9.2.1 We can take the limit as $n \to \infty$ of $f_n(x)$:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{x^n}{1+x^n}\frac{x^{-n}}{x^{-n}} = \lim_{n\to\infty} \frac{1}{x^{-n}+1} = \frac{1}{0+1} = 1 \ ,$$

thus this sequence of functions converges pointwise to 1 for all x. ■

9.2.2 We can see that following:

$$L := \lim_{n\to\infty} n\left(\sqrt[n]{x}-1\right) = \lim_{n\to\infty} \frac{1}{n^{-1}}\left(\sqrt[n]{x}-1\right) = \lim_{n\to\infty} \frac{\sqrt[n]{x}-1}{n^{-1}} = \frac{0}{0} \ ,$$

thus we can apply L'Hôpital's rule:

$$\frac{d}{dn}\left\{n^{-1}\right\} = -n^{-2}$$

$$\frac{d}{dn}\left\{x^{\frac{1}{n}}-1\right\} = \frac{d}{dn}\left\{e^{(1/n)\log x}\right\} = (-n^{-2}\log x)(x^{\frac{1}{n}}) = -\frac{x^{\frac{1}{n}}\log x}{n^2}$$

Thus

$$L = \lim_{n\to\infty} \frac{1}{n^{-2}} \cdot \frac{x^{\frac{1}{n}}\log x}{n^2} = \lim_{n\to\infty} x^{\frac{1}{n}}\log x$$

Since $1/n \to 0$ as $n \to \infty$, we know $L = x^0 \log x = \log x$, thus we obtain our desired equality. For the next part, define $f_n(x) = n\left(\sqrt[n]{x}-1\right)$ for all $n \in \mathbb{N}$ and $f(x) = \log x$. Assuming the sequence uniformly converges, we can make the following observations:

If $f_n(x)$ is continuous over $(0,\infty)$ for all n, then we could conclude that $f(x)$ is continuous over $(0,\infty)$ as a result.

If $f_n(x)$ is differentiable over $(0,\infty)$ for all n, then we would know that $f(x)$ is differentiable over $(0,\infty)$, and that its derivative would be given by

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Finally, if $f_n(x)$ is integrable over $[0,\infty)$ for all n, then we know that $f(x)$ is as well. We could also evaluate $\int_1^2 \log x\, dx$ as

$$\int_1^2 \log x\, dx = \sum_{k=1}^{\infty} \int_1^2 f_k(x)\, dx \ .$$

Note that

$$\int_1^2 f_k(x)\, dx = k\int_1^2 \sqrt[k]{x} - 1\, dx = k\left[\frac{k}{k+1}x^{\frac{k+1}{k}} - x\right]_1^2 = \frac{k}{k+1}\left(\sqrt[k]{2^{k+1}}-1\right) - 1$$

But the sum diverges, so it cannot be equal to the integral, so we must conclude that the sequence of functions f_n does not converge uniformly. ■