Definitions

1.) Given sets X and Y, the *direct product* or *cartesian product* $X \times Y$ is defined as

$$X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$$

2.) Given topological spaces X and Y, the *product topology* $\mathcal{T}_p$ on $X \times Y$ is defined as

$$\mathcal{T}_p = \left\{ U \subseteq X \times Y : U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha \times V_\alpha \right\}$$

given $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ are collections of open sets in X and Y respectively.

3.) Given topological spaces X and Y, the product topology on $X \times Y$ satisfies the following:

i.) The projection maps $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous.

ii.) Given a topological space W and a pair of continuous functions $g_X : W \to X$ and $g_Y : W \to Y$, there exists a unique continuous function $g : W \to X \times Y$ where $g_X = p_X \circ g$ and $g_Y = p_Y \circ g$.

4.) Given a collection of sets $\{X_\alpha\}_{\alpha \in \mathcal{A}}$, the *direct product* $\prod\limits_{\alpha \in \mathcal{A}} X_\alpha$ is defined as

$$\prod_{\alpha \in \mathcal{A}} X_\alpha = \left\{ \{x_\alpha\}_{\alpha \in \mathcal{A}} : x_\alpha \in X_\alpha \right\}$$

5.) Given a collection of topological spaces $\{X_\alpha\}_{\alpha \in \mathcal{A}}$, the product topology on $\prod\limits_{\alpha \in \mathcal{A}} X_\alpha$ satisfies the following:

i.) For all $\alpha \in \mathcal{A}$, the projection map $p_\alpha : \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\alpha$ is continuous.

ii.) Given a topological space W and a collection of continuous functions $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ where $g_\alpha : W \to X_\alpha$, there exists a unique continuous function $g : W \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ where $g_\alpha = p_\alpha \circ g$.

6.) A topological space X is *Hausdorff* if and only if given distinct $x, x' \in X$, there exist open sets U, V in X where $x \in U$, $x' \in V$, and $U \cap V = \varnothing$.

Proofs

a.) A topological space X is Hausdorff if and only if its diagonal is closed under the product topology.

b.) Because this conjecture holds for every example we covered in class, I am choosing to pursue a proof of it.

c.) Let X be a topological space, and suppose it is Hausdorff. Let $D^{\complement}$ denote the complement of the diagonal of X, thus $D^{\complement} = \{(x, x') \in X \times X : x \neq x'\}$. Since X is Hausdorff, we know that for all $(x, x') \in D^{\complement}$, there exist open sets U, V in X where $x \in U$, $x' \in V$, and $U \cap V = \varnothing$, thus $(x, x') \in U \times V$. In addition, since $U \cap V = \varnothing$, $(x, x') \in U \times V \implies x \neq x'$. For each point $\alpha \in D^{\complement}$, assign open sets U_α and V_α that satisfy the Hausdorff property, thus $\alpha \in U_\alpha \times V_\alpha$, thus $D^{\complement} \subseteq \bigcup U_\alpha \times V_\alpha$. Finally, if $\alpha \in U_\alpha \times V_\alpha$, then $\alpha = (x, x')$ where $x \neq x'$, thus $\alpha \in D^{\complement}$, thus $\bigcup U_\alpha \times V_\alpha \subseteq D^{\complement}$, thus $D^{\complement} = \bigcup U_\alpha \times V_\alpha$, thus $D^{\complement}$ is open, and thus the diagonal of X is closed.

Next, suppose the diagonal of X is closed, then $D^{\complement}$ is open, thus there exist collections of open sets $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ where $D^{\complement} = \bigcup U_\alpha \times V_\alpha$. It is clear that for all $\alpha \in \mathcal{A}$, $U_\alpha \times V_\alpha \subseteq D^{\complement}$, thus $(x, x') \in U_\alpha \times V_\alpha \implies x \neq x'$, thus $U_\alpha \cap V_\alpha = \varnothing$. Finally, since $(x, x') \in U_\alpha \times V_\alpha$, we know that $x \in U_\alpha$ and $x' \in V_\alpha$, thus for all distinct $x, x' \in X$, the Hausdorff property is satisfied, thus X is Hausdorff.

Thus we can conclude that X is Hausdorff if and only if the diagonal of X is closed.

$\blacksquare$

d.) This property stuck out to me when we were looking at examples, so not much reworking of ideas was needed before settling on trying to prove it.