Approximations of the Navier Stokes equations in ocean, atmosphere and ice-sheet modeling

Alexander Robinson

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Motivation

Many Earth-system components are essentially dynamic fluids governed by an *equation of state* combined with the laws of

- Conservation of momentum
- Conservation of mass
- Conservation of energy

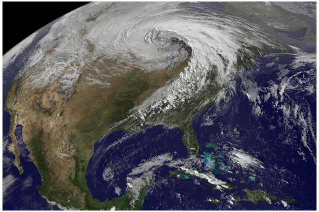
Modeling these systems helps us to understand them and to make predictions.

NASA - Hurricane Dorian

Motivation

The **momentum balance** of all of these systems can be described by the **Navier-Stokes equations**.

Atmosphere



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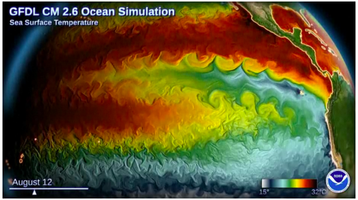
Ocean



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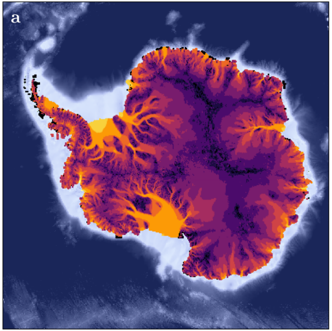
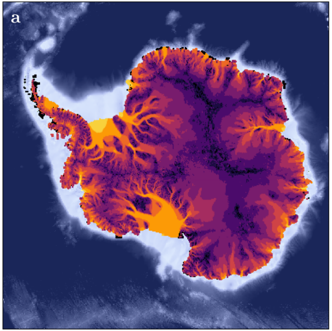
Ice sheets



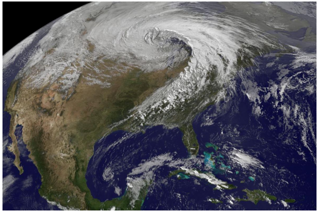
Motivation

The **momentum balance** of all of these systems can be described by the **Navier-Stokes equations**.

Ice sheets



Atmosphere



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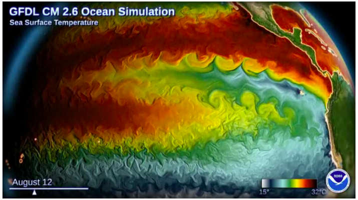
Ocean



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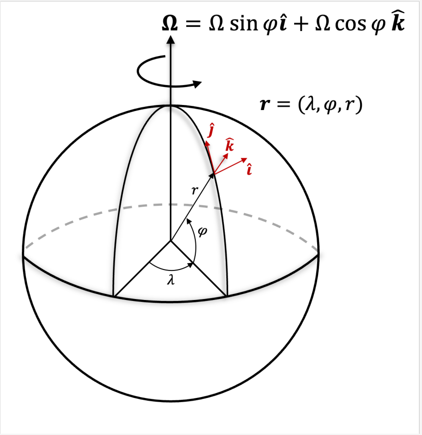
So, why make approximations?

- Computational cost
- Processes
- Scales

Outline

1. Rotation and spherical coordinates
2. Navier-Stokes equations for a rotating frame of reference
3. Scale analysis
4. Atmosphere and ocean dynamics
    - Shallow approximation
    - Hydrostatic approximation
    - Geostrophic approximation
    - Boussinesq approximation (ocean)
5. Ice dynamics
    - Blatter-Pattyn approximation
    - Shallow shelf approximation
    - Shallow ice approximation
6. Summary

Rotation: Coriolis and Centrifugal force



Coriolis parameter: $f = 2\Omega \sin\phi$

The Earth revolves around its axis at the rate $\Omega = 7.2921 \times 10^{-5}$ rad/s, so fluids at the surface operate in a rotating frame of reference.

Coriolis and Centrifugal forces are known as ***apparent forces*** – they only exist because the reference frame is in motion.

The **Coriolis force** *deflects* fluid parcels as a consequence of the Earth's rotation.

The **Centrifugal force** attempts to push fluid parcels away from the axis of rotation, but it is very small compared to gravity.

Full Navier-Stokes equations

We can start with the most general case of the Navier-Stokes equations, defined for a rotating frame of reference.

$$\text{Momentum:} \quad \frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla\Phi + \mathcal{F}$$

$$\text{Continuity:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0$$

$\mathbf{u}$: Velocity, $\mathbf{u} = (u, v, w)$
p: Pressure
ρ: Density
$\Phi = gz$: Gravitational potential
$\mathcal{F}$: Frictional/dissipative forces

Recall definition of total derivative:
$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u}$

Full Navier-Stokes equations

$$\underbrace{\frac{D\mathbf{u}}{Dt}}_{\text{Inertia}} \quad \underbrace{+\, 2\boldsymbol{\Omega} \times \mathbf{u}}_{\text{Rotation}} \quad = \quad \underbrace{-\frac{1}{\rho}\nabla p}_{\text{Pressure gradient}} \quad \underbrace{+\,\nabla\Phi}_{\text{Gravity}} \quad \underbrace{+\,\mathcal{F}}_{\text{Friction/ Dissipation}}$$

Full Navier-Stokes equations by component*

**in spherical coordinates*

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} - 2\Omega v\sin\phi + 2\Omega w\cos\phi = -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} + \mathcal{F}_x$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + 2\Omega u\sin\phi = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \mathcal{F}_y$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} + 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \mathcal{F}_z$$

Inertia
Curvature
Coriolis
Pressure gradient
Gravity
Friction/Dissipation

Full Navier-Stokes equations by component*

**in spherical coordinates*

$$
\begin{aligned}
\frac{Du}{Dt} \quad -\frac{uv\tan\phi}{r} + \frac{uw}{r} \quad -2\Omega v\sin\phi + 2\Omega w\cos\phi \quad &= \quad -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} \qquad\qquad + \mathcal{F}_x \\
\frac{Dv}{Dt} \quad +\frac{u^2\tan\phi}{r} + \frac{vw}{r} \quad +2\Omega u\sin\phi \quad &= \quad -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} \qquad\qquad + \mathcal{F}_y \\
\frac{Dw}{Dt} \quad -\frac{u^2+v^2}{r} \quad +2\Omega u\cos\phi \quad &= \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad -g \quad + \mathcal{F}_z
\end{aligned}
$$

Inertia
Curvature
Coriolis
Pressure gradient
Gravity
Friction/Dissipation

Equations have many terms, how can they be simplified?

↳ **Scale analysis**

Scale analysis

Scaling terms

L	Horizontal distance scale
U	Horizontal velocity scale
T	Time scale $(T = L/U)$
H	Vertical distance scale
W	Vertical velocity scale
$\Delta P/\rho$	Horizontal pressure fluctuation scale

The material derivative $\frac{D}{Dt}$ represents change on the time-scale of motion.
So, its scale is represented as $\left[\frac{D}{Dt}\right] = \frac{1}{T} = \frac{U}{L}$

Some measures of scale

The **Reynolds number (Re)** is a dimensionless number that characterizes the relative importance of inertial forces to viscous forces in a fluid flow.

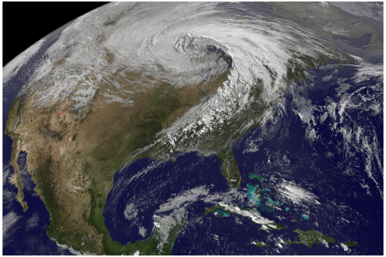
$$\mathrm{Re} = \frac{UL}{\nu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

Meanwhile, the **Rossby number (Ro)** tells us about the relative importance of the Coriolis force.

$$\mathrm{Ro} = \frac{U}{fL} = \frac{\text{inertial forces}}{\text{Coriolis force}}$$

Scaling examples

Mid-latitude storms



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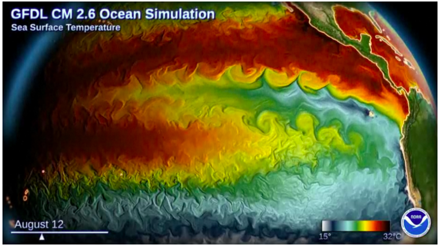
$$U \sim 10 - 20\,\mathsf{m/s}, \qquad L \sim 1000 - 2000\,\mathsf{km}$$
$$f \sim 10^{-4}\,\mathsf{s}^{-1}, \qquad \nu = 10^{-5}\,\mathsf{m}^2\mathsf{s}{-1}$$

$$\mathsf{Re} \sim \mathbf{10^8}, \quad \mathsf{Ro} \sim \mathbf{0.1 - 1}$$

Oceanic mesoscale eddies



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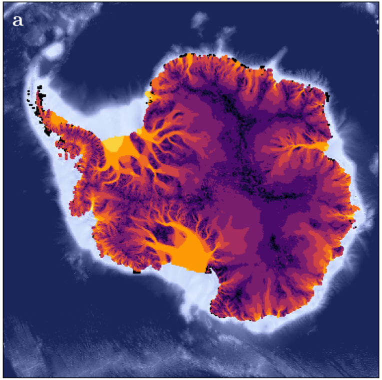
$$U \sim 0.1 - 1\,\mathsf{m/s}, \qquad L \sim 10 - 100\,\mathsf{km}$$
$$f \sim 10^{-4}\,\mathsf{s}^{-1}, \qquad \nu = 10^{-6}\,\mathsf{m}^2\mathsf{s}{-1}$$

$$\mathsf{Re} \sim \mathbf{10^7}, \quad \mathsf{Ro} \sim \mathbf{0.1 - 1}$$

Scaling examples

Ice sheets



$$U \sim 1 - 1000\,\text{m/yr} \quad < 10^{-4}\,\text{m/s}$$
$$L \sim 10^3 - 10^6\,\text{m}$$
$$\nu \sim 10^{10} - 10^{12}\,\text{m}^2\text{s}^{-1}$$

$$\text{Re} \sim \mathbf{10^{-11}}$$

Atmosphere and Ocean Dynamics

Full Navier-Stokes equations by component*

**in spherical coordinates*

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} - 2\Omega v\sin\phi + 2\Omega w\cos\phi = -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + \mathcal{F}_x$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + 2\Omega u\sin\phi = -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} + \mathcal{F}_y$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} + 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \mathcal{F}_z$$

Inertia
Curvature
Coriolis
Pressure gradient
Gravity
Friction/Dissipation

Atmospheric and ocean dynamics

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} - 2\Omega v\sin\phi + 2\Omega w\cos\phi = -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + 2\Omega u\sin\phi = -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} + \nu\nabla^2 v$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} + 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w$$

Viscous momentum diffusion in a Newtonian fluid:

$$\mathcal{F} \to \nu\nabla^2\mathbf{u}$$

Atmospheric and ocean dynamics

Shallow atmosphere/ocean approximation

First, given the radius of the Earth $a = 6370\,\text{km}$, and that $z \ll a$ in the atmosphere and ocean, we can assume a constant radius

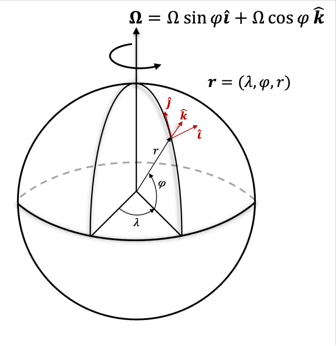
$$r = a = \text{const}$$

And then

$$\frac{\partial}{\partial r} = \frac{\partial z}{\partial r}\frac{\partial}{\partial z} \simeq \frac{\partial}{\partial z}$$

Recall for spherical coordinates:

$$dx = r\cos\phi d\lambda$$
$$dy = r\,d\phi$$



Atmospheric and ocean dynamics

Shallow atmosphere/ocean approximation

$$
\begin{aligned}
\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} - 2\Omega v\sin\phi + 2\Omega w\cos\phi &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\
\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} + 2\Omega u\sin\phi &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\
\frac{Dw}{Dt} - \frac{u^2+v^2}{a} + 2\Omega u\cos\phi &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w
\end{aligned}
$$

Viscous momentum diffusion in a Newtonian fluid:

$$\mathcal{F} \to \nu\nabla^2\mathbf{u}$$

Atmospheric and ocean dynamics

Shallow atmosphere/ocean approximation

$$
\begin{aligned}
\frac{Du}{Dt} &- \frac{uv\tan\phi}{a} + \frac{uw}{a} &&-2\Omega v\sin\phi + 2\Omega w\cos\phi &&= -\frac{1}{\rho}\frac{\partial p}{\partial x} && &&+\nu\nabla^2 u \\
\frac{Dv}{Dt} &+ \frac{u^2\tan\phi}{a} + \frac{vw}{a} &&+2\Omega u\sin\phi &&= -\frac{1}{\rho}\frac{\partial p}{\partial y} && &&+\nu\nabla^2 v \\
\frac{Dw}{Dt} &- \frac{u^2+v^2}{a} &&+2\Omega u\cos\phi &&= -\frac{1}{\rho}\frac{\partial p}{\partial z} &&-g &&+\nu\nabla^2 w
\end{aligned}
$$

Can we eliminate some terms via scale analysis?

Viscous momentum diffusion in a Newtonian fluid:

$$\mathcal{F} \to \nu\nabla^2\mathbf{u}$$

Atmospheric scale analysis for mid-latitude synoptic systems

$L \sim 10^6$ m	Horizontal distance scale		
$U \sim 10$ m s^{-1}	Horizontal velocity scale	$f_0 = 10^{-4}$ s^{-1}	Coriolis parameter
$T \sim 10^5$ s	Time scale $(T = L/U)$	$a = 6370 \times 10^3$ m	Radius of the Earth
$H \sim 10^4$ m	Vertical distance scale	$\nu = 10^{-5}$ m^2 s^{-1}	Kinematic viscosity
$W \sim 10^{-2}$ m s^{-1}	Vertical velocity scale	$P_0 = 10^5$ Pa	Surface air pressure
$\Delta P/\rho \sim 10^3$ m^2 s^{-2}	Horizontal pressure fluctuation scale	$\rho \sim 1$ kg m^{-3}	Density

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Vertical equation

$\frac{Dw}{Dt}$	$-\frac{u^2+v^2}{a}$	$+2\Omega u\cos\phi$	$=$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$-g$	$+\nu\nabla^2 w$
$\frac{UW}{L}$	$\frac{U^2}{a}$	$f_0 U$		$\frac{P_0}{\rho H}$	g	$\frac{\nu W}{H^2}$
10^{-7}	10^{-5}	10^{-3}		10	10	10^{-15}

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Vertical equation

$$\frac{Dw}{Dt} \quad -\frac{u^2+v^2}{a} \quad +2\Omega u\cos\phi \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad -g \quad +\nu\nabla^2 w$$

$$\frac{UW}{L} \quad \frac{U^2}{a} \quad f_0U \qquad \frac{P_0}{\rho H} \quad g \quad \frac{\nu W}{H^2}$$

$$10^{-7} \quad 10^{-5} \quad 10^{-3} \qquad 10 \quad 10 \quad 10^{-15}$$

Atmospheric and ocean dynamics

Hydrostatic approximation

Vertical equation

$$\frac{\partial p}{\partial z} = -\rho g$$

The vertical momentum equation reduces to a balance between the **vertical pressure gradient** and **gravity**. This is known as the **hydrostatic balance**.

With this formulation, pressure $p(z)$ is determined by the weight of the fluid above the current height z.

Integrating from the top of the atmosphere downward, and noting that $p(z=\infty)=0$ then gives the pressure at any level z:

$$p(z) = g\int_z^\infty \rho dz$$

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Horizontal equations

$$\frac{Du}{Dt} \quad -\frac{uv\tan\phi}{a}+\frac{uw}{a} \quad -2\Omega v\sin\phi+2\Omega w\cos\phi \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad +\nu\nabla^2 u$$

$$\frac{Dv}{Dt} \quad +\frac{u^2\tan\phi}{a}+\frac{vw}{a} \quad +2\Omega u\sin\phi \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad +\nu\nabla^2 v$$

$\frac{U^2}{L}$	$\frac{U^2}{a}$	$\frac{UW}{a}$	f_0U	f_0W	$\frac{\Delta P}{\rho L}$	$\frac{\nu U}{H^2}$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-6}	10^{-3}	10^{-12}

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Horizontal equations

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} - 2\Omega v \sin\phi + 2\Omega w \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

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$\frac{U^2}{L}$	$\frac{U^2}{a}$	$\frac{UW}{a}$	$f_0 U$	$f_0 W$	$\frac{\Delta P}{\rho L}$	$\frac{\nu U}{H^2}$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-6}	10^{-3}	10^{-12}

Atmospheric and ocean dynamics

Hydrostatic primitive equations

For flow that is in **hydrostatic balance**,
the momentum and continuity equations become:

Horizontal Momentum: $$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mathcal{F}_x$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \mathcal{F}_y$$

Vertical Momentum: $$\frac{\partial p}{\partial z} = -\rho g$$

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Coriolis parameter: $f = 2\Omega\cos\phi$

Atmospheric and ocean dynamics

Non-hydrostatic conditions

When does the hydrostatic approximation fail?

$\Longrightarrow$ When vertical velocity is not small, like for localized convection and near steep topography.

If these cases are of interest, then it is important to consider the full **non-hydrostatic** equations.

Convection-driven cloud formation

Photo from H. Weller

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Horizontal equations

$$\frac{Du}{Dt} \quad -\frac{uv\tan\phi}{a} + \frac{uw}{a} \quad -2\Omega v\sin\phi + 2\Omega w\cos\phi \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad +\nu\nabla^2 u$$

$$\frac{Dv}{Dt} \quad +\frac{u^2\tan\phi}{a} + \frac{vw}{a} \quad +2\Omega u\sin\phi \quad = \quad -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad +\nu\nabla^2 v$$

$$\frac{U^2}{L} \quad \frac{UW}{a} \quad \frac{U^2}{a} \quad f_0 U \quad f_0 W \quad \frac{\Delta P}{\rho L} \quad \frac{\nu U}{H^2}$$

$$10^{-4} \quad 10^{-5} \quad 10^{-8} \quad 10^{-3} \quad 10^{-6} \quad 10^{-3} \quad 10^{-12}$$

Atmospheric and ocean dynamics

Atmospheric scale analysis for mid-latitude synoptic systems

Horizontal equations

$$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} - 2\Omega v\sin\phi + 2\Omega w\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} + 2\Omega u\sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$\frac{U^2}{L}$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$f_0 U$	$f_0 W$	$\frac{\Delta P}{\rho L}$	$\frac{\nu U}{H^2}$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-6}	10^{-3}	10^{-12}

Atmospheric and ocean dynamics

Geostrophic approximation

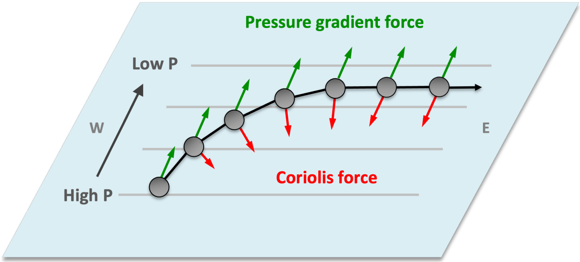
Horizontal equations

$$
\begin{aligned}
\frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega v \sin\phi = 0 \\
\frac{1}{\rho}\frac{\partial p}{\partial y} + 2\Omega u \sin\phi = 0
\end{aligned}
$$

For large-scale mid-latitudinal flows there is an intrinsic balance between the **pressure gradient force** and **Coriolis force**. This is known as the **geostrophic balance** and it leads to air parcels traveling along lines of constant pressure.

Atmospheric and ocean dynamics

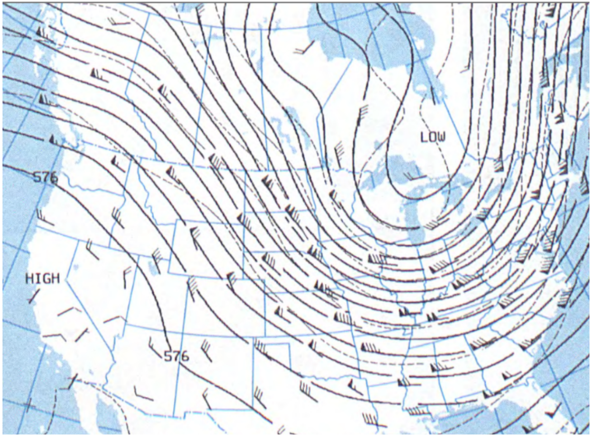
Geostrophic approximation



The geostrophic wind is the natural response of mid-latitudinal atmospheric motions to pressure gradients. It propagates perpendicular to the direction of the pressure gradient force and the Coriolis force (towards the East in the Northern and Southern hemispheres).

Geostrophic wind

1994-01-18 07:00



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Ocean dynamics

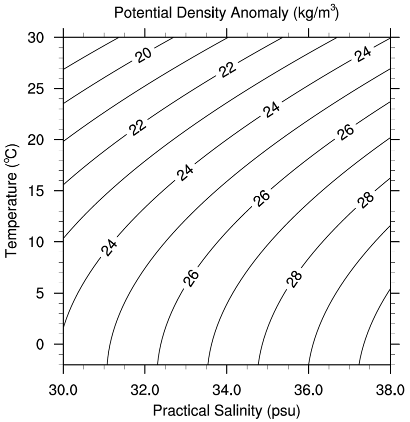
Incompressibility

Seawater is *nearly* incompressible. This means the mass conservation equation can be reduced to:

$$\nabla \cdot \mathbf{u} = 0$$

Seawater density

The density of seawater is a function of temperature, salinity and pressure $\rho = \rho(T, S, p)$. In most cases, the dependence on pressure can be ignored, leaving $\rho = \rho(T, S)$.



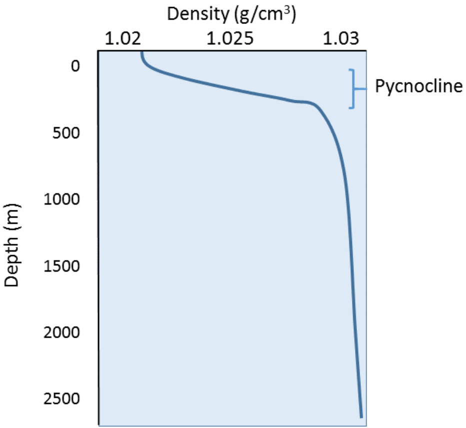
Ullrich, 2020

Ocean dynamics

Boussinesq approximation

The **Boussinesq approximation** assumes that density variations are negligible except where they appear in the buoyancy (gravity) term of the momentum equation.
It is valid when:

- When density variations are small, typically less than a few percent.
- Vertical length scales (H) are much smaller than the density scale height.



Webb, 2023

Ocean dynamics

Hydrostatic Boussinesq approximation

Incompressible flow that is in **hydrostatic balance**,
where density variations assumed to only affect buoyancy.

Horizontal momentum:
$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \mathcal{F}_u$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \mathcal{F}_v$$

Vertical momentum:
$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = -g\frac{\rho'}{\rho_0} \quad [= b]$$

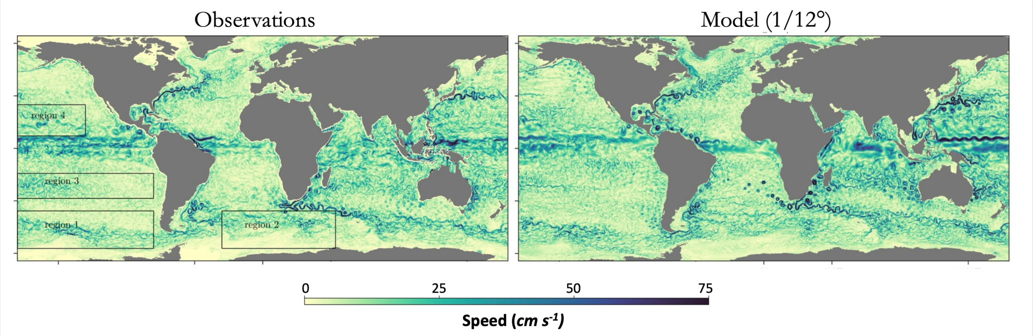
Continuity: $\nabla \cdot \mathbf{u} = 0$

$\rho = \rho_0 + \rho'$,

$\rho_0 = 1000\,\mathrm{kg\,m^{-3}}$

Ocean dynamics

Hydrostatic Boussinesq approximation



Silvestri et al., 2025

Ice-sheet dynamics

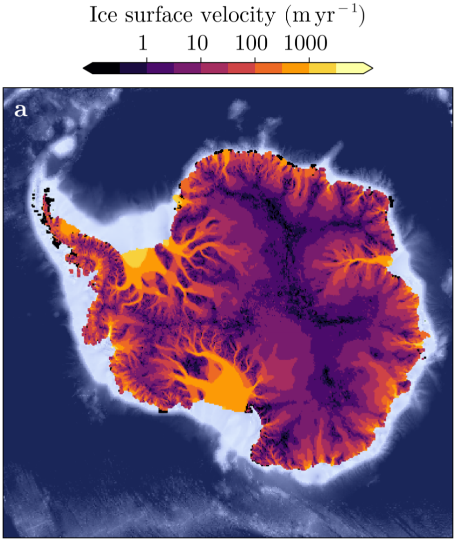
Ice-sheet dynamics

Ice-sheet dynamics are governed by viscous, non-Newtonian, gravitationally driven flow, called **creeping flow** or **Stokes flow**.

This type of flow is very slow and very viscous, so the Reynolds number is **very small**:

$$\mathrm{Re} = \frac{UL}{\nu} \ll 1$$

Recall that in **Non-Newtonian flow**, there is a non-linear relationship between stress and strain.



Ice-sheet dynamics

Incompressible Non-Newtonian Stokes Equation

Momentum (no inertia): $0 = -\nabla p + \nabla \cdot \boldsymbol{\tau} - \rho \mathbf{g}$

Mass conservation: $\nabla \cdot \mathbf{u} = 0$

Ice-sheet dynamics

Incompressible Non-Newtonian Stokes Equation

Momentum (no inertia): $0 = -\nabla p + \nabla \cdot \boldsymbol{\tau} - \rho \mathbf{g}$

Mass conservation: $\nabla \cdot \mathbf{u} = 0$

Constitutive law (non-Newtonian): $\boldsymbol{\tau} = 2\eta(\dot{\varepsilon})\,\mathbf{D}$

Strain-rate tensor: $\mathbf{D} = \dfrac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$

Viscosity (Glen's flow law): $\eta = \dfrac{1}{2} A^{-1/n} \dot{\varepsilon}_e^{(1-n)/n}, \quad A = f(T)$

Ice-sheet dynamics

Incompressible Non-Newtonian Stokes Equation

Momentum (no inertia): $\nabla \cdot \boldsymbol{\tau} = \nabla p + \rho \mathbf{g}$

Mass conservation: $\nabla \cdot \mathbf{u} = 0$

Constitutive law (non-Newtonian): $\boldsymbol{\tau} = 2\eta(\dot{\varepsilon})\,\mathbf{D}$

Strain-rate tensor: $\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$

Viscosity (Glen's flow law): $\eta = \frac{1}{2} A^{-1/n} \dot{\varepsilon}_e^{(1-n)/n}, \quad A = f(T)$

Ice-sheet dynamics

Incompressible Non-Newtonian Stokes Equation

Momentum:

$$\frac{\partial}{\partial x}\left(2\eta\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\left(\eta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)=\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(2\eta\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)=\frac{\partial p}{\partial y}$$

$$+\frac{\partial}{\partial x}\left(\eta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)+\frac{\partial}{\partial z}\left(2\eta\frac{\partial w}{\partial z}\right)=\frac{\partial p}{\partial z}+\rho g$$

Mass conservation: $\nabla\cdot\mathbf{u}=0$

Ice-sheet dynamics

Blatter-Pattyn approximation

- Apply the hydrostatic approximation in the vertical.
- Assumes horizontal gradients of the vertical velocity are small compared to the vertical gradient of the horizontal velocity:
$$\frac{\partial w}{\partial x} \ll \frac{\partial u}{\partial z}, \qquad \frac{\partial w}{\partial y} \ll \frac{\partial v}{\partial z}$$
- Assume a zero normal stress at the surface (stress-free condition).

This eliminate pressure as a prognostic variable and reduces the problem to two equations (solving for u and v) – w is diagnosed from mass conservation.

Ice-sheet dynamics

Blatter-Pattyn approximation

Momentum:

$$\frac{\partial}{\partial x}\left(4\eta\frac{\partial u}{\partial x}+2\eta\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}+\eta\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right) = \rho g\frac{\partial s}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\eta\frac{\partial u}{\partial y}+\eta\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(4\eta\frac{\partial v}{\partial y}+2\eta\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(\eta\frac{\partial v}{\partial z}\right) = \rho g\frac{\partial s}{\partial y}$$

Continuity: $\nabla\cdot\mathbf{u}=0$

Ice-sheet dynamics

Shallow-shelf approximation (SSA)

- Hydrostatic approximation.
- Shallow approximation $(H \ll L) \rightarrow$ vertically integrated.
- Assumes gravitational driving stress is balanced by longitudinal stress.

Momentum: $$\frac{\partial}{\partial x}\left(4\bar{\eta}H\frac{\partial \bar{u}}{\partial x} + 2\bar{\eta}H\frac{\partial \bar{v}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\bar{\eta}H\frac{\partial \bar{u}}{\partial y} + \bar{\eta}H\frac{\partial \bar{v}}{\partial x}\right) = \rho g H\frac{\partial s}{\partial x}$$

$$\frac{\partial}{\partial y}\left(4\bar{\eta}H\frac{\partial \bar{v}}{\partial y} + 2\bar{\eta}H\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial x}\left(\bar{\eta}H\frac{\partial \bar{u}}{\partial y} + \bar{\eta}H\frac{\partial \bar{v}}{\partial x}\right) = \rho g H\frac{\partial s}{\partial y}$$

Continuity: $\nabla \cdot \mathbf{u} = 0$

Ice-sheet dynamics

Shallow-ice approximation (SIA)

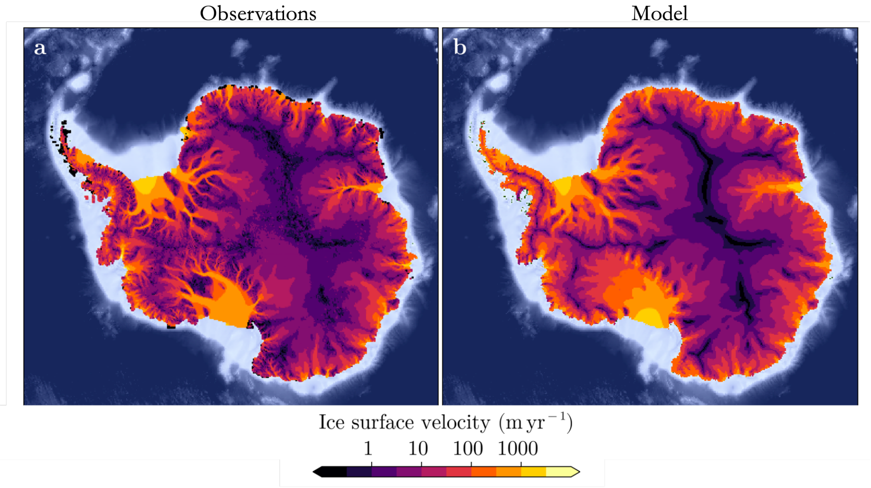
- Hydrostatic approximation.
- Shallow approximation $(H \ll L) \rightarrow$ vertically integrated (sometimes).
- Assumes gravitational driving stress is balanced by vertical shear stress.

Momentum: $$\frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right) = \rho g \frac{\partial s}{\partial x}$$

$$\frac{\partial}{\partial z}\left(\eta \frac{\partial v}{\partial z}\right) = \rho g \frac{\partial s}{\partial y}$$

Continuity: $$\nabla \cdot \mathbf{u} = 0$$

Ice-sheet dynamics



Swierczek-Jereczek et al.

Geophysical fluid dynamics summary

Relevant terms for each system

Term	Atmosphere	Ocean	Ice Sheets
Inertia	X	X	
Curvature	X	X	
Coriolis	X	X	
Pressure Gradient	X	X	X
Gravity	X	X	X
Friction/Dissipation	X	X	X*

*Non-Newtonian fluid

Summary of common approximations

Atmosphere/ocean

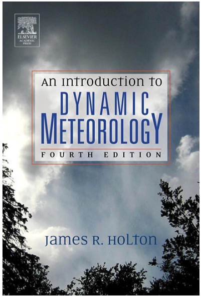
- Shallow atmosphere/ocean approximation
- Hydrostatic approximation
- Geostrophic approximation
- Boussinesq approximation (ocean)

Ice sheets

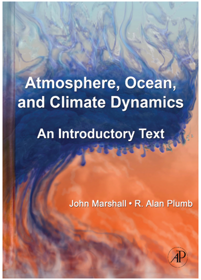
- Blatter-Pattyn approximation
- Shallow-shelf approximation
- Shallow-ice approximation

Useful references

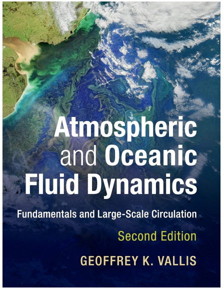
Holton, 2004



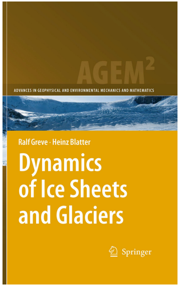
Marshall and Plumb, 2008



Vallis, 2017



Greve and Blatter, 2009



Scale analysis: tropical cyclone

NASA - Hurricane Dorian

The **length scale** is the diameter of the storm. The **horizontal velocity scale** is the typical velocity of the flow. And the **time scale** is approximately the time required for a fluid parcel to move around the storm.

Atmospheric and ocean dynamics

Additional strategies

- Shallow-water approximation – assume horizontal length scale is much larger than vertical, integrate over vertical dimension.
- Operator splitting, e.g. separation into *geostrophic* and *ageostrophic* components – longer timesteps possible for slowly evolving terms, reduced computational needs.

- **Atmosphere**: Use pressure as the vertical coordinate – eliminates explicit density dependence, simplifying the continuity equation, and aligns better with the hydrostatic structure of the atmosphere.
- **Ocean**: Rigid lid approximation – impose a fixed ocean surface, ignores gravitational waves.

Hydrostatic approximation

Although the horizontal atmosphere is in a constant state of motion, vertical velocities are typically fairly small (especially averaged over the large scale).

Definition

A fluid is in **Hydrostatic balance** when external forces (here gravity) are balanced by the pressure-gradient force, and thus vertical acceleration is zero.

The **hydrostatic approximation** assumes that the pressure at any level depends on the weight of the fluid above that level.

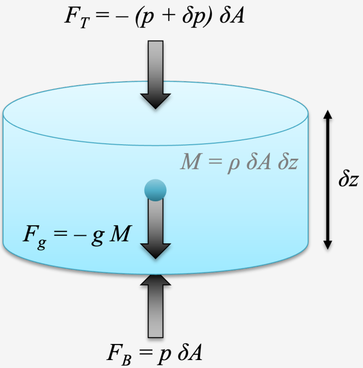
Hydrostatic approximation

Forces acting on a given column of air with no vertical acceleration:

$$F_{\mathrm{T}} = -\left(p + \delta p\right)\delta A$$
$$F_{\mathrm{B}} = p\delta A$$
$$F_{\mathrm{g}} = -gM = -\rho g\delta A\delta z$$

A balance of these forces leads to:

$$F_T + F_B + F_g = \delta p + \rho g \delta z = 0$$



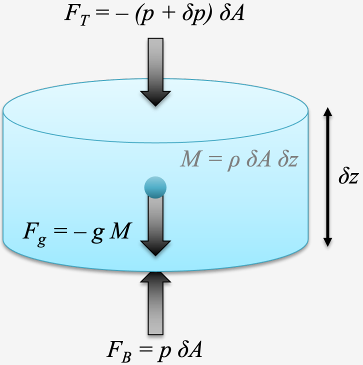
Hydrostatic approximation

$$F_T + F_B + F_g = \delta p + \rho g \delta z = 0$$

Taylor Series: $\delta p \approx \dfrac{\partial p}{\partial z}\delta z$

Hydrostatic balance:

$$\boxed{\frac{\partial p}{\partial z} + \rho g = 0}$$



Hydrostatic approximation

$$F_T + F_B + F_g = \delta p + \rho g \delta z = 0$$

Taylor Series: $\delta p \approx \frac{\partial p}{\partial z}\delta z$

Hydrostatic balance:

$$\boxed{\frac{\partial p}{\partial z} + \rho g = 0}$$

Integrating from the top of the atmosphere downward, and noting that $p(z=\infty)=0$ then gives the pressure at any level z:

$$p(z) = g\int_z^\infty \rho dz$$

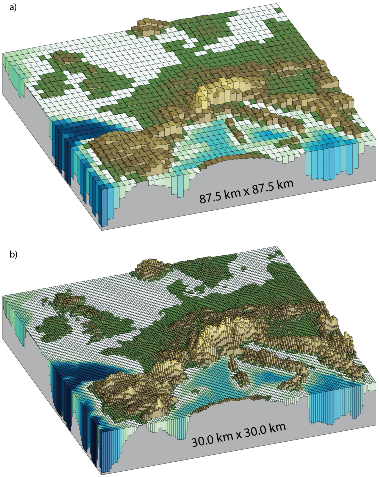
Atmospheric and ocean dynamics

Non-hydrostatic conditions

When does the hydrostatic approximation fail?

$\Longrightarrow$ When vertical velocity is not small, like for localized convection and near steep topography.

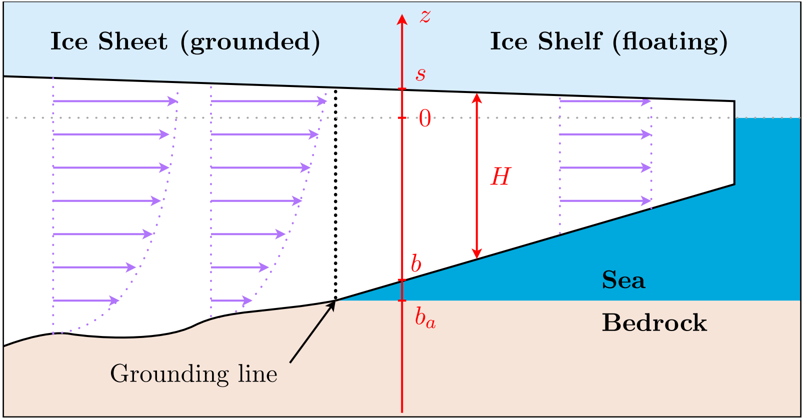
If these cases are of interest, then it is important to consider the full **non-hydrostatic** equations.



Rose, 2020

Ice-sheet dynamics

Flow regimes



Larour et al., 2012