Bilinear pseudodifferential operators and Leibniz-type rules

by

Alexander Thomson

B.S., Missouri State University, 2013

M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Chapter 1

Introduction to Leibniz-type rules

Leibniz-type rules have been extensively studied due to their connections to partial differential equations which model many real world situations such as shallow water waves and fluid flow. In this chapter we introduce some of the definitions and history of the development of Leibniz-type rules which motivated the results to be discussed in chapters 2 and 3 of this manuscript. First consider the Leibniz rule taught in Calculus courses which expresses the derivatives of a product of functions as a linear combination of derivatives of the functions involved; more specifically, for functions f and g sufficiently smooth, it holds that

$$\partial_x^{\alpha}(fg)(x) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1\alpha_2}\partial_x^{\alpha_1} f(x)\partial_x^{\alpha_2} g(x),$$

where $\alpha \in \mathbb{N}_0^n$ and $C_{\alpha_1\alpha_2}$ are appropriate constants. The definitions in of the function spaces used below and multiindices $\alpha \in \mathbb{N}_0^n$ will be discussed in Appendix A A. In particular one term has all the derivatives on f and another with α derivatives on g and

$$\partial_x^{\alpha}(fg)(x) = \partial_x^{\alpha} f(x)g(x) + f(x)\partial_x^{\alpha} g(x) + ...$$

In an analogous way, fractional Leibniz rules give estimates of the smoothness and size of a product of functions in terms of the smoothness and size of the factors. For instance, for f

and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|D^s(fg)\|_{L^r} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}} , \tag{1.0.1}$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, $1 < p_1, p_2, q_1, q_2 \leq \infty$, $1/2 < r \leq \infty$, and $s > N(1/min(r,1) - 1)$ or s is an even natural number. The homogeneous fractional differentiation operator of order s, D^s, is defined as

$$D^s f(x) = \int_{\mathbb{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f. For $s > 0$, the operator D^s is naturally understood as taking s derivatives of its argument. Indeed, in the case $s = 2$, $D^2 f = c\Delta f$, where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian operator. Furthermore, if s is a positive integer,

$$\|D^s f\|_{L^p} \sim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^p} ,$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$.

Another version of (1.0.1) is obtained by using the inhomogeneous sth order fractional differentiation operator J^s:

$$\|J^s(fg)\|_{L^r} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{q_2}} . \tag{1.0.2}$$

Similarly to its homeogenous counterpart, the operator J^s is defined through the Fourier transform as

$$J^s f(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

and can be interpreted as taking derivatives up to order s of f.

The estimates (1.0.1) and (1.0.2) are also known as Kato-Ponce inequalities due to the foundational work of Kato-Ponce[?] , where the estimate (1.0.2) was proved in the case

$1 < r = p_1 = q_2 < \infty$ and $p_2 = q_1 = \infty$, with applications to the Cauchy problem for Euler and Navier-Stokes equations. This result was extended by Gulisashvili-Kon[?], who showed (1.0.1) and (1.0.2) for the cases $s > 0$, $1 < r < \infty$, and $1 < p_1, p_2, q_1, q_2 \leq \infty$ in connection to smoothing properties of Schrödinger semigroups. Grafakos-Oh[?] and Muscalu-Schlag[?] established the cases for $1/2 < r \leq 1$ and the case $r = \infty$ was completed in the work of Bourgain-Li[?] and Grafakos-Maldonado-Naibo[?]. Applications of the estimates (1.0.1) and (1.0.2) to Korteweg-de Vries equations were studied by Christ-Weinstein[?] and Kenig-Ponce-Vega[?].

In the estimates (1.0.1) and (1.0.2) the two functions f and g are related through pointwise multiplication. Throughout the rest of this manuscript we will consider estimates similar to (1.0.1) and (1.0.2) where the two functions are related through a pseudodifferntial operator. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function for $x, \xi, \eta \in \mathbb{R}^n$. We define the *bilinear pseudodifferential operator* associated to σ, T_σ, by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{-2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We refer to σ as the symbol of the operator T_σ. When σ is independent of x we call σ the multiplier of the *bilinear multiplier operator* T_σ.

Throughout this manuscript we will study estimates related to (1.0.1) and (1.0.2) associated to bilinear pseudodifferential operators that are of the form

$$\|D^s T_\sigma(f, g)\|_Z \lesssim \|D^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|D^s g\|_{Y_2}, \tag{1.0.3}$$

$$\|J^s T_\sigma(f, g)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.0.4}$$

for a variety of function spaces X_1, X_2, Y_1, Y_2, and Z.

The estimates (1.0.3) and (1.0.4) have been extensively studied in a variety of settings. In[?], Brummer-Naibo studied Leibniz-type rules in function spaces that admit a molecular decomposition and a φ-transform characterization in the sense of Frazier-Jawerth[?, ?]. In the context of Lebesgue spaces and mixed Lebesgue spaces, estimates of the type (1.0.3) were studied in Hart-Torres-Wu[?] for bilinear multiplier operators with minimal smoothness assumptions on the multipliers. Related mapping properties for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes were studied by Bényi-Torres[?] and Bényi-Nahmod-Torres[?] in the setting of Sobolev spaces, by Bényi[?] in the setting of Besov spaces, and by Naibo[?] and Koezuka-Tomita[?] in the context of Triebel-Lizorkin spaces. Additionally, versions of (1.0.1) and (1.0.2) in weighted Lebesgue spaces were proved in Cruz-Uribe-Naibo[?], while Brummer-Naibo[?] proved (1.0.3) and (1.0.4) in weighted Lebesgue spaces for Coifman-Meyer multiplier operators.

In chapter two we will present Leibniz-type rules in the settings of Besov and Triebel-Lizorkin spaces based in various function spaces. The techniques used are quite flexible and allow the method of proof to be adapted to many different function spaces.

In chapter three we will present Leibniz-type rules in Besov spaces for bilinear pseudodifferential operators symbols of critical order.

Chapter 2

Weighted Leibniz-type rules with applications to scattering properties of PDEs

In this chapter we discuss bilinear multiplier operators associated to Coifman-Meyer multipliers and Leibniz-type rules in the settings of weighted Triebel-Lizorkin and Besov spaces. Additionally we obtain applications of these results to scattering properties of certain systems of parial differential equations. One of the main results in this chapter is in the setting of Triebel-Lizorkin spaces based in weighted Lebesgue spacs and Hardy spaces. It is as follows.

Theorem 2.0.1. *For $m \in \mathbb{R}$, let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leq \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.0.1}$$

If $0 < p, p_1, p_2 \le \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.0.2}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.8) *and* (2.2.9)*; moreover, if $w \in A_\infty$, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.0.3}$$

where $0 < p < \infty$, $0 < q \le \infty$ and $s > \tau_{p,q}(w)$.

In the following sections the function spaces and multipliers used in the hypotheses are defined and discussed. We will then remark on corollaries to Theorem (2.2.1) and their connection to the Leibniz rules in the previous chapter. theorem and the proof of Theorem (2.2.1). The proof is quite flexible and can be readily adapted to Triebel-Lizorkin and Besov spaces based in other function spaces.

2.1 Definitions

2.1.1 Coifman-Meyer Multipliers

The symbols used in Theorem (2.2.1) and results later in this chapter are Coifman-Meyer multipliers. Such multipliers are defined as follows.

Definition 2.1.1. For $m \in \mathbb{R}^n$, a smooth function $\sigma = \sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is a *Coifman-Meyer multiplier* of order m if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$

such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\alpha,\beta}(|\xi| + |\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi, \eta) \in \mathbb{R}^{2n} \setminus \{(0,0)\}. \tag{2.1.4}$$

Operators associated to these multipliers have been widely studied. For instance in Grafakos-Torres? operators associated to Coifman-Meyer multipliers were studied because of their connection to a larger class of operators called Calderón-Zygmund operators. In particular it holds that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

where σ is a Coifman-Meyer multiplier, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $1 < p_1, p_2 < \infty$. We also note that in particular Coifman-Meyer multipliers of order m belong to the bilinear Hörmander class $\dot{BS}^m_{1,1}$. These symbols and operators associated to them will be discussed in the following chapter.

2.1.2 Function spaces

Weighted spaces

Definition 2.1.2. A *weight* $w(x)$ defined on $\mathbb{R}^n$ is a nonnegative, locally integrable function such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$.

Given a weight $w(x)$ and $0 < p < \infty$ we define the weighted Lebesgue space $L^p(w)$ as the space of all measurable functions satisfying

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

In the case that $p = \infty$ we define $L^\infty(w) = L^\infty$.

The specific classes of weights in the hypotheses of our results are Muckenhoupt weights.

Definition 2.1.3. For $1 < p < \infty$ the *Muckenhoupt class* A_p consists of all weights w on $\mathbb{R}^n$

satisfying

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty, \tag{2.1.5}$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. For $p = \infty$ we define $A_\infty = \cup_{1<p} A_p$.

From this definition it follows that $A_p \subset A_q$ when $p \le q$. For a $w \in A_\infty$ we set $\tau_w = \inf\{\tau \in (1, \infty] : w \in A_\tau$. The condition (2.1.5) is motivated by the following fact: for $f \in L^p(w)$ the Hardy-Littlewood maximal function $\mathcal{M}(f)(x)$ is bounded on $L^p(w)$ if and only if $w \in A_p$. That is for for $1 < p < \infty$ $w \in A_p$ if and only if

$$\|\mathcal{M}(f)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Later in this chapter we will use the maximal function $\mathcal{M}_r(f) = (\mathcal{M}(|f|^r))^{\frac{1}{r}}$. By the properties for the Hardy-Littlewood maximal function above it holds that for $0 < r < p$ $\mathcal{M}_r$ is bounded on $L^p(w)$ for $w \in A_{p/r}$ and in this case $0 < r < \frac{p}{\tau_w}$. The following theorem is a vector valued version of the previous statement called a weighted Fefferman-Stein inequality.

Theorem 2.1.1. *If $0 < p < \infty$, $0 < q \le \infty$, $0 < r < \min(p, q)$ and $w \in A_{p/r}$ (i.e. $0 < r < \min(p/\tau_w, q)$), then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)},$$

where the implicit constant depends on r, p, q, and w and the summation in j should be replaced by the supremum in j if $q = \infty$.

For more detail on the Muckenhoupt classes see Grafakos**?** .

Triebel-Lizorkin and Besov spaces

Here we describe the function spaces in which (2.2.1) is based and some properties of such spaces.

Let ψ and φ be functions in $\mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ satisfying the following conditions:

- $\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$,
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ for some $c > 0$,
- $\operatorname{supp}(\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$,
- $|\widehat{\varphi}(\xi)| > c$.

For ψ supported in an annulus and $j \in \mathbb{Z}$ we define the operator $\Delta_j^\psi(f)$ through its Fourier transform as

$$\widehat{\Delta_j^\psi(f)}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$$

For such ψ we define the homogeneous Triebel-Lizorkin and Besov spaces as follows.

Definition 2.1.4. Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The weighted *homogeneous Triebel-Lizorkin space* $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj} |\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

- The weighted *homogeneous Besov space* $\dot{B}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}(w)} = \left(\sum_{j\in\int} (2^{js} \left\|\Delta_j^\psi f\right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty. \tag{2.1.6}$$

Given φ, ψ, S_0^φ, and Δ_j^ψ as above the weighted inhomogeneous Triebel-Lizorkin and Besov spaces are defined as follows.

Definition 2.1.5. Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The weighted *inhomogeneous Triebel-Lizorkin space* $F^s_{p,q}(w)$ is the class of all $f \in$

$\mathcal{S}(\mathbb{R}^n)'$ such that

$$\|f\|_{F^s_{p,q}(w)} = \|S_0^\varphi f\|_{L^p(w)} + \left\| \left(\sum_{j\in\mathbb{N}_0} (2^{sj}|\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

- The weighted *inhomogeneous Besov space* $\dot{B}^s_{p,q}(w)$ is the class of all $f \in \mathcal{S}(\mathbb{R}^n)'$ such that

$$\|f\|_{B^s_{p,q}(w)} = \|S_0^\varphi f\|_{L^p(w)} + \left(\sum_{j\in\mathbb{N}_0} (2^{js} \left\|\Delta_j^\psi f\right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty. \tag{2.1.7}$$

The definitions above are independent of the choice of φ and ψ. These spaces are generally quasi-Banach spaces and if $1 \le p, q < \infty$ they are Banach spaces. These spaces provide a framework to study a variety of other spaces such as Lebesgue, Hardy, Sobolev, and BMO spaces with a unified approach. For a detailed overview of the development of these spaces see Triebel [books 1-3].

For certain s, p, and q as in the definitions (2.1.4) and (2.1.5) Triebel-Lizorkin and Besov spaces coincide with other well known function spaces. For instance we have the following equivalences where the function spaces are equivalent in norm

$$F^0_{p,2}(w) \equiv H^p(w) \text{ for } 0 < p < \infty, \quad w \in A_\infty,$$

$$F^0_{p,2}(w) \equiv L^p(w) \equiv H^p(w) \text{ for } 1 < p < \infty, \quad w \in A_p,$$

$$F^s_{p,2}(w) \equiv \dot{W}^{s,p}(w) \text{ for } 1 < p < \infty, \quad w \in A_p.$$

Additionally by the lifting property of Triebel-Lizorkin and Besov spaces Theorem (2.2.1) and results like it in this chapter can be seen as Leibniz-type rules as in Chapter 1. For weighted Triebel-Lizorkin spaces the lifting property is as follows: for s, p, and q as in (2.1.5) and (2.1.4) and $w \in A_\infty$ we have that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,q}(w)} \simeq \|J^s f\|_{F^0_{p,q}(w)}.$$

The corresponding statement for Besov spaces is: for s, p, and q as in (2.1.5) and (2.1.4) and $w \in A_\infty$ we have that

$$\|f\|_{\dot{B}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{B}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{B^s_{p,q}(w)} \simeq \|J^s f\|_{B^0_{p,q}(w)}.$$

2.2 Weighted Leibniz-type rules

2.2.1 Homogeneous Leibniz-type rules

In the setting of weighted homogeneous Besov and Triebel-Lizorkin spaces we obtain the following Leibniz-type rule. As we will see in the corollaries to this result it improves the Leibniz-type rule (**??**) and has extensions to weighted versions of (**??**).

Theorem 2.2.1. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \le \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \le \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.8}$$

If $0 < p, p_1, p_2 \le \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.9}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.8) *and* (2.2.9)*; moreover, if $w \in A_\infty$, then*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.10}$$

where $0 < p < \infty$, $0 < q \leq \infty$ and $s > \tau_{p,q}(w)$.

We note that if $m \geq 0$ then the above estimates hold for any $f, g \in \mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ when $\mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ is a subspace of the function spaces on the right-hand side. This is the case when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ in (2.2.8) and (2.2.9) and $w \in A_p$ for (2.2.10).

By the lifting property of weighted Besov and Triebel-Lizorkin spaces in section and their relation to weighted Hardy spacees in section (2.1.2) the estimates (2.2.8) and (2.2.9) imply the following Leibniz-type rule for Coifman-Meyer multipliers of order zero.

Corollary 2.2.2. *Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order 0. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.11}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.11)*; moreover, if $w \in A_\infty$, then*

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.12}$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

Corollary (2.2.2) gives estimates related to those in Brummer-Naibo[?] where using different methods the following result was proven

Theorem 2.2.3. *if σ is a Coifman-Meyer multiplier of order 0, $1 < p_1, p_2 \leq \infty$, $\frac{1}{2} < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_p$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that*

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)} . \tag{2.2.13}$$

Moreover, if $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.13).

Corollary (2.2.2) and theorem overlap in the following ways.

- The estimate (2.2.11) allows for $0 < p, p_1, p_2 < \infty$, $w_1, w_2 \in A_\infty$, and the $H^P(w)$ on the left-hand side if $s > \tau_p(w)$. However (2.2.13) requires $1 < p_1, p_2 \leq \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ but allows for the Lebesgue norm on left-hand side when $s > \tau_p$. So (2.2.11) is less restrictive than (2.2.13) in terms of the indices p, p_1, and p_2 and the classes that the weights w_1 and w_2 belong to. However because $\tau_p \leq \tau_p(w)$ (2.2.11) is more restrictive in terms of the range of the regularity s than (2.2.13).

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2.2.2 Inhomogeneous Leibniz-type rules

Appendix A

Title for This Appendix

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