46.) Consider the sequences $x_n = \dfrac{1}{n}$ and $y_n = 0$. For all $n \in \mathbb{N}$, the following is true:

$$0 < \frac{1}{2^n} \leq \frac{1}{n}$$

Since we know that $\dfrac{1}{n} \to 0$ and $0 \to 0$, then by the squeeze theorem we can conclude that $2^{-n} \to 0$. Q.E.D.

80.) You can view subsequences as a more abstract form of function composition. You have x_n, which is a function $f : \mathbb{N} \to \mathbb{R}$, and you have n_k, which is a function $g : \mathbb{N} \to \mathbb{N}$. When you let $y_k = x_{n_k}$, this is equivalent to $y_k = (f \circ g)(k) = f(g(k))$. Since the codomain of g and domain of f are the same, namely $\mathbb{N}$, we know this function composition is well defined.

81.) Let $y_k = x_{2k}$, thus $y_k = (-1)^{2k} = 1$ for all $k \in \mathbb{N}$, thus $y_k \to 1$, thus y_k is convergent.

82.) a.) DNE; if $y_k \preceq x_n$ and $x_n \to L$, then $y_k \to L$ as per theorem 19.

b.) $x_n = n - (-1)^n n$; x_n diverges, but $y_k = x_{2k} = 2k - (-1)^{2k} 2k = 2k - 2k = 0$, $\therefore$ $y_n \to 0$.

91.) a.) $S = \mathbb{N}$ are the friends of x_n.

b.) $S = \{2n\}$ are the friends of y_n.

92.) $S = \{n \in \mathbb{N} : 21 \leq n \leq 56\}$ are the friends of z_n.

114.) Let $S = \{y \in \mathbb{R} : |x - y| < r\}$,

$$|x - y| < r \implies -r < x - y < r \implies -r - x < -y < r - x$$

$$\implies x - r < y < x + r \implies S = (x - r, x + r)$$

Thus $\{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$ Q.E.D.

126.) Suppose $\lim_{x\to 2} 3x+1=7$, then for all $\varepsilon>0$, there exists $\delta>0$ where

$$|x-2|<\delta \implies |3x+1-7|<\varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$|3x+1-7| = |3x-6| = |3(x-2)| = 3\,|x-2| < \varepsilon$$

$$\implies |x-2| < \frac{\varepsilon}{3}$$

Let $\delta = \frac{\varepsilon}{3}$. Manipulating the inequality:

$$|x-2| < \frac{\varepsilon}{3} \implies 3\,|x-2| = |3(x-2)| = |3x-6| = |3x+1-7| < \varepsilon$$

Thus $|x-2|<\delta \implies |3x+1-7|<\varepsilon$, thus $\lim_{x\to 2} 3x+1=7$. Q.E.D.

127.) Suppose $\lim_{x\to 5} x^2=25$, then for all $\varepsilon>0$, there exists $\delta>0$ where

$$|x-5|<\delta \implies \left|x^2-25\right|<\varepsilon$$

We can manipulate the inequality to find a sufficient value for δ:

$$\left|x^2-25\right| = |(x-5)(x+5)| = |x-5|\,|x+5| < \varepsilon$$

$$\implies |x-5| < \frac{\varepsilon}{|x+5|}$$

Let $\delta = \frac{\varepsilon}{|x+5|}$. Manipulating the inequality:

$$|x-5| < \frac{\varepsilon}{|x+5|} \implies |x-5|\,|x+5| = |(x-5)(x+5)| = \left|x^2-25\right| < \varepsilon$$

Thus $|x-5|<\delta \implies |x^2-25|<\varepsilon$, thus $\lim_{x\to 5} x^2=25$. Q.E.D.