Definitions

1.) Given a set P, a partial order on P is a relation $\leq$ on P that satisfies the following conditions:

1. For all $p \in P$, $p \leq p$. (Reflexivity)
2. For all $p, q \in P$, $p \leq q \wedge q \leq p \implies p = q$. (Antisymmetry)
3. For all $p, q, r \in P$, $p \leq q \wedge q \leq r \implies p \leq r$. (Transitivity)

2.) Given a relation R on a set P, R is transitive if for all $p, q, r \in P$, $p \sim q \wedge q \sim r \implies p \sim r$.

3.) The union of a collection of sets $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is defined as the set T where

$$T = \{s : \exists \alpha \in \mathcal{A} \text{ where } s \in S_\alpha\}$$

4.) The intersection of a collection of sets $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is defined as the set T where

$$T = \{s : s \in S_\alpha \text{ for all } \alpha \in \mathcal{A}\}$$

Proofs

a.) Since $\mathcal{P}([2]) = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}, \varnothing\}$, we can see that the only open subsets of $[2]$ are $\{2\}, \{1,2\}, \{0,1,2\}$, and $\varnothing$.

b.) Consider the elements of $\mathcal{P}(\{a,b\})$. Given an open subset $S \subset \mathcal{P}(\{a,b\})$, consider the implications of each element existing in S. Since $\varnothing \subset \{a\}, \varnothing \subset \{b\}$, and $\varnothing \subset \{a,b\}$, we can see that $\varnothing \in S \implies S = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$. Next, since $\{a\} \subset \{a,b\}$ and $\{b\} \subset \{a,b\}$, we can see that $\{a\} \in S \vee \{b\} \in S \implies \{a,b\} \in S$. Finally, $\{a,b\} \in S$ does not necessarily implicate the existance of any other element in S. From this, we find every open subset of $\mathcal{P}(\{a,b\})$ to be $\{\{a,b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\}, \{\{a\}, \{b\}, \{a,b\}\}$, $\{\varnothing, \{a\}, \{b\}, \{a,b\}\}$, and $\varnothing$.

c.) Let S be defined as follows:

$$S = \bigcup_{\alpha \in \mathcal{A}} U_\alpha = \{x : \exists \alpha \in \mathcal{A} \text{ where } x \in U_\alpha\}$$

Let $p, q \in P$ where $p \leq q$ and $p \in S$. We can see that $p \in S \implies \exists \alpha \in \mathcal{A}$ where $p \in U_\alpha$. Since U_α is an open subset of P, and since $p \leq q$, we know that $q \in U_\alpha$, thus $q \in S$, thus $p \in S \wedge p \leq q \implies q \in S$, thus by definition, S is an open subset of P. $\blacksquare$

d.) Let S be defined as follows:

$$S = \bigcap_{\alpha \in \mathcal{A}} U_\alpha = \{x : x \in U_\alpha \text{ for all } \alpha \in \mathcal{A}\}$$

Let $p, q \in P$ where $p \leq q$ and $p \in S$. We can see that $p \in S \implies p \in U_\alpha$ for all $\alpha \in \mathcal{A}$. Since for all α, U_α is an open subset of P, and since $p \leq q$, we know that $q \in U_\alpha$ for all α, thus $q \in S$, thus $p \in S \wedge p \leq q \implies q \in S$, thus by definition, S is an open subset of P. ■

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