Let $\mathcal{C}$ denote the Cantor set and $\mathcal{C}_k$ denote the k^{th} set in the construction of $\mathcal{C}$.

1) $\mathcal{C}$ is totally disconnected and perfect.

Proof: Fix $x, y \in \mathcal{C}$ where $x \neq y$. Since $1/3^k \to 0$ as $k \to \infty$, we can fix a nonnegative integer k where $1/3^k < |x - y|$, thus x and y must lie in disjoint intervals in $\mathcal{C}_k \supset \mathcal{C}$ which shows that $\mathcal{C}$ is totally disconnected.

Next, fix $x \in \mathcal{C}$, then $x \in \mathcal{C}_k$ for all nonnegative k, and thus for each k there exists an integer n_k where $[n_k/3^k, (n_k + 1)/3^k] \subset \mathcal{C}_k$ and

$$\frac{n_k}{3^k} \leq x \leq \frac{n_k + 1}{3^k}.$$

Letting $k \to \infty$, we have that $(n_k + 1)/3^k - n_k/3^k = 1/3^k \to 0$, but since

$$0 \leq x - \frac{n_k}{3^k} \leq \frac{n_k + 1}{3^k} - \frac{n_k}{3^k} = \frac{1}{3^k},$$

we have that $x - n_k/3^k = \left|x - n_k/3^k\right| \to 0$. Since $n_k/3^k \in \mathcal{C}$, we have elements in $\mathcal{C}$ that are arbitrarily close to x, thus x is not an isolated point and $\mathcal{C}$ is perfect. ■

2a) We have that $x \in \mathcal{C}$ if and only if

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where $a_k = 0$ or $a_k = 2$.

Proof: Fix $x \in \mathcal{C}$.

12)

14)

15)