For a function f, denote T_f as the total variation function on f. Additionally, denote P_f and N_f as the positive and negative variation functions on f.

Lemma 1) Let $f : [a, b] \to \mathbb{R}$ be differentiable, then

$$T_f(a, b) = \int_a^b |f'(x)|\, dx.$$

Proof: Since f is differentiable on $[a, b]$, it is also continuous on $[a.b]$, and thus the fundamental theorem of calculus holds for f. Thus, for a fixed partition $t_0, \ldots, t_n$ of $[a, b]$, we have that

$$\sum_{1\le k\le n} |f(t_k) - f(t_{k-1})| = \sum_{1\le k\le n} \left| \int_{t_{k-1}}^{t_k} f'(x)\, dx \right| \le \sum_{1\le k\le n} \int_{t_{k-1}}^{t_k} |f'(x)|\, dx$$
$$= \int_a^b |f'(x)|\, dx.$$

Using this we prove that

$$T_f(a, b) = \sup \sum_{1\le k\le n} |f(t_k) - f(t_{k-1})| \le \sup \int_a^b |f'(x)|\, dx = \int_a^b |f'(x)|\, dx,$$

where the supremum is taken over all partitions of $[a, b]$.

Next, note that f has bounded variation on $[a, b]$ since it is differentiable. Thus, f is the difference of two bounded increasing functions, say $f = g - h$. **Theorem 3.3** allows us to choose $g(x)$ and $h(x)$ as $P_f(a, x) + f(a)$ and $N_f(a, x)$, respectively. We know that f' exists for all $x \in [a, b]$, hence $f' = g' - h'$ does too, implying that g and h are differentiable, and thus continuous. Additionally, g and h are increasing functions, thus g' and h' are non-negative. This allows us to show that

$$\int_a^b |f'(x)|\, dx = \int_a^b |g'(x) - h'(x)|\, dx \le \int_a^b |g'(x)|\, dx + \int_a^b |-h'(x)|\, dx$$
$$= \int_a^b g'(x)\, dx + \int_a^b h'(x)\, dx = g(b) - g(a) + h(b) - h(a)$$
$$= P_f(a, b) + f(a) - P_f(a, a) - f(a) + N_f(a, b) - N_f(a, a) = P_f(a, b) + N_f(a, b) = T_f(a, b),$$

since the variation over an interval of length 0 is always 0. Thus, we have shown that the integral is bounded by the total

11) Fix $a, b > 0$ and define the function f on $[0,1]$ as follows:

$$f(x) = \begin{cases} x^a \sin\left(x^{-b}\right) & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We have that f is of bounded variation on $[0,1]$ if and only if $a > b$. Additionally, if $a = b$, we have that for all $0 < \alpha < 1$ there exists a function g that satisfies the Lipschitz condition for α, i.e. there exists a constant A where

$$|g(x) - g(y)| \leq A\,|x - y|^\alpha$$

for all $x, y \in [0,1]$, but has unbounded variation.

Proof: We compute the derivative of f on $(0,1]$ as

$$f'(x) = ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b}),$$

and by **Lemma 1** we have that

$$T_f(0,1) = \int_0^1 |f'(x)|\, dx = \int_0^1 \left|ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})\right|\, dx$$

$$\leq \int_0^1 \left|ax^{a-1}\sin(x^{-b})\right| - \int_0^1 \left|bx^{a-b-1}\cos(x^{-b})\right|\, dx.$$

$\sin(x)$ is bounded by 1, and ax^{a-1} is non-negative if x is non-negative, thus the left side is bounded by

$$\int_0^1 \left|ax^{a-1}\right|\, dx = \int_0^1 ax^{a-1}\, dx = 1 - 0 = 1,$$

and thus is finite. Additionally, we have that the right side is finite if and only if $a > b$. [1] Thus, $a > b$ implies that the integral of $|f'(x)|$ is finite, thus $T_f(0,1)$ is finite as well, and f has bounded variation on $[0,1]$.

Conversely, assume that f has bounded variation, then, using the reverse triangle inequality, we find

$$T_f(0,1) = \int_a^b |f'(x)|\, dx = \int_0^1 \left|ax^{a-1}\sin(x^{-b}) - bx^{a-b-1}\cos(x^{-b})\right|\, dx$$

$$\geq \int_0^1 \left|\left|ax^{a-1}\sin(x^{-b})\right| - \left|bx^{a-b-1}\cos(x^{-b})\right|\right|\, dx,$$

which, by [1] and the previous observations, is only finite when $a > b$, thus completing the proof. ■

14a) If f is a continuous function on some closed interval $[a, b]$, then the upper right Dini derivative $D^+(f)(x)$ is a measurable function over $[a, b]$.

Proof: We have that

$$D^+(f)(x) = \limsup_{h\to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^+} \sup_{k\in(0,h)} \frac{f(x+k) - f(x)}{k}.$$

Additionally, since f is continuous, we have for all $x \in [a, b]$ and $\varepsilon > 0$ that there exists a rational number r where $|f(x) - f(r)| < \varepsilon$. Thus, it suffices to take the supremum over all rational numbers in $(0, h)$. Since $\mathbb{Q}$ is countable, we can, for fixed h, index the rational numbers in $(0, h)$ over $\mathbb{N}$; let $\{r_n\}_{n\in\mathbb{N}}$ be such an indexing. Also denote $E_h = (0, h) \cap \mathbb{Q}$. Continuous functions are continuous under shifting, subtraction, and division, and thus the function f_n defined for $n \in \mathbb{N}$ as

$$f_n(x) = \frac{f(x + r_n) - f(x)}{r_n}$$

is continuous, and thus measurable. Since we also have

$$\sup_{k\in(0,h)} \frac{f(x+k) - f(x)}{k} = \sup_{k\in E_h} \frac{f(x+k) - f(x)}{k} = \sup_{k\in\mathbb{N}} f_k(x), \tag{1}$$

we see that the leftmost supremum is equal to the supremum of a collection of measurable functions, and is thus measurable. We complete the proof by noting that

$$D^+(f)(x) = \lim_{h\to 0^+} \sup_{k\in(0,h)} \frac{f(x+k) - f(x)}{k} = \lim_{n\to\infty} \sup_{k\in(0,1/n)} \frac{f(x+k) - f(x)}{k}$$

$$= \lim_{n\to\infty} g_n(x),$$

where g_n is the function described in (1) with $h = 1/n$. This shows that $D^+(f)(x)$ is the pointwise limit of a sequence of measurable functions, and thus measurable. ■

Proposition) Fix a function f, and assume that $D^+(g)(x) \leq D_-(g)(x)$ a.e., where $g(x) = -f(-x)$. Then we have that $D^-(f)(x) \leq D_+(f)(x)$ a.e.

Proof: The limit inferior of a sequence is always bounded by the limit superior, thus $D_+(f)(x) \leq D^+(f)(x)$ and $D_-(f)(x) \leq D^-(f)(x)$. Combining this with our assumption on g, we find that $D_+(g)(x) \leq D^-(g)(x)$ a.e. We compute $D_+(g)(x)$ as follows:

$$D_+(g)(x) = \liminf_{h\to 0^+} \frac{g(x+h) - g(x)}{h} = \liminf_{h\to 0^+} -\frac{f(-x-h) - f(-x)}{h}$$

$$= \liminf_{h\to 0^-} -\frac{f(x+h)-f(x)}{h} = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = D^-(f)(x),$$

and similary for $D^-(g)(x)$:

$$D^-(g)(x) = \limsup_{h\to 0^-} \frac{g(x+h)-g(x)}{h} = \limsup_{h\to 0^-} -\frac{f(-x-h)-f(-x)}{h}$$

$$= \limsup_{h\to 0^+} -\frac{f(x+h)-f(x)}{h} = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = D_+(f)(x),$$

thus combined with our previous inequality for g, we find that $D^-(f)(x) \le D_+(f)(x)$ a.e. ■

References

[1] https://math.stackexchange.com/questions/2093423/when-is-int-01xa-b-1-cosx-b-dx-infty. Accessed on 12/11/24.