Section 3.2

17.) Find the characteristic equation and the general solution:

$$y'' + y' - 2y = 0 \implies r^2 + r - 2 = (r+2)(r-1) \implies r_1 = -2,\ r_2 = 1$$

$$\therefore\ y = c_1e^{-2t} + c_2e^t \text{ and } y' = -2c_1e^{-2t} + c_2e^t$$

Solve for c_1 and c_2 given $y(t_0) = 1,\ y'(t_0) = 0$:

$$1 = c_1e^{-2(0)} + c_2e^0 = c_1 + c_2$$

$$0 = -2c_1e^{-2(0)} + c_2e^0 = -2c_1 + c_2$$

$$\therefore\ c_1 = \frac{1}{3},\ c_2 = \frac{2}{3}$$

$$\therefore\ y_1 = \frac{1}{3}e^{-2t} + \frac{2}{3}e^t$$

Now solve for c_1 and c_2 given $y(t_0) = 0,\ y'(t_0) = 1$:

$$0 = c_1e^{-2(0)} + c_2e^0 = c_1 + c_2$$

$$1 = -2c_1e^{-2(0)} + c_2e^0 = -2c_1 + c_2$$

$$\therefore\ c_1 = -\frac{1}{3},\ c_2 = \frac{1}{3}$$

$$\therefore\ y_2 = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

Thus y_1 and y_2 comprise the fundamental set described by theorem 3.2.5

18.) Find the characteristic equation and the general solution:

$$y'' + 4y' + 3y = 0 \implies r^2 + 4r + 3 = (r+3)(r+1) \implies r_1 = -3, r_2 = -1$$

$$y = c_1e^{-3t} + c_2e^{-t} \text{ and } y' = -3c_1e^{-3t} - c_2e^{-t}$$

Solving for c_1 and c_2 given $y(t_0) = 1,\ y'(t_0) = 0$:

$$1 = c_1e^{-3(1)} + c_2e^{-1} = c_1e^{-3} + c_2e^{-1}$$

$$0 = -3c_1e^{-3(1)} - c_2e^{-1} = -3c_1e^{-3} - c_2e^{-1}$$

$$\therefore\ c_1 = -\frac{1}{2e^3},\ c_2 = \frac{3}{2e}$$

$$\therefore\ y_1 = \frac{3}{2}e^{-(t+1)} - \frac{1}{2}e^{-3(t+1)}$$

Now solve for c_1 and c_2 given $y(t_0) = 0,\ y'(t_0) = 1$:

$$0 = c_1e^{-3} + c_2e^{-1}$$

$$1 = -3c_1e^{-3} - c_2e^{-1}$$

$$\therefore\ c_1 = -\frac{e^3}{2},\ c_2 = \frac{e}{2}$$

$$\therefore\ y_2 = \frac{1}{2}e^{-t+1} - \frac{1}{2}e^{-3t+3}$$

Thus y_1 and y_2 comprise the fundamental set described by theorem 3.2.5

Section 3.4

18.) Let $y = v(t)y_1$:

$$y = v(t)y_1 = v(t)t^2,\ y' = v'(t)t^2 + 2tv(t)$$

$$y'' = v''(t)t^2 + 2tv'(t) + 2v(t) + 2tv'(t) = t^2v''(t) + 4tv'(t) + 2v(t)$$

$$\therefore\ t^2y'' - 4ty' + 6y = t^2\left[t^2v''(t) + 4tv'(t) + 2v(t)\right] - 4t\left[v'(t)t^2 + 2tv(t)\right] + 6v(t)t^2$$

$$= t^4v''(t) + 4t^3v'(t) - 4t^3v'(t) + 2t^2v(t) + 6t^2v(t) - 8t^2v(t) = t^4v''(t) = 0$$

Let $w(t) = v'(t)$:

$$t^4v''(t) = t^4w'(t) = 0 \implies w'(t) = 0 \implies w(t) = v'(t) = c_1$$

$$\implies v(t) = c_1t + c_2$$

$$\therefore\ y = (c_1t + c_2)y_1 = c_1t^3 + c_2t^2$$

Thus $y_2 = t^3$ is another solution to the equation.

19.) Let $y = v(t)y_1$:

$$y = tv(t),\ y' = v(t) + tv'(t),\ y'' = v'(t) + v'(t) + tv''(t) = 2v'(t) + tv''(t)$$

$$\therefore\ t^2y'' + 2ty' - 2y = t^2\left[2v'(t) + tv''(t)\right] + 2t\left[v(t) + tv'(t)\right] - 2tv(t)$$

$$= 2t^2v'(t) + t^3v''(t) + 2tv(t) + 2t^2v'(t) - 2tv(t) = t^3v''(t) + 4t^2v'(t)$$

$$= tv''(t) + 4v'(t) = 0$$

Let $w(t) = v'(t)$:

$$tv''(t) + 4v'(t) = tw'(t) + 4w(t) = 0 \implies w'(t) + \frac{4}{t}w(t) = 0$$

$$\mu(t) = e^{\int p(t)dt} = e^{4\ln|t|} = t^4$$

$$\therefore w(t) = v'(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)\ dt = \frac{1}{t^4}\int 0t^4 = \frac{c_1}{t^4}$$

$$\therefore\ v(t) = \int \frac{c_1}{t^4}\ dt = -\frac{c_1}{3t^3} + c_2$$

$$\therefore\ y = v(t)y_1 = \left(c_2 - \frac{c_1}{3t^3}\right)t = c_2t - \frac{c_1t}{3t^3} = c_2t - \frac{c_1}{t^2}$$

Thus $y_2 = t^{-2}$ is another solution to the equation.

20.) Let $y = v(t)y_1$:

$$y = t^{-1}v(t),\ y' = -t^{-2}v(t) + t^{-1}v'(t),\ y'' = 2t^{-3}v(t) - t^{-2}v'(t) - t^{-2}v'(t) + t^{-1}v''(t)$$

$$\therefore\ t^2y'' + 3ty' + y = t^2\left[t^{-1}v''(t) - 2t^{-2}v'(t) + 2t^{-3}v(t)\right] + 3t\left[t^{-1}v'(t) - t^{-2}v(t)\right] + t^{-1}v(t)$$
$$= tv''(t) - 2v'(t) + 2t^{-1}v(t) + 3v'(t) - 3t^{-1}v(t) + t^{-1}v(t) = tv''(t) + v'(t) = 0$$

Let $w(t) = v'(t)$:

$$tv''(t) + v'(t) = tw'(t) + w(t) = 0 \implies w'(t) + \frac{1}{t}w(t) = 0$$

$$\mu(t) = e^{\int p(t)dt} = e^{\ln|t|} = t$$

$$\therefore\ w(t) = v'(t) = \frac{1}{t}\int 0t\ dt = \frac{c_1}{t}$$

$$\therefore\ v(t) = \int \frac{c_1}{t}\ dt = c_1 \ln|t| + c_2$$

$$\therefore\ y = v(t)y_1 = (c_1 \ln|t| + c_2)\, t^{-1} = c_1 t^{-1} \ln|t| + c_2 t^{-1}$$

Thus $y_2 = t^{-1}\ln|t| = t^{-1}\ln t$ is also a solution to the equation.