4.1 Direct Products

7) -

4.2 The Fundamental Theorem of Finitely Generated Abelian Groups

1)

$\lvert G\rvert$	**# of abelian groups**
100	4
576	22
1155	1
42875	10
2704	10

2)

$\lvert G\rvert$	**invariant factors**
270	$2\cdot 3^3\cdot 5; 2\cdot 3^2\cdot 5, 3; 2\cdot 3\cdot 5, 3, 3$
9801	$3^4\cdot 11^2; 3^4\cdot 11, 11; 3^3\cdot 11^2, 3; 3^3\cdot 11, 3\cdot 11;$ $3^2\cdot 11^2, 3^2; 3^2\cdot 11^2, 3, 3; 3^2\cdot 11, 3^2\cdot 11; 3^2\cdot 11, 3\cdot 11, 3;$ $3\cdot 11^2, 3, 3, 3; 3\cdot 11, 3\cdot 11, 3, 3$
320	$2^6\cdot 5; 2^5\cdot 5, 2; 2^4\cdot 5, 2^2; 2^4\cdot 5, 2, 2;$ $2^3\cdot 5, 2^3; 2^3\cdot 5, 2^2, 2; 2^3\cdot 5, 2, 2, 2; 2^2\cdot 5, 2^2, 2^2;$ $2^2\cdot 5, 2^2, 2, 2; 2^2\cdot 5, 2, 2, 2, 2; 2\cdot 5, 2, 2, 2, 2, 2$

3) (Given in same order as the invariant factors).

$\lvert G\rvert$	**elementary divisors**
270	$(2, 5, 27); (2, 3, 5, 9); (2, 3, 5)$
9801	$(81, 121); (11, 11, 81); (3, 27, 121); (3, 11, 11, 27);$ $(9, 9, 121); (3, 3, 9, 121); (9, 9, 11, 11); (3, 3, 9, 11, 11);$ $(3, 3, 3, 3, 121); (3, 3, 3, 3, 11, 11)$
320	$(5, 64); (2, 5, 32); (4, 5, 16); (2, 2, 5, 16)$ $(5, 8, 8); (2, 4, 5, 8); (2, 2, 2, 5, 8); (4, 4, 4, 5);$ $(2, 2, 4, 4, 5); (2, 2, 2, 2, 4, 5); (2, 2, 2, 2, 2, 2, 5)$

4a) The only pair of isomorphic groups is $\mathbb{Z}_9 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_9$.

4b) The only pair of isomorphic groups is $\{2^2, 2 \cdot 3^2\}$ and $\{2^2 \cdot 3^2, 2\}$.

4.4 Recognizing Direct Products

5) If $n \geq 5$, then the commutator subgroup S_n' of S_n is A_n.

Proof: Let $(a\ b\ c)$ be any 3-cycle in S_n, then $(a\ b\ c) = (a\ c)(c\ b)(a\ c)(c\ b) = (a\ c)^{-1}(c\ b)^{-1}(a\ c)(c\ b)$, and thus is a commutator in S_n. Since A_n is generated by the 3-cycles in S_n, we have that $A_n \subseteq S_n'$. Conversely, because $[S_n : A_n] = 2$, we have that S_n/A_n is cyclic and hence abelian, showing that $S_n' \subseteq A_n$ and thus $S_n' = A_n$. ∎

7) Fix a prime p and a non-abelian group P with order p^3, then $P' = Z(P)$.

Proof: Since P is a p-group, we have that $Z(P) \neq \{e\}$, and since P is non-abelian, $|Z(P)| \neq p^3$. We also have that $|Z(P)| \neq p^2$, else $|P/Z(P)|$ would have order p and thus be cyclic, additionally implying that P is abelian, a contradiction. Thus, it must be the case that $|Z(P)| = p$, which implies $|P/Z(P)| = p^2$, showing that $P/Z(P)$ is abelian. Since $P/\{e\} \cong P$ is non-abelian, we have that $Z(P)$ is the smallest normal subgroup of P whose quotient is abelian, thus proving that $Z(P) = P'$. ∎

10) If G is a finite abelian group, then $G \cong S_1 \times \cdots \times S_n$, where each S_i is some Sylow subgroup.

Proof: Since G covered by the S_i, we have that $G = S_1 S_2 \cdots S_n$. Additionally, G is abelian, so each S_i is normal, and since $S_i \cap S_j = \{e\}$ for $i \neq j$, we have that $S_1 S_2 \cdots S_n \cong S_1 \times S_2 \times \cdots \times S_n$. ∎