Solutions

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April 11, 2024

Contents

1 Sound generation by hot perturbated vortex.

Our purpose is to consider the flow that generates sound by entropy inhomogeneties.

1.1 Formulation of the problem.

Let's consider the distributed homogeneous vortex located in a resting infinite space filled with an ideal gas. To create simple entropy inhomogeneities, **mean** flow can be defined in three ways:

1. Semi-constant density:
$$\rho(r) = \begin{cases} \rho_1 = \text{const},\ r \leqslant a \\ \rho_2 = \text{const},\ r > a \end{cases} \tag{1}$$

2. Semi-constant temperature:
$$T(r) = \begin{cases} T_1 = \text{const},\ r \leqslant a \\ T_2 = \text{const},\ r > a \end{cases} \tag{2}$$

3. Semi-constant entropy:
$$S(r) = \begin{cases} S_1 = \text{const},\ r \leqslant a \\ S_2 = \text{const},\ r > a \end{cases} \tag{3}$$

Besides let the pressure at infinity $p(\infty) = p_\infty$ be known and the vorticity distribution be given:
$$\Omega(r) = \begin{cases} \Omega = \text{const},\ r \leqslant a \\ 0,\ r > a. \end{cases}$$

Governing equations are compressible Euler's equations. Since the problem has cylindrical symmetry, it is convenient to work in the corresponding coordinates:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}\, \rho \boldsymbol{u} = 0 \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}, \nabla)\boldsymbol{u} = -\dfrac{\nabla p}{\rho} \\ \rho c_p \dfrac{dT}{dt} - \dfrac{dp}{dt} = 0 \\ p = \rho R T \\ S = c_v \ln\left(\dfrac{p}{\rho^\gamma}\right) \\ \Omega = \operatorname{rot}\, \boldsymbol{u} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{1}{r}\dfrac{\partial \rho u_r r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \rho u_\theta}{\partial \theta} = 0 \\ \dfrac{\partial u_r}{\partial t} + u_r \dfrac{\partial u_r}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \\ \dfrac{\partial u_\theta}{\partial t} + u_r \dfrac{\partial u_\theta}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_\theta u_r}{r} = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial \theta} \\ \rho c_p \left(\dfrac{\partial T}{\partial t} + u_r \dfrac{\partial T}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial T}{\partial \theta}\right) - \dfrac{\partial p}{\partial t} - u_r \dfrac{\partial p}{\partial r} - \dfrac{u_\theta}{r}\dfrac{\partial p}{\partial \theta} = 0 \\ p = \rho R T \\ S = c_v \ln\left(\dfrac{p}{\rho^\gamma}\right) \\ \Omega = \dfrac{1}{r}\left(\dfrac{\partial r u_\theta}{\partial r} - \dfrac{\partial u_r}{\partial \theta}\right) \end{cases} \tag{4}$$

Lets find the mean flow in different formulations of the problem (1), (2), (3), and then solve the corresponding perturbed problems.

1.2 Mean flow. General words.

The equations (4) at steady flow become simple (due to cylindrical symmetry):

$$\begin{cases} \dfrac{u_\theta^2}{r} = \dfrac{1}{\rho}\dfrac{dp}{dr} \\ p = \rho R T \\ S = c_v \ln\left(\dfrac{p}{\rho^\gamma}\right) \\ \Omega = \dfrac{1}{r}\dfrac{d(r u_\theta)}{dr} \end{cases}$$

In all cases of specifying the mean flow, there is the same mean velocity, determined by vorticity:

$$\Omega(r) = \frac{1}{r}\frac{d(r u_\theta)}{dr} \Rightarrow u_\theta = \begin{cases} \dfrac{\Omega r}{2},\ r \leqslant a \\ \dfrac{\Omega a^2}{2r},\ r > a \end{cases}$$

1.3 Mean flow. Density is semi-constant

Solution for this case (using pressure continuity and the conditions at ∞):

$$p(r) = \begin{cases} p_\infty - \rho_2 \dfrac{\Omega^2 a^2}{8} - \rho_1 \dfrac{\Omega^2}{8}(a^2 - r^2),\ r \leqslant a \\ p_\infty - \rho_2 \dfrac{\Omega^2 a^2}{8}\left(\dfrac{a}{r}\right)^2,\ r > a \end{cases} \tag{5}$$

$$T(r) = \frac{p}{\rho R} = \begin{cases} \dfrac{1}{\rho_1 R}\left(p_\infty - \rho_2 \dfrac{\Omega^2 a^2}{8} - \rho_1 \dfrac{\Omega^2}{8}(a^2 - r^2)\right),\ r \leqslant a \\ \dfrac{1}{\rho_2 R}\left(p_\infty - \rho_2 \dfrac{\Omega^2 a^2}{8}\left(\dfrac{a}{r}\right)^2\right),\ r > a \end{cases}$$

$$S(r) = c_v \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$T\nabla S = T\frac{dS}{dr} = \begin{cases} \dfrac{1}{\gamma - 1}\dfrac{\Omega^2 r}{4},\ r \leqslant a \\ \dfrac{1}{\gamma - 1}\dfrac{\Omega^2 a}{4}\left(\dfrac{a}{r}\right)^3,\ r > a \end{cases}$$

1.4 Mean flow. Temperature is semi-constant

Let us express the density from the equation of state and substitute it into the momentum equation.

$$\frac{u_\theta^2}{r} = \frac{c^2}{\gamma}\frac{d\ln p}{dr},$$

where $c^2 = \gamma R T$. Therefore pressure field is:

$$p(r) = \begin{cases} C_1 \exp\left(\dfrac{\gamma}{2} M_1^2 \left(\dfrac{r}{a}\right)^2\right),\ r \leqslant a \\ C_2 \exp\left(-\dfrac{\gamma}{2} M_2^2 \left(\dfrac{a}{r}\right)^2\right),\ r > a \end{cases}$$

where $M_{1,2} = u_{\theta max}/c_{1,2} = \Omega a/(2c_{1,2})$. The constants can be determined using density continuity and the condition at ∞:

$$p(r) = \begin{cases} p_\infty \exp\left\{\frac{\gamma}{2}\left[M_1^2\left(\left(\frac{r}{a}\right)^2 - 1\right) - M_2^2\right]\right\}, \ r \leqslant a \\ p_\infty \exp\left(-\frac{\gamma}{2}M_2^2\left(\frac{a}{r}\right)^2\right), \ r > a \end{cases}$$

$$\rho(r) = \frac{\gamma p}{c^2} = \begin{cases} \frac{\gamma}{c_1^2}p_\infty \exp\left\{\frac{\gamma}{2}\left[M_1^2\left(\left(\frac{r}{a}\right)^2 - 1\right) - M_2^2\right]\right\}, \ r \leqslant a \\ \frac{\gamma}{c_2^2}p_\infty \exp\left(-\frac{\gamma}{2}M_2^2\left(\frac{a}{r}\right)^2\right), \ r > a \end{cases}$$

$$S(r) = c_v \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$T\nabla S = T\frac{dS}{dr} = \begin{cases} -\frac{\Omega^2 r}{4}, \ r \leqslant a \\ -\frac{\Omega^2 a}{4}\left(\frac{a}{r}\right)^3, \ r > a \end{cases}$$

1.5 Mean flow. Entropy is semi-constant

Let us express the density from the expression for entropy and substitute it into the momentum equation.

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{dp}{dr} = \frac{1}{Cp^{1/\gamma}}\frac{dp}{dr} = \frac{\gamma}{C(\gamma-1)}\frac{dp^{(\gamma-1)/\gamma}}{dr}$$

Further solution is carried out as in the case of semi-constant temperature or density.

$$p(r) = \begin{cases} \left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma} - \frac{C_1(\gamma-1)\Omega^2}{8\gamma}(a^2 - r^2)\right)^{\frac{\gamma}{\gamma-1}}, \ r \leqslant a \\ \left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma}\left(\frac{a}{r}\right)^2\right)^{\frac{\gamma}{\gamma-1}}, \ r > a \end{cases}$$

$$\rho(r) = \begin{cases} C_1 \cdot \left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma} - \frac{C_1(\gamma-1)\Omega^2}{8\gamma}(a^2 - r^2)\right)^{\frac{1}{\gamma-1}}, \ r \leqslant a \\ C_2 \cdot \left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma}\left(\frac{a}{r}\right)^2\right)^{\frac{1}{\gamma-1}}, \ r > a \end{cases}$$

$$T(r) = \frac{1}{R}\frac{p}{\rho} = \frac{c_v}{\gamma-1} \cdot \begin{cases} \frac{1}{C_1}\left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma} - \frac{C_1(\gamma-1)\Omega^2}{8\gamma}(a^2 - r^2)\right), \ r \leqslant a \\ \frac{1}{C_2}\left(p_\infty^{(\gamma-1)/\gamma} - \frac{C_2(\gamma-1)\Omega^2 a^2}{8\gamma}\left(\frac{a}{r}\right)^2\right), \ r > a \end{cases}$$

$$T\nabla S = T\frac{dS}{dr} = \begin{cases} 0, \ r \leqslant a \\ 0, \ r > a \end{cases}$$

1.6 Mean flow. Asymptotics for $\Omega \to 0$

It is interesting to consider the limiting case of low flow velocities ($\Omega \to 0$). In the case of semi-constant density, the search for asymptotics is not required; the expression (5) itself is the asymptotics.

1. Semi-constant temperature. Pressure asymptotics:

$$p(r) = \begin{cases} p_\infty \left(1 + \frac{\gamma\Omega^2}{8}\left[\frac{1}{c_1^2}(r^2 - a^2) - \frac{a^2}{c_2^2}\right]\right), & r \leqslant a \\ p_\infty \left(1 - \frac{\gamma\Omega^2}{8}\frac{a^2}{c_2^2}\left(\frac{a}{r}\right)^2\right), & r > a \end{cases} \tag{6}$$

2. Semi-constant entropy. Pressure asymptotics:

$$p(r) = \begin{cases} p_\infty \left(1 - \frac{C_2\Omega^2 a^2}{8p_\infty^{(\gamma-1)/\gamma}} - \frac{C_1\Omega^2}{8p_\infty^{(\gamma-1)/\gamma}}(a^2 - r^2)\right), & r \leqslant a \\ p_\infty \left(1 - \frac{C_2\Omega^2 a^2}{8p_\infty^{(\gamma-1)/\gamma}}\left(\frac{a}{r}\right)^2\right), & r > a \end{cases} \tag{7}$$

Let me remind you that the constants correspond to the given thermodynamic parameters:

$$c^2 = \gamma RT, \quad C = \exp\left(-\frac{S}{c_p}\right).$$

And there are values of temperature $T_{1,2}$ and entropy $S_{1,2}$ such that expressions (6) and (7) coincide with expression (5).

1.7 Perturbations. General words.

Let's consider the parameters of the flow, which is described by equations (4), in the form of the sum of the mean and the disturbed flows:

$$p \to p_0 + p, \quad \rho \to \rho_0 + \rho, \quad T \to T_0 + T, \quad u_\theta \to u_{\theta 0} + u_\theta, \quad u_r \to u_r$$

Let's also assume that the perturbations are small compared to the mean flow. Substitution of these in governing equations gives us:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho_0 u_r}{\partial r} + \frac{\rho_0 u_r}{r} + \frac{1}{r}\frac{\partial \rho_0 u_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial \rho u_{\theta 0}}{\partial \theta} = 0 \\ \frac{\partial u_r}{\partial t} + \frac{u_{\theta 0}}{r}\frac{\partial u_r}{\partial \theta} - \frac{2u_\theta u_{\theta 0}}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} + \frac{\rho}{\rho_0^2}\frac{\partial p_0}{\partial r} \\ \frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_{\theta 0}}{\partial r} + \frac{u_{\theta 0}}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_{\theta 0}}{r} = -\frac{1}{\rho_0 r}\frac{\partial p}{\partial \theta} \\ \rho_0 c_p\left(\frac{\partial T}{\partial t} + \frac{u_{\theta 0}}{r}\frac{\partial T}{\partial \theta}\right) - \frac{\partial p}{\partial t} - \frac{u_{\theta 0}}{r}\frac{\partial p}{\partial \theta} = 0 \\ p = \rho_0 RT + \rho RT_0 \end{cases} \tag{8}$$

If we denote the differential operator as

$$\left(\frac{\partial}{\partial t} + \frac{u_{\theta 0}}{r}\frac{\partial}{\partial \theta}\right) = D$$

then equations (8) transform into

$$\begin{cases} D\rho + \rho_0 \dfrac{\partial u_r}{\partial r} + \left[\left(\dfrac{d\rho_0}{dr} \right) + \dfrac{\rho_0}{r} \right] u_r + \dfrac{\rho_0}{r} \dfrac{\partial u_\theta}{\partial \theta} = 0 \\ Du_r - \dfrac{2u_{\theta 0}}{r} u_\theta + \dfrac{1}{\rho_0} \dfrac{\partial p}{\partial r} - \dfrac{u_{\theta 0}^2}{\rho_0 r} \rho = 0 \\ Du_\theta + \left[\left(\dfrac{du_{\theta 0}}{dr} \right) + \dfrac{u_{\theta 0}}{r} \right] u_r + \dfrac{1}{\rho_0 r} \dfrac{\partial p}{\partial \theta} = 0 \\ Dp = c^2 D\rho, \quad c^2 = \gamma R T_0 \end{cases}$$

to be continuous

1.8 Perturbations. Density is semi-constant.

For simplicity, we assume that there are no density perturbations. So, we can use stream-function:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}.$$

Therefore governing equations (**ADD LINK**) transform into

$$\begin{cases} (-i\omega)\dfrac{(in)}{r}\psi + \dfrac{u_{\theta 0}}{r}\dfrac{(in)^2}{r}\psi + \dfrac{2u_{\theta 0}}{r}\psi' = -\dfrac{p'}{\rho_0}, \\ (i\omega)\psi' + \dfrac{(in)}{r}\psi\left(\dfrac{du_{\theta 0}}{dr}\right) - u_{\theta 0}\dfrac{(in)}{r}\psi' + \dfrac{u_{\theta 0}}{r}\dfrac{(in)}{r}\psi = -\dfrac{(in)}{\rho_0 r}p. \end{cases} \tag{9}$$

Getting rid of pressure, we obtain an equation for the stream function:

$$[(i\omega)r - (in)u_{\theta 0}]\left(\psi'' + \frac{\psi'}{r} - \frac{n^2}{r^2}\psi\right) = 0 \tag{10}$$

Let us consider such flows, where term $(i\omega)r - (in)u_{\theta 0}$ is not zero. Then, solution of (10) is:

$$\psi(r) = \begin{cases} C_1 r^n, & r < a, \\ C_2 r^{-n}, & r \geq a \end{cases}$$

Due to the condition of continuity of the normal velocity component, we conclude that the stream function must also be continuous, therefore

$$C_1 a^n = C_2 a^{-n}$$

Another boundary condition is density continuity. Substituting the stream function into the second equation of the system (9) and equating the pressure inside the vortex and outside it (at $r = a$), we obtain the dispersion relation:

$$\omega_n = \frac{\Omega(n-1)}{1 + (\rho_2/\rho_1)}$$