Definitions

1.) Given a set X, a topology $\mathcal{T}$ on X is a collection of subsets of X such that

1. $\varnothing, X \in \mathcal{T}$
2. Given a collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of elements in $\mathcal{T}$,
$$\bigcup_{\alpha\in\mathcal{A}} U_\alpha$$
is in $\mathcal{T}$.
3. Given a finite collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of elements in $\mathcal{T}$,
$$\bigcap_{\alpha\in\mathcal{A}} U_\alpha$$
is in $\mathcal{T}$.

2.) Given a poset P, the Alexandroff topology $\mathcal{T}$ on P is defined as
$$\mathcal{T} := \{U \subset P : U \text{ is open}\}$$

3.) The standard topology $\mathcal{T}$ on $\mathbb{R}^n$ is defined as
$$\mathcal{T} := \{U \subset \mathbb{R}^n : U \text{ is open}\}$$

4.) Given a set X, the discrete topology $\mathcal{T}$ on X is defined as $\mathcal{T} = \mathcal{P}(X)$.

5.) Given a set X, the trivial topology $\mathcal{T}$ on X is defined as $\mathcal{T} = \{\varnothing, X\}$.

6.) Give a topological space $(X, \mathcal{T})$, $U \subset X$ is open if $U \in \mathcal{T}$.

7.) Given a topological space $(X, \mathcal{T})$, $K \subset X$ is closed if $K^{\complement} \in \mathcal{T}$.

8.) Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we call $f : X \to Y$ continuous if given $U \subset Y$ where U is open, $f^{-1}(U)$ is also open.

9.) Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, we call $f : X \to Y$ a homeomorphism if

1. f is continuous
2. f is a bijection
3. the inverse function f^{-1} is continuous

Proofs

a.) Let X be a set, and consider $\mathcal{P}(X)$. We can clearly see that $\varnothing \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$. Next, let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an arbitrary collection of subsets of X and suppose $x \in \bigcup_{\alpha\in\mathcal{A}} U_\alpha$, thus there exists $\alpha \in \mathcal{A}$ where $x \in U_\alpha \subset X$, thus $x \in X$, thus $\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{P}(X)$. Next, let $\{V_\beta\}_{\beta\in\mathcal{B}}$ be an arbitrary finite collection of subsets of X and suppose $x \in \bigcap_{\beta\in\mathcal{B}} V_\beta$, then for all $\beta \in \mathcal{B}$, $x \in V_\beta \subset X$, thus $x \in X$, thus $\bigcap_{\beta\in\mathcal{B}} V_\beta \in \mathcal{P}(X)$, thus $\mathcal{P}(X)$ is a topology on X. $\blacksquare$

b.) Let $\{\mathcal{T}_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of topologies on X. Because $\mathcal{T}_\alpha$ is a topology on X for all $\alpha \in \mathcal{A}$, we know that $\varnothing, X \in \mathcal{T}_\alpha$ for all α, thus $\varnothing, X \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$. Next, let $X \subset \bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$. We can see that for all $U \in X$, $U \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$, thus $U \in \mathcal{T}_\alpha$ for all α, thus $\bigcup_{U\in X} U \in \mathcal{T}_\alpha$ for all α, and thus $\bigcup_{U\in X} U \in \bigcap_{\alpha\in\mathcal{T}} \mathcal{T}_\alpha$. Finally, let $Y \subset \bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$ be finite. Again, for all $V \in Y$, $V \in \bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$, thus $V \in \mathcal{T}_\alpha$ for all α, thus $\bigcap_{V\in Y} V \in \mathcal{T}_\alpha$ for all α, thus $\bigcap_{V\in Y} V \in \bigcup_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$, thus $\bigcap_{\alpha\in\mathcal{A}} \mathcal{T}_\alpha$ is a topology on X. $\blacksquare$

c.) Because $\mathcal{S} \subset \mathcal{P}(X)$, we know that $\mathcal{S}$ is contained in the discrete topology $\mathcal{P}(X)$ on X. $\blacksquare$

d.) Let $\mathcal{S} \subset \mathcal{P}(X)$ and define $\mathcal{B}$ as follows:

$$\mathcal{B} := \left\{\mathcal{T}' \subset \mathcal{P}(X) : \mathcal{T}' \text{ is a topology on } X \text{ and } \mathcal{S} \subset \mathcal{T}'\right\}$$

By definition, $\mathcal{S} \subset \mathcal{T}'$ for all $\mathcal{T}' \in \mathcal{B}$, thus $\mathcal{S} \subset \bigcap_{\mathcal{T}'\subset\mathcal{B}} \mathcal{T}'$, and thus $\mathcal{S} \subset \mathcal{T}_\mathcal{S}$. Now, let $U \in \mathcal{T}_\mathcal{S}$. By definition, $U \in \mathcal{T}'$ for all $\mathcal{T}' \in \mathcal{B}$, thus for all $\mathcal{T}' \in \mathcal{B}$, we know that $U \in \mathcal{T}_\mathcal{S} \implies U \in \mathcal{T}'$, thus $\mathcal{T}_\mathcal{S} \subset \mathcal{T}'$ for all $\mathcal{T}' \in \mathcal{B}$. $\blacksquare$

e.) When one considers all of the topologies of X that contain S, they may realize that there are almost certainly many different examples of such a topology. Taking the intersection of all of these topologies, however, leaves only what is shared among all such topologies, and thus what is essential for them to contain $\mathcal{S}$, and nothing more. Because of this, when defining $\mathcal{T}_\mathcal{S}$ as such an intersection, it is appropriate to think of $\mathcal{T}_\mathcal{S}$ as the "smallest topology containing $\mathcal{S}$".