13.) Let $x, y \in \mathbb{R}$ such that $x, y > 0$ and $n \in \mathbb{N}$. To show that $x < y \iff x^n < y^n$, we must prove it both ways.

Suppose $x < y$:

$$\begin{aligned} x < y &\implies x - y < 0 \\ &\implies (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1} < 0 \\ &\implies x^n - y^n < 0 \\ &\implies x^n < y^n \end{aligned}$$

Now suppose $x^n < y^n$:

$$\begin{aligned} x^n < y^n &\implies x^n - y^n < 0 \\ &\implies (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1} < 0 \\ &\implies x - y < 0 \\ &\implies x < y \end{aligned}$$

Thus $x < y \iff x^n < y^n$. Q.E.D.

22.) a.) False, as $\sup((0,1)) = 1 \notin (0,1)$

b.) False, as $\sup([0,1]) = 1 \in [0,1]$

c.) True, as $\sup([0,1]) = 1 \in [0,1]$

d.) True, as $\sup((0,1)) = 1 \notin (0,1)$

23.) Let $u, v \in \mathbb{R}$ such that $u = \sup(S)$ and $v = \inf(S)$. For the sake of establishing a contradiction, suppose $v > u$. Since $v > u$, $v > x$ for all $x \in S$, but for $v = \inf(S)$, $v \leq x$ for all $x \in S$, thus $v > u \implies v \neq \inf(S) \Rightarrow\Leftarrow$, thus $v \leq u$. Q.E.D.

24.) a.) $S = \mathbb{R}$; $\mathbb{R} = (-\infty, \infty)$ and thus has no upper bound nor lower bound.

b.) DNE; for $S \subseteq \mathbb{R}$ to be bounded, there must exist $u \in \mathbb{R}$ such that $u = \sup(S)$.

c.) $S = [0,1)$; $\inf(S) = 0 \in S$ and $\sup(S) = 1 \notin S$.

d.) $S = (-\infty, 1]$; $\sup(S)$ exists but $\inf(S)$ does not.

e.) $S = (0,1)$; $\sup(S)$ exists but $\sup(S) \notin S$.

26.) My first exposure to functions was in the context of programming rather than math, specifically in procedural programming languages like C. In this context, functions are often used for their side-effects, rather than being purely functional. When I took discrete math, I was introduced to the mathematical notion of a function, i.e. a mapping between two sets. To me, the distinction between these two types of functions has always been clear as their use cases are quite different.

27.) a.) $S = (0, 1)$; $\sup(S) = 1 \notin S$.

b.) $S = (0, 1)$; $\inf(S) = 0 \notin S$.

c.) DNE; for $u = \sup(S)$, $u \geq x$ for all $x \in S$, but $u < t$ and $t \in S$, thus $u \neq \sup(S)$.

32.) For all $x \in \mathbb{R}$, $|x|$ is defined as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$