52.) Let $M, N \in \mathbb{R}$ where $M \geq x_n$ and $N \geq y_n$ for all $n \in \mathbb{N}$. Consider $z_n = x_n + y_n$. $x_n + y_n \leq M + N$, thus $z_n \leq M + N$ for all $n \in \mathbb{N}$, thus z_n is bounded. Q.E.D. ***

57.) We can show this by induction. For the base case, consider a_1 and a_2:

$$a_1 = 1,\ a_2 = \frac{4+2}{3} = 2$$

Thus $a_1 < a_2 < 4$, thus the base case holds. Now, suppose $a_{n-1} < a_n < 4$, and consider a_{n+1}:

$$a_n = \frac{4-2a_{n-1}}{3},\ a_{n+1} = \frac{4-2a_n}{3}$$
$$a_{n+1} - a_n = \frac{4-2a_n}{3} - \frac{4-2a_{n-1}}{3} = -\frac{2}{3}(a_n + a_{n-1}) ***$$

58.) Let $L, U \subseteq \mathbb{R}$. ***

64.) a.) Let $S = S - S$ and $S \neq \varnothing$.

76.)

i_n							
j_n			X		X	X	
k_n	X						
l_n			X				X

96.) Let $S \subseteq \mathbb{N}$, and consider the cases of S:

Case $S = \mathbb{N}$: Let $x_n = 1$. Since $1 \geq 1$, the friends of x_n are $\mathbb{N}$, and thus S.

Case $S \subset \mathbb{N}$: Let the sequence x_n be defined as follows:

$$x_n = \begin{cases} 1 & n \in S \\ -\frac{1}{n} & n \notin S \end{cases}$$

Since $1 > -\dfrac{1}{n}$ for all $n \in \mathbb{N}$, all n for which $x_n = 1$ are friends of x_n. In addition, for all $m, n \in \mathbb{N}$:

$$m > n \implies \frac{1}{n} > \frac{1}{m} \implies -\frac{1}{n} < -\frac{1}{m}$$

Thus all n for which $x_n = -\dfrac{1}{n}$ cannot be friends of x_n, thus $x_n = 1$ for all friends n of x_n, thusly all $n \in S$ are friends, thus S is the set of all friends of x_n, thus for all $S \subseteq \mathbb{N}$, there exists a sequence such that S is the set of friends of that sequence. Q.E.D.

97.) Let $x_n \to L$, then by theorem 19, $y_k \to L$ for all subsequences y_k of x_n. Similarly, since ***

99.) Let $x_n = n - (-1)^n n$. x_n is unbounded, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, thus $y_k \preceq x_n$ and $y_k \to 0$.

100.) Every cauchy sequence is convergent according to theorem 23, and no convergent sequence can be unbounded.

103.) Let $x_n = \dfrac{1}{n^2}$. Since $x_n \to 0$, x_n is convergent and thus cauchy. Q.E.D.

105.) Since x_n and y_n are cauchy, there exist $A, B \in \mathbb{R}$ where $x_n \to A$ and $y_n \to B$. Since $y_n \neq 0$ for all $n \in \mathbb{N}$, $B \neq 0$. Let $z_n = x_n \,/\, y_n$. According to theorem 14, $z_n = x_n \,/\, y_n \implies z_n \to A \,/\, B$, thus z_n is convergent and thus cauchy. Q.E.D.

127.) For $\lim\limits_{x \to 5} x^2 = 25$, then given $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - 5| < \delta \implies \left|x^2 - 25\right| < \varepsilon$$

Suppose $|x - 5| < 1$, then $|x + 5| < 11$, thus

$$\left|x^2 - 25\right| = |x - 5|\,|x + 5| < 11\,|x - 5| < 11\delta$$

$$11\delta = \varepsilon \implies \delta = \frac{\varepsilon}{11}$$

Let $\delta < \min\left(1, \dfrac{\varepsilon}{11}\right)$:

$$|x - 5| < \delta \implies |x - 5| < \frac{\varepsilon}{11} \implies 11\,|x - 5| < \varepsilon \implies |x - 5|\,|x + 5| < 11\,|x - 5| < \varepsilon$$

$$\implies |x - 5|\,|x + 5| = \left|x^2 - 25\right| < \varepsilon$$

Thus $\lim\limits_{x \to 5} x^2 = 25$. Q.E.D.

128.) For $\lim\limits_{x \to \frac{1}{2}} \dfrac{1}{x} = 2$, then given $\varepsilon > 0$, there must exist $\delta > 0$ where

$$\left|x - \frac{1}{2}\right| < \delta \implies \left|\frac{1}{x} - 2\right| < \varepsilon$$

Suppose $\left|x - \frac{1}{2}\right| < \frac{1}{4}$:

$$\left|x - \frac{1}{2}\right| < \frac{1}{4} \implies -\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4} \implies \frac{1}{4} < x < \frac{3}{4} \implies \frac{4}{3} < \frac{1}{x} < 4 \implies \frac{2}{x} < 8$$

Thus

$$\left|\frac{1}{x} - 2\right| = \left|2 - \frac{1}{x}\right| = \left|\frac{2}{x}\right|\left|x - \frac{1}{2}\right| < 8\delta$$

$$8\delta = \varepsilon \implies \delta = \frac{\varepsilon}{8}$$

Let $\delta = \min\left(\frac{1}{4}, \frac{\varepsilon}{8}\right)$:

$$\left|x - \frac{1}{2}\right| < \delta \implies \left|x - \frac{1}{2}\right| < \frac{\varepsilon}{8} \implies 8\left|x - \frac{1}{2}\right| < \varepsilon \implies \left|\frac{2}{x}\right|\left|x - \frac{1}{2}\right| < 8\left|x - \frac{1}{2}\right| < \varepsilon$$

$$\implies \left|\frac{2}{x}\right|\left|x - \frac{1}{2}\right| = \left|\frac{1}{x} - 2\right| < \varepsilon$$

Thus $\lim_{x \to \frac{1}{2}} \frac{1}{x} = 2$. Q.E.D.

132.) When a function f is continuous at c, then for all vertical windows around $f(c)$, there exists a horizontal window around c such that if c is in the horizontal window, then $f(c)$ is within the vertical window.

133.) For x^2 to be continuous over all $c \in \mathbb{R}$, then for all $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - c| < \delta \implies \left|x^2 - c^2\right| < \varepsilon$$

Suppose $|x - c| < 1$:

$$|x - c| < 1 \implies -1 < x - c < 1 \implies 2c - 1 < x - c + 2c < 2c + 1 \implies |x + c| < 2c + 1$$

Thus

$$\left|x^2 - c^2\right| = |x - c|\,|x + c| < (2c + 1)\delta$$

$$(2c + 1)\delta = \varepsilon \implies \delta = \frac{\varepsilon}{2c + 1}$$

Let $\delta < \min\left(1, \frac{\varepsilon}{2c + 1}\right)$, then

$$|x - c| < \delta \implies |x - c| < \frac{\varepsilon}{2c + 1} \implies |x - c|\,|x + c| < |x - c|\,(2c + 1) < \varepsilon$$

$$\implies \left|x^2 - c^2\right| < \varepsilon$$

Thus x^2 is continuous over all $c \in \mathbb{R}$. Q.E.D.

134.) For $\dfrac{1}{x}$ to be continuous at $x = 4$, then for all $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - 4| < \delta \implies \left|\frac{1}{x} - \frac{1}{4}\right| < \varepsilon$$

Suppose $|x - 4| < 1$

$$|x - 4| < 1 \implies 3 < x < 5 \implies \frac{1}{5} < \frac{1}{x} < \frac{1}{3} \implies \frac{1}{4x} < \frac{1}{12}$$

Thus

$$\left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{1}{4} - \frac{1}{x}\right| = \left|\frac{1}{4x}\right| |x - 4| < \frac{\delta}{12}$$

$$\frac{\delta}{12} = \varepsilon \implies \delta = 12\varepsilon$$

Let $\delta < \min(1, 12\varepsilon)$, then

$$|x - 4| < \delta \implies |x - 4| < 12\varepsilon \implies \frac{1}{12}|x - 4| < \varepsilon \implies \left|\frac{1}{4x}\right| |x - 4| < \varepsilon$$

$$\implies \left|\frac{1}{x} - \frac{1}{4}\right| < \varepsilon$$

Thus $\dfrac{1}{x}$ is continuous at $x = 4$. Q.E.D.

135.) For $|x|$ to be continuous over all $c \in \mathbb{R}$, then for all $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - c| < \delta \implies \big||x| - c\big| < \varepsilon$$

$$***$$

139.) Let $f : \mathbb{R} \to \mathbb{R}$ be 1/2-Hölder, then there exists $C > 0$ where

$$|f(x) - f(y)| \leq C |x - y|^{1/2}$$

***

140.) Let $f(x) = ***$

144.) Let $f : [a, b] \to [a, b]$ be continuous. If $f(a) = a$ or $f(b) = b$, then f has a fixed point. Otherwise, since the codomain of f is $[a, b]$, then $f(a) > a$ and $f(b) < b$. Let $g(x) = x - f(x)$:

$$f(a) > a \implies a - f(a) < 0 \implies g(a) < 0$$

$$f(b) < b \implies b - f(b) > 0 \implies g(b) > 0$$

Since x and $f(x)$ are continuous, $x - f(x)$ is continuous, thus $g(x)$ is continuous. Since $g(a) < 0$, $g(b) > 0$, and $a < b$, then from the intermediate value theorem, there exists $x \in [a, b]$ where $g(x) = 0$, thus $g(x) = x - f(x) = 0 \implies f(x) = x$, thus f has a fixed point. Q.E.D.

145.) Let $f : (a, b) \to (a, b)$ where $f(x) =$.

146.) Since $f(x) = x^3 + x - 10$ is a polynomial, it is continuous. Consider $f(-1)$ and $f(3)$:

$$f(-1) = (-1)^3 + (-1) - 10 = -12$$

$$f(3) = 3^3 + 3 - 10 = 20$$

Thus there exist a, b where $f(a) < 0 < f(b)$, thus from the intermediate value theorem, there exists x where $f(x) = 0$, thus $f(x) = 0$ has at least one real solution. Q.E.D.

147.) awd

149.) $f : D \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$x, y \in D \wedge |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

150.) a.) Consider $f : (3, 5) \to \mathbb{R}$, then ***

b.) Consider $f : (0, 2) \to \mathbb{R}$, then ***