a.) Let P and Q be posets, and $f : P \times Q \to P$ and $g : P \times Q \to Q$ be defined as follows:

$$f(p,q) = p, \;\; g(p,q) = q$$

Let $(p_1, q_1), (p_2, q_2) \in P \times Q$ where $(p_1, q_1) \leq (p_2, q_2)$, thus $p_1 \leq p_2$ and $q_1 \leq q_2$. From this we can see that

$$f(p_1, q_1) = p_1, \;\; f(p_2, q_2) = p_2$$

thus

$$(p_1, q_1) \leq (p_2, q_2) \implies p_1 \leq p_2 \implies f(p_1, q_1) \leq f(p_2, q_2)$$

thus f is a map of posets. Similary, we can see that

$$g(p_1, q_1) = q_1, \;\; g(p_2, q_2) = q_2$$

thus

$$(p_1, q_1) \leq (p_2, q_2) \implies q_1 \leq q_2 \implies g(p_1, q_1) \leq g(p_2, q_2)$$

thus g is a map of posets, thus the projection maps f and g are both maps of posets. ■

b.) Let W be a poset, and consider an arbitrary poset map $h : W \to P \times Q$. Given $w_1, w_2 \in W$ where $w_1 \leq w_2$, we know that $h(w_1) \leq h(w_2)$, thus there exist $p_1, p_2 \in P$ and $q_1, q_2 \in Q$ where $h(w_1) = (p_1, q_1)$, $h(w_2) = (p_2, q_2)$, and $(p_1, q_1) \leq (p_2, q_2)$. Since the projection maps f and g, defined above, are maps of posets, we know that $(p_1, q_1) \leq (p_2, q_2) \implies f(p_1, q_1) \leq f(p_2, q_2)$ and $g(p_1, q_1) \leq g(p_2, q_2)$. From this we can see that $w_1 \leq w_2 \implies h(w_1) \leq h(w_2) \implies f(h(w_1)) \leq f(h(w_2))$ and $g(h(w_1)) \leq g(h(w_2))$, thus the composition of any poset map with the projection maps f and g is a poset map. ■

c.) Given the projection maps f and g as defined above, define j as follows:

$$j : \hom(W, P \times Q) \to \hom(W, P) \times \hom(W, Q)$$

$$\text{where } j(h) = (f(h), g(h)) \text{ given } h : W \to P \times Q$$

From this we can see that composition by the projection maps allows us to define a function j between the two sets.

d.) First we will show that j is injective. Let $h_1, h_2 \in \hom(W, P \times Q)$, and assume that given $w \in W$, we have $j(h_1(w)) = j(h_2(w))$, thus $(f(h_1(w)), g(h_1(w))) = (f(h_2(w)), g(h_2(w)))$, and thus $f(h_1(w)) = f(h_2(w))$ and $g(h_1(w)) = g(h_2(w))$. We know that $h_1(w) = (p_1, q_1)$ and $h_2(w) = (p_2, q_2)$ for some $p_1, p_2 \in P$ and $q_1, q_2 \in Q$, thus $f(p_1, q_1) = f(p_2, q_2)$ and $g(p_1, q_1) = g(p_2, q_2)$, thus $p_1 = p_2$ and $q_1 = q_2$, thus $h_1(w) = h_2(w)$ for all $w \in W$, thus given $h_1, h_2 \in \hom(W, P \times Q)$, $j(h_1(w)) = j(h_2(w)) \implies h_1(w) = h_2(w)$ for all $w \in W$, thus j is injective.

Next we will show that j is surjective. Given $(f(h), g(h)) \in \hom(W, P) \times \hom(W, Q)$ for some $h \in \hom(W, P \times Q)$, we must find $h' \in \hom(W, P \times Q)$ where $j(h') = (f(h), g(h))$. Let $h'(w) = h(w)$ for all $w \in W$, then $j(h') = (f(h'(w)), g(h'(w))) = (f(h(w)), g(h(w)))$, thus for all $(f(h), g(h)) \in \hom(W, P) \times \hom(W, Q)$, there exists $h' \in \hom(W, P \times Q)$ where $j(h') = (f(h), g(h))$, thus j is surjective.

Since j is both injective and surjective, it is bijective. ■