Let $\mathcal{C}$ denote the Cantor set and $\mathcal{C}_k$ denote the k^{th} set in the construction of $\mathcal{C}$.

For any interval I, let $\bar{I}$ denote the closure of that interval.

Example 4) Fix a rectangle $R \subset \mathbb{R}^d$ and let M be the largest side length of R. Since R contains a constant number of $d-1$ dimensional "faces", say n of them, and since each face contains at least $(Mk)^{d-1}$ cubes, we have that $|\mathcal{Q}'| \leq (M^{d-1}n)k^{d-1}$, and thus $|\mathcal{Q}'| = O(k^{d-1})$. ■

1) $\mathcal{C}$ is totally disconnected and perfect.

Proof: Fix $x, y \in \mathcal{C}$ where $x \neq y$. Since $1/3^k \to 0$ as $k \to \infty$, we can fix a nonnegative integer k where $1/3^k < |x-y|$, thus x and y must lie in disjoint intervals in $\mathcal{C}_k \supset \mathcal{C}$ which shows that $\mathcal{C}$ is totally disconnected.

Next, fix $x \in \mathcal{C}$, then $x \in \mathcal{C}_k$ for all nonnegative k, and thus for each k there exists an integer n_k where $[n_k/3^k, (n_k+1)/3^k] \subset \mathcal{C}_k$ and

$$\frac{n_k}{3^k} \leq x \leq \frac{n_k+1}{3^k}.$$

Letting $k \to \infty$, we have that $(n_k+1)/3^k - n_k/3^k = 1/3^k \to 0$, but since

$$0 \leq x - \frac{n_k}{3^k} \leq \frac{n_k+1}{3^k} - \frac{n_k}{3^k} = \frac{1}{3^k},$$

we have that $x - n_k/3^k = \left|x - n_k/3^k\right| \to 0$. Since $n_k/3^k \in \mathcal{C}$, we have elements in $\mathcal{C}$ that are arbitrarily close to x, thus x is not an isolated point and $\mathcal{C}$ is perfect. ■

2a) We have that $x \in \mathcal{C}$ if and only if

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where $a_k = 0$ or $a_k = 2$ for all k.

Proof: Fix $x \in [0,1]$ and suppose x has a suitable ternary representation $\{a_k\}$. If $a_1 = 0$, then we have that $x \in [0, 1/3]$, as we have that

$$\sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3},$$

and thus the sum of the remaining digits cannot surpass $1/3$. The same argument shows that if $a_1 = 2$ then $x \in [2/3, 1]$, and thus $x \in \mathcal{C}_1$. Fix the interval I in $\mathcal{C}_1$ that contains x. Since $|I| = 1/3$, we can scale it by a factor of 3 to obtain an interval with length 1 and endpoints 0 and 1. Likewise, we can drop the a_1 term of x and multiply by 3 to obtain

$$\sum_{n=1}^{\infty} \frac{a_{n+1}}{3^n} = \frac{a_2}{3} + \frac{a_3}{3^2} + \cdots .$$

Applying the same argument as before, we have that $a_2 = 0$ or $a_2 = 2$ implies $x \in \mathcal{C}_2$. Repeating this process infinitely shows that $x \in \mathcal{C}_k$ for all k, and thus $x \in \mathcal{C}$.

Now suppose $x \in [0,1]$ does not have a ternary representation comprising only of 0 and 2, and let k be the smallest integer where $a_k = 1$. We previously established that $x' = \sum_{n=1}^{k-1} a_n/3^n$ exists in $\mathcal{C}$ since the rest of the digits in its ternary representation are 0, thus there exists an interval in $\mathcal{C}_{k-1}$ that contains x'. Since $a_k = 1$, we have that x is contained in the middle third of the interval in $\mathcal{C}_{k-1}$, and thus $x \notin \mathcal{C}_k$, and thus is not in $\mathcal{C}$. ■

2b) Show that the Cantor-Lebesgue function F is well-defined and continuous.

Not a proof, but if we can limit δ less than some negative power of 3, then given $x, y \in \mathcal{C}$ we can ensure that the first n digits of $F(x)$ and $F(y)$ are the same for arbitrarily large n, and thus we can ensure that $|F(x) - F(y)|$ is arbitrarily small.

2c) Show that F is surjective.

Proof: Fix $y \in [0,1]$, then it has a binary representation. Given $x \in \mathcal{C}$, we have that by definition $F(x)$ simply replaces each 2 in the ternary representation of x with a 1 and considers the number in binary, thus if we replace each 1 in the binary representation of y with a 2, we obtain $x \in \mathcal{C}$ where $F(x) = y$, and thus F is surjective. ■

12a) Open discs in $\mathbb{R}^2$ are not the union of disjoint open rectangles.

Proof: Special case of **12b**. ■

12b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Proof: Let $\Omega \subset \mathbb{R}^2$ be an open connected set, and fix an open rectangle R with

$$R = (x_1, x_2) \times (y_1, y_2) \subseteq \Omega.$$

If $\Omega = R$, then $\{R\}$ is a collection of disjoint open rectangles whose union is Ω, and we are finished. Now suppose $\Omega \neq R$, then since Ω is connected, there exists a point p where

$$p = (x, y) \in \Omega \cap \partial R.$$

Clearly $p \notin R$. Also, let R' be an open rectangle disjoint to R where

$$R' = (x_1', x_2') \times (y_1', y_2') \subseteq \Omega,$$

then one or both of the following is possible:

$$x_2' \leq x_1 \quad \text{or} \quad x_1' \leq x_2, \qquad y_2' \leq y_1 \quad \text{or} \quad y_1' \leq y_2.$$

Assume the former, then we have that $x \notin (x_2, x_2')$ since $x_1 \leq x \leq x_1'$, and thus $p \notin R'$. Assuming the latter, we find that $y \notin (y_2, y_2')$ since $y_1 \leq y \leq y_1'$, thus $p \notin R'$. Consequently, any collection of disjoint open rectangles that contains $R \subset \Omega$ has no element that contains $p \in \Omega$, thus Ω cannot be a union of disjoint open rectangles. ■

14a) Fix $E \subseteq \mathbb{R}$, then $J_*(E) = J_*(\bar{E})$.

Proof: Fix a set $E \subset \mathbb{R}$ and let $J_*(E) = M$ for some real M, then there exists a finite collection of intervals $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$ that covers E and whose summed length is M. Without loss of generality, we can assume $\mathcal{I}$ is disjoint.

If E is closed we are finished. Else, let $p \in \partial E$ where $p \notin E$, then there exists $\varepsilon > 0$ where either $(p, p+\varepsilon)$ or $(p-\varepsilon, p)$ is contained in E. Denote this interval I, then for some interval $I_k \in \mathcal{I}$ we have that $I \subseteq I_k$, thus $\bar{I} \subseteq \bar{I}_k$, and thus $p \in \bar{I} \subseteq \bar{I}_k$. Let $\bar{\mathcal{I}}$ be the collection of the closures of the intervals in $\mathcal{I}$, then for all $p \in \bar{E} = \partial E \cup E$, we have that p is contained in some interval in $\bar{\mathcal{I}}$, and thus $\bar{\mathcal{I}}$ covers $\bar{E}$. Since the summed lengths of the intervals in $\bar{\mathcal{I}}$ is M, we have that $J_*(\bar{E}) \leq M$.

Now, to establish a contradiction, suppose we have a collection of intervals $\mathcal{I}'$ that covers $\bar{E}$ whose summed length is $< M$, then it clearly covers E, but then $J_*(E) < M$ which is a contradiction. Thus, we have that $J_*(\bar{E}) = M$. ■

14b) There exists a countable subset $E \subset [0,1]$ where $J_*(E) = 1$ and $m_*(E) = 0$.

Proof: Let $E = [0,1] \cap \mathbb{Q}$. Since $\mathbb{Q}$ is countable, clearly E is too. Thus, we have that $m_*(E) = \sum_{x \in E} m_*(\{x\}) = 0$, since a single point is technically a closed cube with volume 0. Now, to establish a contradiction, suppose $J_*(E) < 1$, then we have some interval $I = [x,y] \subset [0,1]$ where $|I| > 0$ and $I \cap E = \varnothing$, but this is a contradiction since there exists a rational number between any two disinct real numbers, thus we must have that $J_*(E) = 1$ since $[0,1]$ covers E. ■

15) Defining the exterior measure using rectangles is equivalent to defining it using cubes.

Proof: Fix $E \subseteq \mathbb{R}^d$. Since any covering of E with cubes is also a covering with rectangles, we have that $m_*^{\mathcal{R}}(E) \leq m_*(E)$. Suppose a rectangle can be tiled using uniform cubes, then there must be an integer number of cubes on each side, and thus the ratio between any two sides of the rectangle must be rational. Denote such a rectangle as "tileable". Now, let $\{R_j\}$ be a covering of E using rectangles. If each R_j is tileable, then we have a covering of E with cubes and we are finished. Otherwise, fix $\varepsilon > 0$, then for each non-tileable R_j, we can ontain it in a tileable rectangle R_j' where $\left|R_j'\right| - \varepsilon/2^j \leq |R_j|$. This is possible because for each non-rational side length ratio, there exists an arbitrarily close rational number larger than it, and thus we can extend the sides so that their ratios are rational. Note that if R_j is tileable, then we can set $R_j = R_j'$. Thus, we have that

$$\sum_{j=1}^{\infty} |R_j| \geq \sum_{j=1}^{\infty} \left(\left|R_j'\right| - \frac{\varepsilon}{2^j} \right) = \sum_{j=1}^{\infty} \left|R_j'\right| - \varepsilon.$$

Letting $\varepsilon \to 0$, we have that

$$\sum_{j=1}^{\infty} \left|R_j'\right| \leq \sum_{j=1}^{\infty} |R_j| \,.$$

Since any covering of E with tileable rectangles is also a covering with cubes, we have that $m_*(E) \leq m_*^{\mathcal{R}}(E)$, and thus $m_*(E) = m_*^{\mathcal{R}}(E)$. ■