ECON 4190: Industrial Organization **Fall 2021**

Homework 3

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Pledge: On my honor, I pledge that I have neither given nor received help on this assignment
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1. (a) Let L refer to the left-wing candidate and R refer to the right-wing candidate. Simply as a matter of their positions, L will always receive the $\frac{1}{4}$ to their left, and R will always receive the $\frac{1}{8}$ to their right. However, given a uniform distribution we only have to figure out $\hat{x}$, the location of the voter that is indifferent between the candidates, to determine the share each candidate receives.

   Given the general utility function $v_i = r_i - T|x - l_i|$, utilities for a voter for L and R can be determined to equal $v_L = r_1 - T(x - \frac{1}{4})$ and $v_R = r_2 - T(\frac{7}{8} - x)$ respectively. Now we simply solve for x when $v_L = v_R$:

$$\begin{aligned} v_L &= v_R \\ r_1 - T(\hat{x} - \frac{1}{4}) &= r_2 - T(\frac{7}{8} - \hat{x}) \\ 2T(\hat{x}) &= r_1 - r_2 + \frac{9}{8}T \\ \hat{x} &= \frac{r_1 - r_2}{2T} + \frac{9}{16} \end{aligned}$$

   Therefore, L will receive $\hat{x}$ of the vote, and R will receive $1 - \hat{x}$ of the vote.

   (b) Very similar process to before, simply change the function slightly.

$$\begin{aligned} r_1 - T(\hat{x} - \frac{1}{4})^2 &= r_2 - T(\frac{7}{8} - \hat{x})^2 \\ tl; dr& \\ r_1 - r_2 &= T(\hat{x} - \frac{1}{4})^2 - T(\hat{x} - \frac{7}{8})^2 \\ \frac{r_1 - r_2}{T} &= \hat{x}^2 - \frac{\hat{x}}{2} + \frac{1}{16} - \hat{x}^2 + \frac{7}{4}\hat{x} - \frac{49}{64} \\ \frac{r_1 - r_2}{T} + \frac{45}{64} &= \frac{5}{4}\hat{x} \\ \hat{x} &= \frac{4}{5}(\frac{r_1 - r_2}{T}) + \frac{9}{16} \end{aligned}$$

   (c) Once again, we know that at the very least, L will always receive the voters spanning the $\frac{1}{4}$ to their left, and R will always receive the voters spannihg the $\frac{1}{8}$ to their right. Given the changed distribution, we know that they will each receive $\frac{1}{8}$ of the votes at least.

To determine the rest, we need to determine how the votes in the middle are split. Thankfully, we already calculated the location of the indifferent voter $\hat{x}$ in part A to be $\frac{r_1-r_2}{2T} + \frac{9}{16}$ which we can reuse here. From this, we can then calculate how many votes from the "middle" each candidate gets by multiplying the "distance" to the indifferent voter from each candidate by the voter density within that region, $\frac{3}{4} * \frac{1}{\frac{7}{8}-\frac{1}{4}} = \frac{6}{5}$. We know that each candidate also receives $\frac{1}{8}$ from their base, and also since there are only 2 candidates, the number of votes from either candidate can be determined once we know one. We solve for the votes gained by the left candidate as followed:

$$
\begin{aligned}
votes_L &= \frac{6}{5}(\hat{x} - \frac{1}{4}) + \frac{1}{8} \\
&= \frac{6}{5}(\frac{r_1 - r_2}{2T} + \frac{9}{16} - \frac{1}{4}) + \frac{1}{8} \\
&= \frac{6}{5} * \frac{r_1 - r_2}{2T} + \frac{6}{5} * \frac{5}{16} + \frac{1}{8} \\
votes_L(T, r_1, r_2) &= \frac{6}{5} * \frac{r_1 - r_2}{2T} + \frac{1}{2} \\
1 - votes_L = votes_R(T, r_1, r_2) &= \frac{3}{5} * \frac{r_2 - r_1}{T} + \frac{1}{2}
\end{aligned}
$$

(d) Reusing a lot of our work from before, we know the density of the middle region is $\frac{1}{3} * \frac{1}{\frac{7}{8}-\frac{1}{4}} = \frac{8}{15}$

$$
\begin{aligned}
votes_L &= \frac{8}{15}(\frac{r_1 - r_2}{2T} + \frac{9}{16} - \frac{1}{4}) + \frac{1}{3} \\
votes_L(T, r_1, r_2) &= \frac{4}{15}(\frac{r_1 - r_2}{T}) + \frac{1}{2} \\
1 - votes_L = votes_R(T, r_1, r_2) &= \frac{4}{15}(\frac{r_2 - r_1}{T}) + \frac{1}{2}
\end{aligned}
$$

The right candidate (who is further to the right than the left candidate is to the left) gets more votes here than in part C - this indicates that it's especially important for candidates positioned further from the center to have larger bases of support.

2. (a) To find Q_i, we first have to find the indifferent consumers between firm i and firms $i-1$ and $i+1$, called $\hat{x}_L$ and $\hat{x}_R$. Using the utility equation $v_i = r - p_i - T|x - l_i|$, we can solve for both as follows:

$$
\begin{aligned}
r - p_{i-1} - T(\hat{x}_L - \frac{i-1}{n}) &= r - p_i - T(\frac{i}{n} - \hat{x}_L) \\
p_i - p_{i-1} &= T(\hat{x}_L - \frac{i-1}{n}) - T(\frac{i}{n} - \hat{x}_L) \\
&= T\hat{x}_L - \frac{Ti - T}{n} - \frac{Ti}{n} + T\hat{x}_L \\
&= 2T\hat{x}_L - \frac{2Ti - T}{n} \\
2T\hat{x}_L &= p_i - p_{i-1} + \frac{2Ti - T}{n} \\
\hat{x}_L &= \frac{p_i - p_{i-1}}{2T} + \frac{2i - 1}{2n}
\end{aligned}
$$

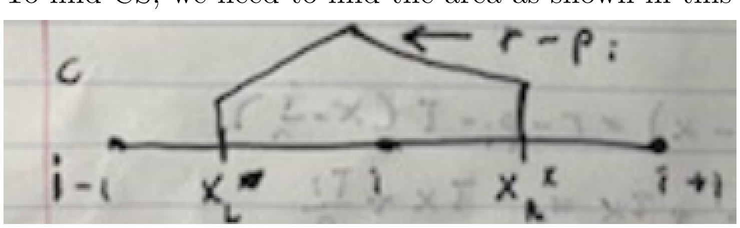
$$
\begin{aligned}
r - p_{i+1} - T(\frac{i+1}{n} - \hat{x}_R) &= r - p_i - T(\hat{x}_R - \frac{i}{n}) \\
p_i - p_{i+1} &= T(\frac{i+1}{n} - \hat{x}_R) - T(\hat{x}_R - \frac{i}{n}) \\
&= \frac{Ti + T}{n} - T\hat{x}_R - T\hat{x}_R + \frac{Ti}{n} \\
&= \frac{2Ti + T}{n} - 2T\hat{x}_R \\
2T\hat{x}_R &= p_{i+1} - p_i + \frac{2Ti - T}{n} \\
\hat{x}_R &= \frac{p_{i+1} - p_i}{2T} + \frac{2i + 1}{2n}
\end{aligned}
$$

Finding the difference $\hat{x}_R$ and $\hat{x}_L$ gives us the share of the market firm i sells to, and since $M = 1$, also gives us the demand Q:

$$
\begin{aligned}
Q = \hat{x}_R - \hat{x}_L &= \frac{p_{i+1} - p_i}{2T} + \frac{2i+1}{2n} - (\frac{p_i - p_{i-1}}{2T} + \frac{2i-1}{2n}) \\
\therefore Q_i(p_i, p_{i-1}, p_{i+1}) &= \frac{p_{i+1} - 2p_i + p_{i-1}}{2T} + \frac{1}{n}
\end{aligned}
$$

(b) If $p_{i-1} = p_{i+1} = p$ then our derived Q_i reduces to $\frac{p - p_i}{T} + \frac{1}{n}$.

(c) To find CS, we need to find the area as shown in this graph:



Doing so requires finding $v(\hat{x}_L)$ and $v(\hat{x}_R)$, the utility gained by the indifferent consumers on both ends of firm i. Given that all non-i firms are charging the same price, we know that the problem must be symmetrical, so we only need to solve for $v(\hat{x}_L)$. *Note: the problem has some ambiguity in its wording regarding whether or not the assumption from part B that all firms except i are charging the same price p still holds. Since it makes the computation simpler, and the problem was presented during discussion as having this assumption, we have also made this assumption here.*

$$\begin{aligned}
v(\hat{x}_L) &= r - p_i - T(\frac{i}{n} - (\frac{p_i - p}{2T} + \frac{2i-1}{2n})) \\
&= r - p_i - \frac{Ti}{n} + \frac{p_i - p}{2} + \frac{Ti}{n} - \frac{T}{2n} \\
v(\hat{x}_L) = v(\hat{x}_R) &= r - p_i + \frac{p_i - p}{2} - \frac{T}{2n}
\end{aligned}$$

This gives us everything we need to solve for consumer surplus:

$$\begin{aligned}
CS &= v(\hat{x}_L) * Q + \frac{1}{2}(r - p_i - v(\hat{x}_L)) * Q = Q(v(\hat{x}_L) * Q + \frac{1}{2}(r - p_i - v(\hat{x}_L))) \\
CS &= (\frac{p - p_i}{T} + \frac{1}{n}) * [(r - p_i + \frac{p_i - p}{2} - \frac{T}{2n}) + \frac{1}{2}(r - p_i - (r - p_i + \frac{p_i - p}{2} - \frac{T}{2n})] \\
CS &= (\frac{p - p_i}{T} + \frac{1}{n}) * [(r - p_i + \frac{p_i - p}{2} - \frac{T}{2n}) + \frac{1}{2}(\frac{T}{2n} - \frac{p_i - p}{2})]
\end{aligned}$$

(d) Setting $p_i = p$ greatly simplifies our equation, which we can also then simply multiply by n firms to find total consumer surplus in the city:

$$\begin{aligned}
CS_i &= (\frac{p - p}{T} + \frac{1}{n}) * [(r - p + \frac{p - p}{2} - \frac{T}{2n}) + \frac{1}{2}(\frac{T}{2n} - \frac{p - p}{2})] \\
CS_i &= \frac{1}{n} * [r - p - \frac{T}{2n} + \frac{T}{4n}] \\
CS_{city} &= n * CS_i = r - p - \frac{T}{2n} + \frac{T}{4n}
\end{aligned}$$