Exercises 13.1

5.)

$$\lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t}\right\rangle = \left\langle \lim_{t\to\infty}\frac{1+t^2}{1-t^2}, \lim_{t\to\infty}\tan^{-1} t, \lim_{t\to\infty}\frac{1-e^{-2t}}{t}\right\rangle$$

$$\lim_{t\to\infty}\frac{1+t^2}{1-t^2} = \lim_{t\to\infty}\frac{2t}{-2t} = \lim_{t\to\infty}\frac{2}{-2} = -1$$

$$\lim_{t\to\infty}\tan^{-1} t = \frac{\pi}{2}$$

$$\lim_{t\to\infty}\frac{1-e^{-2t}}{t} = \lim_{t\to\infty} 2e^{-2t} = \lim_{t\to\infty}\frac{2}{e^{2t}} = 0$$

$$\therefore\ \lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2}, \tan^{-1} t, \frac{1-e^{-2t}}{t}\right\rangle = \left\langle -1, \frac{\pi}{2}, 0\right\rangle$$

Exercises 13.2

11.)

$$\frac{d}{dt}\left\langle t^2, \cos(t^2), \sin^2 t\right\rangle = \left\langle 2t, -2t\sin(t^2), 2\sin t\cos t\right\rangle$$

17.)

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$r'(t) = \left\langle 2t-2, 3, t^2+t\right\rangle$$

$$\|r'(t)\| = \sqrt{(2t-2)^2 + 3^2 + (t^2+t)^2} = \sqrt{4t^2 - 8t + 4 + 9 + t^4 + 2t^3 + t^2}$$

$$= \sqrt{t^4 + 2t^3 + 5t^2 - 8t + 13}$$

$$\therefore\ T(t) = \frac{1}{\sqrt{t^4 + 2t^3 + 5t^2 - 8t + 13}}\left\langle 2t-2, 3, t^2+t\right\rangle$$

$$T(2) = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle$$

23.)

$$r'(t) = \left\langle 2t, \frac{2}{\sqrt{t}}, (2t-1)e^{t^2-t}\right\rangle$$

$$\|r'(t)\| = \sqrt{(2t)^2 + \left(\frac{2}{\sqrt{t}}\right)^2 + ((2t-1)e^{t^2-t})} = \sqrt{4t^2 + \frac{4}{t} + (4t^2-4t+1)(e^{2(t^2-t)})}$$

$$T(2,4,1) = \left\langle \frac{4}{r'(2)}, \frac{1}{r'(4)}, \frac{1}{r'(1)}\right\rangle$$

$$\therefore\ x = 2 + \frac{4}{r'(2)}t,\ y = 4 + \frac{1}{r'(4)},\ z = 1 + \frac{1}{r'(1)}$$

35.)

$$\int_0^2 (t\hat{\imath} - t^3\hat{\jmath} + 3t^5\hat{k})\ dt = \left[\frac{t^2}{2}\hat{\imath} - \frac{t^4}{4}\hat{\jmath} + \frac{t^6}{2}\hat{k}\right]_0^2 = 2i - 4j + 32k$$

39.)

$$\int (\sec^2 t\hat{\imath} + t(t^2+1)^3\hat{\jmath} + t^2 \ln t\hat{k})\ dt$$

$$\int \sec^2 t\hat{\imath}\ dt = \tan t\hat{\imath}$$

$$\int t(t^2+1)^3\hat{\jmath}\ dt = \frac{(t^2+1)^4}{8}\hat{\jmath}$$

$$\int t^2 \ln t\hat{k}\ dt = \left(\frac{t^3}{3}\ln t - \int \frac{t^3}{3t}\right)\hat{k} = \left(\frac{t^3 \ln t}{3} - \frac{t^3}{9}\right)\hat{k}$$

$$\therefore\ I = \tan t\hat{\imath} + \frac{(t^2+1)^4}{8}\hat{\jmath} + \left(\frac{t^3 \ln t}{3} - \frac{t^3}{9}\right)\hat{k} + C$$

41.)

$$r(t) = \int r'(t)\ dt = t^2\hat{\imath} + t^3\hat{\jmath} + \frac{2\sqrt{t^3}}{3}\hat{k} + C$$

$$\hat{\imath} + \hat{\jmath} = \hat{\imath} + \hat{\jmath} + \frac{2}{3}\hat{k} + C \implies C = -\frac{2}{3}\hat{k}$$

$$\therefore\ r(t) = t^2\hat{\imath} + t^3\hat{\jmath} + \left(\frac{2\left(\sqrt{t^3} - 1\right)}{3}\right)\hat{k}$$

Exercises 13.3

1.)

$$\int_{-5}^{5} \|r'(t)\|\ dt = \int_{-5}^{5} \|\langle 1, -3\sin t, 3\cos t\rangle\|\ dt = \int_{-5}^{5} \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\ dt$$

$$= \int_{-5}^{5} \sqrt{1 + 9\sin^2 t + 9\cos^2 t}\ dt = \int_{-5}^{5} \sqrt{1+9}\ dt = \int_{-5}^{5} \sqrt{10}\ dt = \left[\sqrt{10}t\right]_{-5}^{5}$$

$$= 5\sqrt{10} - (-5\sqrt{10}) = 10\sqrt{10}$$

17.) a.)

$$T = \frac{r'}{\|r'\|} = \frac{\langle 1, -3\sin t, 3\cos t\rangle}{\sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}} = \frac{\langle 1, -3\sin t, 3\cos t\rangle}{\sqrt{1 + 9\sin^2 t + 9\cos^2 t}}$$

$$= \frac{\langle 1, -3\sin t, 3\cos t\rangle}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$$

$$= \left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t \right\rangle$$

$$N = \frac{T'}{\|T'\|} = \frac{\left\langle 0, -\frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle}{\sqrt{0^2 + \left(-\frac{3}{\sqrt{10}}\cos t\right)^2 + \left(-\frac{3}{\sqrt{10}}\sin t\right)^2}} = \frac{\left\langle 0, -\frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle}{\sqrt{\frac{9}{10}\cos^2 t + \frac{9}{10}\sin^2 t}}$$

$$= \frac{1}{\sqrt{\frac{9}{10}}}\left\langle 0, -\frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle = \left\langle 0, -\frac{3}{3}\cos t, -\frac{3}{3}\sin t \right\rangle$$

$$= \langle 0, -\cos t, -\sin t\rangle$$

b.)

$$\kappa = \frac{\|T'\|}{\|r'\|} = \frac{\frac{\sqrt{9}}{\sqrt{10}}}{\sqrt{10}} = \frac{\sqrt{9}}{10} = \frac{3}{10}$$

21.)

$$r' = \left\langle 0, 3t^2, 2t \right\rangle,\ r'' = \langle 0, 6t, 2\rangle$$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} = (6t^2 - 12t^2)\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \left\langle -6t^2, 0, 0 \right\rangle$$

$$\|r' \times r''\| = \sqrt{(-6t^2)^2 + 0^2 + 0^2} = \sqrt{36t^4} = 6t^2$$

$$\|r'\|^3 = \left(\sqrt{0^2 + (3t^2)^2 + (2t)^2}\right)^3 = \left(\sqrt{9t^4 + 4t^2}\right)^3 = \left(9t^4 + 4t^2\right)^{\frac{3}{2}}$$

$$\therefore\ \kappa = \frac{6t^2}{\left(9t^4 + 4t^2\right)^{\frac{3}{2}}}$$

47.)

$$T = \frac{r'}{\|r'\|} = \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{(2t)^2 + (2t^2)^2 + 1^2}} = \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \frac{\langle 2t, 2t^2, 1\rangle}{\sqrt{(2t^2+1)^2}}$$

$$= \frac{\langle 2t, 2t^2, 1\rangle}{2t^2+1} = \left\langle \frac{2t}{2t^2+1}, \frac{2t^2}{2t^2+1}, \frac{1}{2t^2+1} \right\rangle$$

$$T\left(1, \frac{2}{3}, 1\right) = \left\langle \frac{2(1)}{2(1)^2+1}, \frac{2(2/3)^2}{2(2/3)^2+1}, \frac{1}{2(1)^2+1} \right\rangle = \left\langle \frac{2}{3}, \frac{8}{17}, \frac{1}{3} \right\rangle$$

$$N = \frac{T'}{\|T'\|} = \frac{\left\langle \frac{2-4t^2}{(2t^2+1)^2}, \frac{2t^2}{(2t^2+1)^2}, -\frac{4t}{(2t^2+1)^2} \right\rangle}{\sqrt{\left(\frac{2-4t^2}{(2t^2+1)^2}\right)^2 + \left(\frac{2t^2}{(2t^2+1)^2}\right)^2 + \left(-\frac{4t}{(2t^2+1)^2}\right)^2}}$$

Exercises 13.4

5.)

$$v(t) = r'(t) = \langle -3\sin t, 2\cos t\rangle$$
$$a(t) = r''(t) = \langle -3\cos t, -2\sin t\rangle$$
$$\|v(t)\| = \|r'(t)\| = \sqrt{(-3\sin t)^2 + (2\cos t)^2} = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{5\sin^2 t + 4}$$

15.)

$$v(t) = \int a(t)\ dt = 2t\hat{\imath} + t^2\hat{k} + C_1$$
$$v(0) = 3\hat{\imath} - \hat{\jmath} = 2(0)\hat{\imath} + (0)^2\hat{k} + C_1 = C_1 \implies C_1 = 3\hat{\imath} - \hat{\jmath}$$
$$\therefore\ v(t) = 2t\hat{\imath} + t^2\hat{k} + 3\hat{\imath} - \hat{\jmath} = (2t+3)\hat{\imath} - \hat{\jmath} + t^2\hat{k}$$
$$r(t) = \int v(t)\ dt = (t^2 + 3t)\hat{\imath} - t\hat{\jmath} + \frac{t^3}{3}\hat{k} + C_2$$
$$r(0) = \hat{\jmath} + \hat{k} = ((0)^2 + 3(0))\hat{\imath} - (0)\hat{\jmath} + \frac{(0)^3}{3}\hat{k} + C_2 = C_2 \implies C_2 = \hat{\jmath} + \hat{k}$$
$$\therefore\ r(t) = (t^2 + 3t)\hat{\imath} - t\hat{\jmath} + \frac{t^3}{3}\hat{k} + \hat{\jmath} + \hat{k} = (t^2 + 3t)\hat{\imath} + (1 - t)\hat{\jmath} + \left(\frac{t^3}{3} + 1\right)\hat{k}$$

23.) Establish initial variables and find $r(t)$:

$$v_0 = 200, \alpha = 60^\circ, g = 9.8$$
$$\therefore\ r(t) = (200\cos(60^\circ))t\hat{\imath} + \left[(200\sin(60^\circ))t - \frac{gt^2}{2}\right]\hat{\jmath} = 100t\hat{\imath} + \left[100\sqrt{3}t - \frac{9.8t^2}{2}\right]\hat{\jmath}$$

Calculating the x-component of $r(t)$ at $t = \dfrac{2v_0\sin\alpha}{g}$ we can find the range:

$$\text{range} = 100\left(\frac{2v_0\sin\alpha}{g}\right) = \frac{20000\sqrt{3}}{9.8} \approx 3534.80\text{m}$$

Calculating the y-component of $r(t)$ at t such that the y-component of $r'(t) = 0$, we can find the maximum height:

$$\left[100\sqrt{3} - 9.8t\right] = 0 \implies t = \frac{100\sqrt{3}}{9.8}$$
$$\therefore\ \text{max height} = 100\sqrt{3}t - \frac{9.8t^2}{2} \approx 1530.61\text{m}$$

To find t at impact, we solve for $t > 0$ when $y = 0$:

$$100\sqrt{3}t - \frac{9.8t^2}{2} = 100\sqrt{3}t - 4.9t^2 = 0 \implies t = \frac{-100\sqrt{3} - \sqrt{30000}}{-9.8} = 35.34$$
$$\therefore\ \text{impact speed} = \|r'(t)\| = \sqrt{100^2 + \left(100\sqrt{3} - 9.8t\right)^2} = 200\text{m/s}$$

27.) Solve for y_{max}:

$$\frac{d}{dt}\left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right] = v_0 \sin\alpha - gt = 0 \implies t = \frac{v_0 \sin\alpha}{g}$$

$$\therefore\ y_{max} = r_y\left(\frac{v_0 \sin\alpha}{g}\right) = (v_0 \sin\alpha)\left(\frac{v_0 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2$$

$$= \frac{v_0^2 \sin^2\alpha}{g} - \frac{g v_0^2 \sin^2\alpha}{2g^2} = \frac{2v_0^2 \sin^2\alpha - v_0^2 \sin^2\alpha}{2g} = \frac{v_0^2 \sin^2\alpha}{2g}$$

$$\implies 2g y_{max} = v_0^2 \sin^2\alpha \implies v_0 = \sqrt{\frac{2g y_{max}}{\sin^2\alpha}} = \frac{\sqrt{2g y_{max}}}{\sin\alpha}$$

Finding our initial values:

$$y_{max} = 1600\text{ft},\ g = 9.8\text{m/s} = 32.144\text{ft/s}$$

$$\therefore\ v_0 = \frac{\sqrt{2(32.144)(1600)}}{\sin(36^\circ)} \approx 545.64\text{ft/s}$$