# Bilinear multiplier operators Leibniz-type rules

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# Leibniz rule

$$\partial_x^\alpha (fg)(x) = \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2} \partial_x^{\alpha_1} f(x) \partial_x^{\alpha_2} g(x)$$
$$= \partial_x^\alpha f(x) g(x) + f(x) \partial_x^\alpha g(x) + \ldots$$

- For $s \geq 0$ set $J^s := (1-\Delta)^{s/2}$ and $D^s := (-\Delta)^{s/2}$ that is,

$$\widehat{J^s(f)}(\xi) = (1+|\xi|^2)^{s/2}\, \hat{f}(\xi), \quad \widehat{D^s(f)}(\xi) = |\xi|^s\, \hat{f}(\xi).$$

- For $1 < p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $s \in 2\mathbb{N}_0$ or $s > n(1/\min(p,1) - 1)$ it holds that

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}},$$

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}}.$$

# Fractional Leibniz rules

For $s \geq 0$ set $J^s := (1-\Delta)^{s/2}$ and $D^s := (-\Delta)^{s/2}$, that is,

$$\widehat{J^s(f)}(\xi) = (1+|\xi|^2)^{s/2}\,\hat{f}(\xi), \quad \widehat{D^s(f)}(\xi) = |\xi|^s\,\hat{f}(\xi).$$

For $1 < p_1, p_2 \leq \infty$, $\frac{1}{2} < p \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $s \in 2\mathbb{N}_0$ or $s > n(1/\min(p,1) - 1)$ it holds that

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}},$$

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}}.$$

# Fractional Leibniz rules

- Case $1 < p < \infty$ :
  - Kato–Ponce, 1988 (for Euler and Navier–Stokes).
  - Christ–Weinstein, 1991 (for KdV).
  - Kenig–Ponce–Vega, 1993 (mixed-norm Lebesgue spaces, for KdV).
  - Gulisashvili–Kon, 1996 (for Schrödinger semigroups).
- Case $\frac{1}{2} < p < \infty$ :
  - Muscalu–Schlag, 2013: homogeneous version.
  - Grafakos–Oh, 2014: homogeneous and inhomogeneous versions.
  - Bernicot–Maldonado–Moen–Naibo, 2014: $s > n$, related to inh. version.
- Case $p = \infty$ :
  - Bourgain–Li, 2014. (Related work by Grafakos–Maldonado–Naibo, 2014.)

# Generalized Leibniz rules

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# Coifman-Meyer multipliers

- $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is a Coifman-Meyer multiplier if for all $\alpha, \beta \in \mathbb{N}_0^n$ if

$$\partial_x^\alpha \partial_y^\beta \sigma(x,y) \lesssim (|x| + |y|)^{-|\alpha|-|\beta|} \text{ for all } (x,y) \neq (0,0).$$

- $T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta$

- If $\sigma$ is a Coifman-Meyer multiplier, $1/p = 1/p_1 + 1/p_2$, and $1 < p_1, p_2 < \infty$

$$||T_\sigma(f,g)||_{L^p(w)} \lesssim ||f||_{L^{p_1}(w)}||g||_{L^{p_2}(w)}.$$

Let $\psi$ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying the conditions

- $\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$
- $|\widehat{\psi}(\xi)| > c$ for all $\xi$ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ and some $c > 0$
- $\widehat{\Delta_j^{\psi} f}(\xi) := \widehat{\psi}(2^{-j}\xi)\widehat{f}(\xi)$

The space $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left( \sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j^{\psi} f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

**Remark:**

- $H^p(w) \simeq \dot{F}^0_{p,2}(w)$ for $0 < p < \infty$ and $w \in A_\infty$.
- $\dot{F}^0_{p,2}(w) \simeq L^p(w) \simeq H^p(w)$ and $\dot{F}^s_{p,2}(w) \simeq \dot{W}^{s,p}(w)$ for $1 < p < \infty$, $s \in \mathbb{R}$ and $w \in A_p$.

# Weighted Leibniz-type rules for C–M multiplier operators

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \le \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

**Theorem 1 (Naibo–T., 2018)**

*Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \le \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.*

*If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|D^s T_\sigma(f,g)\|_{\dot F^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot F^0_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot F^0_{p_2,q}(w_2)}.$$

*If $w_1 = w_2$ then different pairs of $p_1, p_2$ can be used on the right hand sides of the inequality above; moreover, if $w \in A_\infty$, then*

$$\|D^s T_\sigma(f,g)\|_{\dot F^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot F^0_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{\dot F^0_{p,q}(w)}.$$

Lifting property for weighted homogeneous Triebel-Lizorkin spaces: If $0 < p, q < \infty$ and $w \in A_\infty$, it holds that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)}.$$

- Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$. If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,2}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|D^s g\|_{H^{p_2}(w_2)}.$$

- When $w \equiv 1$ estimate 8 extends and improves the Leibniz rule in Lebesgue spaces by allowing $p, \frac{1}{2}$ and a larger quantity on the left-hand side.

# Other settings for Theorem 1 and Corollary ??

- Coifman–Meyer multipliers of order $m$ :
$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi|+|\eta|)^{m-|\alpha+\beta|} \qquad \forall(\xi,\eta)\neq(0,0).$$
The corresponding multiplier operators satisfy
$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|g\|_{\dot F^{s+m}_{p_2,q}(w_2)},$$
as well as versions of the other estimates in Theorem 1 and Corollary **??**.

- Theorem 1 and Corollary **??** hold in other function space settings: weighted homogeneous Besov spaces and weighted inhomogeneous Triebel–Lizorkin and Besov spaces; the latter contexts involve the operator $J^s$.

- Theorem 1 and Corollary **??** hold in homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces based in other function spaces such as variable Lebesgue, weighted Lorrentz, and weighted Morrey spaces.

- These results have applications to scattering properties of certain

Thank you.