Exercises 15.6

3.)

$$I = \int_0^2 \int_0^{z^2} \int_0^{y-z} (2x - y)\, dx\, dy\, dz = \int_0^2 \int_0^{z^2} \left[x^2 - xy\right]_0^{y-z}\, dy\, dz$$

$$= \int_0^2 \int_0^{z^2} (y-z)^2 - (y-z)y\, dy\, dz = \int_0^2 \int_0^{z^2} y^2 - 2yz + z^2 - y^2 + yz\, dy\, dz$$

$$= \int_0^2 \int_0^{z^2} z^2 - yz\, dy\, dz = \int_0^2 \left[yz^2 - \frac{1}{2}y^2 z\right]_0^{z^2} = \int_0^2 z^4 - \frac{1}{2}z^5\, dz$$

$$= \left[\frac{1}{5}z^5 - \frac{1}{12}z^6\right]_0^2 = \frac{32}{5} - \frac{64}{12} = \frac{384}{60} - \frac{320}{60} = \frac{64}{60} = \frac{16}{15}$$

7.)

$$\int_0^\pi \int_0^1 \int_0^{\sqrt{1-z^2}} z \sin x\, dy\, dz\, dx = \int_0^\pi \int_0^1 \left[yz \sin x\right]_0^{\sqrt{1-z^2}}\, dz\, dx$$

$$= \int_0^\pi \int_0^1 z\sqrt{1-z^2} \sin x\, dz\, dx = -\frac{1}{2} \int_0^\pi \int_0^1 \sqrt{u} \sin x\, du\, dx$$

$$= -\frac{1}{2} \int_0^\pi \left[\frac{2}{3}\sqrt{(1-z^2)^3} \sin x\right]_0^1 = \frac{1}{3} \int_0^\pi \sin x\, dx = \left[-\frac{1}{3}\cos x\right]_0^\pi = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

11.)

$$I = \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2 + z^2}\, dx\, dz\, dy = \int_1^4 \int_y^4 \left[\tan^{-1}\left(\frac{x}{z}\right)\right]_0^z\, dz\, dy = \frac{\pi}{4} \int_1^4 \int_y^4\, dz\, dy$$

$$= \frac{\pi}{4} \int_1^4 [z]_y^4\, dy = \frac{\pi}{4} \int_1^4 4 - y\, dy = \frac{\pi}{4} \left[4y - \frac{1}{2}y^2\right]_1^4 = \frac{\pi}{4}\left(16 - 8 - 4 + \frac{1}{2}\right) = \frac{9\pi}{8}$$

19.) First, find bounds for z:

$$2x + y + z = 4 \implies 0 \le z \le 4 - 2x - y$$

Then find bounds for x and y:

$$2(0) + y + 0 = y = 4$$

$$2x + 0 + 0 = 2x = 4 \implies x = 2$$

Thus the triangular region is bounded by the first quadrant and $y = 4 - 2x$, thus

$$0 \le x \le 2,\ 0 \le y \le 4 - 2x$$

Evaluate the integral:

$$I = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} 4 - 2x - y\, dy\, dx$$

$$= \int_0^2 \left[4y - 2xy - \frac{1}{2}y^2\right]_0^{4-2x} dx = \int_0^2 -8x + 16 + 4x^2 - 8x - \frac{1}{2}(4x^2 - 16x + 16)\,dx$$

$$= \int_0^2 2x^2 - 8x + 8\,dx = \left[\frac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \frac{16}{3} - 16 + 16 = \frac{16}{3}$$

21.) First, find bounds for z:

$$x + y = 1 \implies 0 \le z \le 1 - y$$

Then find bounds for x and y:

$$z = 0 \implies y = 1 \implies 1 = x^2 \implies x = \pm 1$$

Thus

$$-1 \le x \le 1,\ x^2 \le y \le 1$$

Evaluate the integral:

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\,dy\,dx = \int_{-1}^1 \int_{x^2}^1 1 - y\,dy\,dx = \int_{-1}^1 \left[y - \frac{1}{2}y^2\right]_{x^2}^1 dx$$

$$= \int_{-1}^1 \frac{1}{2} - x^2 + \frac{1}{2}x^4\,dx = \left[\frac{1}{2}x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_{-1}^1 = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10} = \frac{8}{15}$$

54.)

$$I =$$

Exercises 15.7

1.) a.)

$$x = r\cos\theta = 4\cos(\pi/3) = 2,\ y = r\sin\theta = 4\sin(\pi/3) = 2\sqrt{3},\ z = -2$$

$$\therefore\ (4, \pi/3, -2) \to (2, 2\sqrt{3}, -2)$$

b.)

$$x = 2\cos(-\pi/2) = 0,\ y = 2\sin(-\pi/2) = -2, z = 1$$

$$\therefore\ (2, -\pi/2, 1) \to (0, -2, 1)$$

3.) a.)

$$x = -1,\ y = 1,\ r = \sqrt{x^2 + y^2} = \sqrt{2},\ \theta = \tan^{-1}(-1) = 3\pi/4$$

$$\therefore\ (-1, 1, 1) \to (\sqrt{2}, 3\pi/4, 1)$$

b.)

$$x = -2,\ y = 2\sqrt{3},\ r = \sqrt{4 + 12} = 4,\ \theta = \tan^{-1}(-\sqrt{3}) = 2\pi/3$$

$$\therefore\ (-2, 2\sqrt{3}, 3) \to (4, 2\pi/3, 3)$$

9.) a.)

$$x^2 - x + y^2 + z^2 = r^2 - x - z^2 = r^2 - r\cos\theta - z^2 = 1 \implies z^2 = 1 - r\cos\theta + r^2$$

b.)

$$z = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos(2\theta)$$

17.) Since $x^2 + y^2 = 16$ represents a circle with radius 4, $0 \leq r \leq 4$, and $0 \leq \theta \leq 2\pi$. In addition, $-5 \leq z \leq 4$. Evaluating the integral:

$$I = \int_{-5}^{4}\int_{0}^{2\pi}\int_{0}^{4} r^2\,dr\,d\theta\,dz = \int_{-5}^{4}\int_{0}^{2\pi}\left[\frac{1}{3}r^3\right]_0^4 d\theta\,dz = \int_{-5}^{4}\int_{0}^{2\pi}\frac{64}{3}\,d\theta\,dz$$

$$= \int_{-5}^{4}\left[\frac{64}{3}\theta\right]_0^{2\pi} dz = \int_{-5}^{4}\frac{128\pi}{3}\,dz = \frac{128\pi}{3}(4+5) = 3(128\pi) = 384\pi$$

23.)

24.)

29.)

$$I = \int_0^{2\pi}\int_0^2\int_r^2 zr^2\cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\left[\frac{1}{2}z^2r^2\cos\theta\right]_r^2 dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2}{3}r^3\cos\theta - \frac{1}{10}r^5\cos\theta\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}\cos\theta - \frac{16}{5}\cos\theta\,d\theta = 0 - 0 = 0$$