29.) a.) Since S is bounded below, $x \in S \implies x \geq M$ for some $M \in \mathbb{R}$. Let $l < M$, thus $l < M \leq x$, thus $l < x$ for all $x \in S$, thus l is a lower bound of S, thus $\mathscr{L} \neq \varnothing$. Next, let $x \in S$, thus $x \geq l$ for all lower bounds l of S, thus x is an upper bound of $\mathscr{L}$, thus $\mathscr{L}$ is bounded above. Q.E.D.

b.) Let $w = \sup(\mathscr{L})$, thus $w \geq l$ for all $l \in \mathscr{L}$.

46.) Suppose $2^{-n} \to 0$, then for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ where

$$\left|2^{-n} - 0\right| < \varepsilon$$

Manipulating the inequality:

$$\left|2^{-n} - 0\right| = \left|2^{-n}\right| = 2^{-n} < \varepsilon \implies \frac{1}{\varepsilon} < 2^n$$

Thus

$$\left|2^{-n} - 0\right| = \left|\frac{1}{2^n}\right| < \left|\frac{1}{\frac{1}{\varepsilon}}\right| = |\varepsilon| = \varepsilon$$

Thus $n \geq k \implies |2^{-n} - 0| < \varepsilon$, thus $2^{-n} \to 0$. Q.E.D.

47.) Let $y_n \to 0$, thus for all $\varepsilon_1 > 0$, there exists $k \in \mathbb{N}$ where

$$n \geq k \implies |y_n - 0| < \varepsilon_1$$

Now suppose x_n is a bounded sequence, and $x_n y_n \to 0$, then for all $\varepsilon > 0$,

$$n \geq k \implies |x_n y_n - 0| < \varepsilon$$

Manipulating the inequality:

$$|x_n y_n - 0| = |x_n y_n| = |x_n|\,|y_n - 0| < \varepsilon \implies |y_n - 0| < \frac{\varepsilon}{|x_n|}$$

Let $\varepsilon_1 = \dfrac{\varepsilon}{|x_n|}$, then

$$|y_n - 0| < \frac{\varepsilon}{|x_n|} \implies |x_n|\,|y_n - 0| < \varepsilon \implies |x_n y_n - 0| < \varepsilon$$

Thus $n \geq k \implies |x_n y_n - 0| < \varepsilon$, thus $x_n y_n \to 0$. Q.E.D.

50.) Since $x_n \to A$, then for all $\varepsilon_1 > 0$, there exists $k \in \mathbb{N}$ where

$$n \geq k \implies |x_n - A| < \varepsilon_1$$

Now suppose $cx_n \to cA$, then for all $\varepsilon > 0$,

$$n \geq k \implies |cx_n - cA| < \varepsilon$$

Manipulating the inequality:

$$|cx_n - cA| < |c(x_n - A)| = |c|\,|x_n - A| < \varepsilon$$
$$\implies |x_n - A| < \frac{\varepsilon}{|c|}$$

Let $\varepsilon_1 = \dfrac{\varepsilon}{|c|}$, then

$$|x_n - A| < \frac{\varepsilon}{|c|} \implies |c|\,|x_n - A| < \varepsilon \implies |cx_n - cA| < \varepsilon$$

Thus $n \geq k \implies |cx_n - cA| < \varepsilon$, thus $cx_n \to cA$. Q.E.D.

52.) Let $M, N \in \mathbb{R}$ where $M \geq x_n$ and $N \geq y_n$ for all $n \in \mathbb{N}$. Consider $z_n = x_n + y_n$. $x_n + y_n \leq M + N$, thus $z_n \leq M + N$ for all $n \in \mathbb{N}$, thus z_n is bounded. Q.E.D.

53.) a.) False; many sequences diverge.

b.) True; Since x_n diverges, then we know that $x_n^2 = x_n x_n$ diverges too.

54.) a.) $(-1)^n$

b.) DNE; A sequence cannot converge to a value while having terms that are arbitrarily far from that value.

62.) Assume $x_n \to A$, thus for all $\varepsilon > 0$, there exists k such that

$$n \geq k \implies |x_n - A| < \varepsilon$$

Since $x_n < M$, we know that

$$|x_n - A| < |M - A|$$

***

63.) Since S is nonempty and bounded below, there exists some $v \in \mathbb{R}$ where $v = \inf(S)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence where $x_n = v$, thus $x_n \to \inf(S)$, thus there exists a sequence x_n where $x_n \to \inf(S)$. Q.E.D.

65.) So far, the topic I have had the most trouble grasping is ε-k convergence proofs. I understand the process of constructing the proof, but am still working on understanding the logic.

77.) Let x_n and y_n be sequences where $x_n = y_n = n$. Since n is stricly increasing, so are x_n and y_n. However $x_n - y_n = n - n = 0$, and 0 is not strictly increasing, thus $x_n - y_n$ is not strictly increasing.

79.) A subsequence of x_n is a sequence y_k such that $y_k = x_{n_k}$ for all $k \in \mathbb{N}$, where n_k is a strictly increasing sequence of natural numbers.

80.) You can view subsequences as a more abstract form of function composition. You have x_n, which is a function $f : \mathbb{N} \to \mathbb{R}$, and you have n_k, which is a function $g : \mathbb{N} \to \mathbb{N}$. When you let $y_k = x_{n_k}$, this is equivalent to $y_k = f(g(k))$. Since the codomain of g and domain of f are the same, namely $\mathbb{N}$, we know this function composition is valid.

81.) Let $y_k = x_{2k}$, thus $y_k = (-1)^{2k} = 1$ for all $k \in \mathbb{N}$, thus $y_k \to 1$, thus y_k is convergent.

82.) a.) DNE, if $y_k \preceq x_n$ and $x_n \to L$, then $y_k \to L$ as per theorem 19.

b.) $n - (-1)^n n$, x_n diverges, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, $\therefore y_n \to 0$.

83.) Setting $x_n = y_k$ and solving for n, we can find a suitable n_k:

$$2n - 1 = 8k + 1 \implies 2n = 8k + 2 \implies n = 4k + 1$$

Thus when $n_k = 4k+1$, $x_{n_k} = 2(4k+1) - 1 = 8k + 2 - 1 = 8k + 1 = y_k$, thus $y_k \preceq x_n$. Q.E.D.

84.) Setting $x_n = y_k$ and solving for n, we can find a suitable n_k:

$$2n - 1 = 8k^2 + 24k + 17 \implies 2n = 8k^2 + 24k + 18 \implies n = 4k^2 + 12k + 9$$

Thus when $n_k = 4k^2 + 12k + 9$, $x_{n_k} = 2(4k^2 + 12k + 9) - 1 = 8k^2 + 24k + 17 = y_k$, thus $y_k \preceq x_n$. Q.E.D.

85.) a.) ***

b.) ***

86.) a.) True; Let $n_k = k$, thus $x_k = x_{n_k}$, thus $x \preceq x$.

b.) False; Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

c.) True; Let ***

90.) Since x_n is bounded, $x_n \leq M$ for some $M \in \mathbb{R}$. In addition, since $y \preceq x$, for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $y_k = x_n$, thus $y_k = x_n \leq M$, thus $y_k \leq M$ for all k, thus y_k is bounded. Q.E.D.

91.) a.) $S = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$ are the friends of x_n.

b.) $S = \{1\}$ is the friend of y_n.

92.) $S = \{n \in \mathbb{N} : 21 \leq n \leq 56\}$ are the friends of z_n.

93.) Since x_n is bounded, $x_n \leq M$ for some $M \in \mathbb{R}$. Consider $\overline{x}_n$: ***

113.) $z \in \mathbb{R}$ is a cluster point of S if for all $\varepsilon > 0$ there exists $x \in S$ such that $0 < |x - z| < \varepsilon$.

114.) Let $S = \{y \in \mathbb{R} : |x - y| < r\}$,

$$|x - y| < r \implies -r < x - y < r \implies -r - x < -y < r - x$$

$$\implies x - r < y < x + r \implies S = (x - r, x + r)$$

Thus $\{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$ Q.E.D.

115.) $\mathbb{Z}$ has no cluster points.

116.) 1 is a cluster point of $[0, 1)$.

117.) For 0 to be a cluster point of $[1, 2]$, then for all $\varepsilon > 0$ there exists $a \in [1, 2]$ where $|a - 0| < \varepsilon$. Let $\varepsilon = \frac{1}{2}$, then

$$|a - 0| = |a| < \frac{1}{2} \implies -\frac{1}{2} < a < \frac{1}{2}$$

But since $-\frac{1}{2} < a < \frac{1}{2}$, $a \notin [1, 2]$, thus $a \in [1, 2]$ does not exist for 0, thus 0 is not a cluster point of $[1, 2]$. Q.E.D.

124.) Let $f : E \to \mathbb{R}$, $c \in E'$, and $L \in \mathbb{R}$, then $f(x) \to L$ as $x \to c$, or $\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - L| < \varepsilon$.

125.) Suppose $\lim_{x \to c} f(x) = a$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - c| < \delta \implies |a - a| < \varepsilon$$

Since $|a - a| = |0| = 0 < \varepsilon$, $|a - a| < \varepsilon$, thus $\lim_{x \to c} f(x) = a$. Q.E.D.

126.) Suppose $\lim_{x \to 2} 3x + 1 = 7$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - 2| < \delta \implies |3x + 1 - 7| < \varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$|3x + 1 - 7| = |3x - 6| = |3(x - 2)| = 3\,|x - 2| < \varepsilon$$

$$\implies |x - 2| < \frac{\varepsilon}{3}$$

Let $\delta = \frac{\varepsilon}{3}$, then we know that $|x - 2| < \delta$. Manipulating the inequality:

$$|x - 2| < \frac{\varepsilon}{3} \implies 3\,|x - 2| = |3(x - 2)| = |3x - 6| = |3x + 1 - 7| < \varepsilon$$

Thus $|x - 2| < \delta \implies |3x + 1 - 7| < \varepsilon$, thus $\lim_{x \to 2} 3x + 1 = 7$. Q.E.D.

127.) Suppose $\lim_{x \to 5} x^2 = 25$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x-5| < \delta \implies \left|x^2 - 25\right| < \varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$\left|x^2 - 25\right| = |(x-5)(x+5)| = |x-5|\,|x+5| < \varepsilon$$

$$\implies |x-5| < \frac{\varepsilon}{|x+5|}$$

Let $\delta = \dfrac{\varepsilon}{|x+5|}$, then we know that $|x-5| < \delta$. Manipulating the inequality:

$$|x-5| < \frac{\varepsilon}{|x+5|} \implies |x-5|\,|x+5| = |(x-5)(x+5)| = \left|x^2 - 25\right| < \varepsilon$$

Thus $|x-5| < \delta \implies |x^2 - 25| < \varepsilon$, thus $\lim_{x \to 5} x^2 = 25$. Q.E.D.

128.) Suppose $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$, then for all $\varepsilon > 0$ there exists $\delta > 0$ where

$$\left|x - \frac{1}{2}\right| < \delta \implies \left|\frac{1}{x} - 2\right| < \varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$\left|\frac{1}{x} - 2\right| = ***$$