Lemma 1) Let $f : \mathbb{R}^d \to [0, \infty]$ be a non-negative measurable function and E a subset of $\mathbb{R}^d$ with measure 0, then

$$\int_E f(x)\, dx = 0.$$

Proof: Since f is non-negative and measurable, there exists an increasing sequence of simple functions $\{\phi_k\}_{k\in\mathbb{N}}$ where $\phi_k \to f$ as $k \to \infty$. By definition we can write each ϕ_k as the finite sum $\sum_{i=1}^{n} a_i \chi_{E_i}$ for constants a_i and measurable sets E_i, thus

$$\int_E \phi_k(x)\, dx = \int \left(\sum_{i=1}^{n} a_i \chi_{E_i}(x)\right) \chi_E(x)\, dx = m(E) \left(\sum_{i=1}^{n} a_i \chi_{E_i}(x)\right) = 0$$

for all $k \in \mathbb{N}$, showing that $\int_E \phi_k \to 0$ as $k \to \infty$. By the monotone convergence theorem, we have that $\int_E \phi_k \to \int_E f$, thus $\int_E f = 0$ and we are finished. ■

1, 2) For a fixed number a, define the functions f_a and F_a on $\mathbb{R}^d$ as follows:

$$f_a = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad F_a = \frac{1}{1 + |x|^a},$$

then f_a is integrable if and only if $0 < a < d$, and F_a is integrable if and only if $a > d$.

Proof: Using additivity and **Lemma 1**, we have that

$$\int f_a(x)\, dx = \int_{\mathbb{R}^d \setminus \{0\}} f_a(x)\, dx + \int_{\{0\}} f_a(x)\, dx = \int_{\mathbb{R}^d \setminus \{0\}} f_a(x)\, dx + 0 = \int_{\mathbb{R}^d \setminus \{0\}} f_a(x)\, dx,$$

and so it suffices to show that f_a is integrable over $\mathbb{R}^d \setminus \{0\}$. The same argument shows that this is also true for F_a.

First, we consider f_a. For each $k \in \mathbb{N}_0$, define the disjoint sets E_k as

$$E_k = \left\{x \in \mathbb{R}^d : 2^k \leq f_a(x) < 2^{k+1}\right\},$$

then solving for $|x|$ we see that

$$E_k = \left\{x \in \mathbb{R}^d : 2^{-(k+1)/a} < |x| \leq 2^{-k/a}\right\}.$$

Additionally, if we define the set $E = \left\{x \in \mathbb{R}^d : 2^{-1/a} < |x| < 1\right\}$, we have that $E_k = 2^{-k/a}E$, and thus by relative-dilation invariance we have $m(E_k) = (2^{-k/a})^d m(E)$.

Since $f_a(x) > 0$ implies that $f_a(x) \geq 1$ for all x, we see that

$$\int f_a(x)\,dx = \sum_{k\in\mathbb{N}_0} \int_{E_k} f_a(x)\,dx < \sum_{k\in\mathbb{N}_0} \int_{E_k} 2^{k+1}\,dx = \sum_{k\in\mathbb{N}_0} 2^{k+1} m(E_k)$$

$$= 2m(E) \sum_{k\in\mathbb{N}_0} 2^{k(1-d/a)},$$

but this series only converges when $1 - d/a$ is negative, i.e. when $0 < a < d$. Thus, our assumption on the value of a implies that f_a is integrable.

Conversely, assume f_a is integrable. We just established that

$$\sum_{k\in\mathbb{N}_0} 2^{k+1} m(E_k) \tag{1}$$

only converges when $0 < a < d$, but

$$\sum_{k\in\mathbb{N}_0} 2^{k+1} m(E_k) = 2 \sum_{k\in\mathbb{N}_0} 2^{k} m(E_k) \leq 2 \sum_{k\in\mathbb{N}_0} \int_{E_k} f_a(x)\,dx = 2 \int f_a(x)\,dx,$$

thus (1) converges, which forces $0 < a < d$ and proves the case of f_a.

We now turn our attention to F_a. Redefine E_k and E as

$$E_k = \left\{x \in \mathbb{R}^d : 2^{-(k+1)} \leq F_a(x) < 2^{-k}\right\} \quad \text{and} \quad E = \left\{x \in \mathbb{R}^d : |x| \leq 1\right\}.$$

Since $0 < F_a(x) < 1$ for all $|x| > 0$, we have that $E_k = \varnothing$ for all $k > 0$. Additionally, solving for $|x|$ we see that

$$E_k = \left\{x \in \mathbb{R}^d : \sqrt[a]{2^k - 1} < |x| \leq \sqrt[a]{2^{k+1} - 1}\right\},$$

thus $m(E_k) \leq (\sqrt[a]{2^{k+1} - 1})^d m(E) < (\sqrt[a]{2^{k+1}})^d m(E)$. This shows that

$$\frac{1}{2} \int F_a(x)\,dx = \frac{1}{2} \sum_{k\in\mathbb{N}_0} \int_{E_k} F_a(x)\,dx < \frac{1}{2} \sum_{k\in\mathbb{N}_0} \frac{m(E_k)}{2^k} < m(E) \sum_{k\in\mathbb{N}_0} \frac{2^{d(k+1)/a}}{2^{k+1}}$$

$$= m(E) \sum_{k\in\mathbb{N}_0} 2^{(k+1)(d/a-1)},$$

which converges only when $d/a - 1$ is negative, and thus when $a > d$. Thus we have shown that $a > d$ implies that F_a is integrable.

3) Since $\eta_1(x) = \min\{f(x), \eta(x)\}$, we have for fixed $x \in \mathbb{R}^d$ that $\eta_1(x) = f(x)$ or $\eta_1(x) = \eta(x)$. By assumption g is non-negative, and so in the case where $\eta_1(x) = \eta(x)$, we have that $\eta_2(x) = \eta(x) - \eta_1(x) = \eta(x) - \eta(x) = 0 \leq g(x)$, thus showing that $\eta_2(x) \leq g(x)$. Otherwise, $\eta_1(x) = f(x)$. We have $\eta(x) \leq f(x) + g(x)$ by definition, thus $\eta(x) - f(x) \leq g(x)$, which implies $\eta_2(x) = \eta(x) - \eta_1(x) = \eta(x) - f(x) \leq g(x)$ and $\eta_2(x) \leq g(x)$. ∎

4) Let $\{E_k\}_{k\in\mathbb{N}}$ be a collection of measurable sets in $\mathbb{R}^d$ where $\sum_{k\in\mathbb{N}} m(E_k) < \infty$ and define the set E as

$$E = \left\{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\right\},$$

then $m(E) = 0$.

Proof: If a point x belongs to infinitely many E_k, then $\chi_{E_k}(x) = 1$ for infinitely many k, and thus $\sum_{k\in\mathbb{N}} \chi_{E_k}(x) = \infty$. Conversely, if $\sum_{k\in\mathbb{N}} \chi_{E_k}(x) = \infty$, then $\chi_{E_k}(x) = 1$ for infinitely many k, thus x is contained in infinitely many E_k. By assumption, $\sum_{k\in\mathbb{N}} m(E_k) < \infty$, thus we have

$$\sum_{k\in\mathbb{N}} m(E_k) = \sum_{k\in\mathbb{N}} \int \chi_{E_k} < \infty,$$

and thus by **Corollary 1.10** we have shown that $\sum_{k\in\mathbb{N}} \chi_{E_k} < \infty$ for almost all x, proving $m(E) = 0$. ∎

5) Fix $\varepsilon > 0$, then the function $f : \mathbb{R}^d \to \mathbb{R}$, defined as

$$f(x) = \begin{cases} \frac{1}{|x|^{d+1}} & x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is integrable over all $x \in \mathbb{R}^d$ where $|x| \geq \varepsilon$. We also have that

$$\int_{|x|\geq\varepsilon} f(x)\, dx \leq \frac{C}{\varepsilon}$$

for some constant C.

Proof: For integers $k \geq 0$, define the collection of disjoint sets

$$A_k = \left\{x \in \mathbb{R}^d : 2^k\varepsilon \leq |x| < 2^{k+1}\varepsilon\right\},$$

and let $g : \mathbb{R}^d \to \mathbb{R}$ be a function defined as

$$g(x) = \sum_{k \in \mathbb{N}_0} \frac{1}{(2^k \varepsilon)^{d+1}} \chi_{A_k}(x).$$

Fix $x \in \mathbb{R}^d$ where $|x| \geq \varepsilon$, then for some $k \in \mathbb{N}_0$ we have that $x \in A_k$, thus $|x| \geq 2^k \varepsilon$. Since the A_k are disjoint, we see that

$$g(x) = \frac{1}{(2^k \varepsilon)^{d+1}} \chi_{A_k}(x) = \frac{1}{(2^k \varepsilon)^{d+1}} \geq \frac{1}{|x|^{d+1}} = f(x),$$

showing that $f(x) \leq g(x)$. By monotonicity we have $\int_{|x| \geq \varepsilon} f \leq \int_{|x| \geq \varepsilon} g$, and so it suffices to show that $\int_{|x| \geq \varepsilon} g < \infty$.

Denote $A = \left\{ x \in \mathbb{R}^d : 1 \leq |x| < 2 \right\}$, then we have that $A_k = 2^k \varepsilon A$, and thus by relative-dilation invariance we have $m(A_k) = (2^k \varepsilon)^d m(A)$. Since g is a sum of simple functions, we have by **Corollary 10** that

$$\int_{|x| \geq \varepsilon} g(x)\, dx = \int_{|x| \geq \varepsilon} \sum_{k \in \mathbb{N}_0} \frac{1}{(2^k \varepsilon)^{d+1}} \chi_{A_k}(x)\, dx = \sum_{k \in \mathbb{N}_0} \int_{|x| \geq \varepsilon} \frac{1}{(2^k \varepsilon)^{d+1}} \chi_{A_k}(x)\, dx$$

$$= \sum_{k \in \mathbb{N}_0} \frac{m(A_k)}{(2^k \varepsilon)^{d+1}} = m(A) \sum_{k \in \mathbb{N}_0} \frac{(2^k \varepsilon)^d}{(2^k \varepsilon)^{d+1}} = m(A) \sum_{k \in \mathbb{N}_0} \frac{1}{2^k \varepsilon} = \frac{2m(A)}{\varepsilon},$$

and since $m(A)$ is finite, we have shown that $\int_{|x| \geq \varepsilon} g = C/\varepsilon < \infty$ where $C = 2m(A)$, thus completing the proof. ■