Chapter 10

3.) Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ where $f \mapsto f'$, and let $f, g \in \mathbb{R}[x]$. We can see that $\phi(f) + \phi(g) = f' + g' = (f + g)' = \phi(f + g)$, thus ϕ is a homomorphism. $\blacksquare$

7.) Let $\phi : G \to H$ and $\sigma : H \to K$ be homomorphisms. Let $a, b \in G$, then $(\sigma \circ \phi)(ab) = \sigma(\phi(ab)) = \sigma(\phi(a)\phi(b)) = \sigma(\phi(a))\sigma(\phi(b)) = (\sigma \circ \phi)(a)(\sigma \circ \phi)(b)$, thus $\sigma \circ \phi$ is a homomorphism. We can also show that $\ker \phi$ is a subgroup of $\ker \sigma \circ \phi$. It is obvious that $\ker \phi \subseteq \ker \sigma \circ \phi$, as $a \in \ker \phi \implies \sigma(\phi(a)) = \sigma(e_H) = e_K \implies a \in \ker \sigma \circ \phi$. In addition, the kernel of any homomorphism is a group, thus we know that $\ker \phi$ is a subgroup of $\ker \sigma \circ \phi$. Finally, suppose ϕ and σ are surjective, and that $|G|$ is finite, then $|G| = |\ker \phi| \, |\phi(G)| = |\ker \phi| \, |H|$. We also know that $|G| = |\ker \sigma \circ \phi| \, |\phi(G)| = |\ker \sigma \circ \phi| \, |K|$, thus

$$[\ker \sigma \circ \phi : \ker \phi] = \frac{|\ker \sigma \phi|}{|\ker \phi|} = \frac{|G| \, / \, |K|}{|G| \, / \, |H|} = \frac{|K|}{|H|}$$

Thus the index is finite and given above. $\blacksquare$

9.) Let $a, b \in G \oplus H$ where $a = (g, h)$ and $b = (g', h')$, and consider the map $\phi : G \oplus H \to G$ where $(g, h) \mapsto g$. We can see that $\phi(ab) = \phi(gg', hh') = gg' = \phi(g, h)\phi(g', h')$, thus ϕ is a homomorphism. Next, let $a \in \ker G \oplus H$ where $a = (g, h)$, thus $\phi(a) = \phi(g, h) = g = e_G$, thus $\ker \phi = \{(e_G, h) : h \in H\}$. $\blacksquare$

12.) Let $k, n \in \mathbb{Z}$ where $k \mid n$ and $m \in \langle k \rangle$, and consider the map $\phi : \mathbb{Z}_n / \langle k \rangle \to \mathbb{Z}_k$ where $m\mathbb{Z}_n \mapsto m \pmod{k}$. For $m, m' \in \langle k \rangle$, we can see that $\phi(m\mathbb{Z}_n \circ m'\mathbb{Z}_n) = \phi((mm')\mathbb{Z}_n) = mm' \pmod{k} = \phi(m\mathbb{Z}_n)\phi(m'\mathbb{Z}_n)$, thus ϕ is a homomorphism, and thus $\mathbb{Z}_n / \langle k \rangle \cong \mathbb{Z}_k$. $\blacksquare$

16.) Suppose $\phi : \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is a surjective homomorphism. We know that $|\mathbb{Z}_8 \oplus \mathbb{Z}_2| = |\mathbb{Z}_4 \oplus \mathbb{Z}_4|$, but since $(1, 0) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ has order 8, and since every element in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ has at most order 4, we know that ϕ cannot be surjective. $\blacksquare$