3 Algorithm Implementation

The algorithm has been implemented in three different simulations, the first two using simulated datasets and the last one using a real dataset.

For each simulated dataset, the covariates are randomly drawn from pre-determined Normal distributions and the true parameters are specified beforehand. Given the matrix of covariates X and the vector of true parameters b, the latent variables are then computed as the matrix product $X'b$, to which random standard Normal errors are added. The dependent variables Yi's are then set either equal to 1 or to 0, depending on whether the corresponding latent is above or below 0 respectively. For each generation of the latent, it is then possible to compute the OLS estimators, and, by iterating such process, it is possible to obtain a large sample of OLS estimators for the same true model, from which it is possible to derive the exact distributions of the OLS estimators, which are then used as a reference to evaluate the performance of the algorithms.

For the real dataset, such iteration is obviously not possible, thus the results of the probit regression are used as a reference.

For each dataset, the Metropolis Hastings algorithm, and both the uninformed and informed Gibbs sampler are run using different initial values, to evaluate their robustness, and different priors (when possible), to evaluate their effect on the evolution of the Markov Chain. The criteria for the performance evaluation of the algorithms are the following:

- The speed of convergence of the Markov Chains: the number of burn-in iterations.
- The correctness of the generated distributions: the mean squared errors (MSEs) of their means with respect to the true parameters, and their standard deviation.
- The autocorrelation function of the Markov Chain: if the process converges to stationarity, then the corresponding autocorrelation function should approximately converge to 0 for each parameter.
- The empirical acceptance rate (for the Metropolis Hastings algorithm): to guarantee a proper mixing of the chain, the acceptance rate should stabilize as the process evolves; furthermore, a high acceptance rate would imply a slow convergence.

3.1 Simulations

3.1.1 Simulation 1

The first simulated dataset contains two regressors, and the true parameters are: [beta0 = 2; beta1 = 1; beta2 = -1]. Obviously, the distributions of the OLS estimators are Normally distributed and centred around the true values, as it is possible to see from figures 1,2 and 3, along with their respective standard deviations (0.07, 0.032, 0.032). The first simulation (MCMC1 in the table) starts from relatively close values and relies on a multivariate standard Normal as a prior: as it is possible to see, the process stabilizes quickly, and the distributions approximately coincide with the target ones, with larger standard deviations. Changing the initial values (MCMC2) obviously affects the convergence of the process: the further they are, the more burn-in iterations are necessary before convergence is reached (in this case, approximately 3000). However, they do not seem to affect the quality of the final distribution: both the means and the standard deviations coincide with the preceding simulation. The reason for such late convergence can be identified by considering the acceptance rate: for the first 3000 iterations, it is equal to 1, therefore the process does not mix at all, since all the steps are accepted; indeed, as soon as the rate starts to decline, the process stabilizes, as it is also possible to see by the autocorrelation of the estimates converging to 0.

The algorithm is then run multiple times (from MCMC3 to 6) with different starting values and priors, and it is possible to see that the process converges to stationarity relatively well. In this simulation, different priors (more and less informative) do not lead to significant differences in terms of performance.

On the other hand, MCMC7 is a hypothetical simulation, where, instead of substituting the posterior with the product of the likelihood by the prior, the target posterior distribution is directly inserted in the algorithm. As expected, the generated distributions almost perfectly coincide with the target distributions, both in terms of means and standard deviations.

As it is possible to see from Uninformed Gibbs (UG) 1,2 and 3, the uninformed Gibbs sampler allows for a quicker convergence of the process, even for starting values relatively far from the correct ones (UG 3). The quality of the generated distributions is satisfying: the MSEs are low, while the standard deviations are slightly larger than the true ones.

From Informed Gibbs (IG) 2 and 3, it is possible to identify the effect of a proper prior in the algorithm: given the same starting values ([5,5,5]), IG2 uses a standard Normal distribution as a prior, while IG3 uses exactly the means and the variance-covariance matrix of the target distribution. As it is possible to see, while the speed of convergence is not significantly affected, the distributions generated with the proper prior are almost identical to the correct ones: the means approximately coincide with the true parameters, and the standard deviations are even slightly lower than the true ones.

3.1.2 Simulation 2

The second simulated dataset contains three regressors, and the true parameters are [-1; 2; 3; 0.5]. As it is possible to see from figures 65, 66, 67, and 68, the OLS estimators are still Normally distributed and centred at the true parameters. From MCMC1 and MCMC2, it is possible to see that the algorithm still works properly for relatively close starting values: the MSEs are low, the autocorrelations quickly converge to 0, the acceptance rate quickly stabilizes, and the generated distributions are approximately centred around the true values, with slightly larger standard deviations. MCMC3 and MCMC4 allow to analyse the effect of the prior: given the same starting values, while MCMC3 uses a multivariate standard Normal distribution as a prior, MCMC4 has its prior centred at the true values of the parameters. Compared to MCMC3, MCMC4 process converges more quickly: indeed, the acceptance rate stabilizes at a lower level, implying a more effective mixing as the process evolves. In terms of quality of the generated distributions, there are no significant differences.

The uninformed Gibbs samplers (UG1, UG2 and UG3) still work relatively well: the processes converge relatively quickly, with the autocorrelations converging to 0, and the generated distributions are approximately centred around the true values, with slightly larger standard deviations. Compared to the previous MCMCs, the MSEs are overall slightly higher, however the difference can be considered as negligible.

The informed Gibbs samplers (IG1 and IG2) allow to analyse the effect of a proper prior, given the same starting values: compared to IG1, which relies on a multivariate standard Normal distribution, IG2 uses the means and the variance-covariance matrix of the target distributions. As it is possible to see, IG2 allows for a quicker convergence of the process, and, furthermore, the generated distributions are characterized by a lower standard deviation, which makes them almost identical to the target ones.

3.1.3 Simulation 3: Real Dataset

The real dataset, called "Gaming.csv", contains information about consumers' six driving factors behind the choice of a type of computer ("A" or "B). In particular, the Bernoulli random variable Yi is set equal to 1 if the corresponding consumer's choice is "A", thus the parameters can be imprecisely interpreted as the effect of the regressors on the probability of the consumer choosing "A" over "B". Given the use of a real dataset, it is not possible to generate the true distribution of the OLS estimators, therefore the STATA output of the probit regression is used as a reference for the true values of the parameters (see figure 111).

As it is possible to see from MCMC1, 2, and 3, neither of the processes converges to stationarity: the autocorrelations do not converge to 0, and the generated distributions are thus obviously wrong. The reason can be easily identified by considering the acceptance rate: it is always equal to 1, thus, all the steps are accepted, and the process never starts mixing. Such lack of convergence arises even when using starting values very close to the probit ones, or when using informative priors close to the probit results. Therefore, one possible solution may be to find a different proposal transition probability, which may guarantee a proper mixing of the process.

As it is possible to see from UG1, 2, and 3, uninformed Gibbs samplers processes quickly convergence to stationarity, even for starting values which are relatively far from the probit ones (UG3): the MSEs are very low, and the generated distributions are approximately centred around the probit values.

IG1 and IG2 allow to analyse the effect of a proper prior: while IG1 relies on a multivariate standard Normal distribution, IG2 relies on a more informative prior, closer to the probit values. While in both cases the processes quickly converge to stationarity, the MSE of IG1 is higher than the one of IG2 (2 vs 0.12), mainly due to the generated distribution of beta0, which, in IG1, is centred at a wrong value (1 instead of 4). It is also worth mentioning that the generated distributions by both IGs are characterized by lower standard deviations, when compared to the previous UGs.

3.2 Performance Assessment

From the previous simulations, it is possible to conclude that Gibbs sampler is overall a more effective algorithm with respect to the standard Metropolis Hastings (MH), both in terms of convergence and of running time (at least, in our case). In particular, the number of required burn-in iterations before convergence is lower, which does not compromise the quality of the generated distributions. Without an informative prior, uninformed Gibbs sampler proves to be an effective algorithm, provided that the starting values are not too far from the true ones. Instead, given an informative prior, an informed Gibbs sampler or even a Metropolis Hastings algorithm would be a better choice, since they would lead to a faster convergence and more precise distributions, with lower standard deviations and closer to the true ones. However, the choice of the prior needs to be careful, since, as it is possible to see from simulation 3, the insertion of a wrong prior may lead to the generation of wrong distributions. Finally, for all the algorithms, the starting values clearly represent a determinant factor in the convergence of the process: values too far from the true ones would compromise the effectiveness of the algorithm. With respect to this, it is possible to notice that Gibbs sampler still seems to be more robust to distant starting values compared to MH.

Appendix

SIMULATION 1:

For the sake of clarity, charts can be found in the appendix, referenced accordingly in the tables.

Parameters of the Target Distribution

	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$
Pdf	(1)	(2)	(3)
Mean	(4)		
Covariance matrix	(5)		

Simulation 1 Results:

	Distribution	*Sim. MC*	*MSE*	*Autocorrelation*	*Acc. rate*
MCMC 1	(6)	(7)	(8)	(9)	(10)
MCMC 2	(11)	(12)	(13)	(14)	(15)
MCMC 3	(16)	(17)	(18)	(19)	(20)
MCMC 4	(21)	(22)	(23)	(24)	(25)
MCMC 5	(26)	(27)	(28)	(29)	(30)
MCMC 6	(31) – Target as Prior	(32)	(33)	(34)	(35)
MCMC 7	(36) - Target Posterior	(37)	(38)	(39)	n.d.

	Distribution	*Simulated Markov Chain*	*MSE*	*Autocorrelation*
Uninformed Gibbs 1	(40)	(41)	(42)	(43)
Uninformed Gibbs 2	(44)	(45)	(46)	(47)
Uninformed Gibbs 3	(48)	(49)	(50)	(51)
Uninformed Gibbs 4 (failed convergence)	(52)	(53)	(54)	(55)
Informed Gibbs 1	(56)	(57)	(58)	(59)
Informed Gibbs 2	(60)	(61)	(62)	(63)

SIMULATION 2:

For the sake of clarity, charts can be found in the appendix, referenced accordingly in the tables.

Parameters of the Target Distribution

	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$
Pdf	(65)	(66)	(67)	(68)
Covariance matrix	(69)			
Mean	(70)			

Simulation 2 Results:

	Distribution	*Sim. MC*	*MSE*	*Autocorrelation*	*Acc. rate*
MCMC 1	(71)	(72)	(73)	(74)	(75)
MCMC 2	(76)	(77)	(78)	(79)	(80)

MCMC 3	(81)	(82)	(83)	(84)	(85)
MCMC 4	(86)	(87)	(88)	(89)	(90)

	Distribution	*Simulated Markov Chain*	*MSE*	*Autocorrelation*
Uninformed Gibbs 1	(91)	(92)	(93)	(94)
Uninformed Gibbs 2	(95)	(96)	(97)	(98)
Uninformed Gibbs 3	(99)	(100)	(101)	(102)
Informed Gibbs 1	(103)	(104)	(105)	(106)
Informed Gibbs 2	(107)	(108)	(109)	(110)

SIMULATION 3:

Parameters of the Target Distribution

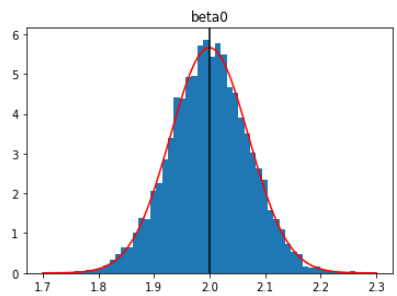
True parameters	(111)
Mean	(112)
Covariance matrix	(113)

Simulation 3 Results:

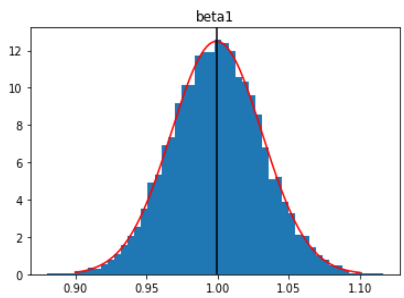
	Distribution	*Simulated Markov Chain*	*MSE*	*Autocorrelation*	*Acc. Rate*
MCMC 1	(114)	(115)	(116)	(117)	(118)
MCMC 2	(119)	(120)	(121)	(122)	(123)
MCMC 3	(124)	(125)	(126)	(127)	(128)

	Distribution	*Simulated Markov Chain*	*MSE*	*Autocorrelation*
Uninformed Gibbs 1	(129)	(130)	(131)	(132)
Uninformed Gibbs 2	(133)	(134)	(135)	n.d.
Uniformed Gibbs 3	(136)	(137)	(138)	(139)
Informed Gibbs 1	(140)	(141)	(142)	(143)
Informed Gibbs 2	(144)	(145)	(146)	(147)

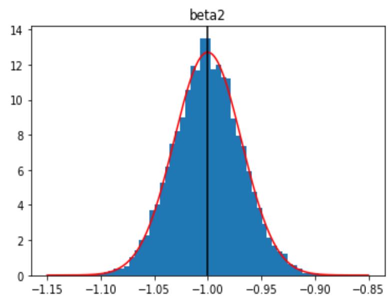
SIMULATION 1



Standard Deviation = 0.07048322382231631



Standard Deviation = 0.031924424686935314



Standard Deviation = 0.031419644461710826

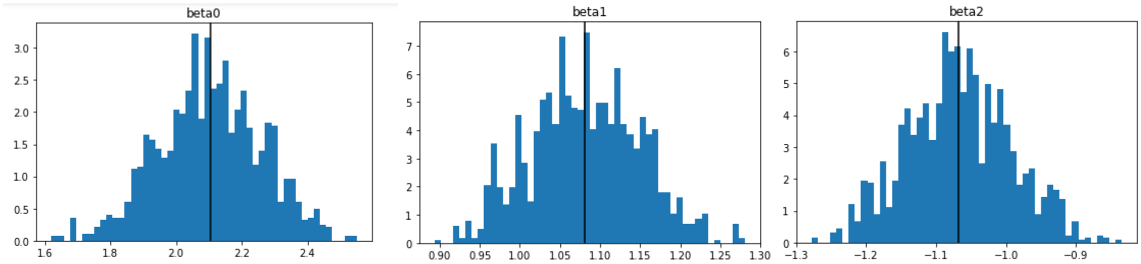
Figure (1) & Figure(2) & Figure (3)

```
The mean of the target is
[ 2.00083596  0.99996139 -1.00022078]
```

(4)

```
The variance-covariance matrix of the target is
[[ 5.07399900e-03 -5.89588163e-05 -2.03148306e-03]
 [-5.89588163e-05  1.01405867e-03  2.76497760e-05]
 [-2.03148306e-03  2.76497760e-05  1.01219678e-03]]
```

(5)




```
Beta0 Standard Deviation = 0.15668209609652772
Beta0 Mean = 2.091830195700063
```

```
Beta1 Standard Deviation = 0.06752490854340534
Beta1 Mean = 1.0792012980943682
```

```
Beta2 Standard Deviation = 0.07365866768229501
Beta2 Mean = -1.0618592952225254
```

Figure (6)

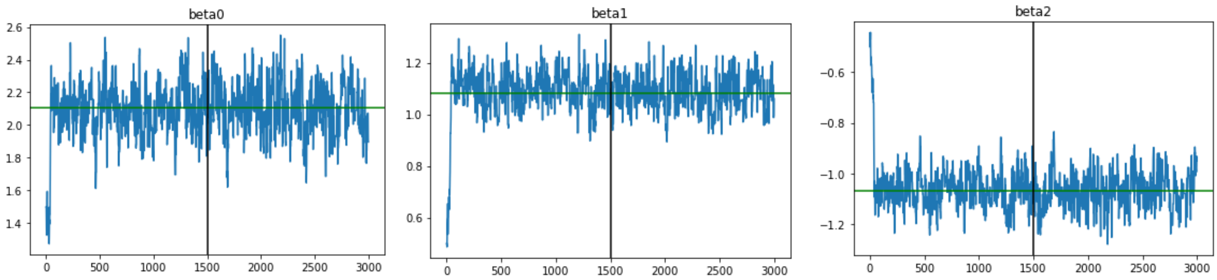


Figure (7)

```
The MSE is 0.006177400095585747
```

Figure (8)

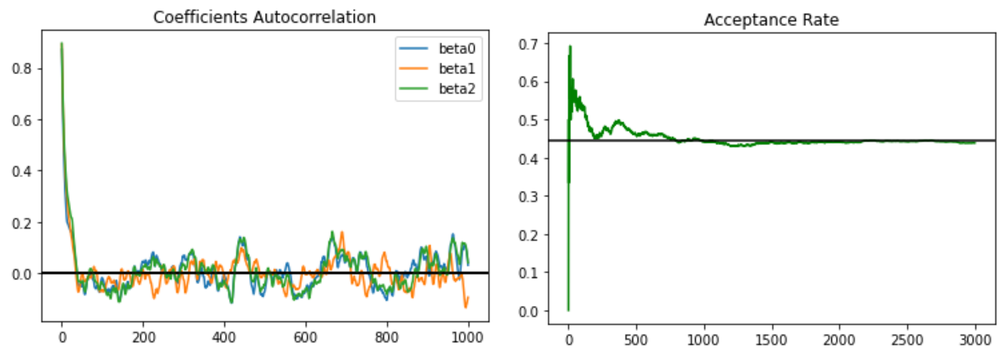
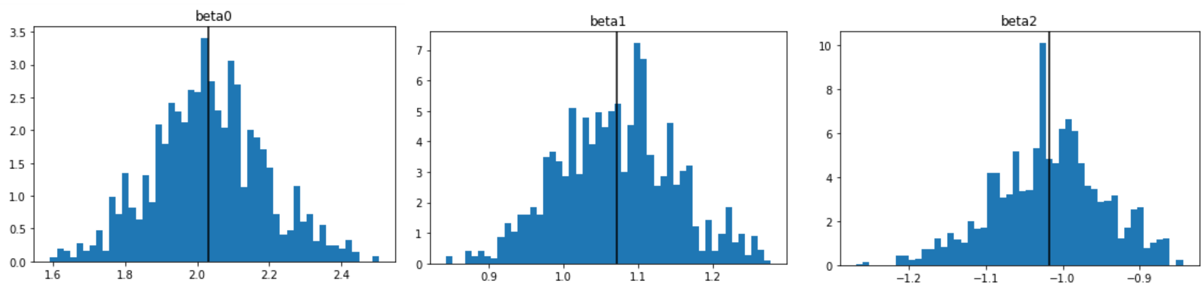


Figure (9) & Figure (10)




```
Beta0 Standard Deviation = 0.15479851961799884
Beta0 Mean = 2.0320639235401563
```

```
Beta1 Standard Deviation = 0.07819618381812934
Beta1 Mean = 1.0708323507783097
```

```
Beta2 Standard Deviation = 0.07133803565175814
Beta2 Mean = -1.0174360143960055
```

Figure (11)

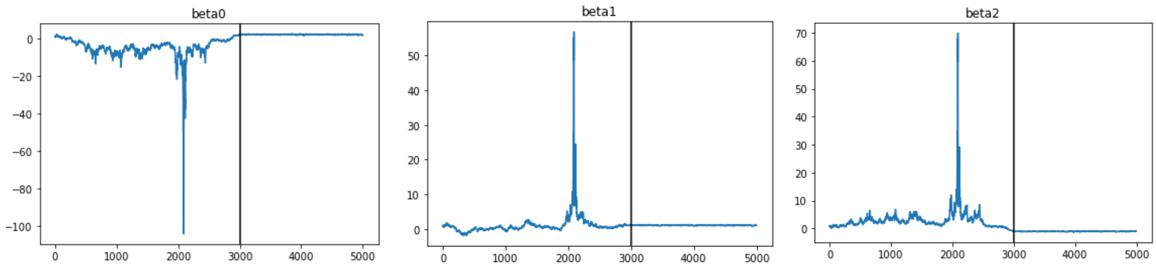


Figure (12)

```
The MSE is 0.0021164439025294053
```

Figure (13)

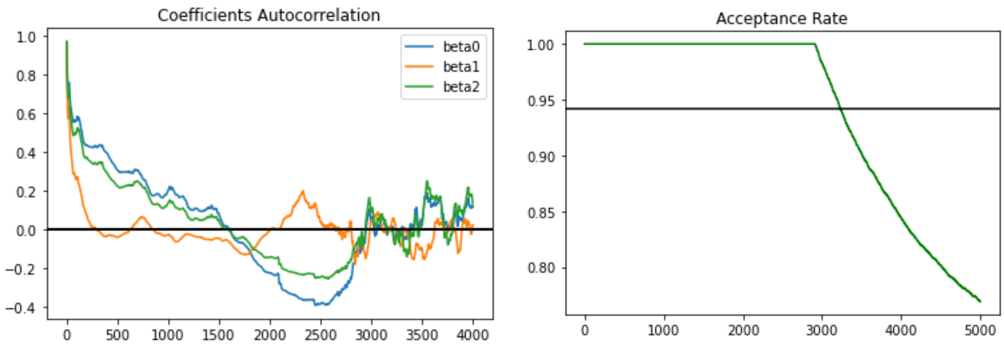
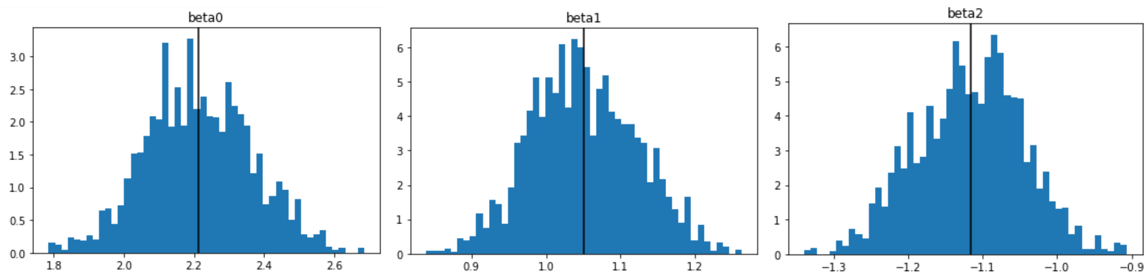


Figure (14) & Figure (15)




```
Beta0 Standard Deviation = 0.15176842728846768
Beta0 Mean = 2.21336770154741
```

```
Beta1 Standard Deviation = 0.07095363840567616
Beta1 Mean = 1.0508763139566708
```

```
Beta2 Standard Deviation = 0.07189859531759614
Beta2 Mean = -1.1164386463534572
```

Figure 16

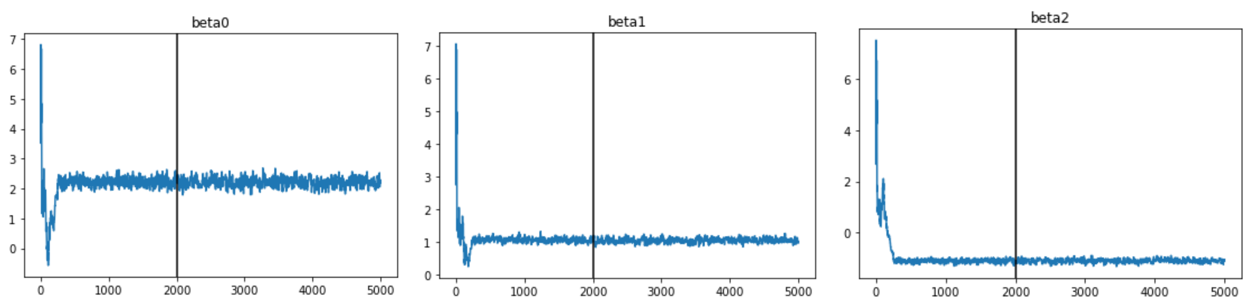


Figure 17

```
The MSE is 0.020557377916689254
```

Figure 18

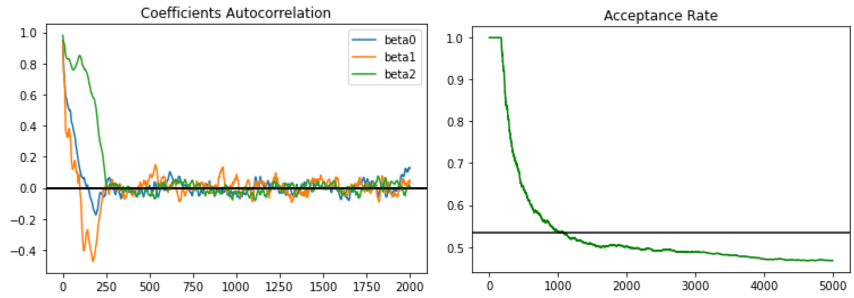
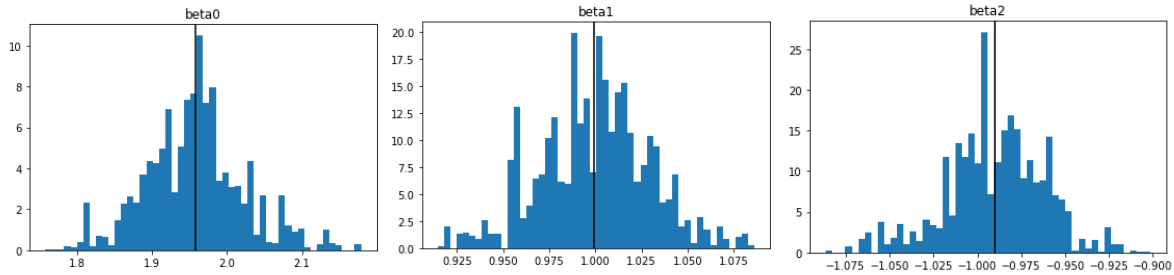


Figure 19 & Figure 20




```
Beta0 Standard Deviation = 0.06512673021865309
Beta0 Mean = 1.957563905866674
```

```
Beta1 Standard Deviation = 0.029899273177338245
Beta1 Mean = 0.9994735509563032
```

```
Beta2 Standard Deviation = 0.028436878337207767
Beta2 Mean = -0.9899238567401232
```

Figure 21

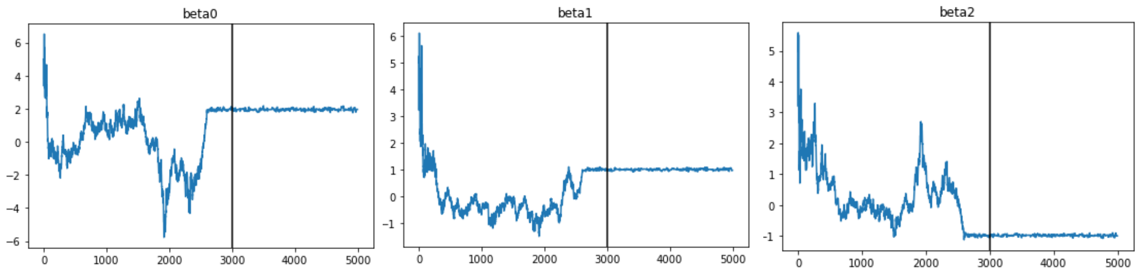


Figure 22

```
The MSE is 0.0006342092989605569
```

Figure 23

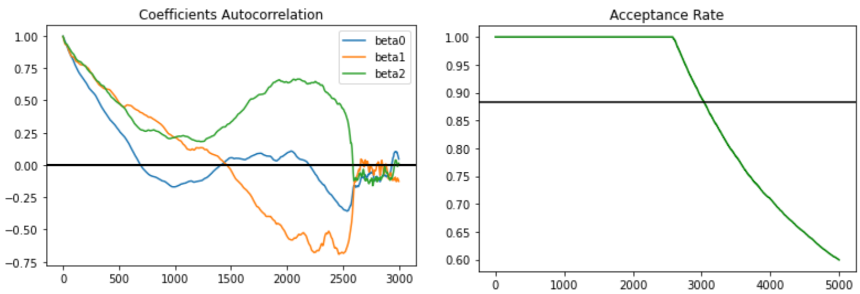
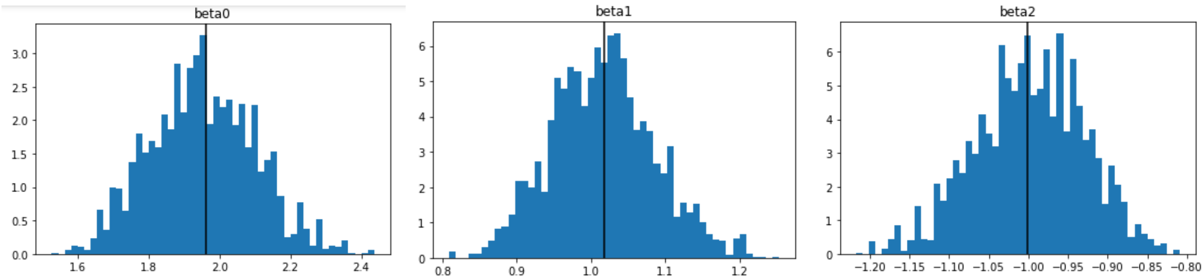


Figure 24 & Figure 25




```
Beta0 Standard Deviation = 0.1473224148004544
Beta0 Mean = 1.9526414837260377
```

```
Beta1 Standard Deviation = 0.06961847949927863
Beta1 Mean = 1.0162236379269332
```

```
Beta2 Standard Deviation = 0.06890090588528201
Beta2 Mean = -0.9979167612521097
```

Figure 26

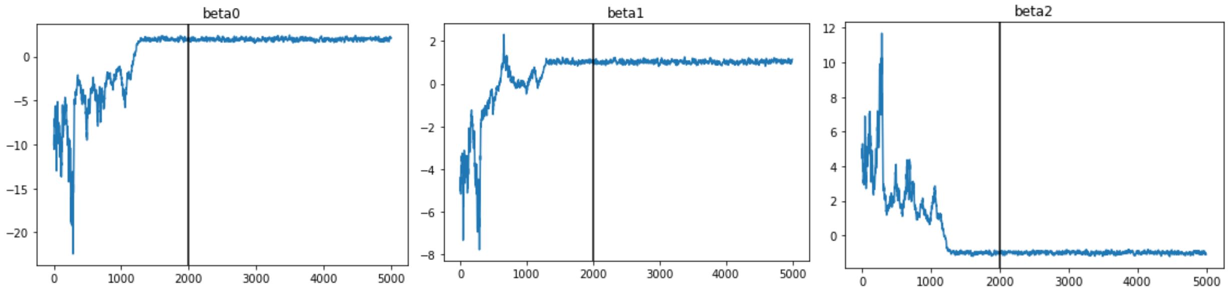


Figure 27

```
The MSE is 0.0008367917916453612
```

Figure 28

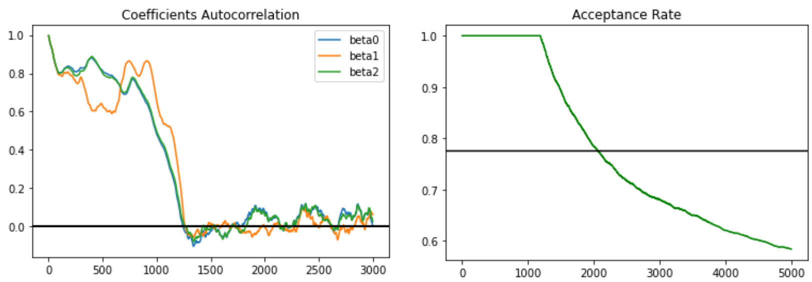
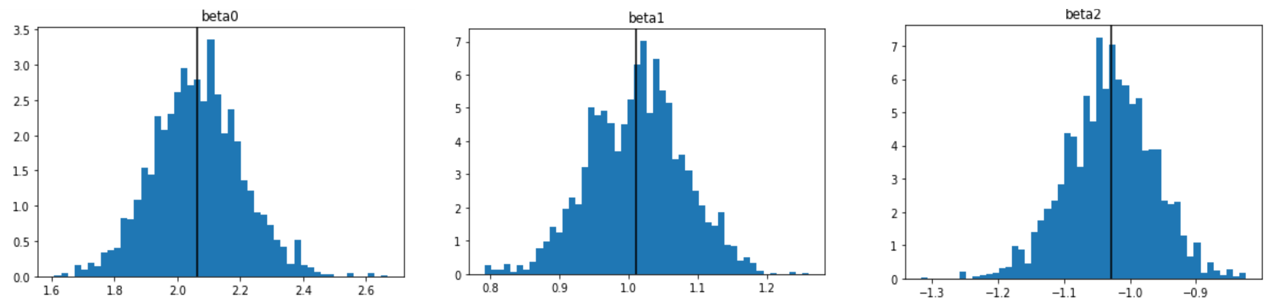


Figure 29 & Figure 30




```
Beta0 Standard Deviation = 0.14144429859500507
Beta0 Mean = 2.0641315858531875
```

```
Beta1 Standard Deviation = 0.06858347965722689
Beta1 Mean = 1.0117142980555711
```

```
Beta2 Standard Deviation = 0.06581164672154301
Beta2 Mean = -1.0300293578974784
```

Figure 31

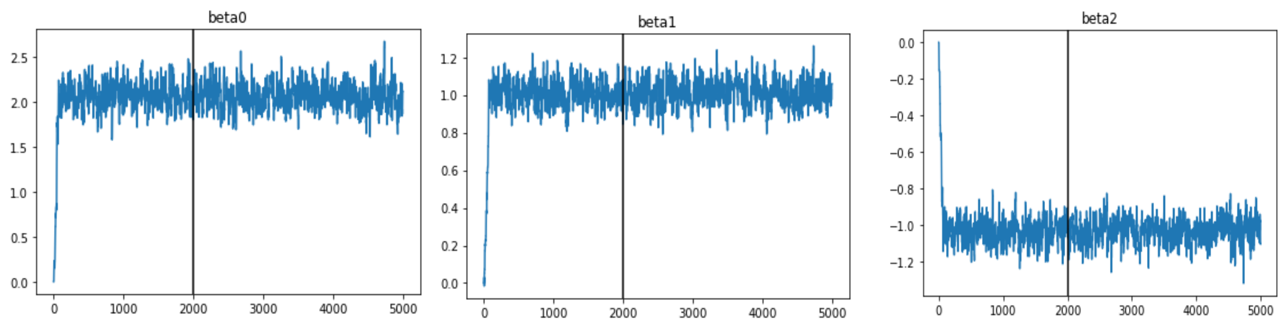


Figure 32

```
The MSE is 0.0017172824729047862
```

Figure 33

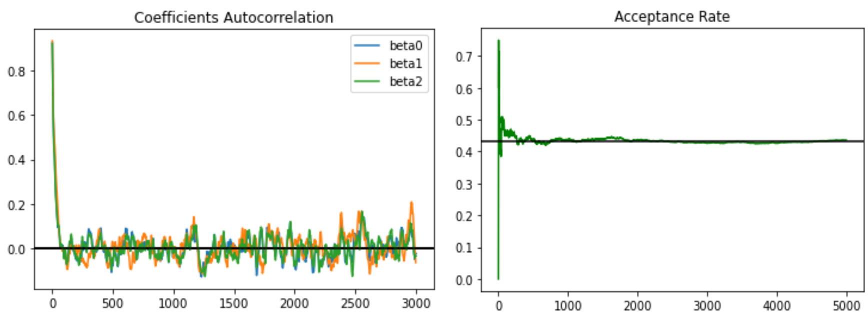
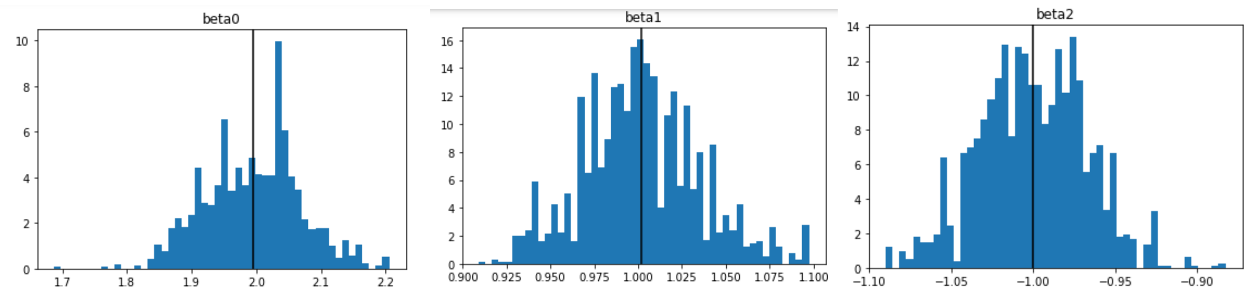


Figure 34 & Figure 35




```
Beta0 Standard Deviation = 0.07431524587729146
Beta0 Mean = 1.99592561248965
```

```
Beta1 Standard Deviation = 0.03315835136655795
Beta1 Mean = 1.0017281570911798
```

```
Beta2 Standard Deviation = 0.03256263920605754
Beta2 Mean = -1.0001630405306075
```

Figure 36

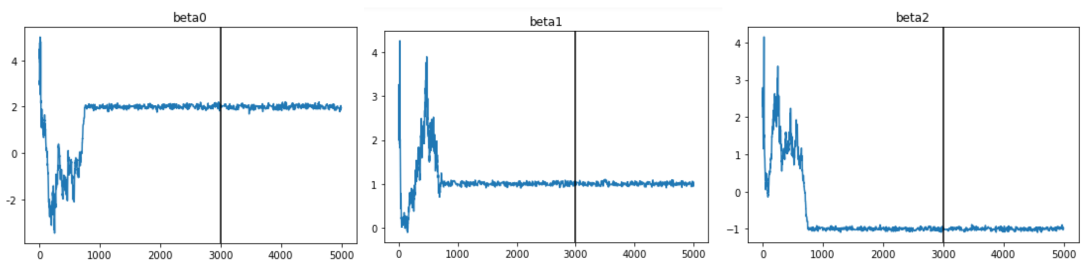


Figure 37

```
The MSE is 6.53791424363712e-06
```

Figure 38

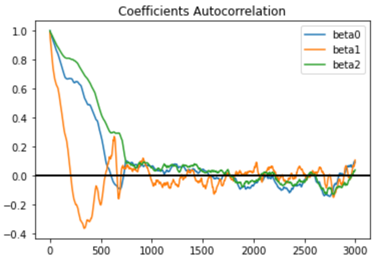
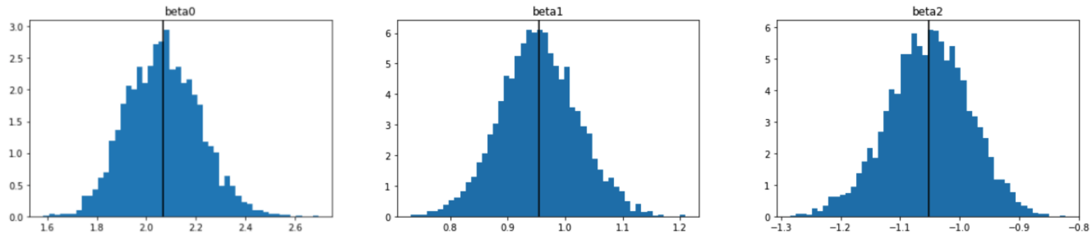


Figure 39




```
Beta0 Standard Deviation = 0.14785732557747l8
Beta0 Mean = 2.0690753165666407
```

```
Beta1 Standard Deviation = 0.06821574803002847
Beta1 Mean = 0.9538875863325201
```

```
Beta2 Standard Deviation = 0.06970833966294367
Beta2 Mean = -1.0532328307046943
```

Figure 40

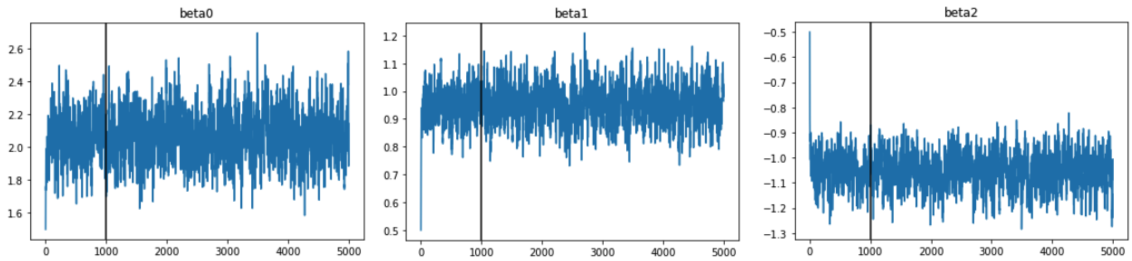


Figure 41

```
The MSE is 0.0032438294392856844
```

Figure 42

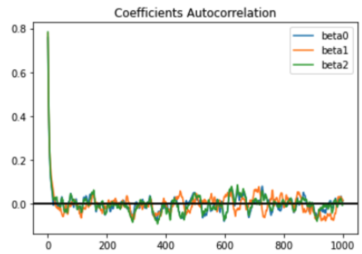
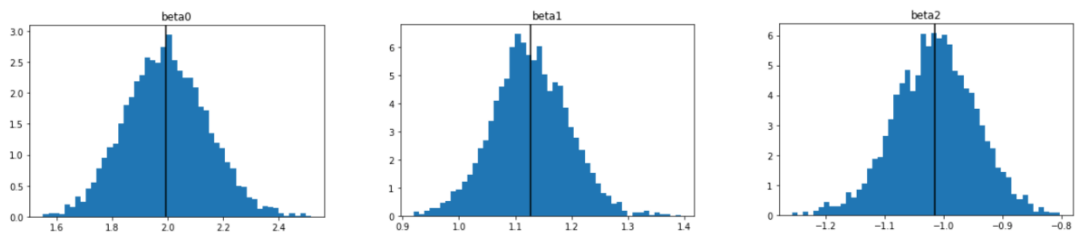


Figure 43




```
Beta0 Standard Deviation = 0.14781396160771312
Beta0 Mean = 1.9958736006907338
```

```
Beta1 Standard Deviation = 0.06740278914351881
Beta1 Mean = 1.126778914144865
```

```
Beta2 Standard Deviation = 0.06881516841675116
Beta2 Mean = -1.0147988165330206
```

Figure 44

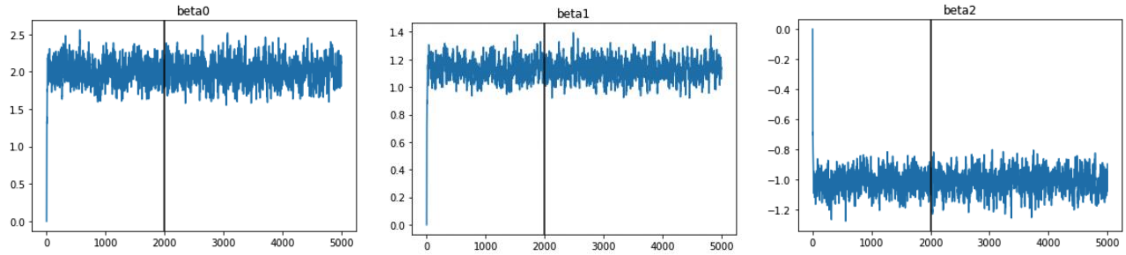


Figure 45

```
The MSE is 0.005436308404596187
```

Figure 46

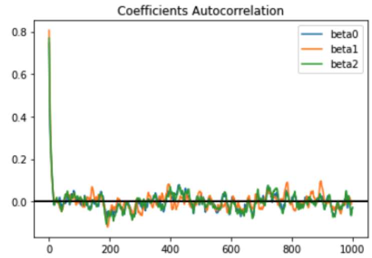
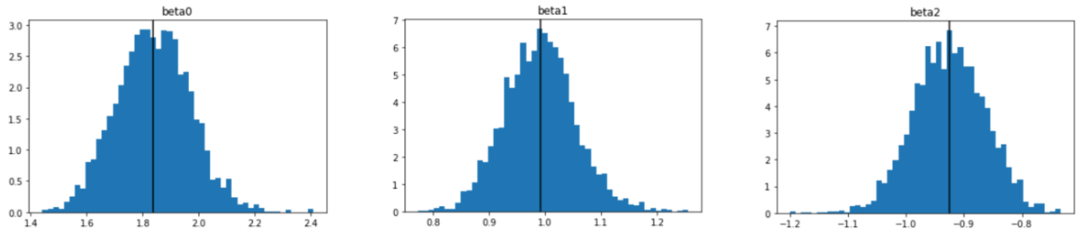


Figure 47




```
Beta0 Standard Deviation = 0.13211168639838528
Beta0 Mean = 1.8381507770173788
```

```
Beta1 Standard Deviation = 0.06409436908107474
Beta1 Mean = 0.9910742597026077
```

```
Beta2 Standard Deviation = 0.062204778307586875
Beta2 Mean = -0.9253396230426282
```

Figure 48

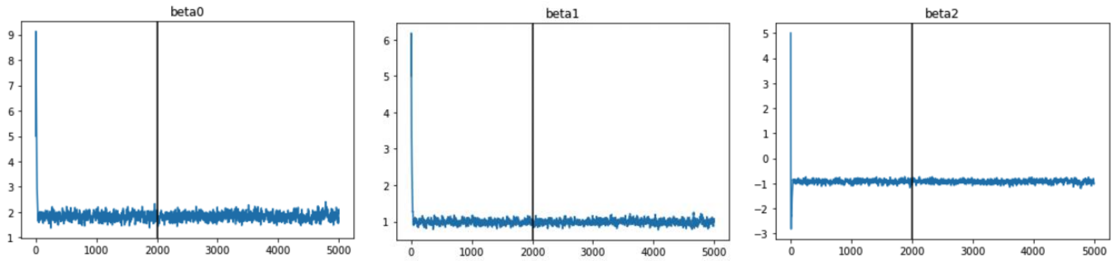


Figure 49

```
The MSE is 0.01061633723578386
```

Figure 50

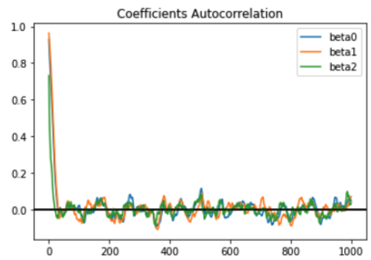


Figure 51

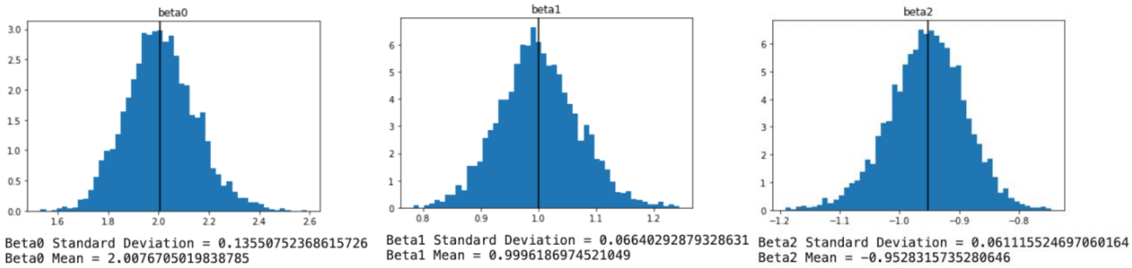


Figure 52

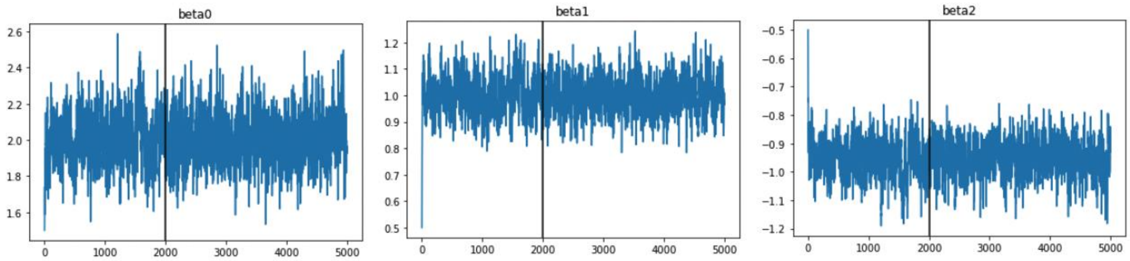


Figure 53

The MSE is 0.0007612808160520314

Figure 54

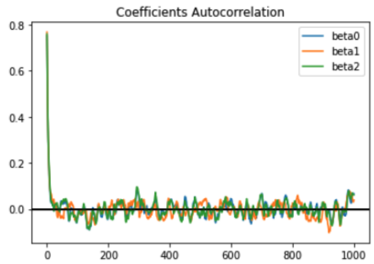


Figure 55

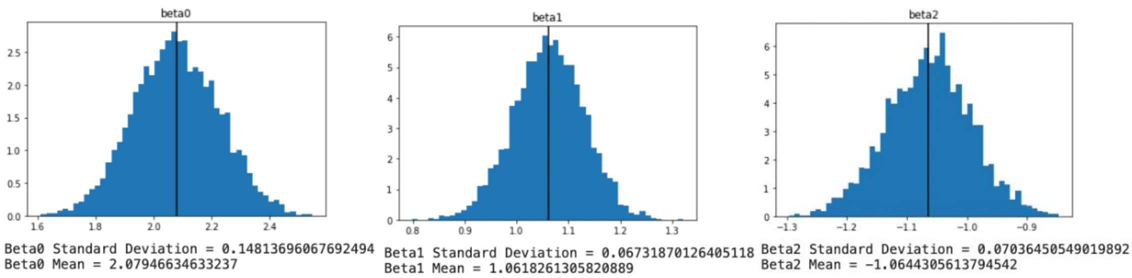


Figure 56

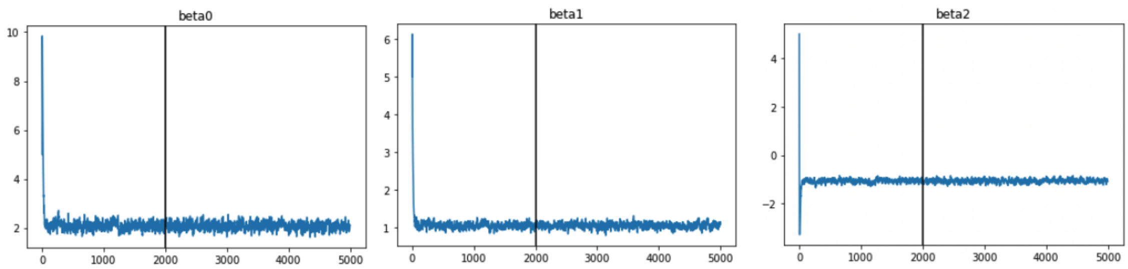


Figure 57

The MSE is 0.004762889287280429

Figure 58

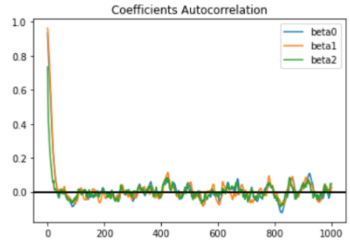
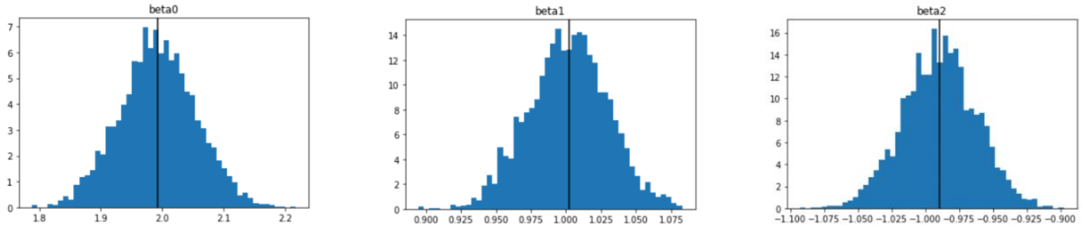


Figure 59



Beta0 Standard Deviation = 0.06187632351163574
Beta0 Mean = 1.9927633663258324

Beta1 Standard Deviation = 0.028262500847828234
Beta1 Mean = 1.001530605833605

Beta2 Standard Deviation = 0.027293133415852714
Beta2 Mean = -0.9893792066447288

Figure 60

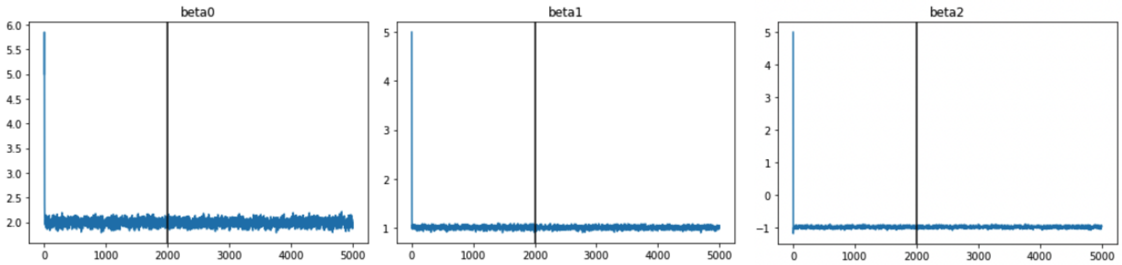


Figure 61

The MSE is 5.5837624215778575e-05

Figure 62

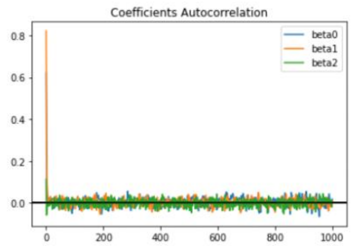


Figure 63

SIMULATION 2

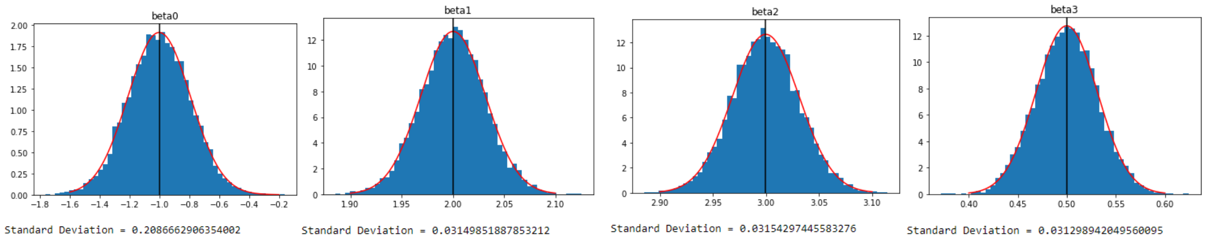


Figure 65-66-67-68

```
The mean of the target is
[-1.00143401  2.00027883  2.99980573  0.49983782]
The variance-covariance matrix of the target is
[[ 4.35459754e-02 -5.00612638e-03  2.98973409e-03 -2.84501362e-03]
 [-5.00612638e-03  9.92255917e-04 -1.41502883e-05 -3.72945259e-06]
 [ 2.98973409e-03 -1.41502883e-05  9.95058743e-04  2.01344582e-05]
 [-2.84501362e-03 -3.72945259e-06  2.01344582e-05  9.79721746e-04]]
```

Figure 69 & Figure 70

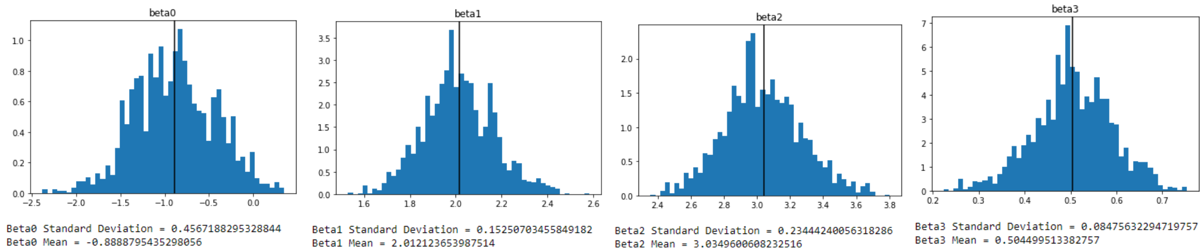


Figure 71

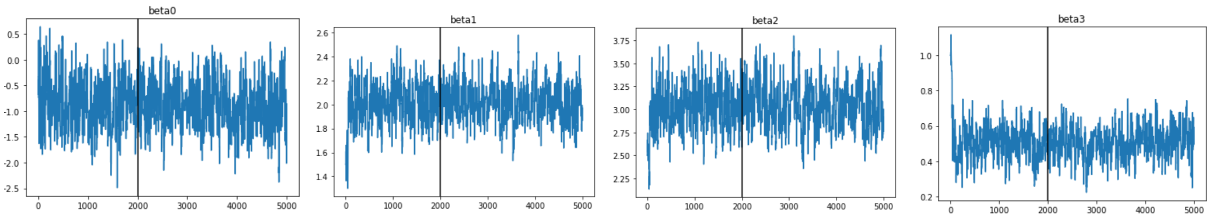
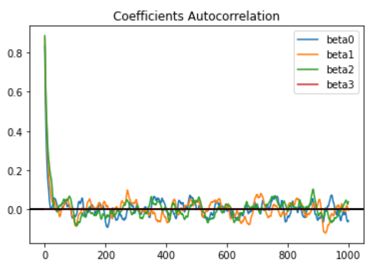


Figure 72

```
The MSE is 0.0034342975764000984
```

Figure 73



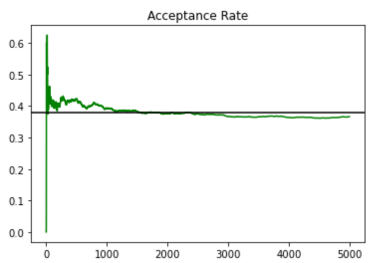


Figure 74 & Figure 75

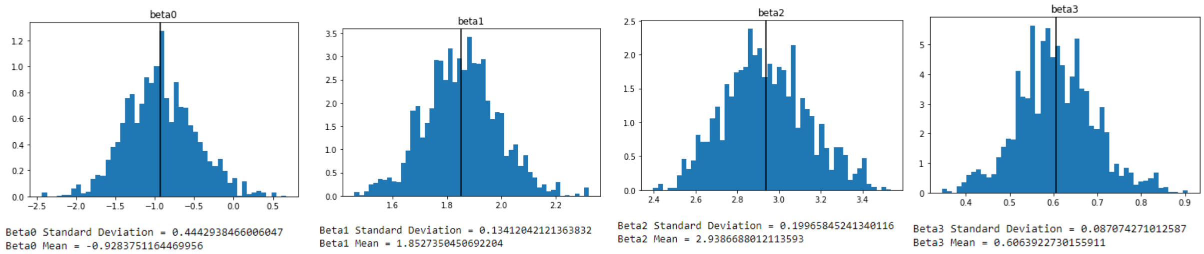


Figure 76

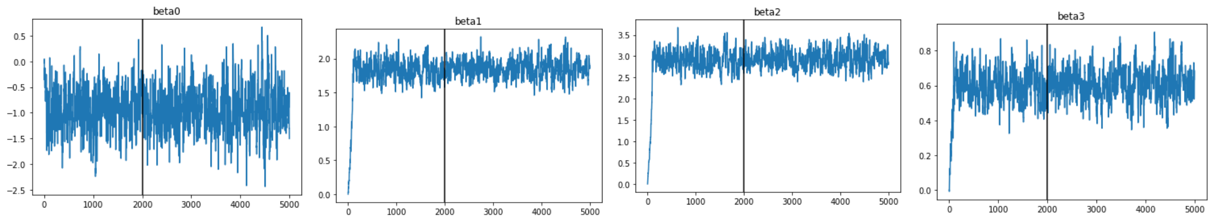


Figure 77

The MSE is 0.010474480649255456

Figure 78

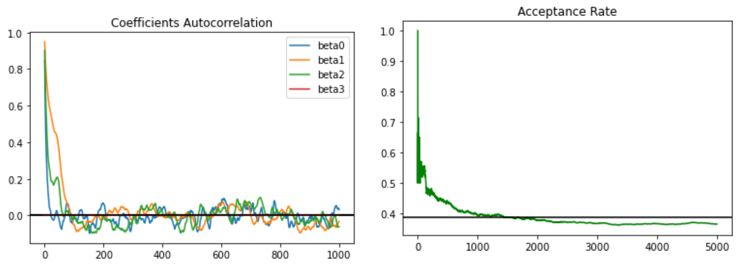


Figure 79 & Figure 80

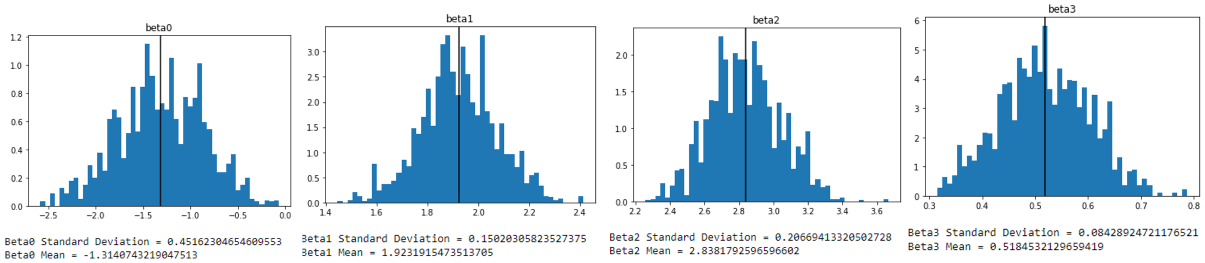


Figure 81

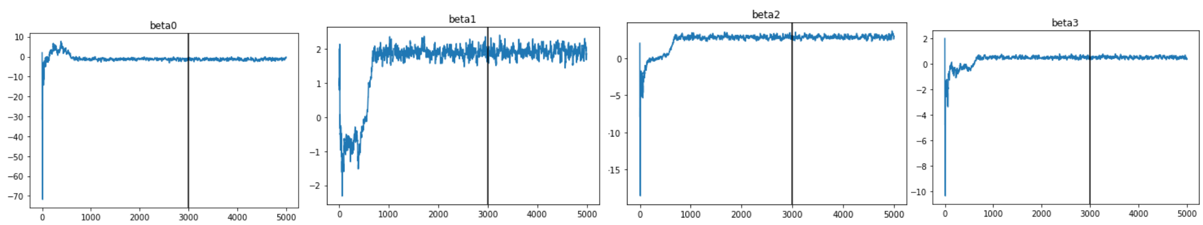
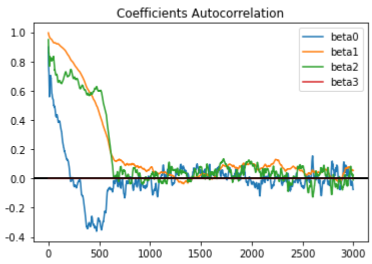


Figure 82

The MSE is 0.03276717278781704

Figure 83



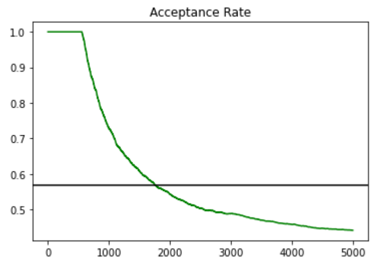
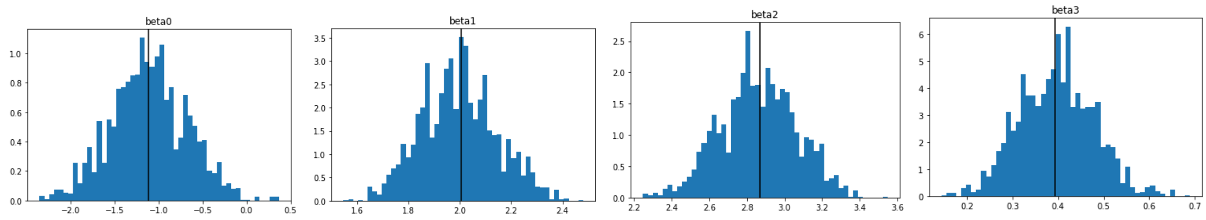


Figure 84 & Figure 85



Beta0 Standard Deviation = 0.42946394319906395
Beta0 Mean = -1.1095834034900953

Beta1 Standard Deviation = 0.14696551549938683
Beta1 Mean = 1.9974132145954948

Beta2 Standard Deviation = 0.20357844638162542
Beta2 Mean = 2.861331049912522

Beta3 Standard Deviation = 0.08216124893725749
Beta3 Mean = 0.3959061252840528

Figure 86

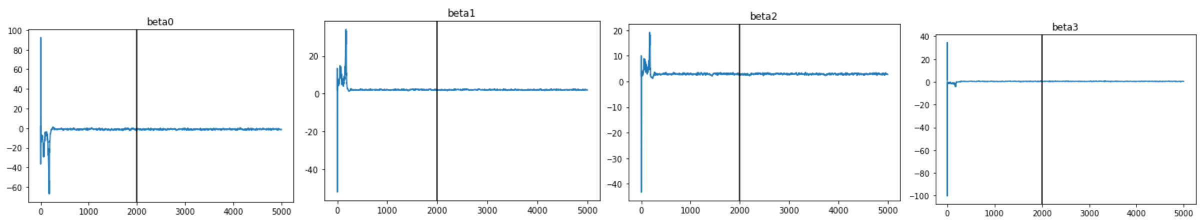
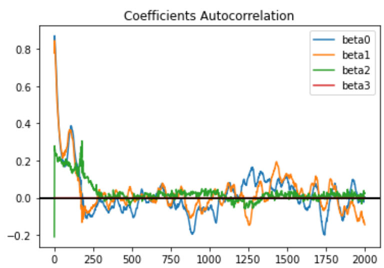


Figure 87

The MSE is 0.010519956562736198

Figure 88



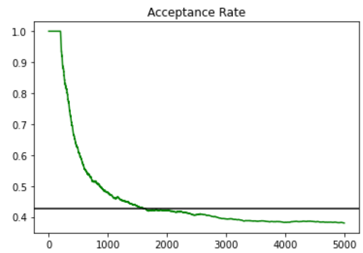
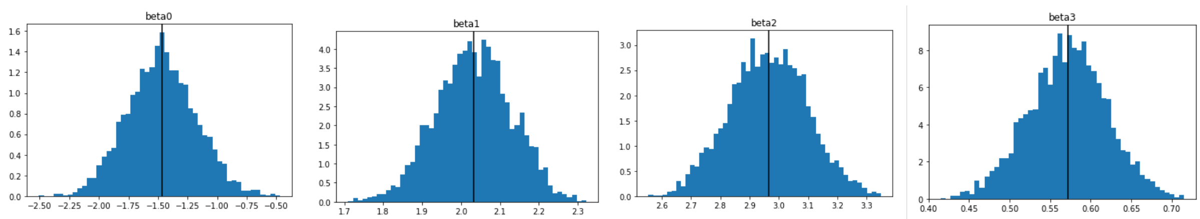


Figure 89 & Figure 90



Beta0 Standard Deviation = 0.2918672377365335
Beta0 Mean = -1.4499270363401364

Beta1 Standard Deviation = 0.09498001125356974
Beta1 Mean = 2.0163559239014037

Beta2 Standard Deviation = 0.1273670303410331
Beta2 Mean = 2.942060323967988

Beta3 Standard Deviation = 0.04769357233992398
Beta3 Mean = 0.566803827851454

Figure 91

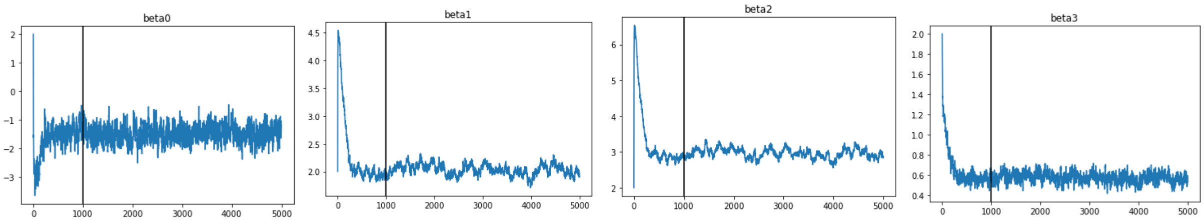


Figure 92

`The MSE is 0.05263040293769704`

Figure 93

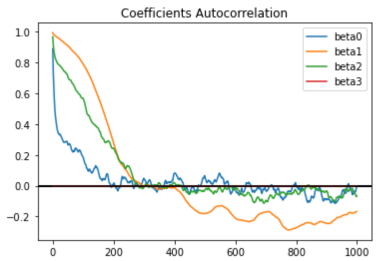
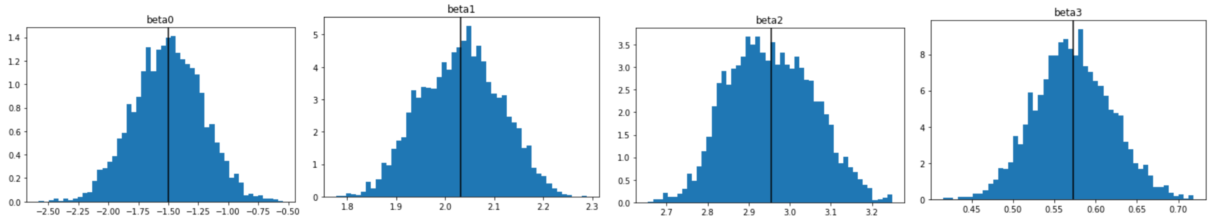


Figure 94




```
Beta0 Standard Deviation = 0.28238793951509844
Beta0 Mean = -1.488214848209536
Beta1 Standard Deviation = 0.08143285483179434
Beta1 Mean = 2.022489289215009
Beta2 Standard Deviation = 0.10443605304973373
Beta2 Mean = 2.9424067615039604
Beta3 Standard Deviation = 0.04647301468151561
Beta3 Mean = 0.5688725643400985
```

Figure 95

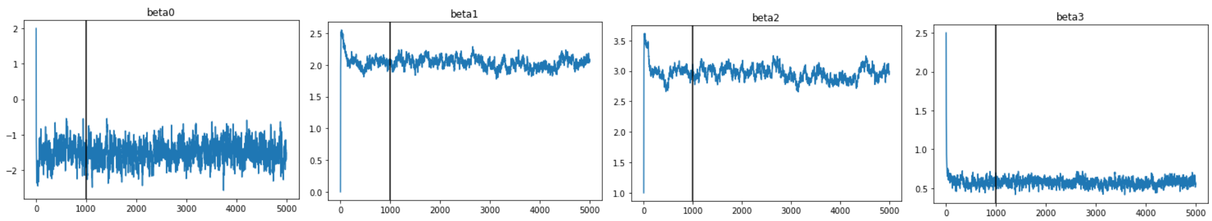


Figure 96

`The MSE is 0.061729979345224846`

Figure 97

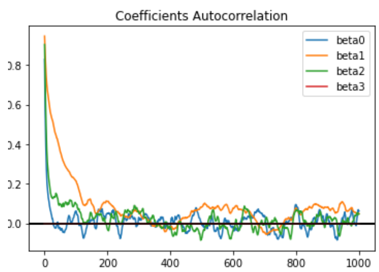
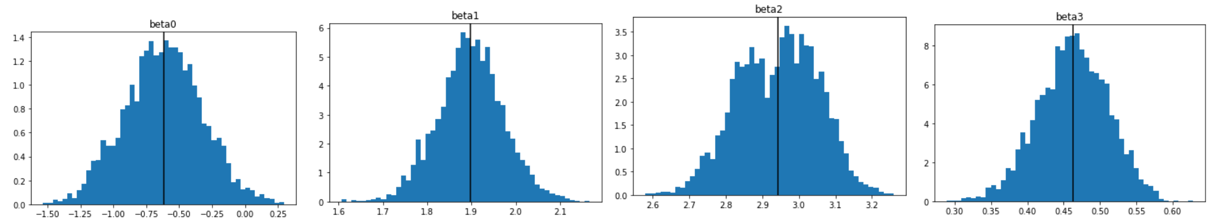


Figure 98




```
Beta0 Standard Deviation = 0.29512260963012177
Beta0 Mean = -0.6191309232099209
```

```
Beta1 Standard Deviation = 0.07462376013813213
Beta1 Mean = 1.8976451615972922
```

```
Beta2 Standard Deviation = 0.10978751219307234
Beta2 Mean = 2.942147618720737
```

```
Beta3 Standard Deviation = 0.048051880346155436
Beta3 Mean = 0.46276749262041844
```

Figure 99

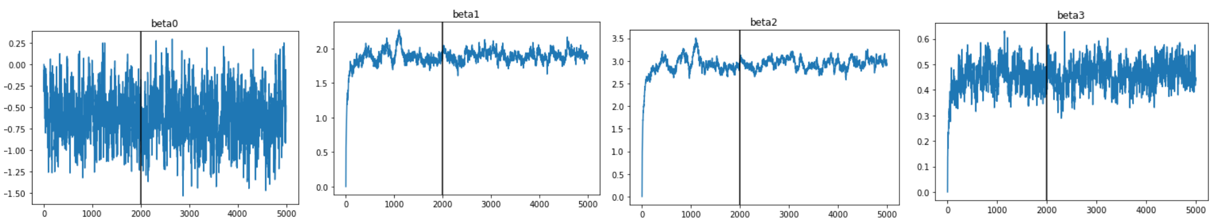


Figure 100

```
The MSE is 0.04006773105620584
```

Figure 101

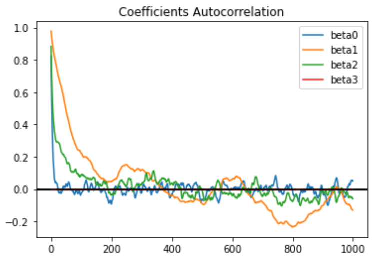
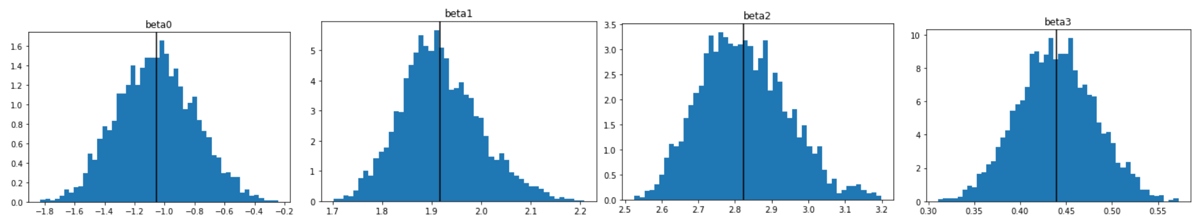


Figure 102




```
Beta0 Standard Deviation = 0.25644263741439977
Beta0 Mean = -1.0551292553614302
```

```
Beta1 Standard Deviation = 0.08011526023371804
Beta1 Mean = 1.9173795750902767
```

```
Beta2 Standard Deviation = 0.11864943509073755
Beta2 Mean = 2.8242134023171213
```

```
Beta3 Standard Deviation = 0.042410755509342396
Beta3 Mean = 0.4393565175916733
```

Figure 103

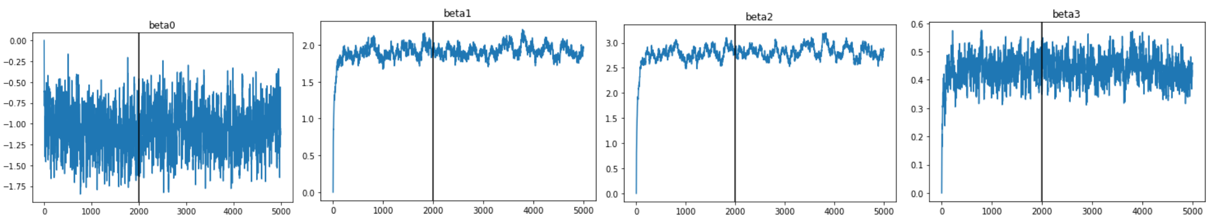


Figure 104

```
The MSE is 0.011110982323125077
```

Figure 105

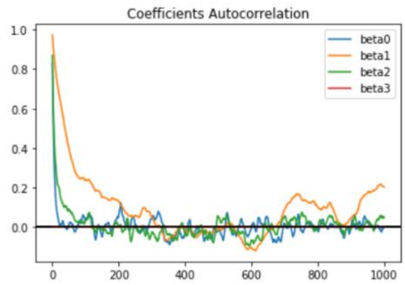


Figure 106

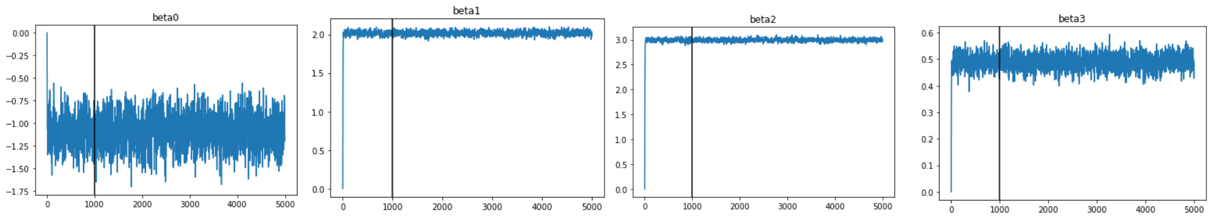
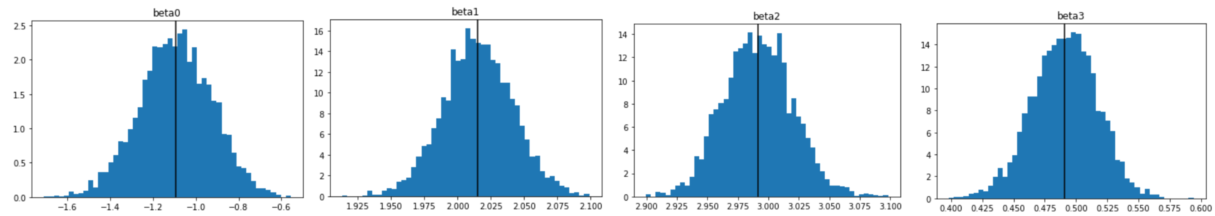


Figure 107




```
Beta0 Standard Deviation = 0.16949540916605477
Beta0 Mean = -1.0903660114485239
Beta1 Standard Deviation = 0.02615740707218465
Beta1 Mean = 2.015482442474003
Beta2 Standard Deviation = 0.028682169978779638
Beta2 Mean = 2.9926917719752866
Beta3 Standard Deviation = 0.0261217392677404
Beta3 Mean = 0.4910095593051972
```

Figure 108

```
The MSE is 0.002134990067705884
```

Figure 109

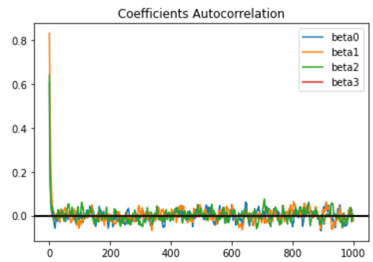


Figure 110

SIMULATION 3

A	Coefficient	Std. err.	z	P>|z|	[95% conf.	interval]
ln_income	-1.14485	.1994539	-5.74	0.000	-1.535773	-.7539278
age	.2783464	.0114136	24.39	0.000	.2559762	.3007166
proportion_gaming_prod~s	-.9335337	.0624765	-14.94	0.000	-1.055986	-.811082
previous_purchase_gami~c	.5112991	.0606246	8.43	0.000	.392477	.6301212
male	-.2623994	.0533403	-4.92	0.000	-.3669445	-.1578543
_cons	4.17425	2.091251	2.00	0.046	.0754736	8.273027

Figure 111

```
The mean of the target is
[ 4.1645595   0.27840926 -0.93369574  0.51139309 -1.14414624 -0.26189784]
```

Figure 112

```
The variance-covariance matrix of the target is
[[ 2.19428383e+00 -6.15020832e-04  2.29795032e-03 -2.79036755e-04
  -2.06062551e-01 -7.34936808e-04]
 [-6.15020832e-04  4.84036027e-05 -8.59878656e-05 -3.43881309e-05
  -7.86469129e-05 -5.94031755e-05]
 [ 2.29795032e-03 -8.59878656e-05  1.72089515e-03  1.36374210e-04
  -5.67827690e-05  1.32812670e-04]
 [-2.79036755e-04 -3.43881309e-05  1.36374210e-04  1.57185628e-03
   7.64963677e-05  3.88976744e-05]
 [-2.06062551e-01 -7.86469129e-05 -5.67827690e-05  7.64963677e-05
   1.97496655e-02  1.59364126e-04]
 [-7.34936808e-04 -5.94031755e-05  1.32812670e-04  3.88976744e-05
   1.59364126e-04  1.42238957e-03]]
```

Figure 113

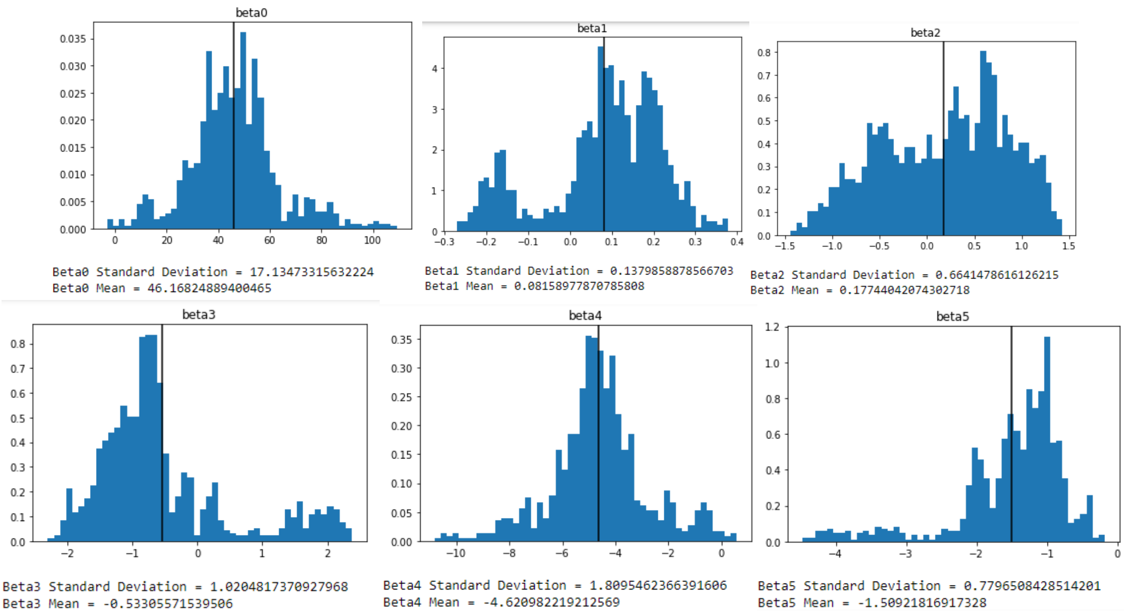


Figure 114

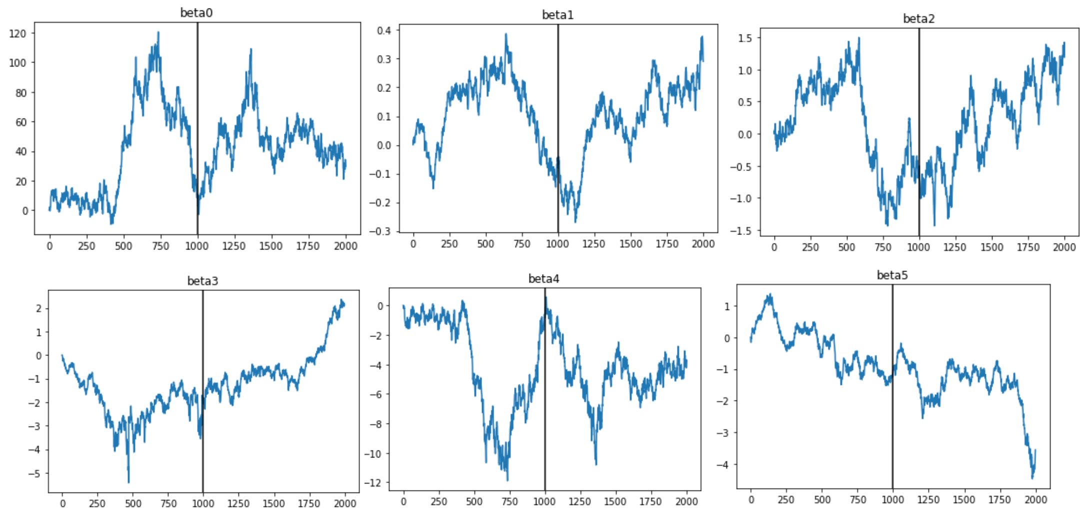
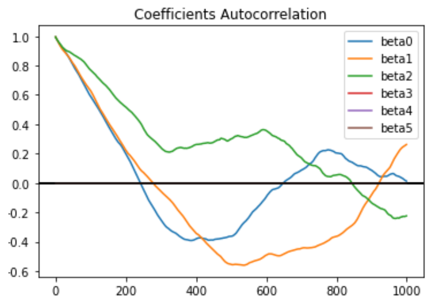


Figure 115

```
The MSE is 296.58294150061596
```

Figure 116



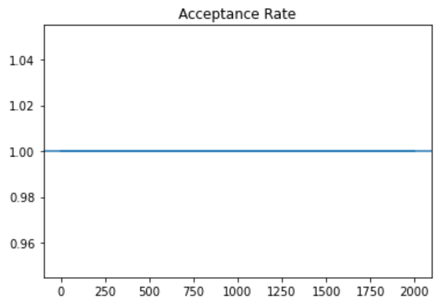
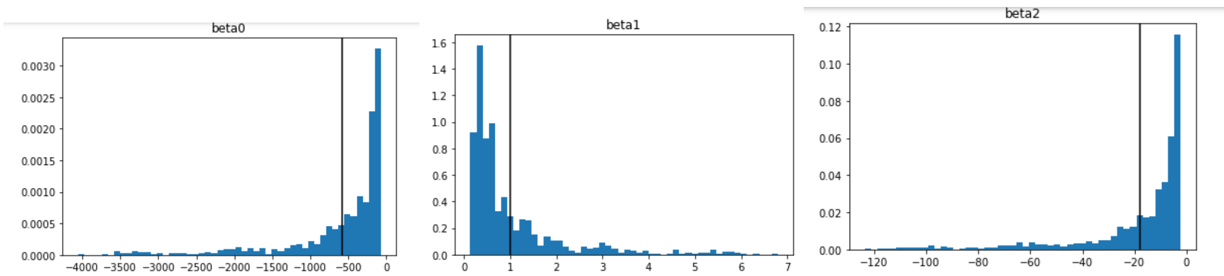


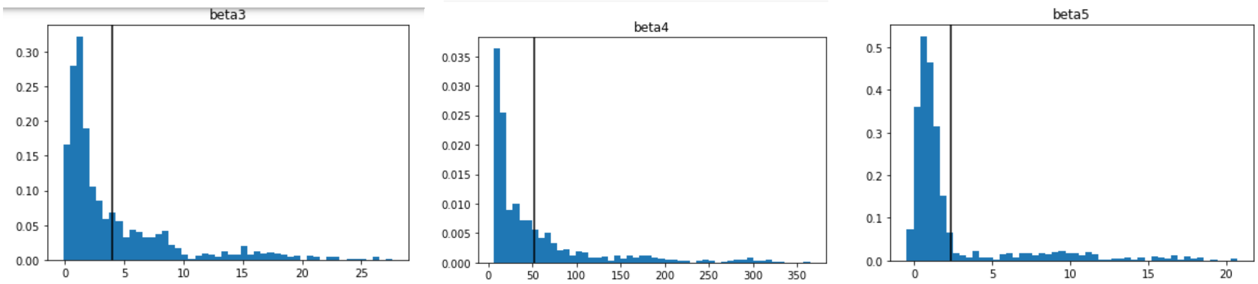
Figure 117 & Figure 118




```
Beta0 Standard Deviation = 704.5882942671996
Beta0 Mean = -579.4168060137417
```

```
Beta1 Standard Deviation = 1.106402075095336
Beta1 Mean = 1.0089584760862413
```

```
Beta2 Standard Deviation = 21.95440703964785
Beta2 Mean = -17.974167362414367
```




```
Beta3 Standard Deviation = 4.807244784340749
Beta3 Mean = 3.9781373377221394
```

```
Beta4 Standard Deviation = 63.50697048209238
Beta4 Mean = 52.00111057268793
```

```
Beta5 Standard Deviation = 3.767649433538754
Beta5 Mean = 2.3244407204754327
```

Figure 119

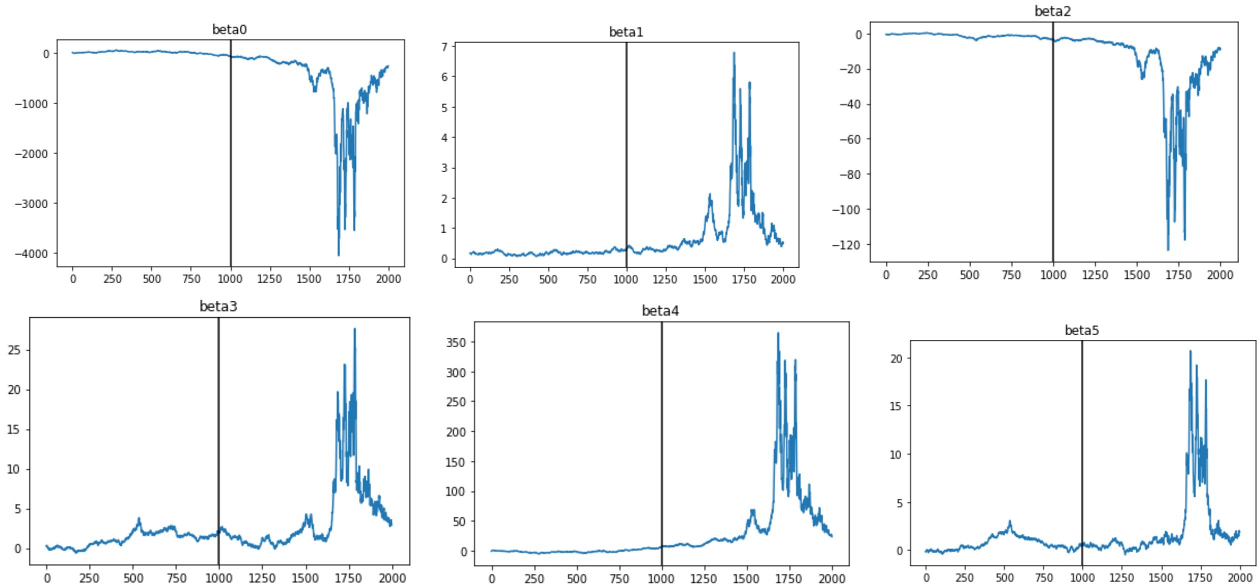
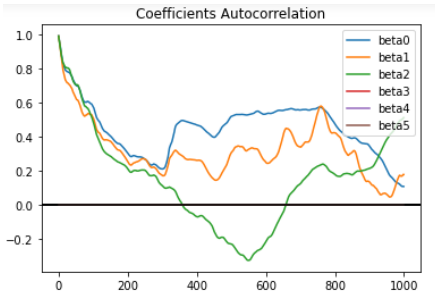


Figure 120

```
The MSE is 57285.44024720429
```

Figure 121



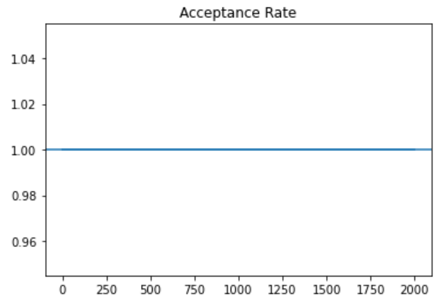


Figure 122 & Figure 123

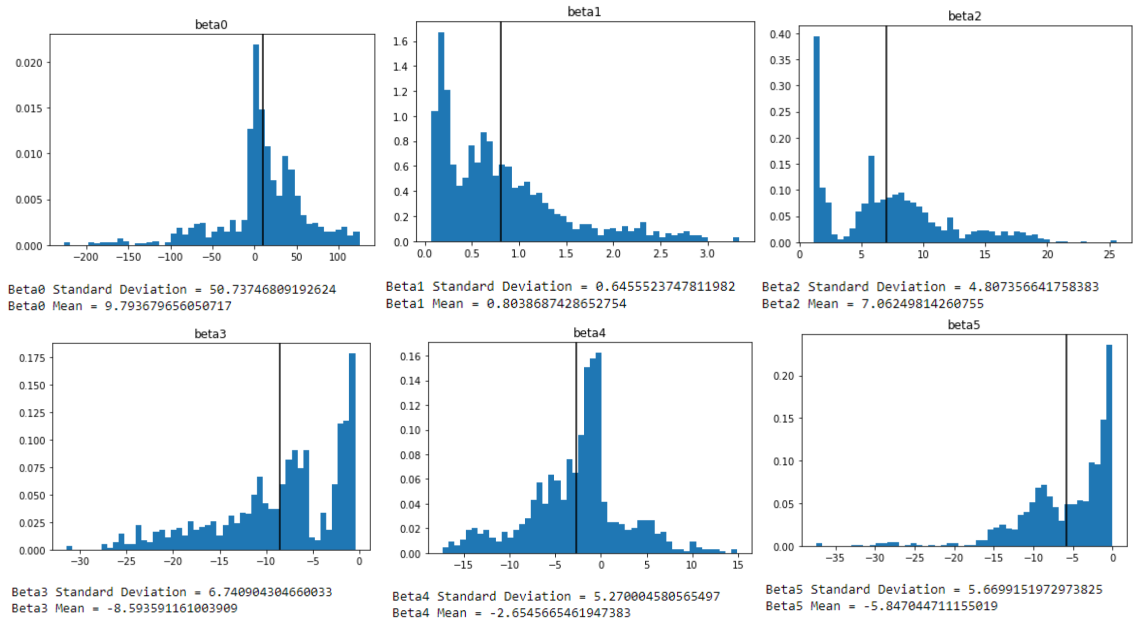


Figure 124

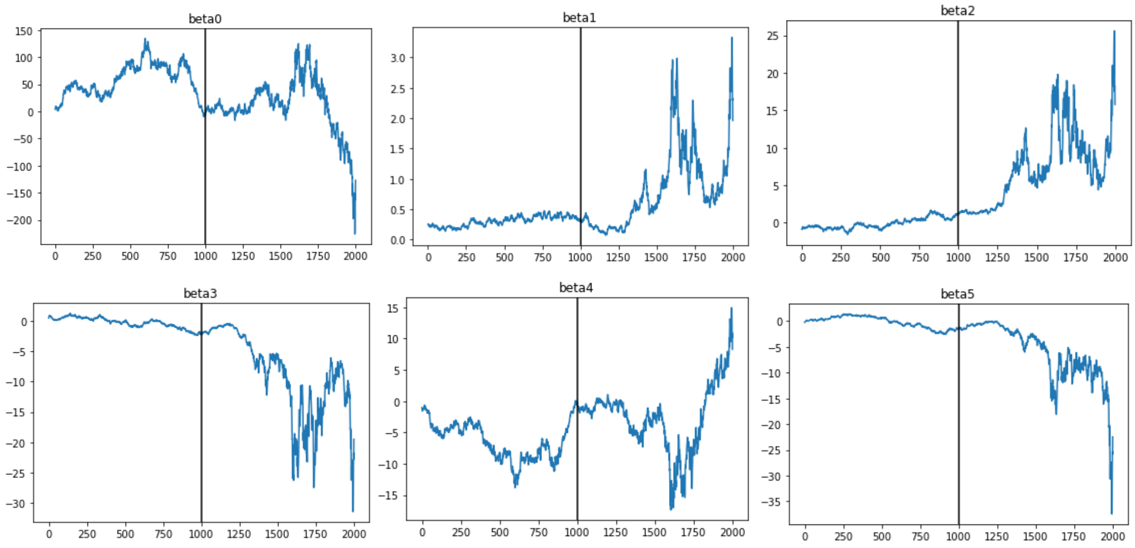


Figure 125

The MSE is 35.359537097721756

Figure 126

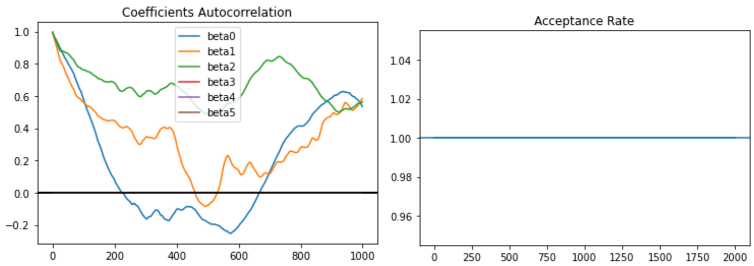
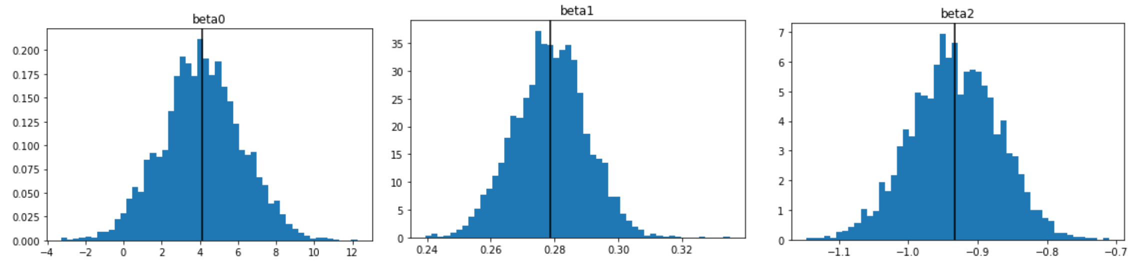


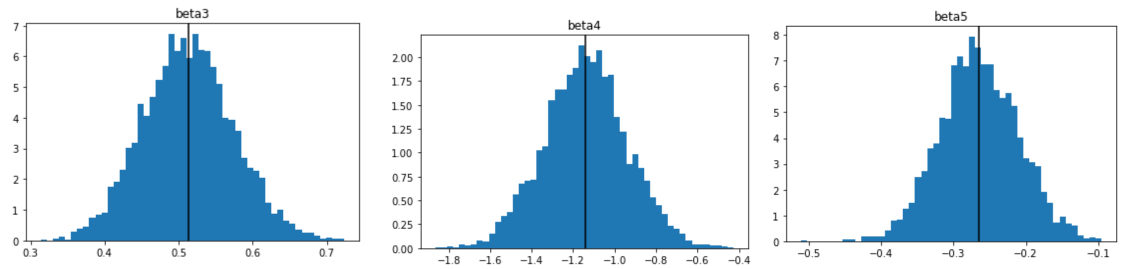
Figure 127 & 128



Beta0 Standard Deviation = 2.1379390758963424
Beta0 Mean = 4.090581469693557

Beta1 Standard Deviation = 0.01133202061377154
Beta1 Mean = 0.27856495505989537

Beta2 Standard Deviation = 0.06288130643918807
Beta2 Mean = -0.9336095049325261



Beta3 Standard Deviation = 0.061257253140462015
Beta3 Mean = 0.5148614792459296

Beta4 Standard Deviation = 0.2044308776070208
Beta4 Mean = -1.1374238864367272

Beta5 Standard Deviation = 0.053607537176056826
Beta5 Mean = -0.2642579610989615

Figure 129

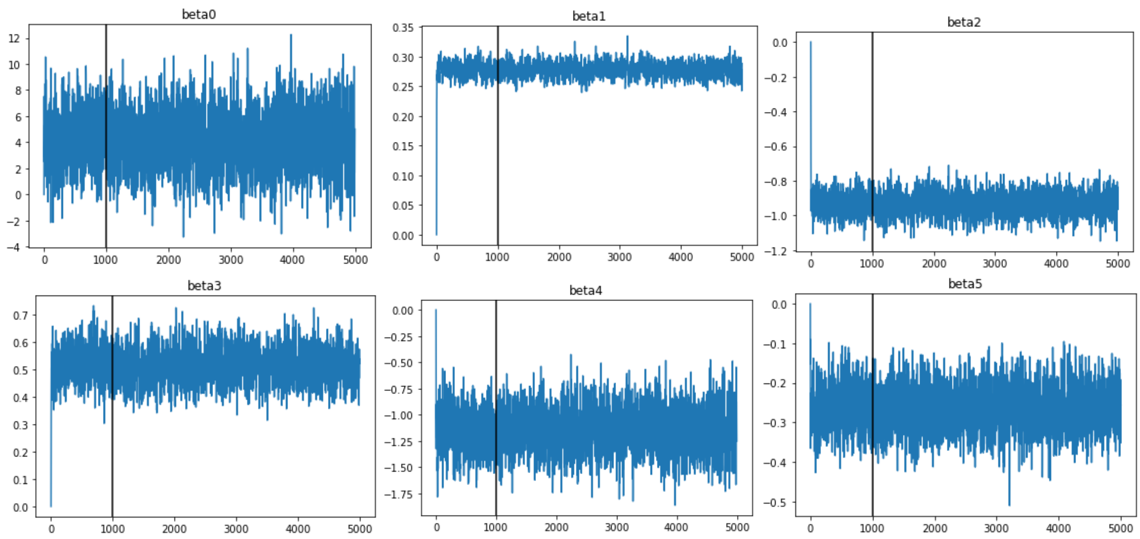


Figure 130

The MSE is 0.001178628072374522

Figure 131

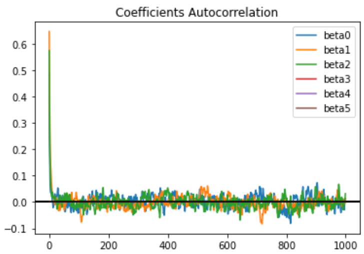
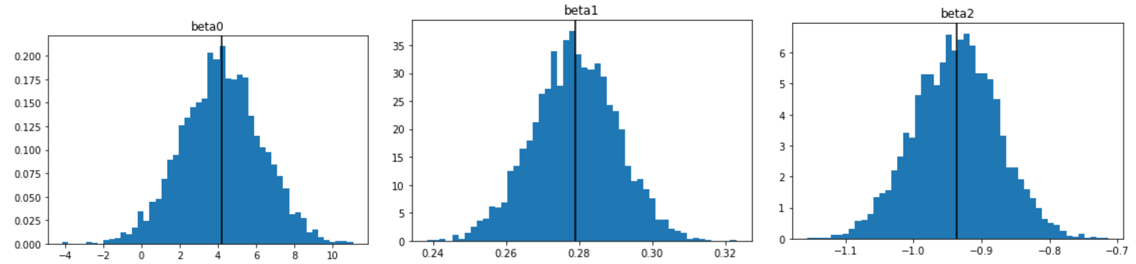


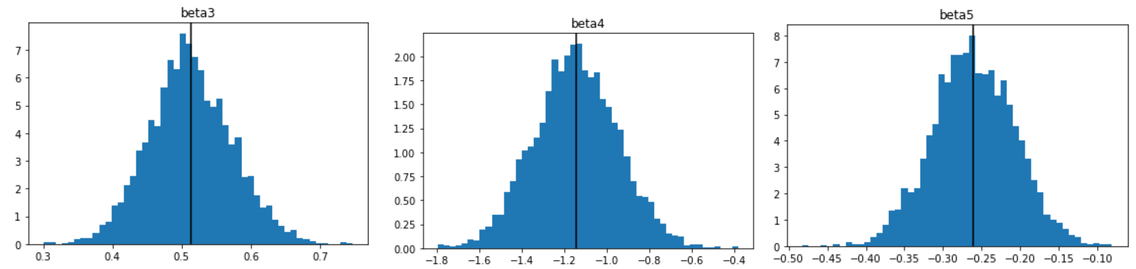
Figure 132



Beta0 Standard Deviation = 2.054468604463455
Beta0 Mean = 4.191483699328729

Beta1 Standard Deviation = 0.011243251601415132
Beta1 Mean = 0.27838346201350633

Beta2 Standard Deviation = 0.06190924052663418
Beta2 Mean = -0.9356885988969058



Beta3 Standard Deviation = 0.05940533243699205
Beta3 Mean = 0.5120055995029921

Beta4 Standard Deviation = 0.19560417996504526
Beta4 Mean = -1.146563254643754

Beta5 Standard Deviation = 0.052357875182726596
Beta5 Mean = -0.26174038184104625

Figure 133

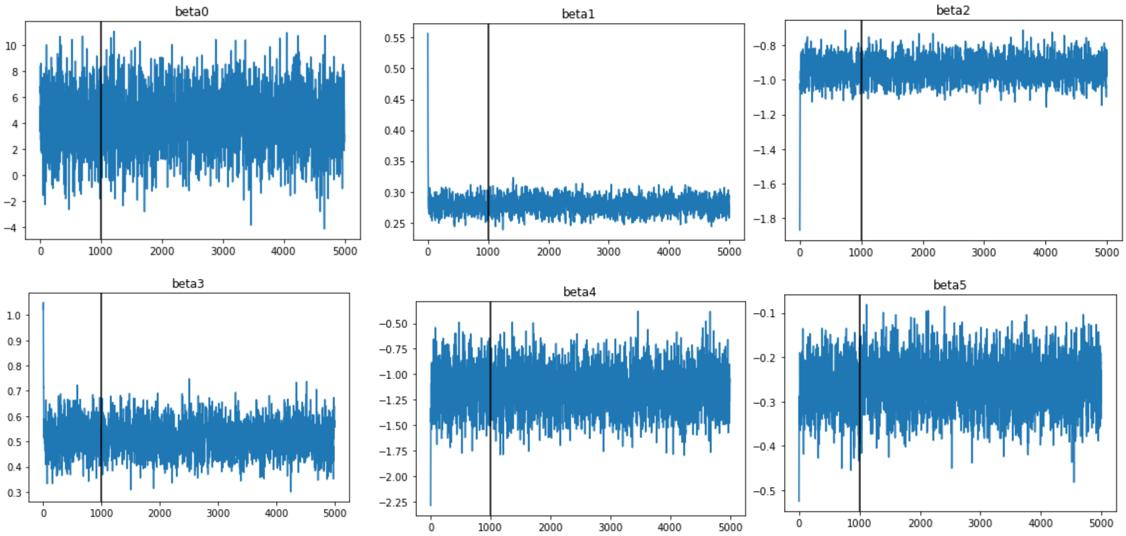


Figure 134

`The MSE is 5.091900722627938e-05`

Figure 135

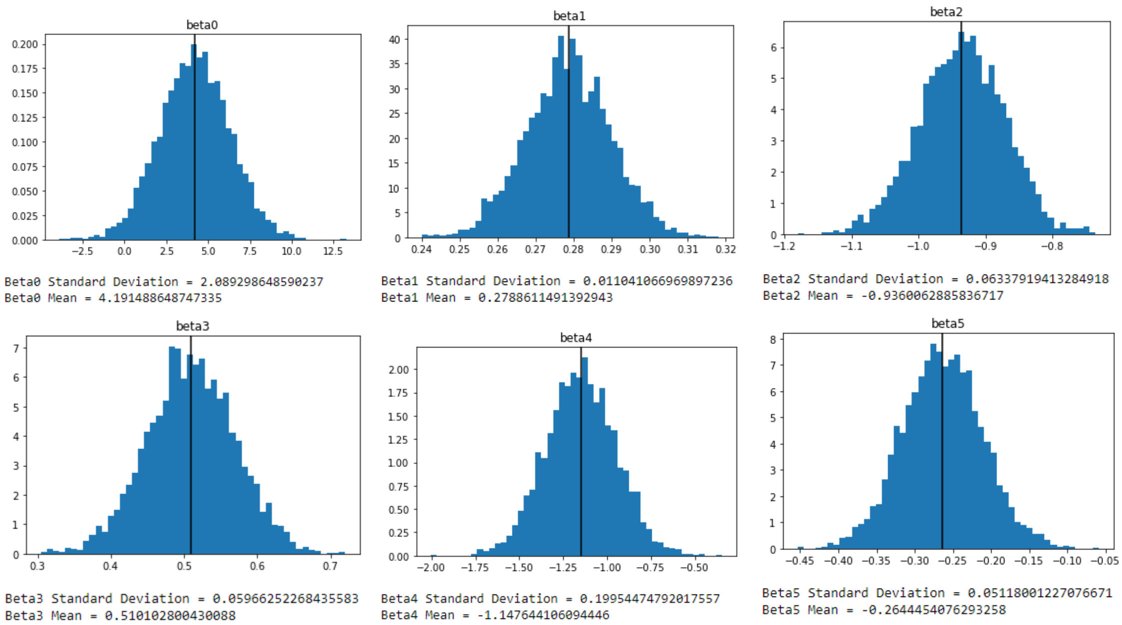


Figure 136

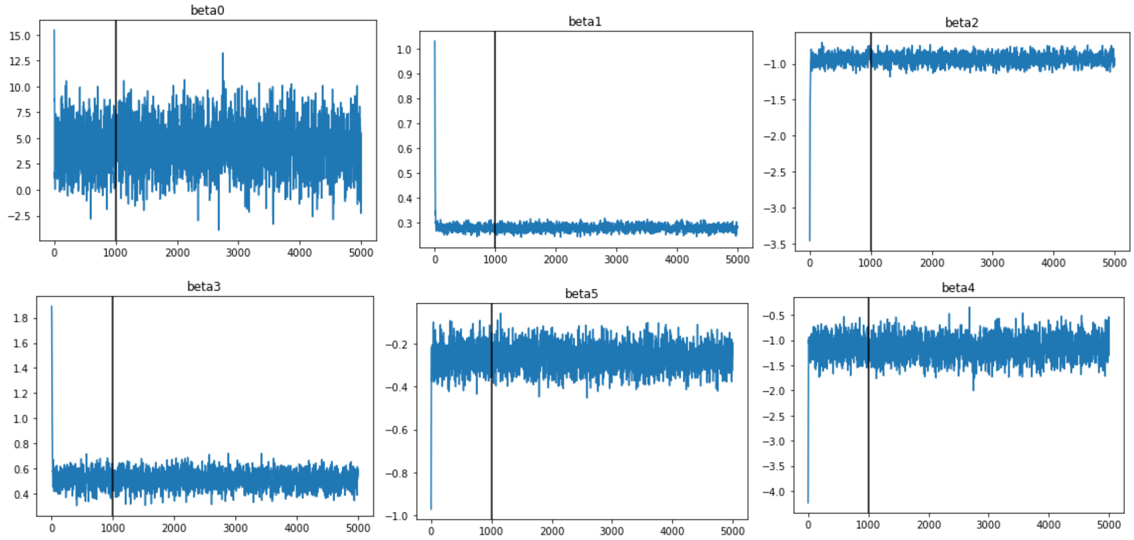


Figure 137

```
The MSE is 5.282899672695802e-05
```

Figure 138

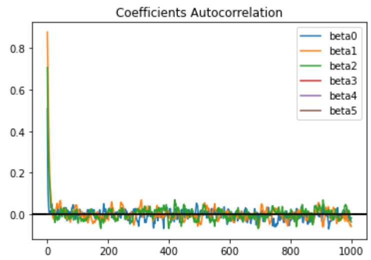
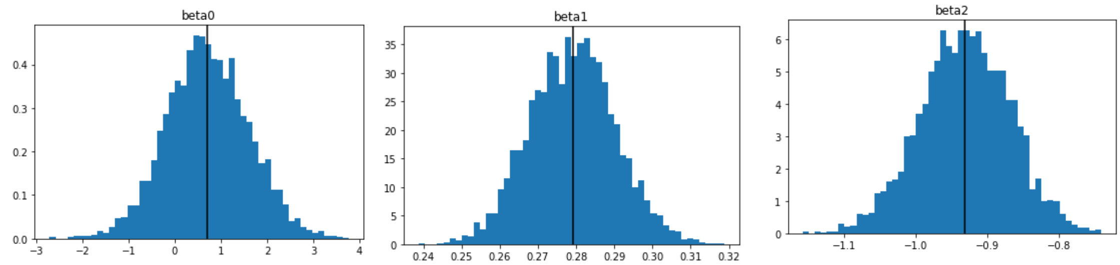


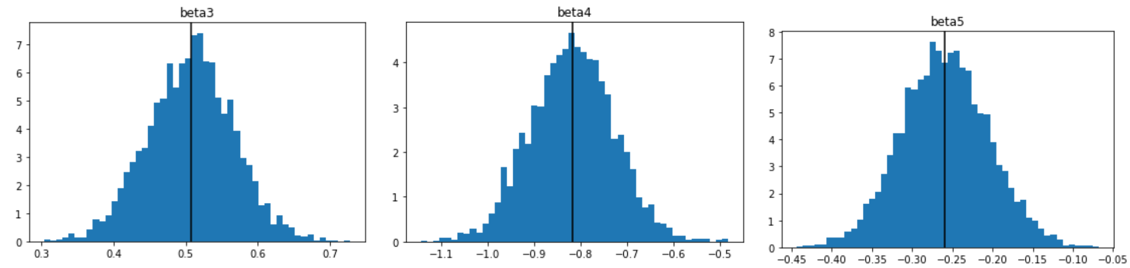
Figure 139




```
Beta0 Standard Deviation = 0.891110743057428
Beta0 Mean = 0.7239960882777328
```

```
Beta1 Standard Deviation = 0.011228194238947603
Beta1 Mean = 0.27926962405643785
```

```
Beta2 Standard Deviation = 0.06309792556292175
Beta2 Mean = -0.9306358015189043
```




```
Beta3 Standard Deviation = 0.05950896703999713
Beta3 Mean = 0.5066344608983268
```

```
Beta4 Standard Deviation = 0.08938323309850373
Beta4 Mean = -0.8211203546343442
```

```
Beta5 Standard Deviation = 0.0541477052735656
Beta5 Mean = -0.25922338637568904
```

Figure 140

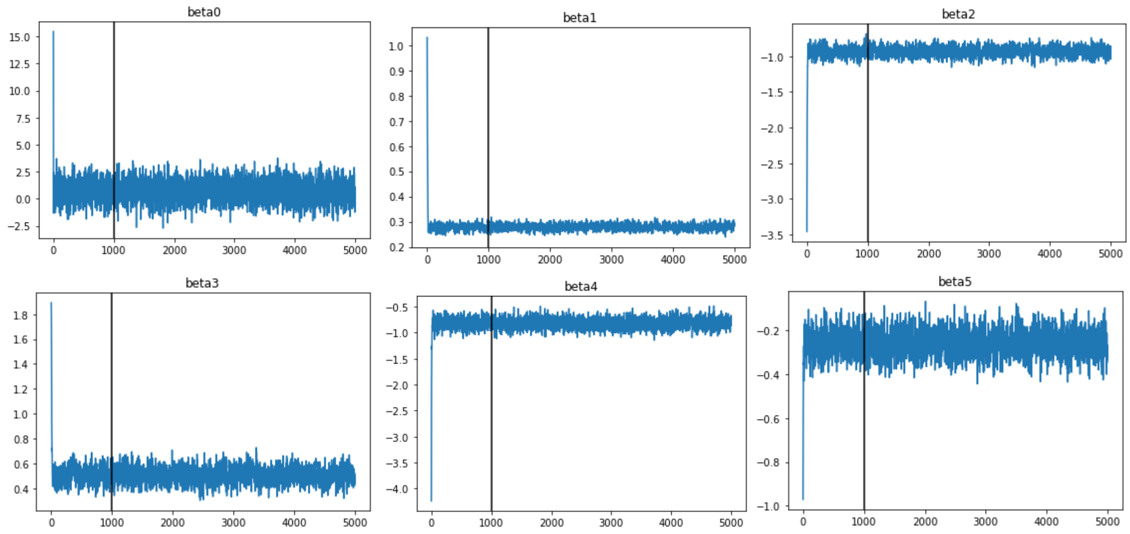


Figure 141

The MSE is 2.001515672453689

Figure 142

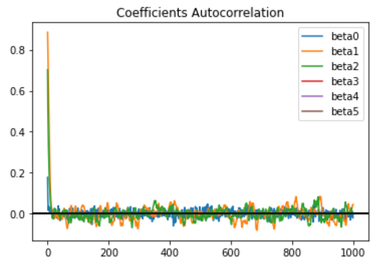


Figure 143

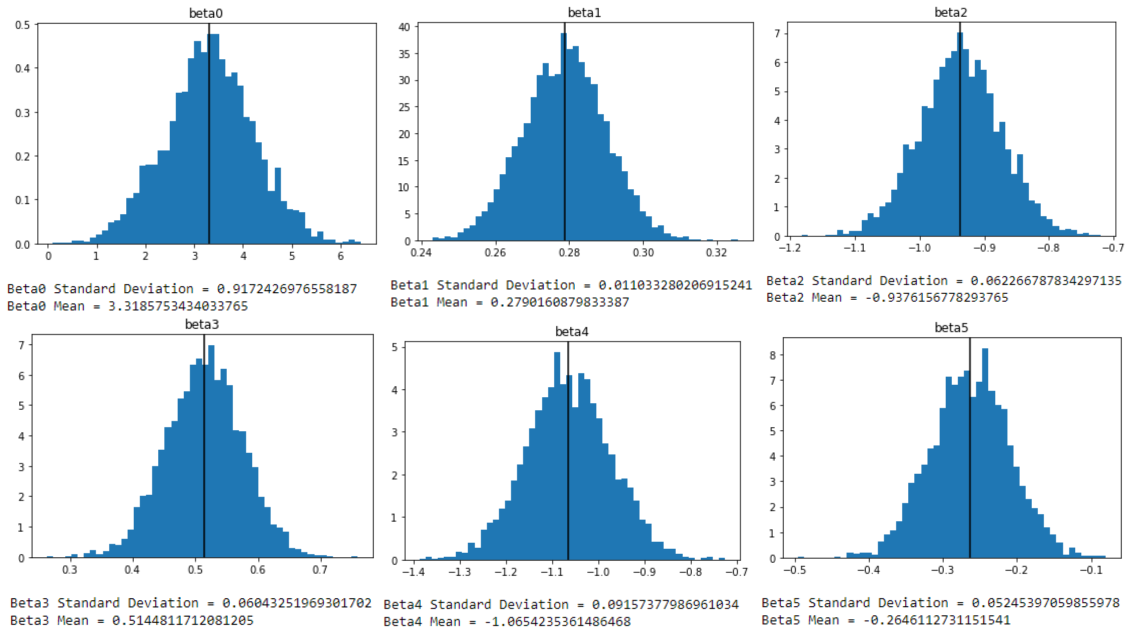


Figure 144

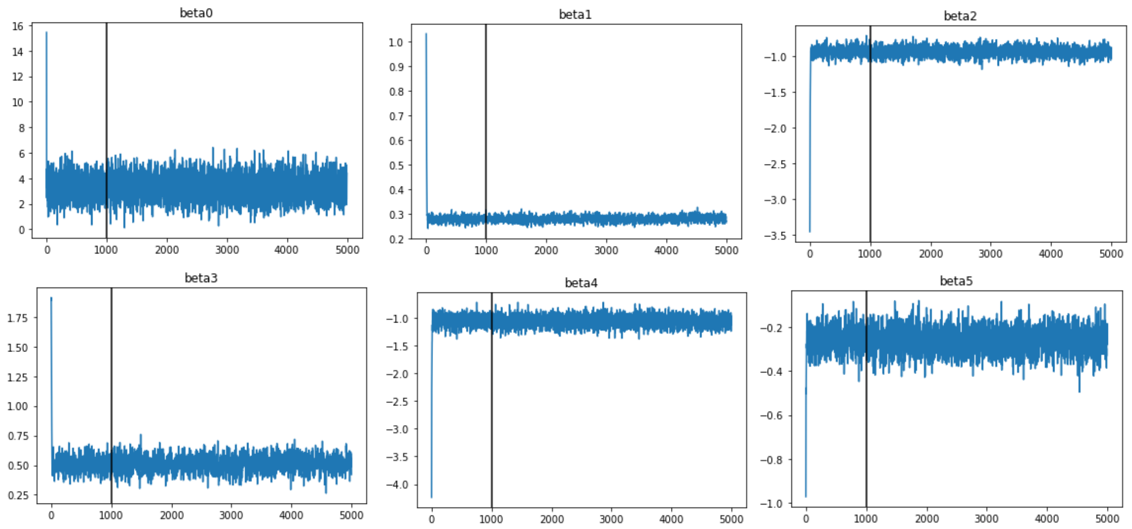


Figure 145

```
The MSE is 0.12308663501442085
```

Figure 146

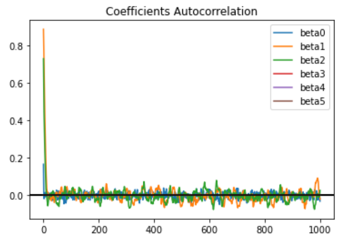


Figure 147