7.3.5.) Using induction, we will show that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$. For the base case, consider $n = 1$:

$$x^n = x^1 = x,\ nx^{n-1} = (1)x^{1-1} = x^0 = 1$$

$$\left.\frac{d}{dx}\right|_{x_0} x = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = \lim_{x \to x_0} 1 = 1$$

Thus the base case holds. Now, assume $\frac{d}{dx}x^n = nx^{n-1}$ holds for n, and consider $n+1$:

$$\left.\frac{d}{dx}\right|_{x_0} x^{n+1} = \lim_{x \to x_0} \frac{x^{n+1} - x_0^{n+1}}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)\left(\sum_{i=0}^{n} x^{n-i}x_0^i\right)}{x - x_0}$$

$$= \lim_{x \to x_0} \sum_{i=0}^{n} x^{n-i}x_0^i = \sum_{i=0}^{n} x_0^n = (n+1)x_0^n$$

Thus the inductive step holds, and thus $\frac{d}{dx}x^n = nx^{n-1}$ holds for all $n \in \mathbb{N}$. ■

7.3.10.) Let f and g be differentiable functions. Using induction we will show that $(fg)^{(n)}(x) = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}(x)g^{(i)}(x)$. For the base case, consider $n = 1$:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) = \sum_{i=0}^{1} \binom{1}{i} f^{(1-i)}(x)g^{(i)}(x)$$

Thus the base case holds. Now, assume $(fg)^{(n)}(x) = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}(x)g^{(i)}(x)$ holds for n, and consider $n + 1$:

$$(fg)^{(n+1)}(x) = \frac{d}{dx}(fg)^{(n)}(x) = \frac{d}{dx} \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}(x)g^{(i)}(x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{d}{dx} f^{(n-i)}(x)g^{(i)}(x) = \sum_{i=0}^{n} \binom{n}{i} \left(f^{(n-i+1)}(x)g^{(i)}(x) + f^{(n-i)}(x)g^{(i+1)}(x)\right)$$

$$= \binom{n}{0} f^{(n+1)}(x)g(x) + \binom{n}{0} f^{(n)}(x)g^{(1)}(x) + \binom{n}{1} f^{(n)}(x)g^{(1)}(x) + \cdots$$

$$= f^{(n+1)}(x)g(x) + \binom{n+1}{1} f^{(n)}(x)g^{(1)}(x) + \binom{n+1}{2} f^{(n-1)}(x)g^{(2)}(x) + \cdots$$

$$+ \binom{n+1}{n} f^{(1)}(x)g^{(n)}(x) + f(x)g^{(n+1)}(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} f^{(n+1-i)}(x)g^{(i)}(x)$$

Thus the induction step holds, and thus we have found a formula for $(fg)^{(n)}(x)$. ■

7.3.22.) Assuming the fact that $\frac{d}{dx}e^x = e^x$, we will show that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{R}$. Given $n \in \mathbb{R}$, we can see the following:

$$x^n = e^{\ln(x^n)}$$

Using this fact, we can find $\frac{d}{dx}x^n$:

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{\ln(x^n)} = e^{\ln(x^n)}\frac{d}{dx}\ln(x^n) = nx^n\frac{d}{dx}\ln x = \frac{nx^n}{x} = nx^{n-1}$$

Thus the identity holds. ■