Chapter 7

1.) Since $H = \{n \in \mathbb{Z} : 3 \mid n\}$, equipped with addition, we can find the left cosets of H in $\mathbb{Z}$. Consider $n \in \mathbb{Z}$:

$$n \equiv 0 \pmod 3 \implies n + H = \{0+0, 0 \pm 3, 0 \pm 6, \cdots\} = \{0, \pm 3, \pm 6, \cdots\} = H$$

$$n \equiv 1 \pmod 3 \implies n + H = \{1+0, 1 \pm 3, 1 \pm 6, \cdots\} = \{\cdots, -5, -2, 1, 4, 7, \cdots\}$$

$$n \equiv 2 \pmod 3 \implies n + H = \{2+0, 2 \pm 3, 2 \pm 6, \cdots\} = \{\cdots, -4, -1, 2, 5, 8, \cdots\}$$

Thus the left cosets of H in $\mathbb{Z}$ are H, $1+H$, and $2+H$.

5.) Since $U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$, the left cosets of $H = \{1, 11\}$ in $U(30)$ are given as follows:

$$1H = \{1, 11\} = H$$

$$7H = \{7, 17\}$$

$$11H = \{11, 1\} = H$$

$$13H = \{13, 23\}$$

$$17H = \{17, 7\} = 7H$$

$$19H = \{19, 29\}$$

$$23H = \{23, 13\} = 13H$$

$$29H = \{29, 19\} = 19H$$

Thus the left cosets of H in $U(30)$ are H, $7H$, $13H$, and $19H$.

6.) Since $|a| = 15$, $H = \langle a^5 \rangle = \{a^5, a^{10}, e\}$. From this, we can find the left cosets of H in $\langle a \rangle$:

$$eH = \{a^5, a^{10}, e\} = H$$

$$aH = \{a^6, a^{11}, a\}$$

$$a^2H = \{a^7, a^{12}, a^2\}$$

$$a^3H = \{a^8, a^{13}, a^3\}$$

$$a^4H = \{a^9, a^{14}, a^4\}$$

$$a^5H = \{a^{10}, e, a^5\} = H$$

$$\vdots$$

Thus the left cosets of H in $\langle a \rangle$ are H, aH, a^2H, a^3H, and a^4H.

15.) By Lagrange's theorem, if H is a subgroup of G, then $|H| \mid 60$, thus the possible orders of H are the divisors of 60:

$$\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

20.) Consider $U(n)$ for $n > 2$. We can see that $(n-1)^2 \equiv n^2 - 2n + 1 \equiv 0 - 0 + 1 \equiv 1 \pmod n$, thus $|n-1| = 2$, thus by Lagrange's theorem, 2 divides $|U(n)|$, thus $|U(n)|$ is even. ■