Vibrations of a uniform Bernoulli beam

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1 Introduction

In this note we discuss the analytic solution of the dynamic bending equation of a uniform unloaded Bernoulli beam. That is

$$\mu\, \ddot{w}(x,t) + EI\, w^{(4)}(x,t) = 0, \qquad w : [0,L] \times \mathbb{R} \to \mathbb{R}, \tag{1}$$

where $\mu, E, I, L > 0$, the double dot denotes the second derivative with respect to t and the upper index (4) stands for the fourth derivative with respect to x. The value $w(x,t)$ is the vertical deflection of the beam at position x and time t. The symbols μ, E, I denote the mass per unit length, the Young modul and the cross sectional moment of inertia, respectively. L is the length of the beam.

2 Standing waves

We seek for solutions of (1) of the form

$$w(x,t) = \sigma(t)\, w(x), \tag{2}$$

where $\sigma(t) = r\,\sin(\omega\, t - \phi)$ with $r, \phi \in \mathbb{R}$ and $\omega > 0$ is a sinusoidal function. These functions can also we written in the form

$$\sigma(t) = \alpha\,\cos(\omega\, t) + \frac{\beta}{\omega}\,\sin(\omega\, t), \qquad \alpha, \beta \in \mathbb{R}.$$

They are precisely the real solutions of the differential equation

$$\ddot{\sigma}(t) + \omega^2\, \sigma(t) = 0. \tag{3}$$

Notice that $\alpha = \sigma(0)$ and $\beta = \dot{\sigma}(0)$. Observe further that $\sigma(t) = \mathrm{Re}(c\, e^{i\omega\, t})$, where $c = \alpha - i\beta/\omega$. The functions (2) represent standing waves (also called vibration modes). The factor $w(x)$ is the shape of the wave and the factor $\sigma(t)$ is the oscillating amplitude. Inserting (2) into (1) yields because of (3) that

$$-\mu\, \omega^2\, \sigma(t)\, w(x) + EI\, \sigma(t)\, w^{(4)}(x) = 0.$$

Dividing by $\sigma(t)$ and reordering terms we get

$$w^{(4)}(x) = \kappa^4\, w(x), \qquad \kappa := \left(\frac{\omega\,\mu}{EI}\right)^{1/4}. \tag{4}$$

This is an eigenvalue equation for the differential operator $\frac{d^4}{dx^4}$ with eigenvalue κ^4 and eigenfunction $w(x)$. The general solution to the differential equation (4) is

$$w(x) \;=\; A\,\cosh(\kappa\, x) + B\,\sinh(\kappa\, x) + C\,\cos(\kappa\, x) + D\,\sin(\kappa\, x), \qquad A, B, C, D \in \mathbb{R}. \tag{5}$$

We will need the derivatives

$$\begin{array}{rcl} w'(x) & = & \kappa\,(A\,\sinh(\kappa\,x) + B\,\cosh(\kappa\,x) - C\,\sin(\kappa\,x) + D\,\cos(\kappa\,x)), \\ w''(x) & = & \kappa^2\,(A\,\cosh(\kappa\,x) + B\,\sinh(\kappa\,x) - C\,\cos(\kappa\,x) - D\,\sin(\kappa\,x)), \\ w'''(x) & = & \kappa^3\,(A\,\sinh(\kappa\,x) + B\,\cosh(\kappa\,x) + C\,\sin(\kappa\,x) - D\,\cos(\kappa\,x)). \end{array} \tag{6}$$

Notice that in (5) the parameter κ can be any positive number. The associated frequency ω then is then given by

$$\omega = \frac{EI}{\mu}\,\kappa^4.$$

The situation changes if we impose boundary conditions on w. Then only a discrete (but infinite) set of values of κ is possible, as we will see in the next sections.

3 The cantilever beam

The unloaded cantilever beam which is horizontally clamped at the left end and free at the right end satisfies the four homogeneous boundary conditions

$$0 = w(0) = w'(0) = w''(L) = w'''(L).$$

The latter two conditions reflect that there is no bending moment and no load at the free end. Using (5) and (6) these four conditions read

$$\begin{array}{rcccl} 0 & = & w(0) & = & A + C, \\ 0 & = & w'(0)/\kappa & = & B + D, \\ 0 & = & w''(L)/\kappa^2 & = & A\,\cosh(\kappa\,L) + B\,\sinh(\kappa\,L) - C\,\cos(\kappa\,L) - D\,\sin(\kappa\,L), \\ 0 & = & w'''(L)/\kappa^3 & = & A\,\sinh(\kappa\,L) + B\,\cosh(\kappa\,L) + C\,\sin(\kappa\,L) - D\,\cos(\kappa\,L). \end{array} \tag{7}$$

This is a homogeneous linear system of equations for A, B, C, D. It can be written in matrix vector form as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh(\kappa\,L) & \sinh(\kappa\,L) & -\cos(\kappa\,L) & -\sin(\kappa\,L) \\ \sinh(\kappa\,L) & \cosh(\kappa\,L) & \sin(\kappa\,L) & -\cos(\kappa\,L) \end{bmatrix}}_{M_\kappa} \underbrace{\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}}_{v}$$

A nontrivial solution $v \neq 0$ to this equation exists if and only if the matrix M_κ is singular. The latter holds if and only if κ is a zero of the so called characteristic function $\chi(\kappa) := \det(M_\kappa)$. However, instead of working with the determinant of M_κ directly we will apply a 2×2 matrix to determine the admissible values of κ. Using the first two equations of (7) we can eliminate the variables $C = -A$ and $D = -B$. Thus,

$$w(x) = A\,(\cosh(\kappa\,x) - \cos(\kappa\,x)) + B\,(\sinh(\kappa\,x) - \sin(\kappa\,x)) \tag{8}$$

and

$$\begin{array}{rcl} 0 = w''(L)/\kappa^2 & = & A\,(\cosh(\kappa\,L) + \cos(\kappa\,L)) + B\,(\sinh(\kappa\,L) + \sin(\kappa\,L)), \\ 0 = w'''(L)/\kappa^3 & = & A\,(\sinh(\kappa\,L) - \sin(\kappa\,L)) + B\,(\cosh(\kappa\,L) + \cos(\kappa\,L)). \end{array}$$

Writing these equations in matrix vector form we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cosh(\kappa\,L) + \cos(\kappa\,L) & \sinh(\kappa\,L) + \sin(\kappa\,L) \\ \sinh(\kappa\,L) - \sin(\kappa\,L) & \cosh(\kappa\,L) + \cos(\kappa\,L) \end{bmatrix}}_{N_\kappa} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{9}$$

A nontrivial solution $[A\ B] \neq [0\ 0]$ exists if and only if

$$\begin{aligned}
0 &= \det(N_\kappa)\\
&= (\cosh(\kappa\,L) + \cos(\kappa\,L))^2 - (\sinh(\kappa\,L) + \sin(\kappa\,L))(\sinh(\kappa\,L) - \sin(\kappa\,L))\\
&= \underbrace{\cosh(\kappa\,L)^2 - \sinh(\kappa\,L)^2}_{=1} + \underbrace{\cos(\kappa\,L)^2 + \sin(\kappa\,L)^2}_{=1} - + 2\,\cosh(\kappa\,L)\,\cos(\kappa\,L)\\
&= 2\,(1 + \cosh(\kappa\,L)\,\cos(\kappa\,L))\\
&= 2\,f(\kappa\,L),
\end{aligned}$$

where

$$f(x) := 1 + \cosh(x)\,\cos(x).$$

The equivalence

$$f(x) = 0 \qquad \Leftrightarrow \qquad \cos(x) = -\frac{1}{\cosh(x)}$$

combined with the fact that $1/\cosh(x) \approx 0$ already for x of moderate size shows that f has infinitely many positive zeros x_j, $j = 1, 2, \ldots$ which approximate the zeros of the cosine function rapidly: $x_j \approx (j-0.5)\pi$. The table below shows the first x_j (computed with Newton's method) and the differences $x_j - (j-0.5)\pi$ with a precision of 5 digits.

j	x_j	$x_j - (j-0.5)\pi$
1	1.8751	0.3043
2	4.6941	−0.0183
3	7.8548	0.0008
4	10.996	0.0000
5	14.137	0.0000

On replacing κL with x_j in (9) we get

$$\begin{bmatrix}0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}\cosh(x_j) + \cos(x_j) & \sinh(x_j) + \sin(x_j)\\ \sinh(x_j) - \sin(x_j) & \cosh(x_j) + \cos(x_j)\end{bmatrix}}_{N} \begin{bmatrix}A\\B\end{bmatrix}. \tag{10}$$

Since N is singular its lower row is a scalar multiple of its upper row. Thus, the pair A, B solves the matrix vector equation if it solves the upper equation. The latter holds if

$$B = -\frac{\cosh(x_j) + \cos(x_j)}{\sinh(x_j) + \sin(x_j)}\,A.$$

Inserting this into (8) we get the following eigenfunction associated with x_j.

$$w_j(x) = A\left[(\cosh(\kappa_j\,x) - \cos(\kappa_j\,x)) - \frac{\cosh(x_j) + \cos(x_j)}{\sinh(x_j) + \sin(x_j)}\,(\sinh(\kappa_j\,x) - \sin(\kappa_j\,x))\right], \tag{11}$$

where $\kappa_j := x_j/L$. Examples are shown in Figure 1. The factor $A \neq 0$ can be chosen arbitrarily. For reasons discussed later we set

$$A := L^{-1/2}. \tag{12}$$

Thus, all standing wave solutions for the cantilever beam are of the form

$$w(x,t) = \left(\alpha_j\,\cos(\omega_j\,t) + \frac{\beta_j}{\omega_j}\,\sin(\omega_j\,t)\right)w_j(x), \qquad \omega_j := \frac{EI}{\mu}\kappa_j^4, \qquad j = 1, 2, \ldots$$

where $\alpha_j, \beta_j \in \mathbb{R}$ are arbitrary.The frequencies ω_j are called eigenfrequencies. By linearity superpositions of standing waves are also solutions to the cantilever beam equation. It turns out (see literature) that all solutions are such superpositions:

$$w(x,t) = \sum_{j=1}^{\infty} \left(\alpha_j \, \cos(\omega_j \, t) + \frac{\beta_j}{\omega_j} \, \sin(\omega_j \, t) \right) \, w_j(x).$$

We now discuss how the amplitudes α_j, β_j can be computed if an initial deflection

$$w_0(x) := w(x,0) = \sum_{j=1}^{\infty} \alpha_j \, w_j(x) \tag{13}$$

and an initial deflection velocity

$$\dot{w}_0(x) := \dot{w}(x,0) = \sum_{j=1}^{\infty} \beta_j \, w_j(x) \tag{14}$$

are given. To this end we introduce an inner product on the space of square integrable functions $u : [0, L] \to \mathbb{R}$ by

$$\langle u_1, u_2 \rangle := \int_0^L u_1(x) \, u_2(x) \, dx.$$

Suppose now, that u_1, u_2 are both 4 times differentiable and satisfy the cantilever boundary conditions. Then partial integration yields

$$\langle u_1^{(4)}, u_2 \rangle = \langle u_1'', u_2'' \rangle = \langle u_1, u_2^{(4)} \rangle.$$

Applying this to a pair of eigenfunctions w_i, w_j, $i \neq j$ we get

$$(\kappa_i - \kappa_j) \, \langle w_i, w_j \rangle = \langle \kappa_i \, w_i, w_j \rangle - \langle w_i, \kappa_j \, w_j \rangle = \langle w_i^{(4)}, w_j \rangle - \langle w_i, w_j^{(4)} \rangle = 0.$$

Since $\kappa_i \neq \kappa_j$ for $i \neq j$ it follows that

$$\langle w_i, w_j \rangle = 0 \qquad \text{for } i \neq j.$$

In words, eigenfunctions to different eigenvalues are orthogonal to each other. Furthermore, a tedious computation shows that (because of (12))

$$\langle w_i, w_i \rangle = 1 \qquad \text{for } i = 1, 2, \ldots$$

Now, we get from (13) and (14) that

$$\begin{aligned} \langle w_i, w_0 \rangle &= \left\langle w_i, \sum_{j=1}^{\infty} \alpha_j \, w_j \right\rangle = \sum_{j=1}^{\infty} \alpha_j \, \langle w_i, w_j \rangle = \alpha_i \\ \langle w_i, \dot{w}_0 \rangle &= \left\langle w_i, \sum_{j=1}^{\infty} \beta_j \, w_j \right\rangle = \sum_{j=1}^{\infty} \beta_j \, \langle w_i, w_j \rangle = \beta_i \end{aligned}$$

In summary the solution to the beam equation

$$\mu \, \ddot{w}(x,t) + EI \, w^{(4)}(t,x) = 0, \qquad w : [0, L] \times \mathbb{R} \to \mathbb{R}$$

with boundary conditions

$$0 = w(0) = w'(0) = w''(L) = w'''(L).$$

and initial value functions

$$w_0(x) = w(x,0), \qquad \dot{w}_0(x) = \dot{w}(x,0)$$

is given by the formula

$$w(x,t) = \sum_{j=1}^{\infty} \left(\langle w_j, w_0 \rangle \, \cos(\omega_j \, t) + \frac{\langle w_j, \dot{w}_0 \rangle}{\omega_j} \, \sin(\omega_j \, t) \right) \, w_j(x)$$

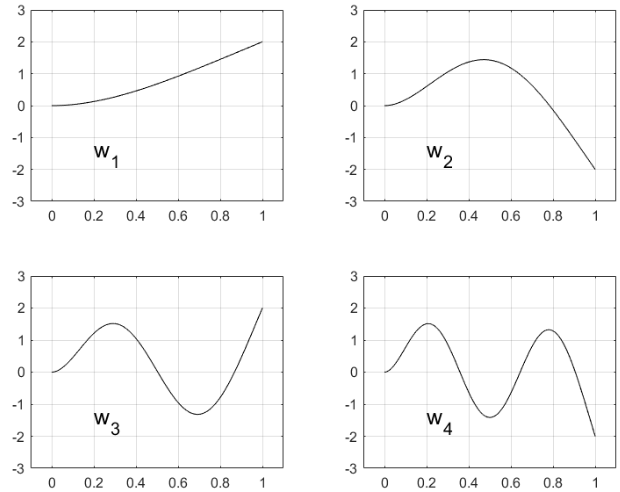

Figure 1: The first four eigenfunctions of the cantilever beam of length $L = 1$

4 The simply supported beam

The simply supported beam satisfies the beam equation and the boundary conditions

$$w(0) = w(L) = w''(0) = w''(L) = 0. \tag{15}$$

The latter two conditions reflect the fact that there is no bending moment at the end points. An eigenfunction

$$w(x) \;=\; A \,\cosh(\kappa\, x) + B\,\sinh(\kappa\, x) + C\,\cos(\kappa\, x) + D\,\sin(\kappa\, x), \qquad A, B, C, D \in \mathbb{R}$$

fulfilles the boundary conditions if

$$\begin{aligned} 0 &= A + C, \\ 0 &= A\,\cosh(\kappa\, L) + B\,\sinh(\kappa\, L) + C\,\cos(\kappa\, L) + D\,\sin(\kappa\, L), \\ 0 &= A - C, \\ 0 &= A\,\cosh(\kappa\, L) + B\,\sinh(\kappa\, L) - C\,\cos(\kappa\, L) - D\,\sin(\kappa\, L) \end{aligned}$$

From the first and third equation it follows that $A = C = 0$. Then by adding the second and fourth equation we conclude that $B = 0$. Thus, $w(x) = D\,\sin(\kappa\, x)$ with $0 = w(L) = D\,\sin(\kappa\, L)$, which implies $D = 0$ or $\sin(\kappa\, L) = 0$. Thus, all eigenfunctions w_j are of the form

$$w_j(x) = D\,\sin(\kappa_j\, x), \qquad \kappa_j = \frac{j\,\pi}{L}, \qquad j = 1, 2, \ldots, \qquad D \neq 0.$$

Choosing $D = L^{-1/2}$ we obtain for the inner product defined in the last section that $\langle w_j, w_j \rangle = 1$. By diect computation or with the same argument as in the last section we get $\langle w_i, w_j \rangle = 0$ for $i \neq j$. The solution formula for an initial value problem $w(x, 0) = w_0(x)$, $\dot{w}(x, 0) = \dot{w}(x)$ with the boundary conditions (15) has the same form as in the last section:

$$w(x, t) = \sum_{j=1}^{\infty} \left(\langle w_j, w_0 \rangle \,\cos(\omega_j\, t) + \frac{\langle w_j, \dot{w}_0 \rangle}{\omega_j}\,\sin(\omega_j\, t) \right) w_j(x), \qquad \omega_j = \frac{EI}{\mu}\kappa_j^4.$$