7.11.1.) a.) For f to be continuous at $x=0$, we must have $f(0)=\lim_{x\to 0} f(x)$, thus

$$f(0)=\lim_{x\to 0}\frac{3^x-2^x}{x}=\lim_{x\to 0}\frac{3^x\ln(3)-2^x\ln(2)}{1}=3^0\ln(3)-2^0\ln(2)=\ln(3)-\ln(2)$$

Thus defining $f(0)=\ln(3)-\ln(2)$ makes f continuous at $x=0$. ■

b.) Now we determine if $f'(0)$ exists given our definition of $f(0)$:

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{3^x-2^x-\ln(3)+\ln(2)}{x}=\frac{\ln(2)-\ln(3)}{0}\implies \text{diverges}$$

Thus $f'(0)$ does not exist given our definition of $f(0)$. ■

c.) We can try computing $f'(0)$ using typical derivative rules as follows:

$$f'(x)=\frac{d}{dx}\left\{\frac{3^x-2^x}{x}\right\}=\frac{x\left(3^x\ln(3)-2^x\ln(2)\right)-3^x+2^x}{x^2}$$

Thus

$$\begin{aligned}\lim_{x\to 0} f'(x)&=\lim_{x\to 0}\frac{x\left(3^x\ln(3)-2^x\ln(2)\right)-3^x+2^x}{x^2}\\
&=\lim_{x\to 0}\frac{3^x\ln(3)+3^x x\ln(3)^2-2^x\ln(2)-2^x x\ln(2)^2-3^x\ln(3)+2^x\ln(2)}{2x}\\
&=\lim_{x\to 0}\frac{3^x x\ln(3)^2-2^x x\ln(2)^2}{2x}=\lim_{x\to 0}\frac{3^x\ln(3)^2+3^x x\ln(3)^3-2^x\ln(2)-2^x x\ln(2)^3}{2}\\
&=\frac{1}{2}\left(3^0\ln(3)^2+3^0(0)\ln(3)^3-2^0\ln(2)-2^0(0)\ln(2)^3\right)=\frac{1}{2}\left(\ln(3)^2-\ln(2)^2\right)\end{aligned}$$

Thus by using the typical rules of calculus and taking a limit, we can reason that $f'(0)=\frac{1}{2}\left(\ln(3)^2-\ln(2)^2\right)$. ■

7.11.5.) Consider $\lim_{x\to 0} f(x)/g(x)$:

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{x^2\sin\left(x^{-1}\right)}{x}=\lim_{x\to 0} x\sin\left(x^{-1}\right)$$

As $x\to 0$, $\sin\left(x^{-1}\right)$ does not converge, but since it is multiplied by x, it vanishes as $x\to 0$, thus $\lim_{x\to 0} f(x)/g(x)=0$. Now, consider $\lim_{x\to 0} f'(x)/g'(x)$:

$$f'(x)=\frac{d}{dx}\left\{x^2\sin\left(x^{-1}\right)\right\}=2x\sin\left(x^{-1}\right)-\cos\left(x^{-1}\right)$$

$$g'(x)=\frac{d}{dx}\{x\}=1$$

Thus

$$\begin{aligned}\lim_{x\to 0}\frac{f'(x)}{g'(x)}&=\lim_{x\to 0}\frac{2x\sin\left(x^{-1}\right)-\cos\left(x^{-1}\right)}{1}=\lim_{x\to 0}2x\sin\left(x^{-1}\right)-\lim_{x\to 0}\cos\left(x^{-1}\right)\\
&=-\lim_{x\to 0}\cos\left(x^{-1}\right)\implies \text{diverges}\end{aligned}$$

Thus while $\lim_{x\to 0} f(x)/g(x)$ exists, $\lim_{x\to 0} f'(x)/g'(x)$ does not. ■