Leibniz-type rules in Triebel-Lizorkin and Besov spaces.

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Chapter 1

Introduction to Leibniz-type rules

Leibniz-type rules have been extensively studied due to their connections to partial differential equations that model many real world situations such as shallow water waves and fluid flow. In this chapter we introduce some of the definitions and history of the development of Leibniz-type rules that motivated the results to be discussed in Chapters 2 and ?? Chapter 3 of this dissertation.

First consider the Leibniz rule taught in Calculus courses, which expresses the derivatives of a product of functions as a linear combination of derivatives of the functions involved; more specifically, for functions f and g sufficiently smooth, it holds that

$$\partial^\alpha(fg)(x) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f(x) \partial^\beta g(x) = \partial^\alpha f(x) g(x) + f(x) \partial^\alpha g(x) + \cdots ,$$

for $\alpha, \beta \in \mathbb{N}_0^n$. In an analogous way, fractional Leibniz rules give estimates of the smoothness and size of a product of functions in terms of the smoothness and size of the factors. For instance, for f and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|D^s(fg)\|_{L^p} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{\tilde{p}_2}} \|D^s g\|_{L^{\tilde{p}_2}} , \tag{1.0.1}$$

where $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$, $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$, $1/2 < p \leqslant \infty$, and $s >$

$n(1/\min(p,1)-1)$ or s is an even whole number. The homogeneous fractional differentiation operator of order s, D^s, is defined as

$$D^s f(x) = \int_{\mathbb{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f. For $s > 0$, the operator D^s is naturally understood as taking s derivatives of its argument. Indeed, in the case $s = 2$, $D^2 f = \frac{-1}{4\pi^2}\Delta f$, where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian operator. Furthermore, if s is a positive integer and $1 < p < \infty$,

$$\|D^s f\|_{L^p} \sim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^p},$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$.

Another version of (1.0.1) is obtained by using the inhomogeneous sth order fractional differentiation operator J^s:

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{L^{p_2}} + \|f\|_{\tilde{p}_1}} \|J^s g\|_{L^{\tilde{p}_2}}. \tag{1.0.2}$$

Similarly to its homeogenous counterpart, the operator J^s is defined through the Fourier transform as

$$J^s f(x) = \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

and can be interpreted as taking derivatives up to order s of f.

The estimates (1.0.1) and (1.0.2) are also known as Kato-Ponce inequalities due to the foundational work of Kato-Ponce [31], where the estimate (1.0.2) was proved in the case $1 < p = p_1 = \tilde{p}_2 < \infty$ and $p_2 = \tilde{p}_1 = \infty$, with applications to the Cauchy problem for Euler and Navier-Stokes equations. This result was extended by Gulisashvili-Kon [25], who showed (1.0.1) and (1.0.2) for the cases $s > 0$, $1 < p < \infty$, and $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$ in connection to smoothing properties of Schrödinger semigroups. Grafakos-Oh [21] and Muscalu-Schlag [41] established the cases for $1/2 < p \leqslant 1$ and the case $p = \infty$ was completed in the work of

Bourgain-Li [7] and Grafakos-Maldonado-Naibo [20]. Applications of the estimates (1.0.1) and (1.0.2) to Korteweg-de Vries equations were studied by Christ-Weinstein [10] and Kenig-Ponce-Vega [32].

In the estimates (1.0.1) and (1.0.2) the two functions f and g are related through pointwise multiplication. In this dissertation we will consider bilinear estimates in the spirit of (1.0.1) and (1.0.2) where the two functions are related through a pseudodifferential operator. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function for $x, \xi, \eta \in \mathbb{R}^n$. We define the *bilinear pseudodifferential operator* associated to σ, T_σ, by

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta. \tag{1.0.3}$$

We call σ the *symbol* of the operator T_σ; when σ is independent of x, σ is also referred to as the *multiplier* of the *bilinear multiplier operator* T_σ. We note that $\sigma \equiv 1$ gives $T_\sigma(f,g) = fg$.

In Chapters 2 and 3 we will present new results on Leibniz-type rules associated to bilinear pseudodifferential operators that are of the form

$$\|D^s T_\sigma(f,g)\|_Z \lesssim \|D^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|D^s g\|_{Y_2}, \tag{1.0.4}$$

$$\|J^s T_\sigma(f,g)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.0.5}$$

for a variety of function spaces X_1, X_2, Y_1, Y_2, and Z. In the particular case that $\sigma \equiv 1$ and X_1, X_2, Y_1, Y_2, Z are appropriate Lebesgue spaces (1.0.4) and (1.0.5) recover (1.0.1) and (1.0.1) respectively.

In Chapter 2 we will discuss Leibniz-type rules (1.0.4) and (1.0.5) in the setting of Besov and Triebel-Lizorkin spaces based on certain quasi-Banach spaces. Such bilinear estimates will be proved for bilinear Coifman-Meyer multiplier operators. A particular case of the results in Chapter 2 is the following fractional Leibniz rule, and its inhomogeneous counterpart, in the context of Hardy spaces:

$$\|D^s(fg)\|_{H^p} \lesssim \|D^s f\|_{H^{p_1}} \|g\|_{H^{p_2}} + \|f\|_{H^{\tilde{p}_1}} \|D^s g\|_{H^{\tilde{p}_2}} \tag{1.0.6}$$

where $0 < p, p_1, \tilde{p}_1, p_2, \tilde{p}_2 < \infty$, $1/p = 1/p_1 + 1/p_2 = 1/\tilde{p}_1 + 1/\tilde{p}_2$. Recalling that $H^p = L^p$ for $1 < p < \infty$, (2.2.22) extends and improves (1.0.1).

Indeed, the inequality (2.2.22) extends the range of p, p_1, $\tilde{p}_1$, p_2, and $\tilde{p}_2$ by allowing $0 < p, p_1, \tilde{p}_1, p_2, \tilde{p}_2 < \infty$ while (1.0.1) requires $1 < p_1, p_2, \tilde{p}_1, \tilde{p}_2 \leqslant \infty$. Additionally (2.2.22) allows for the H^p norm on the left-hand side which is larger than the L^p norm.

The techniques used in the proofs of the results in Chapter 2 are quite flexible and allow us to obtain (1.0.4) and (1.0.5) in Triebel-Lizorkin and Besov spaces based in weighted Lebesque spaces, weighted Lorrentz spaces, weighted Morrey spaces, and variable-exponent Lebesgue spaces. In particular, the proofs make use of Nikol'skij representations of such function spaces. These representations have been used in unweighted settings such as the work of Nikol'skij [43], Meyer [39], Bourdad [6], Triebel [47], and Yamazaki [50].

As an application of the results in Chapter 2 we obtain scattering properties for solutions to certain systems of partial differential equations that involve fractional powers of the Laplacian. Solutions of these systems scatter to functions that can be realized in terms of a Coifman-Meyer multiplier operator acting on appropriate arguments. As a consequence, the main results of Chapter 2 can be applied and lead to estimates associated to the long term behavior of the solutions.

In Chapter **??** we present Leibniz-type rules in Besov and local Hardy spaces for bilinear pseudodifferential operators associated to symbols in bilinear Hörmander classes of critical order. For such symbols we prove bilinear estimates of the form

$$\|D^s T_\sigma(f,g)\|_{B^0_{p,q}} \lesssim \|D^s f\|_{B^0_{p_1,q}} \|g\|_{h^{p_2}} + \|f\|_{h^{p_1}} \|D^s g\|_{B^0_{p_2,q}} \tag{1.0.7}$$

where $B^0_{p,q}$ and h^p denote Besov and Hardy spaces respectively, $0 < p < \infty$, and $0 < p_1, p_2 \leqslant$

∞ are such that $1/p = 1/p_1 + 1/p_2$, $0 < q \leqslant \infty$, and $s > \max\{0, n(1/p - 1)\}$.

The proofs of the results in Chapter **??** (Chapter 3) utilize appropriate spectral decompositions of the symbols, poitwise inequalities in terms of maximal functions, and Nikol'skij representations for Besov spaces. The techniques used are inspired by bilinear techniques used in Naibo [42] and techniques for linear operators in Johnsen [30], Marschall [37], and Park [44].

We close this chapter by referencing serveral works in connection with the study of the bilinear estimates (1.0.4) and (1.0.5). In [9], Brummer-Naibo studied Leibniz-type rules for bilinear pseudodifferential operators with homogeneous symbols and in function spaces that admit a molecular decomposition and a φ-transform characterization in the sense of Frazier-Jawerth [17; 18]. In the context of Lebesgue spaces and mixed Lebesgue spaces, estimates of the type (1.0.4) were studied in Hart-Torres-Wu [27] for bilinear multiplier operators with minimal smoothness assumptions on the multipliers. Related mapping properties for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes were studied by Bényi-Torres [4] and Bényi-Nahmod-Torres [3] in the setting of Sobolev spaces, by Bényi [2] in the setting of Besov spaces, and by Naibo [42] and Koezuka-Tomita [34] in the context of Triebel-Lizorkin spaces. Additionally, versions of (1.0.1) and (1.0.2) in weighted Lebesgue spaces were proved in Cruz-Uribe-Naibo [14], while Brummer-Naibo [8] proved (1.0.4) and (1.0.5) in weighted Lebesgue spaces for Coifman-Meyer multiplier operators.

Chapter 2

Weighted Leibniz-type rules and applications to scattering properties of PDEs

In this chapter we obtain new Leibniz-type rules of the type (1.0.4) and (1.0.5) for bilinear multiplier operators associated to Coifman-Meyer multipliers in the settings of Triebel-Lizorkin and Besov spaces based on quasi-Banach spaces. These results extend and improve the fractional Leibniz rules (1.0.1) and (1.0.2). Additionally, we apply these results to obtain scattering properties of solutions to systems of partial differential equations involving fractional powers of the Laplacian.

We start with some preliminaries in Section 2.1, where we discuss Coifman-Meyer multipliers and Besov and Triebel-Lizorkin spaces based on weighted Lebesgue spaces.

In Section 2.2 we state and prove two of the main results of this chapter on Leibniz-type rules associated to Coifman-Meyer multipliers in the setting of Besov and Triebel-Lizorkin spaces based on weighted Lebesgue spaces. These results are stated as Theorem 2.2.1 and Theorem 2.2.6. We also present several corollaries and connections with related results in the literature and estimates (1.0.1) and (1.0.2). The method of proof used for Theorem

2.2.1 and Theorem 2.2.6 can be adapted to obtain (1.0.4) and (1.0.5) for Coifman-Meyer multipliers in the context of Triebel-Lizorkin and Besov spaces based on other quasi-Banach spaces such as weighted Lorentz, weighted Morrey, and variable-Lebesgue spaces. These results are discussed in Section 2.3.

Finally, in Section 2.4 we present applications of the results in this chapter to scattering properties of solutions to partial differential equations.

2.1 Preliminaries

In this section we set some notation and present definitions and results about weights, the scales of weighted Triebel–Lizorkin, Besov and Hardy spaces, and Coifman–Meyer multiplier operators.

The notations $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are used for the Schwartz class of smooth rapidly decreasing functions defined on $\mathbb{R}^n$ and its dual, the class of tempered distributions on $\mathbb{R}^n$, respectively. $\mathcal{S}_0(\mathbb{R}^n)$ refers to the closed subspace of functions in $\mathcal{S}(\mathbb{R}^n)$ that have vanishing moments of all orders; that is, $f \in \mathcal{S}_0(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} x^\alpha f(x)\, dx = 0$ for all $\alpha \in \mathbb{N}_0^n$. Its dual is $\mathcal{S}_0'(\mathbb{R}^n)$, which coincides with the class of tempered distributions modulo polynomials denoted by $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$. Throughout, all functions are defined on $\mathbb{R}^n$ and therefore we omit $\mathbb{R}^n$ in the notation of the function spaces defined below.

The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is denoted by $\widehat{f}$; in particular, for $f \in L^1$, we use the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx \quad \forall \xi \in \mathbb{R}^n.$$

We know define some notation that will be used throughout this dissertation. If $j \in \mathbb{Z}$ and $h \in \mathcal{S}(\mathbb{R}^n)$, the operator P_j^h is defined so that $\widehat{P_j^h f}(\xi) = h(2^{-j}\xi)\widehat{f}(\xi)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. If $\widehat{h}$ is supported in an annulus centered at the origin we will use the notation Δ_j^h rather than P_j^h; if $\widehat{h}$ is supported in a ball centered at the origin and $\widehat{h}(0) \neq 0$, S_j^h will be used instead of P_j^h. For $y \in \mathbb{R}^n$ the translation operator, denoted by τ_y, is given by $\tau_y h(x) = h(x+y)$ for $x \in \mathbb{R}^n$.

2.1.1 Coifman-Meyer Multipliers

The symbols used in the main results of this chapter are Coifman-Meyer multipliers. Such multipliers are defined as follows.

Definition 2.1.1. Given $m \in \mathbb{R}$, a smooth, complex-valued function $\sigma = \sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, is a *Coifman-Meyer multiplier of order m* if for all multi-indices $\alpha,\beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leqslant C_{\alpha,\beta}(|\xi|+|\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi,\eta) \in \mathbb{R}^{2n}\backslash\{(0,0)\}.$$

We say $\sigma = \sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, is an *inhomogeneous Coifman-Meyer multiplier of order m* if for all multi-indices $\alpha,\beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leqslant C_{\alpha,\beta}(1+|\xi|+|\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi,\eta) \in \mathbb{R}^{2n}. \tag{2.1.1}$$

Bilinear multiplier operators associated to Coifman-Meyer multipliers of order 0 have been well studied. Such operators are examples of bilinear Calderón-Zygmund operators. As a consequence they are bounded in a variety of function spaces; in particular they satisfy

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}},$$

where σ is a Coifman-Meyer multiplier of order 0, $1/p = 1/p_1 + 1/p_2$, and $1 < p_1, p_2 < \infty$. The reader is referred to Coifman-Meyer [11] for various estimates and background information for Coifman-Meyer multiplier operators, and to David-Journé [15], Grafakos-Torres [24], and Kenig-Stein [33] for the development of the Calderón-Zygmund theory. Estimatex in weighted Lebesgue spaces fo bilinear Calderón-Zygmund operators, and in particular for

Coifman-Meyer multiplier operators of order 0, have been obtained in Grafakos-Torres [23], Grafakos-Martell [22], and Lerner et al. [36].

We next describe a decomposition of Coifman-Meyer multiplier operators that will be useful in the proofs of the main results of this chapter. Fix $\Psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\widehat{\Psi}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \ \forall \xi \in \mathbb{R}^n \backslash \{0\};$$

define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\widehat{\Phi}(0) := 1, \quad \widehat{\Phi}(\xi) := \sum_{j\leqslant 0} \widehat{\Psi}(2^{-j}\xi) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}.$$

By the notation previously introduced, if $a, b \in \mathbb{R}^n$, $\Delta_j^{\tau_a\Psi} f$ and $S_j^{\tau_a\Phi} f$ satisfy $\widehat{\Delta_j^{\tau_a\Psi}f}(\xi) = \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot a}\widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{S_j^{\tau_b\Phi}f}(\xi) = \widehat{\tau_b\Phi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot b}\widehat{\Phi}(2^{-j}\xi)\widehat{f}(\xi)$. By the work of Coifman and Meyer in [11], given $N \in \mathbb{N}$ such that $N > n$, it follows that $T_\sigma = T_\sigma^1 + T_\sigma^2$, where, for $f \in \mathcal{S}_0(\mathbb{R}^n)$ ($f \in \mathcal{S}(\mathbb{R}^n)$ if $m \geqslant 0$) and $g \in \mathcal{S}(\mathbb{R}^n)$,

$$T_\sigma^1(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)(x)\,(S_j^{\tau_b\Phi}g)(x) \quad \forall x \in \mathbb{R}^n, \tag{2.1.2}$$

the coefficients $\mathcal{C}_j(a,b)$ satisfy

$$|\mathcal{C}_j(a,b)| \lesssim 2^{jm} \quad \forall a,b \in \mathbb{Z}^n, j \in \mathbb{Z}, \tag{2.1.3}$$

with the implicit constant depending on σ, and an analogous expression holds for T_σ^2 with the roles of f and g interchanged.

If σ is an inhomogeneous Coifman–Meyer multiplier of order m, a similar decomposition to (2.1.2) follows with the summation in $j \in \mathbb{N}_0$ rather than $j \in \mathbb{Z}$, with $\Delta_0^{\tau_a\Psi}$ replaced by $S_0^{\tau_a\Phi}$ and for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Remark 2.1.1. For the formula (2.1.2) and its corresponding counterpart for T_σ^2 to hold, the condition (2.1.1) on the derivatives of σ is only needed for multi-indices α and β such that $|\alpha + \beta| \leqslant 2N$

2.1.2 The scale of weighted Triebel-Lizorkin and Besov spaces

The Leibniz-type rules obtained in two of the main results of this chapter, Theorem 2.2.1 and Theorem 2.2.6, hold in the setting of weighted Triebel-Lizorkin and Besov spaces. In this section we define this scale of spaces and present some of their properties. In particular, we state and prove Nikolśkij representation, which constitute important tools for the proofs of theorems 2.2.1 and 2.2.6.

We start by defining the classes of weights we will be using and present maximal operators and inequalities.

Weighted spaces

A *weight* w defined on $\mathbb{R}^n$ is a locally integrable function such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Given a weight w and $0 < p < \infty$ we define the weighted Lebesgue space $L^p(w)$ as the space of all measurable functions f satisfying

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

In the case that $p = \infty$ we define $L^\infty(w) = L^\infty$.

The specific classes of weights in the hypotheses of the results of this chapter are Muckenhoupt weights, which we next define.

Definition 2.1.2. For $1 < p < \infty$ the *Muckenhoupt class* A_p consists of all weights w on $\mathbb{R}^n$ satisfying

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. For $p = \infty$ we define $A_\infty := \cup_{1<p} A_p$.

From this definition it follows that $A_p \subset A_q$ when $p \leqslant q$. For $w \in A_\infty$ we set $\tau_w = \inf\{\tau \in (1,\infty] : w \in A_\tau\}$.

It can be proved that if $w \in A_p$, $p > 1$, then $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$.

The Muckenhoupt classes arise in the study of boundedness properties of Hardy-Littlewood maximal operators in the setting of weighted Lebesgue spaces, as we next explain. The *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined as

$$\mathcal{M}(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad \forall x \in \mathbb{R}^n, f \in L^1_{loc}(\mathbb{R}^n),$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$. It turns out that the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ if and only if $w \in A_p$. That is, for $1 < p < \infty$, $w \in A_p$ if and only if

$$\|\mathcal{M}(f)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w).$$

Later in this chapter we will use the maximal function

$$\mathcal{M}_r(f) := (\mathcal{M}(|f|^r))^{\frac{1}{r}}. \tag{2.1.4}$$

Since $0 < r < p/\tau_w$ if and only if $0 < r < p$ and $w \in A_{p/r}$, by the boundedness properties for the Hardy-Littlewood maximal operator stated above, it holds that $\mathcal{M}_r$ is bounded on $L^p(w)$ when $w \in A_{p/r}$. This fact is a particular case of the following estimate known as the Fefferman-Stein inequality.

Theorem 2.1.1. *If $0 < p < \infty$, $0 < q \leqslant \infty$, $0 < r < \min(p,q)$ and $w \in A_{p/r}$ (i.e. $0 < r < \min(p/\tau_w, q)$), then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)},$$

where the implicit constant depends on r, p, q, and w and the summation in j should be replaced by the supremum in j if $q = \infty$.

For more details on the Muckenhoupt classes see Muckenhoupt [40] Grafakos [19].

Weighted Triebel-Lizorkin and Besov spaces

Let ψ and φ be functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the following conditions:

- $\operatorname{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$,
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ for some $c > 0$,
- $\operatorname{supp}(\widehat{\varphi}) \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$,
- $|\widehat{\varphi}(\xi)| > c$ for $|\xi| < \frac{5}{3}$ and some $c > 0$.

Definition 2.1.3. (Weighted Homogeneous Triebel-Lizorkin and Besov spaces) Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.

- For $0 < p < \infty$ the *weighted homogeneous Triebel-Lizorkin space* $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

  $$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j \in \mathbb{Z}} (2^{sj} |\Delta^{\psi}_j f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

  with appropriate changes of $q = \infty$.

- For $0 < p \leqslant \infty$ the *weighted homogeneous Besov space* $\dot{B}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

  $$\|f\|_{\dot{B}^s_{p,q}(w)} := \left(\sum_{j \in \mathbb{Z}} (2^{js} \left\| \Delta^{\psi}_j f \right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty,$$

  with appropriate changes of $q = \infty$.

Definition 2.1.4. (Weighted inhomogeneous Triebel-Lizorkin and Besov spaces) Let $s \in \mathbb{R}$ and $0 < q \leqslant \infty$.

- For $0 < p < \infty$ the *weighted inhomogeneous Triebel-Lizorkin space* $F^s_{p,q}(w)$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{p,q}(w)} := \|S_0^{\varphi} f\|_{L^p(w)} + \left\| \left(\sum_{j \in \mathbb{N}} (2^{sj} |\Delta_j^{\psi} f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty,$$

  with appropriate changes if $q = \infty$.

- For $0 < p \leqslant \infty$ the *weighted inhomogeneous Besov space* $\dot{B}^s_{p,q}(w)$ is the class of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(w)} := \|S_0^{\varphi} f\|_{L^p(w)} + \left(\sum_{j \in \mathbb{N}} (2^{js} \left\| \Delta_j^{\psi} f \right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty,$$

  with appropriate changes if $q = \infty$.

The definitions above are independent of the choice of φ and ψ. Triebel-Lizorkin and Besov spaces are quasi-Banach spaces and if $p, q \geqslant 1$ they are Banach spaces. These spaces provide a unified framework that includes a variety of other spaces such as weighted Lebesgue, Hardy, and Sobolev spaces. For instance the following equalities hold with equivalent norms:

$$\dot{F}^0_{p,2}(w) = H^p(w) \text{ for } 0 < p < \infty, \quad w \in A_\infty, \tag{2.1.5}$$

$$\dot{F}^0_{p,2}(w) = L^p(w) \simeq H^p(w) \text{ for } 1 < p < \infty, \quad w \in A_p, \tag{2.1.6}$$

$$\dot{F}^s_{p,2}(w) = \dot{W}^{s,p}(w) \text{ for } 1 < p < \infty, \quad w \in A_p, \tag{2.1.7}$$

$$F_{p,2}(w) = h^p(w) \text{ for } 0 < p < \infty, \quad w \in A_\infty, \tag{2.1.8}$$

where $H^p(w)$ and $\dot{W}^{s,p}(w)$ are, respectively, a weighted Hardy space and a weighted Sobolev space, whose definitions we next recall. Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \Psi(x)\, dx \neq 0$. Given

$0 < p < \infty$, the weighted Hardy space $H^p(w)$ is defined as

$$H^p(w) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^p(w)} < \infty\},$$

where

$$\|f\|_{H^p(w)} := \left\| \sup_{t>0} \left| t^{-n}\Psi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)}.$$

The weighted local Hardy spaces $h^p(w)$ is defined as

$$h^p(w) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{h^p(w)} < \infty\},$$

where

$$\|f\|_{h^p(w)} := \left\| \sup_{0<t<1} \left| t^{-n}\Psi(t^{-1}\cdot) * f \right| \right\|_{L^p(w)}.$$

The weighted Sobolev space $\dot{W}^{s,p}(w)$ is the space of all tempered distributions modulo polynomials such that

$$\|f\|_{\dot{W}^{s,p}(w)} := \|D^s f\|_{L^p(w)} < \infty.$$

For a detailed overview of the development of Besov and Triebel-Lizorkin spaces see Triebel [47] and Qui [45] for the unweighted and weighted settings respectively. We recall that these spaces satisfy the following lifting property: for s, p, and q as in definitions 2.1.3 and 2.1.4 and $w \in A_\infty$ we have that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,q}(w)} \simeq \|J^s f\|_{F^0_{p,q}(w)}, \tag{2.1.9}$$

$$\|f\|_{\dot{B}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{B}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{B^s_{p,q}(w)} \simeq \|J^s f\|_{B^0_{p,q}(w)}. \tag{2.1.10}$$

We end this section by introducing notation used in the statements of the main results in this chapter. For $w \in A_\infty$ and $0 < p, q \leqslant \infty$ denote

$$\tau_{p,q}(w) := n\left(\frac{1}{\min(p/\tau_w, q, 1)} - 1\right) \quad \text{and} \quad \tau_p(w) := n\left(\frac{1}{\min(p/\tau_w, 1)} - 1\right).$$

If $w \equiv 1$, in which case $\tau_w = 1$, we just write $\tau_{p,q}$ and τ_p, respectively. Note that $\tau_{p,2}(w) = \tau_p(w)$, $\tau_{p,q}(w) \geqslant \tau_{p,q}$ and $\tau_p(w) \geqslant \tau_p$ for any $w \in A_\infty$.

Nikol'skij representations for weighted Triebel-Lizorkin and Besov spaces

An important tool for the proof of Theorem 2.2.1 is the Nikol'skij representation for weighted Triebel-Lizorkin and Besov spaces. Here we state a weighted version of [50, Theorem 3.7] (see also [47, Section 2.5.2]). We give the proof of Theorem 2.1.2 at the end of this section after stating and proving several lemmas used in its proof.

Theorem 2.1.2. *Nikol'skij representation For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{Z}.$$

If $w \in A_\infty$, then the following holds:

(i) Let $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$. If $\left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{F}^s_{p,q}(w)$ (in $\mathcal{S}_0'(\mathbb{R}^n)$ if $q = \infty$) and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)},$$

where the implicit constant depends only on n, D, s, p and q. An analogous statement, with $j \in \mathbb{N}_0$, holds true for $F^s_{p,q}(w)$ (when $q = \infty$, the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

(ii) Let $0 < p, q \leqslant \infty$ and $s > \tau_p(w)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{\ell^q(L^p(w))} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$

converges in $\dot{B}^s_{p,q}(w)$ *(in* $\mathcal{S}'_0(\mathbb{R}^n)$ *if* $q = \infty$*) and*

$$\left\| \sum_{j\in\mathbb{Z}} u_j \right\|_{\dot{B}^s_{p,q}(w)} \lesssim \left\| \{2^{js} u_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q(L^p(w))},$$

where the implicit constant depends only on n, D, s, p *and* q. *An analogous statement, with* $j \in \mathbb{N}_0$, *holds true for* $B^s_{p,q}(w)$ *(when* $q = \infty$, *the convergence is in* $\mathcal{S}'(\mathbb{R}^n)$*).*

Before proving Theorem 2.1.2 we state the following lemmas that are used in its proof.

Lemma 2.1.3 (Particular case of Corollary 2.11 in [50])**.** *Suppose* $0 < r \leqslant 1$, $A > 0$, $R \geqslant 1$ *and* $d > n/r$. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that* $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant AR\}$, *it holds that*

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\| (1 + |A\,\cdot\,|)^d \phi \right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n,$$

where the implicit constant is independent of $A, R, \phi,$ *and* f.

Remark 2.1.2. [50, Corollary 2.11] incorrectly states $A^{-n/r}$ instead of A^{-n}. Also, it states $A \geqslant 1$, but the result is true for $A > 0$ as stated in Lemma 2.1.3.

The following lemma is a weighted version of [50, Corollary 2.12 (1)]. We include its brief proof for completeness.

Lemma 2.1.4. *Suppose* $w \in A_\infty$, $0 < p \leqslant \infty$, $A > 0$, $R \geqslant 1$, *and* $d > b > n/\min(1, p/\tau_w)$. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* f *is such that* $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant AR\}$, *it holds that*

$$\|\phi * f\|_{L^p(w)} \lesssim R^{b-n} A^{-n} \left\| (1 + |A\,\cdot\,|^d) \phi \right\|_{L^\infty} \|f\|_{L^p(w)},$$

where the implicit constant is independent of A, R, ϕ *and* f.

Proof. Set $r := n/b < \min(1, p/\tau_w)$. The hypothesis $d > b$ means $d > n/r$ and Lemma 2.1.3 yields

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\| (1 + |A\,\cdot\,|)^d \phi \right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n.$$

Since $r < p/\tau_w$, we have $\|\mathcal{M}_r f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$ and therefore

$$\|\phi * f\|_{L^p(w)} \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\|(1+|A\cdot|)^d \phi\right\|_{L^\infty} \|f\|_{L^p(w)};$$

observing that $1/r - 1 = (b-n)/n$, the desired estimate follows. □

The following lemma is a modified version of [50, Lemma 3.8].

Lemma 2.1.5. *Let $\tau < 0$, $\lambda \in \mathbb{R}$, $0 < q \leqslant \infty$, and $k_0 \in \mathbb{Z}$. Then, for any sequence $\{d_j\}_{j\in\mathbb{Z}} \subset [0,\infty)$ it holds that*

$$\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} \lesssim \left\|\{2^{j\lambda} d_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q},$$

where the implicit constant depends only on k_0, τ, λ and q.

Proof. Suppose first that $0 < q \leqslant 1$. Then,

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} &= \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right)^q \right]^{\frac{1}{q}} \\
&\leqslant \left[\sum_{j\in\mathbb{Z}} \sum_{k=k_0}^{\infty} 2^{\tau q k} 2^{\lambda q(j+k)} d_{j+k}^q \right]^{\frac{1}{q}} = \left[\sum_{k=k_0}^{\infty} 2^{\tau q k} \sum_{j\in\mathbb{Z}} 2^{\lambda q(j+k)} d_{j+k}^q \right]^{\frac{1}{q}} \\
&= \left(\sum_{k=k_0}^{\infty} 2^{\tau q k} \right)^{\frac{1}{q}} \left\|\{2^{j\lambda} d_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q} = C_{k_0,\tau,q} \left\|\{2^{j\lambda} d_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q},
\end{aligned}$$

where in the last equality we have used that $\tau < 0$. If $1 < q < \infty$ we use Hölder's inequality with q and q' to write

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} &\leqslant \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k q/2} 2^{\lambda q(j+k)} d_{j+k}^q \right) \left(\sum_{k=k_0}^{\infty} 2^{\tau k q'/2} \right)^{q/q'} \right]^{\frac{1}{q}} \\
&= C_{k_0,\tau,q} \left\|\{2^{j\lambda} d_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q}.
\end{aligned}$$

The case $q = \infty$ is straightforward. $\square$

We now prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We first prove Theorem 2.1.2 for finite families. We will do this in the homogeneous settings, with the proof in the inhomogeneous settings being similar. Suppose $\{u_j\}_{j\in\mathbb{Z}}$ is such that $u_j = 0$ for all j except those belonging to some finite subset of $\mathbb{Z}$; this assumption allows us to avoid convergence issues since all the sums considered will be finite.

For Part (2.1.2), let D, w, p, q and s be as in the hypotheses. Fix $0 < r < \min(1, p/\tau_w, q)$ such that $s > n(1/r - 1)$; note that the latter is possible since $s > \tau_{p,q}(w)$.

Let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \leqslant 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{Z}.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $\dot{F}^s_{p,q}(w)$ in Section 2.1.2. We have

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.1.11}$$

We will use Lemma 2.1.3 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geqslant -k_0$, we get $R \geqslant 1$.) Fixing $d > n/r$ and applying Lemma 2.1.3, we get

$$\begin{aligned}
|\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\|(1 + |2^j \cdot|)^d 2^{jn}\psi(2^j \cdot)\right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\
&\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y\in\mathbb{R}^n} (1 + |2^j y|)^d |\psi(2^j y)|\right) \mathcal{M}_r(u_{j+k})(x).
\end{aligned}$$

Hence,

$$2^{js}|\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.1.11),

$$2^{js}|\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x).$$

Since $1/r - 1 - s/n < 0$, Lemma 2.1.5 yields

$$\left\| \{2^{js}|\Delta_j^\psi u|\}_{j\in\mathbb{Z}} \right\|_{L^p(w)(\ell^q)} \lesssim \left\| \{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}} \right\|_{L^p(w)(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{Z}}$. Applying the weighted Fefferman-Stein inequality to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{2^{js}u_j\}_{j\in\mathbb{Z}} \right\|_{L^p(\ell^q)}.$$

For Part (2.1.2), let D, w, p, q and s be as in the hypotheses and k_0 be as above. Consider $\Delta_j^\psi u_{j+k}$ in (2.1.11) and apply Lemma 2.1.4 with $\phi(x) = 2^{jn}\psi(2^{-j}x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n/\min(1, p/\tau_w) < b < n+s$; note that such b exists since $s > \tau_p(w)$. We get

$$\left\| \Delta_j^\psi u_{j+k} \right\|_{L^p(w)} \lesssim 2^{(k+k_0)(b-n)} 2^{-jn} \left\| (1+|2^j\cdot|)^d 2^{jn}\psi(2^{-j}\cdot) \right\|_{L^\infty} \|u_{j+k}\|_{L^p(w)} \sim 2^{k(b-n)} \|u_{j+k}\|_{L^p(w)},$$

and setting $p^* := \min(p,1)$ we obtain

$$2^{jsp^*} \left\| \Delta_j^\psi u \right\|_{L^p(w)}^{p^*} \lesssim 2^{jsp^*} \sum_{k=-k_0}^{\infty} \left\| \Delta_j^\psi u_{j+k} \right\|_{L^p(w)}^{p^*} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|_{L^p(w)}^{p^*}.$$

Hence, applying Lemma 2.1.5, it follows that

$$\|u\|_{\dot{B}^s_{p,q}(w)} \lesssim \left\| \left\{ \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|_{L^p(w)}^{p^*} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q/p*}}^{\frac{1}{p^*}} \lesssim \left\| \{2^{js}u_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q(L^p(w))},$$

as desired.

We next show the theorem for any not necessarily finite family. Let $\{u_j\}_{j\in\mathbb{Z}}$, w, p, q, and s be as in the hypotheses. Define $U_N := \sum_{k=-N}^{N} u_j$; since the theorem is true for finite families and, for $M < N$, $\{u_j\}_{M+1\leqslant|j|\leqslant N}$ satisfies the hypotheses of the theorem, we have

$$\|U_N - U_M\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\|\{2^{js}u_j\}_{M+1\leqslant|j|\leqslant N}\right\|_{L^p(w)(\ell^q)}, \tag{2.1.12}$$

where the implicit constant is independent of M, N and the family $\{u_j\}_{j\in\mathbb{Z}}$. The same reasoning can be used to obtain a similar statement for $\dot{B}^s_{p,q}(w)$.

If $0 < q < \infty$, as $M, N \to \infty$, the right-hand side of (2.1.12) tends to zero by the assumption $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^p(w)(\ell^q)} < \infty$ and the dominated convergence theorem; therefore, since $\dot{F}^s_{p,q}(w)$ is complete, $\sum_{j\in\mathbb{Z}} u_j$ converges in $\dot{F}^s_{p,q}(w)$. The same reasoning used to obtain (2.1.12) gives that

$$\|U_N\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\|\{2^{js}u_j\}_{-N\leqslant j\leqslant N}\right\|_{L^p(w)(\ell^q)},$$

where the implicit constant is independent of N and the family $\{u_j\}_{j\in\mathbb{Z}}$. It then follows that

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)},$$

with the implicit constant independent of the family $\{u_j\}_{j\in\mathbb{Z}}$.

Using the same reasoning in the context of homogeneous weighted Besov spaces we obtain

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{B}^s_{p,q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^p(w))}.$$

If $q = \infty$, we use that $\{2^{(s-\varepsilon)j}u_j\}_{j\geqslant 0}$ and $\{2^{(s+\varepsilon)j}u_j\}_{j<0}$ belong to $\ell^1(L^p(w))$ for any $\varepsilon > 0$ and apply Theorem 2.1.2 under the case of finite q to conclude that $\sum_{j=0}^{N} u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{p,1}(w)$ and $\dot{B}^{s+\varepsilon}_{p,1}(w)$, respectively (choosing $\varepsilon > 0$ so that $s - \varepsilon > \tau_{p,q}(w) \geqslant \tau_p(w)$). Therefore, U_N convergence in $\mathcal{S}_0'(\mathbb{R}^n)$. Moreover, by Theorem 2.1.2 applied to the finite sequence $\{u_j\}_{-N\leqslant j\leqslant N}$, we have that $U_N \in \dot{F}^s_{p,\infty}(w)$ and

$$\|U_N\|_{\dot{F}^s_{p,\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{-N\leqslant j\leqslant N}\right\|_{L^p(w)(\ell^\infty)} \leqslant \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^\infty)},$$

with the implicit constant independent of N and $\{u_j\}_{j\in\mathbb{Z}}$. Since $\dot{F}^s_{p,\infty}(w)$ has the Fatou property (see Remark 2.1.3), we conclude that $\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot{F}^s_{p,\infty}(w)$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{p,\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^\infty)}.$$

□

Remark 2.1.3. The last part of the proof of Theorem 2.1.2 uses the Fatou property of Triebel–Lizorkin and Besov spaces. Let $\mathcal{A}$ be a quasi-Banach space such that $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ (or $\mathcal{S}_0(\mathbb{R}^n) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{S}_0'(\mathbb{R}^n)$). The space $\mathcal{A}$ is said to have the Fatou property if for every sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ that converges in $\mathcal{S}'(\mathbb{R}^n)$ ($\mathcal{S}_0'(\mathbb{R}^n)$, respectively), as $j \to \infty$, and that satisfies $\liminf_{j\to\infty} \|f_j\|_{\mathcal{A}} < \infty$, it follows that $\lim_{j\to\infty} f_j \in \mathcal{A}$ and $\|\lim_{j\to\infty} f_j\|_{\mathcal{A}} \lesssim \liminf_{j\to\infty} \|f_j\|_{\mathcal{A}}$, where the implicit constant is independent of $\{f_j\}_{j\in\mathbb{N}}$.

Triebel–Lizorkin and Besov spaces based in weighted Lebesgue spaces posses the Fatou property for any $s \in \mathbb{R}$ and $0 < q \leqslant \infty$ because for $0 < p \leqslant \infty$, $L^p(w)$ satisfies: (1) if $f, g \in L^p(w)$ and $|f| \leqslant |g|$ pointwise a.e., then $\|f\|_{L^p(w)} \lesssim \|g\|_{L^p(w)}$; (2) if $\{f_j\}_{j\in\mathbb{N}} \subset L^p(w)$ and $f_j \geqslant 0$ poinwise a.e., then $\|\liminf_{j\to\infty} f_j\|_{L^p(w)} \lesssim \liminf_{j\to\infty} \|f_j\|_{L^p(w)}$.

2.2 Leibniz-type rules in weighted Triebel-Lizorkin and Besov spaces

2.2.1 Homogeneous Leibniz-type rules

In the setting of weighted homogeneous Besov and Triebel-Lizorkin spaces based in weighted Lebesgue spaces we obtain the following Leibniz-type rules. As we will see in the corollaries to this result it improves the Leibniz-type rule (1.0.2) and has extensions to weighted versions of (1.0.2).

Theorem 2.2.1. *For $m \in \mathbb{R}$, let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.13}$$

If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.14}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.13) and (2.2.14); moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.15}$$

where $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$.

We note that if $m \geqslant 0$ then the above estimates hold for any $f, g \in \mathcal{S}(\mathbb{R}^n)$ when

$\mathcal{S}(\mathbb{R}^n)$ is a subspace of the function spaces on the right-hand side. This is the case when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ in (2.2.13) and (2.2.14) and $w \in A_p$ for (2.2.15).

For $w \in A_\infty$ and $0 < p \leqslant \infty$ we denote $\tau_w = \inf\{\tau \in (1, \infty] : w \in A_p\}$ and $\tau_p(w) := n\left(\frac{1}{min(p/\tau_w, 1)} - 1\right)$. By the lifting property of weighted Besov and Triebel-Lizorkin spaces in Section 2.1.2 and their relation to weighted Hardy spaces (2.1.5), the estimates (2.2.13) and (2.2.14) imply the following Leibniz-type rule for Coifman-Meyer multipliers of order zero.

Corollary 2.2.2. *Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order 0. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.16}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.16); moreover, if $w \in A_\infty$, then

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.17}$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

Corollary 2.2.2 gives estimates related to those in Brummer-Naibo [8], where, using different methods, the following result was proven:

σ is a Coifman-Meyer multiplier of order 0, $1 < p_1, p_2 \leqslant \infty$, $\frac{1}{2} < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_{p,2}(w)$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)}. \tag{2.2.18}$$

Moreover, if $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.18).

Corollary 2.2.2 and the result stated above from Bummer-Naibo compare as follows:

- The estimate (2.2.16) allows for $0 < p, p_1, p_2 < \infty$, $w_1, w_2 \in A_\infty$, and the $H^P(w)$ on the left-hand side if $s > \tau_p(w)$. On the other hand, (2.2.18) requires $1 < p_1, p_2 \leqslant \infty$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$ and the smaller $L^p(w)$ norm on the left-hand side when $s > \tau_{p,2}(w)$. Therefore, (2.2.16) is less restrictive than (2.2.18) in terms of the indices p, p_1, and p_2 and the classes that the weights w_1 and w_2 belong to. However, since $\tau_{p,2}(w) \leqslant \tau_p(w)$, (2.2.16) is more restrictive than (2.2.18) in terms of the range of the regularity s.

- If $s > \tau_p(w)$, $1/2 < p < \infty$, $1 < p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$, then (2.2.16) implies (2.2.18). However if $\tau_p < \tau_p(w)$ then (2.2.16) does not imply (2.2.18) for $\tau_p < s < \tau_p(w)$. We next give examples of weights w_1 and w_2 such that the corresponding weight w satisfies $\tau_p < \tau_p(w)$. Let $1 < p_1 \leqslant p_2 < \infty$ and $w_1(x) = w_2(x) = w(x) = |x|^a$ with $n(r-1) < a < n(p_1 - 1)$ for some $1 < r < p_1$. Then $w \in A_{p_1}$, $A_{p_1} \subset A_{p_2}$, and $w \notin A_r$. This implies that $1 < \tau_w$ which leads to that $\tau_p < \tau_p(w)$.

- The estimate (2.2.18) implies (2.2.17) for $1 < p < \infty$, $w \in A_p$, and $s > \tau_{p,2}(w)$ and gives the endpoint estimate

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^p(w)} + \|f\|_{L^p(w)} \|D^s g\|_{L^\infty}.$$

  However, (2.2.17) allows $0 < p < \infty$ and $w \in A_\infty$ if $s > \tau_p(w)$.

2.2.2 Connection to Kato-Ponce inequalities

Using the lifting properties 2.1.9 and 2.1.10 we obtain the following corollary to Theorem 2.2.1 by setting $\sigma \equiv 1$.

Corollary 2.2.3. *Consider* $0 < p, p_1, p_2 \leqslant \infty$ *such that* $1/p = 1/p_1 + 1/p_2$ *and* $0 < q \leqslant \infty$;

let $w_1, w_2 \in A_\infty$ *and set* $w = w_1^{p/p_1} w_2^{p/p_2}$*. If* $0 < p, p_1, p_2 < \infty$ *and* $s > \tau_{p,q}(w)$*, it holds that*

$$\|fg\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^s_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.19}$$

If $0 < p, p_1, p_2 \leqslant \infty$ *and* $s > \tau_p(w)$*, it holds that*

$$\|fg\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^s_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^s_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.20}$$

where the Hardy spaces $H^{p_1}(w_1)$ *and* $H^{p_2}(w_2)$ *must be replaced by* L^∞ *if* $p_1 = \infty$ *or* $p_2 = \infty$*, respectively.*

If $w_1 = w_2$ *then different pairs of* p_1, p_2 *can be used on the right-hand sides of* (2.2.19) *and* (2.2.20)*; moreover, if* $w \in A_\infty$*, then*

$$\|fg\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^s_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.21}$$

where $0 < p < \infty$*,* $0 < q \leqslant \infty$ *and* $s > \tau_{p,q}(w)$*.*

In particular if we set $q = 2$ and use the connection between weighted Hardy spaces and weighted Triebel-Lizorkin spaces (2.1.5) we obtain the following corollary.

Corollary 2.2.4. *Consider* $0 < p, p_1, p_2 < \infty$ *such that* $1/p = 1/p_1 + 1/p_2$*; let* $w_1, w_2 \in A_\infty$ *and set* $w = w_1^{p/p_1} w_2^{p/p_2}$*. If* $s > \tau_p(w)$*, it holds that*

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.22}$$

If $w_1 = w_2$ *then different pairs of* p_1, p_2 *can be used on the right-hand side of* (2.2.22)*; moreover, if* $w \in A_\infty$*, then*

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n),$$

where $0 < p < \infty$ *and* $s > \tau_p(w)$.

2.2.3 Proof of Theorem 2.2.1

The following lemma will be useful in the proofs of Theorems 2.2.1 and 2.2.6.

Lemma 2.2.5. *Let* $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^n)$ *be such that* $\widehat{\phi_1}$ *and* $\widehat{\phi_2}$ *have compact supports and* $\widehat{\phi_1}\widehat{\phi_2} = \widehat{\phi_1}$. *If* $0 < r \leqslant 1$ *and* $\varepsilon > 0$, *it holds that*

$$\left|P_j^{\tau_a \phi_1} f(x)\right| \lesssim (1+|a|)^{\varepsilon + \frac{n}{r}} \mathcal{M}_r(P_j^{\phi_2} f)(x) \quad \forall x, a \in \mathbb{R}^n, j \in \mathbb{Z}, f \in \mathcal{S}(\mathbb{R}^n).$$

Proof. This estimate is a consequence of Lemma 2.1.3. In view of the supports of $\widehat{\phi_1}$ and $\widehat{\phi_2}$ we have $P_j^{\tau_a\phi_1} f = P_j^{\tau_a\phi_1} P_j^{\phi_2} f$ for $j \in \mathbb{Z}$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Applying Lemma 2.1.3 with $\phi(x) = 2^{nj}\tau_a\phi_1(2^j x)$, $A = 2^j$, $R \geqslant 1$ such that $\operatorname{supp}(\widehat{\phi_2}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant R\}$ and $d = \varepsilon + n/r$, we get

$$\begin{aligned}\left|P_j^{\tau_a \phi_1} f(x)\right| &\lesssim R^{n(\frac{1}{r}-1)} 2^{-jn} \left\|(1+\left|2^j \cdot\right|)^{\varepsilon+\frac{n}{r}} 2^{nj} \tau_a \phi_1(2^j \cdot)\right\|_{L^\infty} \mathcal{M}_r(P_j^{\phi_2} f)(x) \\ &\sim \left\|(1+\left|2^j \cdot\right|)^{\varepsilon+\frac{n}{r}} \tau_a \phi_1(2^j \cdot)\right\|_{L^\infty} \mathcal{M}_r(P_j^{\phi_2} f)(x) \quad \forall x, a \in \mathbb{R}^n, j \in \mathbb{Z}, f \in \mathcal{S}(\mathbb{R}^n).\end{aligned}$$

Since $\phi_1 \in \mathcal{S}(\mathbb{R}^n)$,

$$\left|\tau_a \phi_1(2^j x)\right| = \left|\phi_1(2^j x + a)\right| \lesssim \frac{(1+|a|)^{\varepsilon+\frac{n}{r}}}{(1+|2^j x|)^{\varepsilon+\frac{n}{r}}} \quad \forall x, a \in \mathbb{R}, j \in \mathbb{Z}.$$

Combining these two estimates completes the proof. □

We now prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Consider Φ, Ψ, T_σ^1, T_σ^2, $\{\mathcal{C}_j(a,b)\}_{j\in\mathbb{Z}, a,b\in\mathbb{Z}^n}$ as in Section 2.1.1. Let m, σ, p, p_1, p_2, q, s, w_1, w_2 and w be as in the hypotheses. For ease of notation, p_1 and p_2 will be assumed to be finite; the same proof applies for (2.2.14) if that is not the case, and for (2.2.15).

We next prove (2.2.13) and (2.2.14). Here we will only work with T^1_σ as the estimate for T^2_σ is shown through symmetry. Hence we will prove that

$$\left\|T^1_\sigma(f,g)\right\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} \quad \text{and} \quad \left\|T^1_\sigma(f,g)\right\|_{\dot B^s_{p,q}(w)} \lesssim \|f\|_{\dot B^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}.$$

Moreover, since $\|\sum f_j\|^{\min(p,q,1)}_{\dot F^s_{p,q}(w)} \lesssim \sum \|f_j\|^{\min(p,q,1)}_{\dot F^s_{p,q}(w)}$ and similarly for $\dot B^s_{p,q}(w)$, it suffices to prove that, given $\varepsilon > 0$ there exist $0 < r_1, r_2 \leqslant 1$ such that for all $g \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}_0(\mathbb{R}^n)$ ($f \in \mathcal{S}(\mathbb{R}^n) \cap \dot F^s_{p,q}(w)$ or $f \in \mathcal{S}(\mathbb{R}^n) \cap \dot B^s_{p,q}(w)$ if $m \geqslant 0$), it holds that

$$\left\|T^{a,b}(f,g)\right\|_{\dot F^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} (1+|b|)^{\varepsilon+\frac{n}{r_2}} \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}, \tag{2.2.23}$$

$$\left\|T^{a,b}(f,g)\right\|_{\dot B^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} (1+|b|)^{\varepsilon+\frac{n}{r_2}} \|f\|_{\dot B^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)}, \tag{2.2.24}$$

where

$$T^{a,b}(f,g) := \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)\,(S_j^{\tau_b\Phi} g)$$

and the implicit constants are independent of a and b. We will assume q finite; obvious changes apply if that is not the case.

In view of the supports of Ψ and Φ we have that

$$\operatorname{supp}(\mathcal{F}[\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)\,(S_j^{\tau_b\Phi} g)]) \subset \{\xi \in \mathbb{R}^n : |\xi| \lesssim 2^j\} \quad \forall j \in \mathbb{Z},\ a,b \in \mathbb{Z}^n.$$

For (2.2.23), Theorem 2.1.2(2.1.2), the bound (2.1.3) for $\mathcal{C}_j(a,b)$, and Hölder's inequality imply

$$
\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)}\\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)(x)\,(S_j^{\tau_b\Phi}g)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}\\
&\leqslant \left\|\sup_{j\in\mathbb{Z}}|(S_j^{\tau_b\Phi}g)|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}\\
&\leqslant \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\tau_a\Psi}f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)}.
\end{aligned}
$$

Consider $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ as in Section 2.1.2 such that $\widehat{\varphi}\equiv 1$ on $\mathrm{supp}(\widehat{\Phi})$ and $\widehat{\psi}\equiv 1$ on $\mathrm{supp}(\widehat{\Psi})$. Let $0<r_1<\min(1,p_1/\tau_{w_1},q)$; by Lemma 2.2.5 and the weighted Fefferman-Stein inequality we have that

$$
\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} &\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\mathcal{M}_{r_1}(\Delta_j^{\psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)}\\
&\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}} \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\psi}f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)}\\
&\sim (1+|a|)^{\varepsilon+\frac{n}{r_1}}\,\|f\|_{\dot{F}^{s+m}_{p,q}(w_1)},
\end{aligned}
$$

where the implicit constants are independent of a and f. Next, let $0<r_2<\min(1,p_2/\tau_{w_2})$; by Lemma 2.2.5 and the boundedness properties of the Hardy-Littlewood maximal operator on weighted Lebesgue space we have that

$$
\begin{aligned}
\left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)} &\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}} \left\|\mathcal{M}_{r_2}(\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|)\right\|_{L^{p_2}(w_2)}\\
&\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}} \left\|\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|\right\|_{L^{p_2}(w_2)}\\
&\sim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\,\|g\|_{H^{p_2}(w_2)},
\end{aligned}
$$

where the implicit constants are independent of b and g. Putting all together we obtain (2.2.23).

For (2.2.24), Theorem 2.1.2(2.1.2), the bound (2.1.3) for $\mathcal{C}_j(a,b)$ and Hölder's inequality give

$$\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{B}^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^p(w))} \\
&\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\right\|^q_{L^p(w)}\right)^{\frac{1}{q}} \\
&\leqslant \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\right\|^q_{L^{p_1}(w_1)}\right)^{\frac{1}{q}} \left\|\sup_{k\in\mathbb{Z}}|S_k^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)} \\
&\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}}\,\|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)},
\end{aligned}$$

where in the last inequality we have used Lemma 2.2.5 and the boundedness properties of $\mathcal{M}$ with $0<r_j<\min(1,p_j/\tau_{w_j})$ for $j=1,2$.

It is clear from the proof above that if $w_1=w_2$, then different pairs of p_1,p_2 related to p through the Hölder condition can be used on the right-hand sides of (2.2.13) and (2.2.14); in such case $w=w_1=w_2$. $\square$

We now remark in the condition 2.1.1 and its importance in the proof of Theorem 2.2.1. For convergence purposes, the relations between N in (2.1.2) and the powers $\varepsilon+n/r_1$ and $\varepsilon+n/r_2$ in (2.2.23) and (2.2.24) must be such that $(N-\varepsilon-n/r_1)\,r^*>n$ and $(N-\varepsilon-n/r_2)\,r^*>n$, where $r^*=\min(p,q,1)$. Moreover, r_1 and r_2 were selected so that $0<r_j<\min(1,p_j/\tau_{w_j},q)$ in the context of Triebel–Lizorkin spaces and $0<r_j<\min(1,p_j/\tau_{w_j})$ in the context of Besov spaces. Therefore, if $N>n(1/r^*+1/\min(1,p_1/\tau_{w_1},p_2/\tau_{w_2},q))$ in the Triebel–Lizorkin setting and $N>n(1/r^*+1/\min(1,p_1/\tau_{w_1},p_2/\tau_{w_2}))$ in the Besov setting, ε, r_1 and r_2 can be chosen so that all the conditions above are satisfied. In view of this and Remark 2.1.1, the multiplier σ in Theorem 2.2.1 needs only satisfy (2.1.1) for $|\alpha+\beta|\leqslant 2([n(1/r^*+1/\min(1,p_1/\tau_{w_1},p_2/\tau_{w_2},q))]+1)=\gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the Triebel–Lizorkin case and $|\alpha+\beta| \leqslant 2([n(1/r^* + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))] + 1) = \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the Besov case. An analogous observation follows for the multiplier σ in Theorem 2.2.6 in relation to the condition obtained from (2.1.1) with $|\xi| + |\eta|$ replaced by $1 + |\xi| + |\eta|$.

2.2.4 Inhomogeneous Leibniz-type rules

In this section we obtain Leibniz-type rules for Coifman-Meyer multiplier operators associated to inhomogeneous symbols. Our main result is the inhomogenous counterpart to Theorem 2.2.1.

Theorem 2.2.6. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi, \eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s+m}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{F^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n). \tag{2.2.25}$$

If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{B^s_{p,q}(w)} \lesssim \|f\|_{B^{s+m}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{B^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n), \tag{2.2.26}$$

where the local Hardy spaces $h^{p_1}(w_1)$ and $h^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.25) and (2.2.26); moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{p,q}(w)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

where $0 < p < \infty$, $0 < q \leqslant \infty$ and $s > \tau_{p,q}(w)$.

The proof of Theorem 2.2.6 follows along the same lines as the proof of Theorem 2.2.1.

The corollaries that follow are similar to those in the homogeneous setting. As an example we state the inhomogeneous counterpart to Corrolary 2.2.2.

Corollary 2.2.7. *Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order 0. Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|J^s(T_\sigma(f,g))\|_{h^p(w)} \lesssim \|J^s f\|_{h^{p_1}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|J^s g\|_{h^{p_2}(w_2)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n).$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.16); moreover, if $w \in A_\infty$, then

$$\|J^s(T_\sigma(f,g))\|_{h^p(w)} \lesssim \|J^s f\|_{h^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{h^p(w)} \quad \forall f, g \in \mathcal{S}(\mathbb{R}^n),$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

2.3 Leibniz rules in other functions spaces

The method used to prove Theorem (2.2.1) is quite versatile and can be applied to Triebel-Lizorkin and Besov spaces that are based in other quasi-Banach spaces. Here we highlight the important features of Lebesgue spaces that are necessary for the proof of Theorem (2.2.1) to be adapted to other settings. We also define Triebel-Lizorkin and Besov spaces that are based in other quasi-Banach spaces.

The main features of weighted Triebel-Lizorkin and Besov spaces used in the proof of Theorem (2.2.1) are the following:

(1) there exists $r > 0$ such that $\|f + g\|^r_{F^s_{p,q}(w)} \leqslant \|f\|^r_{F^s_{p,q}(w)} + \|g\|^r_{F^s_{p,q}(w)}$; similarly for the weighted inhomogeneous Besov spaces and the weighted homogeneous Triebel–Lizorkin and Besov spaces;

(2) Hölder's inequality in weighted Lebesgue spaces;

(3) the boundedness properties in weighted Lebesgue spaces of the Hardy–Littlewood maximal operator (for the Besov space setting) and the weighted Fefferman–Stein inequality (for the Triebel–Lizorkin space setting);

(4) Nikol'skij representations for weighted Triebel–Lizorkin and Besov spaces (Theorem 2.1.2).

In the followinig subsections we consider quasi-Banach spaces $\mathcal{X}$ such that properties (1)–(4) hold for the homogeneous and inhomogeneous $\mathcal{X}$-based Triebel-Lizorkin and Besov spaces. Corresponding versions of Theorems 2.2.1 and 2.2.6 hold in Triebel-Lizorkin and Besov spaces based in these spaces. The homogeneous χ-based Triebel-Lizorkin and Besov spaces denoted by $\dot{F}^s_{\mathcal{X},q}$ and $\dot{B}^s_{\mathcal{X},q}$ respectively are defined similarly to the weighted, homogeneous Triebel-Lizorkin and Besov spaces with the $||\cdot||_{L^p(w)}$ quasi-norm replaced with the $||\cdot||_{\mathcal{X}}$ quasi-norm. The inhomogeneous spaces are defined similarly.

2.3.1 Leibniz-type rules in the setting of Lorentz-based Triebel-Lizorkin and Besov spaces

For $0<p<\infty$ and $0<t\leqslant\infty$ or $p=t=\infty$, and an A_∞ weight w defined on $\mathbb{R}^n$, we denote by $L^{p,t}(w)$ the weighted Lorentz space consisting of complex-valued, measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,t}(w)} = \left(\int_0^\infty \left(\tau^{\frac{1}{p}} f^*_w(\tau)\right)^t \frac{d\tau}{\tau}\right)^{\frac{1}{t}} < \infty,$$

where $f^*_w(\tau) = \inf\{\lambda \geqslant 0 : w_f(\lambda) \leqslant \tau\}$ with $w_f(\lambda) = w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$. In the case that $t=\infty$, $\|f\|_{L^{p,t}(w)} = \sup_{t>0} t^{\frac{1}{p}} f^*_w(t)$. We note that for $p=t$ we have $L^{p,p}(w) = L^p(w)$ and $L^{1,\infty}(w)$ is the weighted weak $L^1(w)$ space. For more details on these spaces and their properties see Hunt [28].

We now turn our attention to the analogues to properties ((1))-((4)) in the setting of weighted Lorentz-based Triebel-Lizorkin and Besov spaces.

Property ((1)) follows from the work of Hunt [28] that the quasi norm $||\cdot||_{L^{p,t}(w)}$ is comparable to a quasi-norm $|||\cdot|||_{L^{p,t}(w)}$ that is subadditive. Therefore we have that the norms $|||\cdot|||_{\dot{F}^s_{(p,t),q}}$ and $||\cdot||_{\dot{F}^s_{(p,t),q}}$ are comparable where $|||\cdot|||_{\dot{F}^s_{(p,t),q}}$ is the quasi-norm with $||\cdot||_{L^{p,t}(w)}$ replaced by $|||\cdot|||_{L^{p,t}(w)}$ and for some $r>0$ $|||f+g|||^r_{\dot{F}^s_{(p,t),q}} \leqslant |||f|||^r_{\dot{F}^s_{(p,t),q}} + |||g|||^r_{\dot{F}^s_{(p,t),q}}$.

By using the following version of Hölder's inequality we obtain property ((2)) [Theorem 4.5, [28]]: Let $f,g \in L^{p,t}(w)$. Then for $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}$ it holds that

$$||fg||_{L^{p,t}(w)} \lesssim ||f||_{L^{p_1,t_1}(w)}||g||_{L^{p_2,t_2}(w)}.$$

The corresponding version of property ((3)) is: If $0<p<\infty$, $0<t,q\leqslant\infty$, $0<r<\min(p/\tau_w,q)$ and $0<r\leqslant t$, it holds that

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{M}_r(f_j)|^q\right)^{\frac{1}{q}}\right\|_{L^{p,t}(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{N}_0}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^{p,t}(w)} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in L^{p,t}(w)(\ell^q); \tag{2.3.27}$$

in particular, if $0<r<p/\tau_w$ and $0<r\leqslant t$, it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p,t}(w)} \quad \forall f \in L^{p,t}(w).$$

This is established by the extrapolation theorem [13, Theorem 4.10 and comments on page 70] when $r=1$, $1<p<\infty$, $1\leqslant t\leqslant\infty$ and $1<q\leqslant\infty$. The remaining cases are shown using the previous cases and the following scaling property for Lorentz spaces: For $0<s<\infty$ $\||f|^s\|_{L^{p,t}(w)} = \|f\|^s_{L^{sp,st}(w)}$.

The substitute for property (4) is the following Nikols'kij representation in weighted Lorentz spaces.

Theorem 2.3.1. *For $D>0$, let $\{u_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions* *such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{N}_0.$$

If $w \in A_\infty$, then the following holds:

(i) Let $0 < p < \infty$, $0 < t, q \leqslant \infty$ and $s > \left(\frac{1}{\min(p/\tau_w, t, q, 1)} - 1\right)$. If $\left\|\{2^{js} u_j\}_{j\in\mathbb{Z}}\right\|_{L^{(p,t)}(w)(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{N}_0} u_j$ converges in $F^s_{(p,t),q}(w)$ (in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$) and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{F^s_{(p,t),q}(w)} \lesssim \left\|\{2^{js} u_j\}_{j\in\mathbb{N}_0}\right\|_{L^{(p,t)}(w)(\ell^q)},$$

where the implicit constant depends only on n, D, s, p and q.

(ii) Let $0 < p, q \leqslant \infty$ and $s > \tau_{p,t}(w)$. If $\left\|\{2^{js} u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(L^{(p,t)}(w))} < \infty$, then the series $\sum_{j\in\mathbb{N}_0} u_j$ converges in $B^s_{(p,t),q}(w)$ (in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$) and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{B^s_{(p,t),q}(w)} \lesssim \left\|\{2^{js} u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(L^{(p,t)}(w))},$$

where the implicit constant depends only on n, D, s, p and q.

To prove Theorem 2.3.1 we make use of the following Lemma.

Lemma 2.3.2. *Suppose $w \in A_\infty$, $0 < p \leqslant \infty$, $A > 0$, $R \geqslant 1$, and $d > b > n/\min(1, p/\tau_w, t)$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and f is such that $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant DR\}$, it holds that*

$$\|\phi * f\|_{L^{p,t}(w)} \lesssim R^{b-n} D^{-n} \left\|(1 + |D\cdot|^d)\phi\right\|_{L^\infty} \|f\|_{L^{p,t}(w)},$$

where the implicit constant is independent of A, R, ϕ and f.

Proof. Set $r := n/b < \min(1, p/\tau_w, t)$. The hypothesis $d > b$ means $d > n/r$ and Lemma 2.1.3 yields

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} D^{-n} \left\|(1 + |D\cdot|)^d \phi\right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n.$$

Since $r < p/\tau_w$ and $r < t$, we have $\|\mathcal{M}_r f\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p,t}(w)}$ and therefore

$$\|\phi * f\|_{L^{p,t}(w)} \lesssim R^{n(\frac{1}{r}-1)} D^{-n} \left\|(1+|D\cdot|)^d \phi\right\|_{L^\infty} \|f\|_{L^{p,t}(w)} ;$$

observing that $1/r - 1 = (b-n)/n$, the desired estimate follows. $\square$

We now prove Theorem 2.3.1.

Proof. We first prove the theorem for finite families as in the proof of Theorem 2.1.2. Suppose $\{u_j\}_{j\in\mathbb{N}_0}$ is such that $u_j = 0$ for all j except those belonging to some finite subset of $\mathbb{Z}$; this assumption allows us to avoid convergence issues since all the sums considered will be finite.

For Part 2.3.1, let D, w, p, q, t and s be as in the hypotheses. Fix $0 < r < \min(1, p/\tau_w, q, t)$ such that $s > n(1/r - 1)$; note that the latter is possible since $s > \tau_{p,q}(w)$.

Let $k_0 \in \mathbb{N}_0$ be such that $2^{k_0-1} < D \leqslant 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{N}_0.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $F^s_{(p,t),q}(w)$ in Section 2.1.2. We have

$$\Delta^\psi_j u = \sum_{\ell\in\mathbb{N}_0} \Delta^\psi_j u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta^\psi_j u_\ell = \sum_{k=-k_0}^{\infty} \Delta^\psi_j u_{j+k}. \tag{2.3.28}$$

We now use Lemma 2.1.3 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $D = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geqslant -k_0$, we get $R \geqslant 1$.) Fixing $d > n/r$ and applying Lemma 2.1.3, we get

$$\begin{aligned} |\Delta^\psi_j u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\|(1+|2^j\cdot|)^d 2^{jn}\psi(2^j\cdot)\right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\ &\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y\in\mathbb{R}^n} (1+|2^j y|)^d |\psi(2^j y)|\right) \mathcal{M}_r(u_{j+k})(x). \end{aligned}$$

Hence,

$$2^{js}|\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.3.28),

$$2^{js}|\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x).$$

Since $1/r-1-s/n<0$, Lemma 2.1.5 yields

$$\left\|\{2^{js}|\Delta_j^\psi u|\}_{j\in\mathbb{Z}}\right\|_{L^{p,t}(w)(\ell^q)} \lesssim \left\|\{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p,t}(w)(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{N}_0}$. Applying the weighted Fefferman-Stein inequality for Lorentz spaces to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{F^s_{(p,t),q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{L^{p,t}(\ell^q)}.$$

For Part (2.3.1), let D, w, p, q and s be as in the hypotheses and k_0 be as above. Consider $\Delta_j^\psi u_{j+k}$ in (2.3.28) and apply Lemma 2.3.2 with $\phi(x) = 2^{jn}\psi(2^{-j}x)$, $f = u_{j+k}$, $D = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n/\min(1, p/\tau_w, t) < b < n+s$; note that such b exists since $s > \tau_p(w)$. We get

$$\left\|\Delta_j^\psi u_{j+k}\right\|_{L^{p,t}(w)} \lesssim 2^{(k+k_0)(b-n)}2^{-jn}\left\|(1+|2^j\cdot|)^d 2^{jn}\psi(2^{-j}\cdot)\right\|_{L^\infty}\|u_{j+k}\|_{L^p(w)} \sim 2^{k(b-n)}\|u_{j+k}\|_{L^{p,t}(w)},$$

and setting $p^* := \min(p,t,1)$ we obtain

$$2^{jsp^*}\left\|\Delta_j^\psi u\right\|^{p^*}_{L^{p,t}(w)} \lesssim 2^{jsp^*}\sum_{k=-k_0}^{\infty}\left\|\Delta_j^\psi u_{j+k}\right\|^{p^*}_{L^{p,t}(w)} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*}2^{sp^*(j+k)}\|u_{j+k}\|^{p^*}_{L^{p,t}(w)}.$$

Hence, applying Lemma 2.1.5, it follows that

$$\|u\|_{B^s_{p,t,q}(w)} \lesssim \left\| \left\{ \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|^{p^*}_{L^{p,t}(w)} \right\}_{j\in\mathbb{N}_0} \right\|^{\frac{1}{p^*}}_{\ell^{q/p^*}} \lesssim \left\| \{2^{js}u_j\}_{j\in\mathbb{N}_0} \right\|_{\ell^q(L^{p,t}(w))},$$

as desired.

We know prove the theorem for families that are not necessarily finite. Using the theorem for finite families we have that

$$\|U_N - U_M\|_{F^s_{(p,t),q}} \lesssim \left\| \{2^{js}u_j\}_{M+1\leqslant j\leqslant N} \right\|_{L^{p,t}(w)(\ell^q)}. \tag{2.3.29}$$

Now we apply the following version of the dominated convergence theorem.

Suppose $f_n \to f$ in measure and $|f_n(x)| \leqslant |g(x)|$ for some $g \in L^{p,t}(w)$. Then

$$\lim_{n\to\infty} \|f_n - f\|_{L^{p,t}(w)} = 0.$$

We need to check that the functions $U_{N,M} = \left(\sum_{j=M+1}^{N}(2^{js}u_j)^q\right)^{\frac{1}{q}}$, $U = 0$, and $g = \left(\sum_{j=0}^{\infty}(2^{js}u_j)^q\right)^{\frac{1}{q}}$

satisfy the hypotheses of the theorem. Now we need to check that $f_{N,M} \to f$ in measure. Let $A_N = \{x : \sum_{j=N}^{\infty} |2^{jsq}u_j|^q > \tau^q\}$. Because $\{u_j\} \subset L^{(p,t)}(w)(\ell^q)$ we have that $|A_1| < \infty$. Then since $|A_{N+1}| \leqslant |A_N|$ it follows that $\lim_{N\to\infty}|A_N| = |\cap A_N|$. $\{u_j\} \subset L^{(p,t)}(w)(\ell^q)$ so $u_j \to 0$ as $j \to \infty$. Therefore $f_N \to 0$ in measure. This implies that

$$\|U_N - U_M\|_{F^s_{(p,t),q}} \lesssim \left\| \{2^{js}u_j\}_{M+1\leqslant j\leqslant N} \right\|_{L^{p,t}(w)(\ell^q)} \to 0.$$

Then $\sum_{j=0}^{\infty} u_j$ converges in $F^s_{(p,t),q}$.

Now we consider the case where $q = \infty$. Then $\{2^{j(s-\epsilon)}\}_{j\geqslant 0}$ belongs to $\ell^1(L^{p,t}(w))$ for any $\epsilon > 0$. Then by the case for $q < \infty$ $\sum_{j=0}^{\infty} u_j$ converges in $B^s_{(p,t),1}$ and so it converges in S'.

Then by using the case for finite families applied to $\{u_j\}_{0\leqslant j\leqslant N}$ (this case holds because the lemmas used only depend on the on the indexes considered and maximal function inequalities in Lorentz spaces) we have that

$$\|U_N\|_{F^s_{(p,t),q}} \lesssim ||\{2^{js}u_j\}_{0\leqslant j\leqslant N}||_{F^s_{(p,t),\infty}} \leqslant ||\{2^{js}u_j\}||_{F^s_{(p,t),\infty}}.$$

Then using the Fatou property completes the proof. □

As it was noted in Remark 2.1.3 Triebel-Lizorkin and Besov spaces based in weighted Lorentz spaces posses the Fatou property. As a consequence the proof of Theorem 2.3.1 follows along the sames lines as the proof of Theorem 2.1.2. We know verify properties (i) and (ii) in Remark 2.1.3 for such spaces.

Property (i) is easily verified using the definition of $L^{p,t}(w)$.

For property (ii) it holds that $(\liminf_{n\to\infty} f_n)^* \leqslant \liminf_{n\to\infty} f_n^*$. Then by applying Fatou's Lemma we have property (ii).

With these four properties theorems analogous those earlier in this chapter hold in the setting of weighted Lorentz-based Triebel-Lizorkin and Besov spaces and as an example the analogue to Theorem 2.2.6 in the context of the spaces $F^s_{(p,t),q}(w)$ is below.

Theorem 2.3.3. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. If $w \in A_\infty$, $0 < p, p_1, p_2 < \infty$ and $0 < t, t_1, t_2 \leqslant \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, $0 < q \leqslant \infty$ and $s > \tau_{p,t,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|g\|_{F^{s+m}_{(p_2,t_2),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Different pairs of p_1, p_2 and t_1, t_2 can be used on the right-hand side of the inequality above. Moreover, if $w \in A_\infty$, $0 < p < \infty$, $0 < t, q \leqslant \infty$ and $s > \tau_{p,t,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{(p,t),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

By the Fefferman–Stein inequality (2.3.27) the lifting property $\|f\|_{F^s_{(p,t),q}} \simeq \|J^s f\|_{F^0_{(p,t),q}}$ holds true for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < t, q \leqslant \infty$. Then, under the assumptions of Theorem 2.3.3 we obtain, in particular,

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|J^s g\|_{F^0_{(p_2,t_2),q}(w)};$$

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^0_{(p,t),q}(w)}.$$

These last two estimates supplement the results in [14, Theorem 6.1], where related Leibniz-type rules in Lorentz spaces were obtained.

2.3.2 Leibniz-type rules in the setting of Morrey based Triebel-Lizorkin and Besov spaces

Given $0 < p \leqslant t < \infty$ and $w \in A_\infty$, we denote by $M^t_p(w)$ the weighted Morrey space consisting of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M^t_p(w)} = \sup_{B \subset \mathbb{R}^n} w(B)^{\frac{1}{t}-\frac{1}{p}} \left(\int_B |f(x)|^p \, w(x) \, dx \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all Euclidean balls B contained in $\mathbb{R}^n$. We note that $M^p_p(w) = L^p(w)$. For more details on Morrey spaces see the work Rosenthal–Schmeisser [46] and the references contained therein. The corresponding weighted inhomogeneous Triebel–Lizorkin spaces and inhomogeneous Besov spaces are denoted by $F^s_{[p,t],q}(w)$ and $B^s_{[p,t],q}(w)$, respectively. These Morrey-based Triebel–Lizorkin and Besov spaces are independent of the choice of φ and ψ given in Section 2.1.2 and are quasi-Banach spaces that contain $\mathcal{S}(\mathbb{R}^n)$ (see the works Kozono–Yamazaki [35], Mazzucato [38], Izuki et al. [29] and the references they contain). The corresponding local Hardy spaces are denoted by $h^t_p(w)$.

We now show the analogues to properties (1)–(4) for weighted Morrey spaces.

For (1) we use that for $0 < s < \infty$ $\||f|^s\|_{M^t_p(w)} = \|f\|^s_{M^{st}_{sp}(w)}$ and the corresponding property for weighted Lebesgue spaces to get

$$\|f+g\|^r_{M^t_p(w)} \lesssim \|f\|^r_{M^t_p(w)} + \|g\|^r_{M^t_p(w)}$$

for $r = min(1,p)$. It follows that for $r := min(1,p,q)$

$$\|f+g\|^r_{F^s_{[p,t],q}(w)} \lesssim \|f\|^r_{F^s_{[p,t],q}(w)} + \|g\|^r_{F^s_{[p,t],q}(w)}$$

with similar inequalities for inhomogeneous weighted Morry-based Besov spaces and homogeneous weighted Morry-based Triebel-Lizorkin and Besov spaces.

A version of property (2) follows from Hölder's inequality for weighted Lebesgue spaces. For $0 < p \leqslant t < \infty$, $0 < p_1 \leqslant t_1 < \infty$ and $0 < p_2 \leqslant t_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, then

$$\|fg\|_{M^t_p(w)} \leqslant \|f\|_{M^{t_1}_{p_1}(w)} \|g\|_{M^{t_2}_{p_2}(w)};$$

also, if $0 < p \leqslant t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $w = w_1^{p/p_1} w_2^{p/p_2}$ for weights w_1 and w_2, then

$$\|fg\|_{M^t_p(w)} \leqslant \|f\|_{M^{\frac{p_1 t}{p}}_{p_1}(w_1)} \|g\|_{M^{\frac{p_2 t}{p}}_{p_2}(w_2)}.$$

The analogue to property (3) is the following Fefferman-Stein inequality: Let $0 < p \leqslant t < \infty$, $0 < q \leqslant \infty$ and $0 < r < min(p/\tau_w, q)$, then

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{M^t_p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{M^t_p(w)} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in M^t_p(w)(\ell^q). \tag{2.3.30}$$

If $0 < p \leqslant t < \infty$ and $0 < r < p/\tau_w$, then

$$\|\mathcal{M}_r(f)\|_{M_p^t(w)} \lesssim \|f\|_{M_p^t(w)} \quad \forall f \in M_p^t(w).$$

The case for $r = 1$, $1 < p \leqslant t < \infty$ and $1 < q \leqslant \infty$ are shown using extrapolation and the Fefferman-Stein inequality in weighted Lebesgue spaces. For the extrapolation theorem see [46, Theorem 5.3]. The remaining cases are shown using that for $0 < s < \infty$ $\||f|^s\|_{M_p^t(w)} = \|f\|^s_{M_{sp}^{st}(w)}$ and the previous case.

The Nikol'skij representation for weighted Morrey-based Triebel-Lizorkin and Besov spaces is as follows.

Theorem 2.3.4. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\, 2^j) \quad \forall j \in \mathbb{N}_0.$$

If $w \in A_\infty$, then the following holds:

(i) Let $0 < p \leqslant t < \infty$, $0 < q \leqslant \infty$ and $s > \left(\frac{1}{\min(p/\tau_w, t, q, 1)} - 1\right)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{M_p^t(w)(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{N}_0} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{F^s_{[p,t],q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{M_p^t(w)(\ell^q)},$$

where the implicit constant depends only on n, D, s, p and q.

(ii) Let $0 < p, q \leqslant \infty$ and $s > \tau_{p,t}(w)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\|_{\ell^q(M_p^t(w))} < \infty$, then the series $\sum_{j\in\mathbb{N}_0} u_j$ converges in $B^s_{[p,t],q}(w)$ (in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$) and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{B^s_{[p,t],q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(M_p^t(w))},$$

where the implicit constant depends only on n, D, s, p and q.

The proof of Theorem 2.3.4 uses Lemma 2.1.3, Lemma 2.1.5, and a modified version of Lemma (2.1.4).

Lemma 2.3.5. *Let* $0 < p \leqslant t < \infty$, $A > 0$, $R \geqslant 1$ *and* $d > b > \frac{n}{min(p/\tau_w,1)}$. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* $supp(\widehat{f}) \subset \{\xi : |\xi| \leqslant AR\}$. *Then*

$$\|\phi * f\|_{M_p^t(w)} \lesssim R^{b-n} A^{-n} \left\|(1 + |A \cdot|^d)\phi\right\|_{L^\infty} \|f\|_{M_p^t(w)} \tag{2.3.31}$$

with the implicit constant independent of $R, A, \phi,$ *and* f.

The proof follows along the same lines as Lemma 2.1.3. We now prove Theorem 2.3.4.

Proof. First we will assume that $u_j = 0$ for all but finitely many j. For Part (2.3.4), let D, w, p, q, t and s be as in the hypotheses. Fix $0 < r < \min(1, p/\tau_w, q)$ such that $s > n(1/r - 1)$; note that the latter is possible since $s > \tau_{p,q}(w)$.

Let $k_0 \in \mathbb{N}_0$ be such that $2^{k_0 - 1} < D \leqslant 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell + k_0}) \quad \forall \ell \in \mathbb{N}_0.$$

Define $u = \sum_{\ell \in \mathbb{Z}} u_\ell$ and let ψ be as in the definition of $F^s_{[p,t],q}(w)$ in Section 2.1.2. We have

$$\Delta_j^\psi u = \sum_{\ell \in \mathbb{N}_0} \Delta_j^\psi u_\ell = \sum_{\ell = j - k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k = -k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.3.32}$$

We now use Lemma 2.1.3 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $D = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geqslant -k_0$, we get $R \geqslant 1$.) Fixing $d > n/r$ and applying Lemma 2.1.3, we get

$$\begin{aligned}
|\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r} - 1)} 2^{kn(\frac{1}{r} - 1)} 2^{-jn} \left\|(1 + |2^j \cdot|)^d 2^{jn} \psi(2^j \cdot)\right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\
&\sim 2^{kn(\frac{1}{r} - 1)} \left(\sup_{y \in \mathbb{R}^n} (1 + |2^j y|)^d |\psi(2^j y)| \right) \mathcal{M}_r(u_{j+k})(x).
\end{aligned}$$

Hence,

$$2^{js}|\Delta_j^{\psi} u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.3.32),

$$2^{js}|\Delta_j^{\psi} u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{s(j+k)}\mathcal{M}_r(u_{j+k})(x).$$

Since $1/r - 1 - s/n < 0$, Lemma 2.1.5 yields

$$\left\| \{2^{js}|\Delta_j^{\psi} u|\}_{j\in\mathbb{Z}} \right\|_{M_p^t(w)(\ell^q)} \lesssim \left\| \{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}} \right\|_{M_p^t(w)(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{N}_0}$. Applying the weighted Fefferman-Stein inequality for Morrey spaces to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{F^s_{[p,t],q}(w)} \lesssim \left\| \{2^{js}u_j\}_{j\in\mathbb{N}_0} \right\|_{M_p^t(w)(\ell^q)}.$$

Now we prove the theorem for families with are not necessarily finite. Let $U_N := \sum_{j=0}^{N} u_j$. Then by the finite family case we have that

$$\|u_N\|_{F^s_{[p,t],q}(w)} \lesssim \left\| \{2^{js}u_j\}_{0\leqslant j\leqslant N} \right\|_{M_t^p(w)(\ell^q)} \lesssim \left\| \{2^{js}u_j\} \right\|_{M_t^p(w)(\ell^q)} < \infty.$$

Assume that $1 < q < \infty$. Because $\|\{2^{js}u_j\}\|_{M_t^p(w)(\ell^q)} < \infty$ we have that $\sup_j \|\{2^{js}u_j\}\|_{M_t^p(w)} < \infty$ which falls in the $q = \infty$ case for part (2.3.4). Then $\sum_{j=0}^{\infty} u_j$ converges in S' and from the Fatou property

$$\liminf_{N\to\infty} \|U_N\|_{F^s_{[p,t],q}(w)} \leqslant \|U\|_{F^s_{[p,t],q}(w)}\, v \lesssim \left\| \{2^{js}u_j\} \right\|_{M_t^p(w)(\ell^q)}.$$

If $q = \infty$ then $\{2^{j(s-\epsilon)}u_j\}_{j\in\mathbb{N}_0}$ is in $\ell^1(M_p^t)(w))$ for any $\epsilon > 0$. Then the case for finite q

shows that $\{2^{j(s-\epsilon)}u_j\}_{j\in\mathbb{N}_0}$ converges in $B^{s-\epsilon}_{[p,t],1}(w)$ and we have convergence in $\mathcal{S}'$. Then using the finite family case we have

$$\|U_N\|_{F^s_{[p,t],\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{0\leqslant j\leqslant N}\right\|_{\ell^1(M^t_p)(w)} \leqslant \left\|\{2^{js}u_j\}\right\|_{\ell^1(M^t_p)(w)} < \infty$$

Then after using the Fatou property of $F^s_{[p,t],q}(w)$ we are finished.

Now for part (2.3.4) we first assume that $u_j = 0$ for all but finitely many j. Let D, w, p, q, t and s be as in the hypotheses and k_0 be as above. Consider $\Delta^\psi_j u_{j+k}$ in (2.3.32) and apply Lemma 2.3.5 with $\phi(x) = 2^{jn}\psi(2^{-j}x)$, $f = u_{j+k}$, $D = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n/\min(1, p/\tau_w, t) < b < n + s$; note that such b exists since $s > \tau_p(w)$. We get

$$\left\|\Delta^\psi_j u_{j+k}\right\|_{L^{p,t}(w)} \lesssim 2^{(k+k_0)(b-n)}2^{-jn}\left\|(1+|2^j\cdot|)^d 2^{jn}\psi(2^{-j}\cdot)\right\|_{L^\infty}\|u_{j+k}\|_{M^t_p(w)} \sim 2^{k(b-n)}\|u_{j+k}\|_{L^{p,t}(w)},$$

and setting $p^* := \min(p, q, 1)$ we obtain

$$2^{jsp^*}\left\|\Delta^\psi_j u\right\|^{p^*}_{M^t_p(w)} \lesssim 2^{jsp^*}\sum_{k=-k_0}^{\infty}\left\|\Delta^\psi_j u_{j+k}\right\|^{p^*}_{M^t_p(w)} = \sum_{k=-k_0}^{\infty}2^{k(b-n-s)p^*}2^{sp^*(j+k)}\|u_{j+k}\|^{p^*}_{M^t_p(w)}.$$

Hence, applying Lemma 2.1.5, it follows that

$$\|u\|_{B^s_{[p,t],q}(w)} \lesssim \left\|\left\{\sum_{k=-k_0}^{\infty}2^{k(b-n-s)p^*}2^{sp^*(j+k)}\|u_{j+k}\|^{p^*}_{M^t_p(w)}\right\}_{j\in\mathbb{N}_0}\right\|^{\frac{1}{p^*}}_{\ell^{q/p*}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(M^t_p(w))},$$

as desired.

For a family that is not necessarily finite we apply the finite family case to $U_N - U_M := \sum_{j=M+1}^N u_j$. For finite q this gives us

$$\|U_N - U_M\|_{\dot{B}^s_{[p,t],q}(w)} \lesssim \left\|\{2^{js}u_j\}_{M+1\leqslant j\leqslant N}\right\|_{\ell^q(M^t_p(w))} \leqslant \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(M^t_p(w))} < \infty$$

and by the dominated convergence theorem the left side converges to 0 as $M, N \to \infty$. So

U_N converges in $B^s_{[p,t],q}(w)$. If $q = \infty$ then $\{2^{j(s-\epsilon)}\} \in \ell^1(M^t_p(w))$. By the case for finite q we have that $\sum_{j=0}^N u_j$ converges in $B^{s-\epsilon}_{[p,t],q}(w)$. Therefore it converges in $\mathcal{S}'$. $\square$

We know verify that the Fatou property holds in Triebel-Lizorkin and Besov spaces based in weighted Morrey spaces.

As in Remark 2.1.3 (i) is easily verified using the definition of $\|\cdot\|_{M^p_t(w)}$. Also (ii) holds by applying Fatou's lemma.

2.3.3 Variable Lebesgue spaces

Let $\P_0$ be the collection of measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0 \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

For $p(\cdot) \in \P_0$, the variable-exponent Lebesgue space $L^{p(\cdot)}$ consists of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\} < \infty;$$

such quasi-norm turns $L^{p(\cdot)}$ into a quasi-Banach space (Banach space if $p_- \geqslant 1$). We note that if $p(\cdot) = p$ is constant then $L^{p(\cdot)} \simeq L^p$ with equality of norms and that

$$\left\||f|^t\right\|_{L^{p(\cdot)}} = \|f\|^t_{L^{tp(\cdot)}} \quad \forall\, t > 0. \tag{2.3.33}$$

Let $\mathcal{B}$ be the family of all $p(\cdot) \in \P_0$ such that $\mathcal{M}$, the Hardy–Littlewood maximal operator, is bounded from $L^{p(\cdot)}$ to $L^{p(\cdot)}$. Such exponents satisfy $p_- > 1$ and the following log-Hölder continuity properties

- there exists a constant C_0 such that for all $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$

$$|p(x) - p(y)| \leqslant \frac{C_0}{-log(|x-y|)},$$

- there exist constants C_∞ and p_∞ such that for all $x \in \mathbb{R}^n$

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{log(e + |x|)}.$$

Furthermore if $\tau_0 > 0$ is such that $p(\cdot)/\tau_0 \in \mathcal{B}$ then $p(\cdot)/\tau \in \mathcal{B}$ for $0 < \tau < \tau_0$. Indeed, by Jensen's inequality it holds that $\mathcal{M}(f)^{\tau_0/\tau}(x) \leqslant \mathcal{M}(|f|^{\tau_0/\tau})(x)$ so by 2.3.33 we have that

$$\begin{aligned}
\|\mathcal{M}f\|_{L^{\frac{p(\cdot)}{\tau}}} &\leqslant \left\|\mathcal{M}(|f|^{\tau_0/\tau})^{\tau/\tau_0}\right\|_{L^{\frac{p(\cdot)}{\tau}}} \\
&= \left\|\mathcal{M}(|f|^{\tau_0/\tau})\right\|_{L^{\frac{p(\cdot)}{\tau_0}}}^{\tau/\tau_0} \\
&\leqslant \left\||f|^{\frac{\tau_0}{\tau}}\right\|_{L^{\frac{p(\cdot)}{\tau}}}^{\frac{\tau}{\tau_0}} \\
&= \|f\|_{L^{\frac{p(\cdot)}{\tau}}} .
\end{aligned}$$

We then define

$$\tau_{p(\cdot)} = \sup\{\tau > 0 : \tfrac{p(\cdot)}{\tau} \in \mathcal{B}\}, \quad p(\cdot) \in \P_0^*,$$

where $\P_0^*$ denotes the class of variable exponents in $\P_0$ such that $p(\cdot)/\tau_0 \in \mathcal{B}$ for some $\tau_0 > 0$. Note that $\tau_{p(\cdot)} \leqslant p_-$.

Given $s \in \mathbb{R}$, $0 < q \leqslant \infty$ and $p(\cdot) \in \P_0$, the corresponding inhomogeneous Triebel-Lizorkin and Besov spaces are denoted by $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$, respectively. If $p(\cdot) \in \P_0^*$, these spaces are independent of the functions ψ and φ given in Section 2.1.2 (see Xu [49]), contain $\mathcal{S}(\mathbb{R}^n)$ and are quasi-Banach spaces. If $p(\cdot) \in \mathcal{B}$ and $s > 0$, $F^s_{p(\cdot),2}$ coincides with the variable-exponent Sobolev space $W^{s,p(\cdot)}$ (see Gurka et al. [26] and Xu [48]). More general versions of variable-exponent Triebel–Lizorkin and Besov spaces, where s and q are also allowed to be functions, were introduced in Diening at al. [16] and Almeida–Hästö [1], respectively. The local Hardy space with variable exponent $p(\cdot) \in \P_0$, denoted $h^{p(\cdot)}$, is defined analogously to $h^p(w)$ with the quasi-norm in $L^p(w)$ replaced by the quasi-norm in $L^{p(\cdot)}$.

We now consider the analogues of properties (1)-(4) in the setting of variable Lebesgue based spaces.

For (1) we apply 2.3.33 to get for $r = \min(p_-, q, 1)$

$$\|f+g\|^r_{F^s_{p(\cdot),q}} \leqslant \|f\|^r_{F^s_{p(\cdot),q}} + \|g\|^r_{F^s_{p(\cdot),q}},$$

$$\|f+g\|^r_{B^s_{p(\cdot),q}} \leqslant \|f\|^r_{B^s_{p(\cdot),q}} + \|g\|^r_{B^s_{p(\cdot),q}}.$$

To prove (2) we use [12, Corollary 2.28] and 2.3.33 to get the following version of Hölder's inequality: If $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot) \in \P_0$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ then

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} \quad \forall f \in L^{p_1(\cdot)}, g \in L^{p_2(\cdot)}.$$

The case when $p(\cdot) \in \P_0$ has $p_- \geqslant 1$ is shown in [12, Corollary 2.28]. If $0 < p_- < 1$ then we use 2.3.33 to get

$$\|fg\|_{L^{p(\cdot)}} = \||fg|^{p_-}\|^{\frac{1}{p_-}}_{L^{\frac{p(\cdot)}{p_-}}}$$

and then use the first case since $\frac{p(\cdot)}{p_-} > 1$.

A Fefferman-Stein inequality in variable exponent Lebesgue spaces follows from the discussion in [12, Section 5.6.8] and (2.3.33). For property (3) we have the following: If $p(\cdot) \in \P_0^*$, $0 < q \leqslant \infty$ and $0 < r < \min(\tau_{p(\cdot)}, q)$ then

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^q);$$

in particular, if $0 < r < \tau_{p(\cdot)}$ it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \quad \forall f \in L^{p(\cdot)}.$$

Property (4), the Nikolśkij representation for $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$, for variable exponent Lebesgue spaces is stated below.

Theorem 2.3.6. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that* $\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j)$ *for all $j \in \mathbb{Z}$. Let $p(\cdot) \in ¶_0^*$, $0 < q \leqslant \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^{p(\cdot)}(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $F^s_{p(\cdot),q}$ (in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$) and*

$$\left\|\sum_{j\in\mathbb{N}_0} u_j\right\|_{F^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^q)},$$

where the implicit constant depends only on n, D, s, $p(\cdot)$ and q. An analogous statement holds true for $B^s_{p(\cdot),q}$ with $s > n(1/\min(\tau_{p(\cdot)}, 1) - 1)$.

To prove Theorem 2.3.6 we use the following lemma.

Lemma 2.3.7. *Let $p(\cdot) \in \mathcal{P}_0$, $A > 0$, $R \geqslant 1$ and $d > b > \frac{n}{min(\tau_{p(\cdot)},1)}$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $supp(\widehat{f}) \subset \{\xi : |\xi| \leqslant AR\}$. Then*

$$\|\phi * f\|_{L^{p(\cdot)}} \lesssim R^{b-n}A^{-n}\left\|(1 + |A\cdot|^d)\phi\right\|_{L^\infty}\|f\|_{L^{p(\cdot)}}, \tag{2.3.34}$$

with the implicit constant independent of R, A, ϕ, and f.

We now prove Theorem 2.3.6.

Proof. We first prove the theorem for finite families. Assume that $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ is such that $u_j \equiv 0$ for all but finitely many j. Let D, $p(\cdot)$, q, and s be as in the hypotheses. Fix $0 < r < min(1, p_-, q)$ such that $s > n(1/r - 1)$; note that this is possible since $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$. Let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \leqslant 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{Z}.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $\dot{F}^s_{p(\cdot),q}(w)$. We have

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.3.35}$$

We will use Lemma 2.1.3 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geqslant -k_0$, we get $R \geqslant 1$.) Fixing $d > n/r$ and applying Lemma 2.1.3, we get

$$\begin{aligned}|\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\|(1+|2^j\cdot|)^d 2^{jn}\psi(2^j\cdot)\right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\ &\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y\in\mathbb{R}^n}(1+|2^j y|)^d |\psi(2^j y)|\right) \mathcal{M}_r(u_{j+k})(x).\end{aligned}$$

Hence,

$$2^{js}|\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.3.35),

$$2^{js}|\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x).$$

Since $1/r - 1 - s/n < 0$, Lemma 2.1.5 yields

$$\left\|\{2^{js}|\Delta_j^\psi u|\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^q)} \lesssim \left\|\{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{Z}}$. Applying the weighted Fefferman-Stein inequality to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{\dot{F}^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^q)}.$$

For the space $\dot{B}^s_{p(\cdot),q}$, let D, w, p, q and s be as in the hypotheses and k_0 be as above. Consider $\Delta_j^\psi u_{j+k}$ in (2.1.11) and apply Lemma 2.3.7 with $\phi(x) = 2^{jn}\psi(2^{-j}x)$, $f = u_{j+k}$, $D = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n/\min(1, p/\tau_w) < b < n+s$; note that such b exists since

$s > \tau_p(w)$. We get

$$\left\|\Delta_j^{\psi} u_{j+k}\right\|_{L^{p(\cdot)}} \lesssim 2^{(k+k_0)(b-n)} 2^{-jn} \left\|(1+|2^j \cdot|)^d 2^{jn} \psi(2^{-j} \cdot)\right\|_{L^\infty} \|u_{j+k}\|_{L^{p(\cdot)}} \sim 2^{k(b-n)} \|u_{j+k}\|_{L^{p(\cdot)}},$$

and setting $p^* := \min(p_-, 1)$ we obtain

$$2^{jsp^*} \left\|\Delta_j^{\psi} u\right\|_{L^{p(\cdot)}}^{p^*} \lesssim 2^{jsp^*} \sum_{k=-k_0}^{\infty} \left\|\Delta_j^{\psi} u_{j+k}\right\|_{L^{p(\cdot)}}^{p^*} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|_{L^{p(\cdot)}}^{p^*}.$$

Hence, applying Lemma 2.1.5, it follows that

$$\|u\|_{\dot{B}^s_{p(\cdot),q}} \lesssim \left\|\left\{\sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \|u_{j+k}\|_{L^{p(\cdot)}}^{p^*}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^{q/p^*}}^{\frac{1}{p^*}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^{p(\cdot)})},$$

as desired. Now for families that are not necessarily finite let $U_N := \sum_{j=0}^{N} u_j$. First we assume that $0 < q < \infty$. Then for $M \leqslant N$ we have

$$\|U_N - U_M\|_{\dot{F}^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}_{M+1\leqslant j\leqslant N}\right\|_{L^{p(\cdot)}.(\ell^q)}. \tag{2.3.36}$$

Now we use the following dominated convergence theorem in variable exponent Lebesgue spaces: *Suppose $f_n \to f$ pointwise a.e. and $|f_n(x)| \leqslant |g(x)|$ for some $g \in L^{p(\cdot)}$.. Then $f_n \to f$ in $L^{p(\cdot)}$.*

Applying this theorem with $f_{N,M} = \sum_{j=M+1}^{N} 2^{js} U_j$, $f = 0$, and $g = \sum_{j=0}^{\infty} u_j$ to get

$$\|U_N - U_M\|_{F^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}_{M+1\leqslant j\leqslant N}\right\|_{L^{p(\cdot)}.(\ell^q)} \to 0 \text{ as } M, N \to \infty. \tag{2.3.37}$$

Because $\dot{F}^s_{p(\cdot),q}$ is a quasi-Banach space it is complete so U_N converges in $F^s_{p(\cdot),q}$ and

$$\left\|\sum_{j=0}^{\infty}\right\|_{F^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}\right\|_{L^{p(\cdot)}(\ell^q)}. \tag{2.3.38}$$

If $q = \infty$, we use that $\{2^{(s-\varepsilon)j}u_j\}_{j\geqslant 0}$ and $\{2^{(s+\varepsilon)j}u_j\}_{j<0}$ belong to $\ell^1(L^{p(\cdot)})$ for any $\varepsilon > 0$ and apply Theorem 2.3.6 under the case of finite q to conclude that $\sum_{j=0}^{N} u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{p(\cdot),1}$ and $\dot{B}^{s+\varepsilon}_{p(\cdot),1}$, respectively (choosing $\varepsilon > 0$ so that $s-\varepsilon > n(1/\min(\tau_{p(\cdot)}, 1) - 1)$). Therefore, U_N convergence in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, by applying the finite family case to $\{u_j\}_{-N\leqslant j\leqslant N}$, we have that $U_N \in \dot{F}^s_{p(\cdot),\infty}$ and

$$\|U_N\|_{\dot{F}^s_{p(\cdot),\infty}} \lesssim \left\|\{2^{js}u_j\}_{-N\leqslant j\leqslant N}\right\|_{L^{p(\cdot)}(\ell^\infty)} \leqslant \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^\infty)},$$

with the implicit constant independent of N and $\{u_j\}_{j\in\mathbb{Z}}$. Since $\dot{F}^s_{p(\cdot),\infty}$ has the Fatou property, we conclude that $\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot{F}^s_{p(\cdot),\infty}$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot{F}^s_{p(\cdot),\infty}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^\infty)}.$$

□

To verify the Fatou property as stated in Remark 2.1.3 (i) is clear from the definition of $\|\cdot\|_{L^{p(\cdot)}}$. Item (ii) is shown in in [12, Theorem 2.61].

As a model result the we state the Leibniz type rule for variable exponent Triebel-Lizorkin spaces.

Theorem 2.3.8. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \P^*_0$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $0 < q \leqslant \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p_1(\cdot),q}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|g\|_{F^{s+m}_{p_2(\cdot),q}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Moreover, if $p(\cdot) \in \P^*_0$, $0 < q \leqslant \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p(\cdot),q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{p(\cdot),q}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

2.4 Applications to scattering properties of PDEs

In this section we discuss applications of Theorem 2.2.1 to systems of partial differential equations involving powers of the Laplacian. The systems of partial differential equations that we study are on functions $u = u(t,x)$, $v = v(t,x)$, and $w = w(t,x)$, with $t \geqslant 0$ and $x \in \mathbb{R}^n$, are of the form

$$\begin{cases} \partial_t u = vw, & \partial_t v + a(D)v = 0, \quad \partial_t w + b(D)w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \qquad w(0,x) = g(x), \end{cases} \tag{2.4.39}$$

Here the operators $a(D)$ and $b(D)$ are *linear* Fourier multiplier operators with the symbols $a(\xi)$ and $b(\xi)$ respectively; that is, $\widehat{a(D)f}(\xi) = a(\xi)\widehat{f}(\xi)$ and $\widehat{b(D)f}(\xi) = b(\xi)\widehat{f}(\xi)$. Then formally, without taking issues of convergence into account, we get that

$$v(t,x) = \int_{\mathbb{R}^n} e^{-ta(\xi)}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi. \tag{2.4.40}$$

Indeed, using the system (2.4.39) we obtain

$$\begin{aligned} \partial_t v + a(D)v &= \int_{\mathbb{R}^n} (\widehat{\partial_t v(t,\cdot)}(\xi) + a(\xi)\widehat{v(t,\cdot)}(\xi))e^{2\pi i\xi\cdot x}d\xi \\ &= 0, \end{aligned}$$

so we must have $\widehat{\partial_t v(t,\cdot)}(\xi) + a(\xi)\widehat{v(t,\cdot)}(\xi) = 0$ where the Fourier transform is taken with the variable x. Then by interchanging the Fourier transform with the derivative with respect to t we get $\widehat{v}(t,\cdot)(\xi) = e^{-ta(\xi)}F(\xi)$ for some function F. Setting $t = 0$ and using system (2.4.39) it is apparent that $F(\xi) = \widehat{f}(\xi)$ and by inverting the Fourier transform we have

$$v(t,x) = \int_{\mathbb{R}^n} e^{-ta(\xi)}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

. A similar calculation shows that

$$w(t,x) = \int_{\mathbb{R}^n} e^{-tb(\eta)} \widehat{g}(\eta) e^{2\pi i x\cdot\eta} d\eta.$$

These expressions for v and w yield that

$$u(t,x) = \int_0^t v(s,x)w(s,x)\, ds = \int_{\mathbb{R}^{2n}} \left(\int_0^t e^{-s(a(\xi)+b(\eta))}\, ds \right) \widehat{f}(\xi)\widehat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

Setting $\lambda(\xi,\eta) = a(\xi) + b(\eta)$ and assuming that λ never vanishes, the solution $u(t,x)$ can then be written as the action on f and g of the bilinear multiplier with symbol $\frac{1-e^{-t\lambda(\xi,\eta)}}{\lambda(\xi,\eta)}$, that is,

$$u(t,x) = T_{\frac{1-e^{-t\lambda}}{\lambda}}(f,g)(x). \tag{2.4.41}$$

Following Bernicot–Germain [5, Section 9.4], suppose there exists $u_\infty \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\lim_{t\to\infty} u(t,\cdot) = u_\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n); \tag{2.4.42}$$

then, given a function space X, we say that the solution u of (2.4.39) scatters in the function space X if $u_\infty \in X$.

As an application of Theorems 2.2.1 and 2.2.6 we obtain the following scattering properties for solutions to systems of the type (2.4.39) involving powers of the Laplacian.

For $0 < p_1, p_2, p, q \leqslant \infty$ and $w_1, w_2 \in A_\infty$, set

$$\gamma^{w_1,w_2,tl}_{p_1,p_2,p,q} = 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}, q))] + 1),$$
$$\gamma^{w_1,w_2,b}_{p_1,p_2,p,q} = 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))] + 1).$$

For $\delta > 0$ define

$$\mathcal{S}_\delta = \{(\xi,\eta) \in \mathbb{R}^{2n} : |\eta| \leqslant \delta^{-1}\,|\xi| \text{ and } |\xi| \leqslant \delta^{-1}\,|\eta|\}.$$

Theorem 2.4.1. *Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \geqslant \gamma_{p_1,p_2,p,q}^{w_1,w_2,tl}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \geqslant \gamma_{p_1,p_2,p,q}^{w_1,w_2,b}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}_0(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, & \partial_t v + D^\gamma v = 0, \quad \partial_t w + D^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.43}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.43) scatters in $\dot{F}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.44}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, the solution u of (2.4.43) scatters in $\dot{B}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates

$$\|u_\infty\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.45}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.4.44) and (2.4.45); moreover, if $w \in A_\infty$, then

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \tau_{p,q}(w)$, and the implicit constant is independent of f and g.

For $\delta > 0$

$$\tilde{\mathcal{S}}_\delta = \{(\xi,\eta) \in \mathbb{R}^{2n} : |\eta| \leqslant \delta^{-1}(1+|\xi|^2)^{\frac{1}{2}} \text{ and } |\xi| \leqslant \delta^{-1}(1+|\eta|^2)^{\frac{1}{2}}\}.$$

Theorem 2.4.2. *Consider $0 < p, p_1, p_2 \leqslant \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leqslant \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \geqslant \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \geqslant \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\widetilde{\mathcal{S}}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, & \partial_t v + J^\gamma v = 0, \quad \partial_t w + J^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.46}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.46) scatters in $F^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{F^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.47}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \leqslant \infty$ and $s > \tau_p(w)$, the solution u of (2.4.46) scatters in $B^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates

$$\|u_\infty\|_{B^s_{p,q}(w)} \lesssim \|f\|_{B^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{B^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.48}$$

where the Hardy spaces $h^{p_1}(w_1)$ and $h^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.4.47) and (2.4.48); moreover, if $w \in A_\infty$, then

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$, $0 < q \leqslant \infty$, $s > \tau_{p,q}(w)$, and the implicit constant is independent of f and

g.

Proof of Theorem 2.4.1. We have $a(\xi) = |\xi|^\gamma$ and $b(\eta) = |\eta|^\gamma$; therefore, $\lambda(\xi, \eta) = |\xi|^\gamma + |\eta|^\gamma$. Note that all corresponding integrals for $v(t,x)$, $w(t,x)$ and $u(t,x)$ are absolutely convergent for $t > 0$, $x \in \mathbb{R}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. If we further assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$, the Dominated Convergence Theorem implies that $u(t, \cdot) \to u_\infty$ both pointwise and in $\mathcal{S}'(\mathbb{R}^n)$, where

$$u_\infty(x) = \int_{\mathbb{R}^{2n}} (a(\xi) + b(\eta))^{-1} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta = T_{\lambda^{-1}}(f,g)(x).$$

Indeed, by using the Taylor expansion of $\widehat{f}$ and $\widehat{g}$ we obtain

$$\widehat{f}(\xi) = \sum_{|\alpha| \leqslant \lfloor\gamma\rfloor} \frac{\partial^\alpha \widehat{f}(0)}{\alpha!}\xi^\alpha + \sum_{|\alpha| = \lfloor\gamma\rfloor+1} \frac{\partial^\alpha \widehat{f}(c_1\xi)}{\alpha!}\xi^\alpha$$

and

$$\widehat{g}(\eta) = \sum_{|\alpha| \leqslant \lfloor\gamma\rfloor} \frac{\partial^\alpha \widehat{g}(0)}{\alpha!}\eta^\alpha + \sum_{|\alpha| = \lfloor\gamma\rfloor+1} \frac{\partial^\alpha \widehat{g}(c_2\eta)}{\alpha!}\eta^\alpha$$

for some $0 < c_1, c_2 < 1$. Then by using that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ we have $\partial^\alpha \widehat{f}(0) = 0$ and $\partial^\alpha \widehat{g}(0) = 0$ for all $\alpha \in \mathbb{N}_0$. From this we obtain

$$\left| \frac{1 - e^{-t(|\xi|^\gamma + |\eta|^\gamma)}}{|\xi|^\gamma + |\eta|^\gamma} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)} \right| \leqslant \sum_{|\alpha| = \lfloor\gamma\rfloor+1} \left| \frac{1}{|\xi|^\gamma} \frac{\partial^\alpha \widehat{f}(c_1\xi)}{\alpha!}\xi^\alpha \frac{\partial^\alpha \widehat{g}(c_2\eta)}{\alpha!}\eta^\alpha \right|,$$

and the right-hand side is integrable in ξ and η. So by applying the Dominated Convergence Theorem and letting $t \to \infty$ we get that $u(t, \cdot)$ converges to u_∞ both pointwise and in $\mathcal{S}'$.

Next we note that λ^{-1} is homogeneous of degree $-\gamma$ so $\partial_\xi^\alpha \partial_\eta^\beta \lambda^{-1}$ is homogeneous of degree $-\gamma - |\alpha + \beta|$. That is $\partial_\xi^\alpha \partial_\eta^\beta \lambda^{-1}(r\xi, r\eta) = r^{-\gamma-|\alpha+\beta|} \partial_\xi^\alpha \partial_\eta^\beta \gamma^{-1}(\xi,\eta)$ for any $r > 0$. By letting $r = (|\xi| + |\eta|)^{-1}$ we obtain

$$\partial_\xi^\alpha \partial_\eta^\beta \gamma^{-1}(\xi,\eta) = (|\xi| + |\eta|)^{-\gamma-|\alpha+\beta|} \partial_\xi^\alpha \partial_\eta^\beta \lambda^{-1}\left(\frac{\xi}{(|\xi| + |\eta|)^{-1}}, \frac{\eta}{(|\xi| + |\eta|)^{-1}}\right).$$

We now want to show that λ^{-1} is a Coifman-Meyer multiplier; that is λ^{-1} satisfies (2.1.1).

If γ is an even positive integer then λ^{-1} satisfies the estimates (2.1.1) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Then, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ and therefore the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$.

Let p_1, p_2, p, q, w_1, w_2 be as in the hypotheses. If $\gamma > 0$ and γ is not an even integer, then λ^{-1} satisfies the estimates (2.1.1) with $m = -\gamma$ as long as $\alpha, \beta \in \mathbb{N}_0^n$ are such that $|\alpha| < \gamma$ and $|\beta| < \gamma$; in particular, λ^{-1} satisfies (2.1.1) with $m = -\gamma$ for $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha + \beta| < \gamma$. In view of the remarks following the proof of Theorem 2.2.1, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ if $\gamma \geqslant \gamma_{p_1,p_2,p,q}^{w_1,w_2,tl}$ in the context of Triebel–Lizorkin spaces and if $\gamma \geqslant \gamma_{p_1,p_2,p,q}^{w_1,w_2,b}$ in the context of Besov spaces; as a consequence, the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ for such values of γ.

On the other hand, if $0 < \gamma < \gamma_{p_1,p_2,p,q}^{w_1,w_2,tl}$ in the Triebel-Lizorkin space setting or $0 < \gamma < \gamma_{p_1,p_2,p,q}^{w_1,w_2,b}$ in the Besov space setting, and γ is not an even positive integer, consider $h \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\operatorname{supp}(h) \subset \mathcal{S}_{\delta/2}$ and $h \equiv 1$ on $\mathcal{S}_\delta$. Then, for $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ we have $h(\xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta) = \widehat{f}(\xi)\widehat{g}(\eta)$; therefore, $T_{\lambda^{-1}}(f, g) = T_\Lambda(f, g)$, where $\Lambda(\xi, \eta) = h(\xi, \eta)/(|\xi|^\gamma + |\eta|^\gamma)$. The multiplier Λ verifies (2.1.1) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$ (with constants that depend on δ). Then all estimates from Theorem 2.2.1 hold for T_Λ and therefore the desired estimates follow for u_∞ with constants dependent on δ and independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$. $\square$

Proof of Theorem 2.4.2. We proceed as in the proof of Theorem 2.4.1 with $\lambda(\xi, \eta) = (1 + |\xi|^2)^{\gamma/2} + (1 + |\eta|^2)^{\gamma/2}$ and an application of Theorem 2.2.6. $\square$

Bibliography

[1] A. Almeida and P. Hästö. Besov spaces with variable smoothness and integrability. *J. Funct. Anal.*, 258(5):1628–1655, 2010.

[2] Á. Bényi. Bilinear pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces. *J. Math. Anal. Appl.*, 284(1):97–103, 2003.

[3] Á. Bényi, A. R. Nahmod, and R. H. Torres. Sobolev space estimates and symbolic calculus for bilinear pseudodifferential operators. *J. Geom. Anal.*, 16(3):431–453, 2006.

[4] Á. Bényi and R. H. Torres. Symbolic calculus and the transposes of bilinear pseudodifferential operators. *Comm. Partial Differential Equations*, 28(5-6):1161–1181, 2003.

[5] F. Bernicot and P. Germain. Bilinear oscillatory integrals and boundedness for new bilinear multipliers. *Adv. Math.*, 225(4):1739–1785, 2010.

[6] G. Bourdaud. L^p estimates for certain nonregular pseudodifferential operators. *Comm. Partial Differential Equations*, 7(9):1023–1033, 1982.

[7] J. Bourgain and D. Li. On an endpoint Kato-Ponce inequality. *Differential Integral Equations*, 27(11-12):1037–1072, 2014.

[8] J. Brummer and V. Naibo. Weighted fractional Leibniz-type rules for bilinear multiplier operators. *Preprint*, 2017.

[9] J. Brummer and V. Naibo. Bilinear operators with homogeneous symbols, smooth molecules, and Kato-Ponce inequalities. *Proc. Amer. Math. Soc.*, 146(3):1217–1230, 2018.

[10] M. Christ and M. Weinstein. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal.*, 100(1):87–109, 1991.

[11] R. Coifman and Y. Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société Mathématique de France, Paris, 1978.

[12] D. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.

[13] D. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[14] D. Cruz-Uribe and V. Naibo. Kato-Ponce inequalities on weighted and variable Lebesgue spaces. *Differential Integral Equations*, 29(9-10):801–836, 2016.

[15] Guy David and Jean-Lin Journé. A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. of Math. (2)*, 120(2):371–397, 1984.

[16] L. Diening, P. Hästö, and S. Roudenko. Function spaces of variable smoothness and integrability. *J. Funct. Anal.*, 256(6):1731–1768, 2009.

[17] M. Frazier and B. Jawerth. Decomposition of Besov spaces. *Indiana Univ. Math. J.*, 34(4):777–799, 1985.

[18] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.*, 93(1):34–170, 1990.

[19] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[20] L. Grafakos, D. Maldonado, and V. Naibo. A remark on an endpoint Kato-Ponce inequality. *Differential Integral Equations*, 27(5-6):415–424, 2014.

[21] L. Grafakos and S. Oh. The Kato-Ponce inequality. *Comm. Partial Differential Equations*, 39(6):1128–1157, 2014.

[22] Loukas Grafakos and José María Martell. Extrapolation of weighted norm inequalities for multivariable operators and applications. *J. Geom. Anal.*, 14(1):19–46, 2004.

[23] Loukas Grafakos and Rodolfo H. Torres. Maximal operator and weighted norm inequalities for multilinear singular integrals. *Indiana Univ. Math. J.*, 51(5):1261–1276, 2002.

[24] Loukas Grafakos and Rodolfo H. Torres. Multilinear Calderón-Zygmund theory. *Adv. Math.*, 165(1):124–164, 2002.

[25] A. Gulisashvili and M. Kon. Exact smoothing properties of Schrödinger semigroups. *Amer. J. Math.*, 118(6):1215–1248, 1996.

[26] P. Gurka, P. Harjulehto, and A. Nekvinda. Bessel potential spaces with variable exponent. *Math. Inequal. Appl.*, 10(3):661–676, 2007.

[27] J. Hart, R.H. Torres, and X. Wu. Smoothing properties of bilinear operators and Leibniz-type rules in Lebesgue and mixed Lebesgue spaces. *Trans. Amer. Math. Soc.*, DOI: https://doi.org/10.1090/tran/7312.

[28] R. Hunt. On $L(p, q)$ spaces. *Enseignement Math. (2)*, 12:249–276, 1966.

[29] M. Izuki, Y. Sawano, and H. Tanaka. Weighted Besov-Morrey spaces and Triebel-Lizorkin spaces. In *Harmonic analysis and nonlinear partial differential equations*, RIMS Kôkyûroku Bessatsu, B22, pages 21–60. Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.

[30] J. Johnsen. Domains of pseudo-differential operators: a case for the Triebel-Lizorkin spaces. *J. Funct. Spaces Appl.*, 3(3):263–286, 2005.

[31] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[32] C. Kenig, G. Ponce, and L. Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620, 1993.

[33] Carlos E. Kenig and Elias M. Stein. Multilinear estimates and fractional integration. *Math. Res. Lett.*, 6(1):1–15, 1999.

[34] K. Koezuka and N. Tomita. Bilinear Pseudo-differential Operators with Symbols in $BS^m_{1,1}$ on Triebel–Lizorkin Spaces. *J. Fourier Anal. Appl.*, 24(1):309–319, 2018.

[35] H. Kozono and M. Yamazaki. Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data. *Comm. Partial Differential Equations*, 19(5-6):959–1014, 1994.

[36] Andrei K. Lerner, Sheldy Ombrosi, Carlos Pérez, Rodolfo H. Torres, and Rodrigo Trujillo-González. New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory. *Adv. Math.*, 220(4):1222–1264, 2009.

[37] J. Marschall. Nonregular pseudo-differential operators. *Z. Anal. Anwendungen*, 15(1):109–148, 1996.

[38] A. L. Mazzucato. Besov-Morrey spaces: function space theory and applications to non-linear PDE. *Trans. Amer. Math. Soc.*, 355(4):1297–1364, 2003.

[39] Y. Meyer. Remarques sur un théorème de J.-M. Bony. *Rend. Circ. Mat. Palermo (2)*, (suppl. 1):1–20, 1981.

[40] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[41] C. Muscalu and W. Schlag. *Classical and multilinear harmonic analysis. Vol. II*, volume 138 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[42] V. Naibo. On the bilinear Hörmander classes in the scales of Triebel-Lizorkin and Besov spaces. *J. Fourier Anal. Appl.*, 21(5):1077–1104, 2015.

[43] S. M. Nikol'skiĭ. *Approximation of functions of several variables and imbedding theorems.* Springer-Verlag, New York-Heidelberg., 1975. Translated from the Russian by John M. Danskin, Jr., Die Grundlehren der Mathematischen Wissenschaften, Band 205.

[44] B. J. Park. On the boundedness of pseudo-differential operators on triebel-lizorkin and besov spaces. *Preprint, arXiv:1602.08811v2*, 2016.

[45] Bui Huy Qui. Weighted Besov and Triebel spaces: interpolation by the real method. *Hiroshima Math. J.*, 12(3):581–605, 1982.

[46] M. Rosenthal and H.-J. Schmeisser. The boundedness of operators in Muckenhoupt weighted Morrey spaces via extrapolation techniques and duality. *Rev. Mat. Complut.*, 29(3):623–657, 2016.

[47] H. Triebel. *Theory of function spaces.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2010. Reprint of 1983 edition. Also published in 1983 by Birkhäuser Verlag.

[48] J. Xu. The relation between variable Bessel potential spaces and Triebel-Lizorkin spaces. *Integral Transforms Spec. Funct.*, 19(7-8):599–605, 2008.

[49] J. Xu. Variable Besov and Triebel-Lizorkin spaces. *Ann. Acad. Sci. Fenn. Math.*, 33(2):511–522, 2008.

[50] M. Yamazaki. A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(1):131–174, 1986.