r_0$, the distance at which the two particles can circle each other with constant $r$. [*Hint*: This requires that $dU_{eff}/dr$ be zero.]

(c) By making a Taylor expansion of $U_{eft}(r)$ about the equilibrium point $r_0$ and neglecting all terms in $(r - r_0)^3$ and higher, find the frequency of small oscillations about the circular orbit if the particles are disturbed a little from separation $r_0$.

**iv** Effective potential for closed orbits. Hint: you should be able to show that

$$U''(r_0) = \frac{(n+2)l^2}{\mu {r_0}^4}$$

**8.14** Consider a particle of reduced mass $\mu$ orbiting in a central force with $U = kr^n$ where $kn > 0$.

(a) Explain what the condition $kn > 0$ tells us about the force. Sketch the effective potential energy $U_{eff}$ for the cases that $n = 2, -1$, and $-3$.

(b) Find the radius at which the particle (with given angular momentum $l$) can orbit at a fixed radius. For what values of $n$ is this circular orbit stable? Do your sketches confirm this conclusion?

(c) For the stable case, show that the period of small oscillations about the circular orbit is $\tau_{osc} = \tau_{orb}/\sqrt{n+2}$. Argue that if $\sqrt{n+2}$ is a rational number, these orbits are closed. Sketch them for the cases that $n = 2, -1$, and 7.