4.1 Group Actions and Permutation Representations

1) Let G act on a set A. If $a, b \in A$ where $b = g \cdot a$ for some $g \in G$, then $G_b = gG_a g^{-1}$. Additionally, if G acts transitively on A, then its kernel is $\bigcap_{g \in G} gG_b g^{-1}$ for some $b \in A$.

Proof: Ler $x \in G_b$, then $x \cdot b = (xg) \cdot a = g \cdot a$, thus $(g^{-1}xg) \cdot a = a$ and $g^{-1}xg \in G_a$. Since $x = g(g^{-1}xg)g^{-1}$, we have that $x \in gG_a g^{-1}$, thus $G_b \subseteq gG_a g^{-1}$. Now, let $x \in gG_a g^{-1}$, then $x = gyg^{-1}$ for some $y \in G_a$. Since $b = g \cdot a$, we have that $a = g^{-1} \cdot b$, thus $y \cdot a = (yg^{-1}) \cdot b = g^{-1} \cdot b$, showing that $(gyg^{-1}) \cdot b = b$ and $x = gyg^{-1} \in G_b$. From this we have that $gG_a g^{-1} \subseteq G_b$, thus $G_b = gG_a g^{-1}$.

We know that the kernel of a group action is $\bigcap_{a \in A} G_a$. Fix $b \in A$. Assuming G acts transitively on A, we have for all $a \in A$ that there exists $g \in G$ where $b = g \cdot a$, thus $G_a = gG_b g^{-1}$ for all a. We also have that $a = g^{-1} \cdot b$, thus $gG_b g^{-1} = g(g^{-1}G_a g)g^{-1} = G_a$ for any $g \in G$, thus $\bigcap_{a \in A} G_a = \bigcap_{g \in G} gG_b g^{-1}$ and we are done. ∎

4.2 Groups Acting on Themselves by Left Multiplication

2) Let S_3 act on itself by left multiplication and denote $\phi : S_3 \to S_6$ as the left regular representation of this action. Indexing $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ as $\{1, 2, 3, 4, 5, 6\}$, we have that ϕ is defined as follows:

$$\begin{aligned}
\phi(e) &= e \\
\phi((1\ 2)) &= (1\ 2)(3\ 5)(4\ 6) \\
\phi((1\ 3)) &= (1\ 3)(2\ 5)(4\ 6) \\
\phi((2\ 3)) &= (1\ 4)(2\ 6)(3\ 5) \\
\phi((1\ 2\ 3)) &= (1\ 5\ 6)(2\ 3\ 4) \\
\phi((1\ 3\ 2)) &= (1\ 6\ 5)(2\ 4\ 3)
\end{aligned}$$

5a) Let D_8 act on the set of left cosets of $H = \langle s \rangle$ by left multiplication and index the cosets $\{H, rH, r^2H, r^3H\}$ as $\{1, 2, 3, 4\}$, then the permutation representation of this action $\phi : D_8 \to S_4$ is defined as follows:

$$\begin{aligned}
\phi(e) &= e \\
\phi(r) &= (1\ 2\ 3\ 4) \\
\phi(r^2) &= (1\ 3)(2\ 4) \\
\phi(r^3) &= (1\ 4\ 3\ 2) \\
\phi(s) &= (2\ 4) \\
\phi(sr) &= (1\ 4)(2\ 3) \\
\phi(sr^2) &= (1\ 3) \\
\phi(sr^3) &= (1\ 2)(3\ 4)
\end{aligned}$$

Because ϕ is injective, we deduce that $D_8 \cong \phi(D_8)$, and thus the action is faithful.

8) If H is a subgroup of G with finite index n, then there exists a normal subgroup $K \trianglelefteq G$ where $K \leq H$ and $|G : K| \leq n!$.

Proof: Let G act on the set of cosets of H by left multiplication. This action induces a permutation representation $\phi : G \to S_n$ which is a homomorphism. Using Cayley's theorem, we have that $G/\ker\phi$ is isomorphic to some subgroup of S_n, thus $|G/\ker\phi|$ divides $|S_n| = n!$, and thus $|G : \ker\phi| \leq n!$. Since $K = \ker\phi$ is a normal subgroup of G, we are finished. ■

4.3 Group Actions and Permutation Representations

2a) The conjugacy classes of D_8 are $\{e\}$, $\{r^2\}$, $\{s, r^2 s\}$, $\{rs, r^3 s\}$, and $\{r, r^3\}$.

2c) The conjugacy classes of A_4 are

$$\{e\}$$

$$\{(1\ 2\ 3), (2\ 4\ 3), (1\ 4\ 2), (1\ 3\ 4)\}$$

$$\{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\}$$

$$\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

5) If G is a group where $|G : Z(G)| = n$, then the size of each conjugacy class in G is at most n.

Proof: Fix $a \in G$ where $a \notin Z(G)$, then the size of the conjugacy class of a is $|G : C_G(a)|$. Since $Z(G)$ is contained in $C_G(a)$, we have that

$$\frac{|G|}{|C_G(a)|} \leq \frac{|G|}{|Z(G)|} = n,$$

showing that $|G : C_G(a)| \leq n$. ■

7) The partitions of 3 are $1+1+1$, $1+2$, and 3, with respective cycle representatives e, $(1\ 2)$, and $(1\ 2\ 3)$.

The partitions of 4 are $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4, with respective cycle representatives e, $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$, and $(1\ 2\ 3\ 4)$.

The partitions of 6 are $1+1+1+1+1+1$, $1+1+1+1+2$, $1+1+1+3$, $1+1+4$, $1+5$, $1+1+2+2$, $1+2+3$, $2+2+2$, $2+4$, $3+3$, and 6, with cycle representatives e, $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2\ 3\ 4)$, $(1\ 2\ 3\ 4\ 5)$, $(1\ 2)(3\ 4)$, $(1\ 2)(3\ 4\ 5)$, $(1\ 2)(3\ 4)(5\ 6)$, $(1\ 2)(3\ 4\ 5\ 6)$, $(1\ 2\ 3)(4\ 5\ 6)$, and $(1\ 2\ 3\ 4\ 5\ 6)$.

10) Given $\sigma = (1\ 2\ 3\ 4\ 5)$, $\tau_1 = (2\ 3\ 5\ 4)$, and $\tau_2 = (2\ 5)(3\ 4)$, we have that $\tau_1 \sigma \tau_1^{-1} = \sigma^2$ and $\tau_2 \sigma \tau_2^{-1} = \sigma^{-1}$.

$\mathrm{Aut}(I^{sm})$