Let $E \subseteq \mathbb{R}^d$, $r \in \mathbb{R}$, and $\delta = (\delta_1, \delta_2, \dots, \delta_d) \in \mathbb{R}^d$, then we define the following sets:

$$rE = \{(rx_1, rx_2, \dots, rx_d) : (x_1, x_2, \dots, x_d) \in E\}$$
$$\delta E = \{(\delta_1 x_1, \delta_2 x_2, \dots, \delta_d x_d) : (x_1, x_2, \dots, x_d) \in E\}$$

6) Given an open ball $B \subset \mathbb{R}^d$ centered at x with radius r, we have that $m(B) = r^d m(B_1)$, where B_1 is the unit open ball centered at the origin.

Proof: Let $x \in B_1$, then denoting 0 as the origin we have that $d(x, 0) < 1$, and since $rd(x, 0) = d(rx, 0) < r$, we see that rB_1 is equal to the open ball of radius r centered at the origin. From here it is clear that $B = rB_1 + x$. By translation invariance, we can see that $m(rB_1 + x) = m(rB_1)$, and by the dilation property, we have that $m(rB_1) = r^d m(B_1)$, thus $m(B) = m(rB_1 + x) = r^d m(B_1)$. ■

Lemma 1: Fix a subset $A \subset \mathbb{R}$ and $r > 0$, then $\ell = \inf(A)$ if and only if $r\ell = \inf(rA)$.

Proof: We have that ℓ is a lower bound of A, thus $\ell \leq a$ for all $a \in A$. Since $r > 0$, we also have that $r\ell \leq ra$ for all $ra \in rA$. Now, let k be a lower bound of A, then $k \leq \ell$ by definition, thus $rk \leq r\ell$. Since rk is a lower bound of rA, we have that $r\ell = \inf(rA)$.

The converse is obtained by taking $A' = rA$ and $r' = 1/r$. ■

7) Let $\delta = (\delta_1, \delta_2, \dots, \delta_d) \in \mathbb{R}^d$ with $\delta_i > 0$ and fix a measurable set $E \subseteq \mathbb{R}^d$, then δE is measurable and $m(\delta E) = \delta_1 \delta_2 \cdots \delta_d m(E)$.

Proof: Fix a closed cube $Q = [n_1, n_1'] \times \cdots \times [n_d, n_d'] \subset \mathbb{R}^d$ and let $x = (x_1, \dots, x_d) \in Q$, then $n_i \leq x_i \leq n_i'$ for $1 \leq i \leq d$. Since $\delta_i > 0$, we have that $\delta_i n_i \leq \delta_i x_i \leq \delta_i n_i'$, thus $\delta x \in [\delta_1 n_1, \delta_1 n_1'] \times \cdots \times [\delta_d n_d, \delta_d n_d'] = \delta Q$, and thus

$$|\delta Q| = \prod_{i=1}^{d} |\delta_i n_i - \delta_i n_i'| = \prod_{i=1}^{d} |\delta_i| \, |n_i - n_i'| = \delta_1 \cdots \delta_d \prod_{i=1}^{d} |n_i - n_i'| = \delta_1 \cdots \delta_d \, |Q| \, .$$

For any collection $\mathcal{Q}$ of closed cubes, denote $\delta\mathcal{Q}$ as the set $\{\delta Q : Q \in \mathcal{Q}\}$ and denote δ^{-1} as $(\delta_1^{-1}, \dots, \delta_d^{-1}) \in \mathbb{R}^d$. If $\mathcal{Q}$ covers E, then we have for all $x \in E$ that x is contained in some cube $Q \in \mathcal{Q}$, thus δx is contained in δQ and $\delta\mathcal{Q}$ covers δE. Similarly, since $E = \delta^{-1}\delta E$, if $\mathcal{Q}'$ covers δE then $\delta^{-1}\mathcal{Q}'$ covers E. Thus by **Lemma 1** we have

$$\delta_1 \cdots \delta_d m_*(E) = \delta_1 \cdots \delta_d \inf \sum_{Q \in \mathcal{Q}} |Q| = \inf \sum_{Q \in \mathcal{Q}} \delta_1 \cdots \delta_d \, |Q| = \inf \sum_{Q \in \mathcal{Q}} |\delta Q| \geq m_*(\delta E)$$

and

$$m_*(\delta E) = \inf \sum_{Q \in \mathcal{Q}'} |Q| = \inf \sum_{Q \in \mathcal{Q}'} \delta_1 \cdots \delta_d \left|\delta^{-1} Q\right| = \delta_1 \cdots \delta_d \inf \sum_{Q \in \mathcal{Q}'} \left|\delta^{-1} Q\right| \geq \delta_1 \cdots \delta_d m_*(E),$$

thus we have that $m_*(\delta E) = \delta_1 \cdots \delta_d m_*(E)$ for all $E \subseteq \mathbb{R}^d$. Fix $\varepsilon > 0$ and an open set O where $E \subseteq O$ and $m_*(O \setminus E) < \varepsilon / (\delta_1 \cdots \delta_d)$. Note that $\delta E \subseteq \delta O$ and that δO is open. [1] We have that $m_*(\delta O \setminus \delta E) = m_*(\delta(O \setminus E)) = \delta_1 \cdots \delta_d m_*(O \setminus E) < \varepsilon$, thus δE is measurable. ■

References

[1] https://proofwiki.org/wiki/Dilation_of_Open_Set_in_Topological_Vector_Space_is_Open. Accessed on 9/25/24.