Chapter 14

1.) Let $a \in R$. It is clear that $\langle a \rangle$ is non-empty, as $a = 1a$, thus $a \in \langle a \rangle$. We also know that $ra \in R$ for all $r \in R$, so $\langle a \rangle \subseteq R$. Finally, since R is commutative, we know that $ra = ar$, thus $ar \in \langle a \rangle$, so $\langle a \rangle$ is an ideal. ■

7.) Let $a \in R$. We know that $aR \subseteq R$, and that it is non-empty. In addition, since R is commutative, we have $ar = ra$, so we know that $ra \in aR$, thus aR is an ideal. ■

Given $R = \{2n : n \in \mathbb{Z}\}$, we know that $4R = \{4r : r \in R\} = \{8n : n \in \mathbb{Z}\}$.

28.) Let $a \in R$ where $a \neq 0$. Since R is an ideal, we know that for all $k \in R$, there exists $b \in R$ where $ab = k$. Since $1 \in R$, we know that there exists b where $ab = 1$, thus R has inverses, and is thus a field. ■

Chapter 15

5.) We have $\phi(2)+\phi(3) = 30+45 = 75 \equiv 5 \pmod{10}$, but $\phi(2+3) = \phi(5) = \phi(0) = 0 \pmod{10}$, so ϕ does not preserve addition. ■

20.) Let $\phi : R_1 \to R_2$ be a ring homomorphism, $a \in R_1$ where a is idempotent, and $b \in R_2$ where $\phi(a) = b$. Since $a^2 = a$, we have that $b = \phi(a) = \phi(a * a) = \phi(a)\phi(a) = b * b = b^2$, thus $b = b^2$, thus b is idempotent in R_2, thus ring homomorphisms carry idempotent elements to other idempotent elements. ■