Definitions

1.) Given a set X, a *topology* on X is a subset $\mathcal{T} \subseteq \mathcal{P}(X)$ that satisfies the following properties:

i. $\varnothing, X \in \mathcal{T}$

ii. Given a collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of sets in $\mathcal{T}$, $\bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$

iii. Given a *finite* collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of sets in $\mathcal{T}$, $\bigcap\limits_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$

2.) Given a topological space X, an equivalence relation $\sim$ on X, and the quotient map q on X, the *quotient topology* $\mathcal{T}$ on $X/\sim$ is defined as

$$\mathcal{T} = \left\{U \subseteq X/\sim\, : q^{-1}(U) \text{ is open in } X\right\}$$

3.) Given sets X and Y, the *cartesian product* $X \times Y$ is defined as

$$X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$$

4.) Given topological spaces X and Y, the *product topology* $\mathcal{T}$ on $X \times Y$ is defined as

$$\mathcal{T} = \left\{U \subseteq X \times Y : U = \bigcup_{\alpha\in\mathcal{A}} V_\alpha \times V'_\alpha\right\}$$

where $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{V'_\alpha\}_{\alpha\in\mathcal{A}}$ are collections of open sets in X and Y respectively.

5.) Given sets X and Y, the *projection maps* $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ out of $X \times Y$ are the the respective mappings $(x, y) \mapsto x$ and $(x, y) \mapsto y$.

6.) Given a collection of sets $\{X_\alpha\}_{\alpha\in\mathcal{A}}$, the *direct product* $\prod\limits_{\alpha\in\mathcal{A}} X_\alpha$ is defined as

$$\prod_{\alpha\in\mathcal{A}} X_\alpha = \left\{\{x_\alpha\}_{\alpha\in\mathcal{A}} : x_\alpha \in X_\alpha \text{ for all } \alpha\right\}$$

7.) Given a collection of topological spaces $\{X_\alpha\}_{\alpha\in\mathcal{A}}$, the *product topology* $\mathcal{T}$ on $\prod\limits_{\alpha\in\mathcal{A}} X_\alpha$ is defined as

$$\mathcal{T} = \left\{U \subseteq \prod_{\alpha\in\mathcal{A}} X_\alpha : U = \bigcup \prod_{\alpha\in\mathcal{A}} V_\alpha\right\}$$

where V_α is open in X_α for all α and where $V_\alpha = X_\alpha$ for almost all α.

Proofs

a.) Let $\theta_1, \theta_2 \in \mathbb{R}$ where $\theta_1 \sim \theta_2$ and $\theta_1 \neq \theta_2$, then there exists a nonzero $n \in \mathbb{Z}$ where $\theta_1 - \theta_2 = \pi n$. Consider $\bar{f}([\theta_1])$ and $\bar{f}([\theta_2])$. Since $\theta_1 = \theta_2 + \pi n$, we can see that $f([\theta_1]) = f(\theta_1) = (\cos(2\theta_1), \sin(2\theta_1)) = (\cos(2(\theta_2 + \pi n)), \sin(2(\theta_2 + \pi n))) = (\cos(2\theta_2 + 2\pi n), \sin(2\theta_2 + 2\pi n)) = (\cos(2\theta_2), \sin(2\theta_2)) = f(\theta_2) = f([\theta_2])$, thus the value of $\bar{f}([\theta_1])$ does not depend on the representation of $[\theta_1]$, and thus $\bar{f}$ is well defined. ■

b.) Since $f(\theta) = (\cos(2\theta), \sin(2\theta))$, we know that the components of f are continuous, and thus f is continuous. Also note that $f = \bar{f} \circ q$. Let $U \subseteq Y$ be open, then $f^{-1}(U)$ is open, but $f^{-1}(U) = q^{-1}(\bar{f}^{-1}(U))$, thus $q^{-1}(\bar{f}^{-1}(U))$ is open, thus $\bar{f}^{-1}(U)$ is open in $X/\sim$, thus $\bar{f}$ is continuous. ■

c.) Let $\theta_1, \theta_2 \in \mathbb{R}$ where $f([\theta_1]) = f([\theta_2])$, thus $f(\theta_1) = f(\theta_2)$, thus $(\cos(2\theta_1), \sin(2\theta_1)) = (\cos(2\theta_2), \sin(2\theta_2))$, thus $\cos(2\theta_1) = \cos(2\theta_2)$ and $\sin(2\theta_1) = \sin(2\theta_2)$, thus $\theta_1 = \theta_2$, and thus $\bar{f}$ is injective. ■

d.) Let $(a, b) \in S^1$ and choose $\theta = \cos^{-1}(a)/2 = \sin^{-1}(b)/2$, then $\bar{f}([\theta]) = f(\theta) = (\cos(2\cos^{-1}(a)/2), \sin(2\sin^{-1}(b)/2)) = (a, b)$, thus $\bar{f}$ is surjective. ■

e.) We must also show that the inverse of $\bar{f}$ is continuous.