40.) Let $\varepsilon > 0$, and $n \geq k$ for some $k \in \mathbb{N}$, then

$$\left|\frac{2n-1}{n} - 2\right| < \varepsilon$$

Solving for n, we can find a sufficiently large value for k:

$$\left|\frac{2n-1}{n} - 2\right| = \left|\frac{2n-1-2n}{n}\right| = \left|\frac{-1}{n}\right| = \frac{1}{n} < \varepsilon$$

$$\implies n > \frac{1}{\varepsilon}$$

Now, let $k > \frac{1}{\varepsilon}$, then

$$\left|\frac{2n-1}{n} - 2\right| = \frac{1}{n} \leq \frac{1}{k} < \frac{1}{1/\varepsilon} = \varepsilon$$

$$\therefore \left|\frac{2n-1}{n} - 2\right| < \varepsilon$$

Thus $x_n \to 2$. Q.E.D.

41.) Let $\varepsilon > 0$, and $n \geq k$ for some $k \in \mathbb{N}$, then

$$\left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$$

Solving for n, we can find a sufficiently large value for k:

$$\left|\frac{(-1)^n}{n} - 0\right| = \left|\frac{(-1)^n}{n}\right| = \frac{|(-1)^n|}{|n|} = \frac{1}{n} < \varepsilon$$

$$\implies n > \frac{1}{\varepsilon}$$

Now, let $k > \frac{1}{\varepsilon}$, then

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{k} < \frac{1}{1/\varepsilon} = \varepsilon$$

$$\therefore \left|\frac{(-1)^n}{n} - 0\right| < \varepsilon$$

Thus $x_n \to 0$. Q.E.D.

42.) Let $\varepsilon > 0$, and $n \geq k$ for some $k \in \mathbb{N}$, then

$$\left|\frac{3n+1}{5n-2} - \frac{3}{5}\right| < \varepsilon$$

Solving for n, we can find a sifficiently large value for k:

$$\left|\frac{3n+1}{5n-2} - \frac{3}{5}\right| = \left|\frac{15n+5-15n+6}{25n-10}\right| = \left|\frac{11}{25n-10}\right| = \frac{|11|}{|25n-10|} = \frac{11}{25n-10} < \varepsilon$$

$$\implies \frac{11}{\varepsilon} < 25n - 10 \implies \frac{11}{25\varepsilon} + \frac{2}{5} < n$$

Now, let $k > \dfrac{11}{25\varepsilon} + \dfrac{2}{5}$, then

$$\left|\frac{3n+1}{5n-2} - \frac{3}{5}\right| = \frac{11}{25n-10} \leq \frac{11}{25k-10} < \frac{11}{25\left(\frac{11}{25\varepsilon} + \frac{2}{5}\right) - 10} = \frac{11}{\frac{11}{\varepsilon} + 10 - 10} = \frac{11}{\frac{11}{\varepsilon}} = \varepsilon$$

$$\therefore \left|\frac{3n+1}{5n-2} - \frac{3}{5}\right| < \varepsilon$$

Thus $x_n \to \dfrac{3}{5}$. Q.E.D.

51.) For x_n to be bounded, we must find $M \in \mathbb{R}$ such that $M \geq x_n$ for all $n \in \mathbb{N}$. Consider x_n. When n is odd, $2(-1)^n + 5 = 2(-1) + 5 = 5 - 2 = 3$. When n is even, $2(-1)^n + 5 = 2(1) + 5 = 2 + 7$, thus $x_n = 3$ or $x_n = 7$ for all $n \in \mathbb{N}$. Let $M = 7$, thus $M \geq x_n$ for all $n \in \mathbb{N}$, thus x_n is bounded. Q.E.D.

55.) 1.) $(-\infty, 0]$

2.) DNE

3.) DNE

56.) 4.) $(0, 1)$

5.) $x_n = \dfrac{n}{2n}$

6.) $x_n = n$

7.) $(0, 1)$

61.) Let $\varepsilon > 0$, and $n \geq k$ for some $k \in \mathbb{N}$, then

$$\left|\frac{6n+1}{2n-1} - 3\right| < \varepsilon$$

Solving for n, we can find a sufficiently large value for k:

$$\left|\frac{6n+1}{2n-1} - 3\right| = \left|\frac{6n+1-6n+3}{2n-1}\right| = \left|\frac{4}{2n-1}\right| = \frac{|4|}{|2n-1|} = \frac{4}{2n-1} < \epsilon$$

$$\implies \frac{4}{\varepsilon} < 2n - 1 \implies \frac{4}{2\varepsilon} + \frac{1}{2} < n$$

Now, let $k > \dfrac{4}{2\varepsilon} + \dfrac{1}{2}$, then

$$\left|\frac{6n+1}{2n-1} - 3\right| = \frac{4}{2n-1} \leq \frac{4}{2k-1} < \frac{4}{2\left(\frac{4}{2\varepsilon} + \frac{1}{2}\right) - 1} = \frac{4}{\left(\frac{4}{\varepsilon}\right) + 1 - 1} = \frac{4}{\frac{4}{\varepsilon}} = \varepsilon$$

$$\therefore \ \left|\frac{6n+1}{2n-1} - 3\right| < \varepsilon$$

Thus $x_n \to 3$. Q.E.D.

65.) So far, the topic I have had the most trouble grasping is ε-k convergence proofs. I understand the process of constructing the proof, but am still working on understanding the logic.

72.) We can show that $x_n = \dfrac{(-1)^n}{n}$ is not monotonic by showing that it is neither nonincreasing nor nondecreasing. First, consider x_1 and x_2:

$$x_1 = \frac{(-1)^1}{1} = -\frac{1}{1} = -1$$

$$x_2 = \frac{(-1)^2}{1} = \frac{1}{1} = 1$$

Thus $x_1 < x_2$, thus x_n is not nonincreasing. Next, consider x_2 and x_3:

$$x_2 = 1$$

$$x_3 = \frac{(-1)^3}{1} = -\frac{1}{1} = -1$$

Thus $x_2 > x_3$, thus x_n is not nondecreasing, thus x_n is not monotonic. Q.E.D.

73.) a.) First, assume that x_n is a nondecreasing sequence. From this, we know that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. We can manipulate this inequality as follows:

$$x_n \leq x_{n+1} \implies 0 \leq x_{n+1} - x_n = (\partial x)_n$$

Thus $(\partial x)_n \geq 0$ for all $n \in \mathbb{N}$, thus ∂x is a nonnegative sequence.

Next, assume that ∂x is a nonnegative sequence. From this, we know that $(\partial x)_n = x_{n+1} - x_n \geq 0$ for all $n \in \mathbb{N}$. We can manipulate this inequality as follows:

$$x_{n+1} - x_n \geq 0 \implies x_{n+1} \geq x_n$$

Thus $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$, thus x_n is nondecreasing, thus x_n is nondecreasing if and only if ∂x is nonnegative. Q.E.D.

b.) This reminds me about a video I watched a while back that was about discrete calculus, and the "discrete derivative" resembles the ∂x sequence.

78.) a.) $x_n = n$.

b.) $x_n = -\frac{1}{n}$.

98.) x_n is a cauchy sequence if for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$n, m \geq k \implies |x_n - x_m| < \varepsilon$$

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