46.) Consider the sequences $x_n = \dfrac{1}{n}$ and $y_n = 0$. For all $n \in \mathbb{N}$, the following is true:

$$0 < \frac{1}{2^n} \le \frac{1}{n}$$

Since we know that $\dfrac{1}{n} \to 0$ and $0 \to 0$, then by the squeeze theorem we can conclude that $2^{-n} \to 0$. Q.E.D.

53.) a.) False; many sequences diverge.

b.) False; $(-1)^n$ diverges, but $[(-1)^n]^2 = (-1)^{2n} = 1 \to 1$.

54.) a.) $(-1)^n$

b.) DNE; A sequence cannot converge to a value while having terms that are arbitrarily far from that value.

77.) Let x_n and y_n be sequences where $x_n = y_n = n$. Since n is stricly increasing, so are x_n and y_n. However $x_n - y_n = n - n = 0$, and 0 is not strictly increasing, thus $x_n - y_n$ is not strictly increasing.

79.) A subsequence of x_n is a sequence y_k such that $y_k = x_{n_k}$ for all $k \in \mathbb{N}$, and where n_k is a strictly increasing sequence of natural numbers.

80.) You can view subsequences as a more abstract form of function composition. You have x_n, which is a function $f : \mathbb{N} \to \mathbb{R}$, and you have n_k, which is a function $g : \mathbb{N} \to \mathbb{N}$. When you let $y_k = x_{n_k}$, this is equivalent to $y_k = (f \circ g)(k) = f(g(k))$. Since the codomain of g and domain of f are the same, namely $\mathbb{N}$, we know this function composition is well defined.

81.) Let $y_k = x_{2k}$, thus $y_k = (-1)^{2k} = 1$ for all $k \in \mathbb{N}$, thus $y_k \to 1$, thus y_k is convergent.

82.) a.) DNE; if $y_k \preceq x_n$ and $x_n \to L$, then $y_k \to L$ as per theorem 19.

b.) $x_n = n - (-1)^n n$; x_n diverges, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, $\therefore$ $y_n \to 0$.

83.) Setting $x_n = y_k$ and solving for n, we can find a suitable n_k:

$$2n - 1 = 8k + 1 \implies 2n = 8k + 2 \implies n = 4k + 1$$

Thus when $n_k = 4k+1$, $x_{n_k} = 2(4k+1) - 1 = 8k + 2 - 1 = 8k + 1 = y_k$, thus $y_k \preceq x_n$. Q.E.D.

84.) Setting $x_n = y_k$ and solving for n, we can find a suitable n_k:

$$2n - 1 = 8k^2 + 24k + 17 \implies 2n = 8k^2 + 24k + 18 \implies n = 4k^2 + 12k + 9$$

Thus when $n_k = 4k^2 + 12k + 9$, $x_{n_k} = 2(4k^2 + 12k + 9) - 1 = 8k^2 + 24k + 17 = y_k$, thus $y_k \preceq x_n$. Q.E.D.

90.) Since x_n is bounded, there exists $M \in \mathbb{R}$ where $x_n \leq M$ for all $n \in \mathbb{N}$. Since $y_k \preceq x_n$, $y_k = x_{n_k} \leq M$ for all $k \in \mathbb{N}$, thus y_k is bounded. Q.E.D.

91.) a.) $S = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$ are the friends of x_n.

b.) $S = \{1\}$ is the friend of y_n.

92.) $S = \{n \in \mathbb{N} : 21 \leq n \leq 56\}$ are the friends of z_n.

113.) $z \in \mathbb{R}$ is a cluster point of S if for all $\varepsilon > 0$ there exists $x \in S$ such that $0 < |x - z| < \varepsilon$.

114.) Let $S = \{y \in \mathbb{R} : |x - y| < r\}$,

$$|x - y| < r \implies -r < x - y < r \implies -r - x < -y < r - x$$

$$\implies x - r < y < x + r \implies S = (x - r, x + r)$$

Thus $\{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$ Q.E.D.

115.) $\mathbb{Z}$ has no cluster points.

116.) $[0, 1]$ are the cluster points of $[0, 1)$.

117.) For 0 to be a cluster point of $[1, 2]$, then for all $\varepsilon > 0$ there exists $a \in [1, 2]$ where $|a - 0| < \varepsilon$. Let $\varepsilon = \dfrac{1}{2}$, then

$$|a - 0| = |a| < \frac{1}{2} \implies -\frac{1}{2} < a < \frac{1}{2}$$

But since $-\dfrac{1}{2} < a < \dfrac{1}{2}$, $a \notin [1, 2]$, there exists ε where no $a \in [1, 2]$ satisfies the definition, thus 0 is not a cluster point of $[1, 2]$. Q.E.D.

124.) Let $f : E \to \mathbb{R}$, $c \in E'$, and $L \in \mathbb{R}$, then $f(x) \to L$ as $x \to c$, or $\lim_{x \to c} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta \implies |f(x) - L| < \varepsilon$.

125.) Suppose $\lim_{x \to c} f(x) = a$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - c| < \delta \implies |f(x) - a| < \varepsilon$$

Since $|f(x) - a| = |a - a| = |0| = 0 < \varepsilon$, $|a - a| < \varepsilon$, thus $\lim_{x \to c} f(x) = a$. Q.E.D.

126.) Suppose $\lim_{x\to 2} 3x+1=7$, then for all $\varepsilon>0$, there exists $\delta>0$ where

$$|x-2|<\delta \implies |3x+1-7|<\varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$|3x+1-7| = |3x-6| = |3(x-2)| = 3\,|x-2| < \varepsilon$$

$$\implies |x-2| < \frac{\varepsilon}{3}$$

Let $\delta = \frac{\varepsilon}{3}$. Manipulating the inequality:

$$|x-2| < \frac{\varepsilon}{3} \implies 3\,|x-2| = |3(x-2)| = |3x-6| = |3x+1-7| < \varepsilon$$

Thus $|x-2|<\delta \implies |3x+1-7|<\varepsilon$, thus $\lim_{x\to 2} 3x+1=7$. Q.E.D.

127.) Suppose $\lim_{x\to 5} x^2=25$, then for all $\varepsilon>0$, there exists $\delta>0$ where

$$|x-5|<\delta \implies \left|x^2-25\right|<\varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$\left|x^2-25\right| = |(x-5)(x+5)| = |x-5|\,|x+5| < \varepsilon$$

$$\implies |x-5| < \frac{\varepsilon}{|x+5|}$$

Let $\delta = \frac{\varepsilon}{|x+5|}$. Manipulating the inequality:

$$|x-5| < \frac{\varepsilon}{|x+5|} \implies |x-5|\,|x+5| = |(x-5)(x+5)| = \left|x^2-25\right| < \varepsilon$$

Thus $|x-5|<\delta \implies |x^2-25|<\varepsilon$, thus $\lim_{x\to 5} x^2=25$. Q.E.D.