Leibniz-type rules and applications to scattering properties of PDEs

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Fractional Leibniz rules

$$\partial^\alpha(fg)(x) = \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2}\partial^{\alpha_1}f(x)\partial^{\alpha_2}g(x)$$
$$= \partial^\alpha f(x)g(x) + f(x)\partial^\alpha g(x) + \ldots$$

- For $s \geq 0$ set $D^s := (-\Delta)^{s/2}$. For $s \in 2\mathbb{N}_0$ or $s > n(1/\min(p,1) - 1)$, $1 < p_1, p_2 \leq \infty$, and $1/p = 1/p_1 + 1/p_2$ it holds that

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|D^s g\|_{L^{p_2}}.$$

- Such estimates have applications to PDEs such as Navier-Stokes equations and Korteweg-de Vreis equations (Kato-Ponce '88, Christ-Weinstein '91, Kenig-Ponce-Vega '93).

Fractional Leibniz rules

- Case $1 < p < \infty$:
  - Kato–Ponce, 1988 (for Euler and Navier–Stokes).
  - Christ–Weinstein, 1991 (for KdV).
  - Kenig–Ponce–Vega, 1993 (mixed-norm Lebesgue spaces, for KdV).
  - Gulisashvili–Kon, 1996 (for Schrödinger semigroups).
- Case $\frac{1}{2} < p < \infty$:
  - Muscalu–Schlag, 2013
  - Grafakos–Oh, 2014
  - Bernicot–Maldonado–Moen–Naibo, 2014: $s > n$, for related estimates.
- Case $p = \infty$:
  - Bourgain–Li, 2014. (Related work by Grafakos–Maldonado–Naibo, 2014.)

We will look at estimates of the form

$$\|D^s T_\sigma(f,g)\|_X \lesssim \|D^s f\|_{Y_1}\|g\|_{Z_1} + \|f\|_{Y_2}\|D^s g\|_{Z_2}$$

where X, Y_1, Y_2, Z_1, and Z_2 are weighted function spaces.

The bilinear multiplier operator T_σ is defined as

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

Remark: If $\sigma \equiv 1$ then $T_\sigma(f,g) = fg$.

Coifman-Meyer multipliers

- $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is a Coifman-Meyer multiplier if for all $\alpha, \beta \in \mathbb{N}_0^n$

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|-|\beta|} \text{ for all } (\xi,\eta) \neq (0,0).$$

- Operators associated to Coifman-Meyer multipliers have been extensively studied
  - Coifman-Meyer '78
  - Kenig-Stein '99
  - Grafakos-Torres '02
  - Grafakos-Martell '04
  - Lerner et al. '09

- In particular operators associated to Coifman-Meyer multipliers are bounded in Lebesgue spaces and weighted Lebesgue spaces.

Given a weight w the space $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |\Delta_j f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

where $\{\Delta_j\}$ is a family of Littlewood-Paley operators.

Remark:

- $H^p(w) \simeq \dot{F}^0_{p,2}(w)$ for $0 < p < \infty$ and $w \in A_\infty$.
- $\dot{F}^0_{p,2}(w) \simeq L^p(w) \simeq H^p(w)$ and $\dot{F}^s_{p,2}(w) \simeq \dot{W}^{s,p}(w)$ for $1 < p < \infty$, $s \in \mathbb{R}$ and $w \in A_p$.

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \leq \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

Theorem 1 (Naibo–T., 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \leq \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{\dot{F}^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot{F}^0_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|D^s g\|_{\dot{F}^0_{p_2,q}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand side of the inequality above.

Weighted Leibniz-type rules for C–M multiplier operators

Corollary 2 (Naibo–T., 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta\in\mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0<p,p_1,p_2<\infty$ such that $1/p=1/p_1+1/p_2$. If $w_1,w_2\in A_\infty$, $w=w_1^{p/p_1}w_2^{p/p_2}$ and $s>\tau_{p,2}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{H^p(w)}\lesssim\|D^s f\|_{H^{p_1}(w_1)}\|g\|_{H^{p_2}(w_2)}+\|f\|_{H^{p_1}(w_1)}\|D^s g\|_{H^{p_2}(w_2)}.$$

Remark: When $w=w_1=w_2=1$ and $\sigma\equiv 1$ it holds that

$$\|D^s(fg)\|_{H^p}\lesssim\|D^s f\|_{H^{p_1}}\|g\|_{H^{p_2}}+\|f\|_{H^{p_1}}\|D^s g\|_{H^{p_2}}.$$

This extends and improves the Leibniz rule in the introduction by allowing $0<p,p_1,p_2<\infty$ and a larger norm on the left-hand side.

- Coifman–Meyer multipliers of order m :

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi|+|\eta|)^{m-|\alpha+\beta|} \qquad \forall (\xi,\eta) \neq (0,0).$$

The corresponding multiplier operators satisfy

$$\|D^s T_\sigma(f,g)\|_{\dot{F}^0_{p,q}(w)} \lesssim \left\|D^{s+m} f\right\|_{\dot{F}^0_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \left\|D^{s+m} g\right\|_{\dot{F}^0_{p_2,q}(w_2)}.$$

as well as versions of the other estimates in Theorem 1 and Corollary 2.

- Theorem 1 and Corollary 2 hold in other function space settings:
  - weighted homogeneous Besov spaces and weighted inhomogeneous Triebel–Lizorkin and Besov spaces,
  - homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces based in other function spaces such as variable Lebesgue, weighted Lorentz, and weighted Morrey spaces.

Scattering properties of PDEs

As an application of Theorem 1 we obtain scattering properties of systems of PDEs of the form

$$\begin{cases} \partial_t u = vw, & \partial_t v + D^\gamma v = 0, \quad \partial_t w + D^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases}$$

We say that $u(t,x)$ scatters to u_∞ if

$$\lim_{t\to\infty} u(t,\cdot) = u_\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Then using Theorem 1 we obtain

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|g\|_{\dot{F}^{s-\gamma}_{p_2,q}(w_2)}.$$

Thank you.