Definitions

1.) Given a relation $\sim$ on X, it is an *equivalence relation* if

   i. For all $a \in X$, $a \sim a$.

   ii. For all $a, b \in X$, $a \sim b \implies b \sim a$.

   iii. For all $a, b, c \in X$, $a \sim b \wedge b \sim c \implies a \sim c$.

2.) Given an equivalence relation $\sim$ on X and $x \in X$, the *equivalence class* of x is defined as $[x] = \{y \in X : x \sim y\}$.

3.) Given an equivalence relation $\sim$ on X, $X/{\sim}$ is defined as $X/{\sim} = \{[x] : x \in X\}$

4.) Given an equivalence relation $\sim$ on X, the *quotient map* p is defined as $p : X \to X/{\sim}$ where $x \mapsto [x]$.

5.) Given a topological space X and an equivalence relation $\sim$ on X, the *quotient topology* $\mathcal{T}_q$ on $X/{\sim}$ is defined as $\mathcal{T}_q = \{V \subset X/{\sim} : p^{-1}(V) \text{ is open in } X\}$

Proofs

a.) Suppose $e^{i\theta} = 1$, then $\cos\theta + i\sin\theta = 1$, thus $\sin\theta = 0$ and $\cos\theta = 1$, thus $\theta = 0$.

b.) Suppose $e^{i\theta} = i$, then $\cos\theta + i\sin\theta = i$, thus $\cos\theta = 0$ and $\sin\theta = 1$, thus $\theta = \pi/2$.

c.) Suppose $e^{i\theta} = 1/2 + i\left(\sqrt{3}/2\right)$, then $\cos\theta + i\sin\theta = 1/2 + i\left(\sqrt{3}/2\right)$, thus $\cos\theta = 1/2$ and $\sin\theta = \left(\sqrt{3}/2\right)$, thus $\theta = \pi/3$.

d.) Since $e^{i\theta}$ is simply a rotation by θ, we see that $e^{i\theta_1} = e^{i\theta_2}$ if θ_1 and θ_2 are equivalent angles, that is $\theta_1 - \theta_2 = 2\pi n$ for some $n \in \mathbb{Z}$.

e.) The unit circle, i.e. S^1.

f.) i.) Let $\sim$ be a relation on $\mathbb{R}$ where $a \sim b \iff a - b = 2\pi n$ for some $n \in \mathbb{Z}$. Since $a - a = 0$, we know that $a \sim a$ for all $a \in \mathbb{R}$. Next, assume $a \sim b$, then $a - b = 2\pi n$, thus $b - a = 2\pi(-n)$, thus $a\ b \implies b\ a$. Finally, let $a \sim b$ and $b \sim c$, thus $a - b = 2\pi m$ and $b - c = 2\pi n$ for $m, n \in \mathbb{Z}$, then $a - c = a - b + b - c = 2\pi m - 2\pi n = 2\pi(m - n)$, thus $a \sim c$, thus $\sim$ is an equivalence relation. ■

ii.) Let $f : \mathbb{R}/\!\sim\, \to S^1$ be defined as $f([\theta]) = (\cos\theta, \sin\theta)$. Since $\cos^2\theta + \sin^2\theta = 1$ for all $\theta \in \mathbb{R}$, we know that the image of f is S^1. Let $[\theta_1], [\theta_2] \in \mathbb{R}/\!\sim$ where $f([\theta_1]) = f([\theta_2])$, then $\sin\theta_1 = \sin\theta_2$, thus $\theta_1 \sim \theta_2$, thus $[\theta_1] = [\theta_2]$, thus f is injective. Next, let $(a, b) \in S^1$. Choose $\theta = \cos^{-1}(a) = \sin^{-1}(b)$. This is possible because $(a, b) \in S^1$. We can see that $f([\theta]) = (\cos(\cos^{-1}(a)), \sin(\sin^{-1}(b))) = (a, b)$, thus f is surjective, and thus bijective. ■