0.1 Introduction and main results

In this chapter, we obtain Leibniz-type rules for bilinear pseudodifferential operators associated to symbols in the Hörmander classes of critical order in the setting of Besov and local Hardy spaces.

In this section, we present the bilinear Hörmander classes and state the main results of this chapter. The notation used corresponds with that introduced in Chapter **??**. In particular, L^p, $B^s_{p,q}$ and h^p denoted the unweighted Lebesgue, Besov and local Hardy spaces on $\mathbb{R}^n$, respectively. We recall that $\tau_p = n(\frac{1}{\min(p,1)} - 1)$ for $0 < p < \infty$.

Given $0 \leqslant \delta \leqslant \rho \leqslant 1$ and $m \in \mathbb{R}$, a complex-valued function $\sigma = \sigma(x, \xi, \eta)$, $x, \xi, \eta \in \mathbb{R}^n$, belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if for any multiindices $\alpha, \beta, \gamma \in \mathbb{N}^n_0$ there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\eta \sigma(x, \xi, \eta)| \leqslant C_{\alpha,\beta,\gamma}(1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta+\gamma|)} \quad \forall x, \xi, \eta \in \mathbb{R}^n. \tag{0.1.1}$$

Then for any $\sigma \in BS^m_{\rho,\delta}$, the bilinear pseudodifferential operator T_σ associated to σ is defined as in (**??**).

Bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes have been extensively studied; Lebesgue and Hardy spaces; see Bényi-Bernicot-Maldonado-Naibo-Torres [**?**], Bényi-Chaffee-Naibo [**?**], Bényi-Maldonado-Naibo-Torres [**?**], Bényi-Torres [**?**], Brummer-Naibo [**?**], Herbert-Naibo [**?**], [**?**], Koezuka-Tomita [**?**], Michalowski-Rule-Staubach [**?**], Miyachi-Tomita [**?**], Naibo [**?** **?**], Rodríguez-López-Staubach [**?**], and the references therein.

One fundamental aspect of the study of such symbols is their symbolic calculus for the transposes of operators associated to them. This was established in the works Bényi-Torres [**?**] and Bényi-Maldonado-Naibo-Torres [**?**]. Another important aspect of the study of these operators is their boundedness properties in a variety of function spaces. Operators associated to symbols in $BS^0_{1,\delta}$, $0 \leqslant \delta < 1$, can be realized as Calderón-Zygmund operators.

As a consequence, such operators are bounded from $L^{p_1} \times L^{p_2}$ to L^p for $1 < p_1, p_2 < \infty$ and $1/2 < p < \infty$ related through $1/p = 1/p_1 + 1/p_2$. These operators also satisfy the endpoint mappings $L^\infty \times L^\infty \to BMO$ and $L^1 \times L^1 \to L^{1/2,\infty}$, where BMO is the space of functions with bounded mean oscilation. Operators with symbols in the class $BS^0_{1,1}$ may fail to be bounded in Lebesgue spaces and are better understood in other settings. In Bényi et al. [? ? ?], estimates in Sobolev spaces were obtained for such operators. For results in the settings of Besov and Triebel-Lizorkin spaces see Brummer–Naibo [?], Koezuka–Tomita [?] and Naibo [?]. For $0 < \rho < 1$, unless m is sufficiently negative, the class $BS^m_{\rho,\delta}$ fall outside the bilinear Calderón-Zygmund theory.

Given $0 \leqslant \delta \leqslant \rho < 1$ and $0 < p_1, p_2, p \leqslant \infty$ related by $1/p = 1/p_1 + 1/p_2$, define

$$m(\rho, p_1, p_2) := -n(1-\rho)\max(1/2,\, 1/p_1, 1/p_2,\, 1-1/p, 1/p-1/2).$$

Bényi et al. [?] proved that if $1 \leqslant p_1, p_2, p \leqslant \infty$, $m < m(\rho, p_1, p_2)$ and $\sigma \in BS^m_{\rho,\delta}$ then T_σ is bounded from $L^{p_1} \times L^{p_2}$ to L^p. On the other hand, Miyachi–Tomita [?] proved that if $m > m(\rho, p_1, p_2)$, with $0 < p_1, p_2, p \leqslant \infty$, there are symbols in $BS^m_{\rho,\rho}$ for which the associated bilinear pseudodifferential operators are not bounded from $H^{p_1} \times H^{p_2}$ to L^p (recall that $H^p = L^p$ if $1 < p < \infty$), in the case that $p = \infty$ L^p should be replaced by BMO. As a consequence of these results, the class $BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$ is referred to as a critical class and $m(\rho, p_1, p_2)$ is called a critical order.

We now turn our attention to the critical classes. Miyachi–Tomita [?] showed that the symbols in $BS^{m(0,p_1,p_2)}_{0,0}$ with $0 < p_1, p_2, p \leqslant \infty$ give rise to operators that are bounded from $h^{p_1} \times h^{p_2}$ to h^p (recall that $h^r(\mathbb{R}^n) = L^r(\mathbb{R}^n)$, if $1 < r < \infty$) and h^r should be replaced with bmo if $r = \infty$. In the case that $p_1 = p_2 = \infty$, Naibo [?] proved that if σ is in the critical class $BS^{m(\rho,\infty,\infty)}_{\rho,\delta}$ with $0 \leqslant \delta \leqslant \rho < 1/2$ then T_σ is bounded from $L^\infty \times L^\infty$ to BMO. The theory of boundedness properties in the setting of Lebesgue and Hardy spaces for operators with symbols in the critical classes was completed in Miyachi–Tomita [? ?]: operators with symbols of critical order $m(\rho, p_1, p_2)$, with $0 \leqslant \delta \leqslant \rho < 1$ and $0 < p_1, p_2, p \leqslant \infty$, are bounded from $H^{p_1} \times H^{p_2}$ to L^p, where L^p should be replaced by BMO if $p = \infty$.

In this chapter we prove estimates in the setting of Besov and Hardy spaces for bilinear pseudodifferential operators associated to symbols in the critical classes $BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$. The main result of this chapter is the following theorem.

Theorem 0.1.1. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$, $0 \leqslant \delta \leqslant \rho < 1$ and $\sigma \in BS^{m(\rho,p_1,p_2)}_{\rho,\delta}$. If $s > \tau_p$, then it holds that*

$$\|T_\sigma(f,g)\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|g\|_{B^s_{p_2,q}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{0.1.2}$$

where h^{p_1} and h^{p_2} must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. Moreover, if there exits $\varepsilon > 0$ such that the Fourier transform of $\sigma(\cdot, \xi, \eta)$ is supported outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$, then (0.1.2) holds for any $s \in \mathbb{R}$.

Results related to estimate (0.1.2) were proved for the forbidden class $BS^0_{1,1}$ in Bényi [?], Koezuka-Timita [?], and Naibo [?]. Concerning bilinear pseudodifferential operators with symbols belonging to the subcritical classes $BS^m_{\rho,\delta}$ with $m < m(\rho, p_1, p_2)$ and $1 \leqslant p_1, p_2, p \leqslant \infty$ and in the critical classes $BS^{m(0,p_1,p_2)}_{0,0}$ with $1 < p_1, p_2, p < \infty$, estimate (0.1.2) was shown in Naibo [? , Theorem 1.3] for $s > \tau_p$. Theorem 0.1.1 extends this result to the critical classes and allows for the the regularity s to be in the wider range $(0, \infty)$ under certain assumptions on σ.

The proof of Theorem 0.1.1 uses the fact that operators with symbols in $BS^{(0,p_1,p_2)}_{0,0}$ that are localized at certain dyadic frequencies are bounded in the setting of local Hardy spaces; no other boundedness properties of operators with symbols in the bilinear Hörmander classes are required in the proof. The tools employed are inspired by bilinear techniques in Naibo [?] and linear ones in Johnsen [?], Marschall [?] and Park [?].

As a consequence of Theorem 0.1.1, we obtain Leibniz-type rules for bilinear pseudodifferential operators associated to symbols in a general class $BS^m_{\rho,\delta}$:

Corollary 0.1.2. *Let $0 < p < \infty$ and $0 < p_1, p_2 \leqslant \infty$ be such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$, $0 \leqslant \delta \leqslant \rho < 1$, $m \in \mathbb{R}$ and $\sigma \in BS^m_{\rho,\delta}$; set $\bar{m} = m - m(\rho, p_1, p_2)$. If $s > \tau_p$ then it holds that*

$$\|T_\sigma(f,g)\|_{B^p_{s,q}} \lesssim \|f\|_{B^{p_1}_{s+\bar{m},q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|g\|_{B^{p_2}_{s+\bar{m},q}} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{0.1.3}$$

where h^{p_1} and h^{p_2} must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively. Moreover, if there exits $\varepsilon > 0$ such that the Fourier transform of $\sigma(\cdot, \xi, \eta)$ is supported outside the set $\{\zeta \in \mathbb{R}^n : |\zeta| < \varepsilon(|\xi| + |\eta|)\}$ for all $\xi, \eta \in \mathbb{R}^n$ such that $1/32\,|\xi| \leqslant |\eta| \leqslant 32\,|\xi|$, then (0.1.3) holds for any $s \in \mathbb{R}$.

Remark 0.1.1. If $0 \leqslant \delta \leqslant \rho < 1$, $m < m(\rho, p_1, p_2)$ and $\sigma \in BS^m_{\rho,\delta}$ then T_σ is a smoothing operator since, in such case, $s + \bar{m} < s$ for $s, \bar{m}$ as in the statement of Corollary 0.1.2.

Remark 0.1.2. It will be clear from the proofs that different pairs of p_1, p_2, related to p through the Hölder condition, can be used in each of the terms on the right-hand sides of the estimates in Theorem 0.1.1 and Corollary 0.1.2.

Remark 0.1.3. By the lifting propterty of Besov spaces (**??**), the estimates (0.1.2) and (0.1.3) can be written as

$$\|J^s T_\sigma(f,g)\|_{B^p_{0,q}} \lesssim \|J^s f\|_{B^{p_1}_{\bar{m},q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|J^s g\|_{B^{p_2}_{\bar{m},q}}$$

The organization of the rest of this chapters is as follows. In Section **??**, we prove a maximal inequality for bilinear pseudodifferential operators that will be useful in the proof of Theorem 0.1.1. In Section **??**, we introduce a decomposition for T_σ for $\sigma \in BS^m_{\rho,\delta}$ and prove boundedness properties for the corresponding pieces. Finally, in Section **??**, we combine the results from sections **??** and **??** to conclude the proofs of Theorem 0.1.1 and Corollary 0.1.2.

Note: $\max(0, n(1/p - 1))$ appears in chapters **??** and **??** as $n(1/\min(1,p) - 1)$ or τ_p.

Lemma **??** will be a consequence of the following result from Marchall [**?** , p.118, Proposition 5(a)] and Johnsen [**?** , p.275, Proposition 4.1]:

Proof of Lemma **??**. We have that

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta$$

can be regarded as the restriction to the diagonal in $\mathbb{R}^{2n}$ of the linear pseudodifferential operator

$$T_\Sigma(F)(X) = \int_{\mathbb{R}^{2n}} \Sigma(X,\zeta)\widehat{F}(\zeta)e^{2\pi i X\cdot\zeta}\,d\zeta$$

after setting $\zeta = (\xi,\eta)$ and defining, for $X = (x,y) \in \mathbb{R}^{2n}$,

$$\Sigma(X,\zeta) := \sigma(x,\xi,\eta) \quad \textit{and} \quad F(X) := (f\otimes g)(X) = f(x)g(y).$$

Note that $\Sigma(X,\zeta)$ is in $C^\infty(\mathbb{R}^{2n}\times\mathbb{R}^{2n})$, has polynomial growth in ζ uniformly in X and is supported in $\{\zeta\in\mathbb{R}^{2n} : |\zeta| \leqslant 2^{k_0}\}$ for each $X\in\mathbb{R}^{2n}$; moreover $\widehat{F}(\zeta) = \widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\{\zeta\in\mathbb{R}^{2n} : |\zeta| \leqslant 2^{k_0+1}\}$. Then, (**??**) follows after applying Lemma **??** and (**??**) to $T_\Sigma(F)$ and noticing that

$$\mathcal{M}_r(F)(x,x) \lesssim \mathcal{M}_r(f)(x)\mathcal{M}_r(g)(x) \quad \forall x\in\mathbb{R}^n.$$

□