In $\mathbb{R}^3$, we can define a tetrahedron as a polyhedron having four equilateral triangle faces, each of which is adjacent to the other three. First we will show that the boundary of Δ^3 consists of four equilateral triangles. After, we will show each triangle is adjacent.

Given the following definition of Δ^3:

$$\Delta^3 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i = 1 \text{ and } x_i \geq 0 \right\}$$

let $j \in \{1, 2, 3, 4\}$ and $\mathcal{F}_j$ be defined as follows:

$$\mathcal{F}_j = \left\{ x \in \Delta^3 : i = j \implies x_i = 0 \right\}$$

If we can show bijections $f : \mathcal{F}_j \to \Delta^2$ for all j, then we know that $\mathcal{F}_j \subset \Delta^3$ is an equilateral triangle, and since there are 4 distinct $\mathcal{F}_j$, we can conclude that the boundary of Δ^3 consists of 4 equilateral triangles.

Consider the mapping $f : \mathcal{F}_1 \to \Delta^2$ where $(0, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4)$. We can first show that f is injective. Let $a, b \in \mathcal{F}_1$ where $a = (0, a_1, a_2, a_3)$ and $b = (0, b_1, b_2, b_3)$, and where $f(a) = f(b)$. Since $f(a) = (a_1, a_2, a_3)$ and $f(b) = (b_1, b_2, b_3)$, we can conclude that $(a_1, a_2, a_3) = (b_1, b_2, b_3)$, thus f is injective. Next we can show that f is surjective. Let $a \in \Delta^2$ where $a = (a_1, a_2, a_3)$ and $b \in \mathcal{F}_1$ where $b = (0, a_1, a_2, a_3)$, then $f(b) = (a_1, a_2, a_3) = a$, thus for all $a \in \Delta^2$, there exists $b \in \mathcal{F}_1$ where $f(b) = a$, thus f is a surjective, and thus $f : \mathcal{F}_1 \to \Delta^2$ is a bijection. Without loss of generality, we can conclude that this holds for all j.

To show that any two triangles $\mathcal{F}_j, \mathcal{F}_k$ for $j \neq k$ are adjacent, we can show that their intersection is a line. Consider $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\mathcal{F}_1 = \left\{ (x_1, x_2, x_3, x_4) \in \Delta^3 : x_1 = 0 \right\}$$

$$\mathcal{F}_2 = \left\{ (x_1, x_2, x_3, x_4) \in \Delta^3 : x_2 = 0 \right\}$$

and then $\mathcal{F}_1 \cap \mathcal{F}_2$:

$$\mathcal{F}_1 \cap \mathcal{F}_2 = \{ a : a \in \mathcal{F}_1 \wedge a \in \mathcal{F}_2 \} = \left\{ (x_1, x_2, x_3, x_4) \in \Delta^3 : x_1 = x_2 = 0 \right\}$$

Since $(0, 0, x_3, x_4) \in \Delta^3$, we know that $x_3 + x_4 = 1$, which is the equation of a line, thus we know that $\mathcal{F}_1 \cap \mathcal{F}_2$ are adjacent, and without loss of generality, $\mathcal{F}_j$ and $\mathcal{F}_k$ are adjacent for $j \neq k$.

Since we have shown that the boundary of Δ^3 consists of four equilateral triangles, and that each triangle $\mathcal{F}_j \subset \Delta^3$ is adjacent to the others, we can conclude that according to our definition of a tetrahedron, Δ^3 is a tetrahedron. ■