Framework Project

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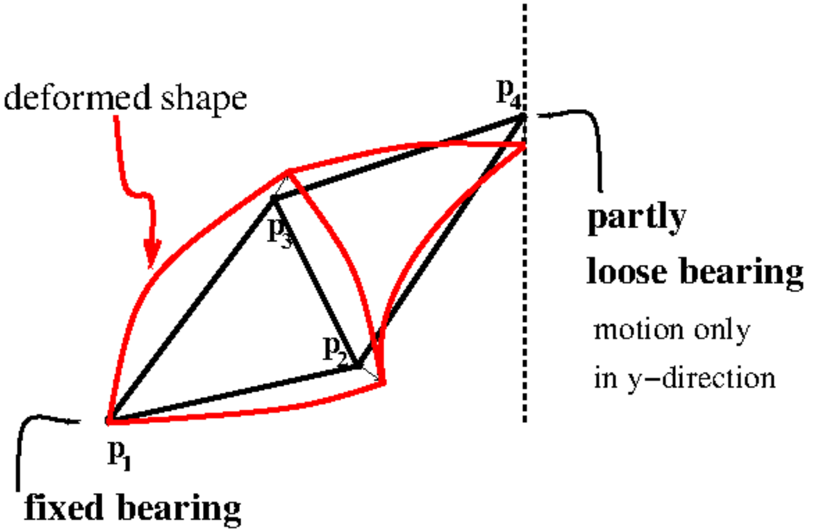
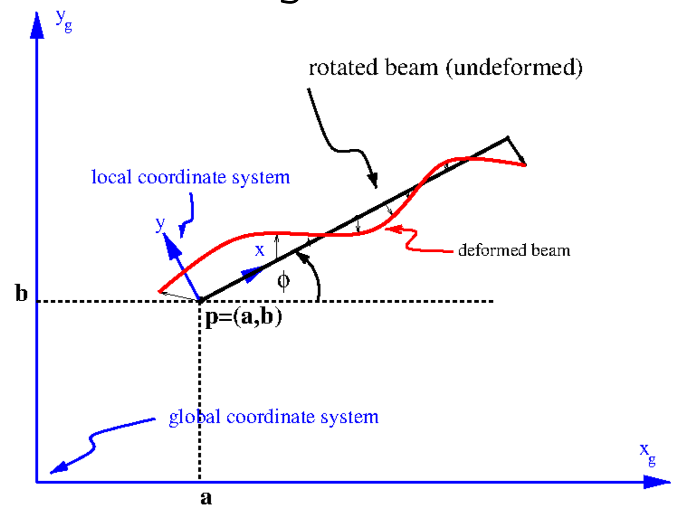
A simple framework:



A single beam:



Curve of deformed beam in local coordinates:

$$\beta(x,t) = \begin{bmatrix} x + v(x,t) \\ w(x,t) \end{bmatrix} = \underset{\substack{\uparrow \\ \text{original position}}}{\begin{bmatrix} x \\ 0 \end{bmatrix}} + \underset{\substack{\uparrow \\ \text{displacement vector}}}{\begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}},$$

$$\underset{\substack{\uparrow \\ \text{equations of motion}}}{\begin{aligned} \mu\,\ddot{v} - (EA\,v')' &= f_1, \ (\text{longitudinal}) \\ \mu\,\ddot{w} + (EI\,w'')'' &= f_2 \ (\text{transversal}) \end{aligned}}$$

Curve of deformed beam in global coordinates:

$$\begin{aligned} \beta_g(x,t) &= p + D_\phi\,\beta(x,t), \qquad \text{with rotation matrix } D_\phi = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} \\ &= \begin{bmatrix} a + \cos(\phi)\,(x + v(x,t)) - \sin(\phi)\,w(x,t) \\ b + \sin(\phi)\,(x + v(x,t)) + \cos(\phi)\,w(x,t) \end{bmatrix}. \end{aligned}$$

Remark on equations of motion:
Transversal equation is bending equation. Longitudinal equation is usual wave equation.

The constraints (boundary and transition conditions)

1. *Bearings:*
   The motion of some beam end points is restricted.
   The at fixed bearing the bending angle may be 0.

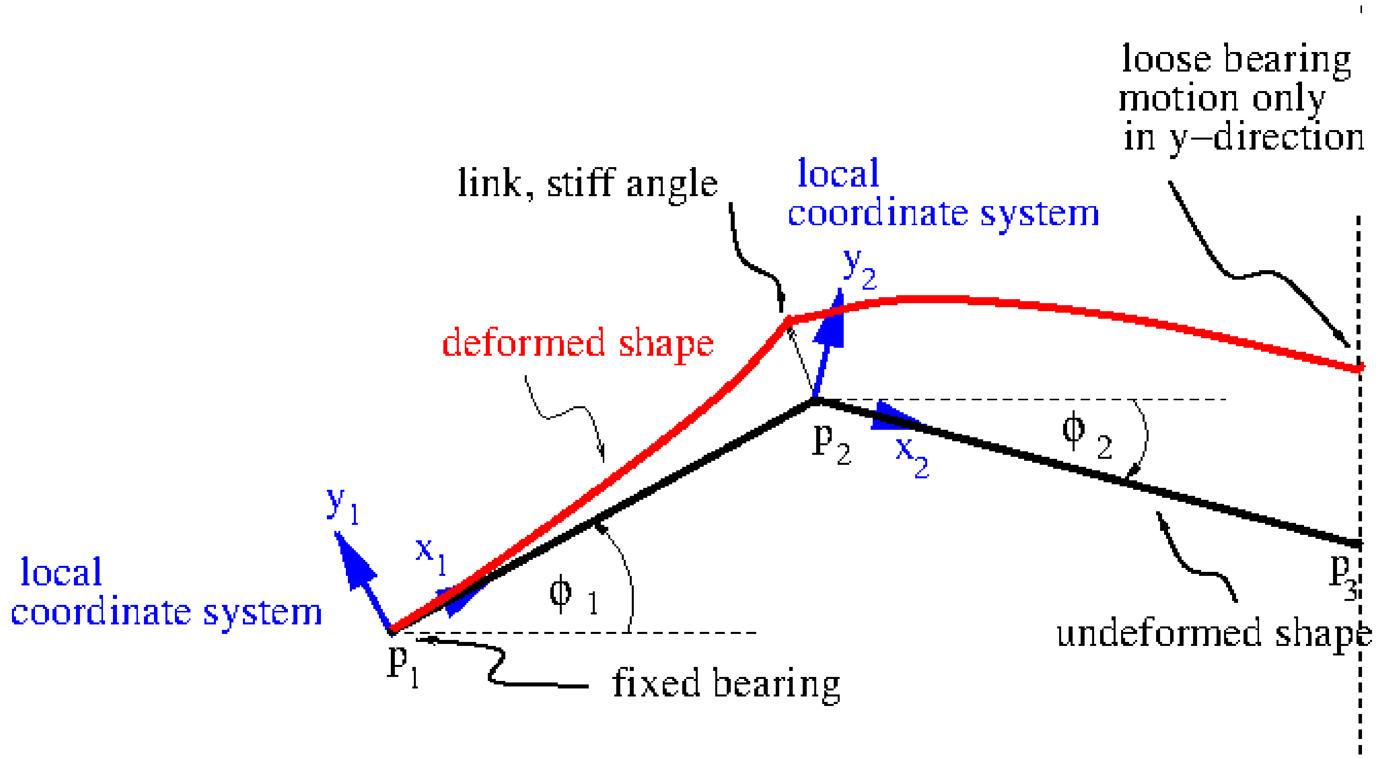
2. *Transition conditions.*
   The end points of some beams should touch everytime.

3. *Stiffness of angles:*
   The angle between touching beams should be constant in time.
   This condition may not be satisfied for all beams.

In order to satisfy these geometric conditions constraining forces and moments have to be present at the end points of the beams.

Mathematical modeling of constraints I

We give an example of a framework of 2 connected beams (see figure).



The constraints are:

1. fixed bearing at point $\mathbf{p_1}$. Movable bearing at point $\mathbf{p_3}$.

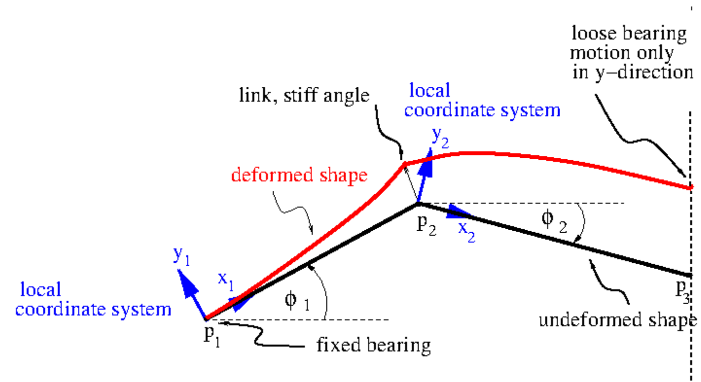
2. Interconnection with fixed angle at point $\mathbf{p_2}$.

The lengths of the beams are L_1 and L_2, respectively.

Mathematical modeling of constraints II:

Linking condition

The linking condition at point $\mathbf{p_2}$ is:



The displacement vector at the end of beam 1
equals
the displacement vector at the beginning of beam 2.

In order to formulate this condition mathematically both displacements have to be written in global cooordinates.

$$\underbrace{\begin{bmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{bmatrix}}_{D_{\phi_1}} \begin{bmatrix} v_1(L_1,t) \\ w_1(L_1,t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\phi_2) & -\sin(\phi_2) \\ \sin(\phi_2) & \cos(\phi_2) \end{bmatrix}}_{D_{\phi_2}} \begin{bmatrix} v_1(0,t) \\ w_1(0,t) \end{bmatrix}.$$

The linking condition then yields the 2 equations

$$\begin{aligned} \cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t) &= \cos(\phi_2)\, v_2(0,t) - \sin(\phi_2)\, w_2(0,t) \\ \sin(\phi_1)\, v_1(L_1,t) + \cos(\phi_1)\, w_1(L_1,t) &= \sin(\phi_2)\, v_2(0,t) + \cos(\phi_2)\, w_2(0,t). \end{aligned}$$

Reordering terms we get

$$\boxed{\begin{aligned} \cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t) - \cos(\phi_2)\, v_2(0,t) + \sin(\phi_2)\, w_2(0,t) &= 0, \\ \sin(\phi_1)\, v_1(L_1,t) + \cos(\phi_1)\, w_1(L_1,t) - \sin(\phi_2)\, v_2(0,t) - \cos(\phi_2)\, w_2(0,t) &= 0. \end{aligned}}$$

Mathematical modeling of constraints III

Further constraints

1. stiffness of angle at point $\mathbf{p_2}$
   (linear approximation):

   $w_1'(L_1, t) - w_2'(0, t) = 0.$

2. Fixed bearing at point $\mathbf{p_1}$:

   $v_1(0, t) = 0, \quad w_1(0, t) = 0, \quad w_1'(0, t) = 0.$

3. movable bearing at point $\mathbf{p_3}$:

   The x-coordinate of the displacement vector at the end of beam 2 is 0:

   $\cos(\phi_2)\, v_2(L_2, t) - \sin(\phi_2)\, w_2(L_2, t) = 0$

Mathematical modeling of constraints IV

List of all constraints:

1. $w_1'(L_1,t) - w_2'(0,t) = 0$
2. $v_1(0,t) = 0$
3. $w_1(0,t) = 0$
4. $w_1'(0,t) = 0$
5. $\cos(\phi_2)\, v_2(L_2,t) - \sin(\phi_2)\, w_2(L_2,t) = 0$
6. $\cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t) - \cos(\phi_2)\, v_2(0,t) + \sin(\phi_2)\, w_2(0,t) = 0$
7. $\sin(\phi_1)\, v_1(L_1,t) + \cos(\phi_1)\, w_1(L_1,t) - \sin(\phi_2)\, v_2(0,t) - \cos(\phi_2)\, w_2(0,t) = 0$

In the next step we associate a vector $\mathbf{c}$ with each constaint.

The information about the disretized framework at time t is stored in a vector $\mathbf{u}(t)$. It contains the values of displacements and slopes at the nodes of each beam.

$$\mathbf{u}(t) = \begin{bmatrix} v_{1,1}(t) \\ \vdots \\ v_{1,n}(t) \\ w_{1,1}(t) \\ \vdots \\ w_{1,2n-1}(t) \\ w_{1,2n}(t) \\ v_{2,1}(t) \\ \vdots \\ v_{2,n}(t) \\ w_{2,1}(t) \\ \vdots \\ w_{2,2n-1}(t) \\ w_{2,2n}(t) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{beam 1} \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{beam 2} \end{array}$$

Constraint are written in terms of the entries of $\mathbf{u}$. See next page.

constraint	constraint in terms of $\mathbf{u}(t)$
$w_1'(L_1,t) - w_2'(0,t) = 0$	$w_{1,2n}(t) - w_{2,2}(t) = 0$
$v_1(0,t) = 0$	$v_{1,1}(t) = 0$
$w_1(0,t) = 0$	$w_{1,1}(t) = 0$
$w_1'(0,t) = 0$	$w_{1,2}(t) = 0$
$\cos(\phi_2)\, v_2(L_2,t) - \sin(\phi_2)\, w_2(L_2,t) = 0$	$\cos(\phi_2)\, v_{2,n}(t) - \sin(\phi_2)\, w_{2,2n-1}(t) = 0$
$\cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t)$ $- \cos(\phi_2)\, v_2(0,t) + \sin(\phi_2)\, w_2(0,t) = 0$	$\cos(\phi_1)\, v_{1,n}(t) - \sin(\phi_1)\, w_{1,2n-1}(t)$ $- \cos(\phi_2)\, v_{2,1}(t) + \sin(\phi_2)\, w_{2,1}(t) = 0$
$\sin(\phi_1)\, v_1(L_1,t) + \cos(\phi_1)\, w_1(L_1,t)$ $- \sin(\phi_2)\, v_2(0,t) - \cos(\phi_2)\, w_2(0,t) = 0$	$\sin(\phi_1)\, v_{1,n}(t) + \cos(\phi_1)\, w_{1,2n-1}(t)$ $- \sin(\phi_2)\, v_{2,1}(t) - \cos(\phi_2)\, w_{2,1}(t) = 0$

Representation of constraints with inner products

Each constraint can be written in the form $\mathbf{c}^T \mathbf{u}(t) = 0$, wher $\mathbf{c}$ is a column vector.

Example 1. Constraint $w_{1,1}(t) = 0$ can be written as

$$0 = w_{1,1}(t) = 1 \cdot w_{1,1}(t) = \underbrace{\begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 \end{bmatrix}}_{\mathbf{c}^T} \underbrace{\begin{bmatrix} v_{1,1}(t) \\ \vdots \\ v_{1,n}(t) \\ w_{1,1}(t) \\ \vdots \\ w_{1,2n-1}(t) \\ w_{1,2n}(t) \\ v_{2,1}(t) \\ \vdots \\ v_{2,n}(t) \\ w_{2,1}(t) \\ \vdots \\ w_{2,2n-1}(t) \\ w_{2,2n}(t) \end{bmatrix}}_{=\mathbf{u}(t)} = \mathbf{c}^T \mathbf{u}(t).$$

The 1 in the vector $\mathbf{c}^T$ has to be put at the right place.

Example 2. The constraint

$$\cos(\phi_1)\, v_{1,n}(t) - \sin(\phi_1)\, w_{1,2n-1}(t) - \cos(\phi_2)\, v_{2,1}(t) + \sin(\phi_2)\, w_{2,1}(t) = 0$$

can be written as

$$\underbrace{\begin{bmatrix} 0 \ldots 0 & \cos(\phi_1) & 0 \ldots 0 & -\sin(\phi_1) & 0 \ldots 0 & -\cos(\phi_2) & 0 \ldots 0 & \sin(\phi_2) & 0 \ldots 0 \end{bmatrix}}_{=\mathbf{c}^T} \underbrace{\begin{bmatrix} v_{1,1}(t) \\ \vdots \\ v_{1,n}(t) \\ w_{1,1}(t) \\ \vdots \\ w_{1,2n-1}(t) \\ w_{1,2n}(t) \\ v_{2,1}(t) \\ \vdots \\ v_{2,n}(t) \\ w_{2,1}(t) \\ \vdots \\ w_{2,2n-1}(t) \\ w_{2,2n}(t) \end{bmatrix}}_{=\mathbf{u}(t)} = 0$$

The constraints are written in the form

$$\mathbf{c}_j^T \mathbf{u}(t) = 0, \qquad j = 1, \ldots, r, \qquad (*)$$

where r=number of constraints.

These equations can be written as 1 matrix vector equation.

Introduce a matrix $\mathbf{C}$ with columns $\mathbf{c}_j$:

$$\mathbf{C} = [\ \mathbf{c}_1\ \mathbf{c}_2\ \ldots \mathbf{c}_r\ ].$$

The equations $(*)$ are equivalent to the identity

$$\mathbf{C}^T \mathbf{u}(t) = \mathbf{0}.$$

Verification:

$$\mathbf{C}^T \mathbf{u}(t) = [\ \mathbf{c}_1\ \mathbf{c}_2\ \ldots \mathbf{c}_r\ ]^T\, \mathbf{u}(t) = \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_r^T \end{bmatrix} \mathbf{u}(t) = \begin{bmatrix} \mathbf{c}_1^T \mathbf{u}(t) \\ \mathbf{c}_2^T \mathbf{u}(t) \\ \vdots \\ \mathbf{c}_r^T \mathbf{u}(t) \end{bmatrix}.$$

Construction of the mass and the stiffness matrix I

To each beam (number j) there belong mass and stiffness matrices for the longitudinal and the transversal displacement. Notation:

$$M_L^{(j)},\ S_L^{(j)},\ M_T^{(j)},\ S_T^{(j)}, \qquad j = 1, \ldots, m = \text{number of beams}$$

Write these into a block diagonal matrix:

$$\mathbf{M} = \begin{bmatrix} M_L^{(1)} & & & & & \\ & M_T^{(1)} & & & & \\ & & M_L^{(2)} & & & \\ & & & M_T^{(2)} & & \\ & & & & \ddots & \\ & & & & & M_T^{(m)} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_L^{(1)} & & & & & \\ & S_T^{(1)} & & & & \\ & & S_L^{(2)} & & & \\ & & & S_T^{(2)} & & \\ & & & & \ddots & \\ & & & & & S_T^{(m)} \end{bmatrix}$$

The associated ODE

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{S}\,\mathbf{u}(t) = \mathbf{f}(t)$$

describes the motion of unconnected and unbeared beams. In order to get the the equation of motion of the framework constraints have to be added. See the next page.

The equation of motion for the discetized framework is

$$\underbrace{\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}}_{M_e} \begin{bmatrix} \ddot{\mathbf{u}}(t) \\ \ddot{\nu}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} S & C \\ C^\top & 0 \end{bmatrix}}_{S_e} \begin{bmatrix} \mathbf{u}(t) \\ \nu(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{a}(t) \end{bmatrix},$$

where

$\mathbf{u}(t)$ is the displacement vector,

$\nu(t)$ is the vector of constraint (reactive) forces and momenta,

$\mathbf{f}(t)$ is the vector of external forces and momenta,

$\mathbf{a}(t)$ is the vector of values of constraints (bearings etc.).

Assignment.

1. Create your own little framework.
   To keep things simple you might model each beam
   by just one finite element.
2. The longitudinal equation of motion for each beam is the standard wave equation of second order. You might use usual hat functions for discretization. For only one element these are linear functions.
3. Perform the same steps as for a single beam: first the static case, then the dynamic one with Newmark method and eigenvalue method.

The following pages are about an old concept and not relevant anymore. You may ignore this material.

Let N be the length (=number of entries) of the vector $\mathbf{u}$, which contains the data of all the beams, $\mathbf{u} \in \mathbb{R}^N$. Among all vectors in $\mathbb{R}^N$ only those represent a possible configuration of the framework which satisfy the condition

$$\mathbf{C}^T\mathbf{u} = \mathbf{0}.$$

These vectors form a subspace $\mathcal{U}$ of $\mathbb{R}^N$:

$$\mathcal{U} := \{ \, \mathbf{u} \in \mathbb{R}^N \mid \mathbf{C}^T\mathbf{u} = 0 \, \}.$$

With this notation we have

$$\mathbf{u} \text{ represents a possible configuation} \quad \Leftrightarrow \quad \mathbf{u} \in \mathcal{U}.$$

The subspace $\mathcal{U}$ has an orthogonal complement $\mathcal{U}^\perp$. This is again a subspace. It contains all vectors of $\mathbb{R}^N$, which are orthogonal to all vectors in $\mathcal{U}$:

$$\mathcal{U}^\perp = \{ \, \mathbf{v} \in \mathbb{R}^N \mid \mathbf{v}^T\mathbf{u} = 0 \text{ für alle } \mathbf{u} \in \mathbf{U} \, \}.$$

Each vextor $\mathbf{x} \in \mathbb{R}^N$ has a decomposition

$$\mathbf{x} = \mathbf{u}_\mathbf{x} + \mathbf{v}_\mathbf{x} \qquad \text{with} \quad \mathbf{u}_\mathbf{x} \in \mathcal{U}, \quad \mathbf{v}_\mathbf{x} \in \mathcal{U}^\perp.$$

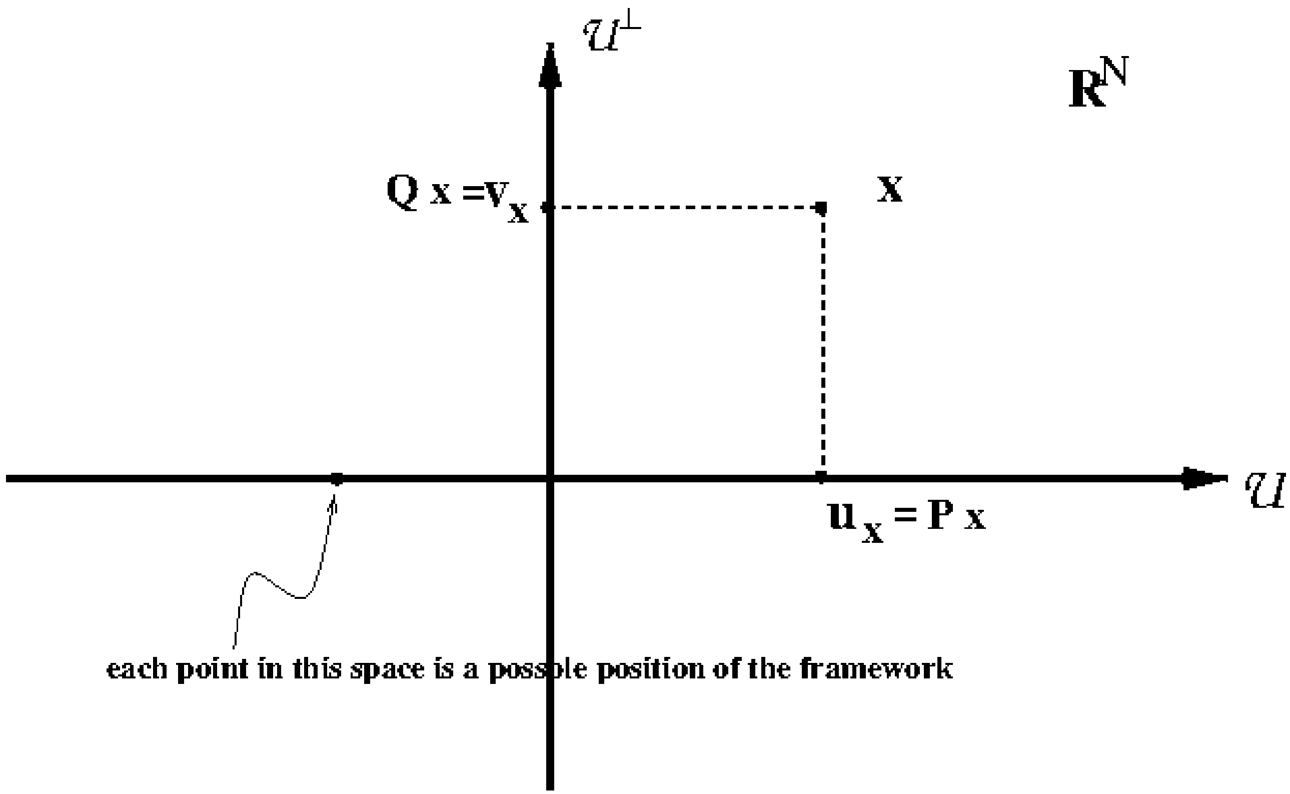
Only the part $\mathbf{u}_\mathbf{x}$ describes a possible configuration of the framework. The vector $\mathbf{u}_\mathbf{x}$ is called the orthogonal projection of $\mathbf{x}$ onto the subspace$\mathcal{U}$. The vector $\mathbf{v}_\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto the subspace $\mathcal{U}^\perp$. Both vectors can be computed via the projection matrices

$$\mathbf{P} = \mathbf{I} - \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T, \qquad \mathbf{Q} = \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T.$$

We have

$$\mathbf{u}_\mathbf{x} = \mathbf{P}\mathbf{x}, \qquad \mathbf{v}_\mathbf{x} = \mathbf{Q}\mathbf{x}.$$

Figure: the geometric situation



The projection matrices $\mathbf{P} = \mathbf{I} - \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T$ and $\mathbf{Q} = \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T$ satisfy the conditions below.

(1) $\mathbf{P} + \mathbf{Q} = \mathbf{I}, \;\; \mathbf{P}^2 = \mathbf{P}, \;\; \mathbf{Q}^2 = \mathbf{Q}, \;\; \mathbf{PQ} = \mathbf{QP} = \mathbf{0}, \;\; \mathbf{P}^T = \mathbf{P}, \;\; \mathbf{Q}^T = \mathbf{Q}.$

(2) $\mathbf{Px} \in \mathcal{U}$ für alle $\mathbf{x} \in \mathbb{R}^N$.

(3) $\mathbf{Pu} = \mathbf{u} \;\; \Leftrightarrow \;\; \mathbf{u} \in \mathcal{U} \;\; \Leftrightarrow \;\; \mathbf{C}^T\mathbf{u} = \mathbf{0}.$

Proof of (2):

$$\mathbf{C}^T(\mathbf{Px}) = \mathbf{C}^T(\mathbf{I} - \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T)\mathbf{x} = \mathbf{C}^T\mathbf{x} - \underbrace{\mathbf{C}^T\mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}}_{=\mathbf{I}}\mathbf{C}^T\mathbf{x} = \mathbf{0} \;\; \Rightarrow \;\; \mathbf{x} \in \mathcal{U}.$$

Construction of the mass and the stiffness matrix II

The equation for the beared framework are

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{S}\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{z}(t), \qquad \mathbf{C}^T\mathbf{u}(t) = 0.$$

$\mathbf{z}(t)$ is the vector of constraining forces an moments which cause the linking and the bearing. The next goal is to eliminate $\mathbf{z}(t)$ from the equations. First, we notice that the condition $\mathbf{C}^T\mathbf{u}(t) = 0$ can be replaced with $\mathbf{P}\mathbf{u}(t) = \mathbf{u}(t)$, where $\mathbf{P} = \mathbf{I} - \mathbf{C}(\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T$ is the associated projection matrix. Thus, the equations of motion can be rewritten as

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{S}\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{z}(t), \qquad \mathbf{P}\mathbf{u}(t) = \mathbf{u}(t).$$

Now, the tool for the elimination of $\mathbf{z}(t)$ is the **Principl of virtual work**. It states that constraining forces do not perform virtual work:

$$\mathbf{z}(t)^T\,\delta\mathbf{u} = 0 \quad \text{for all } \delta\mathbf{u} \in \mathcal{U}. \qquad (*)$$

Here, the vector $\delta\mathbf{u} \in \mathcal{U}$ denotes a virtual displacement. The latter is any displacement which respects the geometric constraints (in linear approximation). The inner product $\mathbf{z}(t)^T\,\delta\mathbf{u}$ is the virtual work. Now, $(*)$ implies that $\mathbf{z}(t) \in \mathcal{U}^\perp$. Hence,

$$\mathbf{P}\,\mathbf{z}(t) = 0.$$

Construction of the mass and the stiffness matrix III

The equation for the beared framework are now

$$\mathbf{M}\,\ddot{\mathbf{u}}(t) + \mathbf{S}\,\mathbf{u}(t) = \mathbf{f}(t) + \mathbf{z}(t), \qquad \mathbf{P}\mathbf{u}(t) = \mathbf{u}(t), \qquad \mathbf{P}\,\mathbf{z}(t) = 0.$$

This ODE is multiplied with $\mathbf{P}$ from the left. Then the constraint forces vanish because of $\mathbf{P}\,\mathbf{z}(t) = 0$, and we obtain

$$\mathbf{PM}\,\ddot{\mathbf{u}}(t) + \mathbf{PS}\,\mathbf{u}(t) = \mathbf{P}\,\mathbf{f}(t).$$

Replace now $\mathbf{u}(t)$ by $\mathbf{Pu}(t)$ to get

$$\mathbf{PMP}\,\ddot{\mathbf{u}}(t) + \mathbf{PSP}\,\mathbf{u}(t) = \mathbf{P}\,\mathbf{f}(t). \qquad (**)$$

By assumption, $\mathbf{Pu}(t) = \mathbf{u}(t)$, thus $\mathbf{u}(t) \in \mathcal{U}$ and $\mathbf{Qu}(t) = 0$. Hence, the solutions do not change if one adds in $(**)$ zu den Matrizen to the matrices an arbitrary multiple of $\mathbf{Q}$:

$$\underbrace{(\mathbf{PMP} + \alpha\,\mathbf{Q})}_{=:\mathbf{M}_*}\,\ddot{\mathbf{u}}(t) + \underbrace{(\mathbf{PSP} + \beta\,\mathbf{Q})}_{=:\mathbf{S}_*}\,\mathbf{u}(t) = \mathbf{P}\,\mathbf{f}(t). \qquad (**)$$

The matrices $\mathbf{M}_*$ and $\mathbf{S}_*$ are the mass and stiffness matrix of the beared framework. A solution of $(**)$ with initial values in $\mathcal{U}$, stays in $\mathcal{U}$ for all time. A numerical solution (e.g. with Newmark) may leave $\mathcal{U}$. This can be corrected by multiplying with $\mathbf{P}$ again.

The terms $\alpha\,\mathbf{Q}$ and $\beta\,\mathbf{Q}$, $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ are needed to mak $\mathbf{M}_*$ and $\mathbf{S}_*$ invertible, so that explicit numerical methods can be applied.