

Exercises 13.1

5.)

$$\begin{aligned}\lim_{t \rightarrow \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \right\rangle &= \left\langle \lim_{t \rightarrow \infty} \frac{1+t^2}{1-t^2}, \lim_{t \rightarrow \infty} \tan^{-1} t, \lim_{t \rightarrow \infty} \frac{1-e^{-2t}}{t} \right\rangle \\ \lim_{t \rightarrow \infty} \frac{1+t^2}{1-t^2} &= \lim_{t \rightarrow \infty} \frac{2t}{-2t} = \lim_{t \rightarrow \infty} \frac{2}{-2} = -1 \\ \lim_{t \rightarrow \infty} \tan^{-1} t &= \frac{\pi}{2} \\ \lim_{t \rightarrow \infty} \frac{1-e^{-2t}}{t} &= \lim_{t \rightarrow \infty} 2e^{-2t} = \lim_{t \rightarrow \infty} \frac{2}{e^{2t}} = 0 \\ \therefore \lim_{t \rightarrow \infty} \left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t} \right\rangle &= \left\langle -1, \frac{\pi}{2}, 0 \right\rangle\end{aligned}$$

Exercises 13.2

11.)

$$\frac{d}{dt} \langle t^2, \cos(t^2), \sin^2 t \rangle = \langle 2t, -2t \sin(t^2), 2 \sin t \cos t \rangle$$

17.)

$$\begin{aligned}T(t) &= \frac{r'(t)}{\|r'(t)\|} \\ r'(t) &= \langle 2t-2, 3, t^2+t \rangle \\ \|r'(t)\| &= \sqrt{(2t-2)^2 + 3^2 + (t^2+t)^2} = \sqrt{4t^2 - 8t + 4 + 9 + t^4 + 2t^3 + t^2} \\ &= \sqrt{t^4 + 2t^3 + 5t^2 - 8t + 13} \\ \therefore T(t) &= \frac{1}{\sqrt{t^4 + 2t^3 + 5t^2 - 8t + 13}} \langle 2t-2, 3, t^2+t \rangle \\ T(2) &= \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle\end{aligned}$$

23.) ***

35.)

$$\int_0^2 (t\hat{i} - t^3\hat{j} + 3t^5\hat{k}) dt = \left[\frac{t^2}{2}\hat{i} - \frac{t^4}{4}\hat{j} + \frac{t^6}{2}\hat{k} \right]_0^2 = 2\hat{i} - 4\hat{j} + 32\hat{k}$$

39.)

$$\begin{aligned}\int (\sec^2 t \hat{i} + t(t^2 + 1)^3 \hat{j} + t^2 \ln t \hat{k}) dt \\ \int \sec^2 t \hat{i} dt = \tan t \hat{i}\end{aligned}$$

$$\begin{aligned}\int t(t^2 + 1)^3 \hat{j} \, dt &= \frac{(t^2 + 1)^4}{8} \hat{j} \\ \int t^2 \ln t \hat{k} \, dt &= \left(\frac{t^3}{3} \ln t - \int \frac{t^3}{3t} \right) \hat{k} = \left(\frac{t^3 \ln t}{3} - \frac{t^3}{9} \right) \hat{k} \\ \therefore I &= \tan t \hat{i} + \frac{(t^2 + 1)^4}{8} \hat{j} + \left(\frac{t^3 \ln t}{3} - \frac{t^3}{9} \right) \hat{k} + C\end{aligned}$$

41.)

$$\begin{aligned}r(t) &= \int r'(t) \, dt = t^2 \hat{i} + t^3 \hat{j} + \frac{2\sqrt{t^3}}{3} \hat{k} + C \\ \hat{i} + \hat{j} &= \hat{i} + \hat{j} + \frac{2}{3} \hat{k} + C \implies C = -\frac{2}{3} \hat{k} \\ \therefore r(t) &= t^2 \hat{i} + t^3 \hat{j} + \left(\frac{2(\sqrt{t^3} - 1)}{3} \right) \hat{k}\end{aligned}$$

Exercises 13.3

1.)

$$\begin{aligned}\int_{-5}^5 \|r'(t)\| \, dt &= \int_{-5}^5 \|(1, -3 \sin t, 3 \cos t)\| \, dt = \int_{-5}^5 \sqrt{1^2 + (-3 \sin t)^2 + (3 \cos t)^2} \, dt \\ &= \int_{-5}^5 \sqrt{1 + 9 \sin^2 t + 9 \cos^2 t} \, dt = \int_{-5}^5 \sqrt{1 + 9} \, dt = \int_{-5}^5 \sqrt{10} \, dt = [\sqrt{10}t]_{-5}^5 \\ &= 5\sqrt{10} - (-5\sqrt{10}) = 10\sqrt{10}\end{aligned}$$

17.) a.)

$$\begin{aligned}T &= \frac{r'}{\|r'\|} = \frac{\langle 1, -3 \sin t, 3 \cos t \rangle}{\sqrt{1^2 + (-3 \sin t)^2 + (3 \cos t)^2}} = \frac{\langle 1, -3 \sin t, 3 \cos t \rangle}{\sqrt{1 + 9 \sin^2 t + 9 \cos^2 t}} \\ &= \frac{\langle 1, -3 \sin t, 3 \cos t \rangle}{\sqrt{1 + 9}} = \frac{1}{\sqrt{10}} \langle 1, -3 \sin t, 3 \cos t \rangle \\ &= \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \sin t, \frac{3}{\sqrt{10}} \cos t \right\rangle \\ N &= \frac{T'}{\|T'\|} = \frac{\left\langle 0, -\frac{3}{\sqrt{10}} \cos t, -\frac{3}{\sqrt{10}} \sin t \right\rangle}{\sqrt{0^2 + \left(-\frac{3}{\sqrt{10}} \cos t\right)^2 + \left(-\frac{3}{\sqrt{10}} \sin t\right)^2}} = \frac{\left\langle 0, -\frac{3}{\sqrt{10}} \cos t, -\frac{3}{\sqrt{10}} \sin t \right\rangle}{\sqrt{\frac{9}{10} \cos^2 t + \frac{9}{10} \sin^2 t}} \\ &= \frac{1}{\sqrt{\frac{9}{10}}} \left\langle 0, -\frac{3}{\sqrt{10}} \cos t, -\frac{3}{\sqrt{10}} \sin t \right\rangle = \left\langle 0, -\frac{3}{3} \cos t, -\frac{3}{3} \sin t \right\rangle \\ &= \langle 0, -\cos t, -\sin t \rangle\end{aligned}$$

b.)

$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{\frac{\sqrt{9}}{\sqrt{10}}}{\sqrt{10}} = \frac{\sqrt{9}}{10} = \frac{3}{10}$$

21.)

$$r' = \langle 0, 3t^2, 2t \rangle, \quad r'' = \langle 0, 6t, 2 \rangle$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} = (6t^2 - 12t^2)\hat{i} + 0\hat{j} + 0\hat{k} = \langle -6t^2, 0, 0 \rangle$$

$$\|r' \times r''\| = \sqrt{(-6t^2)^2 + 0^2 + 0^2} = \sqrt{36t^4} = 6t^2$$

$$\|r'\|^3 = \left(\sqrt{0^2 + (3t^2)^2 + (2t)^2} \right)^3 = \left(\sqrt{9t^4 + 4t^2} \right)^3 = (9t^4 + 4t^2)^{\frac{3}{2}}$$

$$\therefore \kappa = \frac{6t^2}{(9t^4 + 4t^2)^{\frac{3}{2}}}$$

47.)

$$T = \frac{r'}{\|r'\|} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{(2t)^2 + (2t^2)^2 + 1^2}} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \frac{\langle 2t, 2t^2, 1 \rangle}{\sqrt{(2t^2 + 1)^2}}$$

$$= \frac{\langle 2t, 2t^2, 1 \rangle}{2t^2 + 1} = \left\langle \frac{2t}{2t^2 + 1}, \frac{2t^2}{2t^2 + 1}, \frac{1}{2t^2 + 1} \right\rangle$$

$$T \left(1, \frac{2}{3}, 1 \right) = \left\langle \frac{2(1)}{2(1)^2 + 1}, \frac{2(2/3)^2}{2(2/3)^2 + 1}, \frac{1}{2(1)^2 + 1} \right\rangle = \left\langle \frac{2}{3}, \frac{8}{17}, \frac{1}{3} \right\rangle$$

$$N = \frac{T'}{\|T'\|} = \frac{\left\langle \frac{2-4t^2}{(2t^2+1)^2}, \frac{2t^2}{(2t^2+1)^2}, -\frac{4t}{(2t^2+1)^2} \right\rangle}{\sqrt{\left(\frac{2-4t^2}{(2t^2+1)^2} \right)^2 + \left(\frac{2t^2}{(2t^2+1)^2} \right)^2 + \left(-\frac{4t}{(2t^2+1)^2} \right)^2}} \\ = \langle \rangle$$

Exercises 13.4

5.)

$$v(t) = r'(t) = \langle -3 \sin t, 2 \cos t \rangle$$

$$a(t) = r''(t) = \langle -3 \cos t, -2 \sin t \rangle$$

$$\|v(t)\| = \|r'(t)\| = \sqrt{(-3 \sin t)^2 + (2 \cos t)^2} = \sqrt{9 \sin^2 t + 4 \cos^2 t} = \sqrt{5 \sin^2 t + 4}$$

15.)

$$v(t) = \int a(t) dt = 2t\hat{i} + t^2\hat{k} + C_1$$

$$v(0) = 3\hat{i} - \hat{j} = 2(0)\hat{i} + (0)^2\hat{k} + C_1 = C_1 \implies C_1 = 3\hat{i} - \hat{j}$$

$$\therefore v(t) = 2t\hat{i} + t^2\hat{k} + 3\hat{i} - \hat{j} = (2t + 3)\hat{i} - \hat{j} + t^2\hat{k}$$

$$r(t) = \int v(t) dt = (t^2 + 3t)\hat{i} - t\hat{j} + \frac{t^3}{3}\hat{k} + C_2$$

$$r(0) = \hat{j} + \hat{k} = ((0)^2 + 3(0))\hat{i} - (0)\hat{j} + \frac{(0)^3}{3}\hat{k} + C_2 = C_2 \implies C_2 = \hat{j} + \hat{k}$$

$$\therefore r(t) = (t^2 + 3t)\hat{i} - t\hat{j} + \frac{t^3}{3}\hat{k} + \hat{j} + \hat{k} = (t^2 + 3t)\hat{i} + (1 - t)\hat{j} + \left(\frac{t^3}{3} + 1\right)\hat{k}$$

23.) Establish initial variables and find $r(t)$:

$$v_0 = 200, \alpha = 60^\circ, g = 9.8$$

$$\therefore r(t) = (200 \cos(60^\circ))t\hat{i} + \left[(200 \sin(60^\circ))t - \frac{gt^2}{2}\right]\hat{j} = 100t\hat{i} + \left[100\sqrt{3}t - \frac{9.8t^2}{2}\right]\hat{j}$$

Calculating the x -component of $r(t)$ at $t = \frac{2v_0 \sin \alpha}{g}$ we can find the range:

$$\text{range} = 100 \left(\frac{2v_0 \sin \alpha}{g} \right) = \frac{20000\sqrt{3}}{9.8} \approx 3534.80\text{m}$$

Calculating the y -component of $r(t)$ at t such that the y -component of $r'(t) = 0$, we can find the maximum height:

$$\left[100\sqrt{3} - 9.8t\right] = 0 \implies t = \frac{100\sqrt{3}}{9.8}$$

$$\therefore \text{max height} = 100\sqrt{3}t - \frac{9.8t^2}{2} \approx 1530.61\text{m}$$

To find t at impact, we solve for $t > 0$ when $y = 0$:

$$100\sqrt{3}t - \frac{9.8t^2}{2} = 100\sqrt{3}t - 4.9t^2 = 0 \implies t = \frac{-100\sqrt{3} - \sqrt{30000}}{-9.8} = 35.34$$

$$\therefore \text{impact speed} = \|r'(t)\| = \sqrt{100^2 + (100\sqrt{3} - 9.8t)^2} = 200\text{m/s}$$

27.) Solve for y_{\max} :

$$\frac{d}{dt} \left[(v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right] = v_0 \sin \alpha - gt = 0 \implies t = \frac{v_0 \sin \alpha}{g}$$

$$\begin{aligned} \therefore y_{\max} &= r_y \left(\frac{v_0 \sin \alpha}{g} \right) = (v_0 \sin \alpha) \left(\frac{v_0 \sin \alpha}{g} \right) - \frac{1}{2}g \left(\frac{v_0 \sin \alpha}{g} \right)^2 \\ &= \frac{v_0^2 \sin^2 \alpha}{g} - \frac{gv_0^2 \sin^2 \alpha}{2g^2} = \frac{2v_0^2 \sin^2 \alpha - v_0^2 \sin^2 \alpha}{2g} = \frac{v_0^2 \sin^2 \alpha}{2g} \end{aligned}$$

$$\implies 2gy_{max} = v_0^2 \sin^2 \alpha \implies v_0 = \sqrt{\frac{2gy_{max}}{\sin^2 \alpha}} = \frac{\sqrt{2gy_{max}}}{\sin \alpha}$$

Finding our initial values:

$$y_{max} = 1600\text{ft}, \quad g = 9.8\text{m/s} = 32.144\text{ft/s}$$

$$\therefore v_0 = \frac{\sqrt{2(32.144)(1600)}}{\sin(36^\circ)} \approx 545.64\text{ft/s}$$