Level-Set Method to Solve the 2D Fisher–Stefan Model: Numerical Test Cases

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In this document, we provide numerical test cases for our level-set method used to solve the 2D Fisher–Stefan model. We compute solutions on circular domains $\Omega(t)$, and compare our results with Simpson [1], who solved the one-dimensional, radially-symmetric Fisher–Stefan problem on a disc. The numerical methods used are outlined in Simpson [1], and involve a change of variables to map the expanding or shrinking domain to the unit interval. The solutions of Simpson [1] were performed with 1001 grid points, yielding a grid spacing of $\Delta\xi = 0.001$, and a time step size of $\Delta t = 0.0001$. MATLAB code of this implementation is available in a public Github repository. To validate the level-set method, we consider both survival and extinction solutions. In each, we consider the initial condition

$$u\left(x,y,0\right) = \begin{cases} \alpha & \text{if} \quad \sqrt{(x-L_x/2)^2+(y-L_y/2)^2} \leq \beta \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where L_x and L_y are the computational domain widths in the x and y-directions respectively, β is the radius of the circular region $\Omega(0)$, and α is the initial density inside this region.

1 Survival Solutions

We first consider a numerical solution with $\alpha = 0.5$, $\beta = 10$, and $\kappa = 1$, where κ is the Stefan parameter in the Fisher–Stefan model. We compute a numerical solution using our level-set method, with $N_x = N_y = 200$ grid intervals. The solution is computed until $t = 30$, with $L_x = L_y = 50$. To compare, we plot the one-dimensional solution obtained with the method of Simpson [1], with a one-dimensional slice at $y = 25$ of the two-dimensional level-set solution. This is illustrated in Figure 1.1. We obtain good agreement between the two-dimensional level-set solution and the one-dimensional solution using the method of Simpson [1]. For these parameters and initial conditions, the density $u(x,t)$ evolves to a travelling-wave like profile, and spreads at a roughly-constant speed. Furthermore, the level-set method preserves the circular geometry as the solution progresses, as Figure 1.2 shows.

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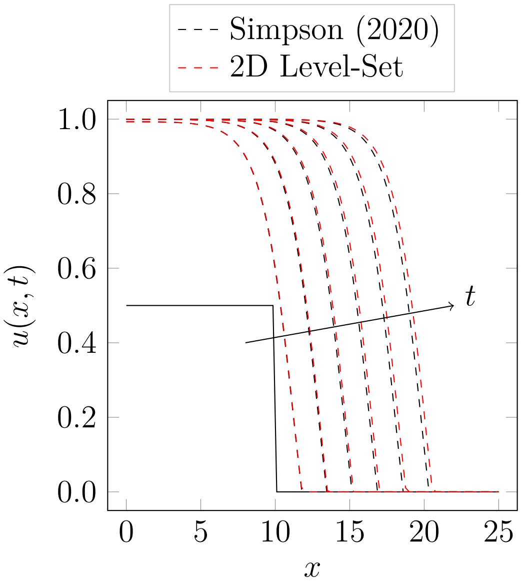
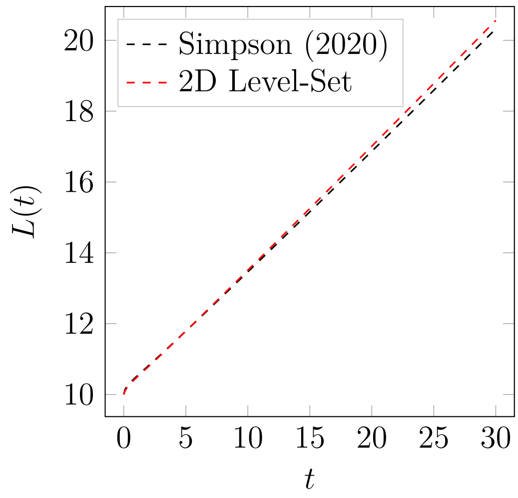


(a) Density $u(x,t)$.

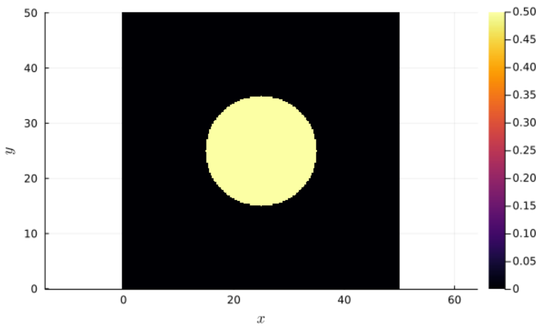


(b) Interface position, $L(t)$.

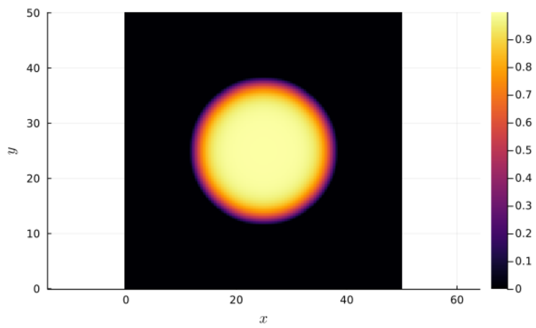
Figure 1.1: Numerical test of the level-set method with $\Delta t = 0.002$, $\Delta x = \Delta y = 0.25$, on a square grid of $L_x = L_y = 50$, *i.e.* a 201×201 spatial grid. Solution computed with $\kappa = 1$, $D = 1$, and $\lambda = 1$. (a) Density $u(x,t)$ plotted at $t \in \{0, 5, 10, 15, 20, 25, 30\}$, where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing t. For the 2D level-set solution, we plot the $y = 25$ slice, $u(x, 25, t)$. (b) Interface position, $L(t)$. For the 2D level-set solution, the position $L(t)$ is the zero level-set position, that is x such that $\phi(x, 25, t) = 0$, which we compute by linear interpolation.

To ensure our results remain valid for different parameters, we computed a solution with the same initial condition as Figure 1.1, but with $\beta = 5$ and $\kappa = 20$. Again, we find good agreement between the radially-symmetric solution using the method of Simpson [1], and the two-dimensional level-set solution. These results are presented in Figure 1.3, and we again find good agreement between the two-dimensional level-set method and the numerical solutions of Simpson [1].

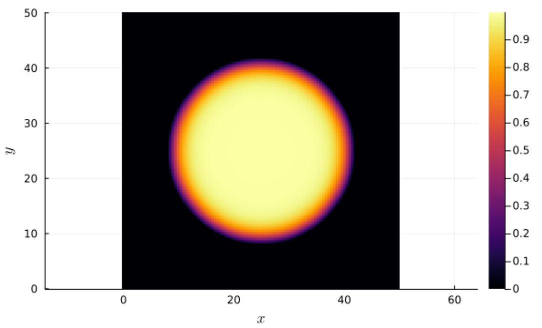
Finally, we consider a solution with $L_x = L_y = 20$, $D = \lambda = 1$, and the smaller value $\kappa = 0.1$. In this solution, we choose $\alpha = 0.5$ and $\beta = 2.4$, such that the initial disc radius is close to the critical radius $R_c \approx 2.4048$ [1]. We then compute a solution to $t = 100$, and present results in Figure 1.4. Like the previous two examples, we obtain good agreement between our level-set solutions and a solution obtained using the method of Simpson [1]. This confirms that our numerical method remains suitable for survival solutions with κ of varying size ($\kappa \in \{0.1, 1, 20\}$).



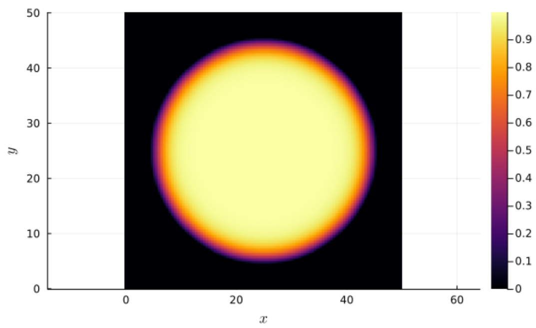
(a) $t = 0$.



(b) $t = 10$.

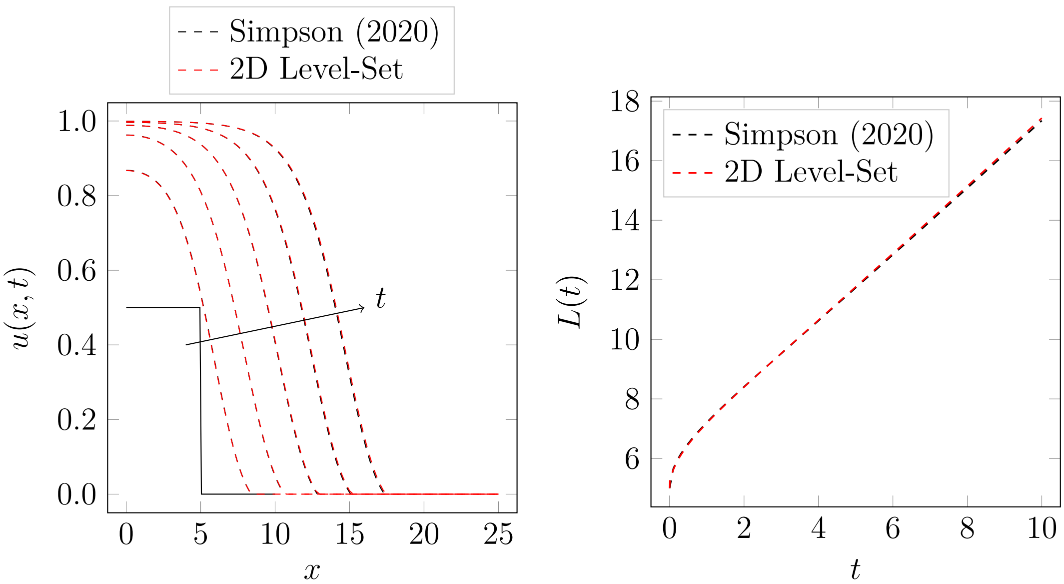


(c) $t = 20$.



(d) $t = 30$.

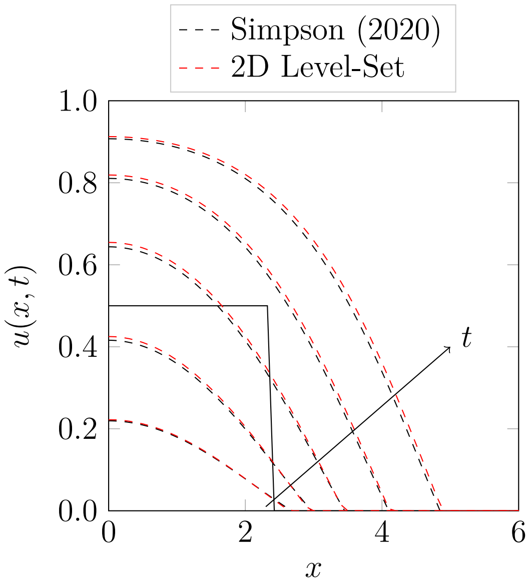
Figure 1.2: Two-dimensional evolution of the density $u(x, y, t)$ in the survival solution of Figure 1.1.



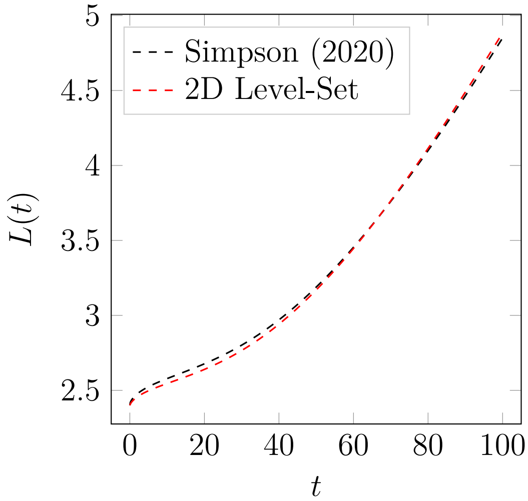
(a) Density $u(x, t)$.

(b) Interface position, $L(t)$.

Figure 1.3: Numerical test of the level-set method with $\Delta t = 0.002$, $\Delta x = \Delta y = 0.25$, on a square grid of $L_x = L_y = 50$, *i.e.* a 201×201 spatial grid. Solution computed with $\kappa = 20$, $D = 1$, and $\lambda = 1$. (a) Density $u(x, t)$ plotted at $t \in \{0, 2, 4, 6, 8, 10\}$, where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing t. For the 2D level-set solution, we plot the $y = 25$ slice, $u(x, 25, t)$. (b) Interface position, $L(t)$. For the 2D level-set solution, the position $L(t)$ is the zero level-set position, that is x such that $\phi(x, 25, t) = 0$, which we compute by linear interpolation.



(a) Density $u(x,t)$.

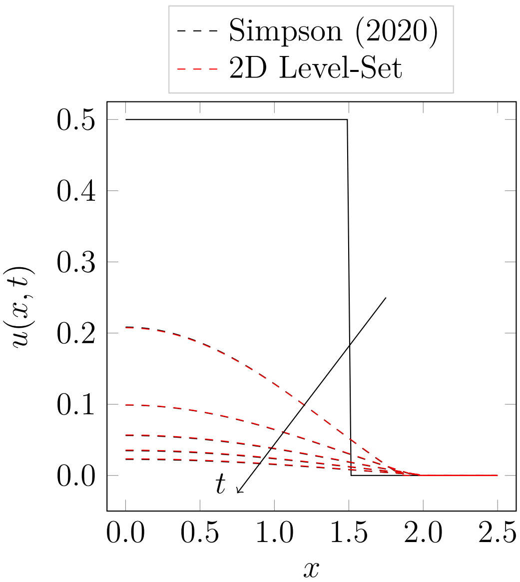


(b) Interface position, $L(t)$.

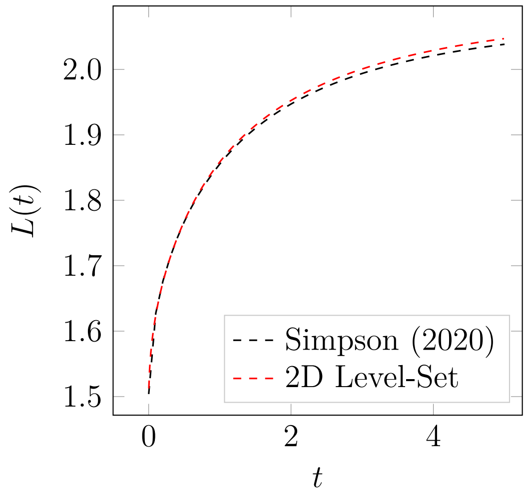
Figure 1.4: Numerical test of the level-set method with $\Delta t = 0.002$, $\Delta x = \Delta y = 0.1$, on a square grid of $L_x = L_y = 20$, *i.e.* a 201×201 spatial grid. Solution computed with $\kappa = 0.1$, $D = 1$, and $\lambda = 1$. (a) Density $u(x,t)$ plotted at $t \in \{0, 20, 40, 60, 80, 100\}$, where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing t. For the 2D level-set solution, we plot the $y = 10$ slice, $u(x, 10, t)$. (b) Interface position, $L(t)$. For the 2D level-set solution, the position $L(t)$ is the zero level-set position, that is x such that $\phi(x, 10, t) = 0$, which we compute by linear interpolation.

2 Extinction Solution

In addition to the survival solutions in §1, the Fisher–Stefan model admits solutions where density decreases with time [1, 2]. This arises from a competition between growth mediated by the source term, and mass loss at the interface due to the Stefan condition. In the radially-symmetric disc problem, survival occurs if at any time the disc radius exceeds the first zero of the zeroth-order Bessel function of the first-kind, $J_0(x)$ [1]. Extinction occurs if the disc radius never exceeds this critical length, $L_c \approx 2.405$. To investigate whether our level-set method is accurate for solutions with extinction, we computed a solution with $\beta = 1.5$. As Figure 2.1 shows, the two-dimensional level-set solution correctly predicts population extinction. In the extinction solution, we obtained best results by solving the velocity extension and reinitialisation PDEs until at least $\tau = 20\Delta\tau$.



(a) Density $u(x,t)$.



(b) Interface position, $L(t)$.

Figure 2.1: Numerical test of the level-set method with $\Delta t = 0.002$, $\Delta x = \Delta y = 0.025$, on a square grid of $L_x = L_y = 5$, *i.e.* a 201×201 spatial grid. Solution computed with $\kappa = 1$, $D = 1$, and $\lambda = 1$. (a) Density $u(x,t)$ plotted at $t \in \{0, 1, 2, 3, 4, 5\}$, where the solid curve denotes the initial condition, and the arrow indicates the direction of increasing t. For the 2D level-set solution, we plot the $y = 2.5$ slice, $u(x, 2.5, t)$. (b) Interface position, $L(t)$. For the 2D level-set solution, the position $L(t)$ is the zero level-set position, that is x such that $\phi(x, 2.5, t) = 0$, which we compute by linear interpolation.

References

[1] M. J. Simpson, "Critical Length for the Spreading–Vanishing Dichotomy in Higher Dimensions", *The ANZIAM Journal* 62 (2020), pp. 3–17, ISSN: 1446-1811, 1446-8735, DOI: 10.1017/S1446181120000103.

[2] M. El-Hachem, S. W. McCue, J. Wang, Y. Du, and M. J. Simpson, "Revisiting the Fisher–Kolmogorov–Petrovsky–Piskunov Equation to Interpret the Spreading–Extinction Dichotomy", *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 475, 20190378 (2019), DOI: 10.1098/rspa.2019.0378.