Leibniz-type rules and applications to scattering properties of PDEs II

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Kato–Ponce inequalities

For $s \geq 0$ set $J^s := (1-\Delta)^{s/2}$ and $D^s := (-\Delta)^{s/2}$, that is,

$$\widehat{J^s(f)}(\xi) = (1+|\xi|^2)^{s/2}\,\hat{f}(\xi), \quad \widehat{D^s(f)}(\xi) = |\xi|^s\,\hat{f}(\xi).$$

Kato–Ponce inequalities

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For $1 < p < \infty$ and $s \geq 0$,

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|J^s g\|_{L^p},$$

$$\|J^s(fg) - fJ^s(g)\|_{L^p} \lesssim \|J^s f\|_{L^p}\|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\left\|J^{s-1}g\right\|_{L^p}.$$

Such estimates played an important role in the treatment by Kato and Ponce (1988) of the Cauchy problem for the Euler and Navier–Stokes equations in the setting of Sobolev spaces.

More generally, it holds that

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}},$$

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}},$$

where

$$1 < p_1, p_2 \leq \infty, \tfrac{1}{2} < p \leq \infty, 1/p = 1/p_1 + 1/p_2,$$
$$s \in 2\mathbb{N}_0 \text{ or } s > \tau_p, \tau_p := n(1/\min(p,1) - 1).$$

(Different choices of p_1 and p_2 can be used on the right-hand sides of the inequalities above.)

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \leq \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

Theorem 1 (N.–Thomson, 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \leq \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|g\|_{\dot{F}^s_{p_2,q}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand sides of the inequality above; moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p,q}(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{\dot{F}^s_{p,q}(w)}.$$

Previous Results

For $s \geq 0$ set $D^s := (-\Delta)^{s/2}$, that is,

$$\widehat{D^s(f)}(\xi) = |\xi|^s \, \hat{f}(\xi).$$

Let $1 < p_1, p_2 < \infty$, $\frac{1}{2} < p < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $s \in 2\mathbb{N}_0$ or $s > n(\frac{1}{min(p,1)} - 1)$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, $w = w_1^{p/p_1} w_2^{p/p_2}$, and σ a Coifman-Meyer multiplier.

Theorem 2 (Cruz-Uribe-Naibo, 2016)

$$\|D^s(fg)\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)}$$

Theorem 3 (Brummer-Naibo, 2017)

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)}$$

Definitions: Muckenhoupt Weights

- For $0 < p < \infty$ and a weight w on $\mathbb{R}^n$, we say that $f \in L^p(w)$ if

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{\frac{1}{p}} < \infty.$$

- For $1 < p < \infty$, a weight w on $\mathbb{R}^n$ is an A_p weight if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty.$$

- $A_\infty = \bigcup_{p>1} A_p$

Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying the conditions

- $\mathrm{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ and some $c > 0$
- $\widehat{\Delta_j^{\psi} f}(\xi) := \widehat{\psi}(2^{-j}\xi)\widehat{f}(\xi)$

For $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $w \in A_\infty$, the homogeneous weighted Triebel-Lizorkin space $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j^{\psi} f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

Definitions: Weighted Hardy spaces

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi(x)\, dx \neq 0$. Given $0 < p < \infty$, the Hardy space $H^p(w)$ is defined as the class of tempered distributions such that

$$\|f\|_{H^p(w)} := \left\| \sup_{0<t<\infty} |t^{-n}\varphi(t^{-1}\cdot) * f| \right\|_{L^p(w)} < \infty.$$

Remark:

- $H^p(w) \simeq \dot{F}^0_{p,2}(w)$ for $0 < p < \infty$ and $w \in A_\infty$.
- $\dot{F}^0_{p,2}(w) \simeq L^p(w) \simeq H^p(w)$ and $\dot{F}^s_{p,2}(w) \simeq \dot{W}^{s,p}(w)$ for $1 < p < \infty$, $s \in \mathbb{R}$ and $w \in A_p$.

New Result

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \le \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

Theorem 4 (Naibo–T., 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \le \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^s_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot F^s_{p_2,q}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand sides of the inequality above; moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^s_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot F^s_{p,q}(w)}.$$

Lifting property for weighted homogeneous Triebel-Lizorkin spaces: If $0 < p, q < \infty$ and $w \in A_\infty$, it holds that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)}.$$

Corollary 5 (N.–Thomson, 2018)

Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,2}(w)$, it holds that

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)}. \tag{1}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand side of (1); moreover, if $w \in A_\infty$, then

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)}. \tag{2}$$

Other settings for Theorem 1 and Corollary 5

- Coifman–Meyer multipliers of order m :

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi|+|\eta|)^{m-|\alpha+\beta|} \qquad \forall(\xi,\eta)\neq(0,0).$$

  The corresponding multiplier operators satisfy

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|g\|_{\dot F^{s+m}_{p_2,q}(w_2)},$$

  as well as versions of the other estimates in Theorem 1 and Corollary 5.

- Theorem 1 and Corollary 5 hold in other function space settings: weighted homogeneous Besov spaces and weighted inhomogeneous Triebel–Lizorkin and Besov spaces; the latter contexts involve the operator J^s.