

7.12.5) Given $f(x) = \ln(1+x)$, we know that for $n > 0$,

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$$

thus

$$f^{(n)}(0) = (-1)^{n-1} (n-1)!$$

From this, we can construct P_n and R_n at $c = 0$ as follows:

$$\begin{aligned} f(x) &= P_n + R_n = \sum_{k=0}^n \left\{ \frac{f^{(k)}(c)}{k!} (x-c)^k \right\} + \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \\ &= \frac{\ln(1)}{0!} x^0 + \sum_{k=1}^n \left\{ \frac{(-1)^{k-1} (k-1)!}{k!} x^k \right\} + (-1)^n \frac{n!}{(n+1)!(1+z)^{n+1}} x^{n+1} \\ &= \sum_{k=1}^n \left\{ \frac{(-1)^{k-1}}{k} x^k \right\} + \frac{(-1)^n}{n+1} \left(\frac{x}{1+z} \right)^{n+1} \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n} x^n + \frac{(-1)^n}{n+1} \left(\frac{x}{1+z} \right)^{n+1} \end{aligned}$$

where $z \in (c, x)$, thus we obtain the desired equation for $f(x)$.

7.12.6) a.) Since $f(0) = 0$, and since 0 is infinitely differentiable, we know $f^{(n)}(0) = 0$ for all n . Now, consider the derivative of $f(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$:

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$$

Thus we know that $e^{-\frac{1}{x^2}}$ is differentiable, and that its derivative is of the form $p(1/x)e^{-\frac{1}{x^2}}$ where p is some polynomial in $1/x$. It is trivial to show that p is differentiable. Next, suppose

$$f^{(n)}(x) = p(1/x)e^{-\frac{1}{x^2}}$$

for some polynomial p in $1/x$, then we can find $f^{(n+1)}(x)$ as follows:

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx} p(1/x) e^{-\frac{1}{x^2}} = p'(1/x) e^{-\frac{1}{x^2}} + \frac{2}{x^3} p(1/x) e^{-\frac{1}{x^2}} \\ &= e^{-\frac{1}{x^2}} \left(p'(1/x) + \frac{2}{x^3} p(1/x) \right) = q(1/x) e^{-\frac{1}{x^2}} \end{aligned}$$

for a similar polynomial q . We have shown that $f^{(n)}$ is differentiable, and since $f^{(n)}$ and $f^{(n+1)}$ take the same form, we know that $f^{(n+1)}$ is also differentiable, thus by induction, $e^{-\frac{1}{x^2}}$ is infinitely differentiable. ■

b.) We can construct P_n at $c = 0$ as follows:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^n \frac{0}{k!} x^k = 0$$

c.) Using Lagrange's form, we can construct R_n as follows:

$$R_n$$

Because we previously determined that $P_n = 0$ for all n ,