Computational Mathematics: Ordinary Differential Equations

Alexander Petrov

April 8, 2024

Contents

1 Main definitions.

Let us consider a problem with given initial conditions (Cauchy's problem):

$$\begin{cases} \mathscr{D}u = f(x,u), \\ u(x_0) = u_0, \\ x \in [0, X], \end{cases} \tag{1}$$

where $\mathscr{D}$ is some differential operator (generally not linear). Our purpose is to discretize domain and equation:

$$\begin{cases} Dy = f_h, \\ y_0 = u^0, \\ \{x_i\}_{i=0}^N : \ h_i = x_{i+1} - x_i \end{cases} \tag{2}$$

Let us denote $[u]_h$ as values of the exact solution $u(x)$ at grid nodes $\{x_i\}_{i=0}^N$, and $h = \max_i h_i$.

Def 1.1. Solution of (2) **converges** to solution of (1) if

$$\|y - [u]_h\|_Y \to 0, \ h \to 0$$

Note 1.1. *It is almost imposible and useless to prove convergense directly. Indeed, if you have* $[u]_h$ *why do you need to solve* (2)*?*

Def 1.2. **Residual** of (2) is:

$$r_h = D[u]_h - f_h$$

Def 1.3. Solution of (2) **approximates** solution of (1) if

$$\|r_h\|_{F_h} \to 0, \ h \to 0.$$

If $\exists\ C_1 > 0 : \ \|r_h\|_{F_h} \leq C_1 h^k$, the number k is called the order of approximation.

Def 1.4. Solution of (2) is **stable** if $\exists\ \varepsilon,\ h_0 : \ \forall h \leq h_0,\ \forall \varepsilon^{(1)}, \varepsilon^{(2)} : \|\varepsilon^{(i)}\| \leq \varepsilon,\ i =$ 1,2 solutions of

$$Dy^{(1)} = f_h + \varepsilon^{(1)}$$
$$Dy^{(2)} = f_h + \varepsilon^{(2)}$$

«do not differ much»:

$$\|y^{(2)} - y^{(1)}\| \leq C_2(\|\varepsilon^{(1)}\| + \|\varepsilon^{(2)}\|)$$

2 Runge-Kutta methods

Def 2.1. Explicit Runge-Ketta method with s stages and defining parameters