Leibniz-type rules and applications to scattering properties of PDEs

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Fractional Leibniz rules

$$\begin{aligned}\partial^\alpha(fg)(x) &= \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2}\partial^{\alpha_1} f(x)\partial^{\alpha_2} g(x)\\ &= \partial_x^\alpha f(x)g(x) + f(x)\partial_x^\alpha g(x) + \ldots\end{aligned}$$

- For $s \geq 0$ set $D^s := (-\Delta)^{s/2}$. For $1 < p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$, and $s \in 2\mathbb{N}_0$ or $s > n(1/\min(p,1) - 1)$ it holds that
$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|D^s g\|_{L^{p_2}}.$$

- Such estimates have applications to PDEs such as Navier-Stokes equations and Korteweg-de Vreis equations(Kato-Ponce '88, Christ-Weinstein '91, Kenig-Ponce-Vega '93).

Fractional Leibniz rules

- Case $1 < p < \infty$:
  - Kato–Ponce, 1988 (for Euler and Navier–Stokes).
  - Christ–Weinstein, 1991 (for KdV).
  - Kenig–Ponce–Vega, 1993 (mixed-norm Lebesgue spaces, for KdV).
  - Gulisashvili–Kon, 1996 (for Schrödinger semigroups).
- Case $\frac{1}{2} < p < \infty$:
  - Muscalu–Schlag, 2013
  - Grafakos–Oh, 2014
  - Bernicot–Maldonado–Moen–Naibo, 2014: $s > n$, for related estimates.
- Case $p = \infty$:
  - Bourgain–Li, 2014. (Related work by Grafakos–Maldonado–Naibo, 2014.)

We will look at estimates of the form

$$\|D^s T_\sigma(f,g)\|_X \lesssim \|D^s f\|_{Y_1}\|g\|_{Z_1} + \|f\|_{Y_2}\|D^s g\|_{Z_2}$$

in weighted function spaces.

The bilinear multiplier operator T_σ is defined as

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

Remark: If $\sigma \equiv 1$ then $T_\sigma(f,g) = fg$.

Coifman-Meyer multipliers

- $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is a Coifman-Meyer multiplier if for all $\alpha, \beta \in \mathbb{N}_0^n$

$$\partial_x^\alpha \partial_y^\beta \sigma(\xi, \eta) \lesssim (|\xi| + |\eta|)^{-|\alpha|-|\beta|} \text{ for all } (\xi, \eta) \neq (0, 0).$$

- Operators associated to Coifman-Meyer multipliers have been extensively studied
  - Coifman-Meyer '78
  - Grafakos-Torres
  - Kenig-Stein
  - Grafakos-Martell
  - Lerner et al.
- If σ is a Coifman-Meyer multiplier, $1/p = 1/p_1 + 1/p_2$, and $1 < p_1, p_2 < \infty$

$$||T_\sigma(f,g)||_{L^p(w)} \lesssim ||f||_{L^{p_1}(w)} ||g||_{L^{p_2}(w)}.$$

Weighted Triebel-Lizorkin Spaces

Given a weight w the space $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

where Δ_j is a Littlewood-Paley operator.

Remark:

- $H^p(w) \simeq \dot{F}^0_{p,2}(w)$ for $0 < p < \infty$ and $w \in A_\infty$.
- $\dot{F}^0_{p,2}(w) \simeq L^p(w) \simeq H^p(w)$ and $\dot{F}^s_{p,2}(w) \simeq \dot{W}^{s,p}(w)$ for $1 < p < \infty$, $s \in \mathbb{R}$ and $w \in A_p$.

For $w \in A_\infty$, let $\tau_w = \inf\{\tau \in [1,\infty) : w \in A_\tau\}$; if $0 < p, q \le \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right).$$

Theorem 1 (Naibo–T., 2018)

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < q \le \infty$ and $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$.

If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,q}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{\dot F^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot F^0_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot F^0_{p_2,q}(w_2)}.$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right hand sides of the inequality above.

Weighted Leibniz-type rules for C–M multiplier operators

Corollary 2

Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$. If $w_1, w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,2}(w)$, it holds that

$$\|D^s T_\sigma(f,g)\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)}\|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)}\|D^s g\|_{H^{p_2}(w_2)}.$$

Case: When $w = w_1 = w_2 = 1$ and $\sigma \equiv 1$ it holds that

$$\|D^s fg\|_{H^p} \lesssim \|D^s f\|_{H^{p_1}}\|g\|_{H^{p_2}} + \|f\|_{H^{p_1}}\|D^s g\|_{H^{p_2}}.$$

This extends and improves the Leibniz rule in the introduction by allowing $0 < p, p_1, p_2 < \frac{1}{2}$ and a larger norm on the left-hand side.

Other settings for Leibniz-type rules

- Coifman–Meyer multipliers of order m :

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \lesssim (|\xi| + |\eta|)^{m-|\alpha+\beta|} \qquad \forall(\xi,\eta) \neq (0,0).$$

  The corresponding multiplier operators satisfy

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot F^{s+m}_{p_2,q}(w_2)},$$

  as well as versions of the other estimates in Theorem 1 and Corollary 2.

- Theorem 1 and Corollary 2 hold in other function space settings: weighted homogeneous Besov spaces and weighted inhomogeneous Triebel–Lizorkin and Besov spaces; the latter contexts involve the operator J^s.

- Theorem 1 and Corollary 2 hold in homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces based in other function spaces such as variable Lebesgue, weighted Lorrentz, and weighted Morrey spaces.

As an application of Theorem 1 we obtain scattering properties of systems of PDEs of the form

$$\begin{cases} \partial_t u = vw, & \partial_t v + D^\gamma v = 0, \quad \partial_t w + D^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases}$$

$$\lim_{t\to\infty} u(t,\cdot) = u_\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n);$$

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s-\gamma}_{p_2,q}(w_2)},$$

Thank you.