If F is a collection of sets, denote $\bigcup F$ and $\bigcap F$ as $\bigcup_{E\in F} E$ and $\bigcap_{E\in F} E$, respectively.

1) Given sets $F_1,\dots,F_n$, there exists a collection of $N = 2^n - 1$ disjoint sets $F_1^*,\dots,F_N^*$ where $\bigcup_{k=1}^n F_k = \bigcup_{k=1}^N F_k^*$.

Proof: Denote $F = \bigcup_{k=1}^n F_k$ and let F^* be the collection of sets of the form $F_1^* \cap \cdots \cap F_n^*$ where F_i^* is either F_i or $F_i^{\complement}$ and excluding the set $F_1^{\complement} \cap \cdots \cap F_n^{\complement}$. Note that $|F^*| = 2^n - 1$. Let A and B be distinct sets in F^*, then for some $1 \le i \le n$, and after possibly switching A and B, we have that $A \subseteq F_i$ and $B \subseteq F_i^{\complement}$, but then $A \cap B \subseteq F_i \cap F_i^{\complement} = \varnothing$, thus A and B are disjoint and F^* is a collection of disjoint sets.

Fix $x \in F$, then we can take I and J to be sets where $I = \{F_i : x \in F_i\}$ and $J = \left\{F_i^{\complement} : x \notin F_i\right\}$, given $1 \le i \le n$; we can see that $x \in \bigcap I \cap \bigcap J$. Since $x \in F_i$ and $x \notin F_i$ are mutually exclusive conditions where exactly one must be true, we have that each F_i is contained in exactly one of I or J, thus I and J are disjoint and $|I| + |J| = n$. We must also have that I is non-empty, since $x \in F$ implies x is contained in some F_i. Thus, $\bigcap I \cap \bigcap J$ takes the form of a set in F^*, showing that x is contained in some set in F^* and $F \subseteq \bigcup F^*$. Now, suppose x is contained in some set in F^*. By the construction of F^*, we have some i where $x \in F_i$, thus $x \in F$, showing that $\bigcup F^* \subseteq F$ and $F = \bigcup F^*$. Since the sets in F^* are disjoint and $|F^*| = 2^n - 1$, we are finished. ■