2.) a.) $\gcd(2, 10) = 2$, $\text{lcm}(2, 10) = 10$

b.) $\gcd(20, 8) = 4$, $\text{lcm}(20, 8) = 40$

c.) $\gcd(12, 40) = 4$, $\text{lcm}(12, 40) = 120$

d.) $\gcd(21, 50) = 1$, $\text{lcm}(21, 50) = 1050$

e.) $\gcd(p^2q^2, pq^3) = pq^2$, $\text{lcm}(p^2q^2, pq^3) = p^2q^3$

7.) **Question:** If a and b are integers and n is a positive integer, prove that $a \equiv b \pmod{n} \iff n \mid a - b$

Solution: Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. Assume $a \equiv b \pmod{n}$, thus $a = b + kn$ for some $k \in \mathbb{Z}$. $a = b + kn \implies a - b = kn \implies n \mid a - b$, thus $a \equiv b \pmod{n} \implies n \mid a - b$.

Next, assume $n \mid a - b$, thus $ln = a - b$ for some $l \in \mathbb{Z}$. $ln = a - b \implies a = b + ln \implies a \equiv b \pmod{n}$, thus $n \mid a - b \implies a \equiv b \pmod{n}$,

Thus $a \equiv b \pmod{n} \iff n \mid a - b$. Q.E.D.

18.) **Question:** Determine $8^{402} \pmod{5}$

Solution: $8^1 \equiv 3 \pmod{5}$, $8^2 \equiv 4 \pmod{5}$, $8^3 \equiv 2 \pmod{5}$, $8^4 \equiv 1 \pmod{5}$, and $8^5 \equiv 3 \pmod{5}$, thus $8^{402} \equiv (8^4)^{100} \times 8^2 \equiv 1^{100} \times 4 \equiv 4 \pmod{5}$.

58.) **Question:** Let S be the set of real numbers. If $a, b \in S$, define $a \sim b$ if $a - b \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation and determine the equivalence classes of S.

Solution: Let $a \in S$. Since $a - a = 0$ and $0 \in \mathbb{Z}$, $a \sim a$ for all $a \in S$, thus $\sim$ is reflexive.

Let $a, b \in S$ and assume $a \sim b$, thus $n = a - b$, for some $n \in \mathbb{Z}$. $n = a - b \implies -n = -(a - b) = b - a$, thus $b - a \in \mathbb{Z}$, thus $b \sim a$, thus $\sim$ is symmetric.

Let $a, b, c \in S$ and assume $a \sim b$ and $b \sim c$, thus $m = a - b$ and $n = b - c$ for some $m, n \in \mathbb{Z}$. We see that $m + n = a - b + b - c = a - c$, thus $a - c \in \mathbb{Z}$, thus $a \sim b \wedge b \sim c \implies a \sim c$, thus $\sim$ is transitive, and thus an equivalence relation. Q.E.D.

Finally, the equivalence classes for $\sim$ are $[x]$ where $0 \leq x < 1$.

60.) **Question:** Let S be the set of integers. If $a, b \in S$, define aRb if $a + b$ is even. Show that R is an equivalence relation and determine the equivalence classes of S.

Solution: Let $a \in S$. Since $a + a = 2a$ is even, aRa for all $a \in S$, thus R is reflexive.

Let $a, b \in S$ and assume $a \sim b$, thus $a + b = 2n$ for some $n \in \mathbb{Z}$. Since integer addition is commutative, $b + a = a + b = 2n$, thus $b + a$ is even, thus $a \sim b \implies b \sim a$, thus R is symmetric.

Let $a, b, c \in S$ and assume $a \sim b$ and $b \sim c$, thus $a + b = 2m$ and $b + c = 2n$ for some $m, n \in \mathbb{Z}$. We see that $2m + 2n = a + 2b + c \implies 2(m + n - b) = a + c$, thus $a + c$ is even, thus $a \sim b \wedge b \sim c \implies a \sim c$, thus R is transitive, and thus an equivalence relation. Q.E.D.

Finally, the equivalence classes for R are [0] and [1].