Denote $\mathbb{N}_0$ as the set $\{n \in \mathbb{Z} : n \geq 0\}$.

28) Fix $E \subseteq \mathbb{R}$ where $m_*(E) > 0$, then for all $0 < \alpha < 1$ there exists an open interval I such that $m_*(E \cap I) \geq \alpha m_*(I)$

Proof: Fix $0 < \alpha < 1$ and $0 < \varepsilon < \alpha$. If $\{I_n\}_{n\in\mathbb{N}}$ is a covering of E using closed intervals, we can expand each interval by less than $\varepsilon/2^n$ to obtain a covering $\{I'_n\}_{n\in\mathbb{N}}$ where

$$\sum_{n\in\mathbb{N}} m_*(I'_n) < \sum_{n\in\mathbb{N}} m_*(I_n) + \varepsilon,$$

thus we can let $\{I_n\}_{n\in\mathbb{N}}$ be a covering of E where

$$\sum_{n\in\mathbb{N}} m_*(I_n) < m_*(E) + \varepsilon/2\alpha.$$

Because each I_n is measurable, we can choose open intervals O_n such that $I_n \subset O_n$ and $m_*(O_n \setminus I_n) < \varepsilon/2^{n+1}\alpha$, thus $m_*(O_n) < m_*(I_n) + \varepsilon/2^{n+1}\alpha$. From this, we can see that

$$\sum_{n\in\mathbb{N}} m_*(O_n) < \sum_{n\in\mathbb{N}} m_*(I_n) + \varepsilon/2^{n+1}\alpha < m_*(E) + \varepsilon/2\alpha + \varepsilon/2\alpha = m_*(E) + \varepsilon/\alpha.$$

To establish a contradiction, suppose we have that $m_*(E\cap I) < \alpha m_*(I)$ for all open intervals I, thus $m_*(E\cap O_n) < \alpha m_*(O_n)$ for all n. Since $\{O_n\}_{n\in\mathbb{N}}$ covers E, we have that $E \subseteq \bigcup_{n\in\mathbb{N}} E\cap O_n \subseteq \bigcup_{n\in\mathbb{N}} O_n$, thus by our assumption we have

$$m_*(E) < \sum_{n\in\mathbb{N}} \alpha m_*(O_n) < \alpha(m_*(E) + \varepsilon/\alpha) = \alpha m_*(E) + \varepsilon.$$

Since ε is arbitrarily small, we find that $m_*(E) < \alpha m_*(E)$, and thus $\alpha > 1$, but this is a contradiction. Thus, there must exist an open interval I where $m_*(E \cap I) \geq \alpha m_*(I)$, and we are finished. ■

37) Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, define $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, then we have that $m(\Gamma) = 0$.

Proof: For $n \in \mathbb{Z}$, let $I_n = [n, n+1]$, then $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} I_n$. Fix $\varepsilon > 0$ and denote ε_n as $\varepsilon/2^{|n|+4}$. Since f is continuous on $\mathbb{R}$, we have that it is uniformly continuous on all I_n, thus for all n we can choose $0 < \delta_n < 1$ and $k_n \in \mathbb{N}$ where $1/(k_n+1) < \delta_n < 1/k_n$, and where $|x-y| < \delta_n \implies |f(x) - f(y)| < \varepsilon_n$ for all $x, y \in I_n$. We have that $1 < (k_n+1)\delta_n < 2$, thus for all n there exists a collection of k_n+1 intervals $I_{n,m} = [x_{n,m}, x_{n,m+1}]$ for $1 \leq m \leq k_n+1$ where $|I_{n,m}| < \delta_n$ and $I_n = \bigcup_{1\leq m\leq k_n+1} I_{n,m}$. For any $(x, f(x)) \in \Gamma$, we have some interval where $x \in I_n$ and thus x is in some $I_{n,m}$. Since $|x - x_{n,m}| < \delta_n$, we have $|f(x) - f(x_{n,m})| < \varepsilon_n$, thus the image $f(I_{n,m})$ is a subset of $[f(x_{n,m}) - \varepsilon_n, f(x_{n,m}) + \varepsilon_n]$ and $(x, f(x)) \in I_{n,m} \times f(I_{n,m})$. Denoting $\Gamma_n = \bigcup_{1\leq m\leq k_n+1} I_{m,n} \times f(I_{m,n})$, we find that

$$m(\Gamma_n) \leq \sum_{1\leq m\leq k_n+1} m(I_{n,m} \times f(I_{m,n})) < \sum_{1\leq m\leq k_n+1} \delta_n \cdot 2\varepsilon_n = (\varepsilon/2^{|n|+3})(k_n+1)\delta_n < \varepsilon/2^{|n|+2}.$$

Since $\Gamma \subseteq \bigcup_{n\in\mathbb{Z}} \Gamma_n$, we have

$$m(\Gamma) \leq \sum_{n\in\mathbb{Z}} m(\Gamma_n) \leq 2\sum_{n\in\mathbb{N}_0} \varepsilon/2^{n+2} = 2\frac{\varepsilon}{2} = \varepsilon,$$

and thus $m(\Gamma) = 0$. ■