Nonlinear Monte Carlo: from American Options to fully nonlinear PDEs

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Outline

The standard model in frictionless markets

• $(\Omega, \mathcal{F}, \mathbb{P})$, W Brownian motion in $\mathbb{R}^d$, $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\} = \mathbb{F}^W$
• The financial market consists of a riskless asset $S^0 \equiv 1$, and a risky asset with price process

$$dS_t \;=\; \mathrm{diag}[S_t]\,(\mu_t dt + \sigma_t dW_t)$$

μ, σ adapted, σ invertible $+\ldots$
• Portfolio Z^i_t : number of shares of asset i in portfolio at time t :

$$\{Z_t, t \geq 0\} \qquad \mathbb{F} - \text{adapted with values in } \mathbb{R}^d$$

• Self-financing condition $\Longrightarrow$ dynamics of portfolio value :

$$dY_t \;=\; Z_t \cdot dS_t$$

Super-hedging problem of $\mathcal{F}_T-$measurable $G \geq 0$

$$V_0 \;:=\; \inf\{Y_0 : \; Y_T \geq G \text{ a.s. for some } Z \in \mathcal{A}\}$$

Solution : the Black-Scholes model

- We may assume $\mu \equiv 0$: equivalent change of measure
- Then for $Y_0 > V_0$, $\mathbb{E}[Y_T] \geq \mathbb{E}[G] \Longrightarrow V_0 \geq \mathbb{E}[G]$
- From the martingale representation in Brownian filtration

$$\hat{Y}_t := \mathbb{E}[G|\mathcal{F}_t] = \mathbb{E}[G] + \int_0^T \phi_t \cdot dW_t = \hat{Y}_0 + \int_0^T \hat{Z}_t \cdot dS_t$$

Since $Y_T = G$, we deduce that $\mathbb{E}[G] \geq V_0$

Hence $V_0 = \mathbb{E}[G]$ and $Y_T = G$ a.s. for some portfolio $\hat{Z} \in \mathcal{A}$

American options

Given the reward process $\{G_t, t \geq 0\}$, find :

$$V_0 \;:=\; \inf\left\{Y_0 :\; Y_t \geq G_t \; 0 \leq t \leq T \text{ a.s. for some } Z \in \mathcal{A}\right\}$$

Reduces to the optimal stopping problem

$$V_0 \;=\; \sup\left\{\mathbb{E}\left[G_\tau\right] :\; \tau \text{ stopping time } \leq T\right\}$$

and can be approximated by the Snell envelope ($t_k := kT/n$) :

$$V_T^n := G_T \quad \text{and} \quad V_{t_k}^n := \max\left\{G_{t_k}, \mathbb{E}\left[V_{t_{k+1}}^n | \mathcal{F}_{t_k}\right]\right\}$$

This is a standard numerical scheme in the financial industry !

Extension to Fully Nonlinear PDEs

• Isolate a diffusion part in the equation :

$$0 = -v_t(t,x) - \frac{1}{2}1\Delta v(t,x) - f\left(x, Dv(t,x), D^2v(t,x)\right)$$

• The Monte Carlo component Let $X_s = x + 1W_{s-t+h}$, $s \geq t-h$, evaluate at (s, X_s), and take expectations :

$$\begin{aligned} 0 &= \mathbb{E}\left[\int_{t-h}^{t} -(v_t + \frac{1}{2}\Delta v)(s, X_s)ds - \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \\ &= v(t-h, x) - \mathbb{E}\left[v(t, X_t) + \int_{t-h}^{t} f\left(., Dv, D^2\right)(s, X_s)ds\right] \end{aligned}$$

• The Finite Differences component Natural approximation

$$\hat{v}(t-h, x) = \mathbb{E}\left[\hat{v}(t, X_t)\right] + h\, f\left(x, \mathbb{E}[D\hat{v}(t, X_t)], \mathbb{E}[D^2\hat{v}(t, X_t)]\right)$$

which yields our numerical scheme by the ibp argument

Intuition From Greeks Calculation

- Using the approximation $f'(x) \sim_{h=0} \mathbb{E}[f'(x+W_h)]$:

$$
\begin{aligned}
f'(x) &\sim \int f'(x+y)\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \int f(x+y)\frac{y}{h}\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \mathbb{E}\left[f(x+W_h)\frac{W_h}{h}\right]
\end{aligned}
$$

- Similarly, by an additional integration by parts :

$$
\begin{aligned}
f''(x) &= \int f(x+y)\frac{y^2-h}{h^2}\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \mathbb{E}\left[f(x+W_h)\left(\frac{W_h^2-h}{h^2}\right)\right]
\end{aligned}
$$

Connection with Finite Differences : $X_h := x + W_h$

- Consider the binomial approximation of the Brownian motion

$$\text{“}W_h \sim \sqrt{h}\left(\frac{1}{2}\delta_{\{1\}} + \frac{1}{2}\delta_{\{-1\}}\right)\text{''} \quad \text{Then :}$$

$$\mathbb{E}\left[\psi'(X_h)\right] = \mathbb{E}\left[\psi(X_h)\frac{W_h}{h}\right] \sim \frac{\psi(x+\sqrt{h}) - \psi(x-\sqrt{h})}{2\sqrt{h}}$$

- With the trinomial approximation of the Brownian motion

$$\text{“}W_h \sim \sqrt{3h}\left(\frac{1}{6}\delta_{\{1\}} + \frac{2}{3}\delta_{\{0\}} + \frac{1}{6}\delta_{\{-1\}}\right)\text{''} \quad \text{Then :}$$

$$\mathbb{E}\left[\psi''(X_h)\right] = \mathbb{E}\left[\psi(X_h)\frac{W_h^2 - h}{h^2}\right] \sim \frac{\psi(x+\sqrt{3h}) - 2\psi(x) + \psi(x-\sqrt{3h})}{3h}$$

A probabilistic numerical scheme for fully nonlinear PDEs

This suggests the following numerical scheme (MC/FD) to compute $Y^n_{t_i} := v^n(t_i, X_{t_i})$, i.e. an approximation of the solution v along the path of the process $X_t := X_0 + 1W_t$ corresponding to the chosen diffusion :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right) , \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i \ , \quad 1 \le i \le n \ , \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Backward SDE : Definition

Find an $\mathbb{F}^W$−**adapted** (Y, Z) satisfying :

$$
\begin{aligned}
Y_t &= G + \int_t^T F_r(Y_r, Z_r)dr - \int_t^T Z_r \cdot dW_r \\
\text{i.e.} \qquad dY_t &= -F_t(Y_t, Z_t)dt + Z_t \cdot dW_t \quad \text{and} \quad Y_T = G
\end{aligned}
$$

where the generator $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, and

$$\{F_t(y, z),\ t \in [0, T]\} \text{ is } \mathbb{F}^W - \text{adapted}$$

If F is Lipschitz in (y, z) uniformly in (ω, t), and $G \in \mathbb{L}^2(\mathbb{P})$, then there is a unique solution satisfying

$$\mathbb{E} \sup_{t \le T} |Y_t|^2 + \mathbb{E} \int_0^T |Z_t|^2 dt < \infty$$

Markov BSDE's

Let $X_.$ be defined by the (forward) SDE

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t$$

and

$$F_t(y,z) = f(t,X_t,y,z), \; f : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$$

$$G = g(X_T) \in \mathbb{L}^2(\mathbb{P}), \;\; g : \mathbb{R}^d \longrightarrow \mathbb{R}$$

If f continuous, Lipschitz in (x,y,z) uniformly in t, then there is a unique solution to the BSDE

$$dY_t = -f(t,X_t,Y_t,Z_t)dt + Z_t\cdot\sigma(t,X_t)dW_t\,, \qquad Y_T = g(X_T)$$

Moreover, there exists a measurable function V :

$$Y_t = V(t,X_t)\,, \qquad 0\le t\le T$$

BSDE's and semilinear PDE's

- By definition,

$$\begin{aligned} Y_{t+h} - Y_t &= V(t+h, X_{t+h}) - V(t, X_t) \\ &= -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

- If $V(t,x)$ is smooth, it follows from Itô's formula that :

$$\begin{aligned} \int_t^{t+h} \mathcal{L}V(r, X_r)dr + \int_t^{t+h} DV(r, X_r) \cdot \sigma(X_r)dW_r \\ = -\int_t^{t+h} f(X_r, Y_r, Z_r)dr + \int_t^{t+h} Z_r \cdot \sigma(X_r)dW_r \end{aligned}$$

where $\mathcal{L}$ is the Dynkin operator associated to X :

$$\mathcal{L}V = V_t + b \cdot DV + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$$

Discrete-time approximation of BSDEs

<Bally-Pagès SPA03, Zhang AAP04, Bouchard-T. SPA04>

Numerical solution of a semi-linear PDE by simulating the associated backward SDE by means of Monte Carlo methods
Start from Euler discretization : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ is given, and

$$Y^n_{t_{i+1}} - Y^n_{t_i} = -f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i + Z^n_{t_i} \cdot \sigma\left(X^n_{t_i}\right)\Delta W_{t_{i+1}}$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ and

$$Y^n_{t_i} = \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \ 0 \le i \le n-1 \ ,$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ and

$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \qquad , \ 0 \le i \le n-1 \\ Z^n_{t_i} &= \left(\Delta t_i\right)^{-1} \mathbb{E}^n_i\left[Y^n_{t_{i+1}} \Delta W_{t_{i+1}}\right] \end{aligned}$$

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$\Longrightarrow$ Discrete-time approximation : $Y^n_{t_n} = g\left(X^n_{t_n}\right)$ and

$$\begin{aligned} Y^n_{t_i} &= \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\right] + f\left(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}\right)\Delta t_i \quad , \; 0 \le i \le n-1 \\ Z^n_{t_i} &= (\Delta t_i)^{-1}\, \mathbb{E}^n_i\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}\right] \end{aligned}$$

$\Longrightarrow$ Similar to numerical computation of **American options**

Discrete-time approximation of BSDEs, continued

$$0 = t_0 < t_1 < \ldots < t_n = T, \; t_i := i\frac{T}{n}$$

Theorem (Zang 04, Bouchard-T. 04) Assume f and g are Lipschitz. Then :

$$\limsup_{n\to\infty} n^{1/2} \left\{ \sup_{0\le t\le 1} \|Y^n_t - Y_t\|_{\mathbb{L}^2} + \|Z^n - Z\|_{\mathbb{H}^2} \right\} < \infty$$

Theorem (Gobet-Labart 06) Under additional conditions :

$$\limsup_{n\to\infty} n\|Y^n_0 - Y_0\|_{\mathbb{L}^2} \quad < \quad \infty$$

Weak error...

Approximation of conditional expectations

Main observation : in our context all conditional expectations are regressions, i.e.

$$\begin{aligned}
\mathbb{E}\left[Y^n_{t_{i+1}}|\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}}|X_{t_i}\right] \\
\mathbb{E}\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}|\mathcal{F}_{t_i}\right] &= \mathbb{E}\left[Y^n_{t_{i+1}}\Delta W_{t_{i+1}}|X_{t_i}\right]
\end{aligned}$$

Classical methods from statistics :
• Kernel regression <Carrière>
• Projection on subspaces of $\mathbb{L}^2(\mathbb{P})$ <Longstaff-Schwarz, Gobet-Lemor-Warin AAP05>

from numerical probabilistic methods
• quantization... <Bally-Pagès SPA03>

Integration by parts <Lions-Reigner 00, Bouchard-T. SPA04>

Simulation of Backward SDE's

1. Simulate trajectories of the forward process X (well understood)

2. Backward algorithm :

$$
\left\| \begin{array}{rcl}
\hat{Y}^n_{t_n} & = & g\left(X^n_{t_n}\right) \\
\hat{Y}^n_{t_{i-1}} & = & \widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, \hat{Y}^n_{t_{i-1}}, \hat{Z}^n_{t_{i-1}}\right)\Delta t_i\ , \quad 1 \le i \le n\ , \\
\hat{Z}^n_{t_{i-1}} & = & \dfrac{1}{\Delta t_i}\,\widehat{\mathbb{E}}^n_{t_{i-1}}\left[\hat{Y}^n_{t_i}\Delta W_{t_i}\right]
\end{array} \right.
$$

(truncation of $\hat{Y}^n$ and $\hat{Z}^n$ needed in order to control the $\mathbb{L}^p$ error)

Simulation of BSDEs : bound on the rate of convergence

Error estimate for the Malliavin-based algorithm

Theorem For $p > 1$:

$$\limsup_{n\to\infty} \max_{0\le i\le n} n^{-1-d/(4p)} N^{1/2p} \left\| \hat{Y}^n_{t_i} - Y^n_{t_i} \right\|_{\mathbb{L}^p} < \infty$$

For the time step $\frac{1}{n}$, and limit case $p = 1$:

rate of convergence of $\frac{1}{\sqrt{n}}$
if and only if
$$n^{-1-\frac{d}{4}} N^{1/2} = n^{1/2}, \quad \text{i.e. } N = n^{3+\frac{d}{2}}$$

Recent developments in Crisan, Manolarakis and T.

Outline

Second order target problems

- Controls set : $Z \in \mathcal{A} = \{\text{Semimartingales}\}$
- Forward process : $X_t := X_0 + \int_0^t \sigma(s, X_s) dW_s$
- The controlled state is defined by

$$dY_t = f(t, X_t, Y_t, Z_t, \Gamma_t)\, dt + Z_t \circ dX_t, \qquad d\langle Z, W\rangle_t = \Gamma_t dt$$

- Given a function g, find

$$V_0 \quad := \quad \inf \left\{ y : Y_T \geq g(X_T)\ \mathbb{P} - a.s. \text{ for some } Z \in \mathcal{A}^0 \right\}$$

Theorem (with M. Soner) V is a viscosity solution of

$$-\frac{\partial V}{\partial t} - \hat{f}\left(t, x, V(t,x), DV(t,x), D^2 V(t,x)\right) \quad = \quad 0$$

where $\hat{f}(., A) := \sup_{\beta \geq 0} f(., A + \beta)$ (elliptic envelope)

Restrictions on the process Z

Let $\mathcal{A}^m_{t,x}$ be the class of all processes Z of the form

$$Z_s \;=\; z + \int_t^s \alpha_r\, dr + \int_t^s \Gamma_r\, dX^{t,x}_r\,, \quad s \in [t,T]$$

where $z \in \mathbb{R}^d$, α and Γ are respectively $\mathbb{R}^d$ and $\mathcal{S}_d(\mathbb{R}^d)$ progressively measurable processes with

$$\max\left\{|Z_s|, \|\alpha\|_b, |\Gamma_s|\right\} \;\leq\; m\left(1+\left|X^{t,x}_s\right|^p\right)\,,$$

$$|\Gamma_r - \Gamma_s| \;\leq\; m\left(1+\left|X^{t,x}_r\right|^p + \left|X^{t,x}_s\right|^p\right)\left(|r-s| + \left|X^{t,x}_r - X^{t,x}_s\right|\right)$$

We shall look for a solution (Y, Z, α, Γ) of (2BSDE) such that

$$Z \;\in\; \mathcal{A}_{t,x} \;:=\; \cup_{m\geq 0}\mathcal{A}^m_{t,x}$$

Second order target problems : a new point of view

- Controls set : $Z \in \mathcal{A} = \{\text{Semimartingales}\}$, f increasing, and

$$dY_t = f(t, X_t, Y_t, Z_t, \Gamma_t)\, dt + Z_t \circ dX_t, \qquad d\langle Z, W\rangle_t = \Gamma_t dt$$

- Given a function g, find

$$V \quad := \quad \inf\left\{ y : Y_T \geq g(X_T)\ \hat{\mathbb{P}} - a.s. \text{ for some } Z \in \mathcal{A} \right\}$$

Define the conjugate function (with respect to γ) :

$$h(t, x, r, p, a) \quad = \quad \sup_{\gamma \in \mathbb{S}_d} \frac{1}{2}\mathrm{Tr}[\gamma a] - f(t, x, r, p, \gamma)$$

For every $a \in \mathrm{dom}(h)$ ($\subset \mathbb{S}_d^+$), introduce the BSDE :

$$Y_t^a \quad := \quad g(X_T) - \int_t^T h(s, X_s^a, Y_s^a, Z_s^a, a_s)\, ds - \int_t^T Z_s^a \cdot dB_s\ \mathbb{P}^a - \text{a.s.}$$

where $d\langle B\rangle_s = a_s ds\ \mathbb{P}^a$−a.s.

Second order target problems : dual formulation

Theorem (with M. Soner, T. and J. Zhang) Under some conditions,

$$V = \Lambda := \sup_a Y_0^a$$

Remark Λ is easily shown to be a viscosity solution of the PDE

The Measures $\mathbb{P}^a$ and $\hat{\mathbb{P}}$

- B Canonical space, $\mathbb{P}$ Wiener measure
- $\mathbb{P}^a$ distribution of the process

$$X_t = \int_0^t a_s^{1/2} dB_s, \quad t \le T$$

- For simple deterministic functions

$$a(t) = \sum_{i=0}^{n-1} a(t_i) \mathbb{1}_{[t_i,t_{i+1})}(t) \qquad n \in \mathbb{N}, t_i, \beta_i \in \mathbb{Q} \text{ (Rationals)}$$

- The set of such simple deterministic functions is infinite but countable : $\{a_i, i \ge 1\}$
- Finally, we define the reference measure

$$\hat{\mathbb{P}} := \sum_{i \ge 1} 2^{-i} \mathbb{P}^{a_i}$$

Outline

Second order BSDEs : Definition

$$f_1(x, y, z, \gamma) := f(x, y, z, \gamma) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T(x)\gamma] \text{ non-decreasing in } \gamma$$

Consider the 2nd order BSDE :

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t \\ dY_t &= -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z_t\sigma(X_t)dW_t\,, \quad Y_T = g(X_T) \end{aligned}$$

where Z is a semimartingale, and the process Γ is defined by

$$d\langle Z, X\rangle_t = \Gamma_t\, dt$$

Question : existence ? uniqueness ? in which class ?

Second order BSDE : Existence

• Consider the fully nonlinear PDE

$$(\mathrm{E}) \qquad v(T,.) = g \text{ and } -\mathcal{L}v - f\left(t,x,v,Dv,D^2v\right) = 0$$

where $\mathcal{L}V = V_t + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^T D^2 V]$

• If (E) has a smooth solution, then

$$\begin{aligned} \bar{Y}_t &= v(t,X_t), & \bar{Z}_t &:= Dv(t,X_t), \\ \bar{\alpha}_t &:= \mathcal{L}Dv(t,X_t), & \bar{\Gamma}_t &:= V_{xx}(t,X_t) \end{aligned}$$

is a solution of (2BSDE), immediate application of Itô's formula

Second order BSDE : Uniqueness Assumptions

Assumption (f) $f \; : \; [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}_d(\mathbb{R}) \longrightarrow \mathbb{R}$ *continuous, Lipschitz in* y *uniformly in* (t,x,z,γ)*, and for some* $C,p>0$ *:*

$$|f(t,x,y,z)| \;\le\; C\;(1+|y|+|x|^p+|z|^p+|\gamma|^p)$$

Assumption (Comp) *If* w *(resp.* u*) :* $[0,T]\times\mathbb{R}^d\longrightarrow\mathbb{R}$ *is a l.s.c. (resp. u.s.c.) viscosity supersolution (resp.subsolution) of (E) with*

$$w(t,x) \;\ge\; -C(1+|x|^p)\,, \quad and \quad u(t,x) \;\le\; C(1+|x|^p)$$

then $w(T,.)\ge u(T,.)$ *implies that* $w\ge u$ *on* $[0,T]\times\mathbb{R}^d$

Scond Order BSDE : The Uniqueness Result

Let Assumptions(Comp) and (f) hold.

Theorem (Cheridito, Soner, T. and Victoir) *For every g with polynomial growth, there is at most one solution with $Z \in \mathcal{A}^0$ of*

$$dY_t = -f(t, X_t, Y_t, Z_t, \Gamma_t)dt + Z_t\sigma(X_t)dW_t, \ d\langle Z, X\rangle_t = \Gamma_t dt$$
$$Y_T = g(X_T) \ \ \mathbb{P} - \text{a.s.}$$

Work in progress with Soner and Zhang : weak existence and uniqueness result using the new point of view with the reference measure $\hat{\mathbb{P}}$

Outline

The probabilistic scheme for fully nonlinear PDEs

By analogy with BSDEs, we consider the following discretization for 2BSDEs :

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right) , \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i\ , \quad 1 \le i \le n\ , \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\,\frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\,\frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

The Convergence Result

Theorem (Fahim-T. 08) (i) *Suppose that f is Lipschitz uniformly in x and $\varepsilon I \leq \nabla_\gamma f \leq \sigma\sigma^{\mathrm{T}}$. Then*

$$Y_0^n(t,x) \longrightarrow v(t,x) \qquad \textit{uniformly on compacts}$$

where v is the unique viscosity solution of the nonlinear PDE.
(ii) *If f is either convex or concave in (y,z,γ), i.e. HJB operator,*

$$ch^{3/8} \leq v - v^h \leq Ch^{1/4}$$

- Convergence result accounting for the Monte Carlo error is available, Rate of convergence in this context is in progress...
- Proof : stability, consistency, monotonicity (Barles-Souganidis 91)
- Bounds on the approximation error (Krylov, Barles-Jacobsen)
- Compare with Bonans and Zidani...

Comments on the 2BSDE algorithm

• in BSDEs the drift coefficient μ of the forward SDE can be changed arbitrarily by Girsanov theorem : importance sampling...

• in 2BSDEs both μ and σ can be changed : previous theorem and numerical results however recommend prudence...

Portfolio optimization

With $U(y) = -e^{-\eta y}$, want to solve :

$$V(t,y) \quad := \quad \sup_Z \mathbb{E}\left[U\left(Y_T\right)\right]$$

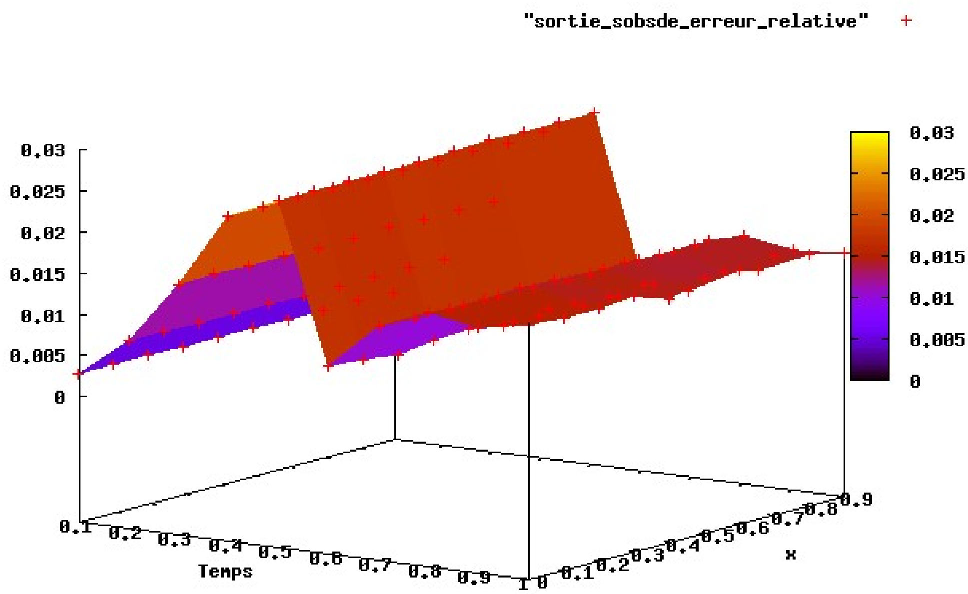
where

$$dY_u = Z_u dS_u \quad \text{and} \quad dS_u = S_u\sigma(dW_u + \lambda du)$$

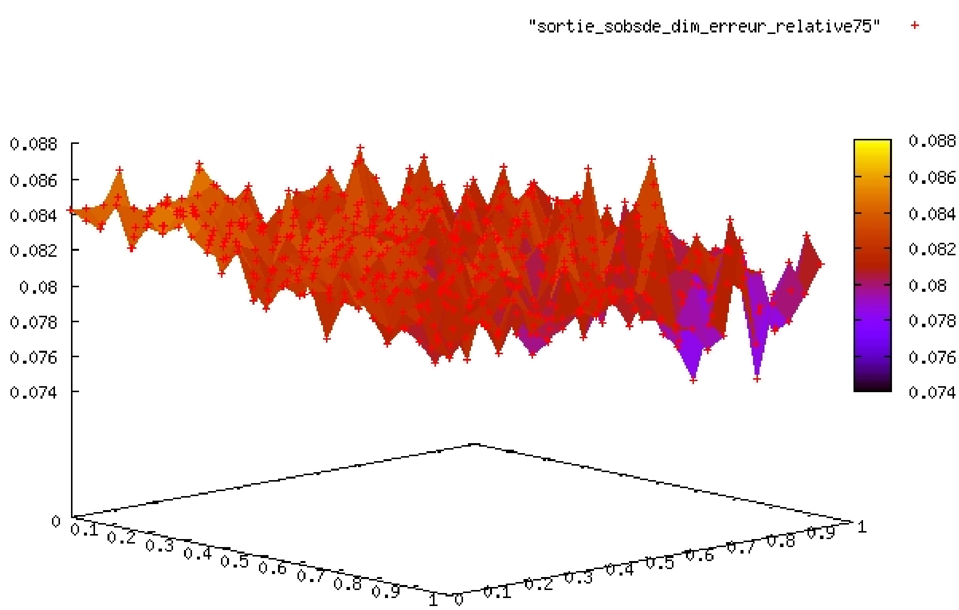
- An explicit solution is available
- V is the characterized by the fully nonlinear PDE

$$-V_t + \frac{1}{2}\lambda^2\frac{(V_y)^2}{V_{yy}} \;=\; 0 \quad \text{and} \quad V(T,.) = U$$

Numerical examples by X. Warin (1)



Numerical examples by X. Warin (2)



Varying the drift of the FSDE

Drift FSDE	Relative error (Regression)
-1	0,0648429
-0,8	0,0676044
-0,6	0,0346846
-0,4	0,0243774
-0,2	0,0172359
0	0,0124126
0,2	0,00880041
0,4	0,00656142
0,6	0,00568952
0,8	0,00637239

Varying the volatility of the FSDE

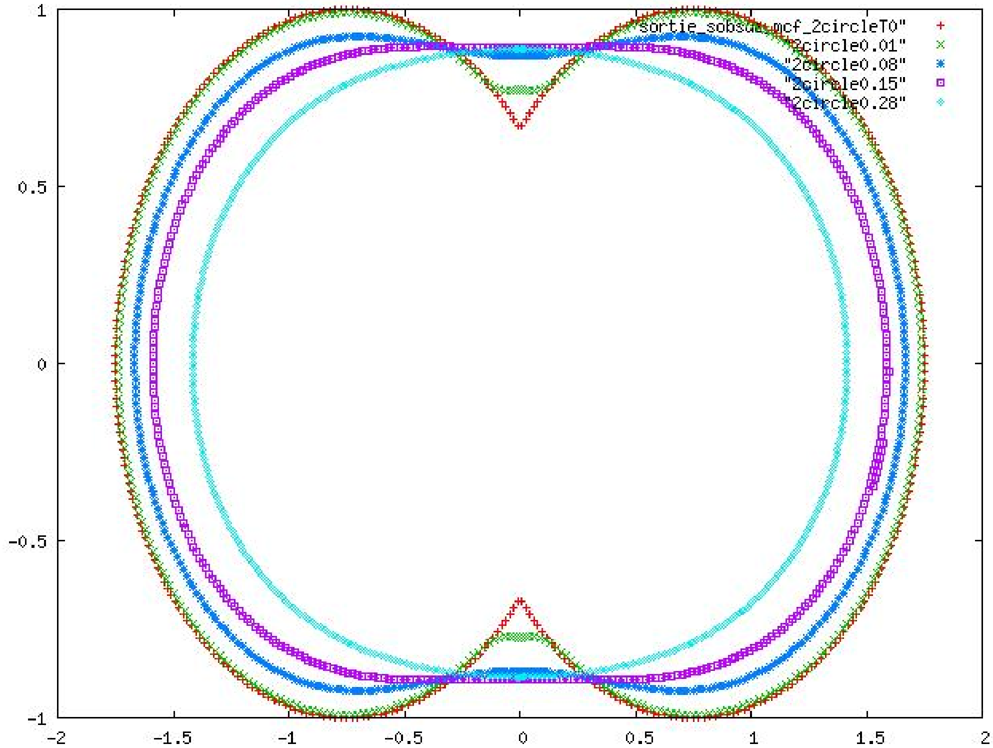
Diffusion FSDE	Relative error (Regression)	Relative error (Quantization)
0,2	0,581541	0,526552
0,4	0,42106	0,134675
0,6	0,0165435	0,0258884
0,8	0,0170161	0,00637319
1 0,	0124126	0,0109905
1,2	0,0211604	0,0209174
1,4	0,0360543	0,0362259
1,6	0,0656076	0,0624566

Mean Curvature Flow : two connected circles (X. Warin)

Mouvement of a surface along the inward normal with speed proportional to the mean curvature (codimension 1) :

$$-u_t - \Delta u - \frac{D^2 u \, Du \cdot Du}{|Du|^2}$$

Mean Curvature Flow : two intersecting circles (X. Warin)



Mean Curvature Flow : two connected circles (X. Warin)