21.) Let $v = \inf(S)$ and $y \in S$ such that $y > v$. For the sake of establishing a contradiction, assume that no element $s \in S$ exists such that $s < y$. Consequently, $y \leq x$ for all $x \in S$, thus y is a lower bound of S, but $y > v$, thus $v \neq \inf(S) \Rightarrow\Leftarrow$, thus for all $y \in S$ such that $y > v$, there exists $s \in S$ such that $s < y$. Q.E.D.

25.) Let $v \in \mathbb{R}$ be a lower bound of B, thus $v \leq x$ for all $x \in B$. Since for all $a \in A$, $a \in B$, $v \leq a$ for all $a \in A$, thus A is bounded below. Now, let $v_b = \inf(B)$ and $v_a = \inf(A)$. Since $v_a \geq v$ for all lower bounds v of A, and since v_b is a lower bound of A, $v_a \geq v_b$, thus $\inf(A) \geq \inf(B)$. Q.E.D.

28.) a.) Let $u \in \mathbb{R}$ be an upper bound of S, thus $u \geq x$ for all $x \in S$, thus $-u \leq -x$ for all $x \in S$, thus $-u \leq y$ for all $y \in -S$, thus $-u$ is a lower bound of $-S$, thus $-S$ is bounded below. Q.E.D.

b.) Let $u = \sup(S)$. Since u is an upper bound of S, $-u$ is a lower bound of $-S$. For the sake of establishing a contradiction, suppose there exists $v \in \mathbb{R}$ such that $-u < v$ and v is a lower bound of $-S$, thus $u > -v$. Since v is a lower bound of $-S$, $v \leq y$ for all $y \in -S$, thus $-v \geq -y$ for all $y \in -S$, thus $-v \geq x$ for all $x \in S$, thus $-v$ is an upper bound of S, but since $u > -v$, $u \neq \sup(S) \Rightarrow\Leftarrow$, thus $-\sup(S) = -u = \inf(-S)$. Q.E.D.

29.) a.) Since $S \neq \varnothing$, $\mathscr{L} \neq \varnothing$. In addition, since S is bounded below, there exists $v \in \mathbb{R}$ such that $v = \inf(S)$, thus $v \geq x$ for all $x \in \mathscr{L}$, thus v is an upper bound of $\mathscr{L}$, thus $\mathscr{L}$ is bounded above. Q.E.D.

b.) Let $w = \sup(\mathscr{L})$. For the sake of establishing a contradiction, suppose there exists $x \in S$ such that $x < w$, thus x is not an upper bound of $\mathscr{L}$, thus there exists $l \in \mathscr{L}$ such that $l > x$, thus l is not a lower bound of S, thus $l \notin \mathscr{L} \Rightarrow\Leftarrow$, thus $x \in S \implies x \geq w$, thus w is a lower bound of S. Q.E.D.

c.) Since $w = \sup(\mathscr{L})$, $w \geq l$ for all $l \in \mathscr{L}$, thus $w \geq l$ for all lower bounds l of S, thus $w = \inf(S)$. Q.E.D.

30.) a.) For unbounded sets, infinite suprema and infima make sense, as for all $x \in \mathbb{R}$, $-\infty < x < \infty$. In addition, there exists no $y \in \mathbb{R}$ such that $y < -\infty$ or $y > \infty$, thus $\sup(S) = \infty$ and $\inf(S) = -\infty$.

b.) If we constrict the empty set to being a subset of $\mathbb{R}$, then we can reason that ∞ and $-\infty$ are vacuously upper and lower bounds of the empty set. Since $\infty > -\infty$, $\inf(\varnothing) = \infty$, and $\sup(\varnothing) = -\infty$.

31.) a.) False; let $S = \{x \in \mathbb{Q} : 0 \leq x < \pi\}$. By definition, all $x \in S$ are rational, but $\sup(S) = \pi$ is irrational.

b.) False; let $S = \{x \in \mathbb{R}\backslash\mathbb{Q} : 0 < x < 3\}$. By definition, all $x \in S$ are irrational, but $\sup(S) = 3$ is rational.

33.) Let $x, y \in \mathbb{R}$, and consider $|x|$:

$$|x| = |x - y + y| \leq |x - y| + |y|$$

$$\implies |x| \leq |x - y| + |y|$$

$$\implies |x| - |y| \leq |x - y|$$

$$\implies (|x| - |y|) - (|x| - |y|) - |x - y| \leq |x| - |y| \leq |x - y|$$

$$\implies 0 - |x - y| = -|x - y| \leq |x| - |y| \leq |x - y|$$

$$\implies \Big||x| - |y|\Big| \leq |x - y|$$

Thus the inequality holds. Q.E.D.

35.) Let S be a bounded set, thus there exist upper and lower bounds $u, v \in \mathbb{R}$ of S, thus $v \leq x \leq u$ for all $x \in S$. Let $M = \max(|u|, |v|)$, thus $M \geq 0$, $M \geq u$ and $-M \leq v$, thus for all $x \in S$, $-M < x < M$, thus $|x| < M$ for some $M \geq 0$.

Now suppose there exists $M \geq 0$ such that $|x| \leq M$ for all $x \in S$, thus $-M \leq x \leq M$, thus $-M \leq x$ and $x \leq M$ for all x, thus $-M$ and M are lower and upper bounds of S respectively, thus S is bounded, thus S is bounded if and only if there exists $M \geq 0$ such that $|x| \leq M$ for all $x \in S$. Q.E.D.

36.) When learning the various integration techniques taught in calculus 2, I noticed that some of them involved manipulating the dx term. I initially found this confusing, as I though that $\frac{d}{dx}$ was a single operator that could not be seperated. I later found out that this was not the case, and that it can often be treated just like a fraction. What helped me to realize this was when I watched a video series that explained calculus in a more intuitive, less purely algebraic way. In this context, dx being a seperate variable simply made sense.

39.) a.) A sequence is defined as a function $x(n)$ such that $x : \mathbb{N} \to \mathbb{R}$.

b.) The sequence $\{x_n\}_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0 : \exists k \in \mathbb{N} : n \geq k \implies |x_n - L| < \epsilon$$

43.) a.) $\lim_{n\to\infty} \frac{1}{10n} = 10$

b.) $\lim_{n\to\infty} \sin n$ diverges

c.) Suppose $x_n \to 15$ and $x_n \to -77$. Since $x_n \to 15$, x_n gets arbitrarily close to 15. Also, since $x_n \to -77$, x_n gets arbitrarily close to -77. However, as x_n gets closer to 15, x_n moves farther from -77, and vice versa, thus x_n cannot get arbitrarily close to both, thus x_n cannot converge to both.