2018) **Problem:** Find all $(a, b) \in \mathbb{N}^2$ where

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}$$

Hints: Find a way to factor the equation such that each factor correlates to a prime factor of some number.

Solution: $(674, 340033)$, $(2018, 1009)$, and $(673, 1358114)$, plus their reverse pairs.

Proof: Clearing denominators, we obtain

$$2018a + 2018b = 3ab \implies 3ab - 2018a - 2018b = 0$$

$$\implies 9ab - 3(2018)a - 3(2018)b = 0\,.$$

Adding 2018^2 to both sides, we can factor the equation as

$$(3a - 2018)(3b - 2018) = 2018^2 = 2^2 1009^2\,.$$

Note that $2018 \equiv 1 \pmod 3$, thus $3a - 2018 \equiv 3b - 2018 \equiv 1 \pmod 3$. The possible factorizations of 2018^2 are $(2)(2 \times 1009^2)$, $(2^2)(1009^2)$, $(2^2 \times 1009)(1009)$, and $(1)(2^2 \times 1009^2)$, but since $2 \not\equiv 1 \pmod 3$, we know that neither $3a - 2018$ nor $3b - 2018$ are equal to it, so we can ignore that factorization. Finally, using the remaining three factorizations, we can solve the following equations for a and b as follows:

$$\begin{aligned} 3a - 2018 &= 2^2, & 3b - 2018 &= 1009^2 \\ 3a - 2018 &= 2^2 \times 1009, & 3b - 2018 &= 1009 \\ 3a - 2018 &= 1, & 3b - 2018 &= 2^2 \times 1009^2 \end{aligned}$$

Thus we obtain $(674, 340033)$, $(2018, 1009)$, and $(673, 1358114)$, as solutions. ■

2017) **Problem:** Let $S \subseteq \mathbb{N}$ be the smallest set where $2 \in S$, $n^2 \in S \implies n \in S$, and $n \in S \implies (n+5)^2 \in S$. Which positive integers are not in S?

Solution: 1 and all n where $n \equiv 0 \pmod 5$.

Proof: It is clear that $1 \notin S$.

2008) **Problem:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function where $f(x, y) + f(y, z) + f(z, x) = 0$ for all $x, y, z \in \mathbb{R}$. Show that $f(x, y) = g(x) - g(y)$ for some function $g : \mathbb{R} \to \mathbb{R}$.

Hints: Since the defining property of f holds for all choices of x, y, and z, any property of f that we derive by fixing some or all of the variables will also hold for all choices.

Solution: Fix $z \in \mathbb{R}$, then $g(x) = f(x, z)$

Proof: Suppose $x = y = z$, then $f(x, y) + f(y, z) + f(z, x) = 3f(x, x) = 0$, thus $f(x, x) = 0$ for all $x \in \mathbb{R}$. Next, suppose $x = z$ and y is free, then $f(x, y) + f(y, z) +$

$f(z,x) = f(x,y) + f(y,x) + f(x,x) = f(x,y) + f(y,z) = 0 \implies f(x,y) = -f(x,y)$ for all $x, y \in \mathbb{R}$. Finally, fix $z \in \mathbb{R}$ and let $g(x) = f(x,z)$, then $f(x,y) + f(y,z) + f(z,x) = f(x,y) + f(y,z) - f(x,z) = f(x,y) + g(y) - g(x) = 0 \implies f(x,y) = g(x) - g(y)$. ■

1999) **Problem:** Find polynomials $f(x)$, $g(x)$, and $h(x)$ where

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & x < -1 \\ 3x + 2 & -1 \leq x \leq 0 \\ 2 - 2x & x > 0 \end{cases}$$

Hints:

Solution:

Proof: Consider the functions $p(x)$ and $q(x)$ defined as follows: