Leibniz-type rules in Triebel-Lizorkin and Besov spaces

by

Alexander Thomson

B.S., Missouri State University, 2013

M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

Abstract

Fractional Leibniz rules have been extensively studied due to their connections to partial differential equations that model many real-world situations such as shallow water waves and fluid flow. Broadly speaking, fractional Leibniz rules provide estimates of the size and smoothness of a product of functions in terms of the size and smoothness of the functions involved. In this dissertation, we obtain new Leibniz-type rules associated to bilinear pseudodifferential operators in a variety of function spaces that satisfy smoothness and size in appropriate ways. Bilinear pseudodifferential combine functions through a symbol and the Fourier transform. When the symbol is identically equal to one, such operators give the product of two functions, and therefore, fractional Leibniz rules are particular cases of the Leibniz-type rules discussed.

The main results of this dissertation concern Leibniz-type rules for operators associated to two classes of symbols: Coifman-Meyer multipliers and symbols in the Bilinear Hörmander classes. Leibniz-type rules for Coifman-Meyer multiplier operators are presented in the setting of Triebel-Lizorkin and Besov spaces based on various quasi-Banach spaces that include weighted Lebesgue, weighted Lorentz, weighted Morrey, and variable-exponent Lebesgue spaces. Such results extend and improve previously known fractional Leibniz rules. As applications, we obtain scattering properties of solutions to certain systems of partial differential equations involving fractional powers of the Laplacian. For operators with symbols in the bilinear Hörmander classes, we obtain Leibniz-type rules in the context of Besov and local Hardy spaces. The tools used in the proofs of the main results include Nikol'skiĭ representations of function spaces, pointwise inequalities for maximal functions, and appropriate spectral decompositions of the symbols of the operators.

Leibniz-type rules in Triebel-Lizorkin and Besov spaces.

by

Alexander Thomson

B.S., Missouri State University, 2013

M.S., Kansas State University, 2015

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

Approved by:

Major Professor
Virginia Naibo

Abstract

Fractional Leibniz rules have been extensively studied due to their connections to partial differential equations that model many real-world situations such as shallow water waves and fluid flow. Broadly speaking, fractional Leibniz rules provide estimates of the size and smoothness of a product of functions in terms of the size and smoothness of the functions involved. In this dissertation, we obtain new Leibniz-type rules associated to bilinear pseudodifferential operators in a variety of function spaces that satisfy smoothness and size in appropriate ways. Bilinear pseudodifferential combine functions through a symbol and the Fourier transform. When the symbol is identically equal to one, such operators give the product of two functions, and therefore, fractional Leibniz rules are particular cases of the Leibniz-type rules discussed.

The main results of this dissertation concern Leibniz-type rules for operators associated to two classes of symbols: Coifman-Meyer multipliers and symbols in the Bilinear Hörmander classes. Leibniz-type rules for Coifman-Meyer multiplier operators are presented in the setting of Triebel-Lizorkin and Besov spaces based on various quasi-Banach spaces that include weighted Lebesgue, weighted Lorentz, weighted Morrey, and variable-exponent Lebesgue spaces. Such results extend and improve previously known fractional Leibniz rules. As applications, we obtain scattering properties of solutions to certain systems of partial differential equations involving fractional powers of the Laplacian. For operators with symbols in the bilinear Hörmander classes, we obtain Leibniz-type rules in the context of Besov and local Hardy spaces. The tools used in the proofs of the main results include Nikol'skiĭ representations of function spaces, pointwise inequalities for maximal functions, and appropriate spectral decompositions of the symbols of the operators.

Table of Contents

1 Introduction to Leibniz-type rules 1

2 Weighted Leibniz-type rules and applications to scattering properties of PDEs . . 6

2.1 Preliminaries 7

2.1.1 Coifman-Meyer Multipliers 8

2.1.2 The scale of weighted Triebel-Lizorkin and Besov spaces 10

2.2 Leibniz-type rules in weighted Triebel-Lizorkin and Besov spaces 21

2.2.1 Homogeneous Leibniz-type rules 21

2.2.2 Proofs of the homogeneous Leibniz-type rules 26

2.2.3 Inhomogeneous Leibniz-type rules 30

2.3 Leibniz-type rules in other functions spaces 32

2.3.1 Leibniz-type rules in the settings of Lorentz-based Triebel–Lizorkin and Besov spaces. 33

2.3.2 Leibniz-type rules in the settings of Morrey-based Triebel–Lizorkin and Besov spaces. 36

2.3.3 Leibniz-type rules in the settings of variable-exponent Triebel–Lizorkin and Besov spaces. 38

2.3.4 Nikol'skiĭ representations of Triebel-Lizorkin and Besov spaces 42

2.4 Applications to scattering properties of PDEs 43

3 Bilinear Hörmander Classes and Leibniz-type rules 49

3.1 Introduction and main results 49

3.2 A maximal inequality for bilinear pseudodiffernetial operators 53

3.3 Decomposition of the operator T_σ and main estimates 55

3.3.1 Estimates for T_{σ^1} 58

3.3.2 Estimates for T_{σ^2} 64

3.3.3 Estimates for T_{σ^3} 71

3.4 Conclusion of the proofs of the main results 75

Bibliography 79

Chapter 1

Introduction to Leibniz-type rules

Fractional Leibniz rules have been extensively studied due to their connections to partial differential equations that model many real world situations such as shallow water waves and fluid flow. In this chapter, we introduce the subject of Leibniz-type rules and describe the main results to be discussed in Chapters 2 and 3 of this dissertation.

First consider the Leibniz rule taught in Calculus courses, which expresses the derivatives of a product of functions as a linear combination of derivatives of the functions involved; more specifically, for functions f and g sufficiently smooth, it holds that

$$\partial^\alpha (fg)(x) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f(x) \partial^\beta g(x) = \partial^\alpha f(x) g(x) + f(x) \partial^\alpha g(x) + \cdots ,$$

for $\alpha, \beta \in \mathbb{N}_0^n$. In an analogous way, fractional Leibniz rules give estimates of the smoothness and size of a product of functions in terms of the smoothness and size of the factors. For instance, for f and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p}_1}} \|D^s g\|_{L^{\tilde{p}_2}} , \tag{1.0.1}$$

where $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$, $1 \leqslant p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$, $1/2 \leqslant p \leqslant \infty$, $s > n(1/\min(p,1) - 1)$ or s is an even whole number, and L^r denotes a Lebesgue space for

$0 < r \leqslant \infty$. The homogeneous fractional differentiation operator of order s, D^s, is defined as

$$D^s f(x) = \int_{\mathbb{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f. For $s > 0$, the operator D^s is naturally understood as taking s derivatives of its argument. Indeed, in the case $s = 2$, $D^2 f = \frac{-1}{4\pi^2} \Delta f$, where $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the Laplacian operator. Furthermore, if s is a positive integer and $1 < p < \infty$,

$$\|D^s f\|_{L^p} \sim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^p},$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$.

Another version of (1.0.1) is obtained by using the inhomogeneous sth order fractional differentiation operator J^s:

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p}_1}} \|J^s g\|_{L^{\tilde{p}_2}}, \tag{1.0.2}$$

where $p_1, p_2, \tilde{p}_1, \tilde{p}_2$, and s satisfy the same conditions as for (1.0.1). Similarly to its homeogenous counterpart, the operator J^s is defined through the Fourier transform as

$$J^s f(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

and can be interpreted as taking derivatives up to order s of f when $s > 0$.

The estimates (1.0.1) and (1.0.2) are also known as Kato-Ponce inequalities due to the foundational work of Kato-Ponce [38], where the estimate (1.0.2) was proved in the case $1 < p = p_1 = \tilde{p}_2 < \infty$ and $p_2 = \tilde{p}_1 = \infty$, with applications to the Cauchy problem for Euler and Navier-Stokes equations. This result was extended by Gulisashvili-Kon [30], who showed (1.0.1) and (1.0.2) for the cases $s > 0$, $1 < p < \infty$, and $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$ in connection to smoothing properties of Schrödinger semigroups. Grafakos-Oh [27] and Muscalu-Schlag [52] established the cases for $1/2 < p \leqslant 1$; the case $p = \infty$ was completed in the work of Bourgain-Li [11] and Grafakos-Maldonado-Naibo [25]; finally, the case $p_1 = 1$, $1 \leqslant p_2 \leqslant \infty$ and $\frac{1}{2} \leqslant p \leqslant 1$ was established by Oh-Wu [58]. Applications of the estimates (1.0.1) and (1.0.2) to Korteweg-de Vries equations were studied by Christ-Weinstein [14] and Kenig-Ponce-Vega [39].

In the estimates (1.0.1) and (1.0.2), the two functions f and g are related through pointwise multiplication. In this dissertation, we will consider bilinear estimates in the spirit of (1.0.1) and (1.0.2) where the two functions are related through a pseudodifferential operator. Let $\sigma(x, \xi, \eta)$ be a smooth, complex-valued function defined for $x, \xi, \eta \in \mathbb{R}^n$. We define the *bilinear pseudodifferential operator* associated to σ, T_σ, by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta. \tag{1.0.3}$$

We call σ the *symbol* of the operator T_σ; when σ is independent of x, σ is also referred to as the *multiplier* of the *bilinear multiplier operator* T_σ. We note that $\sigma \equiv 1$ gives $T_\sigma(f, g) = fg$.

In Chapters 2 and 3 we will present new results on Leibniz-type rules associated to bilinear pseudodifferential operators that are of the form

$$\|D^s T_\sigma(f, g)\|_Z \lesssim \|D^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|D^s g\|_{Y_2}, \tag{1.0.4}$$

$$\|J^s T_\sigma(f, g)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.0.5}$$

for a variety of function spaces X_1, X_2, Y_1, Y_2, and Z. In the particular case that $\sigma \equiv 1$ and X_1, X_2, Y_1, Y_2, and Z are appropriate Lebesgue spaces (1.0.4) and (1.0.5) recover (1.0.1) and (1.0.2) respectively. The main results presented in Chapter 2 will appear in Naibo-Thomson [56], while those discussed in Chapter 3 were published in Naibo-Thomson [55].

In Chapter 2, we will discuss Leibniz-type rules (1.0.4) and (1.0.5) in the setting of Besov and Triebel-Lizorkin spaces based on certain quasi-Banach spaces. Such bilinear estimates will be proved for bilinear Coifman-Meyer multiplier operators. A particular case of the results in Chapter 2 is the following fractional Leibniz rule, and its inhomogeneous counterpart, in the context of Hardy spaces:

$$\|D^s(fg)\|_{H^p} \lesssim \|D^s f\|_{H^{p_1}} \|g\|_{H^{p_2}} + \|f\|_{H^{\tilde{p}_1}} \|D^s g\|_{H^{\tilde{p}_2}}, \tag{1.0.6}$$

where $0 < p, p_1, p_2, \tilde{p}_1, \tilde{p}_2 < \infty$ and $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$. Recalling that $H^p = L^p$ for $1 < p < \infty$, (1.0.6) extends and improves (1.0.1). Indeed, the inequality (1.0.6) extends the range of p, p_1, p_2, $\tilde{p}_1$, $\tilde{p}_2$ by allowing $0 < p, p_1, p_2, \tilde{p}_1, \tilde{p}_2 < \infty$, while (1.0.1) requires $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$. Additionally, (1.0.6) allows for the H^p norm on the left-hand side, which is larger than the L^p norm.

The techniques used in the proofs of the results in Chapter 2 are quite flexible and allow us to obtain (1.0.4) and (1.0.5) in Triebel-Lizorkin and Besov spaces based on weighted Lebesque spaces, weighted Lorrentz spaces, weighted Morrey spaces, and variable-exponent Lebesgue spaces. In particular, the proofs make use of Nikol'skiĭ representations of such function spaces. These representations have been used in unweighted settings in the work of Nikol'skiĭ [57], Meyer [46], Bourdad [10], Triebel [65], and Yamazaki [68].

As an application of the results in Chapter 2 we obtain scattering properties for solutions to certain systems of partial differential equations that involve fractional powers of the Laplacian. Solutions of these systems scatter to functions that can be realized in terms of a Coifman-Meyer multiplier operator acting on appropriate arguments. As a consequence, the main results of Chapter 2 can be applied and lead to estimates associated to the long term behavior of the solutions.

In Chapter 3, we present Leibniz-type rules in Besov and local Hardy spaces for bilinear pseudodifferential operators associated to symbols in bilinear Hörmander classes of critical order. For such symbols we prove bilinear estimates of the form

$$\|D^s T_\sigma(f,g)\|_{B^0_{p,q}} \lesssim \|D^s f\|_{B^0_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|D^s g\|_{B^0_{p_2,q}}, \tag{1.0.7}$$

where $B^0_{p,q}$ and h^p denote Besov and Hardy spaces respectively, $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ are such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$ and $s > n(\frac{1}{\min(p,1)} - 1)$.

The proofs of the results in Chapter 3 utilize appropriate spectral decompositions of the symbols, pointwise inequalities in terms of maximal functions, and Nikol'skiĭ representations for Besov spaces. The techniques used are inspired by bilinear techniques used in Naibo [53] and techniques for linear operators in Johnsen [37], Marschall [44], and Park [59].

We close this chapter by referencing serveral works in connection with the study of the bilinear estimates (1.0.4) and (1.0.5). In [12], Brummer-Naibo studied Leibniz-type rules for bilinear pseudodifferential operators with homogeneous symbols and in function spaces that admit a molecular decomposition and a φ-transform characterization in the sense of Frazier-Jawerth [22; 23]. In the context of Lebesgue spaces and mixed Lebesgue spaces, estimates of the type (1.0.4) were studied in Hart-Torres-Wu [32] for bilinear multiplier operators with minimal smoothness assumptions on the multipliers. Related mapping properties for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes were studied by Bényi-Torres [7] and Bényi-Nahmod-Torres [6] in the setting of Sobolev spaces, by Bényi [2] in the setting of Besov spaces, and by Naibo [53] and Koezuka-Tomita [41] in the context of Triebel-Lizorkin spaces. Additionally, versions of (1.0.1) and (1.0.2) in weighted Lebesgue spaces were proved in Cruz-Uribe-Naibo [18], while Brummer-Naibo [13] proved (1.0.4) and (1.0.5) in weighted Lebesgue spaces for Coifman-Meyer multiplier operators.

Chapter 2

Weighted Leibniz-type rules and applications to scattering properties of PDEs

In this chapter, we obtain new Leibniz-type rules of the type (1.0.4) and (1.0.5) for bilinear multiplier operators associated to Coifman-Meyer multipliers in the settings of Triebel-Lizorkin and Besov spaces based on quasi-Banach spaces. These results extend and improve the fractional Leibniz rules (1.0.1) and (1.0.2). Additionally, we apply these results to obtain scattering properties of solutions to systems of partial differential equations involving fractional powers of the Laplacian.

We start with some preliminaries in Section 2.1, where we discuss Coifman-Meyer multipliers and Besov and Triebel-Lizorkin spaces based on weighted Lebesgue spaces.

In Section 2.2, we state and prove two of the main results of this chapter on Leibniz-type rules associated to Coifman-Meyer multiplier operators in the setting of Besov and Triebel-Lizorkin spaces based on weighted Lebesgue spaces. These results are stated as Theorem 2.2.1 and Theorem 2.2.7. We also present several corollaries and connections with related results in the literature and estimates (1.0.1) and (1.0.2). The method of proof used for

Theorem 2.2.1 and Theorem 2.2.7 can be adapted to obtain (1.0.4) and (1.0.5) for Coifman-Meyer multiplier operators in the context of Triebel-Lizorkin and Besov spaces based on other quasi-Banach spaces such as weighted Lorentz, weighted Morrey, and variable-Lebesgue spaces. These results are discussed in Section 2.3.

Finally, in Section 2.4 we present applications of the results in this chapter to scattering properties of solutions to partial differential equations.

2.1 Preliminaries

In this section, we set some notation and present definitions and results about weights, the scales of weighted Triebel–Lizorkin, Besov and Hardy spaces, and Coifman–Meyer multiplier operators.

The notations $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are used for the Schwartz class of smooth rapidly decreasing functions defined on $\mathbb{R}^n$ and its dual, the class of tempered distributions on $\mathbb{R}^n$, respectively. $\mathcal{S}_0(\mathbb{R}^n)$ refers to the closed subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ that have vanishing moments of all orders; that is, $f \in \mathcal{S}_0(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0$ for all $\alpha \in \mathbb{N}_0^n$. Its dual is $\mathcal{S}_0'(\mathbb{R}^n)$, which coincides with the class of tempered distributions modulo polynomials denoted by $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. Throughout, all functions are defined on $\mathbb{R}^n$ and therefore we omit $\mathbb{R}^n$ in the notation of the function spaces defined below.

The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is denoted by $\widehat{f}$; in particular, for $f \in L^1$, we use the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi\cdot x}\, dx \quad \forall \xi \in \mathbb{R}^n.$$

If $j \in \mathbb{Z}$ and $h \in \mathcal{S}(\mathbb{R}^n)$, the operator P_j^h is defined so that $\widehat{P_j^h f}(\xi) = \widehat{h}(2^{-j}\xi)\widehat{f}(\xi)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. If $\widehat{h}$ is supported in an annulus centered at the origin we will use the notation Δ_j^h rather than P_j^h; if $\widehat{h}$ is supported in a ball centered at the origin and $\widehat{h}(0) \neq 0$, S_j^h will be used instead of P_j^h. For $y \in \mathbb{R}^n$ the translation operator, denoted by τ_y, is given by $\tau_y h(x) = h(x+y)$ for $x \in \mathbb{R}^n$.

The notation $A \lesssim B$ means that $A \leqslant cB$, where c is a constant that may depend on some of the parameters used but not on the functions appearing in the expressions for A and B.

2.1.1 Coifman-Meyer Multipliers

The symbols used in the main results of this chapter are Coifman-Meyer multipliers. Such multipliers are defined as follows.

Definition 2.1.1. Given $m \in \mathbb{R}$, a smooth, complex-valued function $\sigma = \sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is a *Coifman-Meyer multiplier of order* m if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leqslant C_{\alpha,\beta}(|\xi| + |\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi, \eta) \in \mathbb{R}^{2n} \backslash \{(0,0)\}. \tag{2.1.1}$$

We say $\sigma = \sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is an *inhomogeneous Coifman-Meyer multiplier of order* m if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leqslant C_{\alpha,\beta}(1 + |\xi| + |\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi, \eta) \in \mathbb{R}^{2n}. \tag{2.1.2}$$

Bilinear multiplier operators associated to Coifman-Meyer multipliers of order 0 have been well studied. Such operators are examples of bilinear Calderón-Zygmund operators. As a consequence they are bounded in a variety of function spaces; in particular, they satisfy

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

where σ is a Coifman-Meyer multiplier of order 0, $1/p = 1/p_1 + 1/p_2$, and $1 < p_1, p_2 < \infty$. The reader is referred to Coifman-Meyer [15] for various estimates and background information for Coifman-Meyer multiplier operators, and to David-Journé [19], Grafakos-Torres [29], and Kenig-Stein [40] for the development of the Calderón-Zygmund theory. Estimates in weighted Lebesgue spaces for bilinear Calderón-Zygmund operators, and in particular for Coifman-Meyer multiplier operators of order 0, have been obtained in Grafakos-Torres [28], Grafakos-Martell [26], and Lerner et al. [43].

We next describe a decomposition of Coifman-Meyer multiplier operators that will be useful in the proofs of the main results of this chapter. Fix $\Psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\widehat{\Psi}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \ \forall \xi \in \mathbb{R}^n \backslash \{0\};$$

define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\widehat{\Phi}(0) := 1, \quad \widehat{\Phi}(\xi) := \sum_{j\leqslant 0} \widehat{\Psi}(2^{-j}\xi) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}.$$

By the notation previously introduced, if $a, b \in \mathbb{R}^n$, $\Delta_j^{\tau_a\Psi} f$ and $S_j^{\tau_a\Phi} f$ satisfy $\widehat{\Delta_j^{\tau_a\Psi} f}(\xi) = \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot a}\widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{S_j^{\tau_b\Phi} f}(\xi) = \widehat{\tau_b\Phi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot b}\widehat{\Phi}(2^{-j}\xi)\widehat{f}(\xi)$. By the work of Coifman and Meyer in [15], given $N \in \mathbb{N}$ such that $N > n$, it follows that $T_\sigma = T_\sigma^1 + T_\sigma^2$, where, for $f \in \mathcal{S}_0(\mathbb{R}^n)$ ($f \in \mathcal{S}(\mathbb{R}^n)$ if $m \geqslant 0$) and $g \in \mathcal{S}(\mathbb{R}^n)$,

$$T_\sigma^1(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)(x)\,(S_j^{\tau_b\Phi} g)(x) \quad \forall x \in \mathbb{R}^n, \tag{2.1.3}$$

the coefficients $\mathcal{C}_j(a,b)$ satisfy

$$|\mathcal{C}_j(a,b)| \lesssim 2^{jm} \quad \forall a, b \in \mathbb{Z}^n, j \in \mathbb{Z}, \tag{2.1.4}$$

with the implicit constant depending on σ, and an analogous expression holds for T_σ^2 with the roles of f and g interchanged.

If σ is an inhomogeneous Coifman–Meyer multiplier of order m, a similar decomposition to (2.1.3) follows with the summation in $j \in \mathbb{N}_0$ rather than $j \in \mathbb{Z}$, with $\Delta_0^{\tau_a\Psi}$ replaced by

$S_0^{\tau_a\Phi}$ and for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Remark 2.1.2. For the formula (2.1.3) and its corresponding counterpart for T_σ^2 to hold, the condition (2.1.1) on the derivatives of σ is only needed for multi-indices α and β such that $|\alpha + \beta| \leqslant 2N$

2.1.2 The scale of weighted Triebel-Lizorkin and Besov spaces

The Leibniz-type rules obtained in two of the main results of this chapter, Theorem 2.2.1 and Theorem 2.2.7, hold in the settings of weighted Triebel-Lizorkin and Besov spaces. In this section, we define these scales of spaces and present some of their properties. In particular, we state and prove Nikol'skiĭ representations of the spaces, which constitute important tools for the proofs of Theorems 2.2.1 and 2.2.7.

We start by defining the classes of weights we will be using and present maximal operators and inequalities.

A *weight* w defined on $\mathbb{R}^n$ is a locally integrable function such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Given a weight w and $0 < p < \infty$ we define the weighted Lebesgue space $L^p(w)$ as the space of all measurable functions f satisfying

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx\right)^{\frac{1}{p}} < \infty.$$

In the case that $p = \infty$ we define $L^\infty(w) := L^\infty$.

The specific classes of weights in the hypotheses of the results of this chapter are Muckenhoupt weights, which we next define.

Definition 2.1.3. For $1 < p < \infty$, the *Muckenhoupt class* A_p consists of all weights w on $\mathbb{R}^n$ satisfying

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\, dx\right)^{p-1} < \infty,$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. For $p = \infty$ we define $A_\infty := \bigcup_{1<p} A_p$.

From this definition, it follows that $A_p \subset A_q$ when $p \leqslant q$. Moreover, it can be proved that if $w \in A_p$, $p > 1$, then $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. For $w \in A_\infty$ we set

$$\tau_w := \inf\{p \in (1, \infty) : w \in A_p\}.$$

The Muckenhoupt classes arise in the study of boundedness properties of the Hardy-Littlewood maximal operator in the setting of weighted Lebesgue spaces, as we next explain. The *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined as

$$\mathcal{M}(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad \forall x \in \mathbb{R}^n, f \in L^1_{loc},$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$. It turns out that $\mathcal{M}$ is bounded on $L^p(w)$ if and only if $w \in A_p$. That is, for $1 < p < \infty$, $w \in A_p$ if and only if

$$\|\mathcal{M}(f)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \quad \forall f \in L^p(w).$$

Given $0 < r < \infty$ and $f \in L^1_{loc}$, we set $\mathcal{M}_r(f) := (\mathcal{M}(|f|^r))^{\frac{1}{r}}$. By the boundedness properties for the Hardy-Littlewood maximal operator stated above, it holds that $\mathcal{M}_r$ is bounded on $L^p(w)$ for $0 < r < p$ and $w \in A_{p/r}$ (i.e. $0 < r < \frac{p}{\tau_w}$). This fact is a particular case of the following estimate known as the Fefferman-Stein inequality:
If $0 < p < \infty$, $0 < q \leqslant \infty$, $0 < r < \min(p, q)$ and $w \in A_{p/r}$ (i.e. $0 < r < \min(p/\tau_w, q)$), then for all sequences $\{f_j\}_{j \in \mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}, \tag{2.1.5}$$

where the implicit constant depends on r, p, q, and w and the summation in j should be replaced by the supremum in j if $q = \infty$.

For more details on the Muckenhoupt classes see Muckenhoupt [51] and Grafakos [24].

Weighted Triebel-Lizorkin and Besov spaces

Let ψ and φ be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the following conditions:

$$\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}, \tag{2.1.6}$$

$$|\widehat{\psi}(\xi)| > c \text{ for } \frac{3}{5} < |\xi| < \frac{5}{3} \text{ and some } c > 0, \tag{2.1.7}$$

$$\operatorname{supp}(\widehat{\varphi}) \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}, \tag{2.1.8}$$

$$|\widehat{\varphi}(\xi)| > c \text{ for } |\xi| < \frac{5}{3} \text{ and some } c > 0. \tag{2.1.9}$$

Definition 2.1.4. (Weighted homogeneous Triebel-Lizorkin and Besov spaces) Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.

- For $0 < p < \infty$, the *weighted homogeneous Triebel-Lizorkin space* $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj} |\Delta_j^{\psi} f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

  with appropriate changes if $q = \infty$.

- For $0 < p \leqslant \infty$, the *weighted homogeneous Besov space* $\dot{B}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}(w)} := \left(\sum_{j\in\mathbb{Z}} (2^{js} \left\| \Delta_j^{\psi} f \right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty,$$

  with appropriate changes if $q = \infty$.

Definition 2.1.5. (Weighted inhomogeneous Triebel-Lizorkin and Besov spaces) Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.

- For $0 < p < \infty$, the *weighted inhomogeneous Triebel-Lizorkin space* $F^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,q}(w)} := \|S_0^\varphi f\|_{L^p(w)} + \left\| \left(\sum_{j\in\mathbb{N}} (2^{sj}|\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

with appropriate changes if $q = \infty$.

- For $0 < p \leqslant \infty$, the *weighted inhomogeneous Besov space* $B^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(w)} := \|S_0^\varphi f\|_{L^p(w)} + \left(\sum_{j\in\mathbb{N}} (2^{js} \left\|\Delta_j^\psi f\right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty,$$

with appropriate changes if $q = \infty$.

The definitions above are independent of the choice of φ and ψ. Triebel-Lizorkin and Besov spaces are quasi-Banach spaces and, if $p, q \geqslant 1$, they are Banach spaces. These spaces provide a unified framework that includes a variety of other spaces such as weighted Lebesgue, Hardy, and Sobolev spaces. For instance, the following equalities hold with equivalent norms:

$$\dot{F}^0_{p,2}(w) = H^p(w) \text{ for } 0 < p < \infty, w \in A_\infty, \tag{2.1.10}$$

$$F^0_{p,2}(w) = h^p(w) \text{ for } 0 < p < \infty, w \in A_\infty, \tag{2.1.11}$$

$$\dot{F}^0_{p,2}(w) = F^0_{p,2}(w) = L^p(w) = H^p(w) = h^p(w) \text{ for } 1 < p < \infty, w \in A_p, \tag{2.1.12}$$

$$\dot{F}^s_{p,2}(w) = \dot{W}^{s,p}(w) \text{ for } 1 < p < \infty, w \in A_p, \tag{2.1.13}$$

where $H^p(w)$, $h^p(w)$, and $\dot{W}^{s,p}(w)$ are, respectively, a weighted Hardy space, a weighted local Hardy space, and a weighted Sobolev space, whose definitions we next recall. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Given $0 < p < \infty$ and $w \in A_\infty$, the weighted Hardy space $H^p(w)$ is defined as

$$H^p(w) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p(w)} < \infty\},$$

where

$$\|f\|_{H^p(w)} := \left\| \sup_{t>0} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)};$$

the weighted local Hardy spaces $h^p(w)$ is defined as

$$h^p(w) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{h^p(w)} < \infty\},$$

where

$$\|f\|_{h^p(w)} := \left\| \sup_{0<t<1} \left| t^{-n}\phi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)}.$$

For $1 < p < \infty$ and $w \in A_\infty$ the weighted Sobolev space $\dot{W}^{s,p}(w)$ is the space of all tempered distributions modulo polynomials such that

$$\|f\|_{\dot{W}^{s,p}(w)} := \|D^s f\|_{L^p(w)} < \infty.$$

For a detailed overview of the development of Besov and Triebel-Lizorkin spaces see Triebel [65] and Qui [61] for the unweighted and weighted settings respectively. We recall that these spaces satisfy the following lifting property: for s, p, and q as in Definitions 2.1.4 and 2.1.5 and $w \in A_\infty$ we have that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,q}(w)} \simeq \|J^s f\|_{F^0_{p,q}(w)}, \tag{2.1.14}$$

$$\|f\|_{\dot{B}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{B}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{B^s_{p,q}(w)} \simeq \|J^s f\|_{B^0_{p,q}(w)}. \tag{2.1.15}$$

We end this section by introducing notation used in the statements of the main results in this chapter. For $w \in A_\infty$ and $0 < p, q \leqslant \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right) \quad \text{and} \quad \tau_p(w) := n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right). \tag{2.1.16}$$

If $w \equiv 1$, in which case $\tau_w = 1$, we just write $\tau_{p,q}$ and τ_p, respectively. Note that $\tau_{p,2}(w) =$

$\tau_p(w)$, $\tau_{p,q}(w) \geqslant \tau_{p,q}$, and $\tau_p(w) \geqslant \tau_p$ for any $w \in A_\infty$.

Nikol'skiĭ representations for weighted Triebel-Lizorkin and Besov spaces

An important tool for the proofs of Theorem 2.2.1 and Theorem 2.2.7 is the Nikol'skiĭ representation for weighted Triebel-Lizorkin and Besov spaces, stated below as Theorem 2.1.6. Such result is a weighted version of [68, Theorem 3.7].

Given $0 < p, q \leqslant \infty$, and a sequence of functions $\{f_j\}_{j\in\mathbb{Z}}$ defined on $\mathbb{R}^n$, the following notation will be used in this section:

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{L^p(w)(\ell^q)} := \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p(w)}, \quad \|\{f_j\}_{j\in\mathbb{Z}}\|_{\ell^q(L^p(w))} := \left(\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p(w)}^q \right)^{1/q}.$$

Theorem 2.1.6 (Nikol'skiĭ representation)**.** *For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{Z}.$$

If $w \in A_\infty$, then the following holds:

(i) Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^p(w)(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{F}^s_{p,q}(w)$ (in $\mathcal{S}_0'(\mathbb{R}^n)$ if $q = \infty$) and

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{F}^s_{p,q}(w)} \lesssim \|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^p(w)(\ell^q)},$$

where the implicit constant depends only on n, D, s, p and q. An analogous statement, with $j \in \mathbb{N}_0$, holds true for $F^s_{p,q}(w)$ (when $q = \infty$, the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

(ii) Let $0 < p, q \leqslant \infty$ and $s > \tau_p(w)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{\ell^q(L^p(w))} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$

converges in $\dot{B}^s_{p,q}(w)$ *(in* $\mathcal{S}'_0(\mathbb{R}^n)$ *if* $q = \infty$*) and*

$$\left\| \sum_{j \in \mathbb{Z}} u_j \right\|_{\dot{B}^s_{p,q}(w)} \lesssim \left\| \{2^{js} u_j\}_{j \in \mathbb{Z}} \right\|_{\ell^q(L^p(w))},$$

where the implicit constant depends only on n*,* D*,* s*,* p *and* q*. An analogous statement, with* $j \in \mathbb{N}_0$*, holds true for* $B^s_{p,q}(w)$ *(when* $q = \infty$*, the convergence is in* $\mathcal{S}'(\mathbb{R}^n)$*).*

Before proving Theorem 2.1.6 we state several lemmas that are used in its proof.

Lemma 2.1.7 (Particular case of Corollary 2.11 in [68])**.** *Suppose* $0 < r \leqslant 1$*,* $A > 0$*,* $R \geqslant 1$ *and* $d > n/r$*. If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that* $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant AR\}$*, it holds that*

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\| (1 + |A\,\cdot|)^d \phi \right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n,$$

where the implicit constant is independent of A, R, ϕ*, and* f*.*

Remark 2.1.8. [68, Corollary 2.11] incorrectly states $A^{-n/r}$ instead of A^{-n}. Also, it states $A \geqslant 1$, but the result is true for $A > 0$ as stated in Lemma 2.1.7.

The following lemma is a weighted version of [68, Corollary 2.12 (1)]. We include its brief proof for completeness.

Lemma 2.1.9. *Suppose* $w \in A_\infty$*,* $0 < p \leqslant \infty$*,* $A > 0$*,* $R \geqslant 1$*, and* $d > b > n/\min(1, p/\tau_w)$*. If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that* $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant AR\}$*, it holds that*

$$\|\phi * f\|_{L^p(w)} \lesssim R^{b-n} A^{-n} \left\| (1 + |A\,\cdot|^d) \phi \right\|_{L^\infty} \|f\|_{L^p(w)},$$

where the implicit constant is independent of A*,* R*,* ϕ *and* f*.*

Proof. Set $r := n/b < \min(1, p/\tau_w)$. The hypothesis $d > b$ means $d > n/r$ and Lemma 2.1.7 yields

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\| (1 + |A\,\cdot|)^d \phi \right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n.$$

Since $r < p/\tau_w$, we have $\|\mathcal{M}_r f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$ and therefore

$$\|\phi * f\|_{L^p(w)} \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\|(1+|A\cdot|)^d \phi\right\|_{L^\infty} \|f\|_{L^p(w)};$$

observing that $1/r - 1 = (b-n)/n$, the desired estimate follows. $\square$

The following lemma is a modified version of [68, Lemma 3.8].

Lemma 2.1.10. *Let $\tau < 0$, $\lambda \in \mathbb{R}$, $0 < q \leqslant \infty$, and $k_0 \in \mathbb{Z}$. Then, for any sequence $\{d_j\}_{j\in\mathbb{Z}} \subset [0,\infty)$ it holds that*

$$\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} \lesssim \left\| \{2^{j\lambda} d_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q},$$

where the implicit constant depends only on k_0, τ, λ and q.

Proof. Suppose first that $0 < q \leqslant 1$. Then,

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} &= \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right)^q \right]^{\frac{1}{q}} \\
&\leqslant \left[\sum_{j\in\mathbb{Z}} \sum_{k=k_0}^{\infty} 2^{\tau q k} 2^{\lambda q (j+k)} d_{j+k}^q \right]^{\frac{1}{q}} = \left[\sum_{k=k_0}^{\infty} 2^{\tau q k} \sum_{j\in\mathbb{Z}} 2^{\lambda q(j+k)} d_{j+k}^q \right]^{\frac{1}{q}} \\
&= \left(\sum_{k=k_0}^{\infty} 2^{\tau q k} \right)^{\frac{1}{q}} \left\| \{2^{j\lambda} d_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q} = C_{k_0,\tau,q} \left\| \{2^{j\lambda} d_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q},
\end{aligned}$$

where in the last equality we have used that $\tau < 0$. If $1 < q < \infty$ we use Hölder's inequality with q and q' to write

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} &\leqslant \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k q/2} 2^{\lambda q(j+k)} d_{j+k}^q \right) \left(\sum_{k=k_0}^{\infty} 2^{\tau k q'/2} \right)^{q/q'} \right]^{\frac{1}{q}} \\
&= C_{k_0,\tau,q} \left\| \{2^{j\lambda} d_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q}.
\end{aligned}$$

The case $q = \infty$ is straightforward.

$\square$

We now prove Theorem 2.1.6.

Proof of Theorem 2.1.6. We first prove Theorem 2.1.6 for finite families. We will do this in the homogeneous settings, with the proof in the inhomogeneous settings being similar. Suppose $\{u_j\}_{j\in\mathbb{Z}}$ is such that $u_j = 0$ for all j except those belonging to some finite subset of $\mathbb{Z}$; this assumption allows us to avoid convergence issues since all the sums considered will be finite.

For Part (i), let D, w, p, q and s be as in the hypotheses. Fix $0 < r < \min(1, p/\tau_w, q)$ such that $s > n(1/r - 1)$; note that the latter is possible since $s > \tau_{p,q}(w)$.

Let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \leqslant 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{Z}.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $\dot{F}^s_{p,q}(w)$ in Section 2.1.2. We have

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.1.17}$$

We will use Lemma 2.1.7 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geqslant -k_0$, we get $R \geqslant 1$.) Fixing $d > n/r$ and applying Lemma 2.1.7, we get

$$\begin{aligned} |\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\| (1 + |2^j \cdot|)^d 2^{jn} \psi(2^j \cdot) \right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\ &\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y\in\mathbb{R}^n} (1 + |2^j y|)^d |\psi(2^j y)| \right) \mathcal{M}_r(u_{j+k})(x). \end{aligned}$$

Hence,

$$2^{js}|\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.1.17),

$$2^{js}|\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x).$$

Since $1/r - 1 - s/n < 0$, Lemma 2.1.10 yields

$$\left\| \{2^{js}|\Delta_j^\psi u|\}_{j\in\mathbb{Z}} \right\|_{L^p(w)(\ell^q)} \lesssim \left\| \{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}} \right\|_{L^p(w)(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{Z}}$. Applying the weighted Fefferman-Stein inequality to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{2^{js}u_j\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)}.$$

For Part (ii), let D, w, p, q and s be as in the hypotheses and k_0 be as above. Consider $\Delta_j^\psi u_{j+k}$ in (2.1.17) and apply Lemma 2.1.9 with $\phi(x) = 2^{jn}\psi(2^{-j}x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n/\min(1, p/\tau_w) < b < n + s$; note that such b exists since $s > \tau_p(w)$. We get

$$\left\|\Delta_j^\psi u_{j+k}\right\|_{L^p(w)} \lesssim 2^{(k+k_0)(b-n)}2^{-jn}\left\|(1+|2^j\cdot|)^d 2^{jn}\psi(2^{-j}\cdot)\right\|_{L^\infty}\|u_{j+k}\|_{L^p(w)} \sim 2^{k(b-n)}\|u_{j+k}\|_{L^p(w)},$$

and setting $p^* := \min(p, 1)$ we obtain

$$2^{jsp^*}\left\|\Delta_j^\psi u\right\|_{L^p(w)}^{p^*} \lesssim 2^{jsp^*}\sum_{k=-k_0}^{\infty}\left\|\Delta_j^\psi u_{j+k}\right\|_{L^p(w)}^{p^*} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*}2^{sp^*(j+k)}\|u_{j+k}\|_{L^p(w)}^{p^*}.$$

Hence, applying Lemma 2.1.10, it follows that

$$\|u\|_{\dot{B}^s_{p,q}(w)} \lesssim \left\| \left\{ \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|^{p^*}_{L^p(w)} \right\}_{j\in\mathbb{Z}} \right\|^{\frac{1}{p^*}}_{\ell^{q/p^*}} \lesssim \left\| \{2^{js} u_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q(L^p(w))},$$

as desired.

We next show the theorem for families that are not necessarily finite. We work in the setting of homogeneous Triebel-Lizorkin spaces; an analogous reasoning applies to the other contexts. Let $\{u_j\}_{j\in\mathbb{Z}}$, w, p, q, and s be as in the hypotheses. Define $U_N := \sum_{k=-N}^{N} u_j$; since the theorem is true for finite families and, for $M < N$, $\{u_j\}_{M+1\leqslant |j|\leqslant N}$ satisfies the hypotheses of the theorem, we have

$$\|U_N - U_M\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{2^{js} u_j\}_{M+1\leqslant |j|\leqslant N} \right\|_{L^p(w)(\ell^q)}, \tag{2.1.18}$$

where the implicit constant is independent of M, N and the family $\{u_j\}_{j\in\mathbb{Z}}$.

If $0 < q < \infty$, as $M, N \to \infty$, the right-hand side of (2.1.18) tends to zero by the assumption $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^p(w)(\ell^q)} < \infty$ and the dominated convergence theorem; therefore, since $\dot{F}^s_{p,q}(w)$ is complete, $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{F}^s_{p,q}(w)$. The same reasoning used to obtain (2.1.18) gives that

$$\|U_N\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{2^{js} u_j\}_{-N\leqslant j\leqslant N} \right\|_{L^p(w)(\ell^q)},$$

where the implicit constant is independent of N and the family $\{u_j\}_{j\in\mathbb{Z}}$. It then follows that

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{2^{js} u_j\}_{j\in\mathbb{Z}} \right\|_{L^p(w)(\ell^q)},$$

with the implicit constant independent of the family $\{u_j\}_{j\in\mathbb{Z}}$.

If $q = \infty$, we use that $\{2^{(s-\varepsilon)j}u_j\}_{j\geqslant 0}$ and $\{2^{(s+\varepsilon)j}u_j\}_{j<0}$ belong to $\ell^1(L^p(w))$ for any $\varepsilon > 0$ and apply Theorem 2.1.6 under the case of finite q to conclude that $\sum_{j=0}^{N} u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{p,1}(w)$ and $\dot{B}^{s+\varepsilon}_{p,1}(w)$, respectively (choosing $\varepsilon > 0$ so that $s - \varepsilon > \tau_{p,q}(w) \geqslant$

$\tau_p(w)$). Therefore, U_N convergence in $\mathcal{S}_0'(\mathbb{R}^n)$. Moreover, by Theorem 2.1.6 applied to the finite sequence $\{u_j\}_{-N\leqslant j\leqslant N}$, we have that $U_N \in \dot{F}^s_{p,\infty}(w)$ and

$$\|U_N\|_{\dot{F}^s_{p,\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{-N\leqslant j\leqslant N}\right\|_{L^p(w)(\ell^\infty)} \leqslant \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^\infty)},$$

with the implicit constant independent of N and $\{u_j\}_{j\in\mathbb{Z}}$. Since $\dot{F}^s_{p,\infty}(w)$ has the Fatou property (see Section 2.3.4), we conclude that $\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot{F}^s_{p,\infty}(w)$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{p,\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^\infty)}.$$

□

2.2 Leibniz-type rules in weighted Triebel-Lizorkin and Besov spaces

2.2.1 Homogeneous Leibniz-type rules

In this section, we present one of the main results in this chapter (Theorem 2.2.1) about Leibniz-type rules associated to Coifman-Meyer multipliers in the setting of weighted homogeneous Besov and Triebel Lizorkin spaces. We infer corollaries that include extensions and improvements of the fractional Leibniz rules (1.0.1) and we compare the new results to those in the literature.

We remind the reader that the notation $\tau_{p,q}(w)$ and $\tau_p(w)$ for $w \in A_\infty$ is defined in Section 2.1.2 (see 2.1.16); we recall that $\tau_{p,2}(w) = \tau_p(w)$ and we write $\tau_{p,q}$ and τ_p if $w \equiv 1$.

Theorem 2.2.1. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$*

and set $w = w_1^{p/p_1} w_2^{p/p_2}$. *If* $0 < p, p_1, p_2 < \infty$ *and* $s > \tau_{p,q}(w)$, *it holds that*

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot F^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.19}$$

If $0 < p, p_1, p_2 \leqslant \infty$ *and* $s > \tau_p(w)$, *it holds that*

$$\|T_\sigma(f,g)\|_{\dot B^s_{p,q}(w)} \lesssim \|f\|_{\dot B^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot B^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.20}$$

where the Hardy spaces $H^{p_1}(w_1)$ *and* $H^{p_2}(w_2)$ *must be replaced by* L^∞ *if* $p_1 = \infty$ *or* $p_2 = \infty$, *respectively.*

If $w_1 = w_2$ *then different pairs of* p_1, p_2 *can be used on the right-hand sides of* (2.2.19) *and* (2.2.20); *moreover, if* $w \in A_\infty$, *then*

$$\|T_\sigma(f,g)\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot F^{s+m}_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.21}$$

where $0 < p < \infty$, $0 < q \leqslant \infty$ *and* $s > \tau_{p,q}(w)$.

We note that if $m \geqslant 0$ then the above estimates hold for any $f, g \in \mathcal{S}(\mathbb{R}^n)$ when $\mathcal{S}(\mathbb{R}^n)$ is a subspace of the function spaces on the right-hand side. This is the case when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ in (2.2.19) and (2.2.20) and $w \in A_p$ for (2.2.21).

By the lifting properties for weighted homogeneous Besov and Triebel-Lizorkin spaces (2.1.14) and (2.1.15), the estimates (2.2.19), (2.2.20), and (2.2.21) can be respectively written as

$$\|D^s T_\sigma(f,g)\|_{\dot F^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot F^m_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot F^m_{p_2,q}(w_2)}, \tag{2.2.22}$$

$$\|D^s T_\sigma(f,g)\|_{\dot B^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot B^m_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot B^m_{p_2,q}(w_2)}, \tag{2.2.23}$$

$$\|D^s T_\sigma(f,g)\|_{\dot F^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot F^m_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{\dot F^m_{p,q}(w)}. \tag{2.2.24}$$

We next give particular cases of (2.2.22), (2.2.23), and (2.2.24) including extensions and improvements of the fractional Leibniz rules (1.0.1).

The relation (2.1.10) between weighted Triebel-Lizorkin spaces and weighted Hardy spaces imply the following result for Coifman-Meyer multipliers of order 0.

Corollary 2.2.2. *Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order* 0. *Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.25}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.25); moreover, if $w \in A_\infty$, then

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.26}$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

Taking $\sigma \equiv 1$ in Corollary 2.2.2, we obtain weighted fractional Leibniz rules for the product of two functions:

Corollary 2.2.3. *Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.27}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.27); moreover, if $w \in A_\infty$, then

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n),$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

In particular, the case $w_1 = w_2 = 1$ in Corollary 2.2.3 leads to the estimates

$$\|D^s(fg)\|_{H^p} \lesssim \|D^s f\|_{H^{p_1}} \|g\|_{H^{p_2}} + \|f\|_{H^{p_1}} \|D^s g\|_{H^{p_2}} \tag{2.2.28}$$

for $0 < p, p_1, p_2, \tilde{p}_1, \tilde{p}_2 < \infty$, $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$, and $s > n(1/\min(p,1) - 1)$. The estimates (2.2.28) improve and extend (1.0.1). Indeed, the inequality (2.2.28) extends the range of p, p_1, $\tilde{p}_1$, p_2, and $\tilde{p}_2$ by allowing $0 < p, p_1, \tilde{p}_1, p_2, \tilde{p}_2 < \infty$ while (1.0.1) requires $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$. Additionally, (2.2.28) allows for the H^p norm on the left-hand side, which is larger than the L^p norm.

More generally, Theorem 2.2.1 implies the following weighted fractional Leibniz rules for the product of two functions.

Corollary 2.2.4. *Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|D^s(fg)\|_{\dot{F}^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot{F}^0_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot{F}^0_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.29}$$

If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, it holds that

$$\|D^s(fg)\|_{\dot{B}^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot{B}^0_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{\dot{B}^0_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.30}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.29) and (2.2.30); moreover, if $w \in A_\infty$, then

$$\|D^s(fg)\|_{\dot{F}^0_{p,q}(w)} \lesssim \|D^s f\|_{\dot{F}^0_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{\dot{F}^0_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.31}$$

where $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$.

We close this section by comparing Corollaries 2.2.2 and 2.2.3 with other related results in the literature.

Using different methods, the following result was proven in Brummer-Naibo [13, Theorem 1.1]:

If σ is a Coifman-Meyer multiplier of order 0, $1 < p_1, p_2 \leqslant \infty$, $\frac{1}{2} < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_p$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)}. \tag{2.2.32}$$

Moreover, if $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.32).

Corollary 2.2.2 and this result compare as follows:

- The estimate (2.2.25) allows for $0 < p, p_1, p_2 < \infty$, $w_1, w_2 \in A_\infty$, and $H^p(w)$ on the left-hand side if $s > \tau_p(w)$. On the other hand, (2.2.32) requires $1 < p_1, p_2 \leqslant \infty$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, and the smaller $L^p(w)$ norm on the left-hand side when $s > \tau_p$. Therefore, (2.2.25) is less restrictive than (2.2.32) in terms of the indices p, p_1, and p_2 and the classes that the weights w_1 and w_2 belong to. However, since $\tau_p \leqslant \tau_p(w)$, (2.2.25) is more restrictive than (2.2.32) in terms of the range of the regularity s.

- Let $1/2 < p < \infty$, $1 < p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$. If $s > \tau_p(w)$ then (2.2.25) holds and implies (2.2.32); however, if $\tau_p < \tau_p(w)$ then (2.2.32) holds also for $\tau_p < s \leqslant \tau_p(w)$ while (2.2.25) may not be true for such range of s. We next give examples of w_1 and w_2 such that the corresponding weight w satisfies $\tau_p < \tau_p(w)$. Let $1 < p_1 \leqslant p_2 < \infty$ and $w_1(x) = w_2(x) = w(x) = |x|^a$ with $n(r-1) < a < n(p_1 - 1)$ for some $1 < r < p_1$. Then $w \in A_{p_1} \subset A_{p_2}$, and $w \notin A_r$. This implies that $1 < \tau_w$, which leads to $\tau_p < \tau_p(w)$ if $p < \tau_w$.

- The estimate (2.2.32) implies (2.2.26) for $1 < p < \infty$, $w \in A_p$, and $s > \tau_p$ and gives the endpoint estimate

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^p(w)} + \|f\|_{L^p(w)} \|D^s g\|_{L^\infty}.$$

However, (2.2.26) allows $0 < p < \infty$ and $w \in A_\infty$ if $s > \tau_p(w)$.

Corollary 2.2.3 complements some of the estimates obtained through different methods in [18, Theorem 1.1] in the same manner Corollary 2.2.2 complements [13, Theorem 1.1] as explained above; as in that case, Corollary 2.2.3 and [18, Theorem 1.1] have some estimates in common but each of them gives a different set of results.

2.2.2 Proofs of the homogeneous Leibniz-type rules

In this section we prove Theorem 2.2.1. The following lemma will be useful in its proof.

Lemma 2.2.5. *Let $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\phi_1}$ and $\widehat{\phi_2}$ have compact supports and $\widehat{\phi_1}\widehat{\phi_2} = \widehat{\phi_1}$. If $0 < r \leqslant 1$ and $\varepsilon > 0$, it holds that*

$$\left|P_j^{\tau_a\phi_1} f(x)\right| \lesssim (1+|a|)^{\varepsilon+\frac{n}{r}} \mathcal{M}_r(P_j^{\phi_2} f)(x) \quad \forall x, a \in \mathbb{R}^n, j \in \mathbb{Z}, f \in \mathcal{S}(\mathbb{R}^n).$$

Proof. This estimate is a consequence of Lemma 2.1.7. In view of the supports of $\widehat{\phi_1}$ and $\widehat{\phi_2}$ we have $P_j^{\tau_a\phi_1} f = P_j^{\tau_a\phi_1} P_j^{\phi_2} f$ for $j \in \mathbb{Z}$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Applying Lemma 2.1.7 with $\phi(x) = 2^{nj}\tau_a\phi_1(2^j x)$, $A = 2^j$, $R \geqslant 1$ such that $\operatorname{supp}(\widehat{\phi_2}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$ and $d = \varepsilon + n/r$, we get

$$\begin{aligned}\left|P_j^{\tau_a\phi_1} f(x)\right| &\lesssim R^{n(\frac{1}{r}-1)} 2^{-jn} \left\|(1+\left|2^j\cdot\right|)^{\varepsilon+\frac{n}{r}} 2^{nj}\tau_a\phi_1(2^j\cdot)\right\|_{L^\infty} \mathcal{M}_r(P_j^{\phi_2} f)(x) \\ &\sim \left\|(1+\left|2^j\cdot\right|)^{\varepsilon+\frac{n}{r}} \tau_a\phi_1(2^j\cdot)\right\|_{L^\infty} \mathcal{M}_r(P_j^{\phi_2} f)(x) \quad \forall x, a \in \mathbb{R}^n, j \in \mathbb{Z}, f \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

Since $\phi_1 \in \mathcal{S}(\mathbb{R}^n)$,

$$\left|\tau_a\phi_1(2^jx)\right| = \left|\phi_1(2^jx+a)\right| \lesssim \frac{(1+|a|)^{\varepsilon+\frac{n}{r}}}{(1+|2^jx|)^{\varepsilon+\frac{n}{r}}} \quad \forall x, a \in \mathbb{R}, j \in \mathbb{Z}.$$

Combining these two estimates completes the proof. □

We now prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Consider Φ, Ψ, T_σ^1, T_σ^2, $\{\mathcal{C}_j(a,b)\}_{j\in\mathbb{Z},a,b\in\mathbb{Z}^n}$ as in Section 2.1.1. Let m, σ, p, p_1, p_2, q, s, w_1, w_2 and w be as in the hypotheses. For ease of notation, p_1 and p_2 will be assumed to be finite; the same proof applies for (2.2.20) if that is not the case, and for (2.2.21).

We next prove (2.2.19) and (2.2.20). Here we will only work with T_σ^1 as the estimate for T_σ^2 is shown through symmetry. Hence we will prove that

$$\left\|T_\sigma^1(f,g)\right\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} \quad \text{and} \quad \left\|T_\sigma^1(f,g)\right\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}.$$

Moreover, since $\|\sum f_j\|^{\min(p,q,1)}_{\dot{F}^s_{p,q}(w)} \lesssim \sum \|f_j\|^{\min(p,q,1)}_{\dot{F}^s_{p,q}(w)}$ and similarly for $\dot{B}^s_{p,q}(w)$, it suffices to prove that, given $\varepsilon > 0$ there exist $0 < r_1, r_2 \leqslant 1$ such that for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}_0(\mathbb{R}^n)$ ($f \in \mathcal{S}(\mathbb{R}^n) \cap \dot{F}^s_{p,q}(w)$ or $f \in \mathcal{S}(\mathbb{R}^n) \cap \dot{B}^s_{p,q}(w)$ if $m \geqslant 0$), it holds that

$$\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}} \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}, \tag{2.2.33}$$

$$\left\|T^{a,b}(f,g)\right\|_{\dot{B}^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}} \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}, \tag{2.2.34}$$

where

$$T^{a,b}(f,g) := \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)\,(S_j^{\tau_b\Phi} g)$$

and the implicit constants are independent of a and b. We will assume q finite; obvious changes apply if that is not the case.

In view of the supports of Ψ and Φ we have that

$$\operatorname{supp}(\mathcal{F}[\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)\,(S_j^{\tau_b\Phi} g)]) \subset \{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^j\} \quad \forall j \in \mathbb{Z},\ a, b \in \mathbb{Z}^n.$$

For (2.2.33), Theorem 2.1.6 (i), the bound (2.1.4) for $\mathcal{C}_j(a,b)$, and Hölder's inequality imply

$$\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)\,(S_j^{\tau_b\Phi} g)\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)} \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi} f)(x)\,(S_j^{\tau_b\Phi} g)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \\
&\leqslant \left\|\sup_{j\in\mathbb{Z}}|(S_j^{\tau_b\Phi} g)|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi} f)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \\
&\leqslant \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\tau_a\Psi} f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi} g|\right\|_{L^{p_2}(w_2)}.
\end{aligned}$$

Consider $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ as in Section 2.1.2 such that $\widehat{\varphi} \equiv 1$ on $\operatorname{supp}(\widehat{\Phi})$ and $\widehat{\psi} \equiv 1$ on $\operatorname{supp}(\widehat{\Psi})$. Let $0 < r_1 < \min(1, p_1/\tau_{w_1}, q)$; by Lemma 2.2.5 and the weighted Fefferman-Stein inequality (2.1.5) we have that

$$\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi} f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} &\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\mathcal{M}_{r_1}(\Delta_j^{\psi} f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \\
&\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\psi} f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \\
&\sim (1+|a|)^{\varepsilon+\frac{n}{r_1}} \|f\|_{\dot{F}^{s+m}_{p,q}(w_1)},
\end{aligned}$$

where the implicit constants are independent of a and f. Next, let $0 < r_2 < \min(1, p_2/\tau_{w_2})$; by Lemma 2.2.5 and the boundedness properties of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces we have that

$$\begin{aligned}\left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)} &\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|\mathcal{M}_{r_2}(\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|)\right\|_{L^{p_2}(w_2)}\\ &\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|\right\|_{L^{p_2}(w_2)}\\ &\sim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\|g\|_{H^{p_2}(w_2)},\end{aligned}$$

where the implicit constants are independent of b and g. Putting all together we obtain (2.2.33).

For (2.2.34), Theorem 2.1.6 (ii), the bound (2.1.4) for $\mathcal{C}_j(a,b)$, and Hölder's inequality give

$$\begin{aligned}\left\|T^{a,b}(f,g)\right\|_{\dot B^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^p(w))}\\ &\lesssim \left(\sum_{j\in\mathbb{Z}}2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\right\|^q_{L^p(w)}\right)^{\frac{1}{q}}\\ &\leqslant \left(\sum_{j\in\mathbb{Z}}2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\right\|^q_{L^{p_1}(w_1)}\right)^{\frac{1}{q}}\left\|\sup_{k\in\mathbb{Z}}|S_k^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)}\\ &\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}}\|f\|_{\dot B^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)},\end{aligned}$$

where in the last inequality we have used Lemma 2.2.5 and the boundedness properties of $\mathcal{M}$ with $0<r_j<\min(1,p_j/\tau_{w_j})$ for $j=1,2$.

It is clear from the proof above that if $w_1=w_2$, then different pairs of p_1,p_2 related to p through the Hölder condition can be used on the right-hand sides of (2.2.19) and (2.2.20); in such case $w=w_1=w_2$. □

Remark 2.2.6. For convergence purposes, the relations between N in (2.1.3) and the powers $\varepsilon+n/r_1$ and $\varepsilon+n/r_2$ in (2.2.33) and (2.2.34) must be such that $(N-\varepsilon-n/r_1)\,r^*>n$ and $(N-\varepsilon-n/r_2)\,r^*>n$, where $r^*=\min(p,q,1)$. Moreover, r_1 and r_2 were selected so that $0<$

$r_j < \min(1, p_j/\tau_{w_j}, q)$ in the context of Triebel–Lizorkin spaces and $0 < r_j < \min(1, p_j/\tau_{w_j})$ in the context of Besov spaces. Therefore, if $N > n(1/r^* + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}, q))$ in the Triebel–Lizorkin setting and $N > n(1/r^* + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))$ in the Besov setting, ε, r_1 and r_2 can be chosen so that all the conditions above are satisfied. In view of this and Remark 2.1.2, the multiplier σ in Theorem 2.2.1 needs only satisfy (2.1.1) for $|\alpha + \beta| \leqslant 2([n(1/r^* + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}, q))] + 1)$ in the Triebel–Lizorkin case and $|\alpha + \beta| \leqslant 2([n(1/r^* + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))] + 1)$ in the Besov case.

2.2.3 Inhomogeneous Leibniz-type rules

In this section, we obtain Leibniz-type rules for Coifman-Meyer multiplier operators associated to inhomogeneous symbols, which lead to extensions and improvements of the fractional Leibniz rule (1.0.2). Our main result is a counterpart in the inhomogeneous setting to Theorem 2.2.1, which we next state.

Theorem 2.2.7. *For $m \in \mathbb{R}$, let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f, g)\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s+m}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{F^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n). \quad (2.2.35)$$

If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f, g)\|_{B^s_{p,q}(w)} \lesssim \|f\|_{B^{s+m}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{B^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \quad (2.2.36)$$

where the local Hardy spaces $h^{p_1}(w_1)$ and $h^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.35) *and* (2.2.36)*; moreover, if* $w \in A_\infty$*, then*

$$\|T_\sigma(f,g)\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{p,q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n), \tag{2.2.37}$$

where $0 < p < \infty$, $0 < q \leqslant \infty$ *and* $s > \tau_{p,q}(w)$.

By the lifting properties for weighted inhomogeneous Besov and Triebel-Lizorkin spaces (2.1.14) and (2.1.15), the estimates (2.2.35), (2.2.36), and (2.2.37) can be written respectively as

$$\|J^s T_\sigma(f,g)\|_{F^0_{p,q}(w)} \lesssim \|J^s f\|_{F^m_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|J^s g\|_{F^m_{p_2,q}(w_2)}, \tag{2.2.38}$$

$$\|J^s T_\sigma(f,g)\|_{B^0_{p,q}(w)} \lesssim \|J^s f\|_{B^m_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|J^s g\|_{B^m_{p_2,q}(w_2)}, \tag{2.2.39}$$

$$\|J^s T_\sigma(f,g)\|_{F^0_{p,q}(w)} \lesssim \|J^s f\|_{F^m_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^m_{p,q}(w)}. \tag{2.2.40}$$

The proof of Theorem 2.2.7 follows along the same lines as the proof of Theorem 2.2.1. Theorem 2.2.7 implies versions in the inhomogeneous settings of the corollaries presented in Section 2.2.1. As an example we state the inhomogeneous counterpart to Corolarry 2.2.2 and obtain an improvement of the fractional Leibniz rule (1.0.2).

Corollary 2.2.8. *Let* $\sigma(\xi,\eta)$*,* $\xi,\eta \in \mathbb{R}^n$*, be an inhomogeneous Coifman-Meyer multiplier of order* 0*. Consider* $0 < p, p_1, p_2 < \infty$ *such that* $1/p = 1/p_1 + 1/p_2$*; let* $w_1, w_2 \in A_\infty$ *and set* $w = w_1^{p/p_1} w_2^{p/p_2}$*. If* $s > \tau_p(w)$*, it holds that*

$$\|J^s(T_\sigma(f,g))\|_{h^p(w)} \lesssim \|J^s f\|_{h^{p_1}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|J^s g\|_{h^{p_2}(w_2)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n). \tag{2.2.41}$$

If $w_1 = w_2$ *then different pairs of* p_1, p_2 *can be used on the right-hand side of* (2.2.41)*; moreover, if* $w \in A_\infty$*, then*

$$\|J^s(T_\sigma(f,g))\|_{h^p(w)} \lesssim \|J^s f\|_{h^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{h^p(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n), \tag{2.2.42}$$

where $0 < p < \infty$ *and* $s > \tau_p(w)$.

Corollary 2.2.8 applied to the case $\sigma \equiv 1$ gives in particular

$$\|J^s(fg)\|_{h^p(w)} \lesssim \|J^s f\|_{h^{p_1}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|J^s g\|_{h^{p_2}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \quad (2.2.43)$$

which supplements some of the estimates obtained in [18, Theorem 1.1] for J^s. The case $w_1 = w_2 \equiv 1$ of (2.2.43) was obtained in [41] and is an extension and an improvement of (1.0.2); indeed, (2.2.43) allows for $0 < p, p_1, p_2 < \infty$ and, when $1 < p_1, p_2 < \infty$, it improves (1.0.2) by allowing the larger quantity $\|J^s(fg)\|_{h^p}$ on the left-hand side.

Remark 2.2.9. An analogous observation to Remark 2.2.6 follows for the multiplier σ in Theorem 2.2.7 in relation to the condition (2.1.2).

2.3 Leibniz-type rules in other functions spaces

The method used to prove Theorems 2.2.1 and 2.2.7 is quite versatile and can be applied to Triebel-Lizorkin and Besov spaces that are based in other quasi-Banach spaces.

The main features of weighted Triebel-Lizorkin and Besov spaces used in the proofs of Theorems 2.2.1 and 2.2.7 are the following:

(i) there exists $r > 0$ such that $\|f + g\|^r_{F^s_{p,q}(w)} \leqslant \|f\|^r_{F^s_{p,q}(w)} + \|g\|^r_{F^s_{p,q}(w)}$; similarly for the weighted inhomogeneous Besov spaces and the weighted homogeneous Triebel–Lizorkin and Besov spaces;

(ii) Hölder's inequality in weighted Lebesgue spaces;

(iii) the boundedness properties in weighted Lebesgue spaces of the Hardy–Littlewood maximal operator (for the Besov space setting) and the weighted Fefferman–Stein inequality (for the Triebel–Lizorkin space setting);

(iv) Nikol'skiĭ representations for weighted Triebel–Lizorkin and Besov spaces (Theorem 2.1.6).

In the following subsections we consider quasi-Banach spaces $\mathcal{X}$ such that properties (i)–(iv) hold for the homogeneous and inhomogeneous $\mathcal{X}$-based Triebel-Lizorkin and Besov spaces. We show that corresponding versions of Theorems 2.2.1 and 2.2.7 hold in Triebel-Lizorkin and Besov spaces based on these spaces. The homogeneous χ-based Triebel-Lizorkin and Besov spaces, denoted by $\dot{F}^s_{\mathcal{X},q}$ and $\dot{B}^s_{\mathcal{X},q}$ respectively, are defined analogously to the weighted homogeneous Triebel-Lizorkin and Besov spaces with the quasi-norm $||\cdot||_{L^p(w)}$ replaced with the quasi-norm $||\cdot||_{\mathcal{X}}$. The inhomogeneous spaces are defined similarly.

2.3.1 Leibniz-type rules in the settings of Lorentz-based Triebel–Lizorkin and Besov spaces.

Given $0 < p < \infty$ and $0 < t \leqslant \infty$ or $p = t = \infty$, and an A_∞ weight w defined on $\mathbb{R}^n$, we denote by $L^{p,t}(w)$ the weighted Lorentz space consisting of complex-valued, measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,t}(w)} = \left(\int_0^\infty \left(\tau^{\frac{1}{p}} f^*_w(\tau)\right)^t \frac{d\tau}{\tau}\right)^{\frac{1}{t}} < \infty,$$

where $f^*_w(\tau) = \inf\{\lambda \geqslant 0 : w_f(\lambda) \leqslant \tau\}$ with $w_f(\lambda) = w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$; the obvious changes apply if $t = \infty$. It follows that $L^{p,p}(w) = L^p(w)$ for $0 < p \leqslant \infty$. We refer the reader to Hunt [35] for more details about Lorentz spaces.

The corresponding weighted inhomogeneous Triebel–Lizorkin and Besov spaces are denoted by $F^s_{(p,t),q}(w)$ and $B^s_{(p,t),q}(w)$, respectively. These spaces contain $\mathcal{S}(\mathbb{R}^n)$, are independent of the choice of φ and ψ from Section 2.1.2, are quasi-Banach spaces and have appeared in a variety of settings (see Seeger–Trebels [64] and references therein). The space $h^{p,t}(w)$ is defined in the same way as $h^p(w)$ with the quasi-norm in $L^p(w)$ replaced by the quasi-norm in $L^{p,t}(w)$.

We next consider the corresponding properties (i)-(iv) in this context. Regarding property (i), given $0 < p < \infty$, $0 < t, q \leqslant \infty$ and $s \in \mathbb{R}$, it follows that there exist $r > 0$ and a quasi-norm $|||\cdot|||_{L^{p,t}(w)(\ell^q)}$ comparable to $\|\cdot\|_{L^{p,t}(w)(\ell^q)}$ such that $|||\cdot|||^r_{L^{p,t}(w)(\ell^q)}$ is subadditive; this is an adequate substitute for property (i). The quasi-norm $|||\cdot|||_{L^{p,t}(w)(\ell^q)}$ is defined analogously to $\|\cdot\|_{L^{p,t}(w)(\ell^q)}$ by replacing $\|\cdot\|_{L^{p,t}(w)}$ with a comparable quasi-norm $|||\cdot|||_{L^{p,t}(w)}$ for which $|||\cdot|||^r_{L^{p,t}(w)}$ is subadditive (see [35, p. 258, (2.2)]). As for property (ii), weighted Lorentz spaces satisfy a Hölder-type inequality (see [35, Thm 4.5]): Given a weight w in $\mathbb{R}^n$ and indices $0 < p, p_1, p_2 < \infty$ and $0 < t, t_1, t_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, it holds that

$$\|fg\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p_1,t_1}(w)} \|g\|_{L^{p_2,t_2}(w)},$$

where the implicit constant is independent of f and g ($p_1 = t_1 = \infty$, which gives $p = p_2$ and $t_2 = t$, is also allowed). The following boundedness properties of the Hardy–Littlewood maximal operator in weighted Lorentz spaces (property (iii)) hold true: If $0 < p < \infty$, $0 < t, q \leqslant \infty$, $0 < r < \min(p/\tau_w, q)$ and $0 < r \leqslant t$, it holds that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^{p,t}(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p,t}(w)} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in L^{p,t}(w)(\ell^q); \tag{2.3.44}$$

in particular, if $0 < r < p/\tau_w$ and $0 < r \leqslant t$, it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p,t}(w)} \quad \forall f \in L^{p,t}(w).$$

When $r = 1$, $1 < p < \infty$, $1 \leqslant t \leqslant \infty$ and $1 < q \leqslant \infty$, the vector-valued inequality above follows from extrapolation and the weighted Fefferman–Stein inequality in weighted Lebesgue spaces (see [17, Theorem 4.10 and comments on p. 70] for the extrapolation theorem). The rest of the cases follow from the latter and the fact that $\||f|^s\|_{L^{p,t}(w)} = \|f\|^s_{L^{sp,st}(w)}$ for any $0 < s < \infty$. Regarding property (iv), the Nikol'skiĭ representation for $F^s_{(p,t),q}(w)$ and $B^s_{(p,t),q}(w)$ with $w \in A_\infty$ can be stated as in Theorem 2.1.6 with $0 < p < \infty$, $0 < t, q \leqslant \infty$; $s > (1/\min(p/\tau_w, t, q, 1) - 1)$ and $F^s_{p,q}(w)$ replaced with $F^s_{(p,t),q}(w)$ in the Triebel–Lizorkin setting; $s > \tau_{p,t}(w)$ and $B^s_{p,q}(w)$ replaced with $B^s_{(p,t),q}(w)$ in the Besov setting. In the context of $F^s_{(p,t),q}(w)$, the convergence of the series holds in $\mathcal{S}'(\mathbb{R}^n)$ if $t = \infty$ or $q = \infty$ and in $F^s_{(p,t),q}(w)$ otherwise; in the setting of $B^s_{(p,t),q}(w)$, the convergence of the series holds in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$ and in $B^s_{(p,t),q}(w)$ otherwise. The proofs follow parallel steps to those in the proof of Theorem 2.1.6 (see also Section 2.3.4).

As an exemplary result, we next present an analogue to Theorem 2.2.7 in the context of the spaces $F^s_{(p,t),q}(w)$. For $w \in A_\infty$, set $\tau_{p,t,q}(w) := n(1/\min(p/\tau_w, t, q, 1) - 1)$.

Theorem 2.3.1. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. If $w \in A_\infty$, $0 < p, p_1, p_2 < \infty$ and $0 < t, t_1, t_2 \leqslant \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, $0 < q \leqslant \infty$ and $s > \tau_{p,t,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|g\|_{F^{s+m}_{(p_2,t_2),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Different pairs of p_1, p_2 and t_1, t_2 can be used on the right-hand side of the inequality above. Moreover, if $w \in A_\infty$, $0 < p < \infty$, $0 < t, q \leqslant \infty$ and $s > \tau_{p,t,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{(p,t),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

The lifting property $\|f\|_{F^s_{(p,t),q}(w)} \simeq \|J^s f\|_{F^0_{(p,t),q}(w)}$ holds true for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < t, q \leqslant \infty$; this is implied by the Fefferman–Stein inequality (2.3.44) through a proof analogous to that of the lifting property of the standard Triebel–Lizorkin spaces $F^s_{p,q}$. Then, under the assumptions of Theorem 2.3.1 we obtain, in particular,

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|J^s g\|_{F^0_{(p_2,t_2),q}(w)};$$

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^0_{(p,t),q}(w)}.$$

These last two estimates supplement the results in [18, Theorem 6.1], where related Leibniz-type rules in Lorentz spaces were obtained.

2.3.2 Leibniz-type rules in the settings of Morrey-based Triebel–Lizorkin and Besov spaces.

Given $0 < p \leqslant t < \infty$ and $w \in A_\infty$, we denote by $M_p^t(w)$ the weighted Morrey space consisting of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^t(w)} = \sup_{B \subset \mathbb{R}^n} w(B)^{\frac{1}{t}-\frac{1}{p}} \left(\int_B |f(x)|^p \, w(x) \, dx \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all Euclidean balls B contained in $\mathbb{R}^n$; it easily follows that $M_p^p(w) = L^p(w)$. We refer the reader to the work Rosenthal–Schmeisser [63] and the references it contains for more details about weighted Morrey spaces. The corresponding weighted inhomogeneous Triebel–Lizorkin spaces and inhomogeneous Besov spaces are denoted by $F^s_{[p,t],q}(w)$ and $B^s_{[p,t],q}(w)$, respectively. These Morrey-based Triebel–Lizorkin and Besov spaces are independent of the choice of φ and ψ given in Section 2.1.2 and are quasi-Banach spaces that contain $\mathcal{S}(\mathbb{R}^n)$ (see the works Kozono–Yamazaki [42], Mazzucato [45], Izuki et al. [36] and the references they cotain). The corresponding local Hardy spaces are denoted by $h_p^t(w)$.

Property (i) for $F^s_{[p,t],q}(w)$ and $B^s_{[p,t],q}(w)$ is easily verified with $r = \min(p, q, 1)$ using that $\||f|^s\|_{M_p^t(w)} = \|f\|^s_{M_{sp}^{st}(w)}$ for $0 < s < \infty$. Regarding property (ii), we have that if $0 < p \leqslant t < \infty$, $0 < p_1 \leqslant t_1 < \infty$ and $0 < p_2 \leqslant t_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, then

$$\|fg\|_{M_p^t(w)} \leqslant \|f\|_{M_{p_1}^{t_1}(w)} \|g\|_{M_{p_2}^{t_2}(w)};$$

also, if $0 < p \leqslant t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $w = w_1^{p/p_1} w_2^{p/p_2}$ for weights w_1 and w_2, then

$$\|fg\|_{M_p^t(w)} \leqslant \|f\|_{M_{p_1}^{\frac{p_1 t}{p}}(w_1)} \|g\|_{M_{p_2}^{\frac{p_2 t}{p}}(w_2)}.$$

Both inequalities are straightforward consequences of Hölder's inequality for weighted Lebesgue spaces. As for property (iii), it holds that if $0<p\leqslant t<\infty$, $0<q\leqslant\infty$ and $0<r<\min(p/\tau_w,q)$, then

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{M}_r(f_j)|^q\right)^{\frac{1}{q}}\right\|_{M_p^t(w)}\lesssim\left\|\left(\sum_{j\in\mathbb{N}_0}|f_j|^q\right)^{\frac{1}{q}}\right\|_{M_p^t(w)}\qquad\forall\{f_j\}_{j\in\mathbb{N}_0}\in M_p^t(w)(\ell^q);\tag{2.3.45}$$

in particular, if $0<p\leqslant t<\infty$ and $0<r<p/\tau_w$, it holds that

$$\|\mathcal{M}_r(f)\|_{M_p^t(w)}\lesssim\|f\|_{M_p^t(w)}\quad\forall f\in M_p^t(w).$$

When $r=1$, $1<p\leqslant t<\infty$ and $1<q\leqslant\infty$, the vector-valued inequality follows from extrapolation and the weighted Fefferman–Stein inequality for weighted Lebesgue spaces (see [63, Theorem 5.3] for the corresponding extrapolation theorem). The rest of the cases follow from the latter and the fact that $\||f|^s\|_{M_p^t(w)}=\|f\|^s_{M_{sp}^{st}(w)}$ for any $0<s<\infty$. The Nikol'skiĭ representation for $F^s_{[p,t],q}(w)$ and $B^s_{[p,t],q}(w)$ with $w\in A_\infty$ (property (iv)) has an analogous statement to that of Theorem 2.1.6 with parameters $0<p\leqslant t<\infty$, $0<q\leqslant\infty$ and $L^p(w)$ replaced by $M_p^t(w)$. In the setting of $F^s_{[p,t],q}(w)$, the convergence of the series is in $\mathcal{S}'(\mathbb{R}^n)$ for any choice of parameters; in the case of $B^s_{[p,t],q}(w)$, the convergence of the series holds in $\mathcal{S}'(\mathbb{R}^n)$ if $q=\infty$ and in $B^s_{[p,t],q}(w)$ otherwise. A similar proof to that of Theorem 2.1.6 applies (see also Section 2.3.4).

Finally, we next present a counterpart of Theorem 2.2.7 in the context of $F^s_{[p,t],q}(w)$.

Theorem 2.3.2. *For $m\in\mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta\in\mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m.*

(i) If $w\in A_\infty$, $0<p\leqslant t<\infty$, $0<p_1\leqslant t_1<\infty$ and $0<p_2\leqslant t_2<\infty$ are such that $1/p=1/p_1+1/p_2$ and $1/t=1/t_1+1/t_2$, $0<q\leqslant\infty$ and $s>\tau_{p,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{[p,t],q}(w)}\lesssim\|f\|_{F^{s+m}_{[p_1,t_1],q}(w)}\|g\|_{h^{t_2}_{p_2}(w)}+\|f\|_{h^{t_1}_{p_1}(w)}\|g\|_{F^{s+m}_{[p_2,t_2],q}(w)}\quad\forall f,g\in\mathcal{S}(\mathbb{R}^n).$$

Different pairs of p_1, p_2 and t_1, t_2 can be used on the right-hand side of the inequality above. Moreover, if $w \in A_\infty$, $0 < p \leqslant t < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{[p,t],q}(w)} \lesssim \|f\|_{F^{s+m}_{[p,t],q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{[p,t],q}(w)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

(ii) If $w_1, w_2 \in A_\infty$, $w := w_1^{p/p_1} w_2^{p/p_2}$, $0 < p \leqslant t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $s > \tau_{p,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{[p,t],q}(w)} \lesssim \|f\|_{F^{s+m}_{[p_1,p_1 t/p],q}(w_1)} \|g\|_{h^{p_2 t/p}_{p_2}(w_2)} + \|f\|_{h^{p_1 t/p}_{p_1}(w_1)} \|g\|_{F^{s+m}_{[p_2,p_2 t/p],q}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

Applying the lifting property $\|f\|_{F^s_{[p,t],q}(w)} \simeq \|J^s f\|_{F^0_{[p,t],q}(w)}$, valid for $s \in \mathbb{R}$, $0 < p \leqslant t < \infty$ and $0 < q \leqslant \infty$, and under the assumptions of Theorem 2.3.2 we obtain, in particular,

$$\|J^s(fg)\|_{F^0_{[p,t],q}(w)} \lesssim \|J^s f\|_{F^0_{[p_1,t_1],q}(w)} \|g\|_{h^{t_2}_{p_2}(w)} + \|f\|_{h^{t_1}_{p_1}(w)} \|J^s g\|_{F^0_{[p_2,t_2],q}(w)}\,; \tag{2.3.46}$$

$$\|J^s(fg)\|_{F^0_{[p,t],q}(w)} \lesssim \|J^s f\|_{F^0_{[p,t],q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^0_{[p,t],q}(w)}\,;$$

$$\|J^s(fg)\|_{F^0_{[p,t],q}(w)} \lesssim \|J^s f\|_{F^0_{[p_1,p_1 t/p],q}(w_1)} \|g\|_{h^{p_2 t/p}_{p_2}(w_2)} + \|f\|_{h^{p_1 t/p}_{p_1}(w_1)} \|J^s g\|_{F^0_{[p_2,p_2 t/p],q}(w_2)}\,.$$

We refer the reader to [18, Theorem 6.3] for unweighted estimates in Morrey spaces in the spirit of (2.3.46).

2.3.3 Leibniz-type rules in the settings of variable-exponent Triebel–Lizorkin and Besov spaces.

Let $\mathcal{P}_0$ be the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ such that

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0 \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

For $p(\cdot) \in \mathcal{P}_0$, the variable-exponent Lebesgue space $L^{p(\cdot)}$ consists of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\} < \infty;$$

such quasi-norm turns $L^{p(\cdot)}$ into a quasi-Banach space (Banach space if $p_- \geqslant 1$). We note that if $p(\cdot) = p$ is constant then $L^{p(\cdot)} \simeq L^p$ with equality of norms and that

$$\left\||f|^t\right\|_{L^{p(\cdot)}} = \|f\|^t_{L^{tp(\cdot)}} \quad \forall\, t > 0. \tag{2.3.47}$$

We refer the reader to the books Cruz-Uribe–Fiorenza [16] and Diening et al. [20] for more information about variable-exponent Lebesgue spaces.

Let $\mathcal{B}$ be the family of all $p(\cdot) \in \mathcal{P}_0$ such that $\mathcal{M}$, the Hardy–Littlewood maximal operator, is bounded from $L^{p(\cdot)}$ to $L^{p(\cdot)}$. A necessary condition for $p(\cdot) \in \mathcal{B}$ is $p_- > 1$; sufficient conditions for $p(\cdot) \in \mathcal{B}$ include log-Hölder continuity assumptions. Property (2.3.47) and Jensen's inequality imply that if $p(\cdot) \in \mathcal{P}_0$ and $0 < \tau_0 < \infty$ is such that $p(\cdot)/\tau_0 \in \mathcal{B}$ then $p(\cdot)/\tau \in \mathcal{B}$ for $0 < \tau < \tau_0$. We then define

$$\tau_{p(\cdot)} = \sup\{\tau > 0 : \tfrac{p(\cdot)}{\tau} \in \mathcal{B}\}, \quad p(\cdot) \in \mathcal{P}_0^*,$$

where $\mathcal{P}_0^*$ denotes the class of variable exponents in $\mathcal{P}_0$ such that $p(\cdot)/\tau_0 \in \mathcal{B}$ for some $\tau_0 > 0$. Note that $\tau_{p(\cdot)} \leqslant p_-$.

Given $s \in \mathbb{R}$, $0 < q \leqslant \infty$ and $p(\cdot) \in \mathcal{P}_0$, the corresponding inhomogeneous Triebel-Lizorkin and Besov spaces are denoted by $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$, respectively. If $p(\cdot) \in \mathcal{P}_0^*$, these spaces are independent of the functions ψ and φ given in Section 2.1.2 (see Xu [67]), contain $\mathcal{S}(\mathbb{R}^n)$ and are quasi-Banach spaces. If $p(\cdot) \in \mathcal{B}$ and $s > 0$, $F^s_{p(\cdot),2}$ coincides with the variable-exponent Sobolev space $W^{s,p(\cdot)}$ (see Gurka et al. [31] and Xu [66]). More general versions of variable-exponent Triebel–Lizorkin and Besov spaces, where s and q are also allowed to be functions, were introduced in Diening at al. [21] and Almeida–Hästö [1], respectively. The local Hardy space with variable exponent $p(\cdot) \in \mathcal{P}_0$, denoted $h^{p(\cdot)}$, is defined analogously to $h^p(w)$ with the quasi-norm in $L^p(w)$ replaced by the quasi-norm in $L^{p(\cdot)}$.

We next consider properties (i)-(iv) in the variable-exponent setting. Given $p(\cdot) \in \mathcal{P}_0$, $0 < q \leqslant \infty$ and $s \in \mathbb{R}$, property (i) for $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$ with $r = \min(p_-, q, 1)$ follows right away using (2.3.47). Property (ii) is given by the following version of Hölder's inequality in the context of variable-exponent Lebesgue spaces: If $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot) \in \mathcal{P}_0$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ then

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} \quad \forall f \in L^{p_1(\cdot)}, g \in L^{p_2(\cdot)}.$$

For a proof with exponents in $\mathcal{P}_0$ such that $p_- \geqslant 1$ see, for instance, [16, Corollary 2.28]; the general case with exponents in $\mathcal{P}_0$ follows from the latter and (2.3.47). Regarding property (iii) for variable-exponent Lebesgue spaces, the following version of the Fefferman-Stein inequality follows from [16, Section 5.6.8] and (2.3.47): If $p(\cdot) \in \mathcal{P}_0^*$, $0 < q \leqslant \infty$ and $0 < r < \min(\tau_{p(\cdot)}, q)$ then

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j \in \mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \qquad \forall \{f_j\}_{j \in \mathbb{N}_0} \in L^{p(\cdot)}(\ell^q);$$

in particular, if $0 < r < \tau_{p(\cdot)}$ it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \quad \forall f \in L^{p(\cdot)}.$$

Finally, the following version of the Nikol'skiĭ representation for $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$ (property (iv)), can be proved along the lines of the proof of Theorem 2.1.6 (see also Section 2.3.4):

Theorem 2.3.3. *For $D > 0$, let $\{u_j\}_{j \in \mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that $\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j)$ for all $j \in \mathbb{Z}$. Let $p(\cdot) \in \mathcal{P}_0^*$, $0 < q \leqslant \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$. If $\|\{2^{js} u_j\}_{j \in \mathbb{Z}}\|_{L^{p(\cdot)}(\ell^q)} < \infty$, then the series $\sum_{j \in \mathbb{Z}} u_j$ converges in*

$F^s_{p(\cdot),q}$ *(in* $\mathcal{S}'(\mathbb{R}^n)$ *if* $q=\infty$*) and*

$$\Big\| \sum_{j\in\mathbb{N}_0} u_j \Big\|_{F^s_{p(\cdot),q}} \lesssim \big\| \{2^{js} u_j\}_{j\in\mathbb{N}_0} \big\|_{L^{p(\cdot)}(\ell^q)},$$

where the implicit constant depends only on n*,* D*,* s*,* $p(\cdot)$ *and* q*. An analogous statement holds true for* $B^s_{p(\cdot),q}$ *with* $s > n(1/\min(\tau_{p(\cdot)},1)-1)$

We next state a version of Theorem 2.2.7 for variable-exponent Triebel-Lizorkin spaces as a model result.

Theorem 2.3.4. *For* $m\in\mathbb{R}$*, let* $\sigma(\xi,\eta)$*,* $\xi,\eta\in\mathbb{R}^n$*, be an inhomogeneous Coifman-Meyer multiplier of order* m*. If* $p(\cdot),p_1(\cdot),p_2(\cdot)\in\mathcal{P}_0^*$ *are such that* $1/p(\cdot)=1/p_1(\cdot)+1/p_2(\cdot)$*,* $0<q\leqslant\infty$ *and* $s>n(1/\min(\tau_{p(\cdot)},q,1)-1)$*, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p_1(\cdot),q}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|g\|_{F^{s+m}_{p_2(\cdot),q}} \quad \forall f,g\in\mathcal{S}(\mathbb{R}^n).$$

Moreover, if $p(\cdot)\in\mathcal{P}_0^*$*,* $0<q\leqslant\infty$ *and* $s>n(1/\min(\tau_{p(\cdot)},q,1)-1)$*, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p(\cdot),q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{p(\cdot),q}} \quad \forall f,g\in\mathcal{S}(\mathbb{R}^n).$$

The lifting property $\|f\|_{F^s_{p(\cdot),q}} \simeq \|J^s f\|_{F^0_{p(\cdot),q}}$ holds true for $s\in\mathbb{R}$, $p(\cdot)\in\mathcal{P}_0^*$ and $0<q\leqslant\infty$; then, under the assumptions of Theorem 2.3.4 we obtain, in particular,

$$\|J^s(fg)\|_{F^0_{p(\cdot),q}} \lesssim \|J^s f\|_{F^0_{p_1(\cdot),q}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|J^s g\|_{F^0_{p_2(\cdot),q}};$$

$$\|J^s(fg)\|_{F^0_{p(\cdot),q}} \lesssim \|J^s f\|_{F^0_{p(\cdot),q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^0_{p(\cdot),q}}.$$

These last two estimates extend some of the inequalities in [18, Theorem 1.2], where Leibniz-type rules for the product of two functions were proved in variable-exponent Lebesgue spaces through the use of extrapolation techniques.

2.3.4 Nikol'skiĭ representations of Triebel-Lizorkin and Besov spaces

In this section, we remark on the proofs of Theorem 2.1.6 and the corresponding versions in the settings of weighted Lorentz spaces, weighted Morrey spaces, and variable-exponent Lebesgue spaces.

Regarding the proof of Part (i) of Theorem 2.1.6 (for instance, in the inhomogeneous case) the fact that $\left\|\{2^{js}u_j\}_{M+1\leqslant|j|\leqslant N}\right\|_{L^p(w)(\ell^q)}$ converges to zero, as $M,N\to\infty$, when q is finite, allows to conclude that $\sum_{j\in\mathbb{N}_0}u_j$ converges in $F^s_{p,q}(w)$ through the use of (2.1.18). Under the hypothesis of Part (i) for $\mathcal{X}=L^{p,t}(w)$ with $0<p,t<\infty$ or $\mathcal{X}=L^{p(\cdot)}$ with $p(\cdot)\in\mathcal{P}_0$ and q finite, it holds that

$$\left\|\{2^{js}u_j\}_{M+1\leqslant|j|\leqslant N}\right\|_{\mathcal{X}(\ell^q)}\to 0\quad\text{as } M,N\to\infty; \tag{2.3.48}$$

therefore, $\sum_{j\in\mathbb{N}_0}u_j$ converges in $F^s_{(p,t),q}(w)$ and $F^s_{p(\cdot),q}$, respectively.

In the case of Lorentz spaces, the fact (2.3.48) is a consequence of the following dominated convergence type theorem:

Suppose $f_n\to f$ in measure with respect to a weight w and $|f_n(x)|\leqslant|g(x)|$ a.e. for some $g\in L^{p,t}(w)$, $0<p,t<\infty$. Then $\lim\limits_{n\to\infty}\|f_n-f\|_{L^{p,t}(w)}=0$.

In the setting of variable-exponent Lebesgue spaces, (2.3.48) is a consequence of the following dominated convergence type theorem:

Suppose $f_n\to f$ pointwise a.e. and $|f_n(x)|\leqslant|g(x)|$ a.e. for some $g\in L^{p(\cdot)}$, $p(\cdot)\in\mathcal{P}_0$. Then $f_n\to f$ in $L^{p(\cdot)}$.

For the indices for which (2.3.48) does not necessarily hold under the corresponding assumptions in Part (i) ($t=\infty$ or $q=\infty$ when $\mathcal{X}=L^{p,t}(w)$, $0<p\leqslant t<\infty$ and $0<q\leqslant\infty$ when $\mathcal{X}=M^t_p(w)$, $q=\infty$ when $\mathcal{X}=L^{p(\cdot)}$), the convergence of $\sum_{j\in\mathbb{N}_0}u_j$ holds in $\mathcal{S}'(\mathbb{R}^n)$ rather than in $F^s_{(p,t),q}(w)$, $F^s_{[p,t],q}(w)$ or $F^s_{p(\cdot),q}$, respectively. Regarding Part (ii), the counterpart of (2.3.48) is

$$\left\|\{2^{js}u_j\}_{M+1\leqslant|j|\leqslant N}\right\|_{\ell^q(\mathcal{X})}\to 0\quad\text{as } M,N\to\infty,$$

which is always true under the corresponding assumptions of Part (ii) as long as q is finite, in which case the convergence of $\sum_{j\in\mathbb{N}_0} u_j$ holds in the corresponding $\mathcal{X}$-based Besov space. If $q = \infty$, the convergence is in $\mathcal{S}'(\mathbb{R}^n)$ rather than in the $\mathcal{X}$-based Besov space.

The last part of the proof of Theorem 2.1.6 uses the Fatou property of Triebel–Lizorkin and Besov spaces. Let $\mathcal{A}$ be a quasi-Banach space such that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$). The space $\mathcal{A}$ is said to have the Fatou property if for every sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ that converges in $\mathcal{S}'(\mathbb{R}^n)$ ($\mathcal{S}_0'(\mathbb{R}^n)$, respectively), as $j \to \infty$, and that satisfies $\liminf_{j\to\infty} \|f_j\|_{\mathcal{A}} < \infty$, it follows that $\lim_{j\to\infty} f_j \in \mathcal{A}$ and $\|\lim_{j\to\infty} f_j\|_{\mathcal{A}} \lesssim \liminf_{j\to\infty} \|f_j\|_{\mathcal{A}}$, where the implicit constant is independent of $\{f_j\}_{j\in\mathbb{N}}$.

It can be shown, using standard proofs, that Triebel–Lizorkin and Besov spaces based on a quasi-Banach space $\mathcal{X}$ of measurable functions (i.e. $F^s_{\mathcal{X},q}$, $B^s_{\mathcal{X},q}$ and their homogeneous counterparts) posses the Fatou property for any $s \in \mathbb{R}$ and $0 < q \leqslant \infty$ if $\mathcal{X}$ satisfies the following properties: (1) if $f, g \in \mathcal{X}$ and $|f| \leqslant |g|$ pointwise a.e., then $\|f\|_{\mathcal{X}} \lesssim \|g\|_{\mathcal{X}}$; (2) if $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{X}$ and $f_j \geqslant 0$ poinwise a.e., then $\|\liminf_{j\to\infty} f_j\|_{\mathcal{X}} \lesssim \liminf_{j\to\infty} \|f_j\|_{\mathcal{X}}$. Given a weight w, properties (1) and (2) are easily verified for $L^p(w)$ if $0 < p \leqslant \infty$, $L^{p,t}(w)$ if $0 < p < \infty$, $0 < t \leqslant \infty$, $M^t_p(w)$ if $0 < p \leqslant t < \infty$; they also hold for $L^{p(\cdot)}$ if $p(\cdot) \in \mathcal{P}_0$, as shown in [16, Theorem 2.61]. As a consequence, all the Triebel–Lizorkin and Besov spaces considered in the statements of the theorems in Sections 2.2 and 2.3 have the Fatou–Property.

2.4 Applications to scattering properties of PDEs

In this section, we discuss applications of Theorem 2.2.1, Theorem 2.2.7, and their counterparts in Section 2.3 to systems of partial differential equations involving powers of the Laplacian. The systems of partial differential equations that we study are of the form

$$\begin{cases} \partial_t u = vw, & \partial_t v + a(D)v = 0, \quad \partial_t w + b(D)w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \qquad w(0,x) = g(x), \end{cases} \tag{2.4.49}$$

where $u = u(t,x)$, $v = v(t,x)$, and $w = w(t,x)$, $t \geqslant 0$ and $x \in \mathbb{R}^n$. Here the operators $a(D)$ and $b(D)$ are *linear* Fourier multiplier operators associated to the symbols $a(\xi)$ and $b(\xi)$ respectively; that is, $\widehat{a(D)f}(\xi) = a(\xi)\widehat{f}(\xi)$ and $\widehat{b(D)f}(\xi) = b(\xi)\widehat{f}(\xi)$.

Without taking issues of convergence into account, we get that

$$v(t,x) = \int_{\mathbb{R}^n} e^{-ta(\xi)}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi. \tag{2.4.50}$$

Indeed, using the system (2.4.49) we obtain

$$\begin{aligned}\partial_t v(x) + a(D)v(x) &= \int_{\mathbb{R}^n} (\widehat{\partial_t v(t,\cdot)}(\xi) + a(\xi)\widehat{v(t,\cdot)}(\xi))e^{2\pi i\xi\cdot x}d\xi \\ &= 0,\end{aligned}$$

where the Fourier transforms $\widehat{\partial_t v(t,\cdot)}$ and $\widehat{v(t,\cdot)}$ are taken with respect to the variable x; so we must have $\widehat{\partial_t v(t,\cdot)}(\xi) + a(\xi)\widehat{v(t,\cdot)}(\xi) = 0$. By interchanging the Fourier transform with the derivative with respect to t we get $\widehat{v(t,\cdot)}(\xi) = e^{-ta(\xi)}F(\xi)$ for some function F. Setting $t = 0$ and using the system (2.4.49) it follows that $F(\xi) = \widehat{f}(\xi)$; by inverting the Fourier transform we obtain (2.4.50). A similar calculation shows that

$$w(t,x) = \int_{\mathbb{R}^n} e^{-tb(\eta)}\widehat{g}(\eta)e^{2\pi i x\cdot\eta}d\eta.$$

These expressions for v and w yield that

$$u(t,x) = \int_0^t v(s,x)w(s,x)\,ds = \int_{\mathbb{R}^{2n}} \left(\int_0^t e^{-s(a(\xi)+b(\eta))}\,ds\right)\widehat{f}(\xi)\widehat{g}(\eta)\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Setting $\lambda(\xi,\eta) = a(\xi) + b(\eta)$ and assuming that λ never vanishes, the solution $u(t,x)$ can then be written as the action on f and g of the bilinear multiplier with symbol $\frac{1-e^{-t\lambda(\xi,\eta)}}{\lambda(\xi,\eta)}$, that is,

$$u(t,x) = T_{\frac{1-e^{-t\lambda}}{\lambda}}(f,g)(x). \tag{2.4.51}$$

Following Bernicot–Germain [9, Section 9.4], suppose there exists $u_\infty \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\lim_{t\to\infty} u(t,\cdot) = u_\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n); \tag{2.4.52}$$

then, given a function space X, we say that the solution u of (2.4.49) scatters in the function space X if $u_\infty \in X$.

As an application of Theorems 2.2.1 and 2.2.7 we obtain the following scattering properties for solutions to systems of the type (2.4.49) involving powers of the Laplacian.

For $0 < p_1, p_2, p, q \leqslant \infty$ and $w_1, w_2 \in A_\infty$, set

$$\begin{aligned}
\gamma^{w_1,w_2,tl}_{p_1,p_2,p,q} &= 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}, q))] + 1),\\
\gamma^{w_1,w_2,b}_{p_1,p_2,p,q} &= 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))] + 1).
\end{aligned}$$

For $\delta > 0$ define $\mathcal{S}_\delta = \{(\xi,\eta) \in \mathbb{R}^{2n} : |\eta| \leqslant \delta^{-1}|\xi| \text{ and } |\xi| \leqslant \delta^{-1}|\eta|\}$.

Theorem 2.4.1. *Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \geqslant \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \geqslant \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}_0(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, & \partial_t v + D^\gamma v = 0, \quad \partial_t w + D^\gamma w = 0,\\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.53}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.53) scatters in $\dot{F}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.54}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, the solution u of (2.4.53) scatters in $\dot{B}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates

$$\|u_\infty\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.55}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.4.54) and (2.4.55); moreover, if $w \in A_\infty$, then

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \tau_{p,q}(w)$, and the implicit constant is independent of f and g.

For $\delta > 0$ define $\tilde{\mathcal{S}}_\delta = \{(\xi, \eta) \in \mathbb{R}^{2n} : |\eta| \leqslant \delta^{-1}(1 + |\xi|^2)^{\frac{1}{2}}$ and $|\xi| \leqslant \delta^{-1}(1 + |\eta|^2)^{\frac{1}{2}}\}$.

Theorem 2.4.2. *Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \geqslant \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \geqslant \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\widetilde{\mathcal{S}}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, \quad \partial_t v + J^\gamma v = 0, \quad \partial_t w + J^\gamma w = 0, \\ u(0,x) = 0, \quad v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.56}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.56) scatters in $F^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{F^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.57}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, *the solution u of* (2.4.56) *scatters in $B^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates*

$$\|u_\infty\|_{B^s_{p,q}(w)} \lesssim \|f\|_{B^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{B^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.58}$$

where the Hardy spaces $h^{p_1}(w_1)$ and $h^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.4.57) *and* (2.4.58)*; moreover, if $w \in A_\infty$, then*

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \tau_{p,q}(w)$, and the implicit constant is independent of f and g.

Proof of Theorem 2.4.1. We have $a(\xi) = |\xi|^\gamma$ and $b(\eta) = |\eta|^\gamma$; therefore, $\lambda(\xi,\eta) = |\xi|^\gamma + |\eta|^\gamma$. Note that all corresponding integrals for $v(t,x)$, $w(t,x)$ and $u(t,x)$ are absolutely convergent for $t > 0$, $x \in \mathbb{R}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. If we further assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$, the Dominated Convergence Theorem implies that $u(t,\cdot) \to u_\infty$ both pointwise and in $\mathcal{S}'(\mathbb{R}^n)$, where

$$u_\infty(x) = \int_{\mathbb{R}^{2n}} (a(\xi) + b(\eta))^{-1} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta = T_{\lambda^{-1}}(f,g)(x).$$

If γ is an even positive integer then λ^{-1} satisfies the estimates (2.1.1) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Then, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ and therefore the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$.

Let p_1, p_2, p, q, w_1, w_2 be as in the hypotheses. If $\gamma > 0$ and γ is not an even integer, then λ^{-1} satisfies the estimates (2.1.1) with $m = -\gamma$ as long as $\alpha, \beta \in \mathbb{N}_0^n$ are such that $|\alpha| < \gamma$ and $|\beta| < \gamma$; in particular, λ^{-1} satisfies (2.1.1) with $m = -\gamma$ for $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha + \beta| < \gamma$. In view of the Remark 2.2.6, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ if $\gamma \geqslant \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the context of Triebel–Lizorkin spaces and if $\gamma \geqslant \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the context of Besov spaces; as a consequence, the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$

for such values of γ.

On the other hand, if $0 < \gamma < \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the Triebel-Lizorkin space setting or $0 < \gamma < \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the Besov space setting, and γ is not an even positive integer, consider $h \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\mathrm{supp}(h) \subset \mathcal{S}_{\delta/2}$ and $h \equiv 1$ on $\mathcal{S}_\delta$. Then, for $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ we have $h(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta) = \widehat{f}(\xi)\widehat{g}(\eta)$; therefore, $T_{\lambda^{-1}}(f,g) = T_\Lambda(f,g)$, where $\Lambda(\xi,\eta) = h(\xi,\eta)/(|\xi|^\gamma + |\eta|^\gamma)$. The multiplier Λ verifies (2.1.1) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$ (with constants that depend on δ). Then all estimates from Theorem 2.2.1 hold for T_Λ and therefore the desired estimates follow for u_∞ with constants dependent on δ and independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$. $\square$

Proof of Theorem 2.4.2. We proceed as in the proof of Theorem 2.4.1 with $\lambda(\xi,\eta) = (1 + |\xi|^2)^{\gamma/2} + (1 + |\eta|^2)^{\gamma/2}$ and an application of Theorem 2.2.7. $\square$

% +———————————————————————————————+

Chapter 3

Bilinear Hörmander Classes and Leibniz-type rules

3.1 Introduction and main results

In this chapter, we obtain Leibniz-type rules for bilinear pseudodifferential operators associated to symbols in the Hörmander classes of critical order in the setting of Besov and local Hardy spaces.

In this section, we present the bilinear Hörmander classes and state the main results of this chapter. The notation used corresponds with that introduced in Chapter 2. In particular, L^p, $B^s_{p,q}$ and h^p denote the unweighted Lebesgue, Besov and local Hardy spaces on $\mathbb{R}^n$, respectively. We recall that $\tau_p = n(1/\min(p,1) - 1)$ for $0 < p < \infty$.

Given $0 \leqslant \delta \leqslant \rho \leqslant 1$ and $m \in \mathbb{R}$, a complex-valued function $\sigma = \sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for any multiindices $\alpha, \beta, \gamma \in \mathbb{N}^n_0$ there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\eta \sigma(x, \xi, \eta)| \leqslant C_{\alpha,\beta,\gamma}(1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta+\gamma|)} \quad \forall x, \xi, \eta \in \mathbb{R}^n. \tag{3.1.1}$$

Then for $\sigma \in BS^m_{\rho,\delta}$, the bilinear pseudodifferential operator T_σ associated to σ is defined as in (1.0.3).

Bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes have been extensively studied; see Bényi-Bernicot-Maldonado-Naibo-Torres [3], Bényi-Chaffee-Naibo [4], Bényi-Maldonado-Naibo-Torres [5], Bényi–Torres [7; 8], Brummer-Naibo [12], Herbert-Naibo [33; 34], Koezuka-Tomita [41], Michalowski-Rule-Staubach [47], Miyachi-Tomita [48–50], Naibo [53; 54], Rodríguez-López-Staubach [62], and the references therein.

One fundamental aspect of the study of symbols in the bilinear Hörmander classes is the symbolic calculus for the transposes of operators associated to them. This was established in the works Bényi-Torres [7] and Bényi-Maldonado-Naibo-Torres [5]. Another important aspect of the study of these symbols is the boundedness properties of the corresponding pseudodifferential operators in a variety of function spaces. Operators associated to symbols in $BS^0_{1,\delta}$, $0 \leqslant \delta < 1$, can be realized as Calderón-Zygmund operators. As a consequence, such operators are bounded from $L^{p_1} \times L^{p_2}$ to L^p for $1 < p_1, p_2 < \infty$ and $1/2 < p < \infty$ related through $1/p = 1/p_1 + 1/p_2$. These operators also satisfy the endpoint mappings $L^\infty \times L^\infty \to BMO$ and $L^1 \times L^1 \to L^{1/2,\infty}$, where BMO is the space of functions with bounded mean oscilation. Operators with symbols in the class $BS^0_{1,1}$ may fail to be bounded in Lebesgue spaces and are better understood in other settings. In Bényi et al. [6; 7], estimates in Sobolev spaces were obtained for such operators; for results in the settings of Besov and Triebel-Lizorkin spaces see Bényi [2], Brummer–Naibo [12], Koezuka–Tomita [41] and Naibo [53]. For $0 < \rho < 1$, unless m is sufficiently negative, the class $BS^m_{\rho,\delta}$ falls outside the bilinear Calderón-Zygmund theory.

Given $0 \leqslant \delta \leqslant \rho < 1$ and $0 < p_1, p_2, p \leqslant \infty$ related by $1/p = 1/p_1 + 1/p_2$, define

$$m(\rho, p_1, p_2) := -n(1-\rho)\max(1/2,\, 1/p_1, 1/p_2,\, 1 - 1/p, 1/p - 1/2).$$

Bényi et al. [3] proved that if $1 \leqslant p_1, p_2, p \leqslant \infty$, $m < m(\rho, p_1, p_2)$ and $\sigma \in BS^m_{\rho,\delta}$ then T_σ is bounded from $L^{p_1} \times L^{p_2}$ to L^p. On the other hand, Miyachi–Tomita [48] proved that if

$m > m(\rho, p_1, p_2)$, with $0 < p_1, p_2, p \leqslant \infty$, there are symbols in $BS^m_{\rho,\rho}$ for which the associated bilinear pseudodifferential operators are not bounded from $H^{p_1} \times H^{p_2}$ to L^p (recall that $H^r = L^r$ if $1 < r < \infty$); in the case that $p = \infty$, L^p should be replaced by BMO. As a consequence of these results, the class $BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$ is referred to as a critical class and $m(\rho, p_1, p_2)$ is called a critical order.

We now turn our attention to the critical classes. Miyachi–Tomita [48] showed that the symbols in $BS^{m(0,p_1,p_2)}_{0,0}$ with $0 < p_1, p_2, p \leqslant \infty$ give rise to operators that are bounded from $h^{p_1} \times h^{p_2}$ to h^p (recall that $h^r = L^r$ if $1 < r < \infty$), where h^r should be replaced with *bmo* if $r = \infty$. In the case that $p_1 = p_2 = \infty$, Naibo [54] proved that if σ is in the critical class $BS^{m(\rho,\infty,\infty)}_{\rho,\delta}$ with $0 \leqslant \delta \leqslant \rho < 1/2$, then T_σ is bounded from $L^\infty \times L^\infty$ to BMO. The theory of boundedness properties in the setting of Lebesgue and Hardy spaces for operators with symbols in the critical classes was completed in Miyachi–Tomita [49; 50]: operators with symbols of critical order $m(\rho, p_1, p_2)$, with $0 \leqslant \delta \leqslant \rho < 1$ and $0 < p_1, p_2, p \leqslant \infty$, are bounded from $H^{p_1} \times H^{p_2}$ to L^p, where L^p should be replaced by BMO if $p = \infty$.

In this chapter, we prove Leibniz-type rules in the setting of Besov and local Hardy spaces for bilinear pseudodifferential operators associated to symbols in the critical classes $BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$. The main result of this chapter is the following theorem.

Theorem 3.1.1. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$, $0 \leqslant \delta \leqslant \rho < 1$ and $\sigma \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$. If $s > \tau_p$, then it holds that*

$$\|T_\sigma(f, g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|g\|_{B^s_{p_2,q}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{3.1.2}$$

where h^{p_1} and h^{p_2} must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. Moreover, if there exits $\varepsilon > 0$ such that the Fourier transform of $\sigma(\cdot, \xi, \eta)$ is supported outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$, then (3.1.2) holds for any $s \in \mathbb{R}$.

Results related to estimate (3.1.2) were proved for the class $BS^0_{1,1}$ in Bényi [2], Koezuka-

Timita [41], and Naibo [53]. Concerning bilinear pseudodifferential operators with symbols belonging to the subcritical classes $BS^m_{\rho,\delta}$ with $m < m(\rho, p_1, p_2)$ and $1 \leqslant p_1, p_2, p \leqslant \infty$ and to the critical classes $BS^{m(0,p_1,p_2)}_{0,0}$ with $1 < p_1, p_2, p < \infty$, estimate (3.1.2) was shown in Naibo [53, Theorem 1.3] for $s > \tau_p$. Theorem 3.1.1 extends this result to the critical classes and allows for the regularity s to be in the wider range $(0, \infty)$ under certain assumptions on σ.

The proof of Theorem 3.1.1 uses the fact that operators with symbols in $BS^{m(0,p_1,p_2)}_{0,0}$ that are localized at certain dyadic frequencies are bounded in the setting of local Hardy spaces; no other boundedness properties of operators with symbols in the bilinear Hörmander classes are required in the proof. The tools employed are inspired by bilinear techniques in Naibo [53] and linear ones in Johnsen [37], Marschall [44] and Park [59].

As a consequence of Theorem 3.1.1, we obtain Leibniz-type rules for bilinear pseudodifferential operators associated to symbols in a general class $BS^m_{\rho,\delta}$:

Corollary 3.1.2. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$, $0 \leqslant \delta \leqslant \rho < 1$, $m \in \mathbb{R}$ and $\sigma \in BS^m_{\rho,\delta}$; set $\bar{m} = m - m(\rho, p_1, p_2)$. If $s > \tau_p$ then it holds that*

$$\|T_\sigma(f, g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^{s+\bar{m}}_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|g\|_{B^{s+\bar{m}}_{p_2,q}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{3.1.3}$$

where h^{p_1} and h^{p_2} must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. Moreover, if there exits $\varepsilon > 0$ such that the Fourier transform of $\sigma(\cdot, \xi, \eta)$ is supported outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$, then (3.1.3) holds for any $s \in \mathbb{R}$.

Remark 3.1.3. If $0 \leqslant \delta \leqslant \rho < 1$, $m < m(\rho, p_1, p_2)$ and $\sigma \in BS^m_{\rho,\delta}$ then T_σ is a smoothing operator since, in such case, $s + \bar{m} < s$ for $s, \bar{m}$ as in the statement of Corollary 3.1.2.

Remark 3.1.4. It will be clear from the proofs that different pairs of p_1, p_2, related to p through the Hölder condition, can be used in each of the terms on the right-hand sides of the estimates in Theorem 3.1.1 and Corollary 3.1.2.

Remark 3.1.5. By the lifting propterty of Besov spaces (2.1.15), the estimates (3.1.2) and (3.1.3) can be written as

$$\|J^s T_\sigma(f,g)\|_{B^0_{p,q}} \lesssim \|J^s f\|_{B^{\bar m}_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|J^s g\|_{B^{\bar m}_{p_2,q}}.$$

The organization of the rest of this chapter is as follows. In Section 3.2, we prove a maximal inequality for bilinear pseudodifferential operators that will be useful in the proof of Theorem 3.1.1. In Section 3.3, we introduce a decomposition for T_σ with $\sigma \in BS^m_{\rho,\delta}$ and prove boundedness properties for the corresponding pieces. Finally, in Section 3.4, we combine the results from Sections 3.2 and 3.3 to conclude the proofs of Theorem 3.1.1 and Corollary 3.1.2.

3.2 A maximal inequality for bilinear pseudodifferential operators

In this section, we prove the following maximal inequality for bilinear pseudodifferential operators, which will be usefull in the proof of Theorem 3.1.1. We recall that $\mathcal{M}_r f = (\mathcal{M}(|f|^r))^{1/r}$, where $\mathcal{M}$ is the Hardy-Littlewood maximal operator; for functions f and g we set $f \otimes g(x,y) = f(x)g(y)$.

Lemma 3.2.1. *Consider $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let $\sigma = \sigma(x,\xi,\eta)$ be a symbol in $C^\infty(\mathbb{R}^{3n})$ such that for some polynomial $P(\xi,\eta)$,*

$$|\sigma(x,\xi,\eta)| \lesssim P(\xi,\eta) \quad \forall x,\xi,\eta \in \mathbb{R}^n.$$

Suppose there exists $k_0 \in \mathbb{Z}$ such that

$$\operatorname{supp}(\sigma(x,\cdot,\cdot)) \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |\xi| + |\eta| \leqslant 2^{k_0}\} \quad \forall x \in \mathbb{R}^n$$

and

$$\operatorname{supp}(\widehat{f}), \operatorname{supp}(\widehat{g}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{k_0}\}.$$

If $0 < r \leqslant 1$ *and* $\|\sigma(x, 2^{k_0+1}\cdot, 2^{k_0+1}\cdot)\|_{W^{\lfloor 2n/r \rfloor+1,1}(\mathbb{R}^{2n})} \mathcal{M}_r(f \otimes g)(x,y)$ *is locally integrable in* $\mathbb{R}^{2n}$*, it holds that*

$$|T_\sigma(f,g)(x)| \lesssim \|\sigma(x, 2^{k_0+1}\cdot, 2^{k_0+1}\cdot)\|_{W^{\lfloor 2n/r \rfloor+1,1}(\mathbb{R}^{2n})} \mathcal{M}_r(f)(x)\mathcal{M}_r(g)(x) \quad \forall x \in \mathbb{R}^n, \tag{3.2.4}$$

where the implicit constant is independent of σ*,* f*,* g *and* k_0*.*

Lemma 3.2.1 will be a consequence of the following result from Marchall [44, p.118, Proposition 5(a)] and Johnsen [37, p.275, Proposition 4.1]:

Lemma 3.2.A. *Consider* $F \in \mathcal{S}(\mathbb{R}^N)$ *and let* $\Sigma = \Sigma(X,\zeta)$ *be a symbol in* $C^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ *such that for some polynomial* $P(\zeta)$*,*

$$|\Sigma(X,\zeta)| \lesssim P(\zeta) \quad \forall X, \zeta \in \mathbb{R}^N.$$

Suppose there exists $k_0 \in \mathbb{Z}$ *such that*

$$\operatorname{supp}(\Sigma(X,\cdot)) \subset \{\zeta \in \mathbb{R}^N : |\zeta| \leqslant 2^{k_0}\} \quad \forall X \in \mathbb{R}^N \text{ and } \operatorname{supp}(\widehat{F}) \subset \{\zeta \in \mathbb{R}^N : |\zeta| \leqslant 2^{k_0}\}.$$

If $0 < r \leqslant 1$ *and* $\|\Sigma(X, 2^{k_0}\cdot)\|_{W^{\lfloor N/r \rfloor+1,1}(\mathbb{R}^N)} \mathcal{M}_r(F)(X)$ *is locally integrable in* $\mathbb{R}^N$*, it holds that*

$$|T_\Sigma(F)(X)| \lesssim \|\Sigma(X, 2^{k_0}\cdot)\|_{W^{\lfloor N/r \rfloor+1,1}(\mathbb{R}^N} \mathcal{M}_r(F)(X) \quad \forall X \in \mathbb{R}^N, \tag{3.2.5}$$

where the implicit constant is independent of Σ*,* F *and* k_0*.*

Proof of Lemma 3.2.1. We have that

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta$$

can be regarded as the restriction to the diagonal in $\mathbb{R}^{2n}$ of the linear pseudodifferential operator

$$T_\Sigma(F)(X) = \int_{\mathbb{R}^{2n}} \Sigma(X,\zeta)\widehat{F}(\zeta)e^{2\pi i X\cdot\zeta}\,d\zeta$$

after setting $\zeta = (\xi,\eta)$ and defining, for $X = (x,y) \in \mathbb{R}^{2n}$,

$$\Sigma(X,\zeta) := \sigma(x,\xi,\eta) \quad \textit{and} \quad F(X) := (f\otimes g)(X) = f(x)g(y).$$

Note that $\Sigma(X,\zeta)$ is in $C^\infty(\mathbb{R}^{2n}\times\mathbb{R}^{2n})$, has polynomial growth in ζ uniformly in X, and is supported in $\{\zeta \in \mathbb{R}^{2n} : |\zeta| \leqslant 2^{k_0}\}$ for each $X \in \mathbb{R}^{2n}$; moreover $\widehat{F}(\zeta) = \widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\{\zeta \in \mathbb{R}^{2n} : |\zeta| \leqslant 2^{k_0+1}\}$. Then, (3.2.4) follows after applying Lemma 3.2.A and (3.2.5) to $T_\Sigma(F)$ and noticing that

$$\mathcal{M}_r(F)(x,x) \lesssim \mathcal{M}_r(f)(x)\mathcal{M}_r(g)(x) \quad \forall x \in \mathbb{R}^n.$$

□

Remark 3.2.2. We note that $\|\sigma(x,2^{k_0+1}\cdot,2^{k_0+1}\cdot)\|_{W^{\lfloor 2n/r\rfloor+1,1}(\mathbb{R}^{2n})}\mathcal{M}_r(f\otimes g)(x,y)$ is locally integrable in $\mathbb{R}^{2n}$ when $\|\sigma(x,2^{k_0+1}\cdot,2^{k_0+1}\cdot)\|_{W^{\lfloor 2n/r\rfloor+1,1}(\mathbb{R}^{2n})}$ is a bounded function of x since $\mathcal{M}_r(f\otimes g)(x,y)$ is locally integrable in $\mathbb{R}^{2n}$.

3.3 Decomposition of the operator T_σ and main estimates

In the proofs that follow in this chapter we implicitly assume that the symbol σ in the statements of Theorem 3.1.1 and Corollary 3.1.2 has compact support in $\mathbb{R}^{3n}$ and prove estimates for such symbols with constants independent of its support. The following limiting argument then allows us to to prove these results for symbols in the bilinear Hörmander classes without compact support. For $0 \leqslant \delta,\rho \leqslant 1$, $m \in \mathbb{R}$, $\sigma \in BS^m_{\rho,\delta}$ and $0 \leqslant \varepsilon < 1$ let

$\sigma_\epsilon(x,\xi,\eta) = \Psi(\varepsilon x, \varepsilon\xi, \varepsilon\eta)\sigma(x,\xi,\eta)$ for a smooth function Ψ of compact support such that $\Psi(0,0,0) = 1$. It follows that $\sigma_\varepsilon \in BS^m_{\rho,\delta}$ with constants independent of ε and that, as $\varepsilon \to 0$, $T_{\sigma_\varepsilon}(f,g)$ converges to $T_\sigma(f,g)$ in $\mathcal{S}'(\mathbb{R}^n)$ for $f, g \in \mathcal{S}(\mathbb{R}^n)$. Indeed, by the product rule, the facts that $\sigma \in BS^m_{\rho,\delta}$ and $0 < \varepsilon < 1$, and the properties of Ψ, we have

$$\begin{aligned}
|\partial_\xi^\alpha \partial_\eta^\beta \sigma_\varepsilon(x,\xi,\eta)| &\lesssim \sum_{\alpha_0 \leqslant \alpha, \beta_0 \leqslant \beta} \varepsilon^{|\alpha_0|+|\beta_0|} |\partial_\xi^{\alpha_0} \partial_\eta^{\beta_0} \Psi(\varepsilon x, \varepsilon\xi, \varepsilon\eta) \partial_\xi^{\alpha-\alpha_0} \partial_\eta^{\beta-\beta_0} \sigma(x,\xi,\eta)| \\
&\lesssim \sum_{\alpha_0 \leqslant \alpha, \beta_0 \leqslant \beta} (1 + |\xi| + |\eta|)^{m+\delta|\alpha-\alpha_0|-\rho|\beta-\beta_0|} \\
&= (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho|\beta|} \sum_{\alpha_0 \leqslant \alpha, \beta_0 \leqslant \beta} (1 + |\xi| + |\eta|)^{-\delta|\alpha_0|+\rho|\beta_0|} \\
&\lesssim (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho|\beta|},
\end{aligned}$$

for $(x,\xi,\eta) \in \operatorname{supp}(\Psi)$ and with the implicit constants depending only on α, β, and Ψ. Additionally, by the dominated convergence theorem and using that $\Psi(0,0,0) = 1$ we get that $T_{\sigma_\varepsilon}(f,g) \to T_\sigma(f,g)$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. Finally by the Fatou property of Besov spaces the estimates for T_σ follow from the estimates for T_{σ_ε} which are uniform in ε.

We now present the decomposition of T_σ that will be used in the proof of Theorem 3.1.1. Let $\varphi, \varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\check{\varphi}$ and $\widecheck{\varphi_0}$ satisfy (2.1.6)-(2.1.7) and (2.1.8)-(2.1.9) respectively, and assume $\sum_{k\in\mathbb{N}_0} \varphi_k \equiv 1$, where $\varphi_k(\xi) = \varphi(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant \rho < 1$ and $\sigma \in BS^m_{\rho,\delta}$. Denote by $\widehat{\sigma}^1$ the Fourier transform of $\sigma(x,\xi,\eta)$ with respect to x, that is, $\widehat{\sigma}^1(\zeta,\xi,\eta) = \widehat{\sigma(\cdot,\xi,\eta)}(\zeta)$. We perform a spectral decomposition of

$T_\sigma(f,g)$ with $f,g \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}
T_\sigma(f,g)(x) &= \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\
&= \sum_{j,k\in\mathbb{N}_0} \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} \widehat{\sigma}^1(\zeta,\xi,\eta)e^{2\pi i x\cdot\zeta}d\zeta\right)\varphi_k(\xi)\varphi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\
&= \sum_{j,k,\ell\in\mathbb{N}_0} \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} \varphi_\ell(\zeta)\widehat{\sigma}^1(\zeta,\xi,\eta)e^{2\pi i x\cdot\zeta}d\zeta\right)\varphi_k(\xi)\varphi_j(\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\
&= \sum_{j,k,\ell\in\mathbb{N}_0} T_{\sigma_{j,k,\ell}}(f,g)(x),
\end{aligned}$$

where for $j,k,\ell \in \mathbb{N}_0$ we define

$$\sigma_{j,k,\ell}(x,\xi,\eta) := \varphi_k(\xi)\varphi_j(\eta)\int_{\mathbb{R}^n} \varphi_\ell(\zeta)\widehat{\sigma}^1(\zeta,\xi,\eta)e^{2\pi i x\cdot\zeta}\,d\zeta.$$

Using this decomposition we define the following symbols:

$$\begin{aligned}
\sigma^1 &:= \sum_{\ell=4}^{\infty}\sum_{k=0}^{\ell-4}\sum_{j=0}^{k}\sigma_{j,k,\ell}, \quad &\sigma^2 &:= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{\ell=\max(0,k-3)}^{k+3}\sigma_{j,k,\ell}, \quad &\sigma^3 &:= \sum_{k=4}^{\infty}\sum_{j=0}^{k}\sum_{\ell=0}^{k-4}\sigma_{j,k,\ell}, \\
\sigma^4 &:= \sum_{\ell=5}^{\infty}\sum_{j=1}^{\ell-4}\sum_{k=0}^{j-1}\sigma_{j,k,\ell}, \quad &\sigma^5 &:= \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\sum_{\ell=\max(0,j-4)}^{j+3}\sigma_{j,k,\ell}, \quad &\sigma^6 &:= \sum_{j=5}^{\infty}\sum_{k=0}^{j-1}\sum_{\ell=0}^{j-5}\sigma_{j,k,\ell},
\end{aligned}$$

so that $\sigma = \sigma^1 + \sigma^2 + \sigma^3 + \sigma^4 + \sigma^5 + \sigma^6$. Notice that since $j \leqslant k$ in σ_1, σ_2, and σ_3, they are supported on the set $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\eta| \leqslant 4|\xi|\}$. On the other hand σ_4, σ_5, and σ_6 are supported on $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\xi| \leqslant 2|\eta|\}$. By taking the Fourier transform with respect to x we have that $\widehat{\sigma^1(\cdot,\xi,\eta)}$ is supported on $\{\zeta \in \mathbb{R}^n : |\xi| \lesssim |\zeta|\}$, $\widehat{\sigma^2(\cdot,\xi,\eta)}$ is supported on $\{\zeta \in \mathbb{R}^n : |\xi| \sim |\zeta|\}$ and $\widehat{\sigma^3(\cdot,\xi,\eta)}$ is supported on $\{\zeta \in \mathbb{R}^n : |\zeta| \lesssim |\xi|\}$. The supports of $\widehat{\sigma^4(\cdot,\xi,\eta)}$, $\widehat{\sigma^5(\cdot,\xi,\eta)}$, and $\widehat{\sigma^6(\cdot,\xi,\eta)}$ are contained in similar sets with $|\xi|$ replaced by $|\eta|$. The proof of Theorem 3.1.1 will follow from obtaining bounds for T_{σ^j}, $j = 1,2,3,4,5,6$. We will show boundedness properties for T_{σ^1}, T_{σ^2}, and T_{σ^3}; by symmetry, analogous results are obtained for T_{σ^4}, T_{σ^5}, and T_{σ^6}.

We note that $BS^m_{\rho,\delta} \subset BS^m_{\rho,\rho}$ for $m \in \mathbb{R}$ and $0 \leqslant \delta \leqslant \rho \leqslant 1$. With this in mind we will assume that $\rho = \delta$ in the proofs.

3.3.1 Estimates for T_{σ^1}

In this section, we prove the estimates for the operator T_{σ^1}. For $\{\varphi_k\}_{k\in\mathbb{N}_0}$ as in Section 3.3, we set

$$\Phi_k(\xi) := \sum_{j=0}^{k} \varphi_j(\xi) = \varphi_0(2^{-k}\xi) \quad \forall \xi \in \mathbb{R}^n, k \in \mathbb{N}_0,$$

and consider $\widetilde{\varphi}, \widetilde{\varphi}_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\check{\widetilde{\varphi}}$ and $\check{\widetilde{\varphi}}_0$ satisfying conditions (2.1.6) - (2.1.7) and (2.1.8) - (2.1.9), respectively, and such that $\widetilde{\varphi}_0\varphi_0 = \varphi_0$ and $\widetilde{\varphi}\varphi = \varphi$. We then define $\widetilde{\varphi}_k(\xi) = \widetilde{\varphi}(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$ and $\widetilde{\Phi}_k(\xi) = \widetilde{\varphi}_0(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$. It holds that $\widetilde{\varphi}_k\varphi_k = \varphi_k$ and $\widetilde{\Phi}_k\Phi_k = \Phi_k$ for all $k \in \mathbb{N}_0$.

The precise bounds for T_{σ^1} are stated in the following lemma.

Lemma 3.3.1. *Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant \rho < 1$ and $\sigma \in BS^m_{\rho,\delta}$. If $0 < p, p_1, p_2 \leqslant \infty$ are such that $1/p = 1/p_1 + 1/p_2$, $0 < q, \bar{q} \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$, it holds that*

$$\|T_{\sigma^1}(f,g)\|_{B^{s_1}_{p,q}} \lesssim \|f\|_{B^{s_2}_{p_1,\bar{q}}} \|g\|_{h^{p_2}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{3.3.6}$$

where σ^1 is as in Section 3.3 and h^{p_2} must be replaced by L^∞ if $p_2 = \infty$.

We note that in Lemma 3.3.1 there is no restriction on the order m of the symbol and the regularity indices, s_1 and s_2, can be different. In the case that $s = s_1 = s_2$ Lemma 3.3.1 implies the following estimate that is needed for the proof of Theorem 3.1.1:

$$\|T_{\sigma^1}(f,g)\|_{B^{s}_{p,q}} \lesssim \|f\|_{B^{s}_{p_1,q}} \|g\|_{h^{p_2}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

for $s \in \mathbb{R}$ and $\sigma \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$.

Proof. Let m, ρ, p, p_1, p_2, q, $\bar{q}$, s_1, s_2, σ, σ^1, f, and g be as in the statement of the lemma.

For $\ell \in \mathbb{N}_0$ set

$$\sigma_\ell^1 := \sum_{k=0}^{\ell-4} \sum_{j=0}^{k} \sigma_{j,k,\ell},$$

so that $T_{\sigma^1}(f,g) = \sum_{\ell=4}^{\infty} T_{\sigma_\ell^1}(f,g)$. Recalling the definition of Φ_k and $\sigma_{j,k,\ell}$, we have

$$T_{\sigma_\ell^1}(f,g)(x) = \int_{\mathbb{R}^{3n}} \sum_{k=0}^{\ell-4} \varphi_k(\xi)\Phi_k(\eta)\varphi_\ell(\zeta)\widehat{\sigma}^1(\zeta,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta+\zeta)}\,d\zeta\,d\xi\,d\eta,$$

and changing variables, we get

$$T_{\sigma_\ell^1}(f,g)(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \sum_{k=0}^{\ell-4} \varphi_k(\xi)\Phi_k(\eta)\varphi_\ell(\omega-\xi-\eta)\widehat{\sigma}^1(\omega-\xi-\eta,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi\,d\eta \right) e^{2\pi i x\cdot\omega}\,d\omega.$$

This yields

$$\widehat{T_{\sigma_\ell^1}(f,g)}(\omega) = \int_{\mathbb{R}^{2n}} \sum_{k=0}^{\ell-4} \varphi_k(\xi)\Phi_k(\eta)\varphi_\ell(\omega-\xi-\eta)\widehat{\sigma}^1(\omega-\xi-\eta,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi\,d\eta,$$

where the integral effectively takes place when $2^{\ell-1} \leqslant |\omega-\xi-\eta| \leqslant 2^{\ell+1}$ (keep in mind that $\ell \geqslant 4$) as well as $|\xi| \leqslant 2^{\ell-3}$ and $|\eta| \leqslant 2^{\ell-3}$. Consequently, $\widehat{T_{\sigma_\ell^1}(f,g)}$ is supported in $\{\omega \in \mathbb{R}^n : 2^{\ell-2} \leqslant |\omega| \leqslant 2^{\ell+2}\}$. Then, the fact that $T_{\sigma^1}(f,g) = \sum_{\ell=4}^{\infty} T_{\sigma_\ell^1}(f,g)$ and Theorem 2.1.6 with $w \equiv 1$ give

$$\|T_{\sigma^1}(f,g)\|_{B^{s_1}_{p,q}} \lesssim \left(\sum_{\ell=4}^{\infty} 2^{\ell s_1 q} \left\| T_{\sigma_\ell^1}(f,g) \right\|_{L^p}^q \right)^{\frac{1}{q}}. \tag{3.3.7}$$

Recalling the definitions of $\widetilde{\varphi}_k$ and $\widetilde{\Phi}_k$, we have

$$T_{\sigma_\ell^1}(f,g)(x) = \sum_{k=0}^{\ell-4} T_{\sigma_{\ell,k}^1}(\Delta_k^{\check{\widetilde{\varphi}}} f, S_k^{\check{\widetilde{\Phi}}} g)(x),$$

where, to simplify notation, we set $\Delta_0^{\check{\widetilde{\varphi}}} := S_0^{\check{\widetilde{\varphi_0}}}$ and for each k between 0 and $\ell-4$, we set

$$\sigma_{\ell,k}^1(x,\xi,\eta) := \left(\int_{\mathbb{R}^n} \varphi_\ell(\zeta)\widehat{\sigma}^1(\zeta,\xi,\eta)e^{2\pi i x\cdot\zeta}\,d\zeta \right) \varphi_k(\xi)\Phi_k(\eta).$$

Since $\sigma^1_{\ell,k}$ satisfies

$$\left|\sigma^1_{\ell,k}(x,\xi,\eta)\right| \lesssim (1+|\xi|+|\eta|)^m \quad \forall x,\xi,\eta \in \mathbb{R}^n,$$

$\sigma^1_{\ell,k}(x,\cdot,\cdot)$ is supported on $\{(\xi,\eta)\in\mathbb{R}^{2n} : |\xi|+|\eta| \leqslant 2^{k+2}\}$ for all $x\in\mathbb{R}^n$ and $\Delta_k^{\check{\check{\varphi}}} f$ and $S_k^{\check{\check{\Phi}}} g$ are Schwartz functions with Fourier transforms supported in $\{\xi\in\mathbb{R}^n : |\xi| \leqslant 2^{k+1}\}$, we can apply the bilinear inequality (3.2.4) with $0<r\leqslant 1$ to get

$$\begin{aligned}
&|T_{\sigma^1_{\ell,k}}(\Delta_k^{\check{\check{\varphi}}} f, S_k^{\check{\check{\Phi}}} g)(x)| \\
&\quad \lesssim \left\|\sigma^1_{\ell,k}(x,2^{k+3}\cdot,2^{k+3}\cdot)\right\|_{W^{\lfloor 2n/r\rfloor+1,1}(\mathbb{R}^{2n})} \mathcal{M}_r(\Delta_k^{\check{\check{\varphi}}} f)(x)\mathcal{M}_r(S_k^{\check{\check{\Phi}}} g)(x) \quad \forall x\in\mathbb{R}^n.
\end{aligned} \tag{3.3.8}$$

(See Remark 3.2.2 along with (3.3.9) below.)

We next estimate $\left\|\sigma^1_{\ell,k}(x,2^{k+3}\cdot,2^{k+3}\cdot)\right\|_{W^{\lfloor 2n/r\rfloor+1,1}(\mathbb{R}^{2n})}$. For ease of notation we just work with 2^k instead of 2^{k+3}. Notice that

$$\sigma^1_{\ell,k}(x,2^k\xi,2^k\eta) = \varphi(\xi)\varphi_0(\eta)\int_{\mathbb{R}^n} \check{\varphi}_\ell(y)\sigma(x-y,2^k\xi,2^k\eta)\,dy \quad \forall k\in\mathbb{N},$$

with a similar expression for $\sigma^1_{\ell,0}$ obtained by replacing φ with φ_0 in the formula above. For $\ell \geqslant 4$ the function $\check{\varphi}_\ell$ has vanishing moments of every order; if $N\in\mathbb{N}$, we can then write

$$\begin{aligned}
I_{\ell,k}(\xi,\eta) &:= \int_{\mathbb{R}^n} \check{\varphi}_\ell(y)\sigma(x-y,2^k\xi,2^k\eta)\,dy = 2^{n\ell}\int_{\mathbb{R}^n} \check{\varphi}(2^\ell y)\sigma(x-y,2^k\xi,2^k\eta)\,dy \\
&= 2^{n\ell}\int_{\mathbb{R}^n} \check{\varphi}(2^\ell y)\left(\sigma(x-y,2^k\xi,2^k\eta) - \sum_{|\alpha|<N}\frac{1}{\alpha!}(-y)^\alpha\partial_x^\alpha\sigma(x,2^k\xi,2^k\eta)\right)dy \\
&= 2^{n\ell}\int_{\mathbb{R}^n} \check{\varphi}(2^\ell y)\sum_{|\alpha|=N}\frac{N}{\alpha!}(-y)^\alpha\int_0^1(1-t)^{N-1}\partial_x^\alpha\sigma(x-ty,2^k\xi,2^k\eta)\,dy.
\end{aligned}$$

Given multiindices $\beta, \gamma \in \mathbb{N}_0^n$ and using that $\sigma \in BS^m_{\rho,\rho}$, it follows that

$$\begin{aligned}
&|\partial_\xi^\beta \partial_\eta^\gamma I_{\ell,k}(\xi,\eta)| \\
&= 2^{n\ell} 2^{k(|\beta+\gamma|)} \left| \int_{\mathbb{R}^n} \check{\varphi}(2^\ell y) \sum_{|\alpha|=N} \frac{N}{\alpha!}(-y)^\alpha \int_0^1 (1-t)^{N-1} \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x-ty, 2^k\xi, 2^k\eta)\, dt\, dy \right| \\
&\lesssim 2^{k(|\beta+\gamma|)} (1+|2^k\xi|+|2^k\eta|)^{m+\rho N-\rho|\beta+\gamma|} \int_{\mathbb{R}^n} 2^{n\ell} |\check{\varphi}(2^\ell y)||y|^N\, dy \\
&\lesssim 2^{-N\ell} 2^{k(|\beta+\gamma|)} (1+|2^k\xi|+|2^k\eta|)^{m+\rho N-\rho|\beta+\gamma|}.
\end{aligned}$$

Then, for (ξ,η) in the support of $\sigma^1_{\ell,k}(x, 2^k\cdot, 2^k\cdot)$ we get

$$|\partial_\xi^\beta \partial_\eta^\gamma I_{\ell,k}(\xi,\eta)| \lesssim 2^{-N\ell} 2^{k(1-\rho)|\beta+\gamma|} 2^{k(m+\rho N)}.$$

Given $0 < r \leqslant 1$, taking derivatives up to order $\lfloor 2n/r \rfloor + 1$ in (ξ,η) of $\sigma^1_{\ell,k}(x, 2^k\xi, 2^k\eta) = \varphi(\xi)\varphi_0(\eta) I_{\ell,k}(\xi,\eta)$, we obtain

$$\left\| \sigma^1_{\ell,k}(x, 2^k\cdot, 2^k\cdot) \right\|_{W^{\lfloor 2n/r \rfloor+1,1}(\mathbb{R}^{2n})} \lesssim 2^{-N\ell} 2^{k(1-\rho)(\lfloor 2n/r \rfloor+1)} 2^{k(m+\rho N)}. \tag{3.3.9}$$

From (3.3.8), it then follows that for all $x \in \mathbb{R}^n$, we have

$$|T_{\sigma^1_{\ell,k}}(\Delta_k^{\check{\check{\varphi}}} f, S_k^{\check{\check{\Phi}}} g)(x)| \lesssim 2^{-N\ell} 2^{k[(1-\rho)(\lfloor 2n/r \rfloor+1)+m+\rho N]} \mathcal{M}_r(\Delta_k^{\check{\check{\varphi}}} f)(x) \mathcal{M}_r(S_k^{\check{\check{\Phi}}} g)(x).$$

Define $\tilde{p} := \min(1,p)$. For the sake of notation, we will next work with q finite; the case $q = \infty$ can be treated analogously. Recalling that $T_{\sigma^1_\ell}(f,g) = \sum_{k=0}^{\ell-4} T_{\sigma^1_{\ell,k}}(\Delta_k^{\check{\check{\varphi}}} f, S_k^{\check{\check{\Phi}}} g)$, (3.3.7) and the last estimate give

$$\|T_{\sigma^1}(f,g)\|_{B^{s_1}_{p,q}} \lesssim \left(\sum_{\ell=4}^{\infty} 2^{\ell s_1 q}\left\|T_{\sigma^1_\ell}(f,g)\right\|_{L^p}^q\right)^{\frac{1}{q}} \tag{3.3.10}$$

$$\lesssim \left[\sum_{\ell=4}^{\infty} 2^{\ell s_1 q}\left(\sum_{k=0}^{\ell-4} 2^{-N\tilde{p}\ell}2^{k\tilde{p}[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{\tilde{p}}\right)^{\frac{q}{\tilde{p}}}\right]^{\frac{1}{q}}.$$

Next, let us see that, for $N > s_1$ and $0 < \epsilon < N - s_1$, it holds that

$$\sum_{\ell=4}^{\infty} 2^{\ell s_1 q}\left(\sum_{k=0}^{\ell-4} 2^{-N\tilde{p}\ell}2^{k\tilde{p}[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{\tilde{p}}\right)^{\frac{q}{\tilde{p}}}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1-N)+m+s_1+\varepsilon]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{q}. \tag{3.3.11}$$

Indeed, if $0 < q \leqslant \tilde{p}$, we have

$$\sum_{\ell=4}^{\infty} 2^{\ell s_1 q}\left(\sum_{k=0}^{\ell-4} 2^{-N\tilde{p}\ell}2^{k\tilde{p}[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{\tilde{p}}\right)^{\frac{q}{\tilde{p}}}$$

$$\leqslant \sum_{\ell=4}^{\infty} 2^{-(N-s_1)q\ell}\sum_{k=0}^{\ell-4} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{q}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{\ell=k+4}^{\infty} 2^{-(N-s_1)q\ell}\right)2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{q}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}2^{-kq(N-s_1)}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{q},$$

and (3.3.11) follows for any $\varepsilon > 0$. Now, if $\tilde{p} < q < \infty$ and $0 < \varepsilon < N - s_1$, we have

$$\sum_{\ell=4}^{\infty} 2^{\ell s_1 q}\left(\sum_{k=0}^{\ell-4} 2^{-N\tilde{p}\ell}2^{k\tilde{p}[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{\tilde{p}}\right)^{\frac{q}{\tilde{p}}}$$

$$\lesssim \sum_{\ell=4}^{\infty} 2^{-(N-s_1-\varepsilon)q\ell}\left(\sum_{k=0}^{\ell-4} 2^{-\varepsilon\tilde{p}k}2^{k\tilde{p}[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]}\left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}}f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}}g)\right\|_{L^p}^{\tilde{p}}\right)^{\frac{q}{\tilde{p}}}$$

$$
\begin{aligned}
&\lesssim \sum_{\ell=4}^{\infty} 2^{-(N-s_1-\varepsilon)q\ell} \sum_{k=0}^{\ell-4} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}} g)\right\|_{L^p}^q \\
&= \sum_{k=0}^{\infty} \left(\sum_{\ell=k+4}^{\infty} 2^{-(N-s_1-\varepsilon)q\ell} \right) 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}} g)\right\|_{L^p}^q \\
&\lesssim \sum_{k=0}^{\infty} 2^{-(N-s_1-\varepsilon)kq} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1)+m+\rho N]} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)\mathcal{M}_r(S_k^{\check{\tilde{\Phi}}} g)\right\|_{L^p}^q
\end{aligned}
$$

and (3.3.11) follows.

Using (3.3.11), the fact that

$$
|S_k^{\check{\tilde{\Phi}}} g(x)| \leqslant \sup_{0<t\leqslant 1} |t^{-n}\mathcal{F}^{-1}(\widetilde{\varphi}_0)(t^{-1}\cdot) * g(x)| =: g^*(x) \quad \forall k \in \mathbb{N}_0, x \in \mathbb{R}^n, \tag{3.3.12}
$$

and using the fact that $\mathcal{M}_r$ is bounded from L^p to L^p for $0 < r < \min(1,p)$, we can now continue with the inequality (3.3.10) to get

$$
\begin{aligned}
&\|T_{\sigma^1}(f,g)\|_{B^{s_1}_{p,q}} \\
&\lesssim \left[\sum_{k=0}^{\infty} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1-N)+m+s_1+\varepsilon]} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)\right\|_{L^{p_1}}^q \right]^{\frac{1}{q}} \|\mathcal{M}_r(g^*)\|_{L^{p_2}} \\
&\lesssim \left[\sum_{k=0}^{\infty} 2^{kq[(1-\rho)(\lfloor 2n/r\rfloor+1-N)+m+s_1+\varepsilon]} \left\| \Delta_k^{\check{\tilde{\varphi}}} f\right\|_{L^{p_1}}^q \right]^{\frac{1}{q}} \|g^*\|_{L^{p_2}} \\
&\lesssim \|f\|_{B^{(1-\rho)(\lfloor 2n/r\rfloor+1-N)+m+s_1+\varepsilon}_{p_1,q}} \|g\|_{h^{p_2}} ,
\end{aligned}
$$

where, if $p_2 = \infty$, $\|g\|_{h^{p_2}}$ should be replaced with $\|g\|_{L^\infty}$. Since $\rho < 1$, we can choose N large enough so that

$$
(1-\rho)(\lfloor 2n/r\rfloor + 1 - N) + m + s_1 + \varepsilon < s_2,
$$

and obtain $\|f\|_{B^{(1-\rho)(\lfloor 2n/r\rfloor+1-N)+m+s_1+\varepsilon}_{p_1,q}} \leqslant \|f\|_{B^{s_2}_{p_1,\bar{q}}}$ (by the embedding properties of Besov spaces). The proof of Lemma 3.3.1 is then complete.

□

3.3.2 Estimates for T_{σ^2}

In this section, we prove the bounds for T_{σ^2}, which are stated in the following lemma.

Lemma 3.3.2. *Let $m \in \mathbb{R}$, $0 \leqslant \delta \leqslant \rho < 1$ and $\sigma \in BS^m_{\rho,\delta}$. If $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ are such that $1/p = 1/p_1 + 1/p_2$, $0 < q, \bar{q} \leqslant \infty$, and $s_1, s_2 \in \mathbb{R}$, it holds that*

$$\|T_{\sigma^2}(f,g)\|_{B^{s_1}_{p,q}} \lesssim \|f\|_{B^{s_2}_{p_1,\bar{q}}} \|g\|_{h^{p_2}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{3.3.13}$$

where σ^2 is as in Section 3.3 and h^{p_2} must be replaced by L^∞ if $p_2 = \infty$.

Like with Lemma 3.3.1, there is no restriction on the order m of the symbol and the regularity indices can be different on the left and right hand side of (3.3.13). In particular, Lemma 3.3.2 implies the estimate

$$\|T_{\sigma^2}(f,g)\|_{B^{s}_{p,q}} \lesssim \|f\|_{B^{s}_{p_1,\bar{q}}} \|g\|_{h^{p_2}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

with all parameters as in Lemma 3.3.2 and $s \in \mathbb{R}$.

The following two lemmas, whose proofs are presented at the end of this section, will be useful in the proof of Lemma 3.3.2.

Lemma 3.3.3. *Let $\mathcal{K}_k$, $k \in \mathbb{N}_0$, be as in the proof of Lemma 3.3.2. If $J, N \in \mathbb{N}$, it holds that*

$$|\mathcal{K}_k(x,y,z)| \lesssim \frac{2^{-Jk}}{(1+|x-y|+|x-z|)^N} \quad \forall x, y, z \in \mathbb{R}^n, k \geqslant 4. \tag{3.3.14}$$

Lemma 3.3.4. *Given $M > 0$ and $N > M + n$ it holds that*

$$\int_{\mathbb{R}^n} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\, dy \lesssim 2^{kM}(1+|x-w|)^M \quad \forall x, w \in \mathbb{R}^n, k \in \mathbb{N}_0.$$

Proof of Lemma 3.3.2. Let p_1, p_2, p, q, $\bar{q}$, s_1, s_2, m, ρ and σ be as in the hypotheses of the lemma and consider φ_k, Φ_k, $\widetilde{\varphi}_k$ and $\widetilde{\Phi}_k$ as in Section 3.3.1. We assume $q < \infty$; the proof for the case $q = \infty$ is analogous.

Recall that

$$\sigma^2 = \sum_{k=0}^{\infty} \sum_{\ell=\max(0,k-3)}^{k+3} \sum_{j=0}^{k} \sigma_{j,k,\ell}$$

and write $\sigma^2 = \sigma^{2,1} + \sigma^{2,2}$, where

$$\sigma^{2,1} = \sum_{k=0}^{3} \sum_{\ell=0}^{k+3} \sum_{j=0}^{k} \sigma_{j,k,\ell} \quad \text{and} \quad \sigma^{2,2} = \sum_{k=4}^{\infty} \sum_{\ell=k-3}^{k+3} \sum_{j=0}^{k} \sigma_{j,k,\ell}.$$

Notice that the symbol $\sigma^{2,1}$ is supported on $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\xi| \leqslant 2^4 \text{ and } |\eta| \leqslant 2^4\}$ and belongs to any Hörmander class; in particular $\sigma^{2,1} \in BS_{0,0}^{m(0,p_1,p_2)}$ and by Miyachi–Tomita [48, Theorem 1.1], $T_{\sigma^{2,1}}$ is bounded from $h^{p_1} \times h^{p_2}$ to h^p (with h^{p_1} and h^{p_2} replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$). Moreover, the Fourier transform of $T_{\sigma^{2,1}}(f,g)$ is supported on $\{\zeta \in \mathbb{R}^n : |\zeta| \leqslant 2^8\}$. Let $h \in \mathcal{S}(\mathbb{R}^n)$ be compactly supported and identically one on $\{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^4\}$. We then obtain

$$\begin{aligned}
\|T_{\sigma^{2,1}}(f,g)\|_{B^{s_1}_{p,q}} &= \left(\sum_{k=0}^{8} 2^{s_1 k q} \left\| \Delta_k^{\check{\varphi}} T_{\sigma^{2,1}}(f,g) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\
&\lesssim \|T_{\sigma^{2,1}}(f,g)\|_{h^p} = \left\| T_{\sigma^{2,1}}(S_0^{\check{h}} f, g) \right\|_{h^p} \\
&\lesssim \left\| S_0^{\check{h}} f \right\|_{h^{p_1}} \|g\|_{h^{p_2}} \lesssim \|f\|_{B^{s_2}_{p_1,q}} \|g\|_{h^{p_2}} ,
\end{aligned}$$

with h^{p_1} or h^{p_2} replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

We next analyze the operator with symbol $\sigma^{2,2}$; for $k \geqslant 4$ set

$$\sigma^{2,2,k} = \sum_{\ell=k-3}^{k+3} \sum_{j=0}^{k} \sigma_{j,k,\ell},$$

so that $\sigma^{2,2} = \sum\limits_{k=4}^{\infty} \sigma^{2,2,k}$. Let $\mathcal{K}_k$ denote the bilinear kernel of $T_{\sigma^{2,2,k}}$, that is,

$$\mathcal{K}_k(x,y,z) := \int_{\mathbb{R}^{2n}} \sigma^{2,2,k}(x,\xi,\eta) e^{2\pi i \xi \cdot (x-y)} e^{2\pi i \cdot \eta (x-z)} \, d\xi \, d\eta.$$

Note that

$$|\Delta_\nu^{\check{\varphi}} T_{\sigma^{2,2,k}}(f,g)(x)| \leqslant \int_{\mathbb{R}^{3n}} |\widecheck{\varphi_\nu}(x-w)||\mathcal{K}_k(w,y,z)||\Delta_k^{\check{\tilde{\varphi}}} f(y)||S_k^{\check{\tilde{\Phi}}} g(z)|\,dw\,dy\,dz.$$

Given $0 < r < 1$, let $M \in \mathbb{N}$ be such that $M \geqslant n/r$, then

$$\begin{aligned}
|\Delta_k^{\check{\tilde{\varphi}}} f(y)| &= \frac{|\Delta_k^{\check{\tilde{\varphi}}} f(y)|}{(1+2^k|x-y|)^M}(1+2^k|x-y|)^M \\
&\leqslant (1+2^k|x-y|)^M \sup_{y\in\mathbb{R}^n} \frac{|\Delta_k^{\check{\tilde{\varphi}}} f(y)|}{(1+2^k|x-y|)^M} \\
&\lesssim (1+2^k|x-y|)^M \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)(x) \quad \forall x,y \in \mathbb{R}^n, k \in \mathbb{N}_0,
\end{aligned}$$

where for the last inequality we used Peetre's maximal inequality (see Peetre [60] or Triebel [65, p.16, Theorem 1.3.1]). Similarly, and recalling (3.3.12),

$$|S_k^{\check{\tilde{\Phi}}} g(z)| \lesssim (1+2^k|x-z|)^M \mathcal{M}_r(S_k^{\check{\tilde{\Phi}}} g)(x) \leqslant (1+2^k|x-z|)^M \mathcal{M}_r(g^*)(x)$$

for all $x, z \in \mathbb{R}^n$, $k \in \mathbb{N}_0$. Given $J, N \in \mathbb{N}$ with $N > 2(M+n)$, we use the estimates above, the fact that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and Lemmas 3.3.3 and 3.3.4 to obtain

$$\begin{aligned}
|\Delta_\nu^{\check{\varphi}} T_{\sigma^{2,2,k}}(f,g)(x)| &\lesssim 2^{-Jk} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)(x) \mathcal{M}_r(g^*)(x) \\
&\qquad \times \int_{\mathbb{R}^{3n}} \frac{2^{\nu n}}{(1+2^\nu|x-w|)^N} \frac{(1+2^k|x-y|)^M(1+2^k|x-z|)^M}{(1+|w-y|)^{N/2}(1+|w-z|)^{N/2}}\,dw\,dy\,dz \\
&\lesssim 2^{-Jk} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)(x) \mathcal{M}_r(g^*)(x) \\
&\qquad \times \int_{\mathbb{R}^n} \frac{2^{\nu n} 2^{2kM}}{(1+2^\nu|x-w|)^N}(1+|x-w|)^{2M}\,dw \quad \forall x \in \mathbb{R}^n, k, \nu \in \mathbb{N}_0, k \geqslant 4.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^n} \frac{2^{\nu n} 2^{2kM}}{(1+2^\nu|x-w|)^N}(1+|x-w|)^{2M}\,dw \lesssim 2^{2kM} \quad \forall x \in \mathbb{R}^n, k, \nu \in \mathbb{N}_0,$$

we then get

$$|\Delta_\nu^{\check{\varphi}} T_{\sigma^{2,2,k}}(f,g)(x)| \lesssim 2^{-k(J-2M)} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)(x) \mathcal{M}_r(g^*)(x) \quad \forall x \in \mathbb{R}^n, k, \nu \in \mathbb{N}_0, k \geqslant 4. \tag{3.3.15}$$

Using that the Fourier transform of $T_{\sigma^{2,2,k}}(f,g)$ is supported in $\{\zeta \in \mathbb{R}^n : |\zeta| \leqslant 2^{k+5}\}$, choosing $\varepsilon > \max(0, s_1)$ and applying (3.3.15), we have

$$\begin{aligned}
\|T_{\sigma^{2,2}}(f,g)\|_{B^{s_1}_{p,q}} &= \left(\sum_{\nu=0}^{\infty} 2^{\nu s_1 q} \left\| \Delta_\nu^{\check{\varphi}} \left(\sum_{k=4}^{\infty} T_{\sigma^{2,2,k}}(f,g) \right) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\
&\leqslant \left(\sum_{\nu=0}^{\infty} 2^{\nu s_1 q} \left\| \sum_{k=\max(4,\nu-5)}^{\infty} \left| \Delta_\nu^{\check{\varphi}} T_{\sigma^{2,2,k}}(f,g) \right| \right\|_{L^p}^q \right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{\nu=0}^{\infty} 2^{\nu s_1 q} \left\| \sum_{k=\max(4,\nu-5)}^{\infty} 2^{-k(J-2M)} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \mathcal{M}_r(g^*) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{\nu=0}^{\infty} 2^{\nu (s_1-\varepsilon) q} \left\| \sum_{k=\max(4,\nu-5)}^{\infty} 2^{-k(J-2M-\varepsilon)} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \mathcal{M}_r(g^*) \right\|_{L^p}^q \right)^{\frac{1}{q}} \\
&\lesssim \left\| \mathcal{M}_r(g^*) \sum_{k=0}^{\infty} 2^{-k(J-2M-\varepsilon)} \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \right\|_{L^p} \\
&\lesssim \left(\sum_{k=0}^{\infty} 2^{-k(J-2M-\varepsilon)\tilde{p}_1} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \right\|_{L^{p_1}}^{\tilde{p}_1} \right)^{\frac{1}{\tilde{p}_1}} \|\mathcal{M}_r(g^*)\|_{L^{p_2}},
\end{aligned}$$

where $\tilde{p_1} = \min(1, p_1)$. Using that

$$\left(\sum_{k=0}^{\infty} 2^{-k(J-2M-\varepsilon)\tilde{p}_1} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \right\|_{L^{p_1}}^{\tilde{p}_1} \right)^{\frac{1}{\tilde{p}_1}} \lesssim \left(\sum_{k=0}^{\infty} 2^{-k(J-2M-2\varepsilon)q} \left\| \mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f) \right\|_{L^{p_1}}^{q} \right)^{\frac{1}{q}},$$

along with the boundedness properties of $\mathcal{M}_r$ with $0 < r < \min(1, p_1, p_2)$, we obtain

$$\begin{aligned}\|T_{\sigma^{2,2}}(f,g)\|_{B^{s_1}_{p,q}} &\lesssim \left(\sum_{k=0}^{\infty} 2^{-k(J-2M-2\varepsilon)q} \left\|\mathcal{M}_r(\Delta_k^{\check{\tilde{\varphi}}} f)\right\|_{L^{p_1}}^q\right)^{\frac{1}{q}} \|\mathcal{M}_r(g^*)\|_{L^{p_2}} \\ &\lesssim \left(\sum_{k=0}^{\infty} 2^{-k(J-2M-2\varepsilon)q} \left\|\Delta_k^{\check{\tilde{\varphi}}} f)\right\|_{L^{p_1}}^q\right)^{\frac{1}{q}} \|g^*\|_{L^{p_2}} \\ &\lesssim \|f\|_{B^{-J+2M+2\varepsilon}_{p_1,q}} \|g\|_{h^{p_2}} \lesssim \|f\|_{B^{s_2}_{p_1,\tilde{q}}} \|g\|_{h^{p_2}},\end{aligned}$$

provided that we take J large enough so that $-J + 2M + 2\varepsilon < s_2$ and where h^{p_2} should be replaced by L^∞ if $p_2 = \infty$. □

Proof of Lemma 3.3.3. Let $k \in \mathbb{N}$ such that $k \geqslant 4$ and consider J and N as in the hypotheses. Notice that

$$\mathcal{K}_k(x,y,z) = \int_{\mathbb{R}^{4n}} \varphi_k(\xi)\Phi_k(\eta)\Psi_k(\zeta)\sigma(w,\xi,\eta)e^{2\pi i(x-w)\cdot\zeta}e^{2\pi i\xi\cdot(x-y)}e^{2\pi i\eta\cdot(x-z)}\, dw\, d\xi\, d\eta\, d\zeta,$$

where $\Phi_k = \sum_{j=0}^{k} \varphi_j$ is as in Section 3.3.1, Ψ_k is defined as $\Psi_k = \sum_{\ell=k-3}^{k+3} \varphi_\ell$ and the integral effectively takes place where $2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}$, $|\eta| \leqslant 2^{k+1}$ and $2^{k-4} \leqslant |\zeta| \leqslant 2^{k+4}$. Given $M \in \mathbb{N}$, $2M > n$, integration by parts in the w-variable gives

$$\int_{\mathbb{R}^n} \sigma(w,\xi,\eta)e^{-2\pi i w\cdot\zeta}\, dw = \frac{1}{(2\pi i|\zeta|)^{2M}} \int_{\mathbb{R}^n} \Delta_w^M \sigma(w,\xi,\eta)e^{-2\pi i w\cdot\zeta}\, dw. \tag{3.3.16}$$

Integration by parts in the ζ-variable yields

$$\int_{\mathbb{R}^n} \frac{\Psi_k(\zeta)}{|\zeta|^{2M}} e^{2\pi i(x-w)\cdot\zeta}\, d\zeta = 2^{-2Mk} \int_{\mathbb{R}^n} (I - \Delta_\zeta)^M \left(\frac{\Psi_k(\zeta)}{|2^{-k}\zeta|^{2M}}\right) \frac{e^{2\pi i(x-w)\cdot\zeta}}{(1 + 4\pi^2|x-w|^2)^M}\, d\zeta. \tag{3.3.17}$$

Assume first that $|x-y|, |x-z| \leqslant 1$. Then, using that $1 + |\xi| + |\eta| \sim |\xi| \sim |\zeta| \sim 2^k$, that $\sigma \in BS^m_{\rho,\rho}$, (3.3.16) and (3.3.17) we obtain

$$|\mathcal{K}_k(x,y,z)| \lesssim 2^{-2Mk}2^{(m+2\rho M)k}2^{3kn} = 2^{k[3n+m-2(1-\rho)M]}.$$

Then (3.3.14) follows by taking M large enough so that

$$3n + m - 2(1-\rho)M < -J, \tag{3.3.18}$$

which can be done because $\rho < 1$.

Next, assume that $|x-y| > 1$ and $|x-y| \geqslant |x-z|$. Let $j_0 \in \{1, \ldots, n\}$ be such that $|x_{j_0} - y_{j_0}| \sim |x-y|$. After performing (3.3.16) and (3.3.17), integration by parts in the ξ_{j_0}-variable implies

$$\int_{\mathbb{R}^n} \varphi_k(\xi) \Delta_w^M \sigma(w,\xi,\eta) e^{2\pi i \xi\cdot(x-y)}\, d\xi = \frac{(2\pi i)^{-N}}{(x_{j_0} - y_{j_0})^N} \int_{\mathbb{R}^n} \partial_{\xi_{j_0}}^N (\varphi_k(\xi)\Delta_w^M \sigma(w,\xi,\eta)) e^{2\pi i \xi\cdot(x-y)}\, d\xi,$$

and since $1 + |\xi| + |\eta| \sim |\xi| \sim 2^k$ and $\sigma \in BS^m_{\rho,\rho}$,

$$|\partial_{\xi_{j_0}}^N (\varphi_k(\xi)\Delta_w^M \sigma(w,\xi,\eta))| \lesssim 2^{k(m+2\rho M - \rho N)}.$$

Consequently, it holds that

$$|\mathcal{K}_k(x,y,z)| \lesssim \frac{2^{k(m-2(1-\rho)M-\rho N)} 2^{3kn}}{|x-y|^N} \sim \frac{2^{k(3n+m-2(1-\rho)M-\rho N)}}{(1+|x-y|+|x-z|)^N}.$$

The case $|x-z| > 1$ and $|x-z| \geqslant |x-y|$ follows similarly integrating by parts with respect to η and using that $1 + |\xi| + |\eta| \sim |\xi| \sim 2^k$ again. The proof of the lemma is complete by taking M such that $3n + m - 2(1-\rho)M - \rho N < -J$ which is already guaranteed by the choice (3.3.18). $\square$

Proof of Lemma 3.3.4. Fix $w, x \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $M > 0$ and $N > M + n$. We split $\mathbb{R}^n$ into the regions $R_1 := \{y \in \mathbb{R}^n : |x-y| \leqslant 2^{-k}\}$ and $R_2 := \{y \in \mathbb{R}^n : |x-y| > 2^{-k}\}$. Notice that we have $1 + 2^k|x-y| \sim 1$ on R_1 and $1 + 2^k|x-y| \sim 2^k|x-y|$ on R_2. Then we divide R_2 into

$$R_{2,1} := \{y \in R_2 : |w-y| \leqslant 1\} \quad \text{and} \quad R_{2,2} := \{y \in R_2 : |w-y| > 1\},$$

so that we have $1+|w-y|\sim 1$ on $R_{2,1}$ and $1+|w-y|\sim |w-y|$ on $R_{2,2}$. In turn, we split $R_{2,2}$ into the regions

$$R_{2,2,1} := \{y \in R_{2,2} : |x-y| \geqslant 2|x-w|\} \quad \text{and} \quad R_{2,2,2} := \{y \in R_{2,2} : |x-y| < 2|x-w|\}.$$

We then have $\mathbb{R}^n = R_1 \cup R_{2,1} \cup R_{2,2,1} \cup R_{2,2,2}$.

Using that $N > n$, it follows that

$$\int_{R_1} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\,dy \sim \int_{R_1} \frac{dy}{(1+|w-y|)^N} \lesssim 1.$$

If $|x-w| > 2$, on $R_{2,1}$ we have $|x-w| > 2 \geqslant 2|w-y|$, which makes for $|x-y|\sim|x-w|$, and then

$$\int_{R_{2,1}} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\,dy \sim \int_{R_{2,1}} 2^{kM}|x-y|^M\,dy \sim \int_{R_{2,1}} 2^{kM}|x-w|^M\,dy \lesssim 2^{kM}(1+|x-w|)^M,$$

where for the last inequality we used that $R_{2,1} \subset B(w,1)$ and that $1+|x-w| \sim |x-w|$ because $|x-w|>2$. If $|x-w|\leqslant 2$, on $R_{2,1}$ we have $|x-y| \leqslant |x-w|+|w-y| \leqslant 2+1 = 3$ and then (since $k \geqslant 0$)

$$\int_{R_{2,1}} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\,dy \lesssim 2^{kM}\int_{R_{2,1}} \frac{dy}{(1+|w-y|)^N} \sim 2^{kM}|R_{2,1}| \lesssim 2^{kM}.$$

On $R_{2,2,1}$ we have $|x-y| \geqslant 2|x-w|$, which implies $|x-y|\sim|w-y|$, and then

$$\int_{R_{2,2,1}} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\,dy \sim 2^{kM}\int_{R_{2,2,1}} \frac{|w-y|^M}{|w-y|^N}\,dy \lesssim 2^{kM},$$

since $N > M+n$.

Finally, on $R_{2,2,2}$ we have $|x-y| < 2|x-w|$ and then

$$\int_{R_{2,2,2}} \frac{(1+2^k|x-y|)^M}{(1+|w-y|)^N}\,dy \sim \int_{R_{2,2,2}} \frac{2^{kM}|x-y|^M}{|w-y|^N}\,dy$$
$$\lesssim 2^{kM}|x-w|^M \int_{R_{2,2,2}} \frac{dy}{|w-y|^N} \lesssim 2^{kM}|x-w|^M.$$

□

3.3.3 Estimates for T_{σ^3}

In this section, we prove the bounds for T_{σ^3}. Let m, p_1, p_2, p, q, ρ and δ be as in the hypotheses of Theorem 3.1.1 and $\sigma \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$. We decompose σ^3, defined in Section 3.3, as $\sigma^3 = \sigma^{3,1} + \sigma^{3,2}$ where

$$\sigma^{3,1} := \sum_{k=4}^{\infty}\sum_{j=0}^{k-4}\sum_{\ell=0}^{k-4} \sigma_{j,k,\ell} \quad \text{and} \quad \sigma^{3,2} := \sum_{k=4}^{\infty}\sum_{j=k-3}^{k}\sum_{\ell=0}^{k-4} \sigma_{j,k,\ell}.$$

It can be shown that $\sigma^{3,1}, \sigma^{3,2} \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$ and satisfy

$$\operatorname{supp}(\sigma^{3,j}) \subset \{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\eta| \leqslant A|\xi|\},$$

$$\operatorname{supp}(\widehat{\sigma^{3,j}(\cdot,\xi,\eta)}) \subset \{\zeta \in \mathbb{R}^n : |\zeta| \leqslant \frac{1}{4}|\xi|\} \quad \forall \xi,\eta \in \mathbb{R}^n.$$

In the case $j = 1$ we have that $A = 1/4$ and if $j = 2$ then $A = 4$. The following lemma implies that for $s \in \mathbb{R}$,

$$\|T_{\sigma^{3,1}}(f,g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n), \tag{3.3.19}$$

and that for $s > \tau_p$,

$$\|T_{\sigma^{3,2}}(f,g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n). \tag{3.3.20}$$

Lemma 3.3.5. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$ and $0 \leqslant \delta \leqslant \rho < 1$. Consider $\sigma \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$ such that for some positive constant A it satisfies*

$$\operatorname{supp}(\sigma) \subset \{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\eta| \leqslant A\,|\xi|\}, \tag{3.3.21}$$

$$\operatorname{supp}(\widehat{\sigma(\cdot,\xi,\eta)}) \subset \{\zeta \in \mathbb{R}^n : |\zeta| \leqslant A\,|\xi|\} \quad \forall \xi,\eta \in \mathbb{R}^n. \tag{3.3.22}$$

If $0 < A < 1/2$ and $s \in \mathbb{R}$ it holds that

$$\|T_\sigma(f,g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n),$$

where h^{p_2} must be replaced by L^∞ if $p_2 = \infty$; if $A \geqslant \frac{1}{2}$, the inequality above holds for $s > \tau_p$.

Before proving Lemma 3.3.5 we state and prove the following lemma which is useful in its proof.

Lemma 3.3.6. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$ and $0 \leqslant \delta \leqslant \rho < 1$; assume $\{\sigma_k\}_{k\in\mathbb{N}_0}$ is a sequence in $BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$ with constants uniform in k and satisfies*

$$\operatorname{supp}(\sigma_k) \subset \{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\xi| + |\eta| \sim 2^k\},$$

with constants uniform in k, where $|\xi| + |\eta| \sim 2^k$ must be replaced by $|\xi| + |\eta| \lesssim 1$ if $k = 0$. Then

$$\|T_{\sigma_k}(f,g)\|_{L^p} \lesssim \|f\|_{h^{p_1}} \|g\|_{h^{p_2}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n), k \in \mathbb{N}_0,$$

where the local Hardy spaces h^{p_1} and h^{p_2} must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$.

Proof. Let p, p_1, p_2, ρ, σ_k, f and g be as in the hypotheses of the lemma.

Define

$$\Sigma_k(x,\xi,\eta) = \sigma_k(2^{-\rho k}x, 2^{\rho k}\xi, 2^{\rho k}\eta);$$

it easily follows that

$$T_{\sigma_k}(f,g)(x) = T_{\Sigma_k}(f_k, g_k)(2^{\rho k}x), \tag{3.3.23}$$

where $f_k(x) = f(2^{-\rho k}x)$ and $g_k(x) = g(2^{-\rho k}x)$.

We next check that $\Sigma_k \in BS^{m(0,p_1,p_2)}_{0,0}$ with constants uniform in k. Note that $|\xi| + |\eta| \sim 2^{(1-\rho)k}$ for $(x,\xi,\eta) \in \operatorname{supp}(\Sigma_k)$ and $k \in \mathbb{N}$, and $|\xi|+|\eta| \lesssim 1$ for $(x,\xi,\eta) \in \operatorname{supp}(\Sigma_0)$. Using that $\sigma_k \in BS^{m(\rho,p_1,p_2)}_{\rho,\rho}$ with constants uniform in k and assuming $(x,\xi,\eta) \in \operatorname{supp}(\Sigma_k)$, we have

$$\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \Sigma_k(x,\xi,\eta)| &\lesssim (1 + |2^{\rho k}\xi| + |2^{\rho k}\eta|)^{m(\rho,p_1,p_2)+\rho|\alpha|-\rho|\beta+\gamma|} 2^{-\rho k(|\alpha|-|\beta+\gamma|)} \\
&\lesssim 2^{k(m(\rho,p_1,p_2)+\rho|\alpha|-\rho|\beta+\gamma|)} 2^{-\rho k(|\alpha|-|\beta+\gamma|)} = 2^{k(1-\rho)m(0,p_1,p_2)} \\
&\sim (1 + |\xi| + |\eta|)^{m(0,p_1,p_2)}.
\end{aligned}$$

For the sake of notation assume p_1 and p_2 are finite; the argument below works as well replacing h^{p_1} or h^{p_2} with L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. By Miyachi–Tomita [48, Theorem 1.1], we have

$$\|T_{\Sigma_k}(f,g)\|_{h^p} \lesssim \|f\|_{h^{p_1}} \|g\|_{h^{p_2}} \quad \forall k \in \mathbb{N}_0.$$

Since $T_{\Sigma_k}(f,g) \in \mathcal{S}(\mathbb{R}^n)$ for $f,g \in \mathcal{S}(\mathbb{R}^n)$, then $\|T_{\Sigma_k}(f,g)\|_{L^p} \lesssim \|T_{\Sigma_k}(f,g)\|_{h^p}$; therefore

$$\|T_{\Sigma_k}(f,g)\|_{L^p} \lesssim \|f\|_{h^{p_1}} \|g\|_{h^{p_2}} \quad \forall k \in \mathbb{N}_0.$$

Recalling (3.3.23), applying the estimate above and the fact that $\|F(\lambda\cdot)\|_{h^p} \leqslant \lambda^{-\frac{n}{p}} \|F\|_{h^p}$ for $0 < \lambda \leqslant 1$, we then obtain

$$\begin{aligned}
\|T_{\sigma_k}(f,g)\|_{L^p} &= 2^{-\rho k\frac{n}{p}} \|T_{\Sigma_k}(f_k,g_k)\|_{L^p} \lesssim 2^{-\rho k\frac{n}{p}} \|f_k\|_{h^{p_1}} \|g_k\|_{h^{p_2}} \\
&\leqslant 2^{-\rho k\frac{n}{p}} 2^{\rho k\frac{n}{p_1}} \|f\|_{h^{p_1}} 2^{\rho k\frac{n}{p_2}} \|g\|_{h^{p_2}} = \|f\|_{h^{p_1}} \|g\|_{h^{p_2}} \quad \forall k \in \mathbb{N}_0.
\end{aligned}$$

□

Proof of Lemma 3.3.5. Let p, p_1, p_2, q, s, ρ, σ, f and g be as in the hypotheses of the lemma. For the sake of notation, assume p_1 and p_2 are finite; the argument given below works as well with h^{p_1} or h^{p_2} replaced with L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

For $k \in \mathbb{N}_0$, define $\sigma_k(x,\xi,\eta) = \sigma(x,\xi,\eta)\varphi_k(\xi)$, where φ_k is as in Section 3.3; then $T_\sigma = \sum_{k=0}^{\infty} T_{\sigma_k}$. Since $\{\sigma_k\}_{k\in\mathbb{N}_0}$ satisfies the hypotheses of Lemma 3.3.6, we have

$$\|T_{\sigma_k}(f,g)\|_{L^p} \lesssim \|f\|_{h^{p_1}} \|g\|_{h^{p_2}} \quad \forall k \in \mathbb{N}_0. \tag{3.3.24}$$

The conditions on the supports of σ and $\widehat{\sigma}^1$ imply that

$$\begin{aligned}
&\operatorname{supp}(\widehat{T_{\sigma_k}(f,g)}) \subset \{\zeta \in \mathbb{R}^n : |\zeta| \lesssim 2^k\} \quad \text{if } A \geqslant \tfrac{1}{2},\\
&\operatorname{supp}(\widehat{T_{\sigma_k}(f,g)}) \subset \{\zeta \in \mathbb{R}^n : |\zeta| \sim 2^k\} \quad \text{if } 0 < A < \tfrac{1}{2},
\end{aligned}$$

with constants independent of k, f and g (in the second inclusion $|\zeta| \sim 2^k$ must be replaced with $|\zeta| \lesssim 1$ if $k = 0$). Indeed,

$$\begin{aligned}
\widehat{T_{\sigma_k}(f,g)}(\zeta) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \sigma_k(x,\xi,\eta) \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} \, d\xi \, d\eta \right) e^{-2\pi i x\cdot\zeta} \, dx\\
&= \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\widehat{\sigma_k}^1(\zeta-\xi-\eta,\xi,\eta) \, d\xi \, d\eta.
\end{aligned}$$

If $\zeta \in \operatorname{supp}(\widehat{T_{\sigma_k}(f,g)})$, in view of (3.3.21), (3.3.22) and the definition of σ_k, there exist $\xi, \eta \in \mathbb{R}^n$ such that $2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}$ ($|\xi| \leqslant 2$ if $k = 0$), $|\eta| \leqslant A\,|\xi|$ and $|\zeta - \xi - \eta| \leqslant A\,|\xi|$. This leads to

$$|\zeta| \leqslant |\zeta - \xi - \eta| + |\xi| + |\eta| \leqslant (2A+1)\,|\xi| \lesssim 2^k \quad \forall k \in \mathbb{N}_0.$$

and

$$|\zeta| \geqslant |\xi| - |\eta| - |\zeta - \xi - \eta| \geqslant (1-2A)\,|\xi| \geqslant (1-2A)2^{k-1} \quad \forall k \in \mathbb{N}, 0 < A < \tfrac{1}{2}.$$

Applying Theorem 2.1.6 with $w \equiv 1$, recalling the definition of $\widetilde{\varphi}_k$ given at the beginning of Section 3.3.1 and using (3.3.24), we obtain

$$\|T_\sigma(f,g)\|_{B^s_{p,q}} \lesssim \left(\sum_{k=0}^{\infty} 2^{ksq} \|T_{\sigma_k}(f,g)\|^q_{L^p}\right)^{\frac{1}{q}} = \left(\sum_{k=0}^{\infty} 2^{ksq} \left\|T_{\sigma_k}(\Delta_k^{\check{\widetilde{\varphi}}} f, g)\right\|^q_{L^p}\right)^{\frac{1}{q}}$$
$$\lesssim \left(\sum_{k=0}^{\infty} 2^{ksq} \left\|\Delta_k^{\check{\widetilde{\varphi}}} f\right\|^q_{h^{p_1}}\right)^{\frac{1}{q}} \|g\|_{h^{p_2}} \sim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}},$$

where in the last equivalence we have used that the Besov norm can be defined using the corresponding local Hardy space rather than the corresponding Lebesgue space. □

3.4 Conclusion of the proofs of the main results

In this section, we use Lemmas 3.3.1, 3.3.2, and 3.3.5 to prove Theorem 3.1.1. We then prove Corollary 3.1.2.

Proof of Theorem 3.1.1. If $s > \tau_s$, then (3.1.2) is a direct consequence of Lemma 3.3.1, Lemma 3.3.2, the estimates (3.3.19) and (3.3.20) that follow from Lemma 3.3.5 and corresponding versions of those results for σ^4, σ^5 and σ^6.

We next check that (3.1.2) holds for any $s \in \mathbb{R}$ if the support of the Fourier transform of $\sigma(\cdot,\xi,\eta)$ is contained outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$ and for some fixed $\varepsilon > 0$ independent of ξ and η. We first recall that the boundedness properties of the operators T_{σ^1}, T_{σ^2} and $T_{\sigma^{3,1}}$ (and the corresponding operators T_{σ^4}, T_{σ^5} and $T_{\sigma^{6,1}}$) proved in Sections 3.3.1, 3.3.2 and 3.3.3 hold for any $s \in \mathbb{R}$; on the other hand, the boundedness properties for the operator $T_{\sigma^{3,2}}$ (and the corresponding operator $T_{\sigma^{6,2}}$) proved in Section 3.3.3 hold under the condition $s > \tau_p$. Therefore, the desired result will follow from further analyzing $\sigma^{3,2}$ and $\sigma^{6,2}$.

Let $M \in \mathbb{N}$ be such that $M > 4$ and $2^{2-M} < \varepsilon$; consider the following decomposition of

$\sigma^{3,2}$:

$$\sigma^{3,2} = \sum_{k=4}^{M-1} \sum_{j=k-3}^{k} \sum_{\ell=0}^{k-4} \sigma_{j,k,l} + \sum_{k=M}^{\infty} \sum_{j=k-3}^{k} \sum_{\ell=k-M+1}^{k-4} \sigma_{j,k,l} + \sum_{k=M}^{\infty} \sum_{j=k-3}^{k} \sum_{\ell=0}^{k-M} \sigma_{j,k,l}$$
$$=: \sigma^{3,2,1} + \sigma^{3,2,2} + \sigma^{3,2,3}.$$

The operator $T_{\sigma^{3,2,1}}$ can be treated as $T_{\sigma^{2,1}}$ and the operator $T_{\sigma^{3,2,2}}$ can be treated in the same way as $T_{\sigma^{2,2}}$. Therefore $T_{\sigma^{3,2,1}}$ and $T_{\sigma^{3,2,2}}$ satisfy the same estimates as T_{σ^2}. The support of $\widehat{\sigma^{3,2,3}(\cdot,\xi},\eta)$ is contained in $\{\zeta \in \mathbb{R}^n : |\zeta| \leqslant 2^{2-M}|\xi|\} \subset \{\zeta \in \mathbb{R}^n : |\zeta| \leqslant \varepsilon|\xi|\}$ and the support of $\sigma^{3,2,3}$ is contained in $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : \frac{1}{32}|\xi| \leqslant |\eta| \leqslant 4|\xi|\}$. Therefore, $T_{\sigma^{3,2,3}}$ satisfies the same estimates as $T_{\sigma^{3,2}}$ for $s > \tau_p$.

A similar reasoning applies to $\sigma^{6,2} = \sum_{j=5}^{\infty} \sum_{k=j-3}^{j-1} \sum_{\ell=0}^{j-5} \sigma_{j,k,\ell}$. In this case, the corresponding operators with symbols $\sigma^{6,2,1}$ and $\sigma^{6,2,2}$ satisfy the estimates for any $s \in \mathbb{R}$ while the operator with symbol $\sigma^{6,2,3}$ satisfies the estimates for $s > \tau_p$. The support of the Fourier transform of $\sigma^{6,2,3}(\cdot,\xi,\eta)$ is contained in $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon\,|\eta|\}$ and the support of $\sigma^{6,2,3}$ is contained in $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : 1/2\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|\}$.

With the work above and the formulas for the symbols $\sigma^{3,2,3}$ and $\sigma^{6,2,3}$ in terms of σ we have that $\sigma^{3,2,3}$ and $\sigma^{6,2,3}$ are zero if the support of the Fourier transform of $\sigma(\cdot,\xi,\eta)$ is contained outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$. Therefore the desired result follows. $\square$

Before proving Corollary 3.1.2, we set some notation. Let $\Theta \in \mathcal{S}(\mathbb{R}^n)$ be such that $\Theta(\xi) = 1$ for $|\xi| \leqslant 8$ and $\Theta(\xi) = 0$ for $|\xi| \geqslant 16$; define $\chi(\xi,\eta) = \Theta\left(\frac{\eta}{(1+|\xi|^2)^{1/2}}\right)$. It follows that $\chi(\xi,\eta) = 1$ for $|\eta| \leqslant 4$ or $|\eta| \leqslant 4\,|\xi|$ and that the support of χ is contained in the set where $|\eta| \leqslant 16\sqrt{2}$ or $|\eta| \leqslant 32\,|\xi|$. By recalling the supports of σ^j, $j = 1, 2, \ldots, 6$ from Section

3.3 this implies that

$$\sigma^j(x,\xi,\eta)\chi(\xi,\eta) = \sigma^j(x,\xi,\eta) \text{ for } j = 1,2,3,$$
$$\sigma^j(x,\xi,\eta)\chi(\eta,\xi) = \sigma^j(x,\xi,\eta) \text{ for } j = 4,5,6.$$

Furthermore, it can be shown that given any pair of multiindices $\beta,\gamma \in \mathbb{N}_0$, there exists $C_{\beta,\gamma}$ such that

$$\left|\partial_\xi^\beta \partial_\eta^\gamma \chi(\xi,\eta)\right| \leqslant C_{\beta,\gamma}(1+|\xi|+|\eta|)^{-|\beta+\gamma|} \quad \forall \xi,\eta \in \mathbb{R}^n.$$

Therefor $\sigma(x,\xi,\eta)\chi(\xi,\eta)$ and $\sigma(x,\xi,\eta)\chi(\eta,\xi)$ belong to the same Hörmander class as σ.

Proof of Corollary 3.1.2. Let p, p_1, p_2, q, s, ρ, m and $\bar{m}$ be as in the hypotheses and consider $\sigma \in BS^m_{\rho,\rho}$. Let σ^j, $j = 1,\cdots,6$, be as in Section 3.3 and χ as described above.

It easily follows that

$$\begin{aligned} T_\sigma(f,g)(x) =& T_{\Sigma^1}(J^{\bar{m}}f,g)(x) + T_{\Sigma^2}(J^{\bar{m}}f,g)(x) + T_{\Sigma^3}(f,J^{\bar{m}}g)(x) \\ &+ T_{\Sigma^4}(f,J^{\bar{m}}g)(x) + T_{\Sigma^5}(f,J^{\bar{m}}g)(x) + T_{\Sigma^6}(f,J^{\bar{m}}g)(x), \end{aligned}$$

where $\Sigma^j(x,\xi,\eta) := \sigma^j(x,\xi,\eta)(1+|\xi|^2)^{-\bar{m}/2}$ for $j = 1,2,3$ and $\Sigma^j(x,\xi,\eta) := \sigma^j(x,\xi,\eta)(1+|\eta|^2)^{-\bar{m}/2}$ for $j = 4,5,6$.

Note that Σ^1, Σ^2 and Σ^3 are precisely the first three symbols that are obtained through the decomposition described in Section 3.3 corresponding to the symbol $\Sigma^{1,2,3}(x,\xi,\eta) := \sigma(x,\xi,\eta)\chi(\xi,\eta)(1+|\xi|^2)^{-\bar{m}/2}$; likewise, Σ^4, Σ^5 and Σ^6 are exactly the last three symbols that are obtained through the decomposition described in Section 3.3 corresponding to the symbol $\Sigma^{4,5,6}(x,\xi,\eta) := \sigma(x,\xi,\eta)\chi(\eta,\xi)(1+|\eta|^2)^{-\bar{m}/2}$. Since $\bar{m} = m - m(\rho,p_1,p_2)$ and $\sigma \in BS^m_{\rho,\rho}$, it follows that $\Sigma^{1,2,3}$ and $\Sigma^{4,5,6}$ belong to $BS^{m(\rho,p_1,p_2)}_{\rho,\rho}$. Then Lemma 3.3.1, Lemma 3.3.2 and the results from Section 3.3.3, along with their corresponding symmetric versions, can be applied to Σ^1, Σ^2, Σ^3 and Σ^4, Σ^5, Σ^6, respectively, to obtain that

$$\|T_\sigma(f,g)\|_{B^s_{p,q}} \lesssim \|J^{\bar{m}} f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|J^{\bar{m}} g\|_{B^s_{p_2,q}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

where h^{p_1} and h^{p_2} should be replaced with L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. The corollary then follows from the above estimate and the lifting property for Besov spaces. $\square$

Bibliography

[1] A. Almeida and P. Hästö. Besov spaces with variable smoothness and integrability. *J. Funct. Anal.*, 258(5):1628–1655, 2010.

[2] Á. Bényi. Bilinear pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces. *J. Math. Anal. Appl.*, 284(1):97–103, 2003.

[3] Á. Bényi, F. Bernicot, D. Maldonado, V. Naibo, and R. H. Torres. On the Hörmander classes of bilinear pseudodifferential operators II. *Indiana Univ. Math. J.*, 62(6):1733–1764, 2013.

[4] Á. Bényi, L. Chaffee, and V. Naibo. Strongly singular bilinear calderón-zygmund operators and a class of bilinear pseudodifferential operators. *J. Math. Soc. Japan*, 71(2):569–587, 2019.

[5] Á. Bényi, D. Maldonado, V. Naibo, and R. H. Torres. On the Hörmander classes of bilinear pseudodifferential operators. *Integral Equations Operator Theory*, 67(3):341–364, 2010.

[6] Á. Bényi, A. R. Nahmod, and R. H. Torres. Sobolev space estimates and symbolic calculus for bilinear pseudodifferential operators. *J. Geom. Anal.*, 16(3):431–453, 2006.

[7] Á. Bényi and R. H. Torres. Symbolic calculus and the transposes of bilinear pseudodifferential operators. *Comm. Partial Differential Equations*, 28(5-6):1161–1181, 2003.

[8] Á Bényi and R. H. Torres. Almost orthogonality and a class of bounded bilinear pseudodifferential operators. *Math. Res. Lett.*, 11(1):1–11, 2004.

[9] F. Bernicot and P. Germain. Bilinear oscillatory integrals and boundedness for new bilinear multipliers. *Adv. Math.*, 225(4):1739–1785, 2010.

[10] G. Bourdaud. L^p estimates for certain nonregular pseudodifferential operators. *Comm. Partial Differential Equations*, 7(9):1023–1033, 1982.

[11] J. Bourgain and D. Li. On an endpoint Kato-Ponce inequality. *Differential Integral Equations*, 27(11-12):1037–1072, 2014.

[12] J. Brummer and V. Naibo. Bilinear operators with homogeneous symbols, smooth molecules, and Kato-Ponce inequalities. *Proc. Amer. Math. Soc.*, 146(3):1217–1230, 2018.

[13] J. Brummer and V. Naibo. Weighted fractional Leibniz-type rules for bilinear multiplier operators. *Potential Anal.*, 2018, https://doi.org/10.1007/s11118-018-9703-9.

[14] M. Christ and M. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 100(1):87–109, 1991.

[15] R. Coifman and Y. Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société Mathématique de France, Paris, 1978.

[16] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.

[17] D. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[18] D. Cruz-Uribe and V. Naibo. Kato-Ponce inequalities on weighted and variable Lebesgue spaces. *Differential Integral Equations*, 29(9-10):801–836, 2016.

[19] G. David and J-L. Journé. A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. of Math. (2)*, 120(2):371–397, 1984.

[20] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[21] L. Diening, P. Hästö, and S. Roudenko. Function spaces of variable smoothness and integrability. *J. Funct. Anal.*, 256(6):1731–1768, 2009.

[22] M. Frazier and B. Jawerth. Decomposition of Besov spaces. *Indiana Univ. Math. J.*, 34(4):777–799, 1985.

[23] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.*, 93(1):34–170, 1990.

[24] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[25] L. Grafakos, D. Maldonado, and V. Naibo. A remark on an endpoint Kato-Ponce inequality. *Differential Integral Equations*, 27(5-6):415–424, 2014.

[26] L. Grafakos and J. M. Martell. Extrapolation of weighted norm inequalities for multivariable operators and applications. *J. Geom. Anal.*, 14(1):19–46, 2004.

[27] L. Grafakos and S. Oh. The Kato-Ponce inequality. *Comm. Partial Differential Equations*, 39(6):1128–1157, 2014.

[28] L. Grafakos and R. H. Torres. Maximal operator and weighted norm inequalities for multilinear singular integrals. *Indiana Univ. Math. J.*, 51(5):1261–1276, 2002.

[29] L. Grafakos and R. H. Torres. Multilinear Calderón-Zygmund theory. *Adv. Math.*, 165(1):124–164, 2002.

[30] A. Gulisashvili and M. Kon. Exact smoothing properties of Schrödinger semigroups. *Amer. J. Math.*, 118(6):1215–1248, 1996.

[31] P. Gurka, P. Harjulehto, and A. Nekvinda. Bessel potential spaces with variable exponent. *Math. Inequal. Appl.*, 10(3):661–676, 2007.

[32] J. Hart, R. H. Torres, and X. Wu. Smoothing properties of bilinear operators and Leibniz-type rules in Lebesgue and mixed Lebesgue spaces. *Trans. Amer. Math. Soc.*, 370(12):8581–8612, 2018.

[33] J. Herbert and V. Naibo. Bilinear pseudodifferential operators with symbols in Besov spaces. *J. Pseudo-Differ. Oper. Appl.*, 5(2):231–254, 2014.

[34] J. Herbert and V. Naibo. Besov spaces, symbolic calculus, and boundedness of bilinear pseudodifferential operators. In *Harmonic analysis, partial differential equations, complex analysis, Banach spaces, and operator theory. Vol. 1*, volume 4 of *Assoc. Women Math. Ser.*, pages 275–305. Springer, [Cham], 2016.

[35] R. Hunt. On $L(p, q)$ spaces. *Enseignement Math. (2)*, 12:249–276, 1966.

[36] M. Izuki, Y. Sawano, and H. Tanaka. Weighted Besov-Morrey spaces and Triebel-Lizorkin spaces. In *Harmonic analysis and nonlinear partial differential equations*, RIMS Kôkyûroku Bessatsu, B22, pages 21–60. Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[37] J. Johnsen. Domains of pseudo-differential operators: a case for the Triebel-Lizorkin spaces. *J. Funct. Spaces Appl.*, 3(3):263–286, 2005.

[38] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[39] C. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620, 1993.

[40] C. E. Kenig and E. M. Stein. Multilinear estimates and fractional integration. *Math. Res. Lett.*, 6(1):1–15, 1999.

[41] K. Koezuka and N. Tomita. Bilinear Pseudo-differential Operators with Symbols in $BS^m_{1,1}$ on Triebel–Lizorkin Spaces. *J. Fourier Anal. Appl.*, 24(1):309–319, 2018.

[42] H. Kozono and M. Yamazaki. Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. *Comm. Partial Differential Equations*, 19(5-6):959–1014, 1994.

[43] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, and R. Trujillo-González. New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. *Adv. Math.*, 220(4):1222–1264, 2009.

[44] J. Marschall. Nonregular pseudo-differential operators. *Z. Anal. Anwendungen*, 15(1):109–148, 1996.

[45] A. L. Mazzucato. Besov-Morrey spaces: function space theory and applications to non-linear PDE. *Trans. Amer. Math. Soc.*, 355(4):1297–1364, 2003.

[46] Y. Meyer. Remarques sur un théorème de J.-M. Bony. *Rend. Circ. Mat. Palermo (2)*, (suppl. 1):1–20, 1981.

[47] N. Michalowski, D. Rule, and W. Staubach. Multilinear pseudodifferential operators beyond Calderón-Zygmund theory. *J. Math. Anal. Appl.*, 414(1):149–165, 2014.

[48] A. Miyachi and N. Tomita. Calderón-Vaillancourt-type theorem for bilinear operators. *Indiana Univ. Math. J.*, 62(4):1165–1201, 2013.

[49] A. Miyachi and N. Tomita. Bilinear pseudo-differential operators with exotic symbols. *Ann. Inst. Fourier (Grenoble)*, pages 1–17, 2018.

[50] A. Miyachi and N. Tomita. Bilinear pseudo-differential operators with exotic symbols, II. *J. PseudoDiffer. Oper. Appl.*, 2018.

[51] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[52] C. Muscalu and W. Schlag. *Classical and multilinear harmonic analysis. Vol. II*, volume 138 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[53] V. Naibo. On the bilinear Hörmander classes in the scales of Triebel-Lizorkin and Besov spaces. *J. Fourier Anal. Appl.*, 21(5):1077–1104, 2015.

[54] V. Naibo. On the $L^\infty \times L^\infty \to BMO$ mapping property for certain bilinear pseudodifferential operators. *Proc. Amer. Math. Soc.*, 143(12):5323–5336, 2015.

[55] V. Naibo and A. Thomson. Bilinear Hörmander classes of critical order and Leibniz-type rules in Besov and local Hardy spaces. *J. Math. Anal. Appl.*, 473(2):980–1001, 2019.

[56] V. Naibo and A. Thomson. Coifman-Meyer multipliers: Leibniz-type rules and applications to scattering of solutions to PDEs. *To appear in Trans. Amer. Math. Soc.*, 2019.

[57] S. M. Nikol'skiĭ. *Approximation of functions of several variables and imbedding theorems.* Springer-Verlag, New York-Heidelberg., 1975. Translated from the Russian by John M. Danskin, Jr., Die Grundlehren der Mathematischen Wissenschaften, Band 205.

[58] S. Oh and X. Wu. On L^1 endpoint Kato-Ponce inequality. *Math. Res. Lett.*, 2019.

[59] B. J. Park. On the boundedness of pseudo-differential operators on Triebel-Lizorkin and Besov spaces. *J. Math. Anal. Appl.*, 461:544–576, 2018.

[60] J. Peetre. On spaces of Triebel-Lizorkin type. *Ark. Mat.*, 13:123–130, 1975.

[61] Bui Huy Qui. Weighted Besov and Triebel spaces: interpolation by the real method. *Hiroshima Math. J.*, 12(3):581–605, 1982.

[62] S. Rodríguez-López and W. Staubach. Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators. *J. Funct. Anal.*, 264(10):2356–2385, 2013.

[63] M. Rosenthal and H.-J. Schmeisser. The boundedness of operators in Muckenhoupt weighted Morrey spaces via extrapolation techniques and duality. *Rev. Mat. Complut.*, 29(3):623–657, 2016.

[64] A. Seeger and W. Trebels. Embeddings for spaces of Lorentz-Sobolev type. *Math. Ann.*, 2018, https://doi.org/10.1007/s00208-018-1730-8.

[65] H. Triebel. *Theory of function spaces*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of 1983 edition. Also published in 1983 by Birkhäuser Verlag.

[66] J. Xu. The relation between variable Bessel potential spaces and Triebel-Lizorkin spaces. *Integral Transforms Spec. Funct.*, 19(7-8):599–605, 2008.

[67] J. Xu. Variable Besov and Triebel-Lizorkin spaces. *Ann. Acad. Sci. Fenn. Math.*, 33(2):511–522, 2008.

[68] M. Yamazaki. A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(1):131–174, 1986.