# Exercises 14.7

1.) a.) Find $D$:

$$D = f_{xx}f_{yy} - [f_{xy}]^2 = 4(2) - 1^2 = 7$$

Since $D > 0$ and $f_{xx} = 4 > 0$, $(1, 1)$ is a local minimum of $f$.

b.) Find $D$:

$$D = 4(2) - 3^2 = 8 - 9 = -1$$

Since $D < 0$ and $f_{xx} = 4 > 0$, $(1, 1)$ is a saddle point of $f$.

5.) Find the critical points of $f$:

$$f_x = 2x + y = 0,\ f_y = x + 2y + 1 = 0 \implies x + 2y = -1$$

$$\implies 2(-1 - 2y) + y = -2 - 4y + y = -2 - 3y = 0 \implies y = -\frac{2}{3}$$

$$\implies x = \frac{1}{3}$$

Thus there is one critical point at $\left(\frac{1}{3}, -\frac{2}{3}\right)$. Find $D$ at this point:

$$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 1$$

$$\implies D = (2)(2) - 1^2 = 4 - 1 = 3$$

Since $D > 0$ and $f_{xx} > 0$, $\left(\frac{1}{3}, -\frac{2}{3}\right)$ is a local minimum of $f$.

12.) Find the critical points of $f$:

$$f_x = 3x^2 - 6x - 9 = 0,\ f_y = 3y^2 - 6y = 0$$

$$x = (3x + 3)(x - 3) \implies x = -1, 3$$

$$y = \frac{6 \pm \sqrt{36 + 0}}{6} = \frac{6 \pm 6}{6} \implies y = 0, 2$$

Thus there are four critical points at $(-1, 0), (-1, 2), (3, 0)$, and $(3, 2)$. Find $D$ at these points:

$$f_{xx} = 6x - 6,\ f_{yy} = 6y - 6,\ f_{xy} = 0$$

$$D(-1, 0) = -12(-6) - 0^2 = 72$$

$$D(-1, 2) = -12(6) - 0^2 = -72$$

$$D(3, 0) = 12(-6) - 0^2 = -72$$

$$D(3, 2) = 12(6) = 72$$

The properties of each critical point are described in the following table.

| critical point | $D$ and $f_{xx}$ | property |
|---|---|---|
| (-1,0) | $D > 0, f_{xx} < 0$ | local maximum |
| (-1,2) | $D < 0, f_{xx} < 0$ | saddle point |
| (3,0) | $D < 0, f_{xx} > 0$ | saddle point |
| (3,2) | $D > 0, f_{xx} > 0$ | local minimum |

41.) The distance $d$ from $P_0$ to a point on the plane:

$$x + y + z = 1 \implies z = 1 - x - y$$

$$\implies d = \sqrt{(x-2)^2 + y^2 + (z+3)^2} = \sqrt{(x-2)^2 + y^2 + (4-x-y)^2}$$

$$\implies d^2 = (x-2)^2 + y^2 + (4-x-y)^2$$

Now find the critical points:

$$f_x = 2x - 4 - 8 + 2x + 2y = 4x + 2y - 12 = 0 \implies 4x + 2y = 12$$

$$f_y = 2y - 8 + 2x + 2y = 2x + 4y - 8 \implies 2x + 4y = 8$$

$$\implies 2x + 4y = 8 \implies x = 4 - 2y$$

$$\implies 4(4-2y) + 2y = 16 - 8y + 2y = 16 - 6y = 12 \implies y = \frac{2}{3}$$

$$\implies 4x + \frac{4}{3} = 12 \implies 4x = \frac{32}{3} \implies x = \frac{8}{3}$$

Thus there is one critical point at $\left(\frac{8}{3}, \frac{2}{3}\right)$. Find $D$ at this point:

$$f_{xx} = 4,\ f_{yy} = 4,\ f_{xy} = 2$$

$$D = (4)(4) - 2^2 = 16 - 4 = 12$$

Since $D > 0$ and $f_{xx} > 0$, we know there is a local minimum at $\left(\frac{8}{3}, \frac{2}{3}\right)$. Finally, evaluate $d$ at this point to find the minimum distance to the plane:

$$d\left(\frac{8}{3}, \frac{2}{3}\right) = \sqrt{\left(\frac{8}{3} - 2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{12}{9}} = \frac{\sqrt{12}}{3}$$

45.) Setup the equations:

$$x + y + z = 100,\ xyz,\ x, y, z \neq 0$$

Find $f_x$ and $f_y$:

$$x + y + z = 100 \implies z = 100 - x - y$$

$$xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$$

$$f_x = 100y - 2xy - y^2 = y(100 - 2x - y) = 0$$

$$f_y = 100x - x^2 - 2xy = x(100 - 2y - x) = 0$$

Find the critical points:

$$y(100 - 2x - y) = 0 \implies 100 - 2x - y = 0 \implies y = 100 - 2x$$

$$\implies x(100 - 2y - x) = x(100 - 200 + 4x - x) = x(-100 + 3x)$$

$$\implies x = 0, \frac{100}{3}$$

$$\implies y = 100 - \frac{200}{3} = \frac{100}{3}$$

Now find $D\left(\frac{100}{3}, \frac{100}{3}\right)$:

$$f_{xx} = -2y = -\frac{200}{3},\ f_{yy} = -\frac{200}{3},\ f_{xy} = 100 - 2x - y - y = 100 - 100 - \frac{100}{3} = -\frac{100}{3}$$

$$D = \frac{40000}{9} - \frac{10000}{9} = \frac{30000}{9} = \frac{10000}{3}$$

Since $D > 0$ and $f_{xx} < 0$, $\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local minimum. Finally, find $z$:

$$z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$$

Thus $(x, y, z) = \left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$

47.) Setup the equations:

$$x^2 + y^2 + z^2 = r^2,\ (2x)(2y)(2z) = 8xyz,\ x, y, z > 0$$

Find the critical points:

$$x^2 + y^2 + z^2 = r^2 \implies z = \sqrt{r^2 - x^2 - y^2}$$

$$\implies 8xyz = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$f_x = 8y\sqrt{r^2 - x^2 - y^2} - \frac{8x^2y}{\sqrt{r^2 - x^2 - y^2}} = \frac{8y(r^2 - x^2 - y^2) - 8x^2y}{\sqrt{r^2 - x^2 - y^2}}$$

$$= \frac{8y(r^2 - 2x^2 - y^2)}{\sqrt{r^2 - x^2 - y^2}} = 0$$

$$f_y = 8x\sqrt{r^2 - x^2 - y^2} - \frac{8xy^2}{\sqrt{r^2 - x^2 - y^2}} = \frac{8x(r^2 - x^2 - y^2) - 8xy^2}{\sqrt{r^2 - x^2 - y^2}}$$

$$= \frac{8x(r^2 - x^2 - 2y^2)}{\sqrt{r^2 - x^2 - y^2}} = 0$$

$$\implies r^2 - x^2 - 2y^2 = r^2 - 2x^2 - y^2 = 0 \implies r^2 = x^2 + 2y^2 = 2x^2 + y^2$$
$$\implies 2x^2 + 4y^2 = 2r^2 \implies 3y^2 = r^2 \implies y = \frac{r}{\sqrt{3}}$$
$$\implies 2x^2 + y^2 = 2x^2 + \frac{r^2}{3} = r^2 \implies 2x^2 = \frac{2r^2}{3} \implies x = \frac{r}{\sqrt{3}}$$

Thus there is one critical point at $\left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$. Evaluating $f$ at this point gives us the maximum volume:

$$8\left(\frac{r}{\sqrt{3}}\right)\left(\frac{r}{\sqrt{3}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{3}}\right)^2 - \left(\frac{r}{\sqrt{3}}\right)^2} = \frac{8r^2}{3}\sqrt{\frac{3r^2 - r^2 - r^2}{3}} = \frac{8r^2}{3}\sqrt{\frac{r^2}{3}} = \frac{8r^3}{3\sqrt{3}}$$

# Exercises 14.8

3.) Expand the equation:

$$\nabla f = \lambda \nabla \implies \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

Find the critical points:

$$2x = 2\lambda x,\ -2y = 2\lambda y,\ x^2 + y^2 = 1$$

$$2x = 2\lambda x \implies \lambda = 1 \lor x = 0$$

Case: $\lambda = 1$:

$$-2y = 2y \implies -y = y \implies y = 0$$
$$\implies x^2 + y^2 = x^2 = 1 \implies x = \pm 1$$

Case: $x = 0$:

$$x^2 + y^2 = y^2 = 1 \implies y = \pm 1$$

Thus there are four critical points at $(0,-1), (0,1), (-1,0)$, and $(1,0)$. Evaluating $f$ at these points:

$$f(0,-1) = 0 - 1 = -1$$
$$f(0,1) = 0 - 1 = -1$$
$$f(-1,0) = 1 + 0 = 1$$
$$f(1,0) = 1 + 0 = 1$$

Thus there are two minima at $(0,-1)$ and $(0,1)$, and two maxima at $(-1,0)$ and $(1,0)$.

7.) Expand the equation:

$$\nabla f = \lambda \nabla g \implies \langle 2, 2, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

Find the critical points:

$$\implies 2 = 2\lambda x, \implies 2 = 2\lambda y, \implies 1 = 2\lambda z, \; x^2 + y^2 + z^2 = 9$$

$$\implies 2\lambda x = 2\lambda y \implies x = y \implies z = \frac{x}{2} = \frac{y}{2}$$

$$\implies x^2 + y^2 + z^2 = x^2 + x^2 + \left(\frac{x}{2}\right)^2 = \frac{9x^2}{4} = 9 \implies 9x^2 = 36 \implies x = \pm 2$$

$$\implies x^2 + y^2 + z^2 = 4 + y^2 + \left(\frac{y}{2}\right)^2 = 9 \implies \frac{5y^2}{4} = 5 \implies 5y^2 = 20 \implies y = \pm 2$$

$$\implies x^2 + y^2 + z^2 = 4 + 4 + z^2 = 9 \implies z^2 = 1 \implies z = \pm 1$$

Thus there are two critical points at $(2, 2, 1)$ and $(-2, -2, -1)$. Evaluating $f$ at these points:

$$f(2, 2, 1) = 4 + 4 + 1 = 9$$
$$f(-2, -2, -1) = -4 - 4 - 1 = -9$$

Thus there is one minimum at $(-2, -2, -1)$, and one maximum at $(2, 2, 1)$.

9.) Expand the equation:

$$\nabla f = \lambda \nabla g \implies \left\langle y^2 z, 2xyz, xy^2 \right\rangle = \lambda \left\langle 2x, 2y, 2z \right\rangle$$

Find the critical points:

$$\implies y^2 z = 2\lambda x, \; 2xyz = 2\lambda y, \; xy^2 = 2\lambda z, \; x^2 + y^2 + z^2 = 4$$

$$2xyz = 2\lambda y \implies \lambda = xz$$
$$2\lambda x = 2x^2 z = y^2 z \implies y^2 = 2x^2$$
$$2\lambda z = 2xz^2 = xy^2 \implies 2z^2 = 2x^2 \implies z^2 = x^2$$
$$\implies x^2 + y^2 + z^2 = x^2 + 2x^2 + x^2 = 4x^2 = 4 \implies x = \pm 1$$
$$\implies x^2 + y^2 + z^2 = 1 + 2z^2 + z^2 = 4 \implies 3z^2 = 3 \implies z = \pm 1$$
$$\implies x^2 + y^2 + z^2 = 1 + y^2 + 1 = 4 \implies y^2 = 2 \implies y = \pm\sqrt{2}$$

We can check $(1, \pm\sqrt{2}, 1), (-1, \pm\sqrt{2}, 1), (1, \pm\sqrt{2}, -1)$, and $(-1, \pm\sqrt{2}, -1)$ for maxima and minima:

$$f(1, \pm\sqrt{2}, 1) = 2$$
$$f(-1, \pm\sqrt{2}, 1) = -2$$
$$f(1, \pm\sqrt{2}, -1) = -2$$
$$f(-1, \pm\sqrt{2}, -1) = 2$$

Thus there are two minima at $(-1, \pm\sqrt{2}, 1)$ and $(1, \pm\sqrt{2}, -1)$ and two maxima at $(\pm 1, \pm\sqrt{2}, \pm 1)$.