

Proof a Day — 2026

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1 Weeks 1–13

1.1 Week 1 (Jan 1–Jan 7)

2026-01-01

Divisibility

Theorem 1.1. *If $a|c$ and $b|c$ and $\gcd(a, b) = d$, then $ab|cd$. $a, b, c \in \mathbb{Z}$*

Proof. As $d = \gcd(a, b)$, we have that $d = au + bv$ for some $u, v \in \mathbb{Z}$. We now have that $cd = c(au + bv) = cau + cbv$. Since $a|c$ and $b|c$, $c = ax = by$ for some $x, y \in \mathbb{Z}$. Substituting c results in $cd = cau + cbv = (by)au + (ax)bv = (ab)(yu + xv)$. As $u, v, x, y \in \mathbb{Z}$, $yu + xv \in \mathbb{Z}$. By definition of divisibility, $ab|cd$. \square

2026-01-02

Divisibility

Theorem 1.2. *If $a|bc$ and $\gcd(a, b) = 1$, then $a|c$.*

Proof. Assume that $a|bc$ and $\gcd(a, b) = 1$. As a and b are relatively prime, $au + bv = 1$ for some $u, v \in \mathbb{Z}$. If we multiply $au + bv = 1$ by c , then we have $cau + cbv = c$. As $a|bc$, $bc = ax$ for some $x \in \mathbb{Z}$. Now, we have that

$$c = cau + cbv = cau + bcv = cau + (ax)v = a(cu + xv)$$

Since $c, u, x, v \in \mathbb{Z}$ and by the definition of divisibility, $a|c$. \square

2026-01-03

Divisibility

Theorem 1.3. *If $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.*

Proof. By way of contradiction, assume that $\gcd(ab, c) \neq 1$. Thus, there exists a prime p such that

$$p|ab \text{ and } p|c.$$

Since p is prime, $p|a$ or $p|b$. If $p|a$, then $\gcd(a, c) \geq p > 1$, which contradicts the assumption that $\gcd(a, c) = 1$. If $p|b$, then $\gcd(b, c) \geq p > 1$, which contradicts the assumption that $\gcd(b, c) = 1$. Therefore ab and c share no such prime factors p , so $\gcd(ab, c) = 1$. \square

2026-01-04

Rings

Theorem 1.4. *If $a + b = a + c$ in a ring R , then $b = c$.*

Proof. Assume that for $a, b, c \in R$, $a + b = a + c$. If we add $-a$ and by using the associativity property of rings and negatives, we can show that

$$\begin{aligned} -a + (a + b) &= -a + (a + c) \\ (-a + a) + b &= (-a + a) + c \\ 0_R + b &= 0_R + c \\ b &= c \end{aligned}$$

□

2026-01-05

Rings

Theorem 1.5. *If $a \neq 0_R$ and $ab = ac$ in an integral domain R , then $b = c$.*

Proof. If $ab = ac$, then $ab - ac = 0_R$. Thus, $a(b - c) = 0_R$. As R is an integral domain and $a \neq 0_R$, it must be that $b - c = 0_R$. If $b - c \neq 0_R$, then a would be a zero divisor contradicting the zero-product property of integral domains. Thus $b = c$. □

2026-01-06

Rings

Theorem 1.6. *Every field is an integral domain.*

Proof. To show that a field F is an integral domain, it suffices to show that F satisfies the zero product property. Take any $a, b \in F$ such that $ab = 0_F$. We must show that $a = 0_F$ or $b = 0_F$. Assume that $ab = 0_F$. Without loss of generality, if $b = 0_F$, then we are done, so assume that $b \neq 0_F$. As F is a field, b is a unit in F . By definition of a unit, we have that

$$a = a1_F = a(bb^{-1}) = (ab)b^{-1} = 0_Fb^{-1} = 0_F$$

So, for all $a, b \in F$, $a = 0_F$ or $b = 0_F$. □

2026-01-07

Rings

Theorem 1.7. *Every finite integral domain is a field.*

Proof. To show that an integral domain R is a field, it suffices to show that for all $a \in R$ where $a \neq 0_R$, the equation $ax = 1_R$. Let a_1, a_2, \dots, a_n be the n distinct elements in R and $a_t \neq 0$. Consider the products $a_t a_1, a_t a_2, \dots, a_t a_n$. If $a_i \neq a_j$ for $i \neq j$, we must have that $a_t a_i \neq a_t a_j$. Therefore $a_t a_1, a_t a_2, \dots, a_t a_n$ has n distinct elements in R . So, all n elements of R are expressed in the products in some order. So, for some j , $a_t a_j = 1_R$. Therefore the equation $ax = 1_R$ has a solution and R is a field. □

1.2 Week 2 (Jan 8–Jan 14)

2026-01-08

Rings

Theorem 1.8. *Let $f : R \rightarrow S$ be a homomorphism of rings, Then*

- (1) $f(0_R) = 0_S$
- (2) $f(-a) = -f(a) \quad \forall a \in R$
- (3) $f(a - b) = f(a) - f(b) \quad \forall a, b \in R$

Proof. (1) As f is a homomorphism of rings, we have the following:

$$\begin{aligned} f(0_R) + f(0_R) &= f(0_R + 0_R) \\ f(0_R) + f(0_R) &= f(0_R) \quad [0_R + 0_R = 0_R \text{ in } R] \\ f(0_R) + f(0_R) &= f(0_R) + 0_S \quad [f(0_R) + 0_S = f(0_R) \text{ in } S] \\ f(0_R) &= 0_S \end{aligned}$$

(2) Note that: $f(a) + f(-a) = f(a + (-a)) = f(0_R) = 0_S$ by (1) and f is a homomorphism. So, $f(-a)$ is a solution to the equation $f(a) + x = 0_S$. But as S is a ring, there exists only one unique solution to this equation which is $-f(a)$. Therefore by [theorem to be proved at a later date], $f(-a) = -f(a)$.

(3) As f is a homomorphism and by (2), we have the following:

$$\begin{aligned} f(a - b) &= f(a + (-b)) \\ &= f(a) + f(-b) \\ &= f(a) - f(b) \end{aligned}$$

□

2026-01-09

Rings

Theorem 1.9. *Let $f : R \rightarrow S$ be a homomorphism of rings. If R is a ring with identity and f is surjective, then*

- (1) S is a ring with identity $f(1_R)$.
- (2) Whenever u is a unit in R , then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Proof. (1) Let s be some element in S . As f is surjective, there exists some $r \in R$ such that $f(r) = s$. Then we have that

$$s * f(1_R) = f(r)f(1_R) = f(r * 1_R) = f(r) = s$$

Similarly $f(1_R) * s = s$. Therefore $1_S = f(1_R)$.

(2) As u is a unit, there exists some $v \in R$ such that $uv = 1_R = vu$. By (1), we have that $f(u)f(v) = f(uv) = f(1_R) = 1_S$. Similarly, $vu = 1_R$ which implies that $f(v)f(u) = 1_S$. Therefore $f(u)$ is a unit in S with inverse $f(v)$. Said differently $f(u)^{-1} = f(v)$. As v is the inverse of u ($v = u^{-1}$), we see that $f(u)^{-1} = f(v) = f(u^{-1})$. \square

2026-01-10

Groups

Theorem 1.10. *Every ring is an abelian group under addition.*

Proof. The first five axioms for a ring are identical to the five axioms for an abelian group, with addition as the operation, the identity element being 0_R , and the inverse for any element $a \in R$ being $-a$. \square

2026-01-11

Graphs

Theorem 1.11. *If $W = e_1, \dots, e_n$ is a closed walk in D , then W contains a coherent cycle.*

Proof. Consider the subgraph H of edges e_1, \dots, e_n . Thus, no vertex is a source or sink in H . Now let P be a longest path in H . Let v denote the head of P . Because no vertex is a source or sink in H , there is $e = (v, w)$ for some w . Because P has a maximum length $w \in P$, $P \cup (v, w)$ contains a coherent cycle. \square

2026-01-12

Graphs

Theorem 1.12. *If $u, v \in V(D)$ with $u \neq v$ and $W = e_1, \dots, e_n$ is a uv -walk. Then W contains the edges of a uv -path.*

Proof. Base Case: Say $n = 1$. Because W is a uv -walk, its one edge is a uv -path.

Inductive Hypothesis: Assume that for $W = e_1, \dots, e_n$, W contains the edges of a uv -path. We need to show that there is a uv -path if $W = e_1, \dots, e_{n+1}$. Let x_n be the head of e_n . If v is contained within the ux_n -walk, then by the inductive hypothesis we are done. So, assume v is not part of the ux_n -walk. By the inductive hypothesis, the ux_n -walk contains the edges of a ux_n -path. Adding e_{n+1} to this ux_n -path makes a uv -path \square

Theorem 1.13. *A directed graph D is strongly connected if and only if it has no coherent cut.*

Proof. (\Rightarrow) Let D be a directed graph that is strongly connected. By way of contradiction, assume there exists a coherent cut in D defined by $X \subseteq V(D)$, and with all edges in $\delta(X)$ having tails in X without loss of generality. Because all edges in $\delta(D)$ point in the same direction, there would be no way to get from any vertex $v \in V(D) - X$ to any vertex $x \in X$, so D is not strongly connected, which is a contradiction. Thus, D does not have a coherent cut.

(\Leftarrow) Let D be a directed graph that is not strongly connected, so there exists two vertices $u, v \in V(D)$, such that there is no uv -path. Let $V_v \subseteq V(D)$ be the set of all vertices, y , for which there is a yv -path, and $V_u \subseteq V(D)$ be the set of all vertices, x for which there is a ux -path. Let $M = V(D) - (V_v \cup V_u)$. Because there is no uv -path, $V_u \cap V_v = \emptyset$ by Theorem 1.12. Again by Theorem 1.12, all edges in $\delta(V_u)$ must go into V_u by definition of V_u, V_v , and M . Thus, $\delta(V_u)$ is a coherent cut. \square

Theorem 1.14. *Every $e \in E(D)$ is a coherent cycle or a coherent cut.*

Proof. Let D be a directed graph. Let $e \in E(D)$. If e is in a coherent cycle, then we are done, so assume that e is not in a coherent cycle, say $e = (u, v)$. Let $V_v \subseteq V(D)$ be the set of all vertices, y , for which there is a vy -path, $V_u \subseteq V(D)$ be the set of all vertices, x for which there is a xu -path, and let $M = V(D) - (V_v \cup V_u)$. If there were a vu -walk, then by Theorem 1.12, there would be a coherent cycle containing e . So, it must be the case that there is no vu -walk, thus $V_u \cap V_v = \emptyset$. Again by Theorem 1.12, all edges in $\delta(V_u)$ must be going into $V_v \cup M$ by definition of V_u, V_v , and M , creating a coherent cut. \square

Theorem 1.15. *If D is acyclic, then \leq is a partial order set on $V(D)$.*

Proof. Let D be a directed, acyclic graph. To show that \leq is a partial order set on $V(D)$, we must prove the following three properties.

Reflexive: For any $u \in V(D)$, there is a length 0 coherent uu -path, so $u \leq u$.

Anti-Symmetry: Consider $u, v \in V(D)$. By way of contradiction, suppose $u \leq v$ and $v \leq u$, and $u \neq v$. So, there exists a uv -path, P , and a vu -path, Q , by the definition of \leq . Now, $P \cup Q$ is a closed walk in D . So, there is a coherent cycle in $P \cup Q$ by Proposition 2, a contradiction.

Transitivity: Consider $u, v, w \in V(D)$. Suppose $u \leq v$ and $v \leq w$. So there is a vu -path and a wv -path. Because you can get from w to v and from v to u there must be a wu -walk. So, there is a uw -path by Proposition 3. Thus, $u \leq w$. Thus, \leq is a partial order set set on $V(D)$. \square