6) There exists a non-negative continuous function $f : \mathbb{R} \to \mathbb{R}$ that is integrable, but where $\limsup_{x\to\infty} f(x) = \infty$. However, if f is uniformly continuous and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Proof: Let $g(x) = \sum_{k\in\mathbb{N}} k\chi_{I_k}(x)$ where $I_k = [k, k+1/k^3)$. By **Corollary 10**, we have that

$$\int g(x)\,dx = \int \sum_{k\in\mathbb{N}} k\chi_{I_k}(x)\,dx = \sum_{k\in\mathbb{N}} \int k\chi_{I_k}(x)\,dx = \sum_{k\in\mathbb{N}} k \cdot m(I_k) = \sum_{k\in\mathbb{N}} \frac{1}{k^2} < \infty,$$

thus g is integrable. For each k, define the function $g_k(x) = \min\{h_k(x), h'_k(x)\}\,\chi_{I_k}(x)$, where

$$h_k(x) = 2k^4(x-k) \quad \text{and} \quad h'_k(x) = -2k^4(x-k-1/k^3).$$

h_k and h'_k are both lines, and thus continuous. Additionally, the minimum of two continuous functions is continuous, thus g_k is continuous on I_k. We also have that $g_k(k) = g_k(k+1/k^3) = 0$, and since g_k is clearly zero outside of I_k, we have that g_k is continuous everywhere.

We will now show that $g_k(x) \le g(x)$ for all x and k. If $k \le x < k + 1/2k^3$, then

$$h_k(x) < 2k^4(k + 1/2k^3 - k) = 2k^4/2k^3 = k = g(x) < h'_k(x),$$

and if $k + 1/2k^3 \le x < k + 1/k^3$, we have

$$h'_k(x) \le -2k^4(k + 1/2k^3 - k - 1/k^3) = -2k^4/(-2k^3) = k = g(x) \le h_k(x),$$

and since $g(x) = g_k(x) = 0$ for x not contained in any I_k, we have that $g_k(x) \le g(x)$ everywhere.

Now, define the function f as

$$f(x) = \sum_{k\in\mathbb{N}} g_k(x)$$

The I_k are disjoint, and so if $x \in [k, k+1/k^3)$ for some k, then $f(x) = g_k(x)$ and $f(x) = g_m(x) = 0$ for all $m \ne k$. This combined with the fact that $g_k(x)$ is continuous everwhere for all k shows that f is continuous. We also have for all x that $f(x) \le g_k(x) \le g(x)$ for some k, and thus by monotonicity f is integrable. Finally, since $f(k + 1/2k^3) = k$ for arbitrary $k \in \mathbb{N}$, we have that $\limsup_{x\to\infty} f(x) = \infty$, thus completing the first part of the proof. ▲

Next, restrict f to being uniformly continuous and integrable, and for $k \in \mathbb{N}_0$, define the intervals $I_k = [k, k+1]$. The uniform continuity of f guarantees a minimum value over all compact sets, thus we can define $x_k = \min_{x\in I_k} f(x)$. Additionally, since the inervals I_k are almost disjoint, we have that

$$\sum_{k\in\mathbb{N}_0} x_k = \sum_{k\in\mathbb{N}_0} x_k m(I_k) = \sum_{k\in\mathbb{N}_0} \int_{I_k} x_k\,dx \le \sum_{k\in\mathbb{N}_0} \int_{I_k} f(x)\,dx \le \int f(x)\,dx < \infty,$$

thus the sum $\sum_{k\in\mathbb{N}_0} x_n$ must be finite. Suppose, for the sake of contradiction, that $\lim_{x\to\infty} f(x) = L > 0$, then for any $0 < \varepsilon < L$, we can choose $N \in \mathbb{N}$ where $x > N \implies |f(x) - L| < \varepsilon$, thus if $x > N$, we have that $f(x) > L - \varepsilon$. But we have that

$$\infty = \sum_{k>N} L - \varepsilon \le \sum_{k>N} x_k \le \sum_{k\in\mathbb{N}_0} x_k \le \int f(x)\,dx,$$

which is a contradiction. This forces $L = 0$ since f is non-negative, showing that $\lim_{x\to\infty} f(x) = 0$. Defining $g(x) = f(-x)$ and using the immediate result, we can see that $0 = \lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(-x) = \lim_{x\to-\infty} f(x)$, thus $\lim_{|x|\to\infty} f(x) = 0$ and the proof is complete. ■

10) Fix a non-negative measurable function f on $\mathbb{R}^d$, and for $k \in \mathbb{Z}$ define the sets E_k and F_k as follows:

$$E_k = \left\{x \in \mathbb{R}^d : 2^k < f(x)\right\} \quad \text{and} \quad F_k = \left\{x \in \mathbb{R}^d : 2^k < f(x) \le 2^{k+1}\right\},$$

then the following are equivalent:

(a) f is integrable

(b) $\displaystyle\sum_{k\in\mathbb{Z}} 2^k m(E_k) < \infty$

(c) $\displaystyle\sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$

Proof: We will show that (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

First, suppose f is integrable. Using **Corollary 10** we can see that

$$\sum_{k\in\mathbb{Z}} 2^k m(E_k) = \sum_{k\in\mathbb{Z}} \int_{E_k} 2^k\,dx = \sum_{k\in\mathbb{Z}} \int 2^k \chi_{E_k}(x)\,dx = \int \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}(x)\,dx.$$

We also have that $\sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}(x) < 2f(x)$, since if $2^n < f(x) \le 2^{n+1}$, then

$$\sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}(x) = \sum_{k\le n} 2^k = \sum_{k\in\mathbb{N}} 2^{-k} + \sum_{0\le k\le n} 2^k = 1 + 2^{n+1} - 1 = 2^{n+1} < 2f(x),$$

which shows that

$$\int \sum_{k\in\mathbb{Z}} 2^k \chi_{E_k}(x)\,dx < \int 2f(x)\,dx = 2\int f(x)\,dx < \infty,$$

thus proving (a) $\implies$ (b). Next, assume (b). Because $F_k \subset E_k$, we have that

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) < \sum_{k\in\mathbb{Z}} 2^k m(E_k) < \infty,$$

showing that (b) $\implies$ (c). Finally, assume (c). Since f is non-negative, we have that $\{f \neq 0\} = \bigcup_{k\in\mathbb{Z}} F_k$, and since the F_k are disjoint, we have

$$\int f(x)\,dx = \sum_{k\in\mathbb{N}} \int_{F_k} f(x)\,dx \leq \sum_{k\in\mathbb{N}} \int_{F_k} 2^{k+1}\,dx = 2\sum_{k\in\mathbb{N}} 2^k m(F_k) < \infty,$$

thus (c) $\implies$ (a), completing the proof. ■

Corollary) Fix $a \in \mathbb{R}$ and define the functions f_a and g_a on $\mathbb{R}^d$ as

$$f_a(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_a(x) = \begin{cases} |x|^{-a} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases},$$

then f_a is integrable if and only if $0 < a < d$, and g_a is integrable if and only if $a > d$.

Proof: Define the sets F_k with $f(x) = |x|^{-a}$ as before. Solving for $|x|$, we find that

$$F_k = \left\{x \in \mathbb{R}^d : 2^{-(k+1)/a} \leq |x| < 2^{-k/a}\right\}.$$

Additionally, $F_k = 2^{-k/a}F$, where $F = \left\{x \in \mathbb{R}^d : 2^{-1/a} \leq |x| < 1\right\}$, and thus $m(F_k) = (2^{-k/a})^d m(F)$. This combined with the fact that, for f_a, $k < 0$ implies $F_k = \varnothing$, shows that

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) = \sum_{k\in\mathbb{N}_0} 2^k m(F_k) = m(F)\sum_{k\in\mathbb{N}_0} 2^{k(1-d/a)},$$

but this series converges if and only if $1 - d/a$ is negative, i.e. when $0 < a < d$, thus by the previous theorem we have established that f_a is integrable if and only if $0 < a < d$. We also have that, for $g_a(x)$, $k \geq 0$ implies $F_k = \varnothing$, thus

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) = \sum_{k\in\mathbb{N}} \frac{m(F_{-k})}{2^k} = m(F)\sum_{k\in\mathbb{N}} 2^{k(d/a-1)},$$

which converges if and only if $d/a - 1$ is negative, i.e. when $a > d$, and again by the previous theorem we have proven the case of g_a. ■

11) If f is a real-valued function integrable on $\mathbb{R}^d$, and if $\int_E f \geq 0$ for all measurable sets $E \subseteq \mathbb{R}^d$, then $f \geq 0$ a.e.

Proof: Suppose the proposition is false, then there exists some $r > 0$ where the set $E = \{f < -r\}$ has non-zero measure. Since f is integrable and thus measurable, we have that E is also measurable. From this, we see that

$$\int_E f(x)\,dx < \int_E -r\,dx = -r \cdot m(E) < 0,$$

which is a contradiction since E is measurable. Thus, we must have that $m(E) = 0$ for all $r > 0$, showing that $f \geq 0$ a.e. ■

Corollary) If $\int_E f = 0$ for all measurable sets $E \subseteq \mathbb{R}^d$, then $f = 0$ a.e.

Proof: By the previous theorem, we have that $f \geq 0$ a.e. We also have that $\int_E -f = -\int_E f = 0$ for all E, showing that $-f \geq 0$ a.e., and thus $f \leq 0$ a.e. This proves that $0 \leq f \leq 0$ a.e., showing $f = 0$ a.e. ■

12) There exists an integrable function f on $\mathbb{R}^d$ and a sequence of integrable functions $\{f_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}^d$ where, as $n \to \infty$, $\|f - f_n\| \to 0$ but $f_n \to f$ nowhere.

Proof: For $k, m \in \mathbb{N}$, define the sets $I_{k,m}$ as

$$I_{k,m} = \left\{x \in \mathbb{R}^d : (m-1)/2^k \leq |x| < m/2^k\right\}.$$

We will consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined as $f_n = \chi_{I_{k,m}}$, where $f_1 = \chi_{I_{1,1}}$, and for k to increase, m must traverse from 1 to $k2^k$. For example, the sequence starts like this:

n	1	2	3	4	5	6	7
f_n	$\chi_{I_{1,1}}$	$\chi_{I_{1,2}}$	$\chi_{I_{2,1}}$	$\chi_{I_{2,2}}$	$\chi_{I_{2,3}}$	$\chi_{I_{2,4}}$	$\chi_{I_{2,5}}$

n	8	9	10	11	12	13	14
f_n	$\chi_{I_{2,6}}$	$\chi_{I_{2,7}}$	$\chi_{I_{2,8}}$	$\chi_{I_{3,1}}$	$\chi_{I_{3,2}}$	$\chi_{I_{3,3}}$	$\chi_{I_{3,4}}$

Now, define the set $I_r = \left\{x \in \mathbb{R}^d : |x| < r\right\}$ if $r > 0$ and $I_r = \varnothing$ if $r = 0$. Fixing k, we can see that

$$I_{k,m} = I_{m/2^k} \setminus I_{(m-1)/2^k},$$

and since $I_{r_1} \subseteq I_{r_2}$ if $r_1 \leq r_2$, we have that

$$m(I_{k,m}) = m(I_{m/2^k}) - m(I_{(m-1)/2^k}) = m(I_1)\left(\frac{m^d}{2^{kd}} - \frac{(m-1)^d}{2^{kd}}\right)$$

$$= m(I_1)\left(\frac{m^d - (m-1)^d}{2^{kd}}\right).$$

Note that the derivative of $x^d - (x-1)^d$ w.r.t. x is $d(x^{d-1} - (x-1)^{d-1})$, which is positive for all $x > 0$, thus $m^d - (m-1)^d$ is increasing over the positive integers. Using this we see that $m_1 \leq m_2$ implies that $m(I_{k,m_1}) \leq m(I_{k,m_2})$, and thus

$$\int \chi_{I_{k,m_1}}(x)\,dx = m(I_{k,m_1}) \leq m(I_{k,m_2}) = \int \chi_{I_{k,m_2}}(x)\,dx.$$

Additionally, we have that

$$m(I_{k,k2^k}) = m(I_1)\left(\frac{k^d 2^{kd} - (k2^k - 1)^d}{2^{kd}}\right), \tag{1}$$

as well as

$$k^d 2^{kd} - (k2^k - 1)^d = -(d-1)k^{d-1}2^{k(d-1)} + \cdots + (-1)^{d-1}(d-1)k2^k + (-1)^d \leq M2^{k(d-1)},$$

where M is any constant larger than the sum of the binomial coefficients in the expansion of $(k2^k - 1)^d$. Note that M is defined independently of k, thus we can see that (1) is bounded by $M2^{k(d-1)}/2^{kd} = M/2^k$, but this tends to 0 as $k \to \infty$, and thus (1) tends to 0 as well. Consequently, we have that

$$\lim_{n\to\infty} \int f_n(x)\,dx = \lim_{n\to\infty} \int \chi_{I_{\eta_k,\eta_m}}(x)\,dx \leq \lim_{n\to\infty} \int \chi_{I_{\eta_k,\eta_k 2^{\eta_k}}}(x)\,dx$$
$$= \lim_{k\to\infty} m(I_{k,k2^k}) = 0,$$

where η_k and η_m are the corresponding values of k and m for each n taken in the limit. Thus, setting $f(x) = 0$, we have shown that $\|f_n - f\| \to 0$ as $n \to \infty$ for some function f.

We will now show that $\lim_{n\to\infty} f_n(x)$ exists nowhere. Fix $x \in \mathbb{R}^d$. Note that for fixed k, the sets $I_{k,m}$ are disjoint over $1 \leq m \leq k2^k$, and that

$$\bigcup_{1\leq m\leq k2^k} I_{k,m} = \left\{x \in \mathbb{R}^d : |x| < k\right\}.$$

Thus, if $|x| = r$, then for each integer $k > r$, there exists m_k where $x \in I_{k,m_k}$. This means that x is contained in infinitely many of the $I_{k,m}$, hence $f_n(x) = 1$ for infinitely many $n \in \mathbb{N}$. For fixed $k > r$, we have that x is contained in one of $k2^k$ disjoint sets, and thus for each k we have m'_k where $x \notin I_{k,m'_k}$, thus showing that $f_n(x) = 0$ for infinitely many n. This is only possible when $\lim_{n\to\infty} f_n(x)$ does not exist at x, thus completing the proof. ■