6) There exists a positive continuous function $f : \mathbb{R} \to \mathbb{R}$ where f is integrable and $\limsup_{x\to\infty} f(x) = \infty$. However, if f is uniformly continuous and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Proof: Let $g(x) = \sum_{k\in\mathbb{N}} k\chi_{I_k}(x)$, where $I_k = [k, k + 1/k^3)$. Since the I_k are disjoint, we have that

$$\int g(x)\,dx = \sum_{k\in\mathbb{N}} k \cdot m(I_k) = \sum_{k\in\mathbb{N}} \frac{1}{k^2} < \infty,$$

thus g is integrable. For each k, define the function $f_k(x) = \min\{h_k(x), h'_k(x)\}\,\chi_{I_k}(x)$, where

$$h_k(x) = 2k^4(x - k) \quad \text{and} \quad h'_k(x) = -2k^4(x - k - 1/k^3).$$

h_k and h'_k are both lines, and thus continuous. We can see that $f_k(k) = h_k(k) = 0$, Define $f(x) = \sum_{k\in\mathbb{N}} f_k(x)$. Since f is a sum of continuous functions, it is itself continuous.

10)

11) If f is a real-valued function integrable on $\mathbb{R}^d$, and if $\int_E f \geq 0$ for all measurable sets $E \subseteq \mathbb{R}^d$, then $f \geq 0$ a.e. As a corollary, if $\int_E f = 0$ for all E, then $f = 0$ a.e.

Proof: Suppose the proposition is false, then there exists some $r > 0$ where the set $E = \{f < -r\}$ has non-zero measure. Since f is integrable and thus measurable, we have that E is also measurable. From this, we see that

$$\int_E f(x)\,dx < \int_E -r\,dx = -r \cdot m(E) < 0,$$

which is a contradiction since E is measurable. Thus, we must have that $m(E) = 0$ for all $r > 0$, showing that $f \geq 0$ a.e.

As a corollary, suppose $\int_E f = 0$ for all E, then $f \geq 0$ a.e. We also have that $\int_E -f = -\int_E f = 0$ for all E, thus showing that $-f \geq 0$ a.e., revealing $f \leq 0$ a.e. Thus, $0 \leq f \leq 0$ a.e., which proves that $f = 0$ a.e. ■

12)