Low Ply Drawings of Trees

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Abstract. We consider the recently introduced model of *low ply graph drawing*, a straight-line drawing in which the ply-disks of the vertices do not have many common overlaps, which results in a good distribution of the vertices in the plane. The ply-disk of a vertex is the disk centered at it whose radius is half the length of its longest incident edge.

We focus our attention on low-ply drawings of trees. We first consider drawings with constant ply; we prove that they may require exponential area, even for stars, and that they may not even exist for trees with bounded degree. Hence, we turn our attention to logarithmic-ply drawings and show that trees with maximum degree 6 always admit this type of drawings in polynomial area.

1 Introduction

The ply-number of a graph drawing is one of the most recent aesthetic criteria for graph layouts [4]. While traditional aesthetic criteria, such as minimizing edge crossings [3] and optimizing symmetries [5], have been studied for decades, the notion of optimizing a graph layout so that the spheres of influence of each vertex are well distributed is new. Goodrich and Eppstein [6] showed that real-word geographic networks usually have only constant sphere-of-influence overlap, or in the terminology of this paper, constant ply.

More formally, let Γ be a straight-line drawing of a graph G. For a vertex $v \in G$, let the *ply-disk* D_v of v be the open disk with center v and radius r_v that is half of the length of the longest incident edge of v. For a point $q \in \mathbb{R}^2$ in the plane, denote by S_q the set of disks with q in their interior, i.e., $S_q = \{D_v \mid \|v-q\| < r_v\}$.

Definition 1. *The* ply-number *of a straight-line drawing Γ is* $pn(D) = \max\limits_{q\in\mathbb{R}^2} |S_q|$ *and the* ply-number *of a graph G is* $pn(G) = \min\limits_{\Gamma(G)} \max\limits_{q\in\mathbb{R}^2} |S_q|.$

In other words, the ply-number of a drawing describes the maximum number of ply-disks that have a common non-empty intersection.

The problem of computing graph drawings with low ply-number is related to circle-contact representation of graphs, where vertices are interior-disjoint circles in the plane and two vertices are adjacent if the corresponding pair of circles touch each other [7, 8]. Every maximal planar graph has a circle-contact representation [9]. A drawback of such representations is that the sizes of the circles may vary exponentially, making the resulting drawings difficult to read. In *balanced* circle packings and circle-contact representations, the ratio of the maximum and minimum diameters for the set of circles is polynomial in the number of vertices in the graph. Such drawings could be drawn with polynomial area, for instance, where the smallest circle determines the minimum resolution. It is known that trees and planar graphs with bounded tree-depth have balanced circle-contact representation [1]. Breu and Kirkpatrick [2] show that it is NP-complete to test whether a graph has a perfectly-balanced circle-contact representation, in which all circles have the same size, i.e., they are unit disks.

Very recently, Di Giacomo et al. [4] studied low-ply graph drawings and showed that binary trees, stars, and caterpillars have drawings with ply-number 2 (with exponential area), while general trees with height h have drawings with ply-number $h + 1$. Also, they showed that the class of graphs with ply-number 1 coincides with the class of graphs that have a weak contact representation with unit disks, which makes the recognition problem NP-hard for general graphs. On the other hand, testing whether an internally triangulated biconnected planar graph has ply-number 1 can be done in $O(n \log n)$ time. This paper left several natural questions open. Of particular interest are the following two questions:

(i) *Is it possible to draw a binary tree, a star, or a caterpillar in polynomial area with ply-number 2?*

(ii) *While binary trees have constant ply-number, is this true also for trees with larger bounded degree?*

In this paper we provide answers to the two above questions (Section 3). For the first question, we prove an exponential lower bound on the area requirements of constant-ply drawings of stars, and hence of caterpillars. For the second question, we prove that there exist trees with maximum degree 11 that do not have constant ply-number. Motivated by these two negative results, we consider in Section 4 drawings of trees with logarithmic ply-number. In this case, we present an algorithm to construct a drawing of every tree with maximum degree 6 in polynomial area. We give preliminary definitions in Section 2 and discuss some open problems in Section 5.

2   Preliminaries

Let G be a graph. We denote by $\ell(e)$ (by $\ell(u, v)$) the length of an edge $e \in G$ (an edge $(u, v) \in G$) in a straight-line drawing of G. Also, for a path $P = v_1, \ldots, v_m$, we denote by $\ell(P) = \sum_{i=1}^{m-1} \ell(v_i, v_{i+1})$ the total length of its edges. Further, we
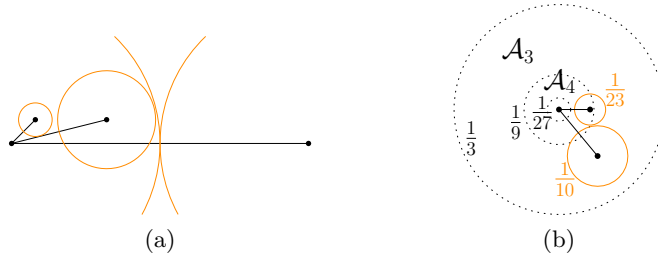
Fig. 1: (a) An exponential-area drawing with ply-number 2 of a star [4]. (b) Illustration for the proof of Theorem 1; two disks belonging to class $\mathcal{T}_3$ are entirely contained inside annuli $\mathcal{A}_3$ and $\mathcal{A}_4$. The number close to each disk is its radius.

denote by D_v the ply-disk in Γ of a vertex $v \in G$ and by r_v the radius of D_v. Finally, we call *constant-ply drawing* (or *log-ply drawing*) a straight-line drawing Γ such that $\mathsf{pn}(\Gamma) = O(1)$ (such that $\mathsf{pn}(\Gamma) = O(\log n)$).

Let T be a tree rooted at a vertex r. The *depth* d_v of a vertex $v \in T$ is the length of the path between v and r; note that $d_r = 0$. The *height* h of T is the maximum depth of a vertex of T.

3 Drawings of Trees with Constant Ply-Number

In this section we consider straight-line drawings of trees with constant ply-number. In Subsection 3.1 we prove that drawings of this type may require exponential area, even for stars and caterpillars, while in Subsection 3.2 we prove that there exist trees with bounded degree that do not admit any of such drawings. These results provide negative answers to two open questions [4].

3.1   Area Lower Bound for Stars

In the original paper on the topic of low-ply drawings [4], it has been shown that a star admits a drawing with ply-number 1 if and only if it has at most six leaves, and that every star admits a drawing with ply-number 2, independently of the number of leaves. The algorithm for the latter result is based on a placement of the leaves at exponentially-increasing distances from the central vertex, which results in a drawing with exponential area; see Fig 1a. In this subsection we prove that this is in fact unavoidable, as we give an exponential lower bound for the area requirements of any constant-ply drawing of a star.

Theorem 1. *Let $K_{1,n-1}$ be an n-vertex star. Any constant-ply drawing of $K_{1,n}$ has exponential area.*

Proof. Let $K_{1,n-1}$ be an n-vertex star with central vertex v, and let Γ be a straight-line drawing of $K_{1,n-1}$ with ply-number p, where $p = O(1)$. We prove

the statement by showing that the ratio of the longest to the shortest edge in Γ is exponential in n. Assume that the longest edge e of Γ has length $\ell(e) = 2$, after a possible scaling of Γ; thus, the largest ply-disk in Γ has radius 1.

For any $i \in \mathbb{N}$ we define $\mathcal{A}_i$ to be the annulus delimited by two circles centered at v with radius 3^{-i+2} and 3^{-i+1}, respectively. Refer to Fig. 1b. Then, we partition the ply-disks of the $n-1$ leaves of $K_{1,n-1}$ into the classes $\mathcal{T}_1, \ldots, \mathcal{T}_k$ in such a way that all the disks with radius in $(3^{-j}, 3^{-j+1}]$ belong to $\mathcal{T}_j$, with $1 \le j \le k$. We observe that every disk in class $\mathcal{T}_j$ is entirely contained inside the annulus $\mathcal{A}_j \cup \mathcal{A}_{j+1}$; see Fig. 1b. However, there can be at most

$$\frac{p|\mathcal{A}_j \cup \mathcal{A}_{j+1}|}{\min_{D \in \mathcal{T}_j} |D|} = \frac{p(\pi 3^{-2j+4} - \pi 3^{-2j})}{\pi 3^{-2j}} = 80p$$

disks in any $\mathcal{A}_j \cup \mathcal{A}_{j+1}$, and hence at most $80p$ disks belong to class $\mathcal{T}_j$. Therefore, $n = 1 + \sum_{j=1}^{k} |T_j| \le 80pk$ implies that the smallest radius of the ply-disk of a vertex in a drawing is at most 3^{-k}. This implies that the ratio between the largest and the smallest ply-disk radii in Γ, and hence between the longest and the shortest edge, is at least $3^k \ge 3^{n/(80p)}$. This concludes the proof. $\qquad\square$

3.2   Large Bounded-Degree Trees

In this section we consider the question on whether bounded-degree trees always admit constant-ply drawings [4]. While the answer to this question is positive for binary trees [4], as they admit drawings with ply-number 2, we prove that this positive result cannot be extended to all bounded-degree trees, and in particular to 10-ary trees, that is, rooted trees with maximum degree 11.

In the following we denote a complete 10-ary tree of height h by T_{10}^h; note that T_{10}^h has 10^h leaves and has 10^d vertices with depth d. The root of a tree T is denoted by $\text{root}(T)$. The rest of the section will be devoted to the proof of the following theorem.

Theorem 2. *For every $M > 0$ there is an integer $h > 0$ such that the ply-number of T_{10}^h is at least M.*

A *branch of T_{10}^h* is a path in T_{10}^h connecting the root with a leaf. Let e and f be two edges of T_{10}^h. Refer to Fig. 2a. We say that e *dominates* f and write $e >_D f$, if e and f lie on a common branch and $\ell(e) \ge 3^{s+1}\ell(f)$, where s is the number of edges on the path between e and f different from e and f. Observe that on each branch of T_{10}^h the relation $>_D$ is transitive. We say that e *first-hand dominates* f and write $e >_{\text{FD}} f$, if the following three conditions are satisfied: (i) f lies on the path connecting e with the root of T_{10}^h, (ii) e dominates f, and (iii) no other edge on the path between e and f dominates f.

Lemma 1. *Let P be a path with edges $f_0, f_1, \ldots, f_p$. Suppose that f_0 dominates each of the edges $f_1, \ldots, f_p$. Let v be the common vertex of the edges f_0 and f_1. Then the edges $f_1, \ldots, f_p$ lie entirely inside the ply-disk D_v of v.*
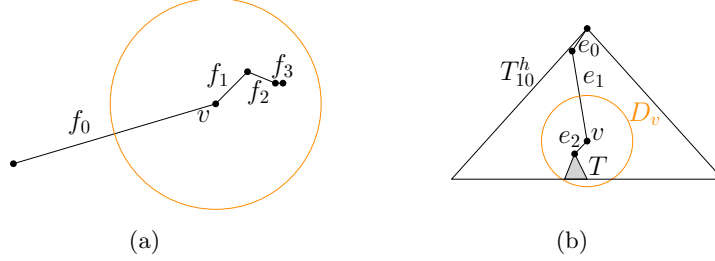
Fig. 2: (a) A path P with $\ell(f_0) = 28$, $\ell(f_1) = 6$, $\ell(f_2) = 4$, and $\ell(f_3) = 1$. Edge f_0 dominates each of f_1, f_2, and f_3, which in fact lie inside D_v. Also, f_0 first-hand dominates f_1 and f_2, but does not first-hand dominate f_3, since f_2 dominates f_3. (b) Illustration for the proof of Lemma 3.

Proof. See Fig. 2a. Since the radius of D_v is at least $\frac{\ell(f_0)}{2}$, it suffices to prove $\ell(f_1) + \cdots + \ell(f_p) < \frac{\ell(f_0)}{2}$. Let $i \in \{1, \ldots, p\}$. Since f_0 dominates f_i, we have $\ell(f_i) \le \frac{\ell(f_0)}{3^i}$. Thus, $\ell(f_1) + \cdots + \ell(f_p) \le \ell(f_0)(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^p}) < \frac{\ell(f_0)}{2}$. □

Lemma 2. *Let $e_1, \ldots, e_M$ be M edges in T_{10}^h such that $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_M$. Then, $\mathsf{pn}(T_{10}^h) \ge M$.*

Proof. By definition, $e_1, \ldots, e_M$ appear in this order, possibly not consecutively, along the same branch of T_{10}^h. Let $\overrightarrow{P}$ be the oriented path that is the subpath of this branch from e_1 to e_M. Since $e_i >_{FD} e_{i+1}$, edge e_i dominates all the edges between e_i and e_{i+1}. Due to the transitivity of $>_D$, each edge e_i dominates all the edges $e_{i+1}, \ldots, e_M$, and hence all the edges appearing after it along $\overrightarrow{P}$.

By Lemma 1, the endvertex v_M of e_M lies inside the ply-disk D_{v_i} of v_i, for each $i = 1, \ldots, M$, where v_i is the last vertex of e_i along $\overrightarrow{P}$. Thus, the M disks $D_{v_1}, \ldots, D_{v_M}$ have non-empty intersection, and the statement follows. □

Consider a vertex v with depth d in T_{10}^h. We say that a vertex $u \ne v$ is a *descendant* of v if the path from $\text{root}(T_{10}(h))$ to u contains v. For any $i = 1, \ldots, h-d$, we denote by $T_{10}^i(v)$ the subtree of T_{10}^h rooted at v induced by v and by all the descendants of v with depth $d+1, d+2, \ldots, d+i$. Note that $T_{10}^i(v)$ is a 10-ary tree of height i, thus it has 10^i leaves. We have the following.

Lemma 3. *Let T' be a subtree of a rooted 10-ary tree T and let P be the path from $\text{root}(T)$ to $\text{root}(T')$. If every edge of T' is dominated by at least one edge of P, then there exists a vertex $v \in P$ such that T' lies completely inside D_v.*
    Consequently, $\mathsf{pn}(T) \ge \mathsf{pn}(T') + 1$.

Proof. Refer to Fig. 2b. Let $e_0, e_1, \ldots, e_t$ be the edges of P in the order in which they appear along P, when P is oriented from $\text{root}(T)$ to $\text{root}(T')$. Let i be an index maximizing the value of $3^i \cdot \ell(e_i)$. Then e_i dominates all the edges $e_{i+1}, e_{i+2}, \ldots, e_t$. Also, due to the choice of i and since any edge of T'
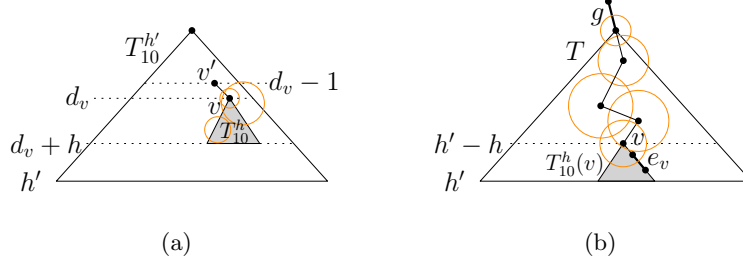
Fig. 3: (a) Illustration for Lemma 4; no edge of $T_{10}^h(v)$ dominates edge (v, v'). (b) Illustration for **Case 2**: for each vertex v with depth $h' - h$, there exists an edge e_v in $T_{10}^h(v)$ that is not dominated by any edge of path from v to the root of T.

is dominated by some edge of P, any edge of T' is dominated by e_i. Let v be the root of T, if $i = 0$, or the common vertex of e_{i-1} and e_i otherwise. Then Lemma 1 can be applied on the path from v to any leaf of T' to show that its subpath from $\mathrm{root}(T')$ to the leaf lies inside D_v, which proves the statement.

As a consequence, we have $\mathsf{pn}(T) \geq \mathsf{pn}(T') + 1$. □

Lemma 4. *Let h, h', M be three positive integers such that $h' \geq h(M - 1) + 1$. If there exists a drawing Γ of $T_{10}^{h'}$ that contains no M edges $e_1, \ldots, e_M$ such that $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_M$, then there exists a vertex v in $T_{10}^{h'}$ with depth $1 \leq d_v \leq h' - h$ such that no edge of $T_{10}^h(v)$ in Γ dominates the edge (v, v'), where v' is the neighbor of v with depth $d_v - 1$. Refer to Fig. 3a.*

Proof. We fix h and proceed by induction on M. If $M = 1$, then there exists no drawing Γ of $T_{10}^{h'}$ satisfying the conditions of the lemma, and thus the statement holds. Suppose now that $M > 1$ and that the lemma holds for $M - 1$. We want to show that the lemma holds for M. Let $h' \geq h(M - 1) + 1$. Suppose that a drawing of $T_{10}^{h'}$ contains no M edges $e_1, \ldots, e_M$ such that $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_M$.

Consider the subtree $T' := T_{10}^{h'-h}(\mathrm{root}(T_{10}^{h'}))$ of $T_{10}^{h'}$, with the same root as $T_{10}^{h'}$, that is induced by the vertices with depth at most $h' - h$. If T' does not contain $M - 1$ edges $e_1, \ldots, e_{M-1}$ such that $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_{M-1}$, then the required vertex v exists by induction. Otherwise, consider $M - 1$ edges $e_2, \ldots, e_M$ in T' such that $e_2 >_{FD} e_3 >_{FD} \cdots >_{FD} e_M$. Let d and $d + 1$, with $d < h' - h$, be the depth of the endvertices v' and v of e_2, respectively, in T' (it is the same depth as they have in $T_{10}^{h'}$). Consider the subtree $T_{10}^h(v)$ of $T_{10}^{h'}$ rooted at v. Suppose, for a contradiction, that there exists an edge in $T_{10}^h(v)$ dominating e_2; then, consider the edge e_1 in $T_{10}^h(v)$ dominating e_2 with the property that no other edge on the path from e_1 to e_2 dominates e_2, that is, e_1 first-hand dominates e_2. Thus, $e_1 >_{FD} e_2 >_{FD} \cdots >_{FD} e_M$, a contradiction. This implies that no edge of $T_{10}^h(v)$ dominates edge e_2, and the statement follows. □

We are now ready to complete the proof of the main result of the section.

Proof. (Proof of Theorem 2) We proceed by induction on M. For $M = 1$ the statement trivially holds.

Suppose now that $M > 1$ and that for $M - 1$ there is an h with the required properties. We need to show that for M there is an h' with the required properties. We set $h' := \max\{h^2 M, Ch(h + M)\}$, where C is a (large) constant to be specified later. We fix a drawing of $T_{10}^{h'}$.

If there are M edges $e_1, \ldots, e_M$ in $T_{10}^{h'}$ such that $e_1 >_{\text{FD}} e_2 >_{\text{FD}} \cdots >_{\text{FD}} e_M$, then Lemma 2 implies $\mathsf{pn}(T_{10}^{h'}) \geq M$. Otherwise, due to Lemma 4 there is a rooted 10-ary subtree T of $T_{10}^{h'}$ with height $\overline{h} \geq \frac{h'}{M}$ such that $\text{root}(T) \neq \text{root}(T_{10}^{h'})$ and no edge of T dominates the first edge on the path from $\text{root}(T)$ to $\text{root}(T_{10}^{h'})$. From now on, we focus on the rooted tree T. In particular, in the following we refer to the depth of a vertex as its depth in T. We distinguish two cases.

In **Case 1** there exists a vertex v with depth $\overline{h} - h$ in T such that every edge of the tree $T_{10}^h(v)$ is dominated by at least one edge of the path from v to $\text{root}(T)$. In this case, Lemma 3 (applied on tree $T_{10}^h(v)$) and the inductive hypothesis show that $\mathsf{pn}(T_{10}^{h'}) \geq \mathsf{pn}(T_{10}^h(v)) + 1 \geq M$.

In **Case 2** there exists no vertex in T with the above properties. Refer to Fig. 3b. Thus, for any vertex v with depth $\overline{h} - h$ in T, the subtree $T_{10}^h(v)$ rooted at v contains at least one edge that is not dominated by any edge of the path from v to $\text{root}(T)$; among these edges of $T_{10}^h(v)$ we choose one, called $e(v)$, whose endvertices have the smallest possible depth. This implies that $e(v)$ is not dominated by any edge of the path P_v from its endvertex u_v to $\text{root}(T)$.

Let g be the first edge from $\text{root}(T)$ to $\text{root}(T_{10}^{h'})$. Note that edges of P_v dominate neither g nor $e(v)$. W.l.o.g., assume $\ell(g) = 1$. Since edges g and $e(v)$ do not dominate each other, we have $1/3^{\overline{h}} < \ell(e(v)) < 3^{\overline{h}}$. Thus, there is a unique integer $k(v) \in \{-\overline{h}, -\overline{h} + 1, \ldots, \overline{h} - 1\}$ such that $\ell(e(v)) \in [3^{k(v)}, 3^{k(v)+1})$.

Let k be a most frequent value of $k(v)$ over all the vertices v with depth $\overline{h} - h$. Since $k(v)$ may have $2\overline{h}$ different values, the set V_k of vertices v at level $\overline{h} - h$ with $k(v) = k$ has size at least $10^{\overline{h}-h}/(2\overline{h})$. Consider now a vertex $v \in V_k$ and the path P_v from $\text{root}(T)$ to u_v. Since no edge of this path dominates g or $e(v)$, we have the following two upper bounds on the length of the i-th edge e_i of the path P_v oriented from $\text{root}(T)$ to u_v:

$$\ell(e_i) \leq 3^i \quad \text{and}$$

$$\ell(e_i) \leq 3^{\overline{h}-i} \cdot \ell(e(v)) < 3^{\overline{h}-i+k+1}.$$

For the latter, we use $\ell(e(v)) < 3^{k+1}$, which follows from the fact that $v \in V_k$. The edges e_i with $i \leq (\overline{h} + k)/2$ have total length at most $\sum_{i=0}^{\lfloor(\overline{h}+k)/2\rfloor} 3^i \leq 3^{(\overline{h}+k)/2+1}$, and the total length of the other edges is at most

$$\sum_{i=\lfloor(\overline{h}+k)/2+1\rfloor}^{\overline{h}} 3^{\overline{h}-i+k+1} = 3^{k+1} \cdot \sum_{i=\lfloor(\overline{h}+k)/2\rfloor+1}^{\overline{h}} 3^{\overline{h}-i}$$

$$= 3^{k+1} \cdot \sum_{j=0}^{\overline{h}-\lfloor (\overline{h}+k)/2 \rfloor -1} 3^j \leq 3^{k+1} \cdot 3^{(\overline{h}-k)/2+1} = 3^{(\overline{h}+k)/2+2}.$$

It follows that the total length of the path P_v is smaller than $12 \cdot 3^{(\overline{h}+k)/2}$.

Thus all the edges $e(v)$, $v \in V_k$, lie in the disk D of radius $12 \cdot 3^{(\overline{h}+k)/2}$ centered at root(T). The area of D is $12^2 \pi 3^{\overline{h}+k}$. Let $v \in V_k$, and let u'_v be the vertex of the path P_v adjacent to u_v. The ply-disk $D_{u'_v}$ contains the disk of radius $3^k/2$ centered at u'_v, which is entirely contained in D. It follows that the region $D_{u'_v} \cap D$ has area at least $\pi(3^k/2)^2 = (\pi/4)3^{2k}$. Therefore there is a point of D lying in at least

$$\frac{|V_k|.(\pi/4)3^{2k}}{\text{area}(D)} \geq \frac{(10^{\overline{h}-h}/(2\overline{h})).(\pi/4)3^{2k}}{12^2\pi 3^{\overline{h}+k}} = \frac{(10/3)^{\overline{h}}/\overline{h}.3^k}{12^2 \cdot 8 \cdot 10^h} \geq \frac{(10/9)^{\overline{h}}/\overline{h}}{12^2 \cdot 8 \cdot 10^h}$$

disks $D_{u'_v}$, with $v \in V_k$.

Since $h' \geq CM(h + \log M)$, we have $\overline{h} \geq C(h + \log M)$. If C is a sufficiently large constant then some point of D lies in at least

$$\frac{(10/9)^{\overline{h}}/\overline{h}}{12^2 \cdot 8 \cdot 10^h} \geq M$$

disks $D_{u'_v}$, with $v \in V_k$, which concludes the proof.   □

4   Log-Ply Drawings of Bounded-Degree Trees in Polynomial Area

Motivated by the fact that constant-ply drawings of stars may require exponential area (Theorem 1) and by the fact that not all the bounded-degree trees admit a constant-ply drawing (Theorem 2), in this section we ask whether allowing a logarithmic ply-number makes it possible to always construct drawings of trees, possibly in polynomial area. We give a first answer by proving in Theorem 3 that this is true for 5-ary trees, that is, trees with maximum degree 6.

We start with some definitions. We define a 2-*drawing* of a path $P = v_1, \dots, v_m$ as a straight-line drawing of P in which all the vertices lie along the same straight-line segment and for each $i = 2, \dots, m$ we have $\frac{\ell(v_{i-1},v_i)}{2} \leq \ell(v_i, v_{i+1}) \leq 2\ell(v_{i-1}, v_i)$; see Fig. 4a. We have the following.

Lemma 5. *A* 2-*drawing of a path* $P = (v_1, \dots, v_n)$ *has ply-number at most 2.*

Proof. Refer to Fig. 4a. For each vertex v_i, we have radius $r_{v_i} \leq \ell(v_i, v_{i+1})$ and $r_{v_i} \leq \ell(v_{i-1}, v_i)$, since $\frac{\ell(v_{i-1},v_i)}{2} \leq \ell(v_i, v_{i+1}) \leq 2\ell(v_{i-1}, v_i)$. This, together with the fact that all the vertices of P lie along the same straight-line segment, implies that the ply-disk D_{v_i} of v_i may only intersect with $D_{v_{i-1}}$ and with $D_{v_{i+1}}$, but not with any of the other disks (note that D_{v_i} may touch $D_{v_{i-2}}$ and $D_{v_{i+2}}$ in a single point, namely the one where vertices v_{i-1} and v_{i+1} lie, respectively), but cannot overlap with them.   □
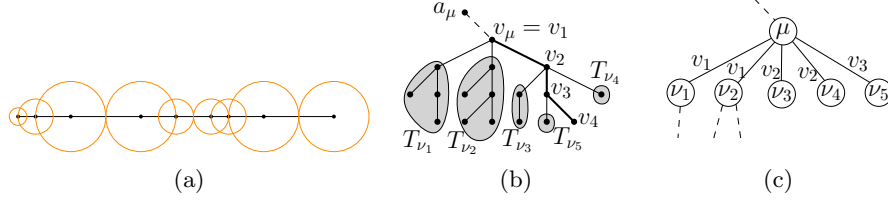
Fig. 4: (a) A 2-drawing of a path. (b) A ternary tree T_μ and the path μ, represented by fat edges, that is a node of the heavy-path tree $\mathcal{T}$; the subtrees $T_{\nu_1}, \ldots, T_{\nu_5}$ obtained when removing μ are inside shaded region. (c) The portion of $\mathcal{T}$ containing nodes μ and its children $\nu_1, \ldots, \nu_5$. The arc of $\mathcal{T}$ between μ and a node ν_i is labeled with the node of μ that is the anchor of ν_i.

Let T be any rooted tree. We define a decomposition tree $\mathcal{T}$ of T, which we call *heavy-path* tree, as follows. Refer to Fig. 4b-4c. Each node $\mu \in \mathcal{T}$ is a path in T between a vertex v_μ of T and a leaf of the subtree T_μ of T rooted at v_μ. At the first step, v_μ is the root of T, T_μ is T, and the path μ we construct is the root of $\mathcal{T}$. To construct μ, we start from v_μ and we always select the child of the current vertex whose subtree contains the largest number of vertices, till reaching a leaf of T_μ. Then, we remove all the vertices of μ from T_μ and their incident edges, hence obtaining a set of subtrees of T_μ. For each of these subtrees T_ν, rooted at a vertex v_ν, we add a new node ν as a child of μ in $\mathcal{T}$ and recursively construct the corresponding path. Since each subtree T_ν has at most half of the vertices of T_μ, the height of the heavy-path tree $\mathcal{T}$ is $O(\log n)$.

Let $\mu = (v_\mu = v_1, \ldots, v_m)$ be any node in $\mathcal{T}$ and let τ be its parent. The vertex of τ that is adjacent to v_μ is the *anchor* a_μ of μ; in order to have an anchor a_μ also when μ is the root of $\mathcal{T}$, we add a dummy vertex to T that is only incident to its root. The proof of Theorem 3 is based on the following algorithm, which we call DRAWPATH, to construct a special 2-drawing of the path P that is the concatenation of edge (a_μ, v_μ) and of path μ.

Let n_μ be the total number of vertices in the subtrees of T_τ whose corresponding paths have a_μ as an anchor. Since T_μ is one of these subtrees, we have that $n_\mu > \sum_{i=1}^m n_i$, where n_i is the total number of vertices in the subtrees $T_{\nu_1}, \ldots, T_{\nu_h}$ of T_μ such that paths $\nu_1, \ldots, \nu_h$ have v_i as anchor. Also, since μ is a path in a heavy-path tree, we have $n_i \le n_\mu/2$ for each $1 \le i \le m$.

Algorithm DRAWPATH starts by initializing $\ell(a_\mu, v_1) = n_1$ and $\ell(v_i, v_{i+1}) = n_i + n_{i+1}$, for each $i = 1, \ldots, m-1$. Then, it visits the edges of P one by one in decreasing order of their length in the current drawing. When an edge (v_i, v_{i+1}), with $1 \le i \le m-1$, is visited, set $\ell(v_{i-1}, v_i) = \max\{\frac{\ell(v_i, v_{i+1})}{2}, \ell(v_{i-1}, v_i)\}$ and $\ell(v_{i+1}, v_{i+2}) = \max\{\frac{\ell(v_i, v_{i+1})}{2}, \ell(v_{i+1}, v_{i+2})\}$. We have the following.

Lemma 6. *Algorithm* DRAWPATH *constructs a 2-drawing Γ of P such that $\ell(a_\mu, v_1) \ge n_1$, $\ell(v_i, v_{i+1}) \ge n_i + n_{i+1}$, for each $i = 1, \ldots, m-1$, and $\ell(P) \le 6n_\mu$.*

Proof. First observe that $\ell(a_\mu, v_1) \geq n_1$ and $\ell(v_i, v_{i+1}) \geq n_i + n_{i+1}$ for each $i = 1, \ldots, m-1$, since this is true already after the initialization and since no operation performed by the algorithm reduces the length of any edge.

Also, the fact that Γ is a 2-drawing can be derived from the operations that are performed when an edge is visited. Note that after an edge has been visited by DRAWPATH, its length is not modified any longer, since the edges are visited in decreasing order of edge lengths and since the length of an edge is modified only if this edge is shorter than one of its adjacent edges.

For the same reason, if an edge (v_i, v_{i+1}), with $1 \leq i \leq m-1$, determines a local maximum in the sequence of edge lengths in Γ (that is, $\ell(v_h, v_{h+1}) \geq \ell(v_{h-1}, v_h)$ and $\ell(v_h, v_{h+1}) \geq \ell(v_{h+1}, v_{h+2})$), then $\ell(v_i, v_{i+1}) = n_i + n_{i+1}$. We use this property to prove that the total length of the edges in Γ is at most $6n_\mu$.

Consider any two edges (v_h, v_{h+1}) and (v_q, v_{q+1}), with $1 \leq h < q \leq m-1$, such that $\ell(v_h, v_{h+1}) = n_h + n_{h+1}$, $\ell(v_q, v_{q+1}) = n_q + n_{q+1}$, and such that $\ell(v_i, v_{i+1}) > n_i + n_{i+1}$ for each $i = h+1, \ldots, q-1$; namely, (v_h, v_{h+1}) (v_q, v_{q+1}) are two edges that have not been modified by algorithm DRAWPATH after the initialization and such that all the edges between them have been modified.

Claim. The total length of the edges in the subpath P' of Γ between v_h and v_{q+1} is at most $2(n_h + n_{h+1}) + 2(n_q + n_{q+1})$.

Proof. Note that there exists no edge in P' different from (v_h, v_{h+1}) and (v_q, v_{q+1}) that determines a local maximum in the sequence of edge lengths, since this would contradict the fact that $\ell(v_i, v_{i+1}) > n_i + n_{i+1}$ for each $i = h+1, \ldots, q-1$. Hence, P' is composed of a sequence of edges starting at (v_h, v_{h+1}) and ending at an edge (v_{j-1}, v_j), with $h < j \leq q-1$, with decreasing edge lengths, and of a sequence of edges starting at (v_j, v_{j+1}) and ending at (v_q, v_{q+1}) with increasing edge lengths. We have $\ell(v_h, v_{h+1}) = n_h + n_{h+1}$ and $\ell(v_q, v_{q+1}) = n_q + n_{q+1}$, by construction. Also, $\sum_{i=h+1}^{j-1} \ell(v_i, v_{i+1}) = \sum_{i=1}^{j-1-h} \frac{n_h + n_{h+1}}{2^i} < n_h + n_{h+1}$, since Γ is a 2-drawing. Analogously, $\sum_{i=j}^{q-1} \ell(v_i, v_{i+1}) < n_q + n_{q+1}$. □

Hence, every edge (v_h, v_{h+1}) such that $\ell(v_h, v_{h+1}) = n_h + n_{h+1}$, together with the possible sequence of edges with increasing (decreasing) edge lengths preceding (following) it, gives a contribution of less than $3(n_h + n_{h+1})$. Since $\sum_{i=1}^{m}(n_i + n_{i+1}) < 2n_\mu$, the total edge length is at most $6n_\mu$. □

We describe an algorithm to construct a log-ply drawing of any rooted n-vertex 5-ary tree T with polynomial area. To simplify the description, we first give the algorithm for ternary trees; we discuss later the extension to 5-ary trees.

Construct the heavy-path tree $\mathcal{T}$ of T. Then, construct a drawing of T recursively according to a bottom-up traversal of $\mathcal{T}$. At each step of the traversal, consider a path $\mu \in \mathcal{T}$. We associate μ with a half-disk D_μ of radius $6^{h-d_\mu}n_\mu$, where h is the height of $\mathcal{T}$ and d_μ is the depth of μ. Refer to Fig. 5a. The goal is to construct a drawing with ply-number at most $2(h - d_\mu + 1)$ of the subtree T_μ rooted at v_μ, augmented with the anchor a_μ of μ and with edge (a_μ, v_μ), inside D_μ in such a way that a_μ lies on the center of D_μ and all the vertices of μ lie along the radius of D_μ that is orthogonal to the diameter delimiting D_μ.
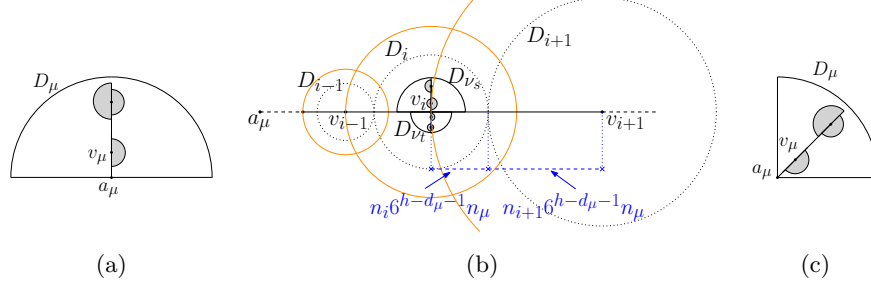
Fig. 5: (a) The half-disk D_μ associated with μ. (b) Illustration for the algorithm for ternary trees. Black dotted circles are the disks D_i in algorithm; orange solid circles are the ply-disks. (c) Using a quarter-disk instead of a half-disk for 5-ary trees.

If $\mu = (v_1, \ldots, v_m)$ is a leaf of $\mathcal{T}$, place a_μ on the center of D_μ and the vertices of μ along the radius of D_μ orthogonal to the diameter delimiting it, so that each edge has length 1. This drawing has ply-number 1 and satisfies the required properties by construction.

If $\mu = (v_1, \ldots, v_m)$ is not a leaf, let $\nu_1, \ldots, \nu_k$ be its children. Assume inductively that for each child ν_j, with $j = 1, \ldots, k$, there exists a drawing with ply-number at most $2(h - d_{\nu_j} + 1)$ inside the half-disk D_{ν_j} with radius $6^{h-d_{\nu_j}} n_{\nu_j}$ with the required properties. We show how to construct a drawing with ply-number at most $2(h - d_\mu + 1)$ of T_μ inside the half-disk D_μ with radius $6^{h-d_\mu} n_\mu$ with the required properties; recall that $d_\mu = d_{\nu_j} - 1$, for each $j = 1, \ldots, k$.

Refer to Fig. 5b. First, apply algorithm DRAWPATH to construct a 2-drawing of the path P composed of μ and of its anchor a_μ such that $\ell(a_\mu, v_1) \geq n_1$, $\ell(v_i, v_{i+1}) \geq n_i + n_{i+1}$, for $i = 1, \ldots, m-1$, and the total length of the edges in P is at most $6n_\mu$. Then, scale the obtained drawing by a factor of $6^{h-d_\mu-1} n_\mu$, which implies that the total length of the edges in P is at most $6^{h-d_\mu} n_\mu$. Hence, it is possible to place the obtained drawing inside D_μ in such a way that a_μ lies on its center and the vertices of μ lie along the radius that is orthogonal to the diameter delimiting D_μ. Further, for each vertex $v_i \in \mu$, consider a disk D_i centered at v_i of diameter $6^{h-d_\mu-1} n_i$. Due to the scaling performed before, no two disks D_i and D_h, with $1 \leq i, h \leq m$, intersect with each other.

Consider now the at most two children ν_s and ν_t of μ whose anchor is v_i; since $d_\mu = d_{\nu_j} - 1$, for each $j = 1, \ldots, k$, and since $n_i = n_{\nu_s} + n_{\nu_t}$, the diameter of the half-disk D_{ν_s} and the one of the half-disk D_{ν_t} are both not larger than the diameter of disk D_i. Thus, we can plug the drawings of T_{ν_s} and T_{ν_t} lying inside D_{ν_s} and D_{ν_t}, which exist by induction, so that the centers of D_{ν_s} and D_{ν_t} coincide with the center of D_i, and the diameters delimiting D_{ν_s} and D_{ν_t} lie along edges (v_{i-1}, v_i) and (v_i, v_{i+1}); Fig. 5b. Hence, the constructed drawing of T_μ lies inside D_μ and satisfies all the required properties.

The ply-number of the 2-drawing of μ constructed by algorithm DRAWPATH is at most 2, by Lemma 5, and it remains the same after the scaling. Also, the

ply-disk of any vertex in T_{ν_s} (in T_{ν_t}) entirely lies inside half-disk D_{ν_s} (half-disk D_{ν_t}) and hence inside disk D_i; thus, it does not overlap with the ply-disk of any vertex in a different subtree. Since the drawing of T_{ν_j}, for each child ν_j of μ, has ply-number at most $2(h - d_{\nu_j} + 1)$, the drawing of T_μ has ply-number at most $2 + 2(h - d_{\nu_j} + 1) = 2(h - d_{\nu_j} + 2) = 2(h - d_\mu + 1)$, given that $d_\mu = d_{\nu_j} + 1$.

At the end of the traversal, when the root ρ of $\mathcal{T}$ has been visited, we have a drawing with ply-number at most $2(h - d_\rho + 1) = O(\log n)$ of $T_\rho = T$ inside the half-disk D_ρ of radius $6^{h-d_\rho} n_\rho = 6^{O(\log n)} n = O(n^2)$, and hence area $O(n^4)$.

In order to extend the algorithm to work for 5-ary trees, we have to be able to fit inside the ply-disk D_i of each vertex $v_i \in \mu$ the drawings of the at most four subtrees T_{ν_j} whose anchor is v_i. Hence, we associate with each node μ a *quarter-disk* D_μ (a sector of a disk with internal angle $\frac{\pi}{2}$; see Fig. 5c), instead of a half-disk, still with radius $6^{h-d_\mu} n_\mu$, and we draw T_μ inside D_μ in such a way that the anchor a_μ of μ lies on the center of D_μ and all the vertices of μ lie along the radius of D_μ along the bisector of D_μ. Also in this case, the ply-disk of each vertex of μ entirely lies inside D_μ. We thus have the following.

Theorem 3. *Every n-vertex 5-ary tree has a drawing with ply $O(\log n)$ and $O(n^4)$ area.*

In order to extend this approach for trees with larger degree, we should use a disk sector D_μ with an internal angle smaller than $\frac{\pi}{2}$. In this case, however, we could not guarantee that the ply-disk of each vertex of μ lies inside D_μ, and thus we could not compute the ply of the subtrees independently of each other.

5   Conclusions and Open Problems

In this work we considered low-ply drawings of trees. We proved that requiring the ply-number to be bounded by a constant is often a somewhat too strong limitation, as these drawings may not exist, even for trees with bounded degree, or may require exponential area. On the positive side, we showed that relaxing the requirement on the ply-number, allowing it to be bounded by a logarithmic function, makes the problem easier, as we gave an algorithm for constructing polynomial-area drawings with this property for trees with maximum degree 6.

Our work leaves several interesting open questions:

First, while it is known that stars, caterpillars, and binary trees admit constant-ply drawings in exponential area [4], we were able to prove that this is unavoidable only for stars and caterpillars; this leaves open the question on the area-requirements of constant-ply drawings of binary trees.

Second, it would be interesting to reduce the gap between binary trees, which alway admit constant-ply drawings, and 10-ary trees, which may not admit any of such drawings. More in general, a characterization of the trees admitting these drawings is a fundamental open question.

Finally, in this paper we provided the first results on log-ply drawings of trees. It would be worth studying which trees (or other classes of graphs) always admit this type of drawings, possibly with polynomial area.

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