Chapter 5

1.) a.) $\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$

b.) $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$

c.) $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$

3.) a.) $(1235)(413) = (15)(234)$

b.) $(13256)(23)(45612) = (124635)$

c.) $(12)(13)(23)(142) = (1423)$

5.) The order of a permutation is given by the least common multiple of the lengths of its disjoint cycles.

a.) $|(124)(357)| = \text{lcm}(3, 3) = 3$

a.) $|(124)(3567)| = \text{lcm}(3, 4) = 12$

c.) $|(124)(35)| = \text{lcm}(3, 2) = 6$

d.) $|(124)(357869)| = \text{lcm}(3, 6) = 6$

e.) $|(1235)(24567)| = \text{lcm}(4, 5) = 20$

f.) $|(345)(245)| = \text{lcm}(3, 3) = 3$

Chapter 6

1.) I am assuming that, in this case, the even integers include negatives, as the non-negative even integers do not form a group under addition. Let $f(n) = 2n$. First we will show that f is injective. Let $a, b \in \mathbb{Z}$ where $f(a) = f(b)$, thus $2a = 2b$, thus $a = b$, thus f is injective. Next, we will show that f is surjective. Let a be an even integer, thus $a = 2n$ for some $n \in \mathbb{Z}$, thus $a/2$ is an integer. From this we can see that $f(a/2) = 2(a/2) = a$, thus f is surjective, and thus bijective. Finally, given $a, b \in \mathbb{Z}$, we can see that $f(a+b) = 2(a+b) = 2a+2b = f(a)+f(b)$, thus f is an isomorphism from the integers to the even integers. ■

6.) Let G, H, and K be groups with operation $*$. Let $f : G \to G$ where $f(g) = g$. Let $a, b \in G$, thus $f(a*b) = a*b = f(a)*f(b)$. In addition, f is trivially bijective, thus f is an isomorphism from G to G, thus $G \cong G$, thus group isomorphism is reflexive.

Next, let $G \cong H$, thus there exists a bijective $f : G \to H$ where $f(a * b) = f(a) * f(b)$ for all $a, b \in G$. Since f is bijective, there exists an inverse function f^{-1} that is also bijective. Let $a, b \in G$, thus $f(a), f(b) \in H$. We can see that $f^{-1}(f(a) * f(b)) = f^{-1}(f(a * b)) = a * b = f^{-1}(f(a)) * f^{-1}(f(b))$, thus f^{-1} is an isomorphism from H to G, thus $H \cong G$, thus group isomorphism is symmetric.

Finally, let $G \cong H$ and $H \cong K$, thus there exist isomorphisms $f : G \to H$ and $g : H \to K$. Let $h = g \circ f$ and let $a, b \in G$, then $h(a*b) = g(f(a*b)) = g(f(a)*f(b)) = g(f(a))*g(f(b)) = h(a) * h(b)$, thus $G \cong K$, thus group isomorphism is transitive.

It follows that group isomorphism is an equivalence relation. ■