

Exercises 15.1

15.)

$$\begin{aligned}\int_1^4 \int_0^2 (6x^2y - 2x) \, dy \, dx &= \int_1^4 [3x^2y^2 - 2xy]_0^2 \, dx = \int_1^4 12x^2 - 4x \, dx \\ &= [4x^3 - 2x^2]_1^4 = 256 - 32 - 4 + 2 = 222\end{aligned}$$

19.)

$$\begin{aligned}\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x) \, dx \, dy &= \int_{-3}^3 [xy + y^2 \sin x]_0^{\pi/2} = \int_{-3}^3 \frac{\pi}{2} y + y^2 \, dy \\ &= \left[\frac{\pi}{4} y^2 + \frac{y^3}{3} \right]_{-3}^3 = \frac{9\pi}{4} + 9 - \frac{9\pi}{4} + 9 = 18\end{aligned}$$

29.)

$$\begin{aligned}\iint_R \frac{xy^2}{x^2 + 1} \, dA &= \int_{-3}^3 \int_0^1 \frac{xy^2}{x^2 + 1} \, dx \, dy = \frac{1}{2} \int_{-3}^3 \int_0^1 \frac{y^2}{u} \, du \, dy = \frac{1}{2} \int_{-3}^3 [y^2 \ln |u|]_0^1 \\ &= \frac{1}{2} \int_{-3}^3 [y^2 \ln |x^2 + 1|]_0^1 = \frac{1}{2} \int_{-3}^3 [y^2 \ln 2] \, dy = \frac{\ln 2}{2} \left[\frac{y^3}{3} \right]_{-3}^3 = \frac{18 \ln 2}{2} = 9 \ln 2\end{aligned}$$

33.)

$$\begin{aligned}\iint_R ye^{-xy} \, dA &= \int_0^3 \int_0^2 ye^{-xy} \, dx \, dy = \int_0^3 [-e^{-xy} dy]_0^2 = \int_0^3 -e^{-2y} + e^0 \, dy \\ &= \int_0^3 1 - e^{-2y} \, dy = \left[y + \frac{e^{-2y}}{2} \right]_0^3 = 3 + \frac{e^{-6}}{2} - \frac{1}{2} = \frac{e^{-6} + 5}{2}\end{aligned}$$

41.) Find the limits:

$$-1 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Evaluate the integral:

$$\begin{aligned}\iint_R 1 + x^2ye^y \, dA &= \int_0^1 \int_{-1}^1 1 + x^2ye^y \, dx \, dy = \int_0^1 \left[x + \frac{1}{3}x^3ye^y \right]_{-1}^1 \, dy \\ &= \int_0^1 1 + \frac{1}{3}ye^y + 1 + \frac{1}{3}ye^y \, dy = \int_0^1 2 + \frac{2}{3}ye^y \, dy = \left[2y + \frac{2}{3}(ye^y - e^y) \right]_0^1 \\ &= 2 + \frac{2}{3} = \frac{8}{3}\end{aligned}$$

Exercises 15.2

3.)

$$\int_0^1 \int_0^y x e^{y^3} dx dy = \int_0^1 \left[\frac{x^2 e^{y^3}}{2} \right]_0^y dy = \int_0^1 \frac{y^2 e^{y^3}}{2} dy = \frac{1}{6} [e^u]_0^1 = \frac{1}{6} [e^{y^3}]_0^1 = \frac{e-1}{6}$$

13.)

$$\iint_D x dA = \int_0^1 \int_0^x x dy dx = \int_0^1 [xy]_0^x dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\iint_D x dA = \int_0^1 \int_y^1 x dx dy = \int_0^1 \left[\frac{x^2}{2} \right]_y^1 dy = \int_0^1 \frac{1}{2} - \frac{y^2}{2} dy = \left[\frac{y}{2} - \frac{y^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

15.)

$$\begin{aligned} \iint_D y dA &= \int_{-1}^2 \int_{y^2}^{y+2} y dx dy = \int_{-1}^2 [xy]_{y^2}^{y+2} dy = \int_{-1}^2 2y + y^2 - y^3 dy \\ &= \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4} \right]_{-1}^2 = 4 + \frac{8}{3} - \frac{16}{4} - 1 + \frac{1}{3} + \frac{1}{4} = 3 - 1 + \frac{1}{4} = \frac{9}{4} \end{aligned}$$

Integrating with respect to x first is easier as it prevents us from having to evaluate two double integrals.

17.)

$$\begin{aligned} \iint_D x \cos y dA &= \int_0^1 \int_0^{x^2} x \cos y dy dx = - \int_0^1 [x \sin y]_0^{x^2} dx = - \int_0^1 x \sin x^2 dx \\ &= - \frac{1}{2} \int_0^1 \sin u du = - \frac{1}{2} [\cos u]_0^1 = - \frac{1}{2} (\cos(1) - 1) = \frac{1 - \cos(1)}{2} \end{aligned}$$

23.)

$$\begin{aligned} \iint_D 3x + 2y dy dx &= \int_0^1 \int_{x^2}^{\sqrt{x}} 3x + 2y dy dx = \int_0^1 [3xy + y^2]_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 3x^{3/2} + x - 3x^3 - x^4 dx = \left[\frac{6x^{5/2}}{5} + \frac{x^2}{2} - \frac{3x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{6}{5} + \frac{1}{2} - \frac{3}{4} - \frac{1}{5} = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

45.)

$$\int_0^1 \int_0^y f(x, y) dx dy = \int_0^1 \int_x^1 f(x, y) dy dx$$

53.)

$$\begin{aligned} \int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} dy dx &= \int_0^1 \int_0^{y^2} \sqrt{y^3 + 1} dx dy = \int_0^1 \left[x \sqrt{y^3 + 1} \right]_0^{y^2} dy \\ &= \int_0^1 y^2 \sqrt{y^3 + 1} dy = \frac{1}{3} \int_0^1 \sqrt{u} du = \frac{1}{3} \left[\frac{2\sqrt{u^3}}{3} \right]_0^1 = \frac{1}{3} \left[\frac{2\sqrt{(y^3 + 1)^3}}{3} \right]_0^1 = \\ &= \frac{1}{3} \left[\frac{2\sqrt{8} - 2}{3} \right] = \frac{2(\sqrt{8} - 1)}{9} \end{aligned}$$

61.) Find area of region of integration:

$$A = \frac{1}{2}bh = \frac{1}{2}(1)(3) = \frac{3}{2}$$

Now evaluate the integral:

$$I = \int_0^1 \int_0^{3x} xy dy dx = \int_0^1 \left[\frac{xy^2}{2} \right]_0^{3x} dx = \int_0^1 \frac{9x^3}{2} dx = \left[\frac{9x^4}{8} \right]_0^1 = \frac{9}{8}$$

Finally, evaluate I/A :

$$I/A = \frac{9}{8} \times \frac{2}{3} = \frac{18}{24} = \frac{3}{4}$$

Exercises 15.3

5.)

$$\int_{\pi/4}^{3\pi/4} \int_1^2 r dr d\theta = \int_{\pi/4}^{3\pi/4} \left[\frac{r^2}{2} \right]_1^2 d\theta = \int_{\pi/4}^{3\pi/4} \frac{3}{2} d\theta = \frac{3}{2} [\theta]_{\pi/4}^{3\pi/4} = \frac{3\pi}{4}$$

7.) Given $x = r \cos \theta$, $y = r \sin \theta$, and $J[x, y] = r$,

$$f(x, y) = Jx^2y = r(r^2 \cos^2 \theta)(r \sin \theta) = r^4 \cos^2 \theta \sin \theta$$

Evaluate the following integral:

$$\begin{aligned} \iint_R x^2 y dA &= \int_0^\pi \int_0^5 r^4 \cos^2 \theta \sin \theta dr d\theta = \int_0^\pi \left[\frac{1}{5} r^5 \cos^2 \theta \sin \theta \right]_0^5 d\theta \\ &= 625 \int_0^\pi \cos^2 \theta \sin \theta d\theta = -625 \left[\frac{1}{3} \cos^3 \theta \right]_0^\pi = 625 \cdot \frac{2}{3} = \frac{1250}{3} \end{aligned}$$

19.)

$$I = \int_0^{2\pi} \int_0^5 r^3 dr d\theta = \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^5 d\theta = \frac{625}{4} \int_0^{2\pi} d\theta = \frac{1250\pi}{4} = \frac{625\pi}{2}$$

29.)

$$I = \int_0^{\pi/2} \int_0^2 r e^{-r^2} dr d\theta = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-r^2} \right]_0^2 d\theta = -\frac{1}{2} \int_0^{\pi/2} e^{-4} - 1 d\theta = \frac{\pi(1 - e^{-4})}{4}$$

Exercises 15.4

17.)

$$I_x = \int_1^3 \int_1^4 y^4 k dy dx = \int_1^3 \left[\frac{1}{5} y^5 k \right]_1^4 dx = \int_1^3 \left(\frac{1023}{5} k \right) dx$$

$$\frac{1023}{5} [xk]_1^3 = \frac{2(1023)}{5} k = 409.2k$$

$$I_y = \int_1^3 \int_1^4 x^2 k y^2 dy dx = \int_1^3 \left[\frac{1}{3} x^2 k y^3 \right]_1^4 dx = \int_1^3 21x^2 k dx$$

$$= 7 [x^3 k]_1^3 = 7(27k - 1k) = 7(26k) = 182k$$

$$I_0 = \int_1^3 \int_1^4 (x^2 + y^2) k y^2 dy dx = I_x + I_y = 591.2k$$

Exercises 15.9

1.)

$$J = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -2 - 4 = -6$$

3.)

$$J = \begin{vmatrix} \cos t & -s \sin t \\ \sin t & s \cos t \end{vmatrix} = s \cos^2 t + s \sin^2 t = s$$

15.) Given the triangle $\{(0,0),(1,2),(2,1)\}$, the transformation $x = 2u + v$ and $y = u + 2v$, and $J[u, v] = 3$, we can evaluate the following integral:

$$I = \iint_R x - 3y dA = 3 \int_0^1 \int_0^1 -u - 5v du dv = 3 \int_0^1 \left[-\frac{1}{2} u^2 - 5uv \right]_0^1 dv$$

$$= 3 \int_0^1 -\frac{1}{2} - 5v dv = 3 \left[-\frac{1}{2} v - \frac{5}{2} v^2 \right]_0^1 = -\frac{3}{2} - \frac{15}{2} = -9$$

- 17.) Given the ellipse $9x^2 + 4y^2 = 36$, and applying the transformation $x = 2u$ and $y = 3v$, we get $u^2 + v^2 = 1$. Converting to polar coordinates, we find $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Evaluating the following integral:

$$\begin{aligned} I &= \iint_R x^2 dA = 6 \int_0^{2\pi} \int_0^1 4r^3 \cos^2 \theta dr d\theta = 6 \int_0^{2\pi} [r^4 \cos^2 \theta]_0^1 d\theta \\ &= 6 \int_0^{2\pi} \cos^2 \theta d\theta = \end{aligned}$$