

# Level-Set Method to Solve the 2D Fisher–Stefan Model: Numerical Test Cases

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In this document, we provide numerical test cases for our level-set method used to solve the 2D Fisher–Stefan model. We compute solutions on circular domains  $\Omega(t)$ , and compare our results with Simpson [1], who solved the one-dimensional, radially-symmetric Fisher–Stefan problem on a disc. The numerical methods used are outlined in Simpson [1], and involve a change of variables to map the expanding or shrinking domain to the unit interval. The solutions of Simpson [1] were performed with 1001 grid points, yielding a grid spacing of  $\Delta\xi = 0.001$ , and a time step size of  $\Delta t = 0.0001$ . MATLAB code of this implementation is available in a public [Github repository](#). To validate the level-set method, we consider both spreading and extinction solutions. In each, we consider the initial condition

$$u(x, y, 0) = \begin{cases} \alpha & \text{if } \sqrt{(x - L_x/2)^2 + (y - L_y/2)^2} \leq \beta \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where  $L_x$  and  $L_y$  are the computational domain widths in the  $x$  and  $y$ -directions respectively,  $\beta$  is the radius of the circular region  $\Omega(0)$ , and  $\alpha$  is the initial density inside this region.

## 1 Spreading Solutions

We first consider a numerical solution with  $\alpha = 0.5$ ,  $\beta = 10$ , and  $\kappa = 1$ , where  $\kappa$  is the Stefan parameter in the Fisher–Stefan model. We compute a numerical solution using our level-set method, with  $N_x = N_y = 200$  grid intervals. The solution is computed until  $t = 30$ , with  $L_x = L_y = 50$ . To compare, we plot the one-dimensional solution obtained with the method of Simpson [1], with a one-dimensional slice at  $y = 25$  of the two-dimensional level-set solution. This is illustrated in Figure 1.1. We obtain good agreement between the two-dimensional level-set solution and the one-dimensional solution using the method of Simpson [1]. For these parameters and initial conditions, the density  $u(x, t)$  evolves to a travelling-wave like profile, and spreads at a roughly-constant speed. Furthermore, the level-set method preserves the circular geometry as the solution progresses, as Figure 1.2 shows.

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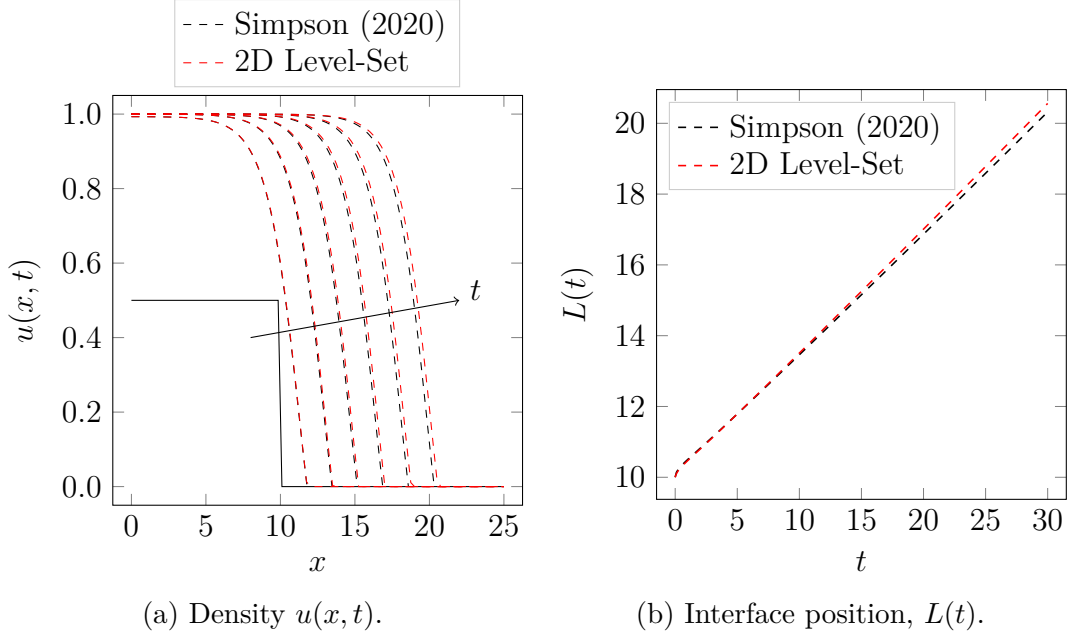


Figure 1.1: Numerical test of the level-set method with  $\Delta t = 0.002$ ,  $\Delta x = \Delta y = 0.25$ , on a square grid of  $L_x = L_y = 50$ , *i.e.* a  $201 \times 201$  spatial grid. Solution computed with  $\kappa = 1$ ,  $D = 1$ , and  $\lambda = 1$ . (a) Density  $u(x, t)$  plotted at  $t \in \{0, 5, 10, 15, 20, 25, 30\}$ , where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing  $t$ . For the 2D level-set solution, we plot the  $y = 25$  slice,  $u(x, 25, t)$ . (b) Interface position,  $L(t)$ . For the 2D level-set solution, the position  $L(t)$  is the zero level-set position, that is  $x$  such that  $\phi(x, 25, t) = 0$ , which we compute by linear interpolation.

To ensure our results remain valid for different parameters, we computed a solution with the same initial condition as Figure 1.1, but with  $\beta = 5$  and  $\kappa = 20$ . Again, we find good agreement between the radially-symmetric solution using the method of Simpson [1], and the two-dimensional level-set solution. These results are presented in Figure 1.3.

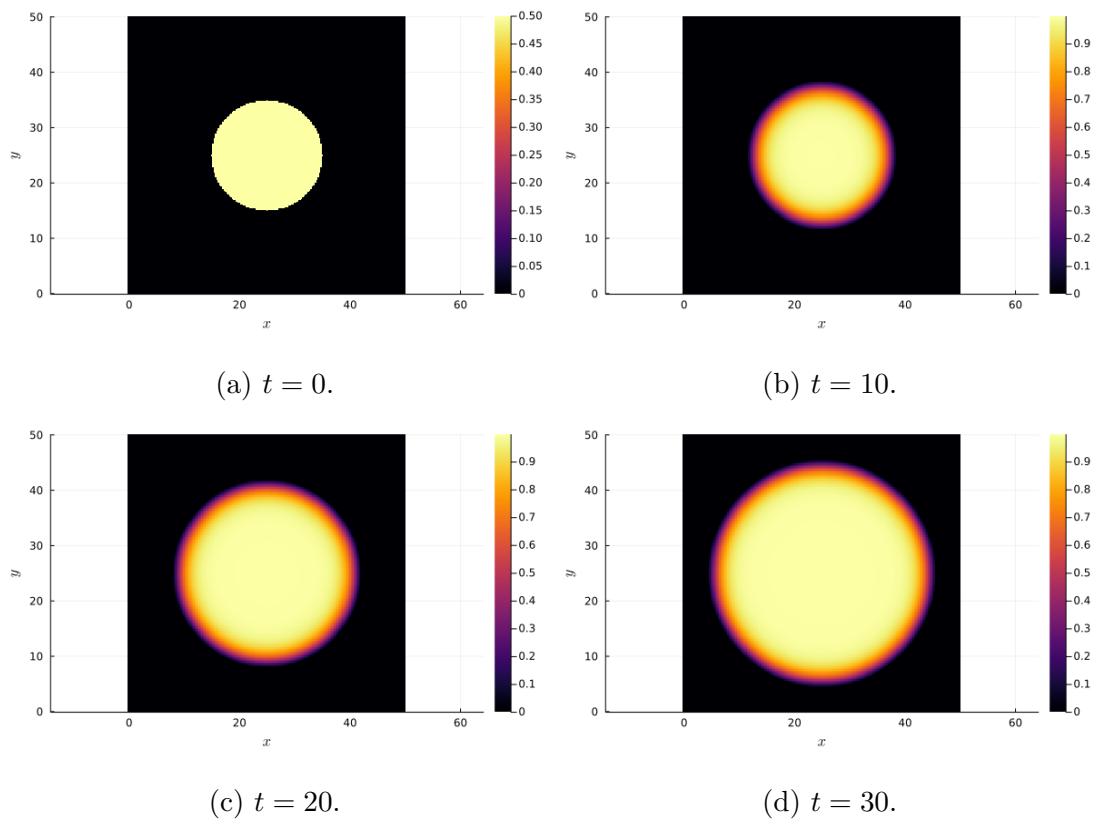


Figure 1.2: Two-dimensional evolution of the density  $u(x, y, t)$  in the spreading solution of Figure 1.1.

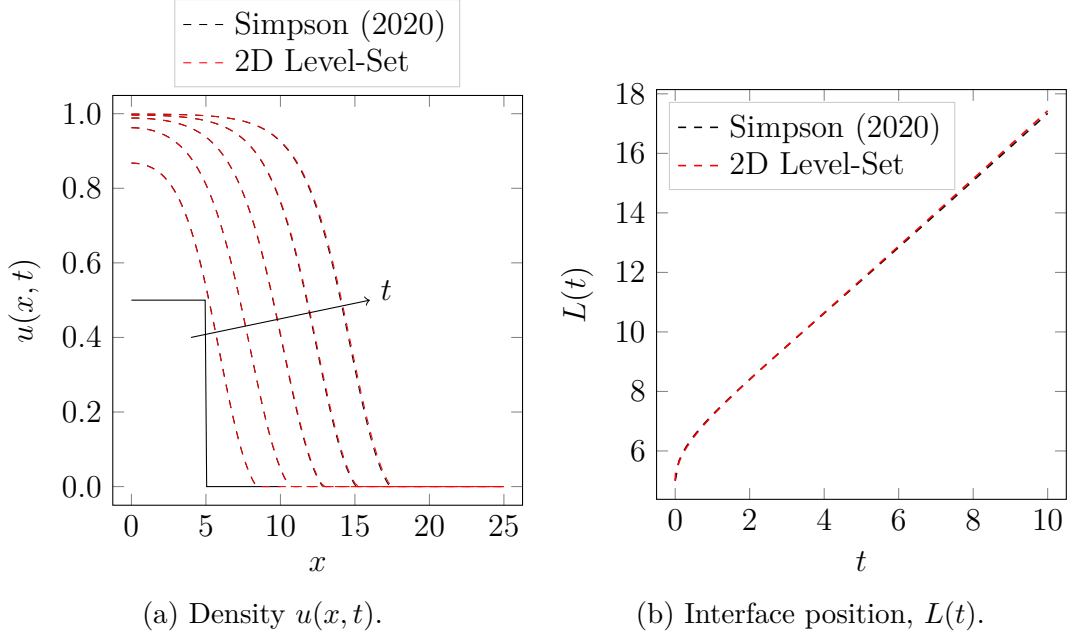


Figure 1.3: Numerical test of the level-set method with  $\Delta t = 0.002$ ,  $\Delta x = \Delta y = 0.25$ , on a square grid of  $L_x = L_y = 50$ , *i.e.* a  $201 \times 201$  spatial grid. Solution computed with  $\kappa = 20$ ,  $D = 1$ , and  $\lambda = 1$ . (a) Density  $u(x, t)$  plotted at  $t \in \{0, 2, 4, 6, 8, 10\}$ , where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing  $t$ . For the 2D level-set solution, we plot the  $y = 25$  slice,  $u(x, 25, t)$ . (b) Interface position,  $L(t)$ . For the 2D level-set solution, the position  $L(t)$  is the zero level-set position, that is  $x$  such that  $\phi(x, 25, t) = 0$ , which we compute by linear interpolation.

## 2 Extinction Solution

In addition to the spreading solutions in §1, the Fisher–Stefan model admits solutions where density decreases with time [1, 2]. This arises from a competition between growth mediated by the source term, and mass loss at the interface due to the Stefan condition. In the radially-symmetric disc problem, spreading occurs if at any time the disc radius exceeds the first zero of the zeroth-order Bessel function of the first-kind,  $J_0(x)$  [1]. Extinction occurs if the disc radius never exceeds this critical length,  $L_c \approx 2.405$ . To investigate whether our level-set method is accurate for solutions with extinction, we computed a solution with  $\beta = 1.5$ . As Figure 2.1 shows, the two-dimensional level-set solution correctly predicts population extinction. In the extinction solution, we obtained best results by solving the velocity extension and reinitialisation PDEs until at least  $\tau = 20\Delta\tau$ .

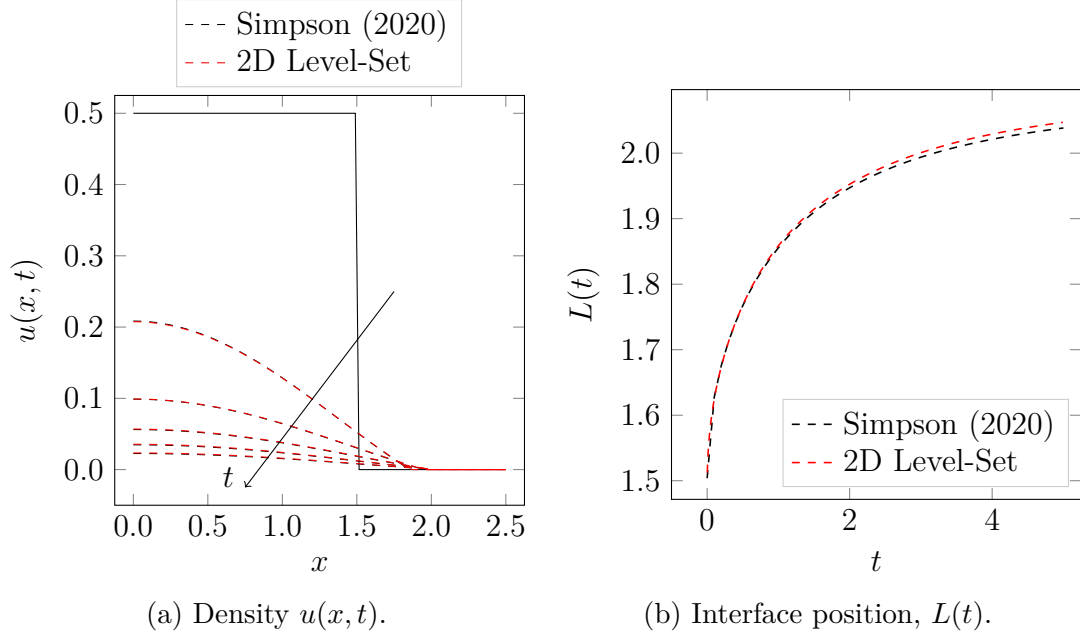


Figure 2.1: Numerical test of the level-set method with  $\Delta t = 0.002$ ,  $\Delta x = \Delta y = 0.025$ , on a square grid of  $L_x = L_y = 5$ , *i.e.* a  $201 \times 201$  spatial grid. Solution computed with  $\kappa = 1$ ,  $D = 1$ , and  $\lambda = 1$ . (a) Density  $u(x, t)$  plotted at  $t \in \{0, 1, 2, 3, 4, 5\}$ , where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing  $t$ . For the 2D level-set solution, we plot the  $y = 2.5$  slice,  $u(x, 2.5, t)$ . (b) Interface position,  $L(t)$ . For the 2D level-set solution, the position  $L(t)$  is the zero level-set position, that is  $x$  such that  $\phi(x, 2.5, t) = 0$ , which we compute by linear interpolation.

## References

- [1] M. J. Simpson, “Critical length for the spreading–vanishing dichotomy in higher dimensions”, *ANZIAM Journal* 62 (2020), pp. 3–17, DOI: [10.1017/S1446181120000103](https://doi.org/10.1017/S1446181120000103).
- [2] M. El-Hachem, S. W. McCue, J. Wang, Y. Du, and M. J. Simpson, “Revisiting the Fisher–Kolmogorov–Petrovsky–Piskunov equation to interpret the spreading–extinction dichotomy”, *Proceedings of the Royal Society of London Series A* 475, 20190378 (2019), DOI: [10.1098/rspa.2019.0378](https://doi.org/10.1098/rspa.2019.0378).