8.2.9 Given $f(x) = x^{-2}$ and a partition π of $[a, b]$, we can calculate the following riemann sum with associated points $\xi_k = \sqrt{x_k x_{k-1}}$:

$$\sum_{k=1}^{1} f(\xi_k)(x_k - x_{k-1}) = \frac{1}{\left(\sqrt{x_1 x_0}\right)^2}(x_1 - x_0) = \frac{x_1 - x_0}{x_1 x_0}$$

Now assume that the following equality holds:

$$\sum_{k=1}^{m} f(\xi_k)(x_k - x_{k-1}) = \frac{x_m - x_0}{x_m x_0}$$

We can compute the sum to $m + 1$ as follows:

$$\sum_{k=1}^{m+1} f(\xi_k)(x_k - x_{k-1}) = \sum_{k+1}^{m} f(\xi_k)(x_k - x_{k-1}) + f(\xi_{m+1})(x_{m+1} - x_m) = \frac{x_m - x_0}{x_m x_0} + \frac{x_{m+1} - x_m}{x_{m+1} x_m}$$

$$= \frac{x_{m+1}x_m - x_{m+1}x_0 + x_{m+1}x_0 - x_m x_0}{x_{m+1}x_m x_0} = \frac{x_m(x_{m+1} - x_0)}{x_{m+1}x_m x_0} = \frac{x_{m+1} - x_0}{x_{m+1}x_0}$$

Thus by induction, we know that

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \frac{x_n - x_0}{x_n x_0} = \frac{b - a}{ab}$$

Which is equal to the definite integral $\int_a^b x^{-2} dx$. ■

8.2.12 By the definition of the definite integral, given a continuous function f defined on $[0, 1]$, the following limit can be expressed as a definite integral:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{n}{k}\right) = \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=1}^{n} f\left(0 + \frac{n}{k}(1-0)\right) = \int_0^1 f(x)dx$$

■