Bilinear pseudodifferential operators and Leibniz-type rules

by

Alexander Thomson

B.S., Missouri State University, 2013

M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2019

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Chapter 1

Introduction to Leibniz-type rules

Leibniz-type rules have been extensively studied due to their connections to partial differential equations which model many real world situations such as shallow water waves and fluid flow. In this chapter we introduce some of the definitions and history of the development of Leibniz-type rules which motivated the results to be discussed in chapters 2 and 3 of this manuscript. First consider the Leibniz rule taught in Calculus courses which expresses the derivatives of a product of functions as a linear combination of derivatives of the functions involved; more specifically, for functions f and g sufficiently smooth, it holds that

$$\partial_x^\alpha (fg)(x) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1\alpha_2} \partial_x^{\alpha_1} f(x) \partial_x^{\alpha_2} g(x),$$

where $\alpha \in \mathbb{N}_0^n$ and $C_{\alpha_1\alpha_2}$ are appropriate constants. The definitions in of the function spaces used below and multiindices $\alpha \in \mathbb{N}_0^n$ will be discussed in Appendix A A. In particular one term has all the derivatives on f and another with α derivatives on g and

$$\partial_x^\alpha (fg)(x) = \partial_x^\alpha f(x) g(x) + f(x) \partial_x^\alpha g(x) + ...$$

In an analogous way, fractional Leibniz rules give estimates of the smoothness and size of a product of functions in terms of the smoothness and size of the factors. For instance, for f

and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, it holds that

$$\|D^s(fg)\|_{L^r} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}} , \tag{1.0.1}$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, $1 < p_1, p_2, q_1, q_2 \leq \infty$, $1/2 < r \leq \infty$, and $s > N(1/min(r,1) - 1)$ or s is an even natural number. The homogeneous fractional differentiation operator of order s, D^s, is defined as

$$D^s f(x) = \int_{\mathbb{R}^n} |\xi|^s \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

where $\widehat{f}$ is the Fourier transform of f. For $s > 0$, the operator D^s is naturally understood as taking s derivatives of its argument. Indeed, in the case $s = 2$, $D^2 f = c\Delta f$, where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian operator. Furthermore, if s is a positive integer,

$$\|D^s f\|_{L^p} \sim \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^p} ,$$

where $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_0^n$.

Another version of (1.0.1) is obtained by using the inhomogeneous sth order fractional differentiation operator J^s:

$$\|J^s(fg)\|_{L^r} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{q_2}} . \tag{1.0.2}$$

Similarly to its homeogenous counterpart, the operator J^s is defined through the Fourier transform as

$$J^s f(x) = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

and can be interpreted as taking derivatives up to order s of f.

The estimates (1.0.1) and (1.0.2) are also known as Kato-Ponce inequalities due to the foundational work of Kato-Ponce[?] , where the estimate (1.0.2) was proved in the case

$1 < r = p_1 = q_2 < \infty$ and $p_2 = q_1 = \infty$, with applications to the Cauchy problem for Euler and Navier-Stokes equations. This result was extended by Gulisashvili-Kon[?] , who showed (1.0.1) and (1.0.2) for the cases $s > 0$, $1 < r < \infty$, and $1 < p_1, p_2, q_1, q_2 \leq \infty$ in connection to smoothing properties of Schrödinger semigroups. Grafakos-Oh[?] and Muscalu-Schlag[?] established the cases for $1/2 < r \leq 1$ and the case $r = \infty$ was completed in the work of Bourgain-Li[?] and Grafakos-Maldonado-Naibo[?] . Applications of the estimates (1.0.1) and (1.0.2) to Korteweg-de Vries equations were studied by Christ-Weinstein[?] and Kenig-Ponce-Vega[?] .

In the estimates (1.0.1) and (1.0.2) the two functions f and g are related through pointwise multiplication. Throughout the rest of this manuscript we will consider estimates similar to (1.0.1) and (1.0.2) where the two functions are related through a pseudodiffernential operator. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function for $x, \xi, \eta \in \mathbb{R}^n$. We define the *bilinear pseudodifferential operator* associated to σ, T_σ, by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{-2\pi i x \cdot (\xi + \eta)} d\xi d\eta.$$

We refer to σ as the symbol of the operator T_σ. When σ is independent of x we call σ the multiplier of the *bilinear multiplier operator* T_σ.

Throughout this manuscript we will study estimates related to (1.0.1) and (1.0.2) associated to bilinear pseudodifferential operators that are of the form

$$\|D^s T_\sigma(f, g)\|_Z \lesssim \|D^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|D^s g\|_{Y_2}, \tag{1.0.3}$$

$$\|J^s T_\sigma(f, g)\|_Z \lesssim \|J^s f\|_{X_1} \|g\|_{Y_1} + \|f\|_{X_2} \|J^s g\|_{Y_2}, \tag{1.0.4}$$

for a variety of function spaces X_1, X_2, Y_1, Y_2, and Z.

The estimates (1.0.3) and (1.0.4) have been extensively studied in a variety of settings. In[?] , Brummer-Naibo studied Leibniz-type rules in function spaces that admit a molecular decomposition and a φ-transform characterization in the sense of Frazier-Jawerth[?, ?]. In the context of Lebesgue spaces and mixed Lebesgue spaces, estimates of the type (1.0.3) were studied in Hart-Torres-Wu[?] for bilinear multiplier operators with minimal smoothness assumptions on the multipliers. Related mapping properties for bilinear pseudodifferential operators with symbols in the bilinear Hörmander classes were studied by Bényi-Torres[?] and Bényi-Nahmod-Torres[?] in the setting of Sobolev spaces, by Bényi[?] in the setting of Besov spaces, and by Naibo[?] and Koezuka-Tomita[?] in the context of Triebel-Lizorkin spaces. Additionally, versions of (1.0.1) and (1.0.2) in weighted Lebesgue spaces were proved in Cruz-Uribe-Naibo[?], while Brummer-Naibo[?] proved (1.0.3) and (1.0.4) in weighted Lebesgue spaces for Coifman-Meyer multiplier operators.

In chapter two we will present Leibniz-type rules in the settings of Besov and Triebel-Lizorkin spaces based in various function spaces. The techniques used are quite flexible and allow the method of proof to be adapted to many different function spaces.

In chapter three we will present Leibniz-type rules in Besov spaces for bilinear pseudodifferential operators symbols of critical order.

Chapter 2

Weighted Leibniz-type rules with applications to scattering properties of PDEs

In this chapter we discuss bilinear multiplier operators associated to Coifman-Meyer multipliers and Leibniz-type rules in the settings of weighted Triebel-Lizorkin and Besov spaces. Additionally we obtain applications of these results to scattering properties of certain systems of parial differential equations. One of the main results in this chapter is in the setting of Triebel-Lizorkin spaces based in weighted Lebesgue spacs and Hardy spaces. It is as follows.

Theorem 2.0.1. *For $m \in \mathbb{R}$, let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leq \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.0.1}$$

If $0 < p, p_1, p_2 \leq \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.0.2}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.10) and (2.2.11); moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.0.3}$$

where $0 < p < \infty$, $0 < q \leq \infty$ and $s > \tau_{p,q}(w)$.

In the following sections the function spaces and multipliers used in the hypotheses are defined and discussed. We will then remark on corollaries to Theorem (2.2.1) and their connection to the Leibniz rules in the previous chapter. theorem and the proof of Theorem (2.2.1). The proof is quite flexible and can be readily adapted to Triebel-Lizorkin and Besov spaces based in other function spaces.

2.1 Definitions

2.1.1 Coifman-Meyer Multipliers

The symbols used in Theorem (2.2.1) and results later in this chapter are Coifman-Meyer multipliers. Such multipliers are defined as follows.

Definition 2.1.1. For $m \in \mathbb{R}^n$, a smooth function $\sigma = \sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, is a *Coifman-Meyer multiplier* of order m if for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ there exists a positive constant $C_{\alpha,\beta}$

such that

$$|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(|\xi|+|\eta|)^{m-(|\alpha|+|\beta|)} \quad \forall (\xi,\eta) \in \mathbb{R}^{2n} \setminus \{(0,0)\}. \tag{2.1.4}$$

Operators associated to these multipliers have been widely studied. For instance in Grafakos-Torres[?] operators associated to Coifman-Meyer multipliers were studied because of their connection to a larger class of operators called Calderón-Zygmund operators. In particular it holds that

$$\|T_\sigma(f,g)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

where σ is a Coifman-Meyer multiplier, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $1 < p_1, p_2 < \infty$. We also note that in particular Coifman-Meyer multipliers of order m belong to the bilinear Hörmander class $\dot{BS}^m_{1,1}$. These symbols and operators associated to them will be discussed in the following chapter.

For the proof of Theorem (2.2.1) we will use a the following decomposition of Coifman-Meyer multipliers. Let σ be a Coifman–Meyer multiplier of order m. Fix $\Psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\widehat{\Psi}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\} \quad \text{ and } \quad \sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1 \ \forall \xi \in \mathbb{R}^n \setminus \{0\};$$

define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\widehat{\Phi}(0) := 1, \quad \widehat{\Phi}(\xi) := \sum_{j \le 0} \widehat{\Psi}(2^{-j}\xi) \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

For $a, b \in \mathbb{R}^n$, $\Delta_j^{\tau_a\Psi} f$ and $S_j^{\tau_a\Phi} f$ satisfy $\widehat{\Delta_j^{\tau_a\Psi}} f(\xi) = \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot a}\widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{S_j^{\tau_b\Phi}} f(\xi) = \widehat{\tau_b\Phi}(2^{-j}\xi)\widehat{f}(\xi) = e^{2\pi i 2^{-j}\xi\cdot b}\widehat{\Phi}(2^{-j}\xi)\widehat{f}(\xi)$. By the work of Coifman and Meyer in[?] , given $N \in \mathbb{N}$ such that $N > n$, it follows that $T_\sigma = T_\sigma^1 + T_\sigma^2$, where, for $f \in \mathcal{S}_0(\mathbb{R}^n)$

($f \in \mathcal{S}(\mathbb{R}^n)$ if $m \geq 0$) and $g \in \mathcal{S}(\mathbb{R}^n)$,

$$T_\sigma^1(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi} f)(x)\,(S_j^{\tau_b\Phi} g)(x) \quad \forall x \in \mathbb{R}^n, \tag{2.1.5}$$

the coefficients $\mathcal{C}_j(a,b)$ satisfy

$$|\mathcal{C}_j(a,b)| \lesssim 2^{jm} \quad \forall a,b \in \mathbb{Z}, j \in \mathbb{Z}, \tag{2.1.6}$$

with the implicit constant depending on σ, and an analogous expression holds for T_σ^2 with the roles of f and g interchanged.

If σ is an inhomogeneous Coifman–Meyer multiplier of order m, a similar decomposition to (2.1.5) follows with the summation in $j \in \mathbb{N}_0$ rather than $j \in \mathbb{Z}$, with $\Delta_0^{\tau_a\Psi}$ replaced by $S_0^{\tau_a\Phi}$ and for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Remark 2.1.1. For the formula (2.1.5) and its corresponding counterpart for T_σ^2 to hold, the condition (2.1.4) on the derivatives of σ is only needed for multi-indices α and β such that $|\alpha + \beta| \leq 2N$

2.1.2 Function spaces

Weighted spaces

Definition 2.1.2. A *weight* $w(x)$ defined on $\mathbb{R}^n$ is a nonnegative, locally integrable function such that $0 < w(x) < \infty$ for almost every $x \in \mathbb{R}^n$.

Given a weight $w(x)$ and $0 < p < \infty$ we define the weighted Lebesgue space $L^p(w)$ as the space of all measurable functions satisfying

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty.$$

In the case that $p = \infty$ we define $L^\infty(w) = L^\infty$.

The specific classes of weights in the hypotheses of our results are Muckenhoupt weights.

Definition 2.1.3. For $1 < p < \infty$ the *Muckenhoupt class* A_p consists of all weights w on $\mathbb{R}^n$ satisfying

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty, \tag{2.1.7}$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^n$ and $|B|$ is the Lebesgue measure of B. For $p = \infty$ we define $A_\infty = \cup_{1<p} A_p$.

From this definition it follows that $A_p \subset A_q$ when $p \leq q$. For a $w \in A_\infty$ we set $\tau_w = \inf\{\tau \in (1, \infty] : w \in A_\tau$. The condition (2.1.7) is motivated by the following fact: for $f \in L^p(w)$ the Hardy-Littlewood maximal function $\mathcal{M}(f)(x)$ is bounded on $L^p(w)$ if and only if $w \in A_p$. That is for for $1 < p < \infty$ $w \in A_p$ if and only if

$$\|\mathcal{M}(f)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}\,.$$

Later in this chapter we will use the maximal function $\mathcal{M}_r(f) = (\mathcal{M}(|f|^r))^{\frac{1}{r}}$. By the properties for the Hardy-Littlewood maximal function above it holds that for $0 < r < p$ $\mathcal{M}_r$ is bounded on $L^p(w)$ for $w \in A_{p/r}$ and in this case $0 < r < \frac{p}{\tau_w}$. The following theorem is a vector valued version of the previous statement called a weighted Fefferman-Stein inequality.

Theorem 2.1.1. *If* $0 < p < \infty$, $0 < q \leq \infty$, $0 < r < \min(p, q)$ *and* $w \in A_{p/r}$ *(i.e.* $0 < r < \min(p/\tau_w, q)$*), then for all sequences* $\{f_j\}_{j\in\mathbb{Z}}$ *of locally integrable functions defined on* $\mathbb{R}^n$*, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)},$$

where the implicit constant depends on r*,* p*,* q*, and* w *and the summation in* j *should be replaced by the supremum in* j *if* $q = \infty$*.*

For more detail on the Muckenhoupt classes see Grafakos**?** .

Triebel-Lizorkin and Besov spaces

Here we describe the function spaces in which (2.2.1) is based and some properties of such spaces.

Let ψ and φ be functions in $\mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ satisfying the following conditions:

- $\text{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$,
- $|\widehat{\psi}(\xi)| > c$ for all ξ such that $\frac{3}{5} < |\xi| < \frac{5}{3}$ for some $c > 0$,
- $\text{supp}(\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$,
- $|\widehat{\varphi}(\xi)| > c$.

For ψ supported in an annulus and $j \in \mathbb{Z}$ we define the operator $\Delta_j^\psi(f)$ through its Fourier transform as

$$\widehat{\Delta_j^\psi(f)}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$$

For such ψ we define the homogeneous Triebel-Lizorkin and Besov spaces as follows.

Definition 2.1.4. Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The weighted *homogeneous Triebel-Lizorkin space* $\dot{F}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}(w)} = \left\| \left(\sum_{j\in\mathbb{Z}} (2^{sj}|\Delta_j^\psi f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

- The weighted *homogeneous Besov space* $\dot{B}^s_{p,q}(w)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}(w)} = \left(\sum_{j\in\int} (2^{js} \left\|\Delta_j^\psi f\right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty. \tag{2.1.8}$$

Given φ, ψ, S_0^φ, and Δ_j^ψ as above the weighted inhomogeneous Triebel-Lizorkin and Besov spaces are defined as follows.

Definition 2.1.5. Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$.

- The weighted *inhomogeneous Triebel-Lizorkin space* $F^s_{p,q}(w)$ is the class of all $f \in \mathcal{S}(\mathbb{R}^n)'$ such that

$$\|f\|_{F^s_{p,q}(w)} = \|S^\varphi_0 f\|_{L^p(w)} + \left\| \left(\sum_{j\in\mathbb{N}_0} (2^{sj}|\Delta^\psi_j f|)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} < \infty.$$

- The weighted *inhomogeneous Besov space* $\dot{B}^s_{p,q}(w)$ is the class of all $f \in \mathcal{S}(\mathbb{R}^n)'$ such that

$$\|f\|_{B^s_{p,q}(w)} = \|S^\varphi_0 f\|_{L^p(w)} + \left(\sum_{j\in\mathbb{N}_0} (2^{js} \left\|\Delta^\psi_j f\right\|_{L^p(w)})^q \right)^{\frac{1}{q}} < \infty. \tag{2.1.9}$$

The definitions above are independent of the choice of φ and ψ. These spaces are generally quasi-Banach spaces and if $1 \leq p, q < \infty$ they are Banach spaces. These spaces provide a framework to study a variety of other spaces such as Lebesgue, Hardy, Sobolev, and BMO spaces with a unified approach. For a detailed overview of the development of these spaces see Triebel [books 1-3].

For certain s, p, and q as in the definitions (2.1.4) and (2.1.5) Triebel-Lizorkin and Besov spaces coincide with other well known function spaces. For instance we have the following equivalences where the function spaces are equivalent in norm

$$\begin{aligned}
&F^0_{p,2}(w) \equiv H^p(w) \text{ for } 0 < p < \infty, \quad w \in A_\infty,\\
&F^0_{p,2}(w) \equiv L^p(w) \equiv H^p(w) \text{ for } 1 < p < \infty, \quad w \in A_p,\\
&F^s_{p,2}(w) \equiv \dot{W}^{s,p}(w) \text{ for } 1 < p < \infty, \quad w \in A_p.
\end{aligned}$$

Additionally by the lifting property of Triebel-Lizorkin and Besov spaces Theorem (2.2.1) and results like it in this chapter can be seen as Leibniz-type rules as in Chapter 1. For weighted Triebel-Lizorkin spaces the lifting property is as follows: for s, p, and q as in (2.1.5) and (2.1.4) and $w \in A_\infty$ we have that

$$\|f\|_{\dot{F}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{F}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{F^s_{p,q}(w)} \simeq \|J^s f\|_{F^0_{p,q}(w)}.$$

The corresponding statement for Besov spaces is: for s, p, and q as in (2.1.5) and (2.1.4) and $w \in A_\infty$ we have that

$$\|f\|_{\dot{B}^s_{p,q}(w)} \simeq \|D^s f\|_{\dot{B}^0_{p,q}(w)} \quad \text{and} \quad \|f\|_{B^s_{p,q}(w)} \simeq \|J^s f\|_{B^0_{p,q}(w)}.$$

Nikol'skij representations for weighted homogeneous and inhomogeneous Triebel-Lizorkin and Besov spaces

An important tool for the proof of Theorem (2.2.1) is the Nikol'skij representation for weighted Triebel-Lizorking and Besov spaces. Here we state a weighted version of[?] Theorem 3.7 (see also[?] Section 2.5.2). For completeness, a sketch of its proof is outlined in Appendix **??**.

Theorem 2.1.2. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{Z}.$$

If $w \in A_\infty$, then the following holds:

2.2 Weighted Leibniz-type rules

2.2.1 Homogeneous Leibniz-type rules

In the setting of weighted homogeneous Besov and Triebel-Lizorkin spaces we obtain the following Leibniz-type rule. As we will see in the corollaries to this result it improves the Leibniz-type rule (**??**) and has extensions to weighted versions of (**??**).

Theorem 2.2.1. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order m. Consider $0 < p, p_1, p_2 \le \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \le \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s+m}_{p_2,q}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.10}$$

If $0 < p, p_1, p_2 \le \infty$ and $s > \tau_p(w)$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s+m}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s+m}_{p_2,q}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.11}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively.

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.10) and (2.2.11); moreover, if $w \in A_\infty$, then

$$\|T_\sigma(f,g)\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s+m}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m}_{p,q}(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.12}$$

where $0 < p < \infty$, $0 < q \le \infty$ and $s > \tau_{p,q}(w)$.

We note that if $m \ge 0$ then the above estimates hold for any $f, g \in \mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ when $\mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)$ is a subspace of the function spaces on the right-hand side. This is the case when $1 < p_1, p_2 < \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ in (2.2.10) and (2.2.11) and $w \in A_p$ for (2.2.12).

By the lifting property of weighted Besov and Triebel-Lizorkin spaces in section and their relation to weighted Hardy spacees in section (2.1.2) the estimates (2.2.10) and (2.2.11) imply the following Leibniz-type rule for Coifman-Meyer multipliers of order zero.

Corollary 2.2.2. *Let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be a Coifman-Meyer multiplier of order* 0. *Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If*

$s > \tau_p(w)$, it holds that

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.13}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.13); moreover, if $w \in A_\infty$, then

$$\|D^s(T_\sigma(f,g))\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f,g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.14}$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

Corollary (2.2.2) gives estimates related to those in Brummer-Naibo[?] where using different methods the following result was proven

Theorem 2.2.3. *if σ is a Coifman-Meyer multiplier of order 0, $1 < p_1, p_2 \leq \infty$, $\frac{1}{2} < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$, $w_2 \in A_{p_2}$, $w = w_1^{p/p_1} w_2^{p/p_2}$ and $s > \tau_p$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ it holds that*

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} + \|f\|_{L^{p_1}(w_1)} \|D^s g\|_{L^{p_2}(w_2)}. \tag{2.2.15}$$

Moreover, if $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.15).

Corollary (2.2.2) and theorem overlap in the following ways.

- The estimate (2.2.13) allows for $0 < p, p_1, p_2 < \infty$, $w_1, w_2 \in A_\infty$, and the $H^P(w)$ on the left-hand side if $s > \tau_p(w)$. However (2.2.15) requires $1 < p_1, p_2 \leq \infty$, $w_1 \in A_{p_1}$, and $w_2 \in A_{p_2}$ but allows for the Lebesgue norm on left-hand side when $s > \tau_p$. So (2.2.13) is less restrictive than (2.2.15) in terms of the indices p, p_1, and p_2 and the classes that the weights w_1 and w_2 belong to. However because $\tau_p \leq \tau_p(w)$ (2.2.13) is more restrictive in terms of the range of the regularity s than (2.2.15).

- If $s > \tau_p(w)$, $1/2 < p < \infty$, $1 < p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$, $w_1 \in A_{p_1}$ and $w_2 \in A_{p_2}$ then (2.2.13) implies (2.2.15). However if $\tau_p < \tau_p(w)$ then (2.2.13) does not imply (2.2.15) for $\tau_p < s < \tau_p(w)$. Here we give examples of weights w_1 and w_2 such that the corresponding weight w satisfies $\tau_p < \tau_p(w)$. Let $1 < p_1 \leq p_2 < \infty$ and $w_1(x) = w_2(x) = w(x) = |x|^a$ with $n(r-1) < a < n(p_1 - 1)$ for some $1 < r < p_1$. Then $w(x) \in A_{p_1}$, $A_{p_1} \subset A_{p_2}$, and $w \notin A_r$. This implies that $1 < \tau_w$ which implies that $\tau_p < \tau_p(w)$.

- The estimate (2.2.15) implies (2.2.14) for $1 < p < \infty$, $w \in A_p$, and $s > \tau_p$ and gives the endpoint estimate

$$\|D^s(T_\sigma(f,g))\|_{L^p(w)} \lesssim \|D^s f\|_{L^\infty(w)} \|g\|_{L^p(w)} + \|f\|_{L^p(w)} \|D^s g\|_{L^\infty}.$$

  However (2.2.14) allows $0 < p < \infty$ and $w \in A_\infty$ if $s > \tau_p(w)$.

2.2.2 Connection to Kato-Ponce inequalities

By setting $\sigma \equiv 1$ we obtain the following Kato-Ponce inequality as a corollary to Theorem (2.2.1).

Corollary 2.2.4. *Consider $0 < p, p_1, p_2 \leq \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \leq \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, it holds that*

$$\|fg\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^s_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.16}$$

If $0 < p, p_1, p_2 \leq \infty$ and $s > \tau_p(w)$, it holds that

$$\|fg\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^s_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^s_{p_2,q}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.17}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, *respectively.*

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.2.16) and (2.2.17); moreover, if $w \in A_\infty$, then

$$\|fg\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^s_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^s_{p,q}(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.2.18}$$

where $0 < p < \infty$, $0 < q \le \infty$ and $s > \tau_{p,q}(w)$.

In particular of we set $q = 2$ and use the connection between weighted Hardy spaces and weighted Triebel-Lizorkin spaces we obtain the following corollary.

Corollary 2.2.5. *Consider $0 < p, p_1, p_2 < \infty$ such that $1/p = 1/p_1 + 1/p_2$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. If $s > \tau_p(w)$, it holds that*

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^{p_1}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|D^s g\|_{H^{p_2}(w_2)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.2.19}$$

If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand side of (2.2.19); moreover, if $w \in A_\infty$, then

$$\|D^s(fg)\|_{H^p(w)} \lesssim \|D^s f\|_{H^p(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|D^s g\|_{H^p(w)} \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n),$$

where $0 < p < \infty$ and $s > \tau_p(w)$.

We note that (2.2.19) extends and improves the inequality (**??**). The inequality (2.2.19) extends the range of p, p_1, and p_2 by allowing $0 < p, p_1, p_2 < \infty$ while (**??**) requires that $1 < p_1, p_2 < \infty$. Additionally (2.2.19) allows for the H^p norm on the left-hand side which is generally larger than $|| \cdot ||_{L^p}$.

2.2.3 Proof of Theorem 2.2.1

Proof of Theorem 2.2.1. Consider Φ, Ψ, T_σ^1, T_σ^2, $\{\mathcal{C}_j(a,b)\}_{j\in\mathbb{Z},a,b\in\mathbb{Z}^n}$ as in Section **??**. Let m, σ, p, p_1, p_2, q, s, w_1, w_2 and w be as in the hypotheses. For ease of notation, p_1 and p_2 will be assumed to be finite; the same proof applies for (2.2.11) if that is not the case, and for (2.2.12).

We next prove (2.2.10) and (2.2.11). By symmetry, it is enough to work with T_σ^1 and prove that

$$\left\|T_\sigma^1(f,g)\right\|_{\dot F^s_{p,q}(w)} \lesssim \|f\|_{\dot F^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)} \quad \text{and} \quad \left\|T_\sigma^1(f,g)\right\|_{\dot B^s_{p,q}(w)} \lesssim \|f\|_{\dot B^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)}.$$

Moreover, since $\|\sum f_j\|_{\dot F^s_{p,q}(w)}^{\min(p,q,1)} \lesssim \sum \|f_j\|_{\dot F^s_{p,q}(w)}^{\min(p,q,1)}$ and similarly for $\dot B^s_{p,q}(w)$, it suffices to prove that, given $\varepsilon>0$ there exist $0<r_1,r_2\le 1$ such that for all $g\in\mathcal{S}(\mathbb{R}^n)$ and $f\in\mathcal{S}_0(\mathbb{R}^n)$ ($f\in\mathcal{S}(\mathbb{R}^n)\cap\dot F^s_{p,q}(w)$ or $f\in\mathcal{S}(\mathbb{R}^n)\cap\dot B^s_{p,q}(w)$ if $m\ge 0$), it holds that

$$\left\|T^{a,b}(f,g)\right\|_{\dot F^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}}\|f\|_{\dot F^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)}, \tag{2.2.20}$$

$$\left\|T^{a,b}(f,g)\right\|_{\dot B^s_{p,q}(w)} \lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}}\|f\|_{\dot B^{s+m}_{p_1,q}(w_1)}\|g\|_{H^{p_2}(w_2)}, \tag{2.2.21}$$

where

$$T^{a,b}(f,g) := \sum_{j\in\mathbb{Z}} \mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)$$

and the implicit constants are independent of a and b. We will assume q finite; obvious changes apply if that is not the case.

In view of the supports of Ψ and Φ we have that

$$\operatorname{supp}(\mathcal{F}[\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)]) \subset \{\xi\in\mathbb{R}^n : |\xi|\lesssim 2^j\} \quad \forall j\in\mathbb{Z},\ a,b\in\mathbb{Z}^n.$$

For (2.2.20), Theorem 2.3.6(**??**), the bound (2.1.6) for $\mathcal{C}_j(a,b)$, and Hölder's inequality imply

$$
\begin{aligned}
\left\|T^{a,b}(f,g)\right\|_{\dot{F}^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\}_{j\in\mathbb{Z}}\right\|_{L^p(w)(\ell^q)} \\
&\lesssim \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)(x)\,(S_j^{\tau_b\Phi}g)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \\
&\leq \left\|\sup_{j\in\mathbb{Z}}|(S_j^{\tau_b\Phi}g)|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \\
&\leq \left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\tau_a\Psi}f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)}.
\end{aligned}
$$

Consider $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ as in Section **??** such that $\widehat{\varphi}\equiv 1$ on $\operatorname{supp}(\widehat{\Phi})$ and $\widehat{\psi}\equiv 1$ on $\operatorname{supp}(\widehat{\Psi})$. Let $0<r_1<\min(1,p_1/\tau_{w_1},q)$; by Lemma **??** and the weighted Fefferman-Stein inequality we have that

$$
\begin{aligned}
\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|(\Delta_j^{\tau_a\Psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} &\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\mathcal{M}_{r_1}(\Delta_j^{\psi}f)|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \\
&\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}\left\|\left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}|\Delta_j^{\psi}f|^q\right)^{\frac{1}{q}}\right\|_{L^{p_1}(w_1)} \\
&\sim (1+|a|)^{\varepsilon+\frac{n}{r_1}}\left\|f\right\|_{\dot{F}^{s+m}_{p,q}(w_1)},
\end{aligned}
$$

where the implicit constants are independent of a and f. Next, let $0<r_2<\min(1,p_2/\tau_{w_2})$; by Lemma **??** and the boundedness properties of the Hardy-Littlewood maximal operator on weighted Lebesgue space we have that

$$
\begin{aligned}
\left\|\sup_{j\in\mathbb{Z}}|S_j^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)} &\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|\mathcal{M}_{r_2}(\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|)\right\|_{L^{p_2}(w_2)} \\
&\lesssim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|\sup_{j\in\mathbb{Z}}|S_j^{\varphi}g|\right\|_{L^{p_2}(w_2)} \\
&\sim (1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|g\right\|_{H^{p_2}(w_2)},
\end{aligned}
$$

where the implicit constants are independent of b and g. Putting all together we obtain (2.2.20).

For (2.2.21), Theorem 2.3.6(**??**), the bound (2.1.6) for $\mathcal{C}_j(a,b)$ and Hölder's inequality give

$$\begin{aligned}\left\|T^{a,b}(f,g)\right\|_{\dot{B}^s_{p,q}(w)} &\lesssim \left\|\{2^{sj}\mathcal{C}_j(a,b)\,(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\}_{j\in\mathbb{Z}}\right\|_{\ell^q(L^p(w))}\\ &\lesssim \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\,(S_j^{\tau_b\Phi}g)\right\|^q_{L^p(w)}\right)^{\frac{1}{q}}\\ &\le \left(\sum_{j\in\mathbb{Z}} 2^{(s+m)qj}\left\|(\Delta_j^{\tau_a\Psi}f)\right\|^q_{L^{p_1}(w_1)}\right)^{\frac{1}{q}}\left\|\sup_{k\in\mathbb{Z}}|S_k^{\tau_b\Phi}g|\right\|_{L^{p_2}(w_2)}\\ &\lesssim (1+|a|)^{\varepsilon+\frac{n}{r_1}}(1+|b|)^{\varepsilon+\frac{n}{r_2}}\left\|f\right\|_{\dot{B}^{s+m}_{p_1,q}(w_1)}\left\|g\right\|_{H^{p_2}(w_2)},\end{aligned}$$

where in the last inequality we have used Lemma **??** and the boundedness properties of $\mathcal{M}$ with $0<r_j<\min(1,p_j/\tau_{w_j})$ for $j=1,2$.

It is clear from the proof above that if $w_1=w_2$, then different pairs of p_1,p_2 related to p through the Hölder condition can be used on the right-hand sides of (2.2.10) and (2.2.11); in such case $w=w_1=w_2$. □

2.2.4 Inhomogeneous Leibniz-type rules

2.3 Leibniz rules in other functions spaces

The method used to prove Theorem (2.2.1) is quite versatile and can be applied to Triebel-Lizorkin and Besov spaces that are based in other quasi-Banach spaces. Here we highlight the important features of Lebesgue spaces that are necessary for the proof of Theorem (2.2.1) to be adapted to other settings. We also define Triebel-Lizorkin and Besov spaces that are based in other quasi-Banach spaces.

The main features of weighted Triebel-Lizorkin and Besov spaces used in the proof of

Theorem (2.2.1) are the following:

In the followinig subsections we consider quasi-Banach spaces $\mathcal{X}$ such that properties (??)–(??) hold for the homogeneous and inhomogeneous $\mathcal{X}$-based Triebel-Lizorkin and Besov spaces. Corresponding versions of Theorems (2.2.1) and ?? hold in Triebel-Lizorkin and Besov spaces based in these spaces. The homogeneous χ-based Triebel-Lizorkin and Besov spaces denoted by $\dot{F}^s_{\mathcal{X},q}$ and $\dot{B}^s_{\mathcal{X},q}$ respectively are defined similarly to the weighted, homogeneous Triebel-Lizorkin and Besov spaces with the $||\cdot||_{L^p(w)}$ quasi-norm replaced with the $||\cdot||_{\mathcal{X}}$ quasi-norm. The inhomogeneous spaces are defined similarly.

2.3.1 Leibniz-type rules in the setting of Lorentz-based Triebel-Lizorkin and Besov spaces

For $0 < p < \infty$ and $0 < t \leq \infty$ or $p = t = \infty$, and an A_∞ weight w defined on $\mathbb{R}^n$, we denote by $L^{p,t}(w)$ the weighted Lorentz space consisting of complex-valued, measurable functions f defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,t}(w)} = \left(\int_0^\infty \left(\tau^{\frac{1}{p}} f^*_w(\tau)\right)^t \frac{d\tau}{\tau}\right)^{\frac{1}{t}} < \infty,$$

where $f^*_w(\tau) = \inf\{\lambda \geq 0 : w_f(\lambda) \leq \tau\}$ with $w_f(\lambda) = w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$. In the case that $t = \infty$, $\|f\|_{L^{p,t}(w)} = \sup_{t>0} t^{\frac{1}{p}} f^*_w(t)$. We note that for $p = t$ we have $L^{p,p}(w) = L^p(w)$ and $L^{1,\infty}(w)$ is the weighted weak $L^1(w)$ space. For more details on these spaces and their properties see Hunt[?] .

We now turn our attention to the analogues to properties (??)-(??) in the setting of weighted Lorrentz-based Triebel-Lizorkin and Besov spaces.

Property (??) follows from the work of Hunt[?] that the quasi norm $||\cdot||_{L^{p,t}(w)}$ is comparable to a quasi-norm $|||\cdot|||_{L^{p,t}(w)}$ that is subadditive. Therefore we have that the norms $|||\cdot|||_{\dot{F}^s_{(p,t),q}}$ and $||\cdot||_{\dot{F}^s_{(p,t),q}}$ are comparable where $|||\cdot|||_{\dot{F}^s_{(p,t),q}}$ is the quasi-norm with $||\cdot||_{L^{p,t}(w)}$ replaced by $|||\cdot|||_{L^{p,t}(w)}$ and for some $r > 0$ $|||f+g|||^r_{\dot{F}^s_{(p,t),q}} \leq |||f|||^r_{\dot{F}^s_{(p,t),q}} + |||g|||^r_{\dot{F}^s_{(p,t),q}}$.

By using the following version of Hölder's inequality we obtain property (**??**) [Theorem 4.5,[?]]: Let $f, g \in L^{p,t}(w)$. Then for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}$ it holds that

$$||fg||_{L^{p,t}(w)} \lesssim ||f||_{L^{p_1,t_1}(w)}||g||_{L^{p_2,t_2}(w)}.$$

The corresponding version of property (**??**) is: If $0 < p < \infty$, $0 < t, q \leq \infty$, $0 < r < \min(p/\tau_w, q)$ and $0 < r \leq t$, it holds that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^{p,t}(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p,t}(w)} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in L^{p,t}(w)(\ell^q); \quad (2.3.22)$$

in particular, if $0 < r < p/\tau_w$ and $0 < r \leq t$, it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p,t}(w)} \quad \forall f \in L^{p,t}(w).$$

This is established by the extrapolation theorem[?] Theorem 4.10 and comments on page 70 when $r = 1$, $1 < p < \infty$, $1 \leq t \leq \infty$ and $1 < q \leq \infty$. The remaining cases are shown using the previous cases and the following scaling property for Lorrentz spaces: For $0 < s < \infty$ $\||f|^s\|_{L^{p,t}(w)} = \|f\|^s_{L^{sp,st}(w)}$.

The substitute for property **??** is the following Nikols'kij representation in weighted Lorrentz spaces.

Theorem 2.3.1. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{N}_0.$$

If $w \in A_\infty$, then the following holds:

For the proof of Theorem 2.3.1 we need the following lemmas.

Lemma 2.3.2 (Particular case of Corollary 2.11 in[?])**.** *Suppose* $0 < r \leq 1$, $A > 0$, $R \geq 1$

and $d > n/r$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and f is such that $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le AR\}$, *it holds that*

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\|(1+|A\,\cdot\,|)^d \phi\right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n,$$

where the implicit constant is independent of A, R, ϕ, and f.

Remark 2.3.1. ? Corollary 2.11 incorrectly states $A^{-n/r}$ instead of A^{-n}. Also, it states $A \ge 1$, but the result is true for $A > 0$ as stated in Lemma 2.3.2.

Lemma 2.3.3. *Suppose $w \in A_\infty$, $0 < p \le \infty$, $A > 0$, $R \ge 1$, and $d > b > n/\min(1, p/\tau_w)$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and f is such that* $\operatorname{supp}(\widehat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \le AR\}$, *it holds that*

$$\|\phi * f\|_{L^p(w)} \lesssim R^{b-n} A^{-n} \left\|(1+|A\,\cdot\,|^d)\phi\right\|_{L^\infty} \|f\|_{L^p(w)},$$

where the implicit constant is independent of A, R, ϕ and f.

Proof. Set $r := n/b < \min(1, p/\tau_w)$. The hypothesis $d > b$ means $d > n/r$ and Lemma 2.3.2 yields

$$|\phi * f(x)| \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\|(1+|A\,\cdot\,|)^d \phi\right\|_{L^\infty} \mathcal{M}_r f(x) \quad \forall x \in \mathbb{R}^n.$$

Since $r < p/\tau_w$, we have $\|\mathcal{M}_r f\|_{L^{p,t}(w)} \lesssim \|f\|_{L^{p,t}(w)}$ and therefore

$$\|\phi * f\|_{L^{p,t}(w)} \lesssim R^{n(\frac{1}{r}-1)} A^{-n} \left\|(1+|A\,\cdot\,|)^d \phi\right\|_{L^\infty} \|f\|_{L^{p,t}(w)};$$

observing that $1/r - 1 = (b-n)/n$, the desired estimate follows. □

The following lemma is a modified version of? Lemma 3.8.

Lemma 2.3.4. *Let $\tau < 0$, $\lambda \in \mathbb{R}$, $0 < q \le \infty$, and $k_0 \in \mathbb{Z}$. Then, for any sequence $\{d_j\}_{j\in\mathbb{Z}} \subset [0,\infty)$ it holds that*

$$\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q} \lesssim \left\| \{2^{j\lambda} d_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q},$$

where the implicit constant depends only on k_0, τ, λ and q.

Proof. Suppose first that $0 < q \leq 1$. Then,

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q} &= \left[\sum_{j \in \mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right)^q \right]^{\frac{1}{q}} \\
&\leq \left[\sum_{j \in \mathbb{Z}} \sum_{k=k_0}^{\infty} 2^{\tau q k} 2^{\lambda q(j+k)} d_{j+k}^q \right]^{\frac{1}{q}} = \left[\sum_{k=k_0}^{\infty} 2^{\tau q k} \sum_{j \in \mathbb{Z}} 2^{\lambda q(j+k)} d_{j+k}^q \right]^{\frac{1}{q}} \\
&= \left(\sum_{k=k_0}^{\infty} 2^{\tau q k} \right)^{\frac{1}{q}} \left\| \{2^{j\lambda} d_j\}_{j \in \mathbb{Z}} \right\|_{\ell^q} = C_{k_0, \tau, q} \left\| \{2^{j\lambda} d_j\}_{j \in \mathbb{Z}} \right\|_{\ell^q},
\end{aligned}$$

where in the last equality we have used that $\tau < 0$. If $1 < q < \infty$ we use Hölder's inequality with q and q' to write

$$\begin{aligned}
\left\| \left\{ \sum_{k=k_0}^{\infty} 2^{\tau k} 2^{\lambda(j+k)} d_{j+k} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q} &\leq \left[\sum_{j \in \mathbb{Z}} \left(\sum_{k=k_0}^{\infty} 2^{\tau k q/2} 2^{\lambda q(j+k)} d_{j+k}^q \right) \left(\sum_{k=k_0}^{\infty} 2^{\tau k q'/2} \right)^{q/q'} \right]^{\frac{1}{q}} \\
&= C_{k_0, \tau, q} \left\| \{2^{j\lambda} d_j\}_{j \in \mathbb{Z}} \right\|_{\ell^q}.
\end{aligned}$$

The case $q = \infty$ is straightforward. □

We now show the proof of Theorem 2.3.1.

Proof. We first establish the theorem for finite families of functions and then extend this result to families that are not necessarily finite. We will do this in the homogeneous settings, with the proof in the inhomogeneous settings being similar. Suppose $\{u_j\}_{j \in \mathbb{Z}}$ is such that $u_j = 0$ for all j except those belonging to some finite subset of $\mathbb{Z}$; this assumption allows us to avoid convergence issues since all the sums considered will be finite.

For Part (**??**), let D, w, p, q and s be as in the hypotheses. Fix $0 < r < \min(1, p/\tau_w, q)$ such that $s > n(1/r - 1)$; note that the latter is possible since $s > \tau_{p,q}(w)$.

Let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \leq 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{Z}.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $\dot{F}^s_{(p,t),q}(w)$ in Section **??**. We have

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.3.23}$$

We will use Lemma 2.3.2 with $\phi(x) = 2^{jn}\psi(2^j x)$, $f = u_{j+k}$, $A = 2^j > 0$, and $R = 2^{k+k_0}$. (Notice that $\operatorname{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geq -k_0$, we get $R \geq 1$.) Fixing $d > n/r$ and applying Lemma 2.3.2, we get

$$\begin{aligned} |\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\| (1+|2^j\cdot|)^d 2^{jn}\psi(2^j\cdot) \right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\ &\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y\in\mathbb{R}^n} (1+|2^j y|)^d |\psi(2^j y)| \right) \mathcal{M}_r(u_{j+k})(x). \end{aligned}$$

Hence,

$$2^{js} |\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x),$$

and then, recalling (2.3.29),

$$2^{js} |\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x).$$

Since $1/r - 1 - s/n < 0$, Lemma 2.3.4 yields

$$\left\| \{2^{js}|\Delta_j^\psi u|\}_{j\in\mathbb{Z}} \right\|_{L^{p,t}(w)(\ell^q)} \lesssim \left\| \{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{Z}} \right\|_{L^{p,t}(w)(\ell^q)}$$

with an implicit constant independent of $\{u_j\}_{j\in\mathbb{Z}}$. Applying the weighted Fefferman-Stein inequality to the right-hand side of the last inequality leads to the desired estimate

$$\|u\|_{\dot{F}^s_{(p,t),q}(w)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p,t}(\ell^q)}.$$

Assume that the theorem is true for finite families. Let $\{u_j\} \subset S'$ be as in the hypothesis. Then using the theorem for finite families we have that

$$\|U_N - U_M\|_{F^s_{(p,t),q}} \lesssim \left\|\{2^{js}u_j\}_{M+1\leq j\leq N}\right\|_{L^{p,t}(w)(\ell^q)}. \tag{2.3.24}$$

Now we apply the following version of the dominated convergence theorem.

Suppose $f_n \to f$ in measure and $|f_n(x)| \leq |g(x)|$ for some $g \in L^{p,t}(w)$. Then

$$\lim_{n\to\infty} \|f_n - f\|_{L^{p,t}(w)} = 0.$$

We need to check that the functions $U_{N,M} = \left(\sum_{j=M+1}^{N}(2^{js}u_j)^q\right)^{\frac{1}{q}}$, $U = 0$, and $g = \left(\sum_{j=0}^{\infty}(2^{js}u_j)^q\right)^{\frac{1}{q}}$ satisfy the hypotheses of the theorem. Now we need to check that $f_{N,M} \to f$ in measure. Let $A_N = \{x : \sum_{j=N}^{\infty}|2^{jsq}u_j|^q > \tau^q\}$. Because $\{u_j\} \subset L^{(p,t)}(w)(\ell^q)$ we have that $|A_1| < \infty$. Then since $|A_{N+1}| \leq |A_N|$ it follows that $\lim_{N\to\infty}|A_N| = |\cap A_N|$. $\{u_j\} \subset L^{(p,t)}(w)(\ell^q)$ so $u_j \to 0$ as $j \to \infty$. Therefore $f_N \to 0$ in measure. This implies that

$$\|U_N - U_M\|_{F^s_{(p,t),q}} \lesssim \left\|\{2^{js}u_j\}_{M+1\leq j\leq N}\right\|_{L^{p,t}(w)(\ell^q)} \to 0.$$

Then $\sum_{j=0}^{\infty} u_j$ converges in $F^s_{(p,t),q}$.

Now we consider the case where $q = \infty$. Then $\{2^{j(s-\epsilon)}\}_{j\geq 0}$ belongs to $\ell^1(L^{p,t}(w))$ for any $\epsilon > 0$. Then by the case for $q < \infty$ $\sum_{j=0}^{\infty} u_j$ converges in $B^s_{(p,t),1}$ and so it converges in S'. Then by using the case for finite families applied to $\{u_j\}_{0\leq j\leq N}$ (this case holds because the lemmas used only depend on the on the indexes considered and maximal function inequalities in Lorrentz spaces) we have that

$$\|U_N\|_{F^s_{(p,t),q}} \lesssim ||\{2^{js}u_j\}_{0\le j\le N}||_{F^s_{(p,t),\infty}} \le ||\{2^{js}u_j\}||_{F^s_{(p,t),\infty}}.$$

Then using the Fatou property finishes the proof. $\square$

With these four properties theorems analogous those earlier in this chapter hold in the setting of weighted Lorrentz-based Triebel-Lizorkin and Besov spaces and as an example the analogue to Theorem **??** in the context of the spaces $F^s_{(p,t),q}(w)$ is below.

Theorem 2.3.5. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta \in \mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. If $w \in A_\infty$, $0 < p, p_1, p_2 < \infty$ and $0 < t, t_1, t_2 \le \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $1/t = 1/t_1 + 1/t_2$, $0 < q \le \infty$ and $s > \tau_{p,t,q}(w)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|g\|_{F^{s+m}_{(p_2,t_2),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Different pairs of p_1, p_2 and t_1, t_2 can be used on the right-hand side of the inequality above. Moreover, if $w \in A_\infty$, $0 < p < \infty$, $0 < t, q \le \infty$ and $s > \tau_{p,t,q}(w)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{(p,t),q}(w)} \lesssim \|f\|_{F^{s+m}_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{(p,t),q}(w)} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

By the Fefferman–Stein inequality (2.3.22) the lifting property $\|f\|_{F^s_{(p,t),q}} \simeq \|J^s f\|_{F^0_{(p,t),q}}$ holds true for $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < t, q \le \infty$. Then, under the assumptions of Theorem 2.3.5 we obtain, in particular,

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p_1,t_1),q}(w)} \|g\|_{h^{p_2,t_2}(w)} + \|f\|_{h^{p_1,t_1}(w)} \|J^s g\|_{F^0_{(p_2,t_2),q}(w)};$$

$$\|J^s(fg)\|_{F^0_{(p,t),q}(w)} \lesssim \|J^s f\|_{F^0_{(p,t),q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{F^0_{(p,t),q}(w)}.$$

These last two estimates supplement the results in[?] Theorem 6.1, where related Leibniz-type rules in Lorentz spaces were obtained.

2.3.2 Morrey spaces

Given $0 < p \leq t < \infty$ and $w \in A_\infty$, we denote by $M_p^t(w)$ the weighted Morrey space consisting of functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^t(w)} = \sup_{B \subset \mathbb{R}^n} w(B)^{\frac{1}{t}-\frac{1}{p}} \left(\int_B |f(x)|^p \, w(x) \, dx \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all Euclidean balls B contained in $\mathbb{R}^n$. We note that $M_p^p(w) = L^p(w)$. For more details on Morrey spaces see the work Rosenthal–Schmeisser**?** and the references contained therein. The corresponding weighted inhomogeneous Triebel–Lizorkin spaces and inhomogeneous Besov spaces are denoted by $F^s_{[p,t],q}(w)$ and $B^s_{[p,t],q}(w)$, respectively. These Morrey-based Triebel–Lizorkin and Besov spaces are independent of the choice of φ and ψ given in Section **??** and are quasi-Banach spaces that contain $\mathcal{S}(\mathbb{R}^n)$ (see the works Kozono–Yamazaki**?** , Mazzucato**?** , Izuki et al.**?** and the references they contain). The corresponding local Hardy spaces are denoted by $h_p^t(w)$.

We now show the analogues to properties **??**–**??** for weighted Morrey spaces.

$$\|f + g\|^r_{M_p^t(w)} \lesssim \|f\|^r_{M_p^t(w)} + \|g\|^r_{M_p^t(w)}$$

for $r = min(1, p)$. It follows that for $r := min(1, p, q)$

$$\|f + g\|^r_{F^s_{[p,t],q}(w)} \lesssim \|f\|^r_{F^s_{[p,t],q}(w)} + \|g\|^r_{F^s_{[p,t],q}(w)}$$

with similar inequalities for inhomogeneous weighted Morry-based Besov spaces and homogeneous weighted Morry-based Triebel-Lizorkin and Besov spaces.

A version of property **??** follows from Hölder's inequality for weighted Lebesgue spaces. For $0 < p \leq t < \infty$, $0 < p_1 \leq t_1 < \infty$ and $0 < p_2 \leq t_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$

and $1/t = 1/t_1 + 1/t_2$, then

$$\|fg\|_{M_p^t(w)} \leq \|f\|_{M_{p_1}^{t_1}(w)} \|g\|_{M_{p_2}^{t_2}(w)};$$

also, if $0 < p \leq t < \infty$, $0 < p_1, p_2 < \infty$ are such that $1/p = 1/p_1 + 1/p_2$ and $w = w_1^{p/p_1} w_2^{p/p_2}$ for weights w_1 and w_2, then

$$\|fg\|_{M_p^t(w)} \leq \|f\|_{M_{p_1}^{\frac{p_1 t}{p}}(w_1)} \|g\|_{M_{p_2}^{\frac{p_2 t}{p}}(w_2)}.$$

The analogue to property **??** is the following Fefferman-Stein inequality: Let $0 < p \leq t < \infty$, $0 < q \leq \infty$ and $0 < r < min(p/\tau_w, q)$, then

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{M_p^t(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{M_p^t(w)} \quad \forall \{f_j\}_{j\in\mathbb{N}_0} \in M_p^t(w)(\ell^q). \tag{2.3.25}$$

If $0 < p \leq t < \infty$ and $0 < r < p/\tau_w$, then

$$\|\mathcal{M}_r(f)\|_{M_p^t(w)} \lesssim \|f\|_{M_p^t(w)} \quad \forall f \in M_p^t(w).$$

The case for $r = 1$, $1 < p \leq t < \infty$ and $1 < q \leq \infty$ are shown using extrapolation and the Fefferman-Stein inequality in weighted Lebesgue spaces. For the extrapolation theorem see[?] Theorem 5.3. The remaining cases are shown using that for $0 < s < \infty$ $\||f|^s\|_{M_p^t(w)} = \|f\|^s_{M_{sp}^{st}(w)}$ and the previous case.

The Nikol'skij representation for weighted Morrey-based Triebel-Lizorkin and Besov spaces is as follows.

Theorem 2.3.6. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that*

$$\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j) \quad \forall j \in \mathbb{N}_0.$$

If $w \in A_\infty$, then the following holds:

The proof of Theorem (2.3.6) uses Lemma 2.3.2, Lemma 2.3.4, and a modified version of Lemma (2.3.3).

Lemma 2.3.7. *Let $0 < p \leq t < \infty$, $A > 0$, $R \geq 1$ and $d > b > \frac{n}{min(p/\tau_w, 1)}$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $supp(\widehat{f}) \subset \{\xi : |\xi| \leq AR\}$. Then*

$$\|\phi * f\|_{M_p^t(w)} \lesssim R^{b-n} A^{-n} \left\|(1 + |A \cdot|^d)\phi\right\|_{L^\infty} \|f\|_{M_p^t(w)} \tag{2.3.26}$$

with the implicit constant independent of $R, A, \phi,$ and f.

The proof follows along the same lines as Lemma 2.3.2.

Proof. Proof of part (ii): Using Lemma 4.3 we have that

$$\left\|\Delta_j^\psi u_{j+k}\right\|_{M_p^t(w)} \lesssim 2^{k(b-n)} \|u_{j+k}\|_{M_p^t(w)}$$

where we have used boundedness of the Hardy-Littlewood maximal function on Morrey spaces when $0 < r < p/\tau_w$. Then by setting $r^* = min(1, p, q)$ and using property (1) for Morrey spaces

$$2^{jsr^*} \left\|\Delta_j^\psi u\right\|_{M_p^t(w)}^{r^*} \lesssim 2^{jsr^*} \sum_{k=-k_0}^{\infty} \left\|\Delta_j^\psi u_{j+k}\right\|_{M_p^t(w)}^{r^*} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)r^*} 2^{sr^*(j+k)} \|u_{j+k}\|_{M_p^t(w)}^{r^*}.$$

Now by applying Lemma A.3 we have

$$\|u\|_{\dot{B}^s_{[p,t],q}(w)} \lesssim \left\| \left\{ \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)r^*} 2^{sr^*(j+k)} \|u_{j+k}\|_{M_p^t(w)}^{p^*} \right\}_{j \in \mathbb{N}_0} \right\|_{\ell^{q/r*}}^{\frac{1}{r^*}} \lesssim \left\| \{2^{js} u_j\}_{j \in \mathbb{N}_0} \right\|_{\ell^q(M_p^t(w))}$$

and part (ii) is shown for the finite family case. For a family that is not necessarily finite we apply the finite family case to $U_N - U_M := \sum_{j=M+1}^{N} u_j$. For finite q this gives us

$$\|U_N - U_M\|_{\dot{B}^s_{[p,t],q}(w)} \lesssim \left\|\{2^{js}u_j\}_{M+1\le j\le N}\right\|_{\ell^q(M^t_p(w))} \le \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{\ell^q(M^t_p(w))} < \infty$$

and by the dominated convergence theorem the left side converges to 0 as $M, N \to \infty$. So U_N converges in $B^s_{[p,t],q}(w)$. If $q = \infty$ then $\{2^{j(s-\epsilon)}\} \in \ell^1(M^t_p(w))$. By the case for finite q we have that $\sum_{j=0}^{N} u_j$ converges in $B^{s-\epsilon}_{[p,t],q}(w)$. Therefore it converges in $\mathcal{S}'$.

Proof of part (i): First we will assume that $u_j = 0$ for all but finitely many j. From Lemma 4.2 we have that

$$|\Delta^{\psi}_j u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})}2^{j(s+k)}\mathcal{M}_r(u_{j+k})(x)$$

where $\Delta^{\psi}_j u = \sum_{\ell\in\mathbb{N}_0}\Delta^{\psi}_j u_\ell$. By Lemma 3.4 and the Fefferman-Stein inequality for Morrey spaces we get

$$\left\|\{2^{js}|\Delta^{\psi}_j u|\}_{j\in\mathbb{N}_0}\right\|_{M^p_t(w)(\ell^q)} \lesssim \left\|\{2^{js}\mathcal{M}_r u_j\}_{j\in\mathbb{N}_0}\right\|_{M^p_t(w)(\ell^q)} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{M^p_t(w)(\ell^q)}$$

for $0 < r < min(\frac{p}{\tau_w}, q)$. Now we prove the theorem for families with are not necessarily finite. Let $U_N := \sum_{j=0}^{N} u_j$. Then by the finite family case we have that

$$\|u_N\|_{F^s_{[p,t],q}(w)} \lesssim \left\|\{2^{js}u_j\}_{0\le j\le N}\right\|_{M^p_t(w)(\ell^q)} \lesssim \left\|\{2^{js}u_j\}\right\|_{M^p_t(w)(\ell^q)} < \infty.$$

Assume that $1 < q < \infty$. Because $\|\{2^{js}u_j\}\|_{M^p_t(w)(\ell^q)} < \infty$ we have that $\sup_j \|\{2^{js}u_j\}\|_{M^p_t(w)} < \infty$ which falls in the $q = \infty$ case for part (ii). Then $\sum_{j=0}^{\infty} u_j$ converges in S' and from the Fatou property

$$\liminf_{N\to\infty} \|U_N\|_{F^s_{[p,t],q}(w)} \le \|U\|_{F^s_{[p,t],q}(w)}\, v \lesssim \left\|\{2^{js}u_j\}\right\|_{M^p_t(w)(\ell^q)}.$$

If $q = \infty$ then $\{2^{j(s-\epsilon)}u_j\}_{j\in\mathbb{N}_0}$ is in $\ell^1(M^t_p)(w))$ for any $\epsilon > 0$. Then the case for finite q shows that $\{2^{j(s-\epsilon)}u_j\}_{j\in\mathbb{N}_0}$ converges in $B^{s-\epsilon}_{[p,t],1}(w)$ and we have convergence in $\mathcal{S}'$. Then using the finite family case we have

$$\|U_N\|_{F^s_{[p,t],\infty}(w)} \lesssim \left\|\{2^{js}u_j\}_{0\le j\le N}\right\|_{\ell^1(M^t_p)(w)} \le \left\|\{2^{js}u_j\}\right\|_{\ell^1(M^t_p)(w)} < \infty$$

Then after using the Fatou property of $F^s_{[p,t],q}(w)$ we are finished. $\square$

2.3.3 Variable Lebesgue spaces

Let $\mathcal{P}_0$ be the collection of measurable functions $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 0 \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty.$$

For $p(\cdot)\in\mathcal{P}_0$, the variable-exponent Lebesgue space $L^{p(\cdot)}$ consists of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda>0 : \int_{\mathbb{R}^n}\left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} < \infty;$$

such quasi-norm turns $L^{p(\cdot)}$ into a quasi-Banach space (Banach space if $p_-\ge 1$). We note that if $p(\cdot)=p$ is constant then $L^{p(\cdot)}\simeq L^p$ with equality of norms and that

$$\left\||f|^t\right\|_{L^{p(\cdot)}} = \|f\|^t_{L^{tp(\cdot)}} \quad \forall\, t>0. \tag{2.3.27}$$

Let $\mathcal{B}$ be the family of all $p(\cdot)\in\mathcal{P}_0$ such that $\mathcal{M}$, the Hardy–Littlewood maximal operator, is bounded from $L^{p(\cdot)}$ to $L^{p(\cdot)}$. Such exponents satisfy $p_->1$ and the following log-Hölder continuity properties

- there exists a constant C_0 such that for all $x,y\in\mathbb{R}^n$, $|x-y|<1/2$

$$|p(x)-p(y)| \le \frac{C_0}{-log(|x-y|)},$$

- there exist constants C_∞ and p_∞ such that for all $x \in \mathbb{R}^n$

$$|p(x) - p_\infty| \le \frac{C_\infty}{log(e + |x|)}.$$

Furthermore if $\tau_0 > 0$ is such that $p(\cdot)/\tau_0 \in \mathcal{B}$ then $p(\cdot)/\tau \in \mathcal{B}$ for $0 < \tau < \tau_0$. Indeed, by Jensen's inequality it holds that $\mathcal{M}(f)^{\tau_0/\tau}(x) \le \mathcal{M}(|f|^{\tau_0/\tau})(x)$ so by 2.3.27 we have that

$$\begin{aligned}
\|\mathcal{M}f\|_{L^{\frac{p(\cdot)}{\tau}}} &\le \left\|\mathcal{M}(|f|^{\tau_0/\tau})^{\tau/\tau_0}\right\|_{L^{\frac{p(\cdot)}{\tau}}} \\
&= \left\|\mathcal{M}(|f|^{\tau_0/\tau})\right\|_{L^{\frac{p(\cdot)}{\tau_0}}}^{\tau/\tau_0} \\
&\le \left\||f|^{\frac{\tau_0}{\tau}}\right\|_{L^{\frac{p(\cdot)}{\tau}}}^{\frac{\tau}{\tau_0}} \\
&= \|f\|_{L^{\frac{p(\cdot)}{\tau}}} .
\end{aligned}$$

We then define

$$\tau_{p(\cdot)} = \sup\{\tau > 0 : \tfrac{p(\cdot)}{\tau} \in \mathcal{B}\}, \quad p(\cdot) \in \mathcal{P}_0^*,$$

where $\mathcal{P}_0^*$ denotes the class of variable exponents in $\mathcal{P}_0$ such that $p(\cdot)/\tau_0 \in \mathcal{B}$ for some $\tau_0 > 0$. Note that $\tau_{p(\cdot)} \le p_-$.

Given $s \in \mathbb{R}$, $0 < q \le \infty$ and $p(\cdot) \in \mathcal{P}_0$, the corresponding inhomogeneous Triebel-Lizorkin and Besov spaces are denoted by $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$, respectively. If $p(\cdot) \in \mathcal{P}_0^*$, these spaces are independent of the functions ψ and φ given in Section **??** (see Xu[?]), contain $\mathcal{S}(\mathbb{R}^n)$ and are quasi-Banach spaces. If $p(\cdot) \in \mathcal{B}$ and $s > 0$, $F^s_{p(\cdot),2}$ coincides with the variable-exponent Sobolev space $W^{s,p(\cdot)}$ (see Gurka et al.[?] and Xu[?]). More general versions of variable-exponent Triebel–Lizorkin and Besov spaces, where s and q are also allowed to be functions, were introduced in Diening at al.[?] and Almeida–Hästö[?] , respectively. The local Hardy space with variable exponent $p(\cdot) \in \mathcal{P}_0$, denoted $h^{p(\cdot)}$, is defined analogously to $h^p(w)$ with the quasi-norm in $L^p(w)$ replaced by the quasi-norm in $L^{p(\cdot)}$.

We now consider the analogues of properties **??**-**??** in the setting of variable Lebesgue based spaces.

For **??** we apply 2.3.27 to get for $r = \min(p_-, q, 1)$

$$\|f+g\|^r_{F^s_{p(\cdot),q}} \leq \|f\|^r_{F^s_{p(\cdot),q}} + \|g\|^r_{F^s_{p(\cdot),q}},$$

$$\|f+g\|^r_{B^s_{p(\cdot),q}} \leq \|f\|^r_{B^s_{p(\cdot),q}} + \|g\|^r_{B^s_{p(\cdot),q}}.$$

To prove **??** we use[?] Corollary 2.28 and 2.3.27 to get the following version of Hölder's inequality: If $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot) \in \mathcal{P}_0$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ then

$$\|fg\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}} \quad \forall f \in L^{p_1(\cdot)}, g \in L^{p_2(\cdot)}.$$

The case when $p(\cdot) \in \mathcal{P}_0$ has $p_- \geq 1$ is shown in[?] Corollary 2.28. If $0 < p_- < 1$ then we use 2.3.27 to get

$$\|fg\|_{L^{p(\cdot)}} = \||fg|^{p_-}\|^{\frac{1}{p_-}}_{L^{\frac{p(\cdot)}{p_-}}}$$

and then use the first case since $\frac{p(\cdot)}{p_-} > 1$.

A Fefferman-Stein inequality in variable exponent Lebesgue spaces follows from the discussion in[?] Section 5.6.8 and (2.3.27). For property **??** we have the following: If $p(\cdot) \in \mathcal{P}_0^*$, $0 < q \leq \infty$ and $0 < r < \min(\tau_{p(\cdot)}, q)$ then

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{M}_r(f_j)|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \lesssim \left\| \left(\sum_{j\in\mathbb{N}_0} |f_j|^q \right)^{\frac{1}{q}} \right\|_{L^{p(\cdot)}} \qquad \forall \{f_j\}_{j\in\mathbb{N}_0} \in L^{p(\cdot)}(\ell^q);$$

in particular, if $0 < r < \tau_{p(\cdot)}$ it holds that

$$\|\mathcal{M}_r(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}} \quad \forall f \in L^{p(\cdot)}.$$

Property **??**, the Nikolśkij representation for $F^s_{p(\cdot),q}$ and $B^s_{p(\cdot),q}$, for variable exponent Lebesgue spaces is stated below.

Theorem 2.3.8. *For $D > 0$, let $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ be a sequence of tempered distributions such that* $\operatorname{supp}(\widehat{u_j}) \subset B(0, D\,2^j)$ *for all $j \in \mathbb{Z}$. Let $p(\cdot) \in \mathcal{P}_0^*$, $0 < q \le \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$. If $\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\|_{L^{p(\cdot)}(\ell^q)} < \infty$, then the series $\sum_{j\in\mathbb{Z}} u_j$ converges in $F^s_{p(\cdot),q}$ (in $\mathcal{S}'(\mathbb{R}^n)$ if $q = \infty$) and*

$$\left\|\sum_{j\in\mathbb{N}_0} u_j\right\|_{F^s_{p(\cdot),q}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{N}_0}\right\|_{L^{p(\cdot)}(\ell^q)},$$

where the implicit constant depends only on n, D, s, $p(\cdot)$ and q. An analogous statement holds true for $B^s_{p(\cdot),q}$ with $s > n(1/\min(\tau_{p(\cdot)}, 1) - 1)$.

Lemma 2.3.9. *Let $p(\cdot) \in \mathcal{P}_0$, $A > 0$, $R \ge 1$ and $d > b > \frac{n}{min(\tau_{p(\cdot)},1)}$. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $supp(\widehat{f}) \subset \{\xi : |\xi| \le AR\}$. Then*

$$\|\phi * f\|_{L^{p(\cdot)}} \lesssim R^{b-n}A^{-n}\left\|(1 + |A\cdot|^d)\phi\right\|_{L^\infty}\|f\|_{L^{p(\cdot)}}, \tag{2.3.28}$$

with the implicit constant independent of R, A, ϕ, and f.

Proof. We first prove the theorem for finite families. Assume that $\{u_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$ is such that $u_j \equiv 0$ for all but finitely many j. Let D, $p(\cdot)$, q, and s be as in the hypotheses. Fix $0 < r < min(1, p_-, q)$ such that $s > n(1/r - 1)$; note that this is possible since $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$. Let $k_0 \in \mathbb{Z}$ be such that $2^{k_0-1} < D \le 2^{k_0}$, then

$$\operatorname{supp}(\widehat{u_\ell}) \subset B(0, 2^\ell D) \subset B(0, 2^{\ell+k_0}) \quad \forall \ell \in \mathbb{Z}.$$

Define $u = \sum_{\ell\in\mathbb{Z}} u_\ell$ and let ψ be as in the definition of $\dot{F}^s_{p(\cdot),q}(w)$. We have

$$\Delta_j^\psi u = \sum_{\ell\in\mathbb{Z}} \Delta_j^\psi u_\ell = \sum_{\ell=j-k_0}^{\infty} \Delta_j^\psi u_\ell = \sum_{k=-k_0}^{\infty} \Delta_j^\psi u_{j+k}. \tag{2.3.29}$$

We will use Lemma 2.3.2 with $\phi(x) = 2^{jn}\psi(2^jx)$, $f = u_{j+k}$, $A = 2^j > 0$, and $R = 2^{k+k_0}$.

(Notice that $\mathrm{supp}(\widehat{u_{j+k}}) \subset B(0, 2^j 2^{k+k_0})$ and, since $k \geq -k_0$, we get $R \geq 1$.) Fixing $d > n/r$ and applying Lemma 2.3.2, we get

$$
\begin{aligned}
|\Delta_j^\psi u_{j+k}(x)| &\lesssim 2^{k_0 n(\frac{1}{r}-1)} 2^{kn(\frac{1}{r}-1)} 2^{-jn} \left\| (1+|2^j \cdot|)^d 2^{jn} \psi(2^j \cdot) \right\|_{L^\infty} \mathcal{M}_r(u_{j+k})(x) \\
&\sim 2^{kn(\frac{1}{r}-1)} \left(\sup_{y \in \mathbb{R}^n} (1+|2^j y|)^d |\psi(2^j y)| \right) \mathcal{M}_r(u_{j+k})(x).
\end{aligned}
$$

Hence,

$$
2^{js} |\Delta_j^\psi u_{j+k}(x)| \lesssim 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x),
$$

and then, recalling (2.3.29),

$$
2^{js} |\Delta_j^\psi u(x)| \lesssim \sum_{k=-k_0}^{\infty} 2^{kn(\frac{1}{r}-1-\frac{s}{n})} 2^{s(j+k)} \mathcal{M}_r(u_{j+k})(x).
$$

Since $1/r - 1 - s/n < 0$, Lemma 2.3.4 yields

$$
\left\| \{ 2^{js} |\Delta_j^\psi u| \}_{j \in \mathbb{Z}} \right\|_{L^p(w)(\ell^q)} \lesssim \left\| \{ 2^{js} \mathcal{M}_r u_j \}_{j \in \mathbb{Z}} \right\|_{L^p(w)(\ell^q)}
$$

with an implicit constant independent of $\{u_j\}_{j \in \mathbb{Z}}$. Applying the weighted Fefferman-Stein inequality to the right-hand side of the last inequality leads to the desired estimate

$$
\|u\|_{\dot{F}^s_{p,q}(w)} \lesssim \left\| \{ 2^{js} u_j \}_{j \in \mathbb{Z}} \right\|_{L^p(\ell^q)}.
$$

For the space $\dot{B}^s_{p(\cdot),q}()$, let D, w, p, q and s be as in the hypotheses and k_0 be as above. Consider $\Delta_j^\psi u_{j+k}$ in (2.3.29) and apply Lemma 2.3.3 with $\phi(x) = 2^{jn} \psi(2^{-j} x)$, $f = u_{j+k}$, $A = 2^j$, $R = 2^{k+k_0}$, $d > b$ and $n / \min(1, p/\tau_w) < b < n + s$; note that such b exists since $s > \tau_p(w)$. We get

$$
\left\| \Delta_j^\psi u_{j+k} \right\|_{L^{p(\cdot)}} \lesssim 2^{(k+k_0)(b-n)} 2^{-jn} \left\| (1+|2^j \cdot|)^d 2^{jn} \psi(2^{-j} \cdot) \right\|_{L^\infty} \|u_{j+k}\|_{L^p(w)} \sim 2^{k(b-n)} \|u_{j+k}\|_{L^{p(\cdot)}},
$$

and setting $p^* := \min(p_-, 1)$ we obtain

$$2^{jsp^*} \left\| \Delta_j^\psi u \right\|_{L^{p(\cdot)}}^{p^*} \lesssim 2^{jsp^*} \sum_{k=-k_0}^{\infty} \left\| \Delta_j^\psi u_{j+k} \right\|_{L^{p(\cdot)}}^{p^*} = \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \left\| u_{j+k} \right\|_{L^{p(\cdot)}}^{p^*}.$$

Hence, applying Lemma 2.3.4, it follows that

$$\|u\|_{\dot{B}^s_{p(\cdot),q}} \lesssim \left\| \left\{ \sum_{k=-k_0}^{\infty} 2^{k(b-n-s)p^*} 2^{sp^*(j+k)} \left\| u_{j+k} \right\|_{L^{p(\cdot)}}^{p^*} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{q/p^*}}^{\frac{1}{p^*}} \lesssim \left\| \{2^{js} u_j\}_{j\in\mathbb{Z}} \right\|_{\ell^q(L^{p(\cdot)})},$$

as desired. Now for families that are not necessarily finite let $U_N := \sum_{j=0}^N u_j$. First we assume that $0 < q < \infty$. Then for $M \le N$ we have

$$\|U_N - U_M\|_{\dot{F}^s_{p(\cdot),q}} \lesssim \left\| \{2^{js} u_j\}_{M+1\le j\le N} \right\|_{L^{p(\cdot)}.(\ell^q)}. \tag{2.3.30}$$

Now we use the following dominated convergence theorem in variable exponent Lebesgue spaces: *Suppose $f_n \to f$ pointwise a.e. and $|f_n(x)| \le |g(x)|$ for some $g \in L^{p(\cdot)}$.. Then $f_n \to f$ in $L^{p(\cdot)}$.*

Applying this theorem with $f_{N,M} = \sum_{j=M+1}^N 2^{js} U_j$, $f = 0$, and $g = \sum_{j=0}^\infty u_j$ to get

$$\|U_N - U_M\|_{F^s_{p(\cdot),q}} \lesssim \left\| \{2^{js} u_j\}_{M+1\le j\le N} \right\|_{L^{p(\cdot)}.(\ell^q)} \to 0 \text{ as } M, N \to \infty. \tag{2.3.31}$$

Because $\dot{F}^s_{p(\cdot),q}$ is a quasi-Banach space it is complete so U_N converges in $F^s_{p(\cdot),q}$ and

$$\left\| \sum_{j=0}^{\infty} \right\|_{F^s_{p(\cdot),q}} \lesssim \left\| \{2^{js} u_j\} \right\|_{L^{p(\cdot)}(\ell^q)}. \tag{2.3.32}$$

If $q = \infty$, we use that $\{2^{(s-\varepsilon)j} u_j\}_{j\ge 0}$ and $\{2^{(s+\varepsilon)j} u_j\}_{j<0}$ belong to $\ell^1(L^{p(\cdot)})$ for any $\varepsilon > 0$ and apply Theorem 2.3.6 under the case of finite q to conclude that $\sum_{j=0}^N u_j$ and $\sum_{j=-N}^{-1} u_j$ converge in $\dot{B}^{s-\varepsilon}_{p(\cdot),1}$ and $\dot{B}^{s+\varepsilon}_{p(\cdot),1}$, respectively (choosing $\varepsilon > 0$ so that $s - \varepsilon > n(1/\min(\tau_{p(\cdot)}, 1) - 1)$). Therefore, U_N convergence in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, by Theorem 2.3.6 applied to the finite sequence $\{u_j\}_{-N\le j\le N}$, we have that $U_N \in \dot F^s_{p(\cdot),\infty}$ and

$$\|U_N\|_{\dot F^s_{p(\cdot),\infty}} \lesssim \left\|\{2^{js}u_j\}_{-N\le j\le N}\right\|_{L^{p(\cdot)}(\ell^\infty)} \le \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^\infty)},$$

with the implicit constant independent of N and $\{u_j\}_{j\in\mathbb{Z}}$. Since $\dot F^s_{p(\cdot),\infty}$ has the Fatou property, we conclude that $\lim_{N\to\infty} U_N = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot F^s_{p(\cdot),\infty}$ and

$$\left\|\sum_{j\in\mathbb{Z}} u_j\right\|_{\dot F^s_{p(\cdot),\infty}} \lesssim \left\|\{2^{js}u_j\}_{j\in\mathbb{Z}}\right\|_{L^{p(\cdot)}(\ell^\infty)}.$$

□

As a model result the we state the Leibniz type rule for variable exponent Triebel-Lizorkin spaces.

Theorem 2.3.10. *For $m \in \mathbb{R}$, let $\sigma(\xi,\eta)$, $\xi,\eta\in\mathbb{R}^n$, be an inhomogeneous Coifman-Meyer multiplier of order m. If $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0^*$ are such that $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$, $0 < q \le \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$, it holds that*

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p_1(\cdot),q}} \|g\|_{h^{p_2(\cdot)}} + \|f\|_{h^{p_1(\cdot)}} \|g\|_{F^{s+m}_{p_2(\cdot),q}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

Moreover, if $p(\cdot) \in \mathcal{P}_0^*$, $0 < q \le \infty$ and $s > n(1/\min(\tau_{p(\cdot)}, q, 1) - 1)$, it holds that

$$\|T_\sigma(f,g)\|_{F^s_{p(\cdot),q}} \lesssim \|f\|_{F^{s+m}_{p(\cdot),q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s+m}_{p(\cdot),q}} \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

2.4 Applications to scattering properties of PDEs

Our applications will be concerned with systems of differential equations on functions $u = u(t,x)$, $v = v(t,x)$ and $w = w(t,x)$, with $t \geq 0$ and $x \in \mathbb{R}^n$, of the form

$$\begin{cases} \partial_t u = vw, & \partial_t v + a(D)v = 0, \quad \partial_t w + b(D)w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \qquad w(0,x) = g(x), \end{cases} \tag{2.4.33}$$

where $a(D)$ and $b(D)$ are (linear) Fourier multipliers with symbols $a(\xi)$ and $b(\xi)$, $\xi \in \mathbb{R}^n$, respectively; that is, $\widehat{a(D)f}(\xi) = a(\xi)\widehat{f}(\xi)$ and $\widehat{b(D)f}(\xi) = b(\xi)\widehat{f}(\xi)$.

As in Bényi et al.**?** Section 2.3, we formally have

$$v(t,x) = \int_{\mathbb{R}^n} e^{-ta(\xi)}\widehat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \quad w(t,x) = \int_{\mathbb{R}^n} e^{-tb(\eta)}\widehat{g}(\eta)\, e^{2\pi i x\cdot\eta}\, d\eta,$$

and

$$u(t,x) = \int_0^t v(s,x)w(s,x)\, ds = \int_{\mathbb{R}^{2n}} \left(\int_0^t e^{-s(a(\xi)+b(\eta))}\, ds\right) \widehat{f}(\xi)\widehat{g}(\eta)\, e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

Setting $\lambda(\xi,\eta) = a(\xi) + b(\eta)$ and assuming that λ never vanishes, the solution $u(t,x)$ can then be written as the action on f and g of the bilinear multiplier with symbol $\frac{1-e^{-t\lambda(\xi,\eta)}}{\lambda(\xi,\eta)}$, that is,

$$u(t,x) = T_{\frac{1-e^{-t\lambda}}{\lambda}}(f,g)(x). \tag{2.4.34}$$

Following Bernicot–Germain**?** Section 9.4, suppose there exists $u_\infty \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\lim_{t\to\infty} u(t,\cdot) = u_\infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n); \tag{2.4.35}$$

then, given a function space X, we say that the solution u of (2.4.33) scatters in the function space X if $u_\infty \in X$.

As an application of Theorems 2.2.1 and **??** we obtain the following scattering properties for solutions to systems of the type (2.4.33) involving powers of the Laplacian.

For $0 < p_1, p_2, p, q \le \infty$ and $w_1, w_2 \in A_\infty$, set

$$\gamma^{w_1,w_2,tl}_{p_1,p_2,p,q} = 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}, q))] + 1),$$
$$\gamma^{w_1,w_2,b}_{p_1,p_2,p,q} = 2([n(1/\min(p,q,1) + 1/\min(1, p_1/\tau_{w_1}, p_2/\tau_{w_2}))] + 1).$$

For $\delta > 0$ define

$$\mathcal{S}_\delta = \{(\xi,\eta) \in \mathbb{R}^{2n} : |\eta| \le \delta^{-1}|\xi| \text{ and } |\xi| \le \delta^{-1}|\eta|\}.$$

Theorem 2.4.1. *Consider $0 < p, p_1, p_2 \le \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \le \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \ge \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \ge \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}_0(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, & \partial_t v + D^\gamma v = 0, \quad \partial_t w + D^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.36}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.36) scatters in $\dot{F}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{F}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.37}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \le \infty$ and $s > \tau_p(w)$, the solution u of (2.4.36) scatters in $\dot{B}^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates

$$\|u_\infty\|_{\dot{B}^s_{p,q}(w)} \lesssim \|f\|_{\dot{B}^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{H^{p_2}(w_2)} + \|f\|_{H^{p_1}(w_1)} \|g\|_{\dot{B}^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.38}$$

where the Hardy spaces $H^{p_1}(w_1)$ and $H^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different pairs of p_1, p_2 can be used on the right-hand sides of (2.4.37) and (2.4.38); moreover, if $w \in A_\infty$, then

$$\|u_\infty\|_{\dot{F}^s_{p,q}(w)} \lesssim \|f\|_{\dot{F}^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$, $0 < q \le \infty$, $s > \tau_{p,q}(w)$, and the implicit constant is independent of f and g.

Theorem 2.4.2. *Consider $0 < p, p_1, p_2 \le \infty$ such that $1/p = 1/p_1 + 1/p_2$ and $0 < q \le \infty$; let $w_1, w_2 \in A_\infty$ and set $w = w_1^{p/p_1} w_2^{p/p_2}$. Fix $\gamma > 0$; if γ is even, or $\gamma \ge \gamma^{w_1,w_2,tl}_{p_1,p_2,p,q}$ in the setting of Triebel–Lizorkin spaces, or $\gamma \ge \gamma^{w_1,w_2,b}_{p_1,p_2,p,q}$ in the setting of Besov spaces, assume $f, g \in \mathcal{S}(\mathbb{R}^n)$; otherwise, assume that $f, g \in \mathcal{S}(\mathbb{R}^n)$ are such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\widetilde{\mathcal{S}}_\delta$ for some $0 < \delta \ll 1$. Consider the system*

$$\begin{cases} \partial_t u = vw, & \partial_t v + J^\gamma v = 0, \quad \partial_t w + J^\gamma w = 0, \\ u(0,x) = 0, & v(0,x) = f(x), \quad w(0,x) = g(x). \end{cases} \tag{2.4.39}$$

If $0 < p, p_1, p_2 < \infty$ and $s > \tau_{p,q}(w)$, the solution u of (2.4.39) scatters in $F^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates:

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{F^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.40}$$

where the implicit constant is independent of f and g. If $0 < p, p_1, p_2 \le \infty$ and $s > \tau_p(w)$, the solution u of (2.4.39) scatters in $B^s_{p,q}(w)$ to a function u_∞ that satisfies the following estimates

$$\|u_\infty\|_{B^s_{p,q}(w)} \lesssim \|f\|_{B^{s-\gamma}_{p_1,q}(w_1)} \|g\|_{h^{p_2}(w_2)} + \|f\|_{h^{p_1}(w_1)} \|g\|_{B^{s-\gamma}_{p_2,q}(w_2)}, \tag{2.4.41}$$

where the Hardy spaces $h^{p_1}(w_1)$ and $h^{p_2}(w_2)$ must be replaced by L^∞ if $p_1 = \infty$ or $p_2 = \infty$, respectively, and the implicit constant is independent of f and g. If $w_1 = w_2$ then different *pairs of p_1, p_2 can be used on the right-hand sides of* (2.4.40) *and* (2.4.41)*; moreover, if* $w \in A_\infty$*, then*

$$\|u_\infty\|_{F^s_{p,q}(w)} \lesssim \|f\|_{F^{s-\gamma}_{p,q}(w)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{F^{s-\gamma}_{p,q}(w)},$$

where $0 < p < \infty$*,* $0 < q \leq \infty$*,* $s > \tau_{p,q}(w)$*, and the implicit constant is independent of f and g.*

Proof of Theorem 2.4.1. Using the notation from Section **??**, we have $a(\xi) = |\xi|^\gamma$ and $b(\eta) = |\eta|^\gamma$; therefore, $\lambda(\xi, \eta) = |\xi|^\gamma + |\eta|^\gamma$. Note that all corresponding integrals for $v(t, x)$, $w(t, x)$ and $u(t, x)$ are absolutely convergent for $t > 0$, $x \in \mathbb{R}^n$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. If we further assume that $f, g \in \mathcal{S}_0(\mathbb{R}^n)$, the Dominated Convergence Theorem implies that $u(t, \cdot) \to u_\infty$ both pointwise and in $\mathcal{S}'(\mathbb{R}^n)$, where

$$u_\infty(x) = \int_{\mathbb{R}^{2n}} (a(\xi) + b(\eta))^{-1} \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi \, d\eta = T_{\lambda^{-1}}(f, g)(x).$$

If γ is an even positive integer then λ^{-1} satisfies the estimates (2.1.4) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Then, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ and therefore the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$.

Let p_1, p_2, p, q, w_1, w_2 be as in the hypotheses. If $\gamma > 0$ and γ is not an even integer, then λ^{-1} satisfies the estimates (2.1.4) with $m = -\gamma$ as long as $\alpha, \beta \in \mathbb{N}_0^n$ are such that $|\alpha| < \gamma$ and $|\beta| < \gamma$; in particular, λ^{-1} satisfies (2.1.4) with $m = -\gamma$ for $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha + \beta| < \gamma$. In view of Remark **??**, all estimates from Theorem 2.2.1 hold for $T_{\lambda^{-1}}$ if $\gamma \geq \gamma^{w_1, w_2, tl}_{p_1, p_2, p, q}$ in the context of Triebel–Lizorkin spaces and if $\gamma \geq \gamma^{w_1, w_2, b}_{p_1, p_2, p, q}$ in the context of Besov spaces; as a consequence, the desired estimates follow for u_∞ with constants independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ for such values of γ.

On the other hand, if $0 < \gamma < \gamma^{w_1, w_2, tl}_{p_1, p_2, p, q}$ in the Triebel-Lizorkin space setting or $0 < \gamma < \gamma^{w_1, w_2, b}_{p_1, p_2, p, q}$ in the Besov space setting, and γ is not an even positive integer, consider $h \in \mathcal{S}(\mathbb{R}^{2n})$ such that $\mathrm{supp}(h) \subset \mathcal{S}_{\delta/2}$ and $h \equiv 1$ on $\mathcal{S}_\delta$. Then, for $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$ we have $h(\xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta) = \widehat{f}(\xi)\widehat{g}(\eta)$; therefore, $T_{\lambda^{-1}}(f, g) = T_\Lambda(f, g)$, where

$\Lambda(\xi, \eta) = h(\xi, \eta)/(|\xi|^\gamma + |\eta|^\gamma)$. The multiplier Λ verifies (2.1.4) with $m = -\gamma$ for all $\alpha, \beta \in \mathbb{N}_0^n$ (with constants that depend on δ). Then all estimates from Theorem 2.2.1 hold for T_Λ and therefore the desired estimates follow for u_∞ with constants dependent on δ and independent of $f, g \in \mathcal{S}_0(\mathbb{R}^n)$ such that $\widehat{f}(\xi)\widehat{g}(\eta)$ is supported in $\mathcal{S}_\delta$. $\square$

Proof of Theorem 2.4.2. We proceed as in the proof of Theorem 2.4.1 with $\lambda(\xi, \eta) = (1 + |\xi|^2)^{\gamma/2} + (1 + |\eta|^2)^{\gamma/2}$ and an application of Theorem **??**. $\square$

Appendix A

Title for This Appendix

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