76.)

i_n							
j_n			X		X	X	
k_n	X						
l_n			X				X

96.) Let $S \subseteq \mathbb{N}$, and consider the cases of S:

Case $S = \mathbb{N}$: Let $x_n = 1$. Since $1 \geq 1$, the friends of x_n are $\mathbb{N}$, and thus S.

Case $S \subset \mathbb{N}$: Let the sequence x_n be defined as follows:

$$x_n = \begin{cases} 1 & n \in S \\ -\frac{1}{n} & n \notin S \end{cases}$$

Since $1 > -\dfrac{1}{n}$ for all $n \in \mathbb{N}$, all n for which $x_n = 1$ are friends of x_n. In addition, for all $m, n \in \mathbb{N}$:

$$m > n \implies \frac{1}{n} > \frac{1}{m} \implies -\frac{1}{n} < -\frac{1}{m}$$

Thus all n for which $x_n = -\dfrac{1}{n}$ cannot be friends of x_n, thus $x_n = 1$ for all friends n of x_n, thusly all $n \in S$ are friends, thus S is the set of all friends of x_n, thus for all $S \subseteq \mathbb{N}$, there exists a sequence such that S is the set of friends of that sequence. Q.E.D.

97.) Let $x_n \to L$, then by theorem 19, $y_k \to L$ for all subsequences y_k of x_n. Similarly, since

99.) Let $x_n = n - (-1)^n n$. x_n is unbounded, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, thus $y_k \preceq x_n$ and $y_k \to 0$.

100.) Every cauchy sequence is convergent according to theorem 23, and no convergent sequence can be unbounded.

103.) Let $x_n = \dfrac{1}{n^2}$. Since $x_n \to 0$, x_n is convergent and thus cauchy. Q.E.D.

105.) Since x_n and y_n are cauchy, there exist $A, B \in \mathbb{R}$ where $x_n \to A$ and $y_n \to B$. Since $y_n \neq 0$ for all $n \in \mathbb{N}$, $B \neq 0$. Let $z_n = x_n \,/\, y_n$. According to theorem 14, $z_n = x_n \,/\, y_n \implies z_n \to A \,/\, B$, thus z_n is convergent and thus cauchy. Q.E.D.

127.) For $\lim\limits_{x\to 5} x^2 = 25$, then given $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x-5| < \delta \implies \left|x^2 - 25\right| < \varepsilon$$

Suppose $|x-5| < 1$, then $|x+5| < 11$, thus

$$\left|x^2 - 25\right| = |x-5|\,|x+5| < 11\,|x-5| < 11\delta$$

$$11\delta = \varepsilon \implies \delta = \frac{\varepsilon}{11}$$

Let $\delta < \min\left(1, \frac{\varepsilon}{11}\right)$:

$$|x-5| < \delta \implies |x-5| < \frac{\varepsilon}{11} \implies 11\,|x-5| < \varepsilon \implies |x-5|\,|x+5| < 11\,|x-5| < \varepsilon$$

$$\implies |x-5|\,|x+5| = \left|x^2-25\right| < \varepsilon$$

Thus $\lim\limits_{x\to 5} x^2 = 25$. Q.E.D.

128.) For $\lim\limits_{x\to \frac{1}{2}} \frac{1}{x} = 2$, then given $\varepsilon > 0$, there must exist $\delta > 0$ where

$$\left|x - \frac{1}{2}\right| < \delta \implies \left|\frac{1}{x} - 2\right| < \varepsilon$$

Suppose $\left|x - \frac{1}{2}\right| < \frac{1}{4}$:

$$\left|x - \frac{1}{2}\right| < \frac{1}{4} \implies -\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4} \implies \frac{1}{4} < x < \frac{3}{4} \implies \frac{4}{3} < \frac{1}{|x|} < 4$$

$$\implies \frac{\left|x - \frac{1}{2}\right|}{|x|} < 4\left|x - \frac{1}{2}\right| < 4\delta$$

$$4\delta = \varepsilon \implies \delta = \frac{\varepsilon}{4}$$

Let $\delta < \min\left(\frac{1}{4}, \frac{\varepsilon}{4}\right)$:

$$\left|x - \frac{1}{2}\right| < \delta \implies \left|x - \frac{1}{2}\right| < \frac{\varepsilon}{4} \implies 4\left|x - \frac{1}{2}\right| < \varepsilon \implies \frac{\left|x - \frac{1}{2}\right|}{|x|} < 4\left|x - \frac{1}{2}\right| < \varepsilon$$

$$\frac{\left|x - \frac{1}{2}\right|}{|x|} = \left|\frac{x - \frac{1}{2}}{x}\right| = 1-$$