Exercises 15.1

15.)

$$\int_1^4 \int_0^2 \left(6x^2y - 2x\right)\, dy\, dx = \int_1^4 \left[3x^2y^2 - 2xy\right]_0^2\, dx = \int_1^4 12x^2 - 4x\, dx$$
$$= \left[4x^3 - 2x^2\right]_1^4 = 256 - 32 - 4 + 2 = 222$$

19.)

$$\int_{-3}^3 \int_0^{\pi/2} (y + y^2\cos x)\, dx\, dy = \int_{-3}^3 \left[xy + y^2 \sin x\right]_0^{\pi/2} = \int_{-3}^3 \frac{\pi}{2}y + y^2\, dy$$
$$= \left[\frac{\pi}{4}y^2 + \frac{y^3}{3}\right]_{-3}^3 = \frac{9\pi}{4} + 9 - \frac{9\pi}{4} + 9 = 18$$

29.)

$$\iint_R \frac{xy^2}{x^2+1}\, dA = \int_{-3}^3 \int_0^1 \frac{xy^2}{x^2+1}\, dx\, dy = \frac{1}{2}\int_{-3}^3 \int_0^1 \frac{y^2}{u}\, du\, dy = \frac{1}{2}\int_{-3}^3 \left[y^2 \ln|u|\right]_0^1$$
$$= \frac{1}{2}\int_{-3}^3 \left[y^2 \ln|x^2+1|\right]_0^1 = \frac{1}{2}\int_{-3}^3 \left[y^2 \ln 2\right]\, dy = \frac{\ln 2}{2}\left[\frac{y^3}{3}\right]_{-3}^3 = \frac{18\ln 2}{2} = 9\ln 2$$

33.)

$$\iint_R ye^{-xy}\, dA = \int_0^3 \int_0^2 ye^{-xy}\, dx\, dy = \int_0^3 \left[-e^{-xy}dy\right]_0^2 = \int_0^3 -e^{-2y} + e^0\, dy$$
$$\int_0^3 1 - e^{-2y}\, dy = \left[y + \frac{e^{-2y}}{2}\right]_0^3 = 3 + \frac{e^{-6}}{2} - \frac{1}{2} = \frac{e^{-6}+5}{2}$$

41.) Find the limits:

$$-1 \le x \le 1,\ 0 \le y \le 1$$

Evaluate the integral:

$$\iint_R 1 + x^2ye^y\, dA = \int_0^1 \int_{-1}^1 1 + x^2ye^y\, dx\, dy = \int_0^1 \left[x + \frac{1}{3}x^3ye^y\right]_{-1}^1\, dy$$
$$= \int_0^1 1 + \frac{1}{3}ye^y + 1 + \frac{1}{3}ye^y\, dy = \int_0^1 2 + \frac{2}{3}ye^y\, dy = \left[2y + \frac{2}{3}\left(ye^y - e^y\right)\right]_0^1$$
$$= 2 + \frac{2}{3} = \frac{8}{3}$$

Exercises 15.2

3.)

$$\int_0^1 \int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \left[\frac{x^2e^{y^3}}{2}\right]_0^y dy = \int_0^1 \frac{y^2e^{y^3}}{2}\,dy = \frac{1}{6}\left[e^u\right]_0^1 = \frac{1}{6}\left[e^{y^3}\right]_0^1 = \frac{e-1}{6}$$

13.)

$$\iint_D x\,dA = \int_0^1 \int_0^x x\,dy\,dx = \int_0^1 [xy]_0^x\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\iint_D x\,dA = \int_0^1 \int_y^1 x\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}\right]_y^1 dx = \int_0^1 \frac{1}{2} - \frac{y^2}{2}\,dx = \left[\frac{y}{2} - \frac{y^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

15.)

$$\iint_D y\,dA = \int_{-1}^2 \int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^2 [xy]_{y^2}^{y+2} = \int_{-1}^2 2y + y^2 - y^3\,dy$$

$$= \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_{-1}^2 = 4 + \frac{8}{3} - \frac{16}{4} - 1 + \frac{1}{3} + \frac{1}{4} = 3 - 1 + \frac{1}{4} = \frac{9}{4}$$

Integrating with respect to x first is easier as it prevents us from having to evaluate two double integrals.

17.)

$$\iint_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = -\int_0^1 [x\sin y]_0^{x^2}\,dx = -\int_0^1 x\sin x^2\,dx$$

$$= -\frac{1}{2}\int_0^1 \sin u\,du = -\frac{1}{2}[\cos u]_0^1 = -\frac{1}{2}(\cos(1) - 1) = \frac{1-\cos(1)}{2}$$

23.)

$$\iint_D 3x + 2y\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 3x + 2y\,dy\,dx = \int_0^1 \left[3xy + y^2\right]_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 3x^{3/2} + x - 3x^3 - x^4\,dx = \left[\frac{6x^{5/2}}{5} + \frac{x^2}{2} - \frac{3x^4}{4} - \frac{x^5}{5}\right] = \frac{6}{5} + \frac{1}{2} - \frac{3}{4} - \frac{1}{5} = 1 - \frac{1}{4} = \frac{3}{4}$$

45.)

$$\int_0^1 \int_0^y f(x,y)\,dx\,dy = \int_0^1 \int_x^1 f(x,y)\,dy\,dx$$

53.)

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dy\,dx = \int_0^1 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \left[x\sqrt{y^3+1}\right]_0^{y^2} dy$$

$$= \int_0^1 y^2\sqrt{y^3+1}\,dy = \frac{1}{3}\int_0^1 \sqrt{u}\,du = \frac{1}{3}\left[\frac{2\sqrt{u^3}}{3}\right]_0^1 = \frac{1}{3}\left[\frac{2\sqrt{(y^3+1)^3}}{3}\right]_0^1 =$$

$$\frac{1}{3}\left[\frac{2\sqrt{8}-2}{3}\right] = \frac{2\left(\sqrt{8}-1\right)}{9}$$

61.) Find area of region of integration:

$$A = \frac{1}{2}bh = \frac{1}{2}(1)(3) = \frac{3}{2}$$

Now evaluate the integral:

$$I = \int_0^1 \int_0^{3x} xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^{3x} dx = \int_0^1 \frac{9x^3}{2}\,dx = \left[\frac{9x^3}{8}\right]_0^1 = \frac{9}{8}$$

Finally, evaluate I/A:

$$I/A = \frac{9}{8} \times \frac{2}{3} = \frac{18}{24} = \frac{3}{4}$$

Exercises 15.3

5.)

$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\,dr\,d\theta = \int_{\pi/4}^{3\pi/4} \left[\frac{r^2}{2}\right]_1^2 d\theta = \int_{\pi/4}^{3\pi/4} \frac{3}{2}\,d\theta = \frac{3}{2}\left[\theta\right]_{\pi/4}^{3\pi/4} = \frac{3\pi}{4}$$

7.) Given $x = r\cos\theta$, $y = r\sin\theta$, and $J[x,y] = r$,

$$f(x,y) = Jx^2y = r(r^2\cos^2\theta)(r\sin\theta) = r^4\cos^2\theta\sin\theta$$

Evaluate the following integral:

$$\iint_R x^2y\,dA = \int_0^\pi \int_0^5 r^4\cos^2\theta\sin\theta\,dr\,d\theta = \int_0^\pi \left[\frac{1}{5}r^5\cos^2\theta\sin\theta\right]_0^5 d\theta$$

$$625\int_0^\pi \cos^2\theta\sin\theta\,d\theta = -625\left[\frac{1}{3}\cos^3\theta\right]_0^\pi = 625\cdot\frac{2}{3} = \frac{1250}{3}$$

19.)
$$I = \int_0^{2\pi} \int_0^5 r^3 \, dr \, d\theta = \int_0^{2\pi} \left[\frac{1}{4}r^4\right]_0^5 d\theta = \frac{625}{4} \int_0^{2\pi} d\theta = \frac{1250\pi}{4} = \frac{625\pi}{2}$$

29.)
$$I = \int_0^{\pi/2} \int_0^2 re^{-r^2} \, dr \, d\theta = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-r^2}\right]_0^2 = -\frac{1}{2} \int_0^{\pi/2} e^{-4} - 1 \, d\theta = \frac{\pi(1 - e^{-4})}{4}$$

Exercises 15.4

17.)
$$I_x = \int_1^3 \int_1^4 y^4 k \, dy \, dx = \int_1^3 \left[\frac{1}{5}y^5 k\right]_1^4 dx = \int_1^3 \left(\frac{1023}{5}k\right) dx$$
$$\frac{1023}{5} \left[xk\right]_1^3 = \frac{2(1023)}{5}k = 409.2k$$
$$I_y = \int_1^3 \int_1^4 x^2 k y^2 \, dy \, dx = \int_1^3 \left[\frac{1}{3}x^2 k y^3\right]_1^4 dx = \int_1^3 21x^2 k \, dx$$
$$= 7\left[x^3 k\right]_1^3 = 7(27k - 1k) = 7(26k) = 182k$$
$$I_0 = \int_1^3 \int_1^4 (x^2 + y^2) k y^2 \, dy \, dx = I_x + I_y = 591.2k$$

Exercises 15.9

1.)
$$J = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -2 - 4 = -6$$

3.)
$$J = \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} = s\cos^2 t + s\sin^2 t = s$$

15.) Given the triangle {(0,0),(1,2),(2,1)}, the transformation $x = 2u + v$ and $y = u + 2v$, and $J[u, v] = 3$, we can evaluate the following integral:

$$I = \iint_R x - 3y \, dA = 3\int_0^1 \int_0^{1-v} -u - 5v \, du \, dv = 3\int_0^1 \left[-\frac{1}{2}u^2 - 5uv\right]_0^{1-v} dv$$
$$= 3\int_0^1 -\frac{1}{2}(1-v)^2 - 5(1-v)v \, dv = \frac{3}{2}\int_0^1 9v^2 - 8v - 1 \, dv = \frac{3}{2}\left[3v^3 - 4v^2 - v\right]_0^1$$
$$= \frac{3}{2}(3 - 4 - 1) = -3$$

17.) Given the ellipse $9x^2 + 4y^2 = 36$, and applying the transformation $x = 2u$ and $y = 3v$, we get $u^2 + v^2 = 1$ and $J[u, v] = 6$. Converting to polar coordinates, we find $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $J[r, \theta] = r$. Evaluating the following integral:

$$I = \iint_R x^2 \, dA = 6 \int_0^{2\pi} \int_0^1 4r^3 \cos^2\theta \, dr \, d\theta = 6 \int_0^{2\pi} \left[r^4 \cos^2\theta\right]_0^1 \, d\theta$$

$$= 6 \int_0^{2\pi} \cos^2\theta \, d\theta = 3 \int_0^{2\pi} 1 + \cos 2\theta \, d\theta = 3 \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 3\,(2\pi + 0 + 0 + 0) = 6\pi$$