Lemma 1: Let $f : \mathbb{R}^d \to [0, \infty]$ be a non-negative measurable function and $E \subset \mathbb{R}^d$ a set with measure 0. Then we have that

$$\int_E f(x)\, dx = 0.$$

Proof: Since $m(E) = 0$, we have that $f\chi_E$ is zero almost everywhere, and thus by [1], we see

$$\int_E f(x)\, dx = \int f(x)\chi_E(x)\, dx = \int |f(x)\chi_E(x)|\, dx = 0.$$

■

Lemma 2: Fix a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and $\varepsilon > 0$. Define a collection of disjoint non-negative intervals $\{I_k\}_{k\in\mathbb{N}_0}$ where $\sup I_k \le \inf I_{k+1}$, and denote $n_k := \inf I_k$ and $n'_k := \sup I_k$. Also define $P_k := f^{-1}(I_k)$ and $P := \bigcup_{k\in\mathbb{N}_0} P_k$. If $\{m(P_k)\}_{k\in\mathbb{N}_0}$ is a monotone sequence, then we have that $\sum_{k\in\mathbb{N}_0} n_k m(P_k) < \infty$ if and only if f is integrable on P.

Proof: Assume the former. Since the P_k are disjoint, we can see that

$$\int_P f(x)\, dx = \sum_{k\in\mathbb{N}_0} \int_{P_k} f(x)\, dx \le \sum_{k\in\mathbb{N}_0} n_k m(P_k) < \infty,$$

showing that f is integrable over P. Conversely, assume f is integrable over P. In the case where $m(P_k)$ is non-decreasing, we have that

$$\infty > \int_P f(x)\, dx \ge \sum_{k\in\mathbb{N}_0} n'_k m(P_k) \ge \sum_{k\in\mathbb{N}} n'_k m(P_k) \ge \sum_{k\in\mathbb{N}} n_{k-1} m(P_{k-1}) = \sum_{k\in\mathbb{N}_0}$$

and thus

$$2\sum_{k\in\mathbb{N}_0} (2^k\varepsilon) m(E_k) = \sum_{k\in\mathbb{N}_0} (2^{k+1}\varepsilon) m(E_k) < \infty,$$

completing the proof ■

1, 2) Define the functions f_a and F_a on $\mathbb{R}^d$ as

$$f_a := \begin{cases} |x|^{-a} & |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad F_a := \frac{1}{1 + |x|^a},$$

then f_a is integrable if and only if $0 < a < d$, and f_a is integrable if and only if $a > d$.

Proof: We have that both f_a and F_a are non-negative and measurable, thus by additivity and **Lemma 1**,

$$\int f_a = \int_{\mathbb{R}^d\setminus\{0\}} f_a + \int_{\{0\}} f_a = \int_{\mathbb{R}^d\setminus\{0\}} f_a + 0 = \int_{\mathbb{R}^d\setminus\{0\}} f_a,$$

and similarly $\int F_a = \int_{\mathbb{R}^d\setminus\{0\}} F_a$. Thus, it suffices to show that these functions are integrable outside any open ball centered at the origin.

We first consider f_a and assume $a \neq 0$. Fix $\varepsilon > 0$, and for integers $k \geq 0$, define the collection of sets

$$I_k = \left\{x \in \mathbb{R}^d : 2^k\varepsilon \leq f_a(x) < 2^{k+1}\varepsilon\right\}.$$

Solving for $|x|$, we see that

$$I_k = \left\{x : (2^{k+1}\varepsilon)^{-1/a} < |x| \leq (2^k\varepsilon)^{-1/a}\right\},$$

and denoting $A = \left\{x : 1 < |x| \leq 2^{1/|a|}\right\}$, we have by relative-dilation invariance that

$$m(I_k) = (2^{k+1}\varepsilon)^{-d/a} m(A),$$

thus

$$\sum_{k\in\mathbb{N}_0} (2^{k+1}\varepsilon) m(I_k) = \sum_{k\in\mathbb{N}_0} (2^{k+1}\varepsilon)^{-(d/a)+1} m(A) = \frac{m(A)\varepsilon^{1-(d/a)}}{2} \sum_{k\in\mathbb{N}} (2^k)^{1-(d/a)},$$

but the rightmost series only converges when $0 < a < d$. Thus, by **Lemma 2**, we have that $\int_{|x|\geq\varepsilon} f_a(x)\,dx$ is bounded by the sum, and thus is integrable when $0 < a < d$. Conversely, suppose f_a is integrable, then again by **Lemma 2**,

$$\sum_{k\in\mathbb{N}_0} (2^k\varepsilon) m(I_k) = \frac{1}{2}\sum_{k\in\mathbb{N}_0} (2^{k+1}\varepsilon) m(I_k) = \frac{m(A)\varepsilon^{1-(d/a)}}{4} \sum_{k\in\mathbb{N}} (2^k)^{1-(d/a)} < \infty,$$

which we previously established to be true only when $0 < a < d$, thus the case for f_a has been proven.

We now turn our attention to F_a. Let $g(x) = x/(1+x)$ and define $h(x) = g(x+r)$ where $r = \varepsilon/(1-\varepsilon)$. We can see that $h(0) = \varepsilon$ and that $h(x) \to 1$ as $x \to \infty$. Fix $0 < \varepsilon < 1$ and define for each $k \in \mathbb{N}_0$ the set

$$J_k = \left\{x \in \mathbb{R}^d : h(k)\varepsilon \leq F_a(x) < h(k+1)\varepsilon\right\}$$

3) Since $\eta_1(x) = \min\{f(x), \eta(x)\}$, we have for fixed $x \in \mathbb{R}^d$ that $\eta_1(x) = f(x)$ or $\eta_1(x) = \eta(x)$. By assumption g is non-negative, and so in the case where $\eta_1(x) = \eta(x)$, we have that $\eta_2(x) = \eta(x) - \eta_1(x) = \eta(x) - \eta(x) = 0 \leq g(x)$, thus showing that

$\eta_2(x) \leq g(x)$. Otherwise, $\eta_1(x) = f(x)$. We have $\eta \leq f + g$ by definition, thus $\eta - f \leq g$, which implies $\eta_2(x) = \eta(x) - \eta_1(x) = \eta(x) - f(x) \leq g(x)$ and $\eta_2(x) \leq g(x)$. ■

4) Let $\{E_k\}_{k\in\mathbb{N}}$ be a collection of measurable sets in $\mathbb{R}^d$ where $\sum_{k\in\mathbb{N}} m(E_k) < \infty$ and define the set E as

$$E = \left\{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\right\},$$

then $m(E) = 0$.

Proof: If a point x belongs to infinitely many E_k, then $\chi_{E_k}(x) = 1$ for infinitely many k, and thus $\sum_{k\in\mathbb{N}} \chi_{E_k}(x) = \infty$. Conversely, if $\sum_{k\in\mathbb{N}} \chi_{E_k}(x) = \infty$, then $\chi_{E_k}(x) = 1$ for infinitely many k, thus x is contained in infinitely many E_k. By assumption, $\sum_{k\in\mathbb{N}} m(E_k) < \infty$, thus we have

$$\sum_{k\in\mathbb{N}} m(E_k) = \sum_{k\in\mathbb{N}} \int \chi_{E_k} < \infty,$$

and thus by **Corollary 1.10** we have shown that $\sum_{k\in\mathbb{N}} \chi_{E_k} < \infty$ for almost all x, proving $m(E) = 0$. ■

5) Fix $\varepsilon > 0$, then the function $f : \mathbb{R}^d \to \mathbb{R}$, defined as

$$f(x) = \begin{cases} \frac{1}{|x|^{d+1}} & x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is integrable over all $x \in \mathbb{R}^d$ where $|x| \geq \varepsilon$. We also have that

$$\int_{|x|\geq\varepsilon} f(x)\,dx \leq \frac{C}{\varepsilon}$$

for some constant C.

Proof: For integers $k \geq 0$, define the collection of disjoint sets

$$A_k = \left\{x \in \mathbb{R}^d : 2^k\varepsilon \leq |x| < 2^{k+1}\varepsilon\right\},$$

and let $g : \mathbb{R}^d \to \mathbb{R}$ be a function defined as

$$g(x) = \sum_{k\in\mathbb{N}_0} \frac{1}{(2^k\varepsilon)^{d+1}} \chi_{A_k}(x).$$

If $|x| \geq \varepsilon$, then x is contained in A_k for some k, thus $|x| \geq 2^k\varepsilon$ and

$$g(x) = \frac{1}{(2^k\varepsilon)^{d+1}}\chi_{A_k}(x) = \frac{1}{(2^k\varepsilon)^{d+1}} \geq \frac{1}{|x|^{d+1}} = f(x),$$

showing that $f(x) \leq g(x)$. By monotonicity, we have $\int_{|x|\geq\varepsilon} f \leq \int_{|x|\geq\varepsilon} g$. It now suffices to show $\int_{|x|\geq\varepsilon} g < \infty$.

Denote $A = \{x \in \mathbb{R}^d : 1 \leq |x| < 2\}$, we have that $A_k = 2^k\varepsilon A$, and thus by relative-dilation invarance we have $m(A_k) = (2^k\varepsilon)^d m(A)$. By **Corollary 10**, we have

$$\int_{|x|\geq\varepsilon} g(x)\,dx = \int_{|x|\geq\varepsilon} \sum_{k\in\mathbb{N}_0} \frac{1}{(2^k\varepsilon)^{d+1}}\chi_{A_k}(x)\,dx = \sum_{k\in\mathbb{N}_0} \int_{|x|\geq\varepsilon} \frac{1}{(2^k\varepsilon)^{d+1}}\chi_{A_k}(x)\,dx$$

$$= \sum_{k\in\mathbb{N}_0} \frac{m(A_k)}{(2^k\varepsilon)^{d+1}} = m(A)\sum_{k\in\mathbb{N}_0} \frac{(2^k\varepsilon)^d}{(2^k\varepsilon)^{d+1}} = m(A)\sum_{k\in\mathbb{N}_0} \frac{1}{2^k\varepsilon} = \frac{2m(A)}{\varepsilon},$$

and since $m(A)$ is finite, we have shown that $\int_{|x|\geq\varepsilon} g = C/\varepsilon < \infty$ where $C = 2m(A)$, thus completing the proof. ■

References

[1] https://proofwiki.org/wiki/Measurable_Function_Zero_A.E._iff_Absolute_Value_has_Zero_Integral. Accessed on 10/30/24.