Computational Fluid Dynamics
http://www.nd.edu/~gtryggva/CFD-Course/

Computational Fluid Dynamics

Lecture 6
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Theory of Partial Differential Equations

Computational Fluid Dynamics
Outline

- Basic Properties of PDE
- Quasi-linear First Order Equations
  - Characteristics
  - Linear and Nonlinear Advection Equations
- Quasi-linear Second Order Equations
  - Classification: hyperbolic, parabolic, elliptic
  - Domain of Dependence/Influence

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Examples of equations

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0$$ Advection

$$\frac{\partial f}{\partial t} - D\frac{\partial^2 f}{\partial x^2} = 0$$ Diffusion

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = 0$$ Wave propagation

Evolution in time

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$ Laplace equation

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Navier-Stokes equations

Diffusion part

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Advection part

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$ Laplace part

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Definitions

- The order of PDE is determined by the highest derivatives
- Linear if no powers or products of the unknown functions or its partial derivatives are present.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f, \qquad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 2xf = 0$$

- Quasi-linear if it is true for the partial derivatives of highest order.

$$f\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial y}\right)^2 = f, \qquad x^2\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y} = f^2$$

Quasi-linear first order partial differential equations

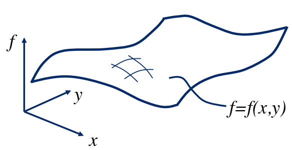
Consider the quasi-linear first order equation

$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=c$$

where the coefficients are functions of x,y, and f, but not the derivatives of f:

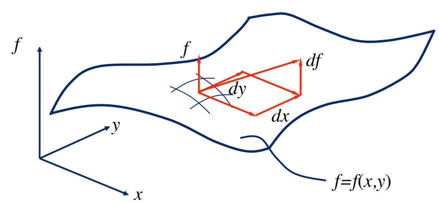
$$a=a(x,y,f)$$
$$b=b(x,y,f)$$
$$c=c(x,y,f)$$

The solution of this equations defines a single valued surface $f(x,y)$ in three-dimensional space:

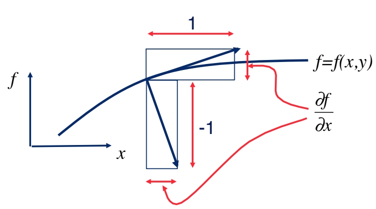


An arbitrary change in f is given by:

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$



The normal vector to the curve $f=f(x,y)$



Same arguments in the y-direction. Thus $\boldsymbol{n}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)$

The original equation and the conditions for a small change can be rewritten as:

$$a\frac{\partial f}{\partial x}+b\frac{\partial f}{\partial y}=c \quad\Rightarrow\quad \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)\cdot(a,b,c)=0$$

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \quad\Rightarrow\quad \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)\cdot(dx,dy,df)=0$$

Normal to the surface

Both (a,b,c) and (dx,dy,df) are in the surface!

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Picking the displacement in the direction of (a,b,c)

$$\Rightarrow \quad (dx, dy, df) = ds(a,b,c)$$

Separating the components

$$\Rightarrow \quad \underbrace{\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b;}_{\frac{dx}{dy} = \frac{a}{b}} \quad \frac{df}{ds} = c;$$

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The three equations specify lines in the x-y plane

$$\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{df}{ds} = c;$$

Characteristics

Given the initial conditions:

$$x = x(s, t_o); \quad y = y(s, t_o); \quad f = f(s, t_o);$$

the equations can be integrated in time

slope $\quad \dfrac{dx}{dy} = \dfrac{a}{b}$

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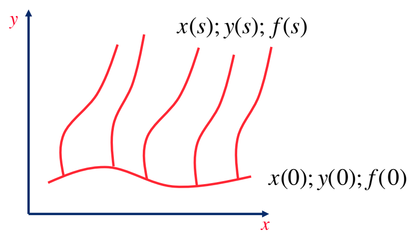
$$\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{df}{ds} = c;$$



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Consider the linear advection equation

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0$$

The characteristics are given by:

$$\frac{dt}{ds} = 1; \quad \frac{dx}{ds} = U; \quad \frac{df}{ds} = 0;$$

or

$$\frac{dx}{dt} = U; \quad df = 0;$$

Which shows that the solution moves along straight characteristics without changing its value

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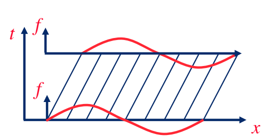
Graphically: $\quad f(x,t) = f_{t=0}(x - Ut)$



Notice that these results are specific for:

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0$$

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The solution is therefore:

$f(x,t) = g(x - Ut)$ where $g(x) = f(x, t=0)$

This can be verified by direct substitution:

Set $\quad \eta(x,t) = x - Ut$

Then $\quad \dfrac{\partial f}{\partial t} = \dfrac{\partial g}{\partial \eta}\dfrac{\partial \eta}{\partial t} = \dfrac{\partial g}{\partial \eta}(-U)$ and $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial \eta}\dfrac{\partial \eta}{\partial x} = \dfrac{\partial g}{\partial \eta}(1)$

Substitute into the original equation

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \cancel{\frac{\partial g}{\partial \eta}}(-U) + U\cancel{\frac{\partial g}{\partial \eta}} = 0$$

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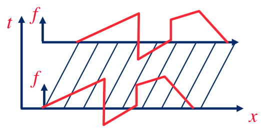
Discontinuous initial data

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0 \qquad f(x,t) = g(x - Ut)$$

Since the solution propagates along characteristics completely independently of the solution at the next spatial point, there is no requirement that it is differentiable or even continuous



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Add a source $\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = -f$

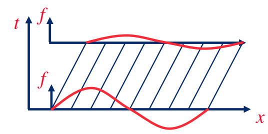
The characteristics are given by:

$$\frac{dt}{ds} = 1; \quad \frac{dx}{ds} = U; \quad \frac{df}{ds} = -f;$$

or $\frac{dx}{dt} = U; \quad \frac{df}{dt} = -f$

$$\Rightarrow f = f(0)e^{-t}$$

Moving wave with
decaying amplitude



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Consider a nonlinear (quasi-linear) advection equation

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0$$

The characteristics are given by:

$$\frac{dt}{ds} = 1; \; \frac{dx}{ds} = f; \; \frac{df}{ds} = 0;$$

or

$$\frac{dx}{dt} = f; \qquad df = 0;$$

The slope of the characteristics depends on the value of $f(x,t)$.

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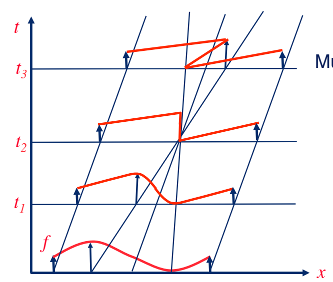

Multi-valued solution

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Why unphysical solutions?

- Because mathematical equation neglects some physical process (dissipation)

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} - \varepsilon\frac{\partial^2 f}{\partial x^2} = 0 \qquad \text{Burgers Equation}$$

Additional condition is required to pick out the physically Relevant solution

Correct solution is expected from Burgers equation with $\varepsilon \to 0$

Entropy Condition

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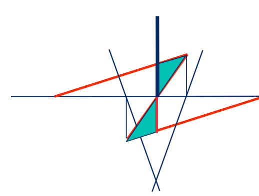
Constructing physical solutions

$$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0$$

In most cases the solution is not allowed to be multiple valued and the “physical solution” must be reconstructed using conservation of f



The discontinuous solution propagates with a shock speed that is different from the slope of the characteristics on either side

Quasi-linear Second order partial differential equations

Consider

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=d$$

where

$$a=a(x,y,f,f_x,f_y)$$
$$b=b(x,y,f,f_x,f_y)$$
$$c=c(x,y,f,f_x,f_y)$$
$$d=d(x,y,f,f_x,f_y)$$

First write the second order PDE as a system of first order equations

Define $v=\dfrac{\partial f}{\partial x}$ and $w=\dfrac{\partial f}{\partial y}$

then

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=d \quad\longrightarrow\quad a\frac{\partial v}{\partial x}+b\frac{\partial v}{\partial y}+c\frac{\partial w}{\partial y}=d$$

The second equation is obtained from

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x} \quad\longrightarrow\quad \frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}=0$$

Thus

$$a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial x\partial y}+c\frac{\partial^2 f}{\partial y^2}=d$$

Is equivalent to:

$$a\frac{\partial v}{\partial x}+b\frac{\partial v}{\partial y}+c\frac{\partial w}{\partial y}=d$$
$$\frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}=0$$

Any high-order PDE can be rewritten as a system of first order equations!

where $v=\dfrac{\partial f}{\partial x}$ $\quad w=\dfrac{\partial f}{\partial y}$

$$a\frac{\partial v}{\partial x}+b\frac{\partial v}{\partial y}+c\frac{\partial w}{\partial y}=d \qquad \frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}=0$$

In matrix form

$$\begin{pmatrix}\partial v/\partial x\\ \partial w/\partial x\end{pmatrix}+\begin{pmatrix}b/a & c/a\\ -1 & 0\end{pmatrix}\begin{pmatrix}\partial v/\partial y\\ \partial w/\partial y\end{pmatrix}=\begin{pmatrix}d/a\\ 0\end{pmatrix}$$

or $\quad \mathbf{u}_x+\mathbf{A}\mathbf{u}_y=\mathbf{s} \qquad \mathbf{u}=\begin{pmatrix}v\\ w\end{pmatrix}$

Are there lines in the x-y plane, along which the solution is determined by an ordinary differential equation?

The total derivative is

$$\frac{dv}{dx}=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial x}=\frac{\partial v}{\partial x}+\alpha\frac{\partial v}{\partial y} \qquad \text{where} \qquad \alpha=\frac{\partial y}{\partial x}$$

Rate of change of v with x, along the line y=y(x)

If there are lines (determined by α) where the solution is governed by ODE's, then it must be possible to rewrite the equations in such a way that the result contains only α and the total derivatives.

Add the original equations:

$$l_1\left(\frac{\partial v}{\partial x}+\frac{b}{a}\frac{\partial v}{\partial y}+\frac{c}{a}\frac{\partial w}{\partial y}\right)+l_2\left(\frac{\partial w}{\partial x}-\frac{\partial v}{\partial y}\right)=l_1\frac{d}{a}$$

Is this ever equal to

$$l_1\left(\frac{\partial v}{\partial x}+\alpha\frac{\partial v}{\partial y}\right)+l_2\left(\frac{\partial w}{\partial x}+\alpha\frac{\partial w}{\partial y}\right)=l_1\frac{d}{a}$$

For some l' s and α

Compare the terms:

$$l_1\left(\boxed{\frac{\partial v}{\partial x}}+\frac{b}{a}\frac{\partial v}{\partial y}+\frac{c}{a}\frac{\partial w}{\partial y}\right)+l_2\left(\boxed{\frac{\partial w}{\partial x}}-\frac{\partial v}{\partial y}\right)=\boxed{l_1\frac{d}{a}}$$

$$l_1\left(\boxed{\frac{\partial v}{\partial x}}+\alpha\frac{\partial v}{\partial y}\right)+l_2\left(\boxed{\frac{\partial w}{\partial x}}+\alpha\frac{\partial w}{\partial y}\right)=\boxed{l_1\frac{d}{a}}$$

Therefore, we must have:

$$l_1\frac{b}{a}-l_2=l_1\alpha$$

$$l_1\frac{c}{a}=l_2\alpha$$

Characteristic lines exist if:

$$l_1\frac{b}{a}-l_2=l_1\alpha$$

$$l_1\frac{c}{a}=l_2\alpha$$

Or, in matrix form:

$$\begin{pmatrix} b/a-\alpha & -1 \\ c/a & -\alpha \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The original equation is:

$$\begin{pmatrix} \partial v/\partial x \\ \partial w/\partial x \end{pmatrix}+\underbrace{\begin{pmatrix} b/a & c/a \\ -1 & 0 \end{pmatrix}}_{A}\begin{pmatrix} \partial v/\partial y \\ \partial w/\partial y \end{pmatrix}=\begin{pmatrix} d/a \\ 0 \end{pmatrix}$$

Rewrite

$$\begin{pmatrix} b/a-\alpha & -1 \\ c/a & -\alpha \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as

$$\underbrace{\begin{pmatrix} b/a & -1 \\ c/a & 0 \end{pmatrix}}_{A^T}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}-\begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \left(A^T-\alpha I\right)l=0$$

The equation has a solution only if the determinant is zero

$$\begin{pmatrix} b/a-\alpha & -1 \\ c/a & -\alpha \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant is: $\left|A^T-\alpha I\right|=0$

$$-\alpha\left(\frac{b}{a}-\alpha\right)+\frac{c}{a}=0$$

Or, solving for α

$$\alpha=\frac{1}{2a}\left(b\pm\sqrt{b^2-4ac}\right)$$

$$\alpha=\frac{1}{2a}\left(b\pm\sqrt{b^2-4ac}\right)$$

$b^2-4ac>0$ Two real characteristics

$b^2-4ac=0$ One real characteristics

$b^2-4ac<0$ No real characteristics

Computational Fluid Dynamics

Examples

Computational Fluid Dynamics
Examples

$$\frac{\partial^2 f}{\partial x^2} - c^2 \frac{\partial^2 f}{\partial y^2} = 0$$

Comparing with the standard form

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = d$$

shows that $a = 1; \quad b = 0; \quad c = -c^2; \quad d = 0;$

$$b^2 - 4ac = 0^2 + 4 \cdot 1 \cdot c^2 = 4c^2 > 0$$

Hyperbolic

Computational Fluid Dynamics
Examples

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Comparing with the standard form

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = d$$

shows that $a = 1; \quad b = 0; \quad c = 1; \quad d = 0;$

$$b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0$$

Elliptic

Computational Fluid Dynamics
Examples

$$\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial y^2}$$

Comparing with the standard form

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = d$$

shows that $a = 0; \quad b = 0; \quad c = -D; \quad d = 0;$

$$b^2 - 4ac = 0^2 + 4 \cdot 0 \cdot D = 0$$

Parabolic

Computational Fluid Dynamics
Examples

Wave equation	$\frac{\partial^2 f}{\partial x^2} - c^2 \frac{\partial^2 f}{\partial y^2} = 0$	Hyperbolic
Diffusion equation	$\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial y^2}$	Parabolic
Laplace equation	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$	Elliptic

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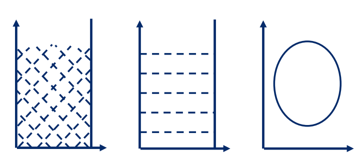
$$\alpha = \frac{\partial y}{\partial x} \qquad \alpha = \frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right)$$



Hyperbolic　　Parabolic　　Elliptic

Summary

Why is the classification Important?

1. Initial and boundary conditions
2. Different physics
3. Different numerical method apply

Navier-Stokes equations

Parabolic part

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Hyperbolic part

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Elliptic equation

The Navier-Stokes equations contain three equation types that have their own characteristic behavior

Depending on the governing parameters, one behavior can be dominant

The different equation types require different solution techniques

For inviscid compressible flows, only the hyperbolic part survives

The Wave Equation and Advection

$$\frac{\partial^2 f}{\partial t^2}-c^2\frac{\partial^2 f}{\partial x^2}=0$$

First write the equation as a system of first order equations

Introduce $\quad u=\frac{\partial f}{\partial t};\quad v=\frac{\partial f}{\partial x};$

yielding

$$\frac{\partial u}{\partial t}-c^2\frac{\partial v}{\partial x}=0$$
$$\frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0$$

from the pde

since $\quad \frac{\partial^2 f}{\partial t\partial x}=\frac{\partial^2 f}{\partial x\partial t}$

Computational Fluid Dynamics
Wave equation

To find the characteristics

$$\frac{\partial u}{\partial t} - c^2\frac{\partial v}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0$$
$$\rightarrow \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = 0$$

$$\rightarrow \det\left(\mathbf{A}^T - \alpha\mathbf{I}\right) = \det\begin{bmatrix} -\alpha & -1 \\ -c^2 & -\alpha \end{bmatrix} = \left(\alpha^2 - c^2\right) = 0$$

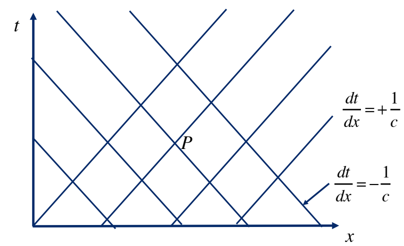
$$\Rightarrow \alpha = \pm c$$

We can also use

$\alpha = \frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right)$ with $b = 0; \quad a = 1; \quad c = -c^2$

Computational Fluid Dynamics
Wave equation

Two characteristic lines $\frac{dt}{dx} = +\frac{1}{c}; \ \frac{dt}{dx} = -\frac{1}{c}$



Computational Fluid Dynamics
Wave equation

To find the solution we need to find the eigenvectors

$$\Rightarrow \alpha = \pm c$$

$$l_1\left(\frac{\partial u}{\partial t} - c^2\frac{\partial v}{\partial x} = 0\right)$$
$$+l_2\left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0\right)$$
$$\rightarrow \begin{bmatrix} -\alpha & -1 \\ -c^2 & -\alpha \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = 0$$

Take l_1=1

For $\alpha = +c$ $\quad -c\, l_1 - l_2 = 0 \quad \Rightarrow \quad l_2 = -c$

For $\alpha = -c$ $\quad +c\, l_1 - l_2 = 0 \quad \Rightarrow \quad l_2 = +c$

Computational Fluid Dynamics
Wave equation

For $\alpha = +c$ $\quad l_1 = 1 \quad l_2 = -c$

$$1\left(\frac{\partial u}{\partial t} - c^2\frac{\partial v}{\partial x} = 0\right) - c\left(\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0\right)$$

Add the equations

$$\rightarrow \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) - c\left(\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x}\right) = 0$$

$$\rightarrow \frac{du}{dt} - c\frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = +c$$

Relation between the total derivative on the characteristic

Similarly:

For $\alpha = -c$ $\quad l_1 = 1 \quad l_2 = +c$

$$\frac{du}{dt} + c\frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -c$$

Computational Fluid Dynamics
Wave equation

$$\frac{du}{dt} - c\frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = +c$$

$$\frac{du}{dt} + c\frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -c$$

For constant c

$$\rightarrow \frac{dr_1}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = +c \quad \text{where} \quad r_1 = u - cv$$

$$\frac{dr_2}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -c \quad \text{where} \quad r_2 = u + cv$$

r_1 and r_2 are called the Rieman invariants

Computational Fluid Dynamics
Wave equation

The general solution can therefore be written as:

$$f(x,t) = r_1(x - ct) + r_2(x + ct)$$

where

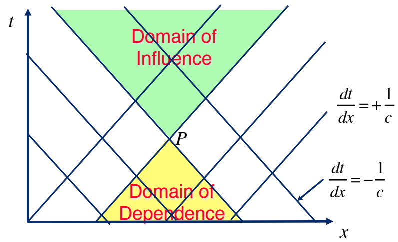
$$r_1(x) = \left[\frac{\partial f}{\partial t} - c\frac{\partial f}{\partial x}\right]_{t=0}$$

$$r_2(x) = \left[\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x}\right]_{t=0}$$

Can also be verified by direct substitution

Computational Fluid Dynamics
Wave equation-general

Two characteristic lines $\frac{dt}{dx} = +\frac{1}{c};\ \frac{dt}{dx} = -\frac{1}{c}$



Computational Fluid Dynamics

Ill-posed problems

Computational Fluid Dynamics
Ill-posed Problems

Consider the initial value problem:

$$\frac{\partial^2 f}{\partial t^2} = -\frac{\partial^2 f}{\partial x^2}$$

This is simply Laplace's equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$$

which has a solution if $\partial f/\partial t$ or f are given on the boundaries

Computational Fluid Dynamics
Ill-posed Problems

Here, however, this equation appeared as an initial value problem, where the only boundary conditions available are at $t = 0$. Since this is a second order equation we will need two conditions, which we may assume are that f and $\partial f/\partial t$ are specified at $t = 0$.

Computational Fluid Dynamics
Ill-posed Problems

The general solution can be written as:

$$f(x,t) = \sum_k a_k(t) e^{ikx}$$

where the a's depend on the initial conditions

Look for solutions of the type:

$$f = a_k(t) e^{ikx}$$

Substitute into: $\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0$

to get: $\frac{d^2 a_k}{dt^2} = k^2 a_k$

Computational Fluid Dynamics
Ill-posed Problems

$$\frac{d^2 a_k}{dt^2} = k^2 a_k$$

General solution

$$a_k(t) = Ae^{kt} + Be^{-kt}$$

A, B determined by initial conditions $a(0);\quad ika(0)$

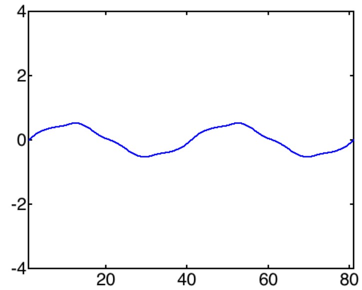
Generally, both A and B are non-zero

Therefore: $a(t) \to \infty$ as $t \to \infty$

Ill-posed Problem

Computational Fluid Dynamics
Ill-posed Problems

Long wave with short wave perturbations



Computational Fluid Dynamics
Ill-posed Problems

Similarly, it can be shown that the diffusion equation with a negative diffusion coefficient

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}; \quad D < 0$$

has solutions with unbounded growth rate for high wave number modes and is therefore an ill-posed problem

Computational Fluid Dynamics
Ill-posed Problems

Ill-posed problems generally appear when the initial or boundary data and the equation type do not match.

Frequently arise because small but important higher order effects have been neglected

Ill-posedness generally manifests itself in the exponential growth of small perturbations so that the solution does not "depend continuously on the initial data"

Inviscid vortex sheet rollup, multiphase flow models and some viscoelastic constitutive models are examples of problems that exhibit ill-posedness.

Computational Fluid Dynamics

Stability in terms of Fluxes

Computational Fluid Dynamics

We can do a similar analysis for the diffusion equation

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad \text{where} \qquad F = -D\frac{\partial f}{\partial x}$$

The finite volume approximation is

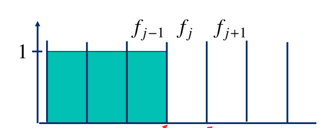
$$\frac{d}{dt}(hf_j) = F_{j-1/2} - F_{j+1/2} \qquad F_{j+1/2} = -D\frac{f_{j+1} - f_j}{h}$$

Thus, we update:

$$f_j^{n+1} = f_j^n + \frac{\Delta t D}{h^2}(f_{j+1}^n - 2f_j^n - f_{j-1}^n)$$

Computational Fluid Dynamics
Stability in terms of fluxes

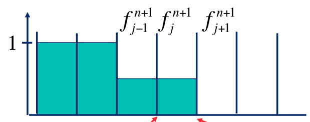
Consider the following initial conditions:



$$F_{j-1/2} = -D\frac{f_j - f_{j-1}}{h} = \frac{D}{h} \qquad F_{j+1/2} = -D\frac{f_{j+1} - f_j}{h} = 0$$

During one time step, $\Delta tD/h$ of f flows into cell j, but nothing flow out of it. Eventually cell j-1 becomes empty and cell j becomes full.

Computational Fluid Dynamics
Stability in terms of fluxes



$$F_{j-1/2} = \frac{D}{h} \qquad F_{j+1/2} = 0$$

It seems reasonable to limit Δt in such a way that we stop when both cells are equally full.

$$f_{j-1}^n + \frac{\Delta t D}{h^2}(f_j^n - 2f_{j-1}^n + f_{j-2}^n) = f_j^n + \frac{\Delta t D}{h^2}(f_{j+1}^n - 2f_j^n + f_{j-1}^n)$$

Since $f_{j-2}^n = f_{j-1}^n = 1$ and $f_j^n = f_{j+1}^n = 0$ we get:

$$1 + \frac{\Delta t D}{h^2}(0 - 2 + 1) = 0 + \frac{\Delta t D}{h^2}(0 - 0 + 1)$$ or: $$\frac{\Delta t D}{h^2} = 2$$ as maximum Δt

Computational Fluid Dynamics

Advection Equation:

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad F = Uf$$

The finite volume approximation is

$$\frac{d}{dt}(hf_j) = F_{j-1/2} - F_{j+1/2} \qquad F_{j+1/2} = Uf_{j+1/2}$$

Approximate the fluxes by the average values of f to the left and right.

$$f_{j+1/2} \approx \tfrac{1}{2}(f_{j+1} + f_j)$$

Computational Fluid Dynamics

The fluxes are therefore

$$F_{j+1/2} = \frac{1}{2}U(f_{j+1} + f_j)$$

Then approximate the time derivative by

$$\frac{d}{dt}f_j \approx \frac{1}{\Delta t}(f_j^{n+1} - f_j^n)$$

So that

$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n)$$

Computational Fluid Dynamics
Stability in terms of fluxes

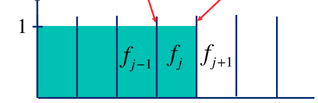
By considering the fluxes, it is easy to see why the centered difference approximation is always unstable.

Consider the following initial conditions:

$$F_{j-1/2} = \frac{U}{2}(f_{j-1}^n + f_j^n) = 1.0 \qquad F_{j+1/2} = \frac{U}{2}(f_j^n + f_{j+1}^n) = 0.5$$

$U = 1$

$\frac{\Delta t}{h} = 0.5$



$$f_j^{n+1} = f_j^n - \frac{\Delta t}{h}(F_{j+1/2}^n - F_{j-1/2}^n) = 1.0 - 0.5(0.5 - 1) = 1.25$$

So cell j will overflow immediately!

Computational Fluid Dynamics
http://www.nd.edu/~gtryggva/CFD-Course/

Summary

Characteristics for 1st order PDEs

Classification of second order PDEs

The wave equation and advection

Ill posed problems

Fluxes and stability

Computational Fluid Dynamics
Summary

In the next several lectures we will discuss numerical solutions techniques for each class:

Advection and Hyperbolic equations, including solutions of the Euler equations

Parabolic equations

Elliptic equations

Then we will consider advection/diffusion equations and the special considerations needed there.

Finally we will return to the full Navier-Stokes equations