3.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3x^2 - 2xy + 2\right] = -2x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[6y^2 - x^2 + 3\right] = -2x$$

Integrate M and N:

$$\int 3x^2 - 2xy + 2 \; dx = x^3 - x^2 y + 2x + h(y)$$

$$\int 6y^2 - x^2 + 3 \; dy = 2y^3 - x^2 y + 3y + g(x)$$

$$\therefore \; g(x) = x^3 + 2x, h(y) = 2y^3 + 3y$$

$$\therefore \; f(x,y) = x^3 - x^2 y + 2x + 2y^3 + 3y = C$$

Which is the implicit solution.

9.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[2x - y\right] = -1 = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2y - x\right] = -1$$

Integrate M and N:

$$\int 2x - y \; dx = x^2 - xy + h(y)$$

$$\int 2y - x \; dy = y^2 - xy + g(x)$$

$$\therefore \; g(x) = x^2, h(y) = y^2$$

$$\therefore \; f(x,y) = x^2 - xy + y^2 = C$$

Solve for C given $y(1) = 3$:

$$1^2 - 3(1) + 1^2 = 1 - 3 + 1 = -1 = C$$

$$\therefore \; f(x,y) = x^2 - xy + y^2 = C \implies y^2 - xy + x^2 - C = 0$$

$$\implies y = \frac{x \pm \sqrt{x^2 - 4(x^2 - C)}}{2} = \frac{x \pm \sqrt{-3x^2 + 4}}{2}$$

Which is the particular solution.

11.) Take mixed derivatives of M and N:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 + bx^2y\right] = 2xy + bx^2, \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x^3 + x^2y\right] = 3x^2 + 2xy$$

Thus $b = 3$, and the equation is exact. Integration M and N:

$$\int xy^2 + 3x^2y\ dx = \frac{x^2y^2}{2} + x^3y + h(y)$$

$$\int x^3 + x^2y\ dy = x^3y + \frac{x^2y^2}{2} + g(x)$$

$$\therefore\ g(x) = h(y) = 0$$

$$\therefore\ f(x,y) = x^3y + \frac{x^2y^2}{2} = C$$

Which is the implicit solution.

15.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[x^2y^3\right] = 3x^2y^2 \neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x + xy^2\right] = 1 + y^2$$

Since the equation is not exact, multiply by given integrating factor $\mu(x,y) = \dfrac{1}{xy^3}$:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2y^3}{xy^3}\right] = \frac{\partial}{\partial y}[x] = 0 = \frac{\partial \mu N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x + xy^2}{xy^3}\right] = \frac{\partial}{\partial x}\left[\frac{1}{y^3} + \frac{1}{y}\right] = 0$$

Thus the equation is exact. Integrate μM and μN:

$$\int x\ dx = \frac{x^2}{2} + h(y)$$

$$\int \frac{1}{y^3} + \frac{1}{y}\ dy = \ln|y| - \frac{1}{2y^2} + g(x)$$

$$\therefore\ g(x) = \frac{x^2}{2}, h(y) = \ln|y| - \frac{1}{2y^2}$$

$$\therefore\ f(x,y) = \frac{x^2}{2} + \ln|y| - \frac{1}{2y^2} = C$$

Which is the implicit solution.

21.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[y] = 1 \neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2xy - e^{-2y}\right] = 2y$$

Thus the equation is not exact. Find integrating factor μ:

$$\frac{M_y - N_x}{N} = \frac{1-2y}{2xy - e^{-2y}} \text{ is not a function of } x$$

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}$$

$$\therefore\ \mu(y) = e^{\int 2 - \frac{1}{y} dy} = e^{2y - \ln|y|} = \frac{e^{2y}}{y}$$

Multiply the equation by μ and solve:

$$\mu M = \frac{ye^{2y}}{y} = e^{2y}, \mu N = \frac{e^{2y}(2xy - e^{-2y})}{y} = 2xe^{2y} - \frac{1}{y}$$

$$\int e^{2y}\ dx = xe^{2y} + h(y)$$

$$\int 2xe^{2y} - \frac{1}{y}\ dy = xe^{2y} - \ln|y| + g(x)$$

$$\therefore\ g(x) = 0, h(y) = -\ln|y|$$

$$\therefore\ f(x,y) = xe^{2y} - \ln|y| = C$$

Which is the implicit solution.

1.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[2xy^2 + 2y\right] = 4xy + 2 = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2x^2y + 2x\right] = 4xy + 2$$

Integration M and N:

$$\int 2xy^2 + 2y\ dx = x^2y^2 + 2xy + h(y)$$

$$\int 2x^2y + 2x\ dy = x^2y^2 + 2xy + g(x)$$

$$\therefore\ g(x) = h(y) = 0$$

$$\therefore\ f(x,y) = x^2y^2 + 2xy = C \implies y = \frac{-2x \pm \sqrt{4x^2 - 4(-Cx^2)}}{2x^2}$$

$$= -\frac{1}{x} \pm \frac{\sqrt{x^2(1+C)}}{x^2} = \pm\frac{\sqrt{1+C}}{x} - \frac{1}{x}$$

Which is the explicit solution.

2.) Check equation for exactness:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[e^x \sin y - 2y \sin x\right] = e^x \cos y - 2 \sin x$$

$$= \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[e^x \cos y + 2 \cos x\right] = e^x \cos y - 2 \sin x$$

Integrate M and N:

$$\int e^x \sin y - 2y \sin x \; dx = e^x \sin y + 2y \cos x + h(y)$$

$$\int e^x \cos y + 2 \cos x \; dy = e^x \sin y + 2y \cos x + g(x)$$

$$\therefore \; g(x) = h(y) = 0$$

$$\therefore \; f(x,y) = e^x \sin y + 2y \cos x = C$$

Which is the implicit solution.