7.6.8.) Define an interval $[a, b]$, and let f be continuous on $[a, b]$ and differentiable on (a, b). In addition, suppose f satisfies the Lipschitz condition, thus for all $x, y \in [a, b]$, there exists $M \in \mathbb{R}$ where

$$|f(x) - f(y)| \leq M\,|x - y|$$

thus

$$\left|\frac{f(x) - f(y)}{x - y}\right|$$

By the mean value theorem, for all $c \in (a, b)$, we can find $x, y \in [a, b]$ where

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

thus

$$\left|f'(c)\right| = \left|\frac{f(x) - f(y)}{x - y}\right| \leq M$$

thus f' is bounded on $[a, b]$. ■

7.6.20.) Let f and g be continuous on $[a, b]$ and differentiable on (a, b), then according to the Cauchy mean value theorem, there exists $c \in (a, b)$ where

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

Consider the following determinant:

$$\begin{vmatrix} f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \\ f'(c) & g'(c) & 0 \end{vmatrix} = -f(a)g'(c) + g(a)f'(c) + f(b)g'(c) - f'(c)g(b) = 0$$

$$\implies g'(c)\,(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

Thus the conclusion of Cauchy's mean value theorem aligns with the given determinant form. ■

7.7.1.) Let $f(x) = (1 - x)e^x$ and consider f':

$$f' = \frac{d}{dx}(1 - x)e^x = -xe^x$$

Since $f'(0) = 0$, 0 is an inflection point for f. Consider f' for $x \in (0, 1)$:

$$0 < x < 1 \implies 0 < xe^x < e^x \implies 0 > -xe^x > -e^x$$

Thus $f'(x) < 0$ given $x \in (0, 1)$, thus f is strictly decreasing on $(0, 1)$, and since $f(0) = 1$, we know that $f(x) = (1 - x)e^x \leq 1$ for $x \in [0, 1)$. Next, consider f' for $x < 0$:

$$x < 0 \implies xe^x < 0$$

Thus $f'(x) < 0$ given $x < 0$, thus f is strictly decreasing on $(-\infty, 0)$, thus $f(x) = (1-x)e^x \leq 1$ for $x < 0$. Since $f(x) \leq 1$ for all $x < 1$, we know that the ratio between e^x and $\frac{1}{1-x}$ is less than or equal to 1 for all $x < 1$, thus the inequality $e^x \leq \frac{1}{1-x}$ holds. ■