A quick lemma that will be useful for this proof:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)\,. \tag{$*$}$$

Proof: Let A, B, C, D be sets, and let $(x, y) \in (A \times B) \cap (C \times D)$, then $(x, y) \in A \times B$ and $(x, y) \in C \times D$, thus $x \in A$ and $x \in C$, and $y \in B$ and $y \in D$, thus $x \in A \cap C$ and $y \in B \cap D$, thus $(x, y) \in (A \cap C) \times (B \cap D)$. These statements are biconditional, so the converse is also proven. ■

Let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies defined in the problem. If $\mathcal{T}$ and $\mathcal{T}'$ are equal, then a subset $U \subseteq A \times B$ is open in $\mathcal{T}$ if and only if it is open in $\mathcal{T}'$. Let U be open in $\mathcal{T}$, then there exist collections $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of open sets in A and B respectively where

$$U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha\,.$$

Since for all $\alpha \in \mathcal{A}$, U_α and V_α are open in A and B respectively, we know that for each α, there exist open sets $U'_\alpha \subseteq X$ and $V'_\alpha \subseteq Y$where $U_\alpha = A \cap U'_\alpha$ and $V_\alpha = B \cap V'_\alpha$. From this, it is clear that

$$U = \bigcup_{\alpha \in \mathcal{A}} (A \cap U'_\alpha) \times (B \cap V'_\alpha)\,,$$

thus by $(*)$, we have

$$U = \bigcup_{\alpha \in \mathcal{A}} (A \times B) \cap (U'_\alpha \times V'_\alpha) = (A \times B) \cap \bigcup_{\alpha \in \mathcal{A}} U'_\alpha \times V'_\alpha\,.$$

Since $\{U'_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V'_\alpha\}_{\alpha \in \mathcal{A}}$ are collections of open sets in X and Y respectively, we know that $\bigcup U'_\alpha \times V'_\alpha$ is open in $X \times Y$, thus by definition, U is open in the subspace topology on $A \times B$, and thus U is open in $\mathcal{T}'$.

Conversely, let $U \subseteq A \times B$ be open in $\mathcal{T}'$, then there exist collections of open sets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ in X and Y respectively where

$$U = (A \times B) \cap \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha\,,$$

thus by $(*)$,

$$U = \bigcup_{\alpha \in \mathcal{A}} (A \times B) \cap (U_\alpha \times V_\alpha) = \bigcup_{\alpha \in \mathcal{A}} (A \cap U_\alpha) \times (B \cap V_\alpha)\,.$$

We know that for each α, $A \cap U_\alpha$ and $B \cap V_\alpha$ are open in A and B respectively. Finally, let $\{U'_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V'_\alpha\}_{\alpha \in \mathcal{A}}$ be collections of sets where $U'_\alpha = A \cap U_\alpha$ and $V'_\alpha = A \cap V_\alpha$. It is clear that

$$U = \bigcup_{\alpha \in \mathcal{A}} U'_\alpha \times V'_\alpha\,,$$

and since for all α, U'_α and V'_α are open in A and B respectively, we know that U is open in the product topology on $A \times B$, thus U is open in $\mathcal{T}$, and thus $\mathcal{T} = \mathcal{T}'$. ■