Definitions

1.) Given a set X, a *topology* on X is a collection of sets $\mathcal{T} \subset \mathcal{P}(X)$ such that

i.) $\varnothing, X \in \mathcal{T}$

ii.) Given a collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of sets in $\mathcal{T}$, $\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in T$

iii.) Given a finite collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of sets in $\mathcal{T}$, $\bigcap_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$

2.) Given a set X and a topology $\mathcal{T}$ on X, $(X, \mathcal{T})$ is a *topological space*.

3.) Given a topological space $(X, \mathcal{T})$, an *open subset* of X is a subset $U \subset X$ where $U \in \mathcal{T}$.

4.) Given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ the function $f : X \to Y$ is *continuous* if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

5.) Given a set X, the collection $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of subsets of X is a *cover* of X if $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = X$

6.) Given a cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X, $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an *open cover* if U_α is open in X for all $\alpha \in \mathcal{A}$.

7.) Given a cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X, a *finite subcover* of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a cover $\{U_\beta\}_{\beta\in\mathcal{B}}$ of X where $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ is finite.

8.) Given a topological space $(X, \mathcal{T})$, it is *compact* if for all open covers $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X, there exists a finite subcover $\{U_\beta\}_{\beta\in\mathcal{B}}$ of $\{U_\alpha\}_{\alpha\in\mathcal{A}}$.

Proof

Assume f is continuous and let $V \subset Y$ be open. Fix $y \in V$, thus there exists $\varepsilon > 0$ where $\mathrm{Ball}(y, \varepsilon) \subset V$. Since $\mathrm{Ball}(y, \varepsilon)$ is open, $f^{-1}(\mathrm{Ball}(y, \varepsilon))$ is open, thus there exists $\delta > 0$ where $\mathrm{Ball}(x, \delta) \subset f^{-1}(\mathrm{Ball}(y, \varepsilon))$, thus for all $x' \in \mathrm{Ball}(x, \delta)$ we know that $x' \in f^{-1}(\mathrm{Ball}(y, \varepsilon))$, and thus $f(x') \in \mathrm{Ball}(y, \varepsilon)$.

Assuming (b), fix $\varepsilon > 0$, thus for some $\delta > 0$, $x' \in \mathrm{Ball}(x, \delta) \implies f(x') \in \mathrm{Ball}(y, \varepsilon)$ for arbitrary $x \in X$ and $y \in Y$. We can see that $f(x') \in \mathrm{Ball}(y, \varepsilon) \implies x' \in f^{-1}(\mathrm{Ball}(y, \varepsilon))$, thus $x' \in \mathrm{Ball}(x, \delta) \implies x' \in f^{-1}(\mathrm{Ball}(y, \varepsilon))$, thus $\mathrm{Ball}(x, \delta) \subset f^{-1}(\mathrm{Ball}(y, \varepsilon))$. Let $V \subset Y$ be open where $\mathrm{Ball}(y, \varepsilon) \subset V$, thus $f^{-1}(\mathrm{Ball}(y, \varepsilon) \subset f^{-1}(V))$, thus $\mathrm{Ball}(x, \delta) \subset f^{-1}(V)$. Since x is arbitrary, we know that $\mathrm{Ball}(x, \delta) \subset f^{-1}(V)$ for all $x \in f^{-1}(V)$, thus $f^{-1}(V)$ is open.

It follows that (a) and (b) are equivalent. $\blacksquare$