Chapter 1

5.) In D_n for $n \geq 3$, let e be the identity element, r be a counter-clockwise rotation by $(360/n)^\circ$, and s be a reflection across the vertical axis. These elements of D_n form a generating set of D_n, as every symmetry in D_n is represented by one of the following group elements:

$$\{e, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$$

From this set we can conclude that D_n has $2n$ elements, or $|D_n| = 2n$.

15.) Considering the symmetries of a nonsquare rectangle, we find the identity element e, the 180° counter-clockwise rotation r, and the reflection across the vertical axis s. These elements form the symmetries of the rectangle with the following set:

$$\{e, r, s, sr\}$$

Using this set we can construct the cayley table for the symmetries of the rectangle as follows:

	e	r	s	sr
e	e	r	s	sr
r	r	e	sr	s
s	s	sr	e	r
sr	sr	s	r	e

Chapter 2

7.)

1. Let a and b be odd integers, thus there exist $m, n \in \mathbb{Z}$ where $a = 2m + 1$ and $b = 2n+1$. We can see that $a+b = 2m+1+2n+1 = 2m+2n+2 = 2(m+n+1)$, which is even, thus the odd integers are not closed under addition, and thus are not a group under addition.

2. For the sake of establishing a contradiction, let a and b be odd integers where $a + b = a$, thus for some $m, n \in \mathbb{Z}$:

$$2m + 1 + 2n + 1 = 2m + 2n + 2 = 2m + 1$$

$$\implies 2n + 1 = 0 \implies 2n = -1 \implies n = -\frac{1}{2}$$

Thus $n \notin \mathbb{Z}$. $\Rightarrow\Leftarrow$ Thus no such a, b exist in the odd integers, thus the odd integers lack an identity element under addition, and thus are not a group under addition.

14.) 1. $(ab)^3 = (ab)(ab)(ab) = ababab$

2. $(ab^{-2}c)^{-2} = (ab^{-2}c)^{-1}(ab^{-2}c)^{-1} = (c^{-1}b^2a^{-1})(c^{-1}b^2a^{-1}) = c^{-1}b^2a^{-1}c^{-1}b^2a^{-1}$

33.)

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Chapter 3

1.) a. $|\mathbb{Z}_{12}| = |\{[0], [1], [2], \ldots, [11]\}| = 12$

$|[0]| = 1, |[1]| = 12, |[2]| = 6, |[3]| = 4, |[4]| = 3, |[5]| = 12, |[6]| = 2, |[7]| = 12$
$|[8]| = 3, |[9]| = 4, |[10]| = 6, |[11]| = 12$

b. $|U(10)| = |\{1, 3, 7, 9\}| = 4$

$|1| = 1, |3| = 4, |7| = 4, |9| = 2$

c. $|U(12)| = |\{1, 5, 7, 11\}| = 4$

$|1| = 1, |5| = 2, |7| = 2, |11| = 2$

d. $|U(20)| = |\{1, 3, 7, 9, 11, 13, 17, 19\}| = 8$

$|1| = 1, |3| = 4, |7| = 4, |9| = 2, |11| = 2, |13| = 4, |17| = 4, |19| = 2$

e. $|D_4| = |\{e, r, r^2, r^3, s, sr, sr^2, sr^3\}| = 8$

$|e| = 1, |r| = 4, |r^2| = 2, |r^3| = 4, |s| = 2, |sr| = 2, |sr^2| = 2, |sr^3| = 2$

One might notice that the order of any element of a group never exceeds the order of the group itself.

4.) Consider a group $(G, *)$ and let $a, a^{-1} \in G$ where $|a| = n$ for $n \in \mathbb{N}$. We can see that $a^n = e \implies (a^{-1})^n * a^n = (a^{-1})^n = a^{-n} * a^n = a^{-n} = e$, thus $(a^{-1})^n = e$. Now consider $m \in \mathbb{N}$ where $m < n$, then $a^m \neq e \implies a^{-m} * a^m \neq a^{-m} \implies a^{-m} \neq e$. From this we can conclude that $|a| = |a^{-1}|$. ■

7.) Consider a group $(G, *)$ and let $a, b, c \in G$ where $|a| = 6$ and $|b| = 7$. First, we can see that $(a^4 * c^{-2} * b^4)^{-1} = b^{-4} * c^2 * a^{-4}$. Since $|a| = 6$, we can see that $a^{-4} = e * a^{-4} = a^6 * a^{-4} = a^2$, and since $|b| = 7$, we can see that $b^{-4} = e * b^{-4} = b^7 * b^{-4} = b^3$. From this we can conclude that $(a^4 * c^{-2} * b^4)^{-1} = b^3 * c^2 * a^2$. ■

13.) Consider a group $(G, *)$ and let $a, x, x^{-1} \in G$ where $|a| = n$ for $n \in \mathbb{N}$. We can see the following:

$$\left(xax^{-1}\right)^n = \underbrace{\left(xax^{-1}\right) \cdot \left(xax^{-1}\right) \cdot \ldots \cdot \left(xax^{-1}\right)}_{n \text{ times}}$$

$$= xa\left(x^{-1}x\right)a\left(x^{-1}x\right)a \cdot \ldots \cdot a\left(x^{-1}x\right)ax^{-1} = xa^nx^{-1} = xex^{-1} = xx^{-1} = e$$

Thus $(xax^{-1})^n = e$. Now consider $m \in \mathbb{N}$ where $m < n$, thus $a^m \neq e$. Knowing this, we can see the following:

$$\left(xax^{-1}\right)^m = \underbrace{\left(xax^{-1}\right) \cdot \left(xax^{-1}\right) \cdot \ldots \cdot \left(xax^{-1}\right)}_{m \text{ times}}$$

$$= xa\left(x^{-1}x\right)a\left(x^{-1}x\right)a \cdot \ldots \cdot a\left(x^{-1}x\right)ax^{-1} = xa^mx^{-1} \neq xex^{-1} = xx^{-1} = e$$

$$\implies \left(xax^{-1}\right)^m \neq e$$

From this we can conclude that $|a| = |xax^{-1}|$. ■

14.) Consider a group $(G, *)$ and let $a \in G$ be the only element in G with order 2, thus $a^2 = e$. Let $b, x, x^{-1} \in G$ where $b = xax^{-1}$, and consider b^2:

$$b^2 = \left(xax^{-1}\right)\left(xax^{-1}\right) = xa\left(x^{-1}x\right)ax^{-1} = xa^2x^{-1} = xex^{-1} = xx^{-1} = e$$

If a is the only element in G with order 2, then $b = a$, thus $xax^{-1} = a \implies xax^{-1}x = xa = ax$, thus $a \in Z(G)$. ■