2.1 Definitions and Examples

2a) We have that (1 2)(1 3) = (1 3 2), thus closure isn't satisfied. We also have no identity element.

2b) We have that $(sr^2)(sr^2) = (sr^2)(r^{-2}s) = s^2 = e$, thus closure isn't satisfied. We also have no identity element.

6) Let G be an abelian group. Since $|e| = 1$, we have that the set of torsion elements in G is non-empty. Now, suppose $x, y \in G$ have finite orders m and n respectively, then mn is also finite and $(xy)^{mn} = x^{mn}y^{mn} = (x^m)^n(y^n)^m = e$, thus $|xy| \leq mn$ and xy is a torsion element and closure is satisfied. Additionally, $|x^{-1}| = |x| = n$, thus x^{-1} is a torsion element and inverses exist. Thus, the torsion elements of G satisfy the subgroup criterion. ■

9) Since all matrices in $\mathrm{SL}_n(F)$ have non-zero determinant, we have that $\mathrm{SL}_n(F) \subset \mathrm{GL}_n(F)$. Let $A, B \in \mathrm{SL}_n(F)$, thus $\det(A) = \det(B) = 1$. Since $\det(AB) = \det(A)\det(B) = 1$, we have closure. We also have that A is invertible, thus there exists A^{-1} where $AA^{-1} = E$, but $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(E) = 1$, thus $\det(A^{-1}) = \det(E)/\det(A) = 1$, thus $A^{-1} \in \mathrm{SL}_n(F)$ and inverses are satisfied. Consequently, $\mathrm{SL}_n(F) \leq \mathrm{GL}_n(F)$. ■

10a) Fix a group G and subgroups $H, K \leq G$. Clearly $H \cap K$ is non-empty since they both contain e. Let $x, y \in H \cap K$, thus $x, y \in H$ and $x, y \in K$. By closure, we have that $xy \in H$ and $xy \in K$, thus $xy \in H \cap K$. We also have that $x^{-1} \in H$ and $x^{-1} \in K$, thus $x^{-1} \in H \cap K$, thus $H \cap K$ satisfies the subgroup criterion. ■

2.2 Centralizers and Normalizers, Stabilizers and Kernels

2) Given a group G, we have $C_G(Z(G)) = \{g \in G : gag^{-1} = a \text{ for all } a \in Z(G)\}$, but since every element in G commutes with every element in $Z(G)$, we have that $C_G(Z(G)) = G$. Since $C_G(A) \subseteq N_G(A)$ for all subsets $A \subseteq G$, we have $G \subseteq N(Z(G))$, thus $G = N_G(Z(G))$. ■

5a) We have (1 2 3)(1 3 2) = (1), so $C_G(A) = A$.

6a) Fix a subgroup $H \leq G$. Since H is a group, it suffices to show that $H \subseteq N_G(H)$. Fix $a \in H$, then $aH = H$, thus $aHa^{-1} = (aH)a^{-1} = Ha^{-1} = H$, which shows that $a \in N_G(H)$, thus $H \subseteq N_G(H)$. ■

6b) Let G be an abelian group and $H \leq G$ a subgroup, then H is also abelian and $C_G(H) = G$, thus $H \subseteq C_G(H)$. Since H is a group, we have $H \leq C_G(H)$. ■

2.3 Cyclic Groups and Cyclic Subgroups

1) We have that $\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \langle 9\rangle, \langle 15\rangle \leq \mathbb{Z}/45\mathbb{Z}$. If $m \mid n$, then $\langle m\rangle \leq \langle n\rangle$.

3) Since $48 = 2^4 \times 3$, any number ≤ 48 without a 2 or 3 in its prime factorization generates $\mathbb{Z}/48\mathbb{Z}$.

12a For all n, we have $(0,0)^n = (0,0)$, $(1,1)^n = (0,0)$ or $(1,1)$, $(0,1)^n = (0,1)$ or $(0,0)$, and $(1,0)^n = (1,0)$ or $(0,0)$, thus no element in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generates the group. ■

12b) Fix $(a,k) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$, then there exists no $n \in \mathbb{N}$ where $(a,k)^n = (a,2k)$, so this group is not cyclic. ■

19) Fix a group H where $h \in H$ and suppose $\phi : \mathbb{Z} \to H$ is a homomorphism where $\phi(1) = h$. Clearly $\phi(0) = e$. If $n > 0$, then $\phi(n) = \phi(1 + 1 + \cdots + 1) = \phi(1)\phi(1)\cdots\phi(1) = h^n$, and if $n < 0$ we have $\phi(n) = \phi(-|n|) = \phi(|n|)^{-1} = (h^n)^{-1} = h^{-n}$. Consquently, we are forced to define ϕ as $n \mapsto h^n$, thus ϕ is unique. ■