Bilinear pseudodifferential operators and Leibniz-type rules

by

Alexander Thomson

B.S., Missouri State University, 2013

M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

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Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction to Leibniz-type rules

Leibniz-type rules have been extensively studied due to their connections to partial differential equations which model many real world situations such as shallow water waves and fluid flow. In this chapter we introduce some of the definitions and history of the development of Leibniz-type rules which motivated the results to be discussed in chapters 2 and 3 of this manuscript. First consider the Leibniz rule taught in Calculus courses which expresses the derivatives of a product of functions as a linear combination of derivatives of the functions involved; more specifically, for functions f and g sufficiently smooth, it holds that

$$\partial_x^\alpha (fg)(x) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1\alpha_2} \partial_x^{\alpha_1} f(x) \partial_x^{\alpha_2} g(x),$$

where $\alpha \in \mathbb{N}_0^n$ and $C_{\alpha_1\alpha_2}$ are appropriate constants. The definitions in of the function spaces used below and multiindices $\alpha \in \mathbb{N}_0^n$ will be discussed in Appendix A A. In particular one term has all the derivatives on f and another with α derivatives on g and

$$\partial_x^\alpha (fg)(x) = \partial_x^\alpha f(x) g(x) + f(x) \partial_x^\alpha g(x) + ...$$

In an analogous way, fractional Leibniz rules give estimates of the smoothness and size of a product of functions in terms of the smoothness and size of the factors. For instance, for f

and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|D^s(fg)\|_{L^r} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}}\,, \tag{1.0.1}$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, $1 < p_1, p_2, q_1, q_2 \leq \infty$, $1/2 < r \leq \infty$, and $s > N(1/min(r,1) - 1)$ or s is an even natural number. The homogeneous fractional differentiation operator of order s, D^s, is defined as

$$D^s f(x) = \int_{\mathbb{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f. For $s > 0$, the operator D^s is naturally understood as taking s derivatives of its argument. Indeed, in the case $s = 2$, $D^2 f = c\Delta f$, where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian operator. Furthermore, if s is a positive integer,

$$\|D^s f\|_{L^p} \sim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^p}\,,$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$.

Another version of (1.0.1) is obtained by using the inhomogeneous sth order fractional differentiation operator J^s:

$$\|J^s(fg)\|_{L^r} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{q_2}}\,. \tag{1.0.2}$$

Similarly to its homeogenous counterpart, the operator J^s is defined through the Fourier transform as

$$J^s f(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

and can be interpreted as taking derivatives up to order s of f.

The estimates (1.0.1) and (1.0.2) are also known as Kato-Ponce inequalities due to the foundational work of Kato-Ponce[?] , where the estimate (1.0.2) was proved in the case

$1 < r = p_1 = q_2 < \infty$ and $p_2 = q_1 = \infty$, with applications to the Cauchy problem for Euler and Navier-Stokes equations. This result was extended by Gulisashvili-Kon[?], who showed (1.0.1) and (1.0.2) for the cases $s > 0$, $1 < r < \infty$, and $1 < p_1, p_2, q_1, q_2 \leq \infty$ in connection to smoothing properties of Schrödinger semigroups. Grafakos-Oh[?] and Muscalu-Schlag[?] established the cases for $1/2 < r \leq 1$ and the case $r = \infty$ was completed in the work of Bourgain-Li[?] and Grafakos-Maldonado-Naibo[?]. Applications of the estimates (1.0.1) and (1.0.2) to Korteweg-de Vries equations were studied by Christ-Weinstein[?] and Kenig-Ponce-Vega[?].

In the estimates (1.0.1) and (1.0.2) the two functions f and g are related through pointwise multiplication. Throughout the rest of this manuscript we will consider estimates similar to (1.0.1) and (1.0.2) where the two functions are related through a pseudodiffferntial operator. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function for $x, \xi, \eta \in \mathbb{R}^n$. We define the *bilinear pseudodifferential operator* associated to σ, T_σ, by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{-2\pi i x\cdot(\xi+\eta)}d\xi d\eta.$$

We refer to σ as the symbol of the operator T_σ. When σ is independent of x we call σ the multiplier of the *bilinear multiplier operator* T_σ.

Throughout this manuscript we will study estimates related to (1.0.1) and (1.0.2) associated to bilinear pseudodifferential operators that are of the form

$$\|D^s T_\sigma(f, g)\|_Z \lesssim \|D^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|D^s g\|_{Y_2}, \tag{1.0.3}$$

$$\|J^s T_\sigma(f, g)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.0.4}$$

for a variety of function spaces X_1, X_2, Y_1, Y_2, and Z.

The estimates (1.0.3) and (1.0.4) have been extensively studied in a variety of settings. In[?], Brummer-Naibo studied Leibniz-type rules in function spaces that admit a molecular decomposition and a φ-transform characterization in the sense of Frazier-Jawerth[?,?]. In the context of Lebesgue spaces and mixed Lebesgue spaces, estimates of the type (1.0.3) were studied in Hart-Torres-Wu[?] for bilinear multiplier operators with minimal smoothness assumptions on the multipliers. Related mapping properties for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes were studied by Bényi-Torres[?] and Bényi-Nahmod-Torres[?] in the setting of Sobolev spaces, by Bényi[?] in the setting of Besov spaces, and by Naibo[?] and Koezuka-Tomita[?] in the context of Triebel-Lizorkin spaces. Additionally, versions of (1.0.1) and (1.0.2) in weighted Lebesgue spaces were proved in Cruz-Uribe-Naibo[?], while Brummer-Naibo[?] proved (1.0.3) and (1.0.4) in weighted Lebesgue spaces for Coifman-Meyer multiplier operators.

In chapter two we will present Leibniz-type rules in the settings of Besov and Triebel-Lizorkin spaces based in various function spaces. The techniques used are quite flexible and allow the method of proof to be adapted to many different function spaces.

In chapter three we will present Leibniz-type rules in Besov spaces for bilinear pseudodifferential operators symbols of critical order.

Chapter 2

Bilinear multiplier operators and applications to scattering properties of certain systems of PDEs

2.1 Introduction

In this chapter we discuss pseudodifferential operators associated to Coifman-Meyer multipliers in the settings of weighted Triebel-Lizorkin and Besov spaces. Additionally we obtain applications of these results to scattering properties of certains systems of partial differential equations. To do this we first focus our attention to the following theorem which serves as a model for the other results later in this chapter.

Theorem 2.1.1. *For $m \in \mathbb{R}$, let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leq \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.1.1}$$

If $0 < p, p_1, p_2 \le \infty$ *and* $s > \tau_p(w)$*, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.1.2}$$

where the Hardy spaces $H^{p_1}(w_1)$ *and* $H^{p_2}(w_2)$ *must be replaced by* L^∞ *if* $p_1 = \infty$ *or* $p_2 = \infty$*, respectively.*

If $w_1 = w_2$ *then different pairs of* p_1, p_2 *can be used on the right-hand sides of* (2.1.1) *and* (2.1.2)*; moreover, if* $w \in A_\infty$*, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.1.3}$$

where $0 < p < \infty$, $0 < q \le \infty$ *and* $s > \tau_{p,q}(w)$.

2.2 Definitions

2.2.1 Coifman-Meyer Multipliers

The symbols used in Theorem (2.1.1) and results later in this chapter are Coifman-Meyer multipliers. Such multipliers are defined as follows.

Definition 2.2.1. For $m \in \mathbb{R}^n$, a smooth function $\sigma = \sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is a Coifman-Meyer multipliers of order m if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi,\eta) \in \mathbb{R}^{2n} \setminus \{(0,0)\}. \tag{2.2.4}$$

An introduction to these multipliers and boundedness properties of operators associated to them can be found in the pioneering work Coifman-Meyer[?] . Further work was done in Grafakos-Torres[?] where Coifman-Meyer multipliers were studied in their connection to Calderón-Zygmund operators, which are a more general than the class of operators associated to Coifman-Meyer multipliers. In particular they obtain that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|f\|_{L_{p_1}} \|g\|_{L^{p_2}}$$

where σ is a Coifman-Meyer multiplier, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $1 < p_1, p_2 < \infty$.

2.2.2 Weighted spaces

Definition 2.2.2. A *weight* $w(x)$ defined on $\mathbb{R}^n$ is a nonnegative, locally integrable function such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$.

Given a weight $w(x)$ and $0 < p < \infty$ we define the weighted Lebesgue spaces $L^p(w)$ as the spaces of measurable functions satisfying

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

In the case that $p = \infty$ the space $L^\infty(w) = L^\infty$.

The specific classes of weights that are in the hypotheses of our results are *Muckenhoupt weights*.

Definition 2.2.3. For $1 < p < \infty$ the *Muckenhoupt weight class* A_p is the collection of weights satisfying

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B.

From this definition it follows that $A_p \subset A_q$ when $p \leq q$. For $p = \infty$ the class $A_\infty = \cup_{p>1} A_p$ and for $w \in A_\infty$ we set $\tau_w = inf\{\tau \in (1,\infty] : w \in A_\tau\}$. The condition (2.2.3) is motivated by the fact that the Hardy-Littlewood maximal function $\mathcal{M}(f)(x)$ is bounded on $L^p(w)$. That is for $1 < p < \infty$ w/inA_p if and only if

$$\|\mathcal{M}(f)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Later in this chapter we use the maximal function $\mathcal{M}_r(f) = (\mathcal{M}(|f|^r))^{\frac{1}{r}}$ and by using the properties above for the Hardy-Littlewood maximal function we have that for $0 < r < p$ $\mathcal{M}_r$ is bounded on $L^p(w)$ for $w \in A_{p/r}$ and in this case $0 < r < p/tau_w$. We will also use a weighted Fefferman-Stein inequality that is the vector-valued version of the previous statement statement.

Theorem 2.2.1. *If $0 < p < \infty$, $0 < q \leq \infty$, $0 < r < \min(p,q)$ and $w \in A_{p/r}$ (i.e. $0 < r < \min(p/\tau_w, q)$), then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)},$$

where the implicit constant depends on r, p, q, and w and the summation in j should be replaced by the supremum in j if $q = \infty$.

For more details on the Muckenhoupt classes see Grafakos**?** .

2.3 Function spaces

In this subsection we detail the function spaces that are to be used throughout the rest of the chapter. In particular we focus on the Triebel-Lizorkin and Besov spaces based in quasi-Banach spaces and Hardy spaces.

2.3.1 Triebel-Lizorkin and Besov spaces

Let ψ and φ be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the following conditions:

- *supp*$(\widehat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}\}$,
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ and some $c > 0$,
- $\mathrm{supp}(\widehat{\varphi}) \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$,

- $|\widehat{\varphi}(\xi)| > c \quad$ for all ξ such that $|\xi| < \frac{5}{3}$ and some $c > 0$.

For any $\psi, f \in \mathcal{S}(\mathbb{R}^n)$ such that ψ is supported in an annulus we define the operator $\Delta_j^\psi f$ in terms of its fourier transform as $\widehat{\Delta^\psi f}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$. Similarly if $\varphi, f \in \mathcal{S}(\mathbb{R}^n)$ such that φ is supported in a ball we define the operator $S_j^\varphi f$ as $\widehat{S_j^\varphi f}(\xi) = \varphi(2^{-j}\xi)\widehat{f}(\xi)$.

Definition 2.3.1. Let χ be a quasi-Banach space , $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The *homogeneous χ-based Triebel-Lizorkin space* $\dot{F}^s_{\chi,q}$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that
$$\|f\|_{\dot{F}^s_{\chi,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

- The *inhomogeneous χ-based Triebel-Lizorkin space* $F^s_{\chi,q}$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$\|f\|_{F^s_{\chi,q}} = \|S_0^\varphi f\|_\chi + \left\| \left(\sum_{j\in\mathbb{N}} (2^{sj}|\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_\chi < \infty.$$

Here we note that if $\Delta_j^\psi f = 0$ then the tempered distribution $\widehat{f}$ must be supported at the origin and is therefore f must be a polynomial. So for $\|\cdot\|_{\dot{F}^s_{\chi,q}}$ to be a quasi-norm we must consider tempered distributions modulo polynomials as in the definition. In a similar way we define homogeneous and inhomogeneous χ-based Besov spaces.

Definition 2.3.2. Let χ be a quasi-Banach space , $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The *homogeneous χ-based Besov space* $B^s_{\chi,q}$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$\|f\|_{\dot{B}^s_{\chi,q}} = \left(\sum_{j\in\mathbb{Z}} (2^{sj} \left\| \Delta_j^\psi f \right\|_\chi)^q \right)^{\frac{1}{q}}. \tag{2.3.5}$$

- The *inhomogeneous χ-based Besov space* $B^s_{\chi,q}$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that
$$\|f\|_{B^s_{\chi,q}} = \|S_0^\varphi f\|_\chi + \left(\sum_{j\in\mathbb{N}} (2^{sj} \left\| \Delta_j^\psi f \right\|_\chi)^q \right)^{\frac{1}{q}}. \tag{2.3.6}$$

The definitions above are independent of the choice of ψ and φ. Additionally the Triebel-Lizorkin and Besov spaces above are quasi-Banach spaces.

As an example we note that in the model result (2.1.1) $\chi = L^p(w)$. In this case we write $\|f\|_{\dot{F}^s_{L^p,q}} = \|f\|_{\dot{F}^s_{p,q}}$ and $\|f\|_{\dot{B}^s_{L^p,q}} = \|f\|_{\dot{B}^s_{p,q}}$ with similar notation for the inhomogeneous Triebel-Lizorkin and Besov spaces. We will also make use of the lifting properties of weighted Lebesgue-based Besov and Triebel-Lizorkin spaces: for any p, q, and s as in the definitions above and $w \in A_\infty$ it follows that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,q}(w)} \simeq \|J^s f\|_{F^0_{p,q}(w)},$$

and an analagous statement holds in Besov spaces.

Appendix A

Title for This Appendix

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