

- 29.) a.) Since $S \neq \emptyset$ and is bounded below, there exists $l \in \mathbb{R}$ where $l \leq x$ for all $x \in S$, thus $l \in \mathcal{L}$, thus $\mathcal{L} \neq \emptyset$. Q.E.D.
- b.) Let $w = \sup(\mathcal{L})$, thus $w \geq l$ for all $l \in \mathcal{L}$. For the sake of establishing a contradiction, suppose $w > x$ for some $x \in S$. Let $m = \text{mid}(w, x)$, thus $w > m > x$. Since $m > x$ for some $x \in S$, m cannot be a lower bound of S , thus w cannot be a lower bound of S , thus $w \neq \sup(\mathcal{L}) \Rightarrow \Leftarrow$, thus $w \leq x$ for all $x \in S$, thus $w = \sup(\mathcal{L})$ is a lower bound of S , thus $w \in \mathcal{L}$. Q.E.D.
- c.) Since $w = \sup(\mathcal{L})$, $w \geq l$ for all $l \in \mathcal{L}$ and thus all lower bounds l of S , thus $w = \inf(S)$. Q.E.D.

33.) Consider $|x|$:

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

Now consider $|y|$:

$$|y| = |y - x + x| \leq |y - x| + |x| \implies |y| - |x| \leq |y - x| = |x - y|$$

$$\implies |x| - |y| \geq -|x - y|$$

Since $-|x - y| \leq |x| - |y| \leq |x - y|$, $||x| - |y|| \leq |x - y|$, thus the inequality holds. Q.E.D.

- 37.) Since $b \in \mathbb{B} \implies b \geq a$ for all $a \in A$, b is an upper bound of A for all $b \in B$. Since $A \neq \emptyset$ and is bounded above, there exists $u \in \mathbb{R}$ where $u = \sup(A)$. For the sake of establishing a contradiction, suppose $u > b$ for some $b \in B$. Let $m = \text{mid}(u, b)$, thus $u > m > b$. Since $m > b$ for some $b \in B$, $m > a$ for all $a \in A$, thus m is an upper bound of A , but since $u > m$, $u \neq \sup(A) \Rightarrow \Leftarrow$, thus $u \leq b$ for all $b \in B$, thus u is a lower bound of B , and thus $u = \sup(A) \leq \inf(B)$. Q.E.D.

38.) Suppose S is uniformly discrete. For the sake of establishing a contradiction, suppose S has no maximal element, thus $x \in S \implies (x + \varepsilon) \in S$ for some $\varepsilon > 0$. Let $x \in S$, thus there exists $\varepsilon > 0$ where $(x + \varepsilon) \in S$. Since $x, (x + \varepsilon) \in S$ and $x, (x + \varepsilon) \in (x - 2\varepsilon, x + 2\varepsilon)$, then $\{x, x + \varepsilon\} \subseteq S \cap (x - 2\varepsilon, x + 2\varepsilon)$. Let $\varepsilon_0 = 2\varepsilon$, thus for some $\varepsilon_0 > 0$, $S \cap (x - \varepsilon_0, x + \varepsilon_0) \neq \{x\}$, thus S is not uniformly discrete $\implies \Leftarrow$, thus if S is uniformly discrete, then S has a maximal element $m \in S$. Since $m \geq x$ for all $x \in S$, m is an upper bound of S . Since $m \in S$, then for all $u < m$, there exists $x \in S$ where $x > u$, thus u cannot be an upper bound of S , thus $m \leq u$ for all upper bounds u of S , thus $m = \sup(S)$, and thus $\sup(S) \in S$. Q.E.D.

45.) Since $y_n \rightarrow B$, then for all $\varepsilon_0 > 0$, there exists $k_0 \in \mathbb{N}$ where

$$n \geq k_0 \implies |y_n - B| < \varepsilon_0$$

Let $\lambda = |B|/2$. Since $B \neq 0$, there exists $k_1 \in \mathbb{N}$ where

$$n \geq k_1 \implies |y_n| > \lambda$$

Let $n \geq \max(k_0, k_1)$, then

$$\left| \frac{1}{y_n} - \frac{1}{B} \right| = \frac{|y_n - B|}{|y_n| |B|} < \frac{\varepsilon_0}{\lambda |B|}$$

$$\frac{\varepsilon_0}{\lambda |B|} = \varepsilon \implies \varepsilon_0 = \varepsilon |B| \lambda$$

Let $\varepsilon_0 = \varepsilon |B| \lambda$ and $n \geq \max(k_0, k_1)$, then

$$|y_n - B| < \varepsilon_0 \implies |y_n - B| < \varepsilon |B| \lambda \implies \frac{|y_n - B|}{|y_n| |B|} < \frac{|y_n - B|}{\lambda |B|} < \varepsilon$$

$$\implies \frac{|y_n - B|}{|y_n| |B|} = \left| \frac{1}{y_n} - \frac{1}{B} \right| < \varepsilon$$

Thus if $y_n \rightarrow B$ and $B \neq 0$, then $1/y_n \rightarrow 1/B$. Q.E.D.

47.) Since $y_n \rightarrow 0$, then for all $\varepsilon_0 > 0$, there exists $k \in \mathbb{R}$ where

$$n \geq k \implies |y_n - 0| < \varepsilon_0$$

Since x_n is bounded, there exists $M \in \mathbb{R}$ where $M \geq |x_n|$ for all $n \in \mathbb{N}$. Let $\varepsilon_0 = \varepsilon/M$ and $n \geq k$:

$$\begin{aligned} |y_n - 0| < \varepsilon_0 &\implies |y_n - 0| < \frac{\varepsilon}{M} \leq \frac{\varepsilon}{|x_n|} \implies |y_n - 0| < \frac{\varepsilon}{|x_n|} \\ &\implies |x_n| |y_n - 0| = |x_n y_n - 0| < \varepsilon \end{aligned}$$

Thus if x_n is bounded and $y_n \rightarrow 0$, then $x_n y_n \rightarrow 0$. Q.E.D.

48.) ***

49.)

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1 - 5n^2 + 40x^3 + 2n^{-2}}{4 - 12n - 2n^3} &= \lim_{n \rightarrow \infty} \frac{n^{-3}}{n^{-3}} \cdot \frac{1 - 5n^2 + 40x^3 + 2n^{-2}}{4 - 12n - 2n^3} \\ &= \lim_{n \rightarrow \infty} \frac{x^{-3} - 5n^{-1} + 40 + 2n^{-5}}{4n^{-3} - 12n^{-2} - 2} = \frac{\lim_{n \rightarrow \infty} \frac{1}{n^3} - \frac{5}{n} + 40 + \frac{2}{n^5}}{\lim_{n \rightarrow \infty} \frac{4}{n^3} - \frac{12}{n^2} - 2} = \frac{0 - 0 + 40 + 0}{0 - 0 - 2} \\ &= \frac{40}{-2} = -20 \end{aligned}$$

50.) Since $x_n \rightarrow A$, then for all $\varepsilon_0 > 0$, there exists $k \in \mathbb{N}$ where

$$n \geq k \implies |x_n - A| < \varepsilon_0$$

Let $\varepsilon_0 = \varepsilon/|c|$, then

$$|x_n - A| < \varepsilon_0 \implies |x_n - A| < \frac{\varepsilon}{|c|} \implies |c| |x_n - A| = |cx_n - cA| < \varepsilon$$

Thus if $x_n \rightarrow A$, then $cx_n \rightarrow cA$. Q.E.D.

105.) Suppose $y_n = 1/n$. $y_n \neq 0$ for all $n \in \mathbb{N}$, yet $y_n \rightarrow 0$. Since y_n converges, it is also cauchy. Because $y_n \rightarrow 0$, x_n/y_n does not converge, thus $z_n = x_n/y_n$ does not converge, thus z_n is not cauchy. Q.E.D.

109.) a.) A cauchy sequence is a sequence whose terms, past a certain point, get arbitrarily close to each other. A type-C sequence is a sequence whose terms, past a certain point, remain constant.

b.) Since $n \neq m \implies 1/n \neq 1/m$, there exists no N such that $n, m \geq N \implies |x_n - x_m| < \varepsilon$ for all $\varepsilon > 0$, thus $x_n = 1/n$ is not type-C.

c.) Let $n \in \mathbb{N}$ be fixed, and consider y_n, y_{n+1} , and y_{n+2} :

$$|y_n - y_{n+1}| = 2, |y_n - y_{n+2}| = 0$$

Since the distance between any two terms of y_n is either 0 or 2, it cannot be less than all $\varepsilon > 0$, thus y_n is not type-C.

d.) ***

e.) Since any type-C sequence eventually reaches a point where its terms remain constant, we know that every type-C sequence converges to this constant. Since it converges, it is also cauchy.

f.) $1/n$ is a cauchy sequence, but not a type-C sequence, thus not every cauchy sequence is type-C.

110.) Let $y_n = 2^{-n}$ and $z_n = -2^{-n}$. We know that $\lim_{n \rightarrow \infty} 2^{-n} = 0$, thus $\lim_{n \rightarrow \infty} -2^{-n} = -\lim_{n \rightarrow \infty} 2^{-n} = -0 = 0$, thus $y_n \rightarrow 0$ and $z_n \rightarrow 0$. Now consider x_n . For all $n \in \mathbb{N}$, the following is true:

118.) ***

121.) $E = \{1\}$

122.) $E = \{x_n\} \cup \{y_n\} \cup \{z_n\}$ where $x_n = 1/n$, $y_n = (n+1)/n$, and $z_n = (2n+1)/n$.

123.) $E = \bigcup_{k \in \mathbb{N}} \left\{ \frac{n}{kn+1} \right\}_{n \in \mathbb{N}}$ where $\left\{ \frac{n}{kn+1} \right\}_{n \in \mathbb{N}}$ is a sequence given some value of k .

135.) For $|x|$ to be continuous over all $c \in \mathbb{R}$, then for all $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - c| < \delta \implies ||x| - |c|| < \varepsilon$$

By the reverse triangle inequality, we know that $||x| - |c|| \leq |x - c|$, thus

$$||x| - |c|| \leq |x - c| < \delta$$

Let $\delta = \varepsilon$, then

$$||x| - |c|| < \varepsilon$$

Thus $|x|$ is continuous for all $c \in \mathbb{R}$. Q.E.D.

137.) $g(x) = \lfloor x \rfloor$ is continuous over all $x \in \mathbb{R} - \mathbb{Z}$.

139.) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $1/2$ -Hölder. For f to be continuous over all \mathbb{R} , then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Let $c = y$ and $\delta = (\varepsilon/C)^2$:

$$|x - y| < \delta \implies |x - y| < \left(\frac{\varepsilon}{C}\right)^2 \implies |x - y|^{1/2} < \frac{\varepsilon}{C} \implies C|x - y|^{1/2} < \varepsilon$$

Since f is $1/2$ -Hölder, $|f(x) - f(y)| < C|x - y|^{1/2}$, thus $|f(x) - f(y)| < \varepsilon$, thus f is continuous over all \mathbb{R} . Q.E.D.

150.) a.) ***

b.) ***