Chapter 1

Let the prime factorization of an integer n be written as follows:

$$n = \prod_{p \text{ prime}} p_i^{x_i},$$

where p_i is the i^{th} prime and $x_i \geq 0$.

For problems 1 through 6, lowercase latin letters $a, b, \ldots, y, z$ represent integers.

1.) *Claim:* If $(a, b) = 1$, and $c \mid a$ and $d \mid b$, then $(c, d) = 1$.

Proof: Since $(a, b) = 1$, there exist x and y where $ax + by = 1$. We also know that $c \mid a$ and $b \mid d$, so there exists m and n where $a = cm$ and $b = dn$, thus $ax + by = c(mx) + d(ny) = 1$, thus $(c, d) = 1$. ■

2.) *Claim:* If $(a, b) = (a, c) = 1$, then $(a, bc) = 1$.

Proof: Since $(a, b) = (a, c) = 1$, there exist x_1, x_2, y_1, and y_2 where $ax_1 + by_1 = 1$ and $ax_2 + cy_2 = 1$. We can see that

$$1 = (ax_1 + by_1)(ax_2 + cy_2) = (a^2x_1x_2 + abx_2y_1 + acx_1y_2 + bcy_1y_2)$$

$$= a(ax_1x_2 + bx_2y_1 + cx_1y_2) + bc(y_1y_2)\,,$$

so $(a, bc) = 1$. ■

3.) *Claim:* If $(a, b) = 1$, then $(a^n, b^k) = 1$ for all n and k.

Proof: We can take the prime factorizations of a and b:

$$a = \prod p_i^{x_i} \quad \text{and} \quad b = \prod p_i^{y_i}\,,$$

where p_i are the primes, and x_i and y_i are integers that depend on p_i. Further, the prime factorizations for a^n and b^k are

$$a^n = \left(\prod p_i^{x_i}\right)^n = \prod p_i^{x_i^n}$$

and

$$b^k = \left(\prod p_i^{y_i}\right)^k = \prod p_i^{y_i^k}$$

Since $(a, b) = 1$, we know that $\min\{x_i, y_i\} = 0$ for all i, thus $\min\left\{x_i^n, y_i^k\right\} = 0$, thus $(a^n, b^k) = 1$. ■

4.) *Claim:* If $(a, b) = 1$, then $(a + b, a - b)$ is either 1 or 2.

Proof: Let $d = (a + b, a - b)$, then $d \mid a + b$ and $d \mid a - b$, so $a + b = dm$ and $a - b = dn$ for some m and n. We have $a + b = a - b + 2b = dn + 2b$, so $d \mid dn + 2b$, and thus $d \mid 2b$. Suppose b is even, then a is odd, since $(a, b) = 1$, thus $a + b$ is odd, thus d divides an odd number. But b is even, so $2b$ is even, thus d divides an even number too, thus $d = 1$. Finally, suppose b is odd. If a is even we are done as before, but if a is odd ***

5.) ***

6.) *Claim:* If $(a, b) = 1$ and $d \mid a + b$, then $(a, d) = (b, d) = 1$.

Proof: Since $(a, b) = 1$, we know that $ax + by = 1$ for some x and y, and since $d \mid a + b$, $a + b = dm$ for some m. We have $a + b = dm \implies b = dm - a$, thus $ax + by = ax + (dm - a)y = ax + dmy - ay = a(x - y) + d(my) = 1$, thus $(a, d) = 1$. A similar argument shows that $(b, d) = 1$ as well. ■

7.) ***

8.) *Claim:* Every integer can be represented as a^2b, where a and b are unique positive integers and b is squarefree.

Proof: We obtain the unique prime factorization of n as follows:

$$n = \prod_{\text{primes}} p_i^{x_i} \prod_{\text{primes}} p_j^{x_j},$$

where powers x_i of p_i are all odd, and powers x_j of p_j are all even. From this we can see that

$$n = p_i \prod_{\text{primes}} p_i^{x_i - 1} \prod_{\text{primes}} p_j^{x_j}$$

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