Definitions

1.) Given a topological space X, a *continuous path* in X is a continuous mapping $\gamma : [0,1] \to X$. Given two points $x, x' \in X$, a path γ connects them if $\gamma(0) = x$ and $\gamma(1) = x'$.

2.) A topological space X is *path-connected* if for all $x, x' \in X$, there exists a continuous path that connects x and x'.

3.) Let X be a topological space and $\sim$ be an equivalence relation on X defined as follows:

$$x \sim x' \iff x \text{ is path connected to } x' \text{ in } X$$

We define $\pi_0(X)$ to be the topological space $X/{\sim}$.

4.) $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n \iff m = n$.

Proofs

a.) Let $[x] = [x']$ for some $x, x' \in X$, then there exists a continuous path γ in X where $\gamma(0) = x$ and $\gamma(1) = x'$. Let $y, y' \in Y$ where $y = f(x)$ and $y' = f(x')$, and define $\gamma' = f \circ \gamma$. We know f and γ are continuous, thus γ' is continuous. In addition, we know that $\gamma'(0) = f(\gamma(0)) = f(x) = y$ and $\gamma'(1) = f(\gamma(1)) = f(x') = y'$, thus γ' is a path that connects y and y', thus $y \sim y' \implies [y] = [y'] \implies [f(x)] = [f(x')]$, thus $f_\sharp$ is a well defined function. ■

b.) Let $x \in X$, then $(g \circ f)_\sharp([x]) = [(g \circ f)(x)] = [g(f(x))]$. We can also see that $(g_\sharp \circ f_\sharp)([x]) = g_\sharp(f_\sharp([x])) = g_\sharp([f(x)]) = [g(f(x))]$, thus $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. ■

c.) Let $[x] \in X/{\sim}$, then $(\mathrm{id}_X)_\sharp([x]) = [x]$. We can also see that $\mathrm{id}_{\pi_0(X)}([x]) = [x]$, thus $(\mathrm{id}_X)_\sharp = \mathrm{id}_{\pi_0(X)}$. ■

d.) Let $[x], [x'] \in X/{\sim}$ where $f_\sharp([x]) = f_\sharp([x'])$, then we know that $[f(x)] = [f(x')]$, thus $f(x) \sim f(x')$, thus there exists a path γ in Y that connects $f(x)$ and $f(x')$. Using a similar argument to before, it is easy to show that $\gamma' = f^{-1} \circ \gamma$ is a path in X that connects x to x', thus $x \sim x'$, thus $[x] = [x']$, thus $f_\sharp$ is injective. Next, let $[y] \in Y/{\sim}$, then $f_\sharp([f^{-1}(y)]) = [f(f^{-1}(y))] = [y]$, thus $f_\sharp$ is surjective, and thus bijective. ■