Second Order with Constant Coefficients

$$ay'' + by' + cy = 0 \implies ar^2 + br + c = 0$$

Case 3: Equal Roots

$$r = r_1 = r_2 \implies r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ y_1 = e^{rt}$$

Reduction of Order

$$y_2 = v(t)y_1 = ve^{rt}$$

$$\therefore\ ay_2'' + by_2' + cy_2 = 0$$

$$y_2' = v'e^{rt} + rve^{rt}$$

$$y_2'' = v''e^{rt} + 2rv'e^{rt} + r^2ve^{rt}$$

$$\therefore\ a(v''e^{rt} + 2rv'e^{rt} + r^2ve^{rt}) + b(v'e^{rt} + rve^{rt}) + cve^{rt} = 0$$

$$\implies a(v'' + 2rv' + r^2v) + b(v' + rv) + cv = 0$$

$$\implies av'' + v'(2ar + b) + v(ar^2 + br + c) + = 0$$

$$\implies av'' + v'(2ar + b) = av'' + v'\left[2a\left(\frac{-b}{2a} + b\right)\right] = 0$$

$$\implies av'' = v'' = 0$$

$$v = \int\int 0\ dt = c_1t + c_2$$

$$\therefore\ y = vy_1 = (c_1t + c_2)e^{rt} = c_1te^{rt} + c_2e^{rt}$$

Which is the general solution.

Theorem 3.2.1 Existence & Uniqueness

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t);\ y(t_0) = y_0;\ y'(t_0) = y_0'$$

Where p, q, and g are continuous on the open interval I that contains point t_0, then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists on the interval I.

e.g.) Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t-3)t = 0;\; y(1) = 2;\; y'(1) = 1$$

is certain to exist.
Solution:

$$y'' + \frac{1}{(t-3)}y' + \frac{1}{t}y = 0$$
$$\therefore\; p(t) = \frac{1}{t-3},\; q(t) = \frac{1}{t},\; g(t) = 0$$

p is continuous when $t \neq 3$, q is continuous when $t \neq 0$, g is always continuous, thus $I = (0, 3)$.

Principle of Super Position

Given solutions y_1 and y_2 to the following different equation

$$y'' + p(t)y' + q(t)y = 0$$

$c_1y_1 + c_2y_2$ is a solution for all $c_1, c_2 \in \mathbb{R}$.

Theorem

$c_1y_1 + c_2y_2$ is a general solution

Proof

Apply initial conditions:

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$
$$w = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$$

Wronskian of solutions y_1 and y_2

$$W[y_1, y_2] = w = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

If $w \neq 0$ for y_1, y_2, then they are a fundamental set

Theorem 3.2.4

Let y_1, y_2 be solutions to the following differential equation

$$y'' + p(t)y' + q(t)y = 0$$

Then $y = c_1y_1 + c_2y_2$ is a basis for the solution family of the equation.

Proof

Let $\phi(t)$ be a solution to the equation, and t_0 be a point where $W[y_1, y_2] \neq 0$. Let $y_0 = \phi(t_0)$, $y'_0 = \phi'(t_0)$, thus the equation is satisfied. Q.E.D.