Chapter 12

12.) Let $a, b, c \in R$ where a is a unit. Suppose $b \mid c$, then $c = bd$ for some $d \in R$. We know that $d = d(aa^{-1}) = a(da^{-1})$, thus $c = ba(da^{-1})$, and thus $ab \mid c$. Next, suppose $ab \mid c$, then $c = abd$ for some $d \in R$. We previous established that $d = ada^{-1}$, thus $c = abd = ab(ada^{-1}) = b(a^2da^{-1} = b(ad)$, thus $b \mid c$. ■

16.) Let $n, a \in R$. We can see that $n * (-a) = n * ((-1) * a) = (n * (-1)) * a = (-n) * a = ((-1) * n) * a = (-1) * (n * a) = -(n * a)$. ■

Chapter 13

3.) Let R be a commutative ring with cancellation, and let $a, b \in R$ where $a, b \neq 0$. For the sake of establishing a contradiction, suppose $ab = 0$, then $ab = 0 = a0$, thus $b = 0$, $\Rightarrow\Leftarrow$ Thus if $a, b \neq 0$, then $ab \neq 0$. ■

4.) The zero-divisors of $\mathbb{Z}_{20}$ are $2, 4, 5, 6, 8, 10, 12, 14, 15, 16$ and 18, as

$$2 * 10 \equiv 4 * 15 \equiv 5 * 8 \equiv 6 * 10 \equiv 12 * 5 \equiv 16 * 10 \equiv 0 \pmod{20}$$

We can see that the zero-divisors are all elements of $\mathbb{Z}_{20}$ that are not units. ■

5.) Let $a \in \mathbb{Z}_n$ where $a \neq 0$, and suppose $a \notin U_n$, then $\gcd(a, n) = d$ for some $d \geq 2$. Since $d \mid n$, let $k = n/d$. We know that $k \neq 0$ because $a \neq 0$, thus $gcd(a, n) \neq 0$. Finally, since $d \mid a$, we know that $a = dl$ for some $l \in \mathbb{Z}$, thus $ak = dln/d = ln \equiv 0 \pmod{n}$, thus since $a, k \neq 0$, we know that a is a zero-divisor. ■