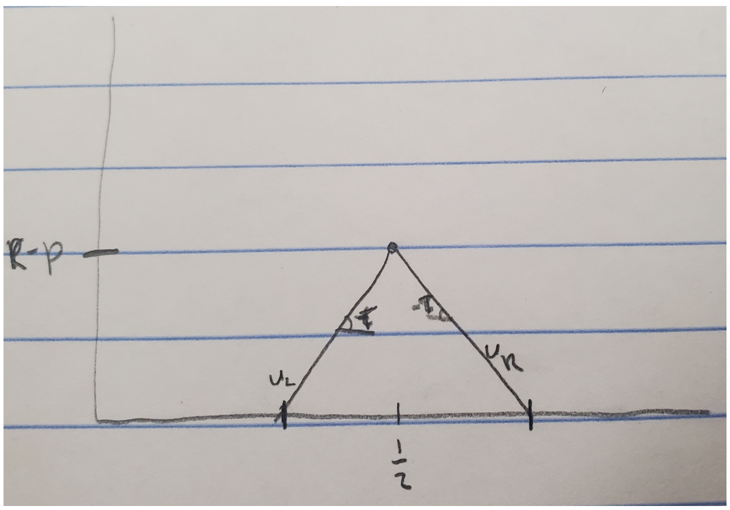
ECON 4190: Industrial Organization **Fall 2021**

Homework 2

Collaborators: Alex Shen (as5gd), Sean Velhagen (spv5hq), Max Bresticker (mtb9sex)

Pledge: On my honor, I pledge that I have neither given nor received help on this assignment
Signature: *Alex Shen, Sean Velhagen, Max Bresticker*

a. A profit-maximizing firm will seek to max $\pi(Q) = PQ - C(Q)Q$, which we can reduce to $\pi(Q) = PQ$ because Firm 1 has no costs. Since Firm 1 has control over its price P, we first attempt to calculate $Q(P)$, or the quantity of consumers that will buy at a given price. To do this we can graph the consumer's utility against location, which yields this:



We represent the utility of all consumers to the "left" of Firm 1 (i.e. $x < \frac{1}{2}$) with v_l and all consumers to the "right" as v_r. They can be represented by the following equations:

$$v_l = R_1 - p + T(x - \frac{1}{2})$$
$$v_r = R_1 - p - T(x - \frac{1}{2})$$

To find Q, we would multiply the fraction of the market that is covered by the total mass M, but since $M = 1$ in this case, we simply have to find the distance between the x-intercepts of v_r and v_l: $\hat{x_r}$ and $\hat{x_l}$.

First solve for $\hat{x_r}$ and $\hat{x_l}$:

$$v_l = 0 = R_1 - p + T(x - \frac{1}{2}) \Leftrightarrow \hat{x_l} = \frac{p - R_1}{T} + \frac{1}{2}$$
$$v_r = R_1 - p - T(x - \frac{1}{2}) \Leftrightarrow \hat{x_r} = \frac{R_1 - p}{T} + \frac{1}{2}$$

Then find the difference:

$$
\begin{aligned}
Q(p) = \hat{x}_r - \hat{x}_l &= \frac{R_1 - p}{T} + \frac{1}{2} - (\frac{p - R_1}{T} + \frac{1}{2}) \\
&= \frac{R_1 - p}{T} - \frac{p - R_1}{T} \\
&= \frac{2R_1 - 2p}{T} \\
Q(p) &= \frac{2}{T}(R_1 - p)
\end{aligned}
$$

Now we return to the profit function and maximize it to find the maximum price:

$$
\begin{aligned}
max_p \pi = p * \frac{2}{T}(R_1 - p) &= \frac{2R_1 p}{T} - \frac{2p^2}{T} \\
\frac{\partial \pi}{\partial p} = \frac{2R_1}{T} - \frac{4p}{T} &= 0 \\
\therefore p^* &= \frac{R_1}{2}
\end{aligned}
$$

Knowing price makes it easy to calculate maximum profit:

$$
\begin{aligned}
\pi^* &= PQ = P * \frac{2}{T}(R_1 - P) \\
&= \frac{R_1}{2}\frac{2}{T}(R_1 - \frac{R_1}{2}) \\
&= \frac{R_1}{T} * \frac{R_1}{2} \\
\therefore \pi^* &= \frac{R_1^2}{2T}
\end{aligned}
$$

We also need to address the edge case where the market is completely covered; in that case, instead of using our previous derivations, we can use a much simpler form: since we know that the consumers at 0 and 1 are indifferent, we can use $v_l(0) = 0$ to solve for p^* in this case (which also gives us profit, because Q would be exactly 1):

$$
\begin{aligned}
v_l(0) = R_1 - p + T(-\frac{1}{2}) &= 0 \\
p^* &= R_1 - \frac{T}{2} \\
\pi^* &= R_1 - \frac{T}{2}
\end{aligned}
$$

Then Firm 1 would simply pick whichever scenario (uncovered vs. covered market) has a higher profit based on the parameters.

Solution: $p^* = \frac{R_1}{2}$**,** $\pi^* = \frac{R_1^2}{2T}$ **when** $\frac{R_1^2}{2T} > R_1 - \frac{T}{2}$**, otherwise** $p^* = \pi^* = R_1 - \frac{T}{2}$

b. The problem is very similar to before - to determine the uncovered market quantity but we simply replace the $\frac{1}{2}$ with $\frac{1}{4}$:

$$v_l = R_1 - p + T(x - \frac{1}{4}) = 0 \Leftrightarrow \hat{x_l} = \frac{p - R_1}{T} + \frac{1}{4}$$
$$v_r = R_1 - p - T(x - \frac{1}{4}) = 0 \Leftrightarrow \hat{x_r} = \frac{R_1 - p}{T} + \frac{1}{4}$$
$$Q(p) = \frac{R_1 - p}{T} + \frac{1}{4} - (\frac{p - R_1}{T} + \frac{1}{4}) = 2(\frac{R_1 - p_1}{T})$$

Since this is the same $Q(p)$ from part (a), we know that $p^* = \frac{R_1}{2}$, $\pi^* = \frac{R_1^2}{2T}$ when the market is uncovered.

If Firm 1 wants to cover all the way to the left, then Q is entirely determined by $\hat{x_r} = \frac{R_1 - p}{T} + \frac{1}{4}$. This gives us the profit function $\pi = p(\frac{R_1 - p}{T} + \frac{1}{4})$, which we can maximize to solve for p^*, and therefore π^*:

$$\pi(p) = p(\frac{R_1 - p}{T} + \frac{1}{4}) = \frac{R_1}{T}p - \frac{1}{T}p^2 + \frac{1}{4}p$$
$$\frac{\partial \pi}{\partial p} = \frac{R_1}{T} - \frac{2}{T}p + \frac{1}{4} = 0$$
$$0 = 4R_1 - 8p + T$$
$$p^* = \frac{1}{2}R_1 + \frac{1}{8}T$$

If Firm 1 wants to cover the entire market, we can calculate price simply by finding what price makes a customer located at 1 indifferent (since 1 is further than 0, any price acceptable to someone at 1 will also work for 0). The distance from $\frac{1}{4}$ to 1 is $\frac{3}{4}$, so $p^* = R_1 - \frac{3}{4}T$. Since the market is fully covered in this case, $Q = 1$, so $\pi^* = R_1 - \frac{3}{4}T$ as well.

Solution: there are 3 strategies Firm 1 can choose to employ (uncovered, covered to the left, fully covered); the profit-maximizing price and profit will be determined by which of these strategies has the highest profit based on R_1 and T.

c. There is only one "side" Firm 1 has to deal with in this scenario, so the equation becomes easy: $v(x) = R_1 - p - T(1 - x)$

When the market is uncovered, we solve for the location of an indifferent consumer at $\hat{x}$, which we then use to solve for the total quantity Q (i.e. the distance between 1 and $\hat{x}$):

$$v(\hat{x}) = R_1 - p - T(1 - \hat{x}) = 0 \Leftrightarrow \hat{x} = \frac{p - R_1}{T} + 1 = 0$$
$$\therefore Q(p) = 1 - \hat{x} = 1 - (\frac{p - R_1}{T} + 1) = \frac{R_1 - p}{T}$$

Given $\pi = Q(p)p$, we can solve for the profit-maximizing p:

$$\pi = (\frac{R_1 - p}{T})p = \frac{R_1}{T}p - \frac{1}{T}p^2$$
$$\frac{\partial \pi}{\partial p} = \frac{R_1}{T} - \frac{2}{T}p \Longleftrightarrow p^* = \frac{R_1}{2}$$
$$\therefore \pi^* = \frac{R_1}{2}(\frac{R_1}{2}\frac{1}{T}) = \frac{R_1^2}{4T}$$

When the market is fully covered, set price such that the consumer at 0 will purchase: $p = R_1 - T$. Since $Q = 1$ when the market is fully covered, this is also the value of profit.

Solution: $p^* = R_1 - T$**,** $\pi^* = R_1 - T$ **when** $R_1 - T > \frac{R_1^2}{4T}$**, otherwise** $p^* = \frac{R_1}{2}, \pi^* = \frac{R_1^2}{4T}$

NOTE: in parts (a), (b), and (c), price/profit increase when R_1 increase and decrease when T decreases.

d. Firm 1 will want to be located at $\frac{1}{2}$. For any given combination of the exogenous variables R_1 and T that determine our profit-maximizing price and profit, the only effect that changing the location of the firm will have is affecting the travel distance for its customers. Positioning at $\frac{1}{2}$ minimizes the maximum distance a consumer has to travel; moving by any distance ϵ to be closer to a consumer increases the distance to another consumer by ϵ.