Bilinear multiplier operators Leibniz-type rules

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\begin{aligned}
\partial_x^\alpha(fg)(x) &= \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2}\partial_x^{\alpha_1} f(x)\partial_x^{\alpha_2} g(x) \\
&= \partial_x^\alpha f(x)g(x) + f(x)\partial_x^\alpha g(x) + \ldots
\end{aligned}
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Fractional Leibniz rules

For $s \geq 0$ set $J^s := (1-\Delta)^{s/2}$ and $D^s := (-\Delta)^{s/2}$, that is,

$$\widehat{J^s(f)}(\xi) = (1+|\xi|^2)^{s/2}\,\hat{f}(\xi), \quad \widehat{D^s(f)}(\xi) = |\xi|^s\,\hat{f}(\xi).$$

For $1 < p_1, p_2 \leq \infty$, $\frac{1}{2} < p \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $s \in 2\mathbb{N}_0$ or $s > n(1/\min(p,1) - 1)$ it holds that

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}},$$

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}}.$$

Fractional Leibniz rules

- Case $1 < p < \infty$:
  - Kato–Ponce, 1988 (for Euler and Navier–Stokes).
  - Christ–Weinstein, 1991 (for KdV).
  - Kenig–Ponce–Vega, 1993 (mixed-norm Lebesgue spaces, for KdV).
  - Gulisashvili–Kon, 1996 (for Schrödinger semigroups).
- Case $\frac{1}{2} < p < \infty$:
  - Muscalu–Schlag, 2013: homogeneous version.
  - Grafakos–Oh, 2014: homogeneous and inhomogeneous versions.
  - Bernicot–Maldonado–Moen–Naibo, 2014: $s > n$, related to inh. version.
- Case $p = \infty$:
  - Bourgain–Li, 2014. (Related work by Grafakos–Maldonado–Naibo, 2014.)

Coifman-Meyer multipliers

- $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is a Coifman-Meyer multiplier if for all $\alpha, \beta \in \mathbb{N}_0^n$ if

$$\partial_x^\alpha \partial_y^\beta \sigma(x, y) \lesssim (|x| + |y|)^{-|\alpha|-|\beta|} \text{ for all } (x, y) \neq (0, 0).$$

- $T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta$
- Boundedness properties of operators associated to Coifman-Meyer multipliers are well known. In the setting of weighted Lebesgue spaces

$$||T_\sigma(f, g)||_{L^p(w)} \lesssim ||f||_{L^{p_1}(w)} ||g||_{L^{p_2}(w)}$$

where σ is a Coifman-Meyer multiplier, $1/p = 1/p_1 + 1/p_2$, and $1 < p_1, p_2 < \infty$.

Weighted Triebel-Lizorkin Spaces

Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying the conditions

- $\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ and some $c > 0$
- $\widehat{\Delta_j^{\psi} f}(\xi) := \widehat{\psi}(2^{-j}\xi)\widehat{f}(\xi)$

For $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $w \in A_\infty$, the space $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j^{\psi} f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

Remark:

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \leq \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

Theorem 1 (Naibo–T., 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \leq \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{\dot{F}^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot{F}^0_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|D^s g\|_{\dot{F}^0_{p_2,q}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand sides of the inequality above; moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p,q}(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{\dot{F}^s_{p,q}(w)}.$$

Lifting property for weighted homogeneous Triebel-Lizorkin spaces: If $0 < p, q < \infty$ and $w \in A_\infty$, it holds that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)}.$$

Corollary 2 (Naibo–T., 2018)

Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,2}(w)$, it holds that

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|D^s g\|_{H^{p_2}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand side of (2); moreover, if $w \in A_\infty$, then

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|D^s g\|_{H^p(w)}.$$

Other settings for Theorem 1 and Corollary 2

- Coifman–Meyer multipliers of order m :

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi|+|\eta|)^{m-|\alpha+\beta|} \qquad \forall (\xi,\eta) \neq (0,0).$$

  The corresponding multiplier operators satisfy

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)},$$

  as well as versions of the other estimates in Theorem 1 and Corollary 2.

- Theorem 1 and Corollary 2 hold in other function space settings: weighted homogeneous Besov spaces and weighted inhomogeneous Triebel–Lizorkin and Besov spaces; the latter contexts involve the operator J^s.