11.) Quantifiers are important because they allow us to make more rigorous mathematical statements. For example, we could state that $x < y$, but without context, the exact meaning of this statement is ambiguous. However, using quantifiers, we can provide context. For example, $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$. This gives context to the original statement and allows us to fully understand it.

12.) *

13.) Assume $x, y \in \mathbb{R}$, $x, y > 0$, and $n \in \mathbb{N}$. Because it is a biconditional statement, we must prove it both ways:

a.) Assume $x < y$. We can manipulate the inequality as follows:

$$
\begin{aligned}
x &< y \\
nx &< ny \\
\ln(x^n) &< \ln(y^n) \\
e^{\ln(x^n)} &< e^{\ln(y^n)} \\
x^n &< y^n
\end{aligned}
$$

Thus $x < y \implies x^n < y^n$.

b.) Assume $x^n < y^n$ and manipulate as follows:

$$
\begin{aligned}
x^n &< y^n \\
\ln(x^n) &< \ln(y^n) \\
nx &< ny \\
x &< y
\end{aligned}
$$

Thus $x^n < y^n \implies x < y$.

Thus $x < y \iff x^n < y^n$. Q.E.D.

14.) a.) Assume $x \in \mathbb{R}$ and $0 < x < 1$. Multiplying each term by x^n gives us $0(x^n) < x(x^n) < 1(x^n) \implies 0 < x^{n+1} < x^n$, thus $x^{n+1} < x^n$. Q.E.D.

b.) Assume $x \in \mathbb{R}$ and $x > 1$. Multiplying both terms by x^n gives us $x(x^n) > 1(x^n) \implies x^{n+1} > x^n$. Q.E.D.

15.) Consider the base case where $n = 1$, $2^n = 2^1 = 2 \geq 1$, thus the base case holds. Next, suppose the inequality holds for n, and consider $n + 1$. $n + 1 \leq n + n \leq 2^n + 2^n = 2(2^n) = 2^{n+1}$, thus $n + 1 \leq 2^{n+1}$, thus the induction step holds. Q.E.D.

16.) a.) $s \in \mathbb{R}$ is an upper bound of G if for all $x \in G$, $s \geq x$.

b.) G is bounded if there exist $u, v \in \mathbb{R}$ such that u is an upper bound of G and v is a lower bound of G.

c.) $n \in \mathbb{R}$ is the infimum of G if n is a lower bound of G and for all $v \in \mathbb{R}$ such that v is a lower bound of G, $n \geq v$.

17.) Three upper bounds for S are 11, 12, and 13. Three lower bounds for S are 1, 0, and -1. $\sup(S) = 11$, as 11 is an upper bound of S and $11 < u$ for all upper bounds u of S. Finally, $\inf(S) = 1$, as 1 is a lower bound of S and $1 > v$ for all lower bounds v of S.

18.) Let $v \in \mathbb{R}$ be a lower bound of B, thus $v \leq x$ for all $x \in B$. Since $A \subseteq B$, we know that for all $a \in A$, $a \in B$, thus for all $a \in A$, $v \leq a$, thus v is a lower bound of A. Q.E.D.

19.) Let $s \in \mathbb{R}$ such that $s = \sup(A)$. Since $s = \sup(A)$, $s \leq u$ for all upper bounds u of A. Consequently, for $u \in \mathbb{R}$ to be an upper bound of A, $u \geq s$, thus the set of all upper bounds of A is $\{u \in \mathbb{R} : u \geq s\} = [\sup(A), \infty)$. Q.E.D.

20.) $S = [-2, 5] = \{x \in \mathbb{R} : -2 \leq x \leq 5\}$. Consider -2: -2 is by definition a lower bound of S as $-2 \leq x$ for all $x \in S$. Next, for establishing a contradiction, suppose v is a lower bound of S and $v > -2$, but since $-2 \in S$, $v \not\leq x$ for all $x \in S$, thus $v > -2$ cannot be a lower bound of $S \Rightarrow\Leftarrow$, thus $\inf(S) = -2$. Q.E.D.