Physics 111 Section 1
Problem Set 08
October 5, 2016

Central Forces: Kepler Orbits

Help:

i Convince yourself once and for all of this surprising and historically important result: particles in bound Kepler orbits follow an elliptical path with the sun at a focus. Your ancestors spent lifetimes trying to understand the celestial bodies, and how you are among the select few who do.

8.16 We have proved in (8.49) that any Kepler orbit can be written in the form $r(\phi) = c/1 + \epsilon\cos\phi)$, where $c > 0$ and $\epsilon \geq 0$. For the case that $0 \leq \epsilon < 1$, rewrite this equation in rectangular coordinates (x, y) and prove that the equation can be cast in the form (8.51), which is the equation of an ellipse. Verify the values of the constants given in (8.52).

(8.49):

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}$$

(8.51):

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

(8.52):

$$a = \frac{c}{1-\epsilon^2}, \qquad b = \frac{c}{\sqrt{1-\epsilon^2}}, \qquad \text{and } d = a\epsilon$$

ii Now convince yourself once and for all that particles in unbound Kepler orbits follow parabolic or hyperbolic trajectories.

8.30 The general Kepler orbit is given in polar coordinates by (8.49). Rewrite this in Cartesian coordinates for the cases that $\epsilon = 1$ and $\epsilon > 1$. Show that if $\epsilon = 1$, you get the parabola (8.60), and if $\epsilon > 1$, the hyperbola (8.61). For the latter, identify the constants α, β, and δ in terms of c and ϵ.

(8.60):

$$y^2 = c^2 - 2cx$$

(8.61):

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$$

iii A repulsive inverse-square law

8.13 Consider the motion of two particles subject to a *repulsive* inverse-square force (for example, two positive charges). Show that this system has no states with $E < 0$ (as measured in the CM frame), and that in all states with $E > 0$, the relative motion follows a hyperbola. Sketch a typical orbit. [*Hint*: You can follow closely the analysis of Sections 8.6 and 8.7 except that you must reverse the force; probably the simplest way to do this is to change the sign of γ in (8.44) and all subsequent equations (so that $F(r) = +\gamma/r^2$) and then keep γ itself positive. Assume $\ell \neq 0$.]