Definitions

1.) Given a set X, a *metric* d on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties:

i.) For all $x, x' \in X$, $x = x' \iff d(x, x') = 0$. *Non-degeneracy*

ii.) For all $x, x' \in X$, $d(x, x') = d(x', x)$. *Symmetry*

iii.) For all $x, x', x'' \in X$, $d(x, x'') \leq d(x, x') + d(x', x'')$. *Triangle Inequality*

2.) Given a metric d on X, $r \in \mathbb{R}_{>0}$, and $x \in X$, the *open ball of radius r centered at x* is defined as follows:

$$\text{Ball}(x, r) = \{x' \in X : d(x, x') < r\}$$

3.) Given a metric d on X, the *metric topology* on X is defined as follows:

$$\mathcal{T}_d \subseteq \mathcal{P}(X) = \left\{ U \subseteq X : U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \right\}$$

Where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of open balls in X.

4.) A *metric space* (X, d) is a set X equipped with a metric d.

5.) The *taxi cab* metric on $\mathbb{R}^n$ is defined as follows:

$$d_{taxi}(x, x') = \sum_{i=1}^{n} |x_i - x'_i|$$

6.) The l^∞ metric on $\mathbb{R}^n$ is defined as follows:

$$d_{l^\infty}(x, x') = \max_{i=1,2,\dots,n} |x_i - x'_i|$$

7.) The *standard* metric on R^n is defined as follows:

$$d_{std}(x, x') = \sqrt{\sum_{i=1}^{n} (x_i - x'_i)^2}$$

8.) The *discrete* metric on a set X is defined as follows:

$$d_{discrete}(x, x') = \begin{cases} 1 & x \neq x' \\ 0 & x = x' \end{cases}$$

9.) Given metric spaces X and Y, an *isometry* between them is a bijection $f : X \to Y$ where for all $x, x' \in X$, we have $d_Y(f(x), f(x')) = d_X(x, x')$.

10.) Given a metric space X, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a function $x : \mathbb{N} \to X$. We say that x_n converges to a point $L \in X$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ where

$$n > N \implies d(x_n, L) < \varepsilon$$

Proofs

a.) Let W be a compact topological space, and $\sim$ an equivalence relation on W. Consider the quotient space $W/{\sim}$ and quotient map $q : W \to W/{\sim}$, and let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of $W/{\sim}$. Since U_α is open in $W/{\sim}$ for all $\alpha \in \mathcal{A}$, we know that $q^{-1}(U_\alpha)$ is open. Let $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of open sets in X where $V_\alpha = q^{-1}(U_\alpha)$. Since $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of $W/{\sim}$, we know that $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of W, and since W is compact, there exists a finite subcover $\{V_\beta\}_{\beta\in\mathcal{B}}$ of W where $\mathcal{B} \subseteq \mathcal{A}$. Finally, let $\{U_\beta\}_{\beta\in\mathcal{B}}$ be a collection of open sets in $W/{\sim}$ where $U_\beta = q(V_\beta)$. Since $\{V_\beta\}_{\beta\in\mathcal{B}}$ is a cover of W, we know that $\{U_\beta\}_{\beta\in\mathcal{B}}$ is a cover of $W/{\sim}$, and since $\mathcal{B} \subseteq \mathcal{A}$ and is finite, we know that it is a finite subcover of $W/{\sim}$, thus $W/{\sim}$ is compact. ■

b.) Let X be a Hausdorff topological space and equip $A \subseteq X$ with the subspace topology. Let $a, a' \in A$ be distinct points. Since $A \subseteq X$, a and a' are also distinct points in X, thus there exist open sets U, U' in X where $a \in U$, $a' \in U'$, and $U \cap U' = \varnothing$. It is clear that $a \in A \cap U$ and $a' \in A \cap U'$. We also know that by definition $A \cap U$ and $A \cap U'$ are open in A. Finally, note that $(A \cap U) \cap (A \cap U') = A \cap (U \cap U') = A \cap \varnothing = \varnothing$, thus the two sets are distinct, and thus A is Hausdorff. ■