Problems

5.1.8. Let $\varepsilon > 0$ be given and consider the following:

$$\left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon \implies \left|\sqrt{x} - \sqrt{x_0}\right| \left|\sqrt{x} + \sqrt{x_0}\right| = |x - x_0| < \varepsilon \left|\sqrt{x} + \sqrt{x_0}\right|$$

Since $\sqrt{x}$ is only defined for $x \geq 0$, we know that $x, x_0 \geq 0$. Suppose $|x - x_0| < 1$, then

$$|x - x_0| < 1 \implies x - x_0 < 1 \implies x < x_0 + 1 \implies \sqrt{x} < \sqrt{x_0 + 1}$$

$$\implies \sqrt{x} + \sqrt{x_0} = \left|\sqrt{x} + \sqrt{x_0}\right| < \sqrt{x_0 + 1} + \sqrt{x_0}$$

Thus we can define $\delta = \min\left\{1, \varepsilon\left(\sqrt{x_0 + 1} + \sqrt{x_0}\right)\right\}$. From this we can conclude the following:

$$|x - x_0| < \delta \implies \left|\sqrt{x} - \sqrt{x_0}\right| \left|\sqrt{x} + \sqrt{x_0}\right| < \varepsilon\left(\sqrt{x_0 + 1} + \sqrt{x_0}\right)$$

$$\implies \left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon\left(\frac{\sqrt{x_0 + 1} + \sqrt{x_0}}{\left|\sqrt{x} + \sqrt{x_0}\right|}\right) < \varepsilon\left(\frac{\sqrt{x_0 + 1} + \sqrt{x_0}}{\sqrt{x_0 + 1} + \sqrt{x_0}}\right) = \varepsilon$$

Thus $|x - x_0| < \delta \implies \left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon$, thus $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$. Q.E.D.

5.1.16 awd