Computational Fluid Dynamics

Lecture 4
January 30, 2017

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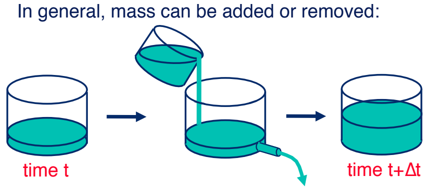
The Equations Governing Fluid Motion

Derivation of the equations governing fluid flow in integral form
- Conservation of Mass
- Conservation of Momentum
- Conservation of Energy

Differential form
Summary
Incompressible flows
Inviscid compressible flows

Conservation of Mass

In general, mass can be added or removed:



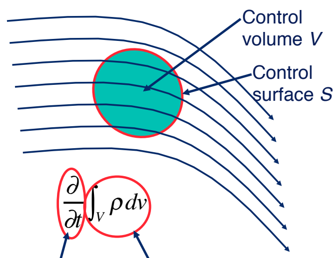
The conservation law must be stated as:

Final Mass = Original Mass + Mass Added - Mass Removed

Rate of increase of mass = Net influx of mass

We now apply this statement to an arbitrary control volume in an arbitrary flow field

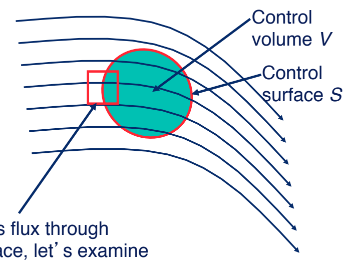
The rate of change of total mass in the control volume is given by:



$$\frac{\partial}{\partial t}\int_V \rho\, dv$$

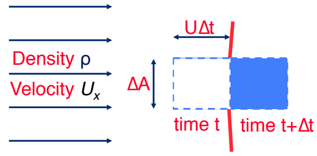
Rate of change

Total mass



To find the mass flux through the control surface, let's examine a small part of the surface where the velocity is normal to the surface

Select a small rectangle outside the boundary such that during time Δt it flows into the CV:



Mass flow through the boundary ΔA during Δt is:

$$\boxed{\rho U_x \Delta t \Delta A}$$

where $U_x = -\boldsymbol{u}\cdot\boldsymbol{n}$

The sign of the normal component of the velocity determines whether the fluid flows in or out of the control volume. We will take the outflow to be positive:

$\boldsymbol{u}\cdot\boldsymbol{n} < 0$ Inflow

The net in-flow through the boundary of the control volume is therefore:

Negative since this is net inflow

$$-\oint_S \rho\boldsymbol{u}\cdot\boldsymbol{n}\, ds$$

Integral over the boundary

Mass flow normal to boundary

Conservation of Mass Equation in Integral Form

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho\, dv}_{\text{Rate of change of mass}} = \underbrace{-\oint_S \rho\boldsymbol{u}\cdot\boldsymbol{n}\, ds}_{\text{Net inflow of mass}}$$

Conservation of momentum

Computational Fluid Dynamics
Conservation of momentum

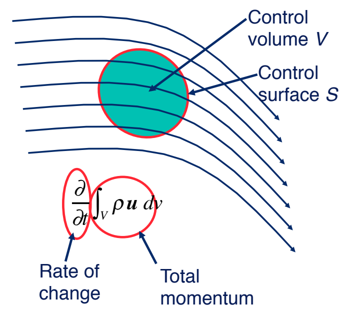
Momentum per volume: $\rho \boldsymbol{u}$

Momentum in control volume: $\int_V \rho \boldsymbol{u}\, dv$

Rate of increase of momentum	=	Net influx of momentum	+	Body forces	+	Surface forces

Computational Fluid Dynamics
Rate of change of momentum

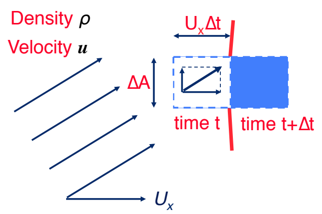
The rate of change of momentum in the control volume is given by:



$$\frac{\partial}{\partial t}\int_V \rho \boldsymbol{u}\, dv$$

Computational Fluid Dynamics
In/out flow of momentum

Select a small rectangle outside the boundary such that during time Δt it flows into the CV:



Momentum flow through the boundary ΔA during Δt is:

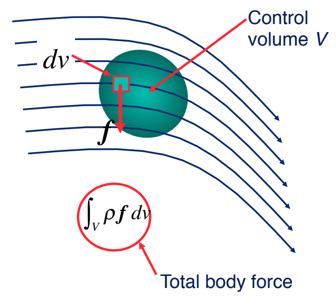
$$(\rho \boldsymbol{u}) U_x \Delta t \Delta A$$

$U_x = -\boldsymbol{u}\cdot\boldsymbol{n}$ so the net influx of momentum per unit time (divided by Δt) is:

$$-\oint \rho \boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n}\, ds$$

Computational Fluid Dynamics
Body force

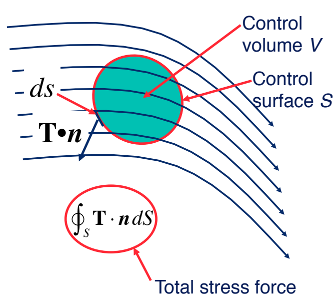
Body forces, such as gravity act on the fluid in the control volume.



$$\int_V \rho \boldsymbol{f}\, dv$$

Computational Fluid Dynamics
Viscous force

The viscous force is given by the dot product of the normal with the stress tensor **T**



$$\oint_S \mathbf{T}\cdot\boldsymbol{n}\, dS$$

Computational Fluid Dynamics
Viscous flux of momentum

Stress Tensor

$$\mathbf{T} = (-p + \lambda\nabla\cdot\boldsymbol{u})\mathbf{I} + 2\mu\mathbf{D}$$

Deformation Tensor

$$\mathbf{D} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)$$

Whose components are:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

For incompressible flows

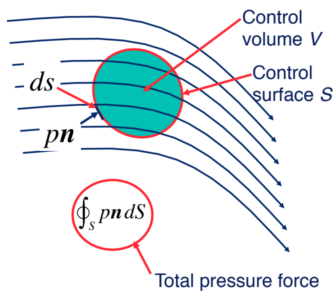
$$\nabla\cdot\boldsymbol{u} = 0$$

And the stress tensor is

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$$

Computational Fluid Dynamics
Pressure force

The pressure produces a normal force on the control surface. The total force is:



Computational Fluid Dynamics
Conservation of Momentum

Gathering the terms

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, dv = -\oint \rho \boldsymbol{u}\boldsymbol{u}\cdot \boldsymbol{n}\, ds - \oint \mathbf{T}\boldsymbol{n}\, ds + \int \rho \boldsymbol{f}\, dv$$

Substitute for the stress tensor $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$

$$\underbrace{\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, dv}_{\text{Rate of change of momentum}} = \underbrace{-\oint \rho \boldsymbol{u}\boldsymbol{u}\cdot \boldsymbol{n}\, ds}_{\text{Net inflow of momentum}} \underbrace{- \oint p\boldsymbol{n}\, ds}_{\text{Total pressure}} + \underbrace{\oint 2\mu\mathbf{D}\cdot\boldsymbol{n}\, ds}_{\text{Total viscous force}} + \underbrace{\int \rho \boldsymbol{f}\, dv}_{\text{Total body force}}$$

Computational Fluid Dynamics

Conservation of energy

Computational Fluid Dynamics
Conservation of energy

The energy equation in integral form

$$\overbrace{\frac{\partial}{\partial t}\int \rho\left(e + \frac{1}{2}u^2\right) dv}^{\text{Rate of change of kinetic+internal energy}} = \overbrace{-\oint \rho\left(e + \frac{1}{2}u^2\right)\boldsymbol{u}\cdot\boldsymbol{n}\, ds}^{\text{Net inflow of kinetic+internal energy}}$$

$$+ \underbrace{\int_V \boldsymbol{u}\cdot\boldsymbol{f}\, dv}_{\text{Work done by body forces}} + \underbrace{\oint \boldsymbol{n}\cdot(\boldsymbol{u}\mathbf{T})\, ds}_{\text{Net work done by the stress tensor}} \underbrace{- \oint \boldsymbol{n}\cdot\mathbf{q}\, ds}_{\text{Net heat flow}}$$

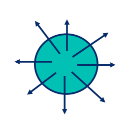
Computational Fluid Dynamics

Differential Form of the Governing Equations

Computational Fluid Dynamics
Differential form

The Divergence or Gauss Theorem can be used to convert surface integrals to volume integrals

$$\int_V \nabla\cdot\mathbf{a}\, dv = \oint_S \mathbf{a}\cdot\boldsymbol{n}\, ds$$



Start with the integral form of the mass conservation equation

$$\frac{\partial}{\partial t}\int_V \rho\, dv = -\oint_S \rho \boldsymbol{u}\cdot \boldsymbol{n}\, ds$$

Using Gauss' s theorem

$$\oint_S \rho \boldsymbol{u}\cdot \boldsymbol{n}\, ds = \int_V \nabla\cdot(\rho \boldsymbol{u})\, dv$$

The mass conservation equations becomes

$$\frac{\partial}{\partial t}\int_V \rho\, dv = -\int_V \nabla\cdot(\rho \boldsymbol{u})\, dv$$

Rearranging

$$\frac{\partial}{\partial t}\int_V \rho\, dv + \int_V \nabla\cdot(\rho \boldsymbol{u})\, dv = 0$$

Since the control volume is fixed, the derivative can be moved under the integral sign

$$\int_V \frac{\partial \rho}{\partial t} dv + \int_V \nabla\cdot(\rho \boldsymbol{u}) dv = 0$$

Rearranging

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) \right) dv = 0$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) \right) dv = 0$$

This equation must hold for ANY control volume, no matte what shape and size. Therefore the integrand must be equal to zero

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) = 0$$

Expanding the divergence term:

$$\nabla\cdot(\rho \boldsymbol{u}) = \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u}$$

The mass conservation equation becomes:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) = \frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u}$$

or

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0$$

where

$$\frac{D()}{Dt} = \frac{\partial()}{\partial t} + \boldsymbol{u}\cdot\nabla()$$

Convective derivative

The differential form of the momentum equation is derived in the same way:

Start with

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, dv = \int \rho \boldsymbol{f}\, dv + \oint (\boldsymbol{n}\mathbf{T} - \rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n}))ds$$

Rewrite as:

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, dv = \int \rho \boldsymbol{f}\, dv + \int \nabla\cdot(\mathbf{T} - \rho\boldsymbol{u}\boldsymbol{u})dv$$

To get:

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} = \rho \boldsymbol{f} + \nabla\cdot(\mathbf{T} - \rho\boldsymbol{u}\boldsymbol{u})$$

Using the mass conservation equation the advection part can be rewritten:

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla\cdot\rho\boldsymbol{u}\boldsymbol{u} =$$

$$\boldsymbol{u}\frac{\partial \rho}{\partial t} + \boldsymbol{u}\nabla\cdot\boldsymbol{u}\rho + \rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} =$$

$$\boldsymbol{u}\left(\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{u}\rho\right) + \rho\frac{D\boldsymbol{u}}{Dt} = \rho\frac{D\boldsymbol{u}}{Dt}$$

=0, by mass conservation

Computational Fluid Dynamics
Differential form

The momentum equation equation

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} = \rho \boldsymbol{f} + \nabla \cdot (\mathbf{T} - \rho \boldsymbol{u}\boldsymbol{u})$$

Can therefore be rewritten as

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{f} + \nabla \cdot \mathbf{T}$$

where

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right)$$

Computational Fluid Dynamics
Differential form

The energy equation equation can be converted to a differential form in the same way. It is usually simplified by subtracting the "mechanical energy"

Computational Fluid Dynamics
Differential form

The "mechanical energy equation" is obtained by taking the dot product of the momentum equation and the velocity:

$$\boldsymbol{u} \cdot \left(\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \rho \boldsymbol{f} + \nabla \cdot \mathbf{T} \right)$$

The result is:

$$\rho \frac{\partial}{\partial t}\left(\frac{u^2}{2} \right) = -\rho \boldsymbol{u} \cdot \nabla \left(\frac{u^2}{2} \right) + \rho \boldsymbol{u} \cdot f_b + \boldsymbol{u} \cdot (\nabla \cdot \mathbf{T})$$

The "mechanical energy equation"

Computational Fluid Dynamics
Differential form

Subtract the "mechanical energy equation" from the energy equation

The final result is

$$\rho \frac{De}{Dt} - \mathbf{T} \cdot \nabla \boldsymbol{u} + \nabla \cdot \mathbf{q} = 0$$

Using the constitutive relation presented earlier for the stress tensor and Fourie's law for the heat conduction:

$$\mathbf{T} = (-p + \lambda \nabla \cdot \boldsymbol{u})\mathbf{I} + 2\mu \mathbf{D}$$
$$\mathbf{q} = -k \nabla T$$

Computational Fluid Dynamics
Differential form

The final result is

$$\rho \frac{De}{Dt} + p \nabla \cdot \boldsymbol{u} = \Phi + \nabla \cdot k \nabla T$$

where

$$\Phi = \lambda (\nabla \cdot \boldsymbol{u})^2 + 2\mu \mathbf{D} \cdot \mathbf{D}$$

is the dissipation function and is the rate at which work is converted into heat.

Generally we also need:

$$p = p(e, \rho); \quad T = T(e, \rho);$$

and equations for $\mu, \quad \lambda, \quad k$

Computational Fluid Dynamics

Summary of governing equations

Computational Fluid Dynamics

Integral Form

$$\frac{\partial}{\partial t}\int_v \rho\, dv = -\oint_s \rho \boldsymbol{u}\cdot\boldsymbol{n}\, ds$$

$$\frac{\partial}{\partial t}\int \rho \boldsymbol{u}\, dv = \int \rho \boldsymbol{f}\, dv + \oint (\boldsymbol{n}\mathbf{T} - \rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n}))\, ds$$

$$\frac{\partial}{\partial t}\int \rho(e+\frac{1}{2}u^2)\, dv = \int \boldsymbol{u}\cdot\rho\boldsymbol{f}\, dv + \oint \boldsymbol{n}\cdot(\boldsymbol{u}\mathbf{T} - \rho(e+\frac{1}{2}u^2) - \boldsymbol{q})\, ds$$

Computational Fluid Dynamics

Conservative Form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

$$\frac{\partial\rho\boldsymbol{u}}{\partial t} = \rho\boldsymbol{f} + \nabla\cdot(\mathbf{T} - \rho\boldsymbol{u}\boldsymbol{u})$$

$$\frac{\partial}{\partial t}\rho(e+\frac{1}{2}u^2) = \nabla\cdot\left(\rho(e+\frac{1}{2}u^2)\boldsymbol{u} - \boldsymbol{u}\mathbf{T} + \boldsymbol{q}\right)$$

Convective Form

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0$$

$$\rho\frac{D\boldsymbol{u}}{Dt} = \rho\boldsymbol{f} + \nabla\cdot\mathbf{T}$$

$$\rho\frac{De}{Dt} = \mathbf{T}\nabla\cdot\boldsymbol{u} - \nabla\cdot\boldsymbol{q}$$

Computational Fluid Dynamics

Special Cases

Compressible inviscid flows
Incompressible flows
Stokes flow
Potential flows

$\left.\begin{array}{l}\text{Stokes flow}\\ \text{Potential flows}\end{array}\right\}$ Not in this course!

Computational Fluid Dynamics

Inviscid compressible flows

Computational Fluid Dynamics

$$\frac{\partial\mathbf{U}}{\partial t} + \frac{\partial\mathbf{E}}{\partial x} + \frac{\partial\mathbf{F}}{\partial y} + \frac{\partial\mathbf{G}}{\partial z} = 0$$

Where we have taken $\mathbf{T} = -p\mathbf{I}$

$$\mathbf{U} = \begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{pmatrix}\quad \mathbf{E} = \begin{pmatrix}\rho u\\ \rho u^2 + p\\ \rho uv\\ \rho uw\\ (\rho E + p)u\end{pmatrix}\quad \mathbf{F} = \begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2 + p\\ \rho vw\\ (\rho E + p)v\end{pmatrix}\quad \mathbf{G} = \begin{pmatrix}\rho u\\ \rho uw\\ \rho vw\\ \rho w^2 + p\\ (\rho E + p)u\end{pmatrix}$$

$$p = \rho RT;\quad e = c_v T;\quad h = c_p T;\quad \gamma = c_p / c_v$$

$$c_v = \frac{R}{\gamma - 1};\quad p = (\gamma - 1)\rho e;\quad T = \frac{(\gamma-1)e}{R};$$

Assuming Ideal Gas

Computational Fluid Dynamics

Incompressible flow

Computational Fluid Dynamics

Incompressible flows:

$$\frac{D\rho}{Dt} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0 \longrightarrow \nabla\cdot\boldsymbol{u} = 0$$

Navier-Stokes equations (conservation of momentum)

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} = -\nabla\cdot\rho\boldsymbol{u}\boldsymbol{u} - \nabla\boldsymbol{P} + \rho\boldsymbol{f}_b + \nabla\cdot\mu\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right)$$

For constant viscosity

$$\rho\frac{\partial}{\partial t}\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\boldsymbol{P} + \rho\boldsymbol{f}_b + \mu\nabla^2\boldsymbol{u}$$

Computational Fluid Dynamics
Objectives:

The 2D Navier-Stokes Equations for incompressible, homogeneous flow:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Computational Fluid Dynamics

Incompressibility (conservation of mass)

$$\oint_S \boldsymbol{u}\cdot\boldsymbol{n}\,ds = 0$$

Navier-Stokes equations (conservation of momentum)

$$\frac{\partial}{\partial t}\int_V u dV = -\oint_S u\boldsymbol{u}\cdot\boldsymbol{n}dS + \nu\oint_S \nabla u\cdot\boldsymbol{n}dS - \tfrac{1}{\rho}\oint_S p n_x dS$$

$$\frac{\partial}{\partial t}\int_V v dV = -\oint_S v\boldsymbol{u}\cdot\boldsymbol{n}dS + \nu\oint_S \nabla v\cdot\boldsymbol{n}dS - \tfrac{1}{\rho}\oint_S p n_y dS$$

Computational Fluid Dynamics
Pressure and Advection

Advection:

Newton's law of motion: In the absence of forces, a fluid particle will move in a straight line

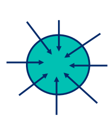
The role of pressure:

Needed to accelerate/decelerate a fluid particle: Easy to use if viscous forces are small

Needed to prevent accumulation/depletion of fluid particles: Use if there are strong viscous forces

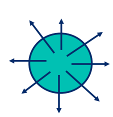
Computational Fluid Dynamics
Pressure

The pressure opposes local accumulation of fluid. For compressible flow, the pressure increases if the density increases. For incompressible flows, the pressure takes on whatever value necessary to prevent local accumulation:



High Pressure

$\nabla\cdot\boldsymbol{u} < 0$

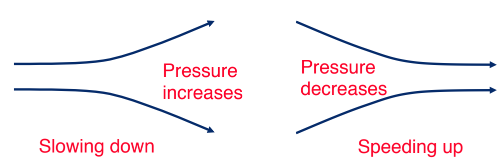


Low Pressure

$\nabla\cdot\boldsymbol{u} > 0$

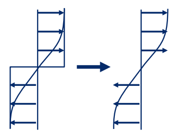
Computational Fluid Dynamics
Pressure

Increasing pressure slows the fluid down and decreasing pressure accelerates it



Computational Fluid Dynamics
Viscosity

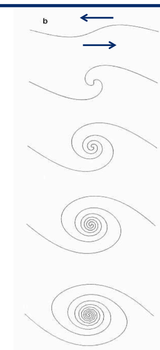
Diffusion of fluid momentum is the result of friction between fluid particles moving at uneven speed. The velocity of fluid particles initially moving with different velocities will gradually become the same. Due to friction, more and more of the fluid next to a solid wall will move with the wall velocity.



Computational Fluid Dynamics
Instability and unsteadyness

For all but the lowest Reynolds numbers, the fluid flow is unstable and unsteady, forming transient whorls and vortices that greatly increase the transfer of momentum over what viscosity alone can do.

The Kelvin Helmholtz instability of a slip line between flows in different directions is one of the fundamental ways in which steady flow becomes unstable.



Computational Fluid Dynamics

Nondimensional Numbers—the Reynolds number

Computational Fluid Dynamics
Nondimensional Numbers

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\tilde{u},\tilde{v}=\frac{u,v}{U};\quad \tilde{x},\tilde{y}=\frac{x,y}{L};\quad \tilde{t}=\frac{Ut}{L}$$

$$\frac{U^2}{L}\left(\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}\right)=-\frac{1}{L\rho}\frac{\partial P}{\partial\tilde{x}}+\frac{U}{L^2}\frac{\mu}{\rho}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)$$

$$\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=-\frac{L}{U^2}\frac{1}{L\rho}\frac{\partial P}{\partial\tilde{x}}+\frac{L}{U^2}\frac{U}{L^2}\frac{\mu}{\rho}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)$$

Computational Fluid Dynamics
Nondimensional Numbers

$$\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=-\frac{L}{U^2}\frac{1}{L\rho}\frac{\partial P}{\partial\tilde{x}}+\frac{L}{U^2}\frac{U}{L^2}\frac{\mu}{\rho}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)$$

$$\frac{L}{U^2}\frac{1}{L\rho}\frac{\partial P}{\partial\tilde{x}}=\frac{1}{\rho U^2}\frac{\partial P}{\partial\tilde{x}}=\frac{\partial\tilde{P}}{\partial\tilde{x}}$$ where $\tilde{P}=\frac{P}{\rho U^2}$

$$\frac{L}{U^2}\frac{U}{L^2}\frac{\mu}{\rho}=\frac{\mu}{UL\rho}=\frac{1}{\mathrm{Re}}$$ where $\mathrm{Re}=\frac{\rho UL}{\mu}$

Computational Fluid Dynamics
Nondimensional Numbers

The momentum equations are:

$$\frac{\partial\tilde{u}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=-\frac{\partial\tilde{P}}{\partial\tilde{x}}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right)$$

$$\frac{\partial\tilde{v}}{\partial\tilde{t}}+\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}=-\frac{\partial\tilde{P}}{\partial\tilde{y}}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}+\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right)$$

The continuity equation is unchanged

$$\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\partial\tilde{v}}{\partial\tilde{y}}=0$$

Computational Fluid Dynamics
Outline

Derivation of the equations governing fluid flow in integral form
- Conservation of Mass
- Conservation of Momentum
- Conservation of Energy

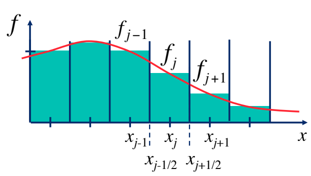
Differential form
Summary
Incompressible flows
Inviscid compressible flows

Computational Fluid Dynamics

Integral versus differential form of the governing equations

Computational Fluid Dynamics
Finite Volume Approximations

When using FINITE VOLUME approximations, we work directly with the integral form of the conservation principles. The average values of f in a small volume are stored



Computational Fluid Dynamics
Finite Volume Approximations

To derive an equation that governs the evolution of the average value in each cell, consider:

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Integrating the equation over a small control volume of size h= $\Delta x = x_{j+1} - x_j$

$$\int_{\Delta x} \frac{\partial f}{\partial t} dx + \int_{\Delta x} \frac{\partial F}{\partial x} dx = 0$$

yields:

$$\frac{d}{dt} \int_{\Delta x} f \, dx + F_{j+1/2} - F_{j-1/2} = 0$$

The volume is fixed in space

Computational Fluid Dynamics
Finite Volume Approximations

or:

$$\frac{d}{dt}(hf_j) = F_{j-1/2} - F_{j+1/2}$$

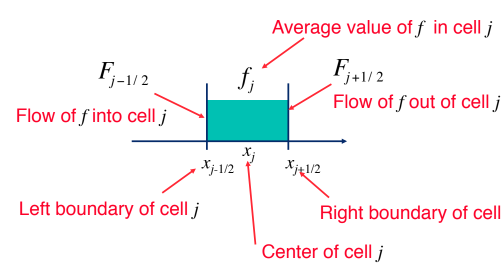
where the average in each cell is defined by:

$$f_j = \frac{1}{h} \int_{\Delta x} f \, dx$$

$F_{j-1/2}$ is the flow of f into cell j
$F_{j+1/2}$ is the flow of f out off cell j
F is usually called the flux function

Computational Fluid Dynamics
Finite Volume Approximations

Notation:



Computational Fluid Dynamics
Finite Volume Approximations

The exact form of F depends on the problem:

$$F = Uf \qquad \text{Advection}$$

$$F = -D\frac{\partial f}{\partial x} \qquad \text{Diffusion}$$

$$F = Uf - D\frac{\partial f}{\partial x} \qquad \text{Advection/Diffusion}$$

Computational Fluid Dynamics
Finite Volume Approximations

The statement

$$\frac{d}{dt}(hf_j) = F_{j-1/2} - F_{j+1/2}$$

Is exactly the original conservation law

The rate of increase of the average value of f is equal to the inflow into the control volume minus the outflow from the control volume

Rate of increase in f = inflow of f -outflow of f

Computational Fluid Dynamics
Finite Volume Approximations

Example: advection/diffusion equation

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = D\frac{\partial^2 f}{\partial x^2}$$

Rewrite:

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0$$

were

$$F = Uf - D\frac{\partial f}{\partial x}$$

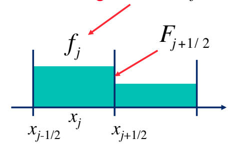
Assume here that U is a constant

Computational Fluid Dynamics
Finite Volume Approximations

For each cell

$$\frac{d}{dt}(hf_j) = F_{j-1/2} - F_{j+1/2}$$

Average value of f in cell j

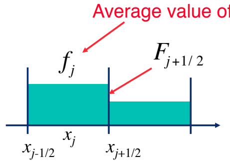


$$f_j = \frac{1}{h}\int_{\Delta x} f\,dx$$

$$F_{j+1/2} = Uf_{j+1/2} - D\left.\frac{\partial f}{\partial x}\right|_{j+1/2}$$

Computational Fluid Dynamics
Finite Volume Approximations

Average value of f in cell j



Need to find

$$F_{j+1/2} = Uf_{j+1/2} - D\left.\frac{\partial f}{\partial x}\right|_{j+1/2}$$

Approximate:

$$f_{j+1/2} \approx \tfrac{1}{2}(f_{j+1} + f_j) \quad \text{and} \quad \left.\frac{\partial f}{\partial x}\right|_{j+1/2} \approx \frac{f_{j+1} - f_j}{h}$$

f at a cell boundary

gradient of f at a cell boundary

Computational Fluid Dynamics
Finite Volume Approximations

Approximate:

$$F_{j+1/2} = Uf_{j+1/2} - D\left.\frac{\partial f}{\partial x}\right|_{j+1/2} \approx \frac{1}{2}U(f_{j+1} + f_j) - D\left(\frac{f_{j+1} - f_j}{h}\right)$$

$$F_{j-1/2} = Uf_{j-1/2} - D\left.\frac{\partial f}{\partial x}\right|_{j-1/2} \approx \frac{1}{2}U(f_j + f_{j-1}) - D\left(\frac{f_j - f_{j-1}}{h}\right)$$

and

$$\frac{d}{dt}f_j \approx \frac{1}{\Delta t}(f_j^{n+1} - f_j^n) \qquad \text{time derivative of } f$$

Substituting:

$$\frac{h}{\Delta t}(f_j^{n+1}-f_j^n)=$$

$$-U\left(\tfrac{1}{2}(f_{j+1}+f_j)-\tfrac{1}{2}(f_j+f_{j-1})\right)+D\left(\frac{f_{j+1}-f_j}{h}-\frac{f_j-f_{j-1}}{h}\right)$$

Or, rearranging the terms:

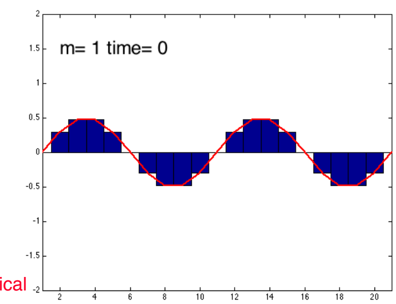
$$\frac{f_j^{n+1}-f_j^n}{\Delta t}+\frac{U}{2h}(f_{j+1}-f_{j-1})=\frac{D}{h^2}\left(f_{j+1}-2f_j+f_{j-1}\right)$$

Which is exactly the same as the finite difference equation if we take the average value to be the same as the value in the center of the cell

Example:

U=1;
D=0.05;
k=1

N=21
Δt=0.05



Exact

Numerical

The finite volume point of view has two main advantages over the finite difference point of view

1. Working directly with the conservation principle (no implicit assumption of smoothness and differentiability)

2. Easier visualization of how the solution is updated (the fluxes have a physical meaning)

Analysis of accuracy and stability is, however, usually easier using the finite difference point of view

Most physical laws are based on CONSERVATION principles: In the absence of explicit sources or sinks, f is neither created nor destroyed.

For a control volume fixed in space, we can state the conservation of f as:

Amount of f in a control volume after a given time interval Δt **=** Amount of f in the control volume at the beginning of the time interval Δt **+** Amount of f that flows into the control volume during Δt **-** Amount of f that flows out of the control volume during Δt

This can be restated in rate form as:

Rate of increase of f in a control volume during a given time interval Δt **=** Net rate of flow of f into the control volume during the time interval Δt

The mathematical equivalent of this statement is:

$$\frac{d}{dt}\int_{CV} f\,dv=-\oint_{CS} f\boldsymbol{u}\cdot\boldsymbol{n}ds$$

While we can derive a partial differential expression from this equation, often it is better to work directly with the conservation principle in integral form