Property 5.ii) If f and g are finite valued measurable functions, then $f+g$ is measurable.

Proof: We will show that

$$\{f+g>a\}=M$$

where

$$M=\bigcup_{r\in\mathbb{Q}}\{f>a-r\}\cap\{g>r\}$$

and conclude that $\{f+g>a\}$ is a countable union of measurable sets, and thus measurable. It is clear that $M\subseteq\{f+g>a\}$ since if $x\in M$, then $f(x)>a-r$ and $g(x)>r$ for some $r\in\mathbb{Q}$, thus implying that $f(x)+g(x)>a$. Conversely, suppose $x\in\{f+g>a\}$, then $f(x)+g(x)>a$ and $g(x)>a-f(x)$. We can choose $r\in\mathbb{Q}$ where $a-f(x)<r<g(x)$, thus obtaining $g(x)>r$ and $f(x)-a>-r \implies f(x)>a-r$, which shows that $x\in M$, thus $\{f+g>a\}=M$ and we are finished. ■

18) Every measurable function is the limit a.e. of a sequence of continuous functions.

Proof: Without loss of generality, we let f be non-negative, since $f=f^+-f^-$. We must also assume that f is finite a.e. Let $Q_n\subseteq\mathbb{R}^d$ denote the closed cube with side length n, and define the function $f_n:\mathbb{R}^d\to\mathbb{R}$ as follows:

$$f_n(x)=\begin{cases} f(x) & x\in Q_n \text{ and } f(x)\le n\\ n & x\in Q_n \text{ and } f(x)>n\\ 0 & x\notin Q_n\end{cases}$$

We previously established that $f_n\to f$ everywhere f is finite. Since $g(x)=\min\{n,f(x)\}$ is measurable for all n [1], and since $f_n(x)=\min\{n,f(x)\}\,\chi_{[0,1]}$, we have that f_n is measurable. Fix $\varepsilon>0$, then since f_n is finite valued for all n, we can use **theorem 4.5** to obtain closed sets $F_n\subseteq Q_n$ with $m(Q_n\setminus F_n)<\varepsilon$ and where the restriction $f_n|_{F_n}$ is continuous. By the **Tietze extension theorem** [2], there exists an extension g_n of $f_n|_{F_n}$ that is continuous on $\mathbb{R}^d$, and since $g_n(x)=f_n(x)$ if $x\in F_n$, we have that $g_n\to f$ everywhere it is finite. Finally, as $\varepsilon\to 0$, $m(Q_n\setminus F_n)\to 0$, thus the set of points where f_n doesn't converge to f has measure 0. ■

22) There exists no everywhere continuous function $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=\chi_{[0,1]}(x)$ a.e.

Proof: Fix a function $f:\mathbb{R}\to\mathbb{R}$ with E denoting the set of points $x\in\mathbb{R}$ where $f(x)\neq\chi_{[0,1]}(x)$, and assume $m(E)=0$. To establish a contradiction, suppose f is continuous on $\mathbb{R}$.

First we consider the case where $0\notin E$, then we have that $f(0)=1$. Fix $0<\varepsilon<1$, then since f is continuous at $x=0$ we can choose $\delta>0$ where $x\in(-\delta,\delta)\implies f(x)\in(1-\varepsilon,1+\varepsilon)$. If $-\delta<x<0$, then we have that $f(x)>1-\varepsilon>0=\chi_{[0,1]}(x)$, thus if $x\in(-\delta,0)$ we have $f(x)\neq\chi_{[0,1]}(x)$. This implies that $m(E)\ge m((-\delta,0))>0$ which is a contradiction.

Next, consider the case where $0\in E$, then $f(0)\neq 1$. Fix $0<\varepsilon<|1-f(0)|$ and choose $\delta>0$ where $x\in(-\delta,\delta)\implies f(x)\in(f(0)-\varepsilon,f(0)+\varepsilon)$. Let $0<x<\delta$. If $f(0)>1$, then $f(x)>f(0)-\varepsilon>1=\chi_{[0,1]}(x)$, and if $f(0)<1$, then $f(x)<f(0)+\varepsilon<1=\chi_{[0,1]}(x)$. Thus, if $x\in(0,\delta)$, then $f(x)\neq\chi_{[0,1]}(x)$, which again implies that $m(E)\ge m((0,\delta))>0$, thus obtaining a contradiction.

Since each case leads to a contradiction, we know that f cannot be continuous everywhere, and we are finished. ■

References

[1] https://proofwiki.org/wiki/Pointwise_Minimum_of_Measurable_Functions_is_Measurable. Accessed on 10/5/24.

[2] https://proofwiki.org/wiki/Tietze_Extension_Theorem. Accessed on 10/6/24.