Chapter 8

3.) Let G and H be groups with respective identity elements e_G and e_H. Let $f : G \to G \bigoplus \{e_H\}$ where $g \mapsto (g, e_H)$. We can show that f is bijective. Let $g_1, g_2 \in G$ where $f(g_1) = f(g_2)$, thus $(g_1, e_H) = (g_2, e_H)$, thus $g_1 = g_2$, thus f is injective. Next, for all $(g, e_H) \in G \bigoplus \{e_H\}$, $f(g) = (g, e_H)$, thus f is surjective, and thus bijective. Finally, let $g_1, g_2 \in G$, thus $f(g_1g_2) = (g_1g_2, e_H) = (g_1, e_H)(g_2, e_H) = f(g_1)f(g_2)$, thus f is an isomorphism from G to $G \bigoplus \{e_H\}$, thus $G \cong G \bigoplus \{e_H\}$. A similar argument shows that $h \mapsto (h, e_G)$ is an isomorphism from H to $H \bigoplus \{e_G\}$, thus $H \cong H \bigoplus \{e_G\}$. $\blacksquare$

6.) Consider $(1, 1) \in \mathbb{Z}_8 \bigoplus \mathbb{Z}_2$. We can see that the order of this element is $\text{lcm}(|1|, |1|) = \text{lcm}(7, 1) = 7$. However, it is clear that there are no elements with order 7 in $\mathbb{Z}_4 \bigoplus \mathbb{Z}_4$, thus $\mathbb{Z}_8 \bigoplus \mathbb{Z}_2 \ncong \mathbb{Z}_4 \bigoplus \mathbb{Z}_4$. $\blacksquare$

14.) Given D_n, we know that the order of the rotation subgroup R is n, and the order of the reflection subgroup S is 2. Because both of these numbers divide $2n$, which is the order of D_n, we know that $R \bigoplus S \ncong D_n$. $\blacksquare$

20.) Since $4 \mid 12$ and $9 \mid 18$, we can find a subgroup of $\mathbb{Z}_{12} \bigoplus \mathbb{Z}_{18}$ that is isomorphic to $\mathbb{Z}_9 \bigoplus \mathbb{Z}_4$. We can see that $\langle 3 \rangle$ is a subgroup of $\mathbb{Z}_{12}$ with order 4, and also that $\langle 2 \rangle$ is a subgroup of $\mathbb{Z}_{18}$ with order 9, thus $\langle 2 \rangle \bigoplus \langle 3 \rangle \cong \mathbb{Z}_9 \bigoplus \mathbb{Z}_4$. $\blacksquare$

55.) Consider $(a, b) \in \mathbb{Z}_m \bigoplus \mathbb{Z}_n$. Since $|(a, b)| = \text{lcm}(|a|, |b|)$, and since $|a| \mid m$ and $|b| \mid n$, we know that $\text{lcm}(|a|, |b|) \mid \text{lcm}(m, n)$, thus $|(a, b)| \mid \text{lcm}(m, n)$. $\blacksquare$