Section 3.5

1.) Find the homogeneous solution:

$$y'' - 2y' - 3y = 0 \implies r^2 - 2r - 3 = (r-3)(r+1) \implies r = 3, -1$$

$$\therefore y_h = c_1 e^{3t} + c_2 e^{-t}$$

Now find Y:

$$g(t) = 3e^{2t} \implies Y = Ae^{2t}$$

$$\therefore \; Y' = 2Ae^{2t}, \; Y'' = 4Ae^{2t}$$

Substituting y with Y:

$$Y'' - 2Y' - 3Y = 4Ae^{2t} - 4Ae^{2t} - 3Ae^{2t} = 3e^{2t} \implies 4A - 4A - 3A = 3$$

$$\implies -3A = 3 \implies A = -1$$

$$\therefore y_h + Y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$$

Which is the general solution.

2.) Find the homogeneous solution:

$$y'' - y' - 2y = 0 \implies r^2 - r - 2 = (r-2)(r+1) \implies r = 2, -1$$

$$\therefore \; y_h = c_1 e^{2t} + c_2 e^{-t}$$

Now find Y:

$$g(t) = -2t + 4t^2 \implies Y = A + Bt + Ct^2$$

$$\therefore Y' = B + 2Ct, \; Y'' = 2C$$

Substituting y with Y:

$$Y'' - Y' - 2Y = 2C - B - 2Ct - 2A - 2Bt - 2Ct^2 = -2Ct^2 - 2Bt - 2Ct - 2A - B + 2C$$

$$\therefore \; -2Ct^2 = 4t^2 \implies C = -2$$

$$\therefore \; -2Bt - 2Ct = -2t \implies -2Bt + 4t = -2t \implies -2Bt = -6t \implies B = 3$$

$$\therefore \; -2A - B + 2C = -2A - 3 - 4 = 0 \implies -2A = 7 \implies A = -\frac{7}{2}$$

$$\therefore \; y_h + Y = c_1 e^{2t} + c_2 e^{-t} - \frac{7}{2} + 3t - 2t^2$$

Which is the general solution.

5.) Find the homogeneous solution:

$$y'' + 2y' = 0 \implies r^2 + 2r = r(r+2) \implies r = 0, -2$$
$$\implies y_h = c_1 e^0 + c_2 e^{-2t} = c_1 + c_2 e^{-2t}$$

Now find Y:

$$g(t) = 3 + 4\sin(2t) \implies Y = At + B\sin(2t) + C\cos(2t)$$
$$\therefore\ Y' = A + 2B\cos(2t) - 2C\sin(2t),\ Y'' = -4B\sin(2t) - 4C\cos(2t)$$

Substituting y with Y:

$$Y'' + 2Y' = -4B\sin(2t) - 4C\cos(2t) + 2A + 4B\cos(2t) - 4C\sin(2t)$$
$$\therefore\ 2A = 3 \implies A = \frac{3}{2}$$
$$\therefore\ -4B - 4C = 4 \implies -B - C = 1 \implies C = -B - 1$$
$$\therefore\ 4B - 4C = 4B - 4(-B-1) = 4B + 4B + 4 = 0 \implies 8B = -4 \implies B = -\frac{1}{2}$$
$$\therefore\ -4B - 4C = 2 - 4C = 4 \implies C = -\frac{2}{4} = -\frac{1}{2}$$
$$\therefore y_h + Y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t)$$

Which is the general solution.

10.) Find the homogeneous solution:

$$y'' + y' + 4y = 0 \implies r^2 + r + 4 \implies r = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm \sqrt{15}i}{2}$$
$$\implies \lambda = -\frac{1}{2},\ \mu = \sqrt{15}$$
$$\therefore\ y_h = c_1 e^{-t/2}\cos\left(\sqrt{15}t\right) + c_2 e^{-t/2}\sin\left(\sqrt{15}t\right)$$

Now find Y:

$$g(t) = 2\sinh t = e^t - e^{-t} \implies Y = Ae^t + Be^{-t}$$
$$\therefore\ Y' = Ae^t - Be^{-t},\ Y'' = Ae^t + Be^{-t}$$

Substituting y with Y:

$$Y'' + Y' + 4Y = Ae^t + Be^{-t} + Ae^t - Be^{-t} + 4Ae^t + 4Be^{-t} = 6Ae^t + 4Be^{-t}$$
$$\therefore\ 6A = 1 \implies A = \frac{1}{6}$$
$$\therefore\ 4B = -1 \implies B = -\frac{1}{4}$$
$$\therefore\ y_h + Y = c_1 e^{-t/2}\cos\left(\sqrt{15}t\right) + c_2 e^{-t/2}\sin\left(\sqrt{15}t\right) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}$$

Which is the general solution.

11.) Find the homogeneous solution:

$$y'' + y' - 2y = 0 \implies r^2 + r - 2 = (r+2)(r-1) \implies r = -2, 1$$

$$\therefore\ y_h = c_1 e^t + c_2 e^{-2t}$$

Now find Y:

$$g(t) = 2t \implies Y = A + Bt$$

$$\therefore\ Y' = B,\ Y'' = 0$$

Substituting y with Y:

$$Y'' + Y' - 2Y = 0 + B - 2A - 2Bt = 2t$$

$$\therefore\ B - 2A = 0 \implies B = 2A$$

$$\therefore\ -2B = 2 \implies B = -1 \implies A = -\frac{1}{2}$$

$$\therefore\ y_h + Y = c_1 e^t + c_2 e^{-2t} - \frac{1}{2} - t$$

$$\therefore\ (y_h + Y)' = c_1 e^t - 2c_2 e^{-2t} - 1$$

Which is the general solution. We can solve for c_1 and c_2 given initial conditions $y(0) = 0$ and $y'(0) = 1$:

$$0 = c_1 e^0 + c_2 e^{-2(0)} - \frac{1}{2} - 0 = c_1 + c_2 - \frac{1}{2} \implies \frac{1}{2} = c_1 + c_2$$

$$1 = c_1 e^0 - 2c_2 e^{-2(0)} - 1 \implies 2 = c_1 - 2c_2 \implies c_1 = 2c_2 + 2$$

$$\implies \frac{1}{2} = 2c_2 + 2 + c_2 = 3c_2 + 2 \implies -\frac{3}{2} = 3c_2 \implies c_2 = -\frac{1}{2}$$

$$\implies \frac{1}{2} = c_1 - \frac{1}{2} \implies c_1 = 1$$

$$\therefore\ y_h + Y = e^t - \frac{1}{2}e^{-2t} - \frac{1}{2} - t$$

Which is the particular solution.

16.) Given $y_h = c_1 + c_2 e^{-3t}$, and given $g(t) = 2t^4 + t^2 e^{-3t} + \sin(3t)$, we can determine the form of the particular solution to be as follows:

$$Y = t(A + Bt + Ct^2 + Dt^3 + Et^4) + t(F + Gt + Ht^2)e^{-3t} + I\sin(3t) + J\cos(3t)$$

19.) Given $y_h = c_1 + c_2 e^{-4t}$, and given $g(t) = t^2 \sin(2t) + (6t + 7)\cos(2t)$, we can determine the form of the particular solution to be as follows:

$$Y = t(A + Bt + Ct^2)(D\sin(2t) + E\cos(2t)) + t(F + Gt + Ht^2)(I\sin(2t) + J\cos(2t))$$

$$= t(A + Bt + Ct^2)\sin(2t) + t(D + Et + Ft^2)\cos(2t)$$