3.3 The Isomorphism Theorems

1) Fix any $c \in \mathbb{F}_q$ where $c \neq 0$, then we have for all $M \in \mathrm{SL}_n(\mathbb{F}_q)$ that $cM \in \mathrm{GL}_n(\mathbb{F}_q)$ and $c^{-1}cM \in \mathrm{SL}_n(\mathbb{F}_q)$, which shows a bijection between the elements in $\mathrm{SL}_n(\mathbb{F}_q)$ and $c \cdot \mathrm{SL}_n(\mathbb{F}_q)$. We also have that if $c_1, c_2 \in \mathbb{F}_q$ are distinct, then $\det(c_1 M) \neq \det(c_2 M)$, thus since $\mathbb{F}_q$ has $q-1$ non-zero choices for c, we have that $|\mathrm{GL}_n(\mathbb{F}_q)/\mathrm{SL}_n(\mathbb{F}_q)| = q-1$. ■

3) Since $H \trianglelefteq G$, we have that $N_G(H) = G$, thus any subgroup K of G is a subgroup of $N_G(H)$, thus by the second isomorphism theorem, we have that $KH \leq G$ and $H \trianglelefteq KH$, and by Lagrange's theorem we have $|G : H| = |G : KH|\,|KH : H|$. Since $|G : H| = p$ is prime, we must have that $|G : KH|$ is p or 1. If it is p, then $|KH : H| = 1$ which shows that $KH = H$, thus $K \leq H$. Otherwise, if $|G : KH| = 1$, then $|KH : H| = p$, and again by the second isomorphism theorem we have $K/K \cap H \cong KH/H$ which implies $|K : K \cap H| = |KH : H| = p$, and we are finished. ■

4) Given groups A and B, fix normal subgroups $C \trianglelefteq A$ and $D \trianglelefteq B$. Clearly $C \times D \leq A \times B$. Let $(c, d) \in C \times D$ and $(a, b) \in A \times B$, then we have that $aca^{-1} \in C$ and $bdb^{-1} \in D$. Since $(a, b)^{-1} = (a^{-1}, b^{-1})$, we have that $(a, b) \circ (c, d) \circ (a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}) \in C \times D$, thus showing that $C \times D \trianglelefteq A \times B$.

Since $C \times D$ is normal, it is the kernel of some homomorphism $\phi : A \times B \to (A \times B)/(C \times D)$. By the first isomorphism theorem, we have that $(A \times B)/\ker\phi \cong \phi(A \times B)$. For each $(aC, bD) \in (A/C) \times (B/D)$, we have that $\phi((a, b)) = (a, b)(C \times D) = (aC, bD)$, thus ϕ is surjective and $\phi(A \times B) = (A/C) \times (B/D)$, showing that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.
■

3.4 Composition Series and the Hölder Program

1) Let G be an abelian simple group and $x \in G$ be a non-identity element. We must have that $\langle x \rangle = G$, otherwise it would be a normal subgroup that is both proper and non-trivial, and thus G would not be simple. Now, suppose $|G|$ is infinite, then $\langle x^n \rangle$ is a proper normal subgroup for any $n \in \mathbb{Z}$, thus $|G|$ must be finite. Finally, if $n > 1$ is a proper divisor of $|G|$, then $\langle x^n \rangle$ is a proper normal subgroup, thus $|G|$ must have no proper divisors > 1, thus $|G|$ is prime. Since G is also cyclic, we have that $G \cong \mathbb{Z}/p\mathbb{Z}$. ■

2) By the subgroup lattice for D_8 provided in the text, we have that the decomposition series for D_8 are as follows:

$$1 \trianglelefteq \langle s \rangle \trianglelefteq \left\langle s, r^2 \right\rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \left\langle sr^2 \right\rangle \trianglelefteq \left\langle s, r^2 \right\rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \left\langle r^2 \right\rangle \trianglelefteq \left\langle s, r^2 \right\rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \left\langle r^2 \right\rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \left\langle r^2 \right\rangle \trianglelefteq \left\langle rs, r^2 \right\rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle rs \rangle \trianglelefteq \left\langle rs, r^2 \right\rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle sr \rangle \trianglelefteq \left\langle rs, r^2 \right\rangle \trianglelefteq D_8$$

4) Fix a finite abelian group G. If $|G| = 1$, we have that $\{e\} \leq \{e\}$, thus the base case holds. Now fix a positive integer n and suppose the proposition holds for all groups where $|G| < n$, and let k be a divisor of n. If k is prime, then by Cauchy's theorem there exists $x \in G$ with $|x| = k$, and thus $|\langle x\rangle| = k$. Otherwise, some prime $p < k$ divides k, and again we can choose $x \in G$ with $|\langle x\rangle| = p$. Since k/p divides $|G/\langle x\rangle|$, we can use the induction hypothesis and the fourth isomorphism theorem to obtain a subgroup $H/\langle x\rangle \leq G/\langle x\rangle$ with order k/p and $H \leq G$. By Lagrange's theorem we find that $k/p = |H/\langle x\rangle| = |H|/p$, thus $|H| = k$ and we are finished. ■

3.5 Transpositions and the Alternating Group

1) σ is even and τ is odd, thus $\sigma\tau$ is odd.

3) For any $\sigma \in S_n$, we can take its decomposition into transpositions, thus without loss of generality we can take σ to be a transposition in S_n, say $(i\ j)$ for integers $1 \leq i < j \leq n$. Let $\sigma' \in S_n$ be defined as $(i\ i+1)(i+1\ i+2)\cdots(j-1\ j)(j-2\ j-1)\cdots(i\ i+1)$. We have that $\sigma'(i)$ brings i to $i+1$, then $i+2$, until we get to j, then doesn't touch j for the rest of the transpositions, thus $\sigma'(i) = j$. Likewise, j isn't touched until $(j-1\ j)$, and then cascades through the rest of the transpositions until it reaches i, thus $\sigma'(j) = i$ and $\sigma = \sigma'$. We can repeat this process for each transposition in a general element of S_n, thus we are finished. ■

9) We have that $\{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is the subgroup of A_4 with order 4. The mapping $e \mapsto (0,0)$, $(1\ 2)(3\ 4) \mapsto (0,1)$, $(1\ 3)(2\ 4) \mapsto (1,0)$, $(1\ 4)(2\ 3) \mapsto (1,1)$ is an isomorphism, so the subgroup is isomorphic to V_4.