Definitions

1.) Let A and B be sets. We say that A is a "subset" of B, and write $A \subset B$ if $a \in A \implies a \in B$ holds for all $a \in A$.

2.) Let A be a set. We call $\mathcal{P}(A)$ the "power set" of A, and define it as $\mathcal{P}(A) = \{B : B \subset A\}$.

3.) We call $\varnothing$ the "empty set" and define it where $|\varnothing| = 0$.

4.) We call S^2 the 2-sphere and define it as follows:

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 = 1 \right\}$$

It represents the boundary points of a closed ball in $\mathbb{R}^3$.

5.) Given a set A, $\mathcal{P}(A)$ represents the "power set" of A.

6.) We call Δ^2 the 2-simplex and define it as follows:

$$\Delta^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i = 1 \right\}$$

It represents an equilateral triangle with side length $\sqrt{2}$ in $\mathbb{R}^3$.

Proofs

a.) Let A be an arbitrary set, and G be the set of all functions $g : A \to \{0, 1\}$. Also, let $f : \mathcal{P}(A) \to G$ be a mapping defined as follows:

$$\text{Given } S \in \mathcal{P}(A), \text{ define } f(S) \text{ as } (f(S))(a) = \begin{cases} 1 & \text{if } a \in S \\ 0 & \text{otherwise} \end{cases}$$

First we will show that f is injective. Let $S_1, S_2 \in \mathcal{P}(A)$ where $f(S_1) = f(S_2)$. We can see that $s \in S_1 \implies (f(S_1))(s) = 1 \implies (f(S_2))(s) = 1 \implies s \in S_2$. Without loss of generality, we also see that $s \in S_2 \implies s \in S_1$, thus $s \in S_1 \iff s \in S_2 \implies S_1 = S_2$, thus $f(S_1) = f(S_2) \implies S_1 = S_2$, thus f is injective.

Next we will show that f is surjective. Let $g \in G$ and $S \in \mathcal{P}(A)$ where $S = \{a \in A : g(a) = 1\}$. Since $a \in S \implies g(a) = 1$ and $a \notin S \implies g(a) = 0$, we know that $f(S) = g$, thus for any g we can find S where $f(S) = g$, thus f is surjective.

Since f is both injective and surjective, it is bijective, thus we have found a bijection from $\mathcal{P}(A)$ to G. Q.E.D.

b.) Let A be an arbitrary set, and consider an arbitrary mapping $f : A \to \mathcal{P}(A)$. Also, let $S \in \mathcal{P}(A)$ where $S = \{a \in A : a \notin f(a)\}$. For the sake of establishing a contradiction, suppose f is surjective, then there exists $b \in A$ where $f(b) = S$. Assume that $b \in S$, then $b \notin f(b)$. However, since $f(b) = S$, $b \in S \implies b \in f(b) \implies b \notin S$. $\Rightarrow\Leftarrow$ Instead, assume that $b \notin S$, then $b \in f(b)$, and thus $b \in S$. $\Rightarrow\Leftarrow$ Since $b \in S \wedge b \notin S$ is a contradictionl we can conclude that f is not surjective, and thus not bijective, thus any mapping $f : A \to \mathcal{P}(A)$ is not bijective. Q.E.D.