Proof a Day — 2026

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1 Weeks 1–13

1.1 Week 1 (Jan 1–Jan 7)

2026-01-01 Divisibility

Theorem 1.1. *If* $a|c$ *and* $b|c$ *and* $\gcd(a,b) = d$*, then* $ab|cd$*.* $a, b, c \in \mathbb{Z}$

Proof. As $d = \gcd(a,b)$, we have that $d = au + bv$ for some $u, v \in \mathbb{Z}$. We now have that $cd = c(au + bv) = cau + cbv$. Since $a|c$ and $b|c$, $c = ax = by$ for some $x, y \in \mathbb{Z}$. Substituing c results in $cd = cau + cbv = (by)au + (ax)bv = (ab)(yu + xv)$. As $u, v, x, y \in \mathbb{Z}$, $yu + xv \in \mathbb{Z}$. By definition of divisibility, $ab|cd$. □

2026-01-02 Divisibility

Theorem 1.2. *If* $a|bc$ *and* $\gcd(a,b) = 1$*, then* $a|c$*.*

Proof. Assume that $a|bc$ and $\gcd(a,b) = 1$. As a and b are relatively prime, $au + bv = 1$ for some $u, v \in \mathbb{Z}$. If we multiply $au + bv = 1$ by c, then we have $cau + cbv = c$. As $a|bc$, $bc = ax$ for some $x \in \mathbb{Z}$. Now, we have that

$$c = cau + cbv = cau + bcv = cau + (ax)v = a(cu + xv)$$

Since $c, u, x, v \in \mathbb{Z}$ and by the definition of divisibility, $a|c$. □

2026-01-03 gcd

Theorem 1.3. *If* $\gcd(a,c) = 1$ *and* $\gcd(b,c) = 1$*, then* $\gcd(ab,c) = 1$*.*

Proof. By way of contradiction, assume that $\gcd(ab,c) \neq 1$. Thus, there exists a prime p such that

$$p|ab \text{ and } p|c.$$

Since p is prime, $p|a$ or $p|b$. If $p|a$, then $\gcd a, c \geq p > 1$, which contradiscts the assumption that $\gcd(a,c) = 1$. If $p|b$, then $\gcd b, c \geq p > 1$, which contradicts the assumption that $\gcd(b,c) = 1$. Therefore ab and c share no such prime factors p, so $\gcd(ab,c) = 1$. □