Definitions

1.) Given two points x and x' in a topological space X, a *continuous path* in X from x to x' is a continuous function $\gamma : [0,1] \to X$ where $\gamma(0) = x$ and $\gamma(1) = x'$. ‡ú

2.) A topological space X is *path-connected* if for all $x, x' \in X$, there exists a continuous path between them.

3.) Define the equivalence relation $\sim$ on a topological space X as follows:

$$x \sim x' \iff x \text{ and } x' \text{ are path-connected}\,.$$

We define $\pi_0(X)$ as the quotient space $X/\!\sim$.

4.) The *invariance of domain* theorem states that $\mathbb{R}^m \cong \mathbb{R}^n \iff m = n$.

5.) A topological space X is *connected* if for any set in X that is both open and closed, that set is either X or $\varnothing$.

Proof

We know that $\mathbb{R}^n \cup \{*\}$ is compact. In addition, since $\mathbb{R}^{n+1}$ is Hausdorff and $S^n \subset \mathbb{R}^{n+1}$, we know that S^n is Hausdorff. We also know that the stereographic projection $p : S^n \backslash \{0, \dots, 0, 1\} \to \mathbb{R}^n$ is bijective, so consider its inverse p^{-1}, and define a function $f : \mathbb{R}^n \cup \{*\} \to S^n$ as follows:

$$f(x) = \begin{cases} (0, \dots, 0, 1) \in S^n & x = * \\ p^{-1}(x) & \text{otherwise} \end{cases}$$

We can readily see that f is bijective. Let U be an open set in S^n, thus $U = S^n \cap V$ where V is open in $\mathbb{R}^n$, thus $f^{-1}(U) = f^{-1}(S^n \cap V) = f^{-1}(S^n) \cap f^{-1}(V)$. Since $V \subseteq \mathbb{R}^n$, we know that $* \notin V$, so $f^{-1}(V) = p(V)$, and since p is a homeomorphism, we know that $V' = p(V)$ is open in $\mathbb{R}^n$, thus $f^{-1}(S^n) \cap f^{-1}(V) = (\mathbb{R}^n \cup \{*\}) \cap V' = (\mathbb{R}^n \cap V') \cup (V' \cap \{*\}) = \mathbb{R}^n \cap V' = V'$, which is open in $\mathbb{R}^n$, thus f is continuous, and thus by theorem 15.5.1, a homeomorphism. ■