Chapter 4: Hilbert Spaces

5) There exists a function $f \in L^1(\mathbb{R}^d)$ where $f \notin L^2(\mathbb{R}^d)$, and a function $g \in L^2(\mathbb{R}^d)$ where $g \in L^1(\mathbb{R}^d)$.

However, if f is a measurable function supported on a set E of finite measure, and if $f \in L^2(\mathbb{R}^d)$, we have that $f \in L^1(\mathbb{R}^d)$ and

$$\|f\|_{L^1(\mathbb{R}^d)} \leq m(E)^{1/2} \|f\|_{L^2(\mathbb{R}^d)}.$$

Furthermore, if f is bounded, say by M, and if $f \in L^1(\mathbb{R}^d)$, then $f \in L^2(\mathbb{R}^d)$ and

$$\|f\|_{L^2(\mathbb{R}^d)} \leq M^{1/2} \|f\|_{L^1(\mathbb{R}^d)}^{1/2}.$$

Proof: (1)

(2)

Finally, suppose f is

6) The following sets of functions are dense subspaces of $L^2(\mathbb{R}^d)$:

(a) Simple functions

(b) Continuous functions of compact support

Proof: Fix $\varepsilon > 0$ and a simple function f on $\mathbb{R}^d$. Using the canonical representation of f, we obtain disjoint measurable sets $E_1, E_2, \ldots, E_n$ of finite measure and constants $a_1, a_2, \ldots, a_n$ in $\mathbb{R}$ where

$$f(x) = \sum_{1 \leq k \leq n} a_k \chi_{E_k}(x).$$

Also denote $E = \bigcup_{k=1}^n E_k$. We will now show that the function g, defined as

$$g(x) = f(x) - \delta \chi_E(x)$$

where $0 < \delta < \sqrt{\varepsilon / m(E)}$, is a simple function, and that $\|f - g\|_{L^2} < \varepsilon$.

We first note that

$$\delta \chi_E(x) = \sum_{1 \leq k \leq n} \delta \chi_{E_k}(x)$$

since the E_k are disjoint. This leads to

$$g(x) = f(x) - \delta \chi_E(x) = \sum_{1 \leq k \leq n} a_k \chi_{E_k}(x) - \sum_{1 \leq k \leq n} \delta \chi_{E_k}(x) = \sum_{1 \leq k \leq n} (a_k - \delta) \chi_{E_k}(x),$$

thus showing that g is a simple function. Finally, we compute $\|f-g\|_{L^2}$ as follows:

$$\|f-g\|_{L^2} = \int |f(x)-g(x)|^2\,dx = \int |f(x)-f(x)+\delta\chi_E(x)|^2\,dx = \int |\delta\chi_E(x)|^2\,dx$$

$$= \int |\delta|^2\,|\chi_E(x)|^2\,dx = \int \delta^2\chi_E(x)\,dx = \int_E \delta^2\,dx < \int_E \frac{\varepsilon}{m(E)}\,dx = m(E)\frac{\varepsilon}{m(E)} = \varepsilon,$$

thus proving (a).

We now let f be a continuous function of compact support.