Chapter 10

3.) Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ where $f \mapsto f'$, and let $f, g \in \mathbb{R}[x]$. We can see that $\phi(f) + \phi(g) = f' + g' = (f+g)' = \phi(f+g)$, thus ϕ is a homomorphism. ■

7.) Let $\phi : G \to H$ and $\sigma : H \to K$ be homomorphisms. Let $a, b \in G$, then $(\sigma \circ \phi)(ab) = \sigma(\phi(ab)) = \sigma(\phi(a)\phi(b)) = \sigma(\phi(a))\sigma(\phi(b)) = (\sigma\circ\phi)(a)(\sigma\circ\phi)(b)$, thus $\sigma\circ\phi$ is a homomorphism. We can also show that $\ker\phi$ is a subgroup of $\ker\sigma\circ\phi$. It is obvious that $\ker\phi \subseteq \ker\sigma\circ\phi$, as $a \in \ker\phi \implies \sigma(\phi(a)) = \sigma(e_H) = e_K \implies a \in \ker\sigma\circ\phi$. In addition, the kernel of any homomorphism is a group, thus we know that $\ker\phi$ is a subgroup of $\ker\sigma\circ\phi$. Finally, ***

9.) Let $a, b \in G \oplus H$ where $a = (g, h)$ and $b = (g', h')$, and consider the map $\phi : G \oplus H \to G$ where $(g, h) \mapsto g$. We can see that $\phi(ab) = \phi(gg', hh') = gg' = \phi(g, h)\phi(g', h')$, thus ϕ is a homomorphism. Next, let $a \in \ker G \oplus H$ where $a = (g, h)$, thus $\phi(a) = \phi(g, h) = g = e_G$, thus $\ker\phi = \{(e_G, h) : h \in H\}$. ■

12.) Let $k, n \in \mathbb{Z}$ where $k \mid n$ and $m \in \langle k\rangle$, and consider the map $\phi : \mathbb{Z}_n/\langle k\rangle \to \mathbb{Z}_k$ where $m\mathbb{Z}_n \mapsto m \pmod{k}$. For $m, m' \in \langle k\rangle$, we can see that $\phi(m\mathbb{Z}_n \circ m'\mathbb{Z}_n) = \phi((mm')\mathbb{Z}_n) = mm' \pmod{k} = \phi(m\mathbb{Z}_n)\phi(m'\mathbb{Z}_n)$, thus ϕ is a homomorphism, and thus $\mathbb{Z}_n/\langle k\rangle \cong \mathbb{Z}_k$. ■

16.)