7.12.5) Given $f(x) = \ln(1+x)$, we know that for $n > 0$,

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

thus

$$f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

From this, we can construct P_n and R_n at $c = 0$ as follows:

$$f(x) = P_n + R_n = \sum_{k=0}^{n}\left\{\frac{f^{(k)}(c)}{k!}(x-c)^k\right\} + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$$= \frac{\ln(1)}{0!}x^0 + \sum_{k=1}^{n}\left\{\frac{(-1)^{k-1}(k-1)!}{k!}x^k\right\} + (-1)^n\frac{n!}{(n+1)!(1+z)^{n+1}}x^{n+1}$$

$$= \sum_{k=1}^{n}\left\{\frac{(-1)^{k-1}}{k}x^k\right\} + \frac{(-1)^n}{n+1}\left(\frac{x}{1+z}\right)^{n+1}$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\left(\frac{x}{1+z}\right)^{n+1}$$

where $z \in (c, x)$, thus we obtain the desired equation for $f(x)$. To estimate R_n in the interval $[0, 1/10]$, we set $x = 1/10$, and pick $z \in [0, 1/10]$, say $z = 0.05$, thus

$$R_n = \frac{(-1)^n}{n+1}\left(\frac{1/10}{1+0.05}\right)^{n+1} \approx \frac{(-1)^n}{n+1}(0.095)^{n+1}$$

Given large n, this estimate becomes very small, thus we can conclude that P_n is a good approximation for f on the interval $[0, 1/10]$.

■

7.12.6) a.) Consider the derivative of $f(x) = e^{-\frac{1}{x^2}}$:

$$f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}$$

Thus we know that $e^{-\frac{1}{x^2}}$ is differentiable when $x \neq 0$, and that its derivative is of the form $p(1/x)e^{-\frac{1}{x^2}}$ where p is some polynomial in $1/x$. It is trivial to show that p is differentiable. Next, suppose

$$f^{(n)}(x) = p(1/x)e^{-\frac{1}{x^2}}$$

for some polynomial p in $1/x$, then we can find $f^{(n+1)}(x)$ as follows:

$$f^{(n+1)}(x) = \frac{d}{dx}p(1/x)e^{-\frac{1}{x^2}} = p'(1/x)e^{-\frac{1}{x^2}} + \frac{2}{x^3}p(1/x)e^{-\frac{1}{x^2}}$$

$$= e^{-\frac{1}{x^2}}\left(p'(1/x) + \frac{2}{x^3}p(1/x)\right) = q(1/x)e^{-\frac{1}{x^2}}$$

for a similar polynomial q. We have shown that $f^{(n)}$ is differentiable, and since $f^{(n)}$ and $f^{(n+1)}$ take the same form, we know that $f^{(n+1)}$ is also differentiable, thus by induction, $e^{-\frac{1}{x^2}}$ is infinitely differentiable.

To find $f^{(n)}(0)$, we can take the following limit:

$$\lim_{x \to 0} f^{(n)}(x) = \lim_{x \to 0} p(1/x) e^{-\frac{1}{x^2}} = \lim_{x \to 0} p(1/x) \lim_{x \to 0} e^{-\frac{1}{x^2}}$$

As $x \to 0$, we know that $p(1/x) \to \infty$ and $e^{-\frac{1}{x^2}} \to 0$, but as x gets closer to 0, $e^{-\frac{1}{x^2}}$ shrinks faster than $p(1/x)$ grows, thus the product tends to 0, thus the limit is 0, and thus $f^{(n)}(0) = 0$ for all n.

■

b.) We can construct P_n at $c = 0$ as follows:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{0}{k!} x^k = 0$$

Thus $P_n = 0$ for all n.

c.) Using Lagrange's form, we can construct R_n at $c = 0$ as follows:

$$R_n = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

Because we previously determined that $P_n = 0$ for all n, we know that this error term dominates for all n, thus we can conclude that P_n is not a good approximation for f.

■