7.9.3 a.) Using the product rule, we can find $F'(x)$ and $G'(x)$ for all $x \neq 0$:

$$F'(x) = -\frac{3x^2}{x^4}\cos\left(\frac{1}{x^3}\right) + 2x\sin\left(\frac{1}{x^3}\right) = 2x\sin\left(\frac{1}{x^3}\right) - \frac{3}{x^2}\cos\left(\frac{1}{x^3}\right)$$

$$G'(x) = \frac{3x^2}{x^4}\sin\left(\frac{1}{x^3}\right) + 2x\cos\left(\frac{1}{x^3}\right) = \frac{3}{x^2}\sin\left(\frac{1}{x^3}\right) + 2x\cos\left(\frac{1}{x^3}\right)$$

Since $F'(x)$ and $G'(X)$ are defined for all $x \neq 0$, we know that F is differentiable on all $x \neq 0$. Next, consider $F'(0)$ and $G'(0)$:

$$F'(0) = \lim_{x\to 0}\frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin\left(\frac{1}{x^3} - 0\right)}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x^3}\right)$$

$$G'(0) = \lim_{x\to 0}\frac{G(x) - G(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\cos\left(\frac{1}{x^3}\right) - 0}{x} = \lim_{x\to 0} x\cos\left(\frac{1}{x^3}\right)$$

Since each term vanishes to 0 as $x \to 0$, we know that both limits are 0, thus both limits are defined, and thus F and G are differentiable on $\mathbb{R}$. ■

b.) Consider FG':

$$FG' = x^2\sin\left(\frac{1}{x^3}\right)\left(\frac{3}{x^2}\sin\left(\frac{1}{x^3}\right) + 2x\cos\left(\frac{1}{x^3}\right)\right)$$

$$= 3\sin^2\left(\frac{1}{x^3}\right) + 2x^3\sin\left(\frac{1}{x^3}\right)\cos\left(\frac{1}{x^3}\right)$$

It is clear that $3\sin^2\left(1/x^3\right)$ is bounded. As $x \to 0$, $2x^3\sin(1/x^3)\cos(1/x^3)$ vanishes to 0, and as $x \to \pm\infty$, $1/x^3 \to 0$, thus $\sin(1/x^3) \to 0$ and $\cos(1/x^3) \to 1$, thus $2x^3\sin(1/x^3)\cos(1/x^3)$ again vanishes to zero, and thus is bounded. Since the sum of two bounded functions is bounded, we know that FG' is bounded. Similarly, consider GF':

$$GF' = x^2\cos\left(\frac{1}{x^3}\right)\left(2x\sin\left(\frac{1}{x^3}\right) - \frac{3}{x^2}\cos\left(\frac{1}{x^3}\right)\right)$$

$$= 2x^3\sin\left(\frac{1}{x^3}\right)\cos\left(\frac{1}{x^3}\right) - 3\cos^2\left(\frac{1}{x^3}\right)$$

Again, it is clear that $-3\cos^2(1/x^3)$ is bounded. We also previously showed that $2x^3\sin(1/x^3)\cos(1/x^3)$ is bounded, thus GF' is bounded. ■

c.) Consider $F(0)G'(0) - F'(0)G(0)$:

$$F(0)G'(0) - F'(0)G(0) = F(0)(0) - (0)G(0) = 0$$

Next, let $x \neq 0$ and consider $F(x)G'(x) - F'(x)G(x)$:

$$F(x)G'(x) - F'(x)G(x)$$

$$= 3\sin^2\left(\frac{1}{x^3}\right) + 2x^3\sin\left(\frac{1}{x^3}\right)\cos\left(\frac{1}{x^3}\right) - 2x^3\sin\left(\frac{1}{x^3}\right)\cos\left(\frac{1}{x^3}\right) + 3\cos^2\left(\frac{1}{x^3}\right)$$

$$= 3\left(\sin^2\left(\frac{1}{x^3}\right) + \cos^2\left(\frac{1}{x^3}\right)\right) = 3(1) = 3$$

It follows that

$$F(x)G'(x) = F'(x)G(x) = \begin{cases} 3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\blacksquare$

d.) By looking at the graph of both FG' and GF', it is clear that the derivative at $x = 0$ does not exist, as the functions themselves are undefined at $x = 0$, thus neither FG' nor GF' are differentiable on $\mathbb{R}$. $\blacksquare$