Exercises 14.4

1.) Find f_x and f_y:

$$f_x = 4x,\ f_y = 2y - 5$$

Now find f_x and f_y at $(1, 2)$:

$$f_x(1,2) = 4(1) = 4,\ f_y(1,2) = 2(2) - 5 = 4 - 5 = -1$$

Now find the equation of the plane:

$$z + 4 = 4(x-1) - 1(y-2) \implies z = 4x - 4 - y + 2 - 4$$

$$\implies z = 4x - y - 6$$

11.) Find f_x and f_y:

$$f_x = \ln(xy-5) + \frac{xy}{xy-5},\ f_y = \frac{x^2}{xy-5}$$

Now find f, f_x, and f_y at $(2, 3)$:

$$f(2,3) = 1 + 2\ln[(2)(3) - 5] = 1 + 2\ln(1) = 1$$

$$f_x(2,3) = \ln(6-5) + \frac{(2)(3)}{(2)(3)-5} = 0 + \frac{6}{6-5} = 6$$

$$f_y(2,3) = \frac{(2)^2}{(2)(3)-5} = \frac{4}{6-5} = 4$$

Now find the linearization of f:

$$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) = 1 + 6(x-2) + 4(y-3)$$

$$\implies L(x,y) = 6x + 4y - 23$$

15.) Find f_x and f_y:

$$f_x = \frac{4y}{1+(xy)^2},\ f_y = \frac{4x}{1+(xy)^2}$$

Now find f, f_x, and f_y at $(1, 1)$:

$$f(1,1) = 4\arctan(1) = \frac{4\pi}{4} = \pi$$

$$f_x(1,1) = \frac{4(1)}{1+1} = 2,\ f_y(1,1) = \frac{4(1)}{1+1} = 2$$

Now find the linearization of f:

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = \pi + 2(x-1) + 2(y-1)$$

$$\implies L(x,y) = 2x + 2y + \pi - 4$$

31.) Find dz:

$$dz = f_x\,dx + f_y\,dy = 10x\,dx + 2y\,dy$$

Now find dx and dy:

$$dx = 1.05 - 1 = 0.05,\ \ dy = 2.1 - 2 = 0.1$$

Thus

$$dz = 10(1)(0.05) + 2(2)(0.1) = 0.5 + 0.4 = 0.9$$

And

$$\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5(1)^2 - (2)^2 = 0.9225$$

Exercises 14.6

7.) a.)

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{1}{y}, -\frac{x}{y^2} \right\rangle$$

b.)

$$\nabla f(2,1) = \left\langle \frac{1}{1}, -\frac{2}{1^2} \right\rangle = \langle 1, -2 \rangle$$

c.)

$$D_u f(2,1) = f_x(2,1)\,\frac{3}{5} + f_y(2,1)\,\frac{4}{5} = \frac{3}{5(1)} - \frac{4(2)}{5(1)^2} = -\frac{5}{5} = -1$$

13.) Find f_s and f_t:

$$f_s = \sqrt{t},\ \ f_t = \frac{s}{2\sqrt{t}}$$

Now find the unit vector in the direction of $\overline{v}$:

$$\|\overline{v}\| = 5 \implies \hat{v} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

Thus

$$D_v f(2,4) = \frac{2\sqrt{4}}{\sqrt{5}} - \frac{2}{2\sqrt{5(4)}} = \frac{16-2}{4\sqrt{5}} = \frac{14}{4\sqrt{5}} = \frac{7}{2\sqrt{5}}$$

19.) Find f_x and f_y:

$$f_x = \frac{y}{2\sqrt{xy}},\ f_y = \frac{x}{2\sqrt{xy}}$$

Now find $\overline{v}$:

$$\overline{v} = \langle 5-2, 4-8\rangle = \langle 3, -4\rangle$$

Now find the unit vector in the direction of $\overline{v}$:

$$\|\overline{v}\| = 5 \implies \hat{v} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

Thus

$$D_v f(2,8) = \frac{3(8)}{5(2)\sqrt{(2)(8)}} - \frac{4(2)}{5(2)\sqrt{(2)(8)}} = \frac{24-8}{10(4)} = \frac{16}{40} = \frac{2}{5}$$

21.) Find ∇f then $\nabla f(4,1)$:

$$\nabla f = \left\langle \frac{4y}{2\sqrt{x}}, 4\sqrt{x} \right\rangle,\ \overline{v} = \nabla f(4,1) = \left\langle \frac{4(1)}{2\sqrt{4}}, 4\sqrt{4} \right\rangle = \langle 1, 8\rangle$$

Now find $\|\overline{v}\|$:

$$\|\overline{v}\| = \sqrt{1^2+8^2} = \sqrt{65}$$

Thus the maximum rate of change of f at $(4,1)$ is $\sqrt{65}$ in the direction of $\langle 1, 8\rangle$.

43.) Find ∇f:

$$f_x = y^2z^3,\ f_y = 2xyz^3,\ f_z = 3xy^2z^2$$
$$\implies \nabla f = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle$$

Now find $\nabla f(2,2,1)$:

$$\nabla f(2,2,1) = \langle 4, 8, 24\rangle$$

Now find the equation of the tangent plane:

$$4(x-2) + 8(y-2) + 24(z-1) = 4x + 8y + 24z = 48$$

$$\implies x + 2y + 6z = 12$$

Finally, parametrize P_0 and $\nabla f(2,2,1)$ to find the equation of the normal line:

$$P_0 + t\nabla f(2,2,1) = \langle 2,2,1\rangle + t\langle 4,8,24\rangle = \langle 2+4t, 2+8t, 1+24t\rangle$$

$$\implies \frac{x-2}{4} = \frac{y-2}{8} = \frac{z-1}{24}$$

45.) Find ∇f:

$$f_x = 1 + yze^{xyz},\ f_y = 1 + xze^{xyz},\ f_z = 1 + xye^{xyz}$$
$$\implies \nabla f = \langle 1 + yze^{xyz}, 1 + xze^{xyz}, 1 + xze^{xyz} \rangle$$

Now find $\nabla f(0,0,1)$:

$$\nabla f(0,0,1) = \langle 1,1,1 \rangle$$

Now find the equation of the plane:

$$x + y + (z-1) \implies x + y + z = 1$$

Finally, parametrize P_0 and $\nabla f(0,0,1)$ to find the equation of the normal line:

$$P_0 + t\nabla f(0,0,1) = \langle 0,0,1 \rangle + t\,\langle 1,1,1 \rangle = \langle t, t, 1+t \rangle$$

$$\implies x = y = z - 1$$