Chapter 20

1.) The elements in $\mathbb{Q}\left(\sqrt[3]{5}\right)$ all take the form $a + b\left(\sqrt[3]{5}\right)$, where a and b are rational numbers.

3.) We can factor $x^3 - 1$ as $(x-1)(x^2+x+1)$. Since $x-1$ has a root in $\mathbb{Q}$, it suffices to construct a splitting field for x^2+x+1. We find the roots of x^2+x+1 to be $-(1/2) \pm i\left(\sqrt{3}/2\right)$, thus the extension field $\mathbb{Q}\left(i\sqrt{3}\right)$ contains all the roots of x^2+x+1, and thus all the roots of x^3-1, thus it is a splitting field of x^3-1 over $\mathbb{Q}$. ■

4.) We can factor x^4+1 as $(x^2+i)(x^2-i)$, thus the roots of x^4-1 are $\pm\sqrt{i}$ and $\pm\sqrt{-i}$, thus the extension field $\mathbb{Q}\left(\sqrt{i}, \sqrt{-i}\right)$ is a splitting field of x^4+1 over $\mathbb{Q}$. ■

11.) The elements in $\mathbb{Q}(\pi)$ all take the form $a + b\pi$, where a and b are rational numbers. Note that since π is transcendental, $\mathbb{Q}(\pi)$ is not a splitting field for any polynomial in $\mathbb{Q}[x]$.

35.) For a root of $f(x) = x^{p^n} - x$ to be multiple, then that root would also be a root of the derivative $f'(x) = p^n x^{p^n} - 1$. Since this polynomial exists in a field with characteristic $p \neq 0$, we know that $p^n = 0$, so $f'(x) = 0x^0 - 1 = -1$, thus $f'(x)$ has no roots, and thus no common roots with $f(x)$, thus the roots of $f(x)$ are distinct. ■