65.) So far, the topic I have had the most trouble understanding is $\varepsilon - \delta$ limit proofs.

66.) a.) $x_n = n$

b.) DNE; a sequence cannot converge to a value while also having terms arbitrarily far from that value.

68.) Let $P(n)$ propose that $1 - y_1 - y_2 - \cdots - y_n = (1 - x_1)(1 - x_2)\cdots(1 - x_n)$. For the base case, consider $P(1)$:

$$y_1 = x_1 \implies 1 - y_1 = 1 - x_1$$

Thus $P(1)$ holds. For the induction step, assume $P(n)$ and consider $P(n+1)$:

$$1 - y_1 - y_2 - \cdots - y_n = (1 - x_1)(1 - x_2)\cdots(1 - x_n)$$

$$\begin{aligned}\implies 1 - y_1 - y_2 - \cdots - y_n - y_{n+1} &= (1 - x_1)(1 - x_2)\cdots(1 - x_n) - y_{n+1} \\ &= (1 - x_1)(1 - x_2)\cdots(1 - x_n) - x_{n+1}(1 - x_1)(1 - x_2)\cdots(1 - x_n) \\ &= (1 - x_1)(1 - x_2)\cdots(1 - x_n)(1 - x_{n+1})\end{aligned}$$

Thus $P(n)$ holds for all $n \in \mathbb{N}$. Q.E.D.

69.) a.) $\{x_n\}_{n\in\mathbb{N}}$ is nondecreasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

b.) $\{x_n\}_{n\in\mathbb{N}}$ is strictly decreasing if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.

70.) Consider x_n and x_{n+1}:

$$x_{n+1} - x_n = 2(n+1) + 6 - (2n + 6) = 2n + 2 + 6 - 2n - 6 = 2$$

$$\implies x_{n+1} - x_n = 2 > 0 \implies x_{n+1} > x_n$$

Thus $x_{n+1} > x_n$ for all $n \in \mathbb{N}$, thus x_n is strictly increasing. Q.E.D.

71.) Consider x_n and x_{n+1}:

$$x_{n+1} - x_n = 5^{-(n+1)} - 5^{-n} = \frac{1}{5^{n+1}} - \frac{1}{5^n} = \frac{1}{5^{n+1}} - \frac{5}{5^{n+1}} = -\frac{4}{5^{n+1}} < 0$$

$$\implies x_{n+1} - x_n = -\frac{4}{5^{n+1}} < 0 \implies x_{n+1} < x_n$$

Thus $x_{n+1} < x_n$ for all $n \in \mathbb{N}$, thus x_n is strictly decreasing. Q.E.D.

74.)

a_n	X	X	X	X			X
b_n	X	X		X		X	X
c_n	X						
d_n	X	X					

75.)

e_n	X	X	X		X		X
f_n			X		X		X
g_n							
h_n			X				X

76.)

i_n							
j_n			X		X	X	
k_n							
l_n			X				X

85.) a.) True; Let x_n be a bounded sequence, thus $x_n \leq M$ for some $M \in \mathbb{R}$. Let $y_k \preceq x_n$, thus for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ where $y_k = x_n$, thus $y_k \leq M$ for all $k \in \mathbb{N}$, thus y_k is bounded.

b.) True; Since x_n is monotonic, all terms x_n maintain monotonicity with all x_m where $m > n$, thus if $y_k \preceq x_n$, then y_k is monotonic.

91.) a.) $S = \mathbb{N}$ are the friends of x_n.

b.) $S = \{2n\}$ are the friends of y_n.

92.) $S = \{n \in \mathbb{N} : 1 \leq n \leq 36\}$ are the friends of z_n.

99.) Let $x_n = n - (-1)^n n$. x_n is unbounded, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, thus $y_k \preceq x_n$ and $y_k \to 0$.

100.) Every cauchy sequence is convergent according to theorem 23, and no convergent sequence can be unbounded.

101.) Every cauchy sequence is convergent according to theorem 23.

102.) $x_n = \dfrac{\sin x}{x}$ is not monotonic, but $x_n \to 0$, thus x_n is convergent and thus cauchy.

103.) Let $x_n = \dfrac{1}{n^2}$. Since $x_n \to \dfrac{\pi^2}{6}$, x_n is convergent and thus cauchy. Q.E.D.

104.) Let x_n and y_n be cauchy sequences, thus $x_n \to A$ and $y_n \to B$ for some $A, B \in \mathbb{R}$. Let $z_n = x_n y_n$, thus $z_n \to AB$, thus z_n is convergent and thus cauchy. Q.E.D.

105.) Similarly, let $z_n = \dfrac{x_n}{y_n}$ where $y_n \neq 0$, thus $z_n \to \dfrac{A}{B}$, thus z_n is convergent and thus cauchy. Q.E.D.

114.) Let $S = \{y \in \mathbb{R} : |x - y| < r\}$,

$$y \in S \implies |x - y| < r \implies -r < x - y < r \implies -r - x < -y < r - x$$

$$\implies x - r < y < x + r \implies y \in (x - r, x + r)$$

Thus $S \subseteq (x - r, x + r)$. Next, consider $(x - r, x + r)$:

$$y \in (x - r, x + r) \implies x - r < y < x + r \implies r - x > -y > -r - x$$

$$\implies -r < x - y < r \implies |x - y| < r \implies y \in S$$

Thus $(x - r, x + r) \subseteq S$, thus $S = (x - r, x + r)$. Q.E.D.

126.) awd

127.) Suppose $\lim_{x \to 5} x^2 = 25$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - 5| < \delta \implies \left|x^2 - 25\right| < \varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$\left|x^2 - 25\right| = |(x-5)(x+5)| = |x-5|\,|x+5| < \varepsilon$$

$$\implies |x - 5| < \frac{\varepsilon}{|x+5|}$$

Let $\delta = \dfrac{\varepsilon}{|x+5|}$. Manipulating the inequality:

$$|x - 5| < \frac{\varepsilon}{|x+5|} \implies |x-5|\,|x+5| = |(x-5)(x+5)| = \left|x^2 - 25\right| < \varepsilon$$

Thus $|x - 5| < \delta \implies |x^2 - 25| < \varepsilon$, thus $\lim_{x \to 5} x^2 = 25$. Q.E.D.