Section 3.6

4.) First find the homogeneous solution:

$$y'' + y = 0 \implies r = \pm i$$

$$\implies y_h = c_1 \sin t + c_2 \cos t$$

Let u_1 and u_2 be functions of t, then the particular solution takes the form

$$Y = u_1 y_1 + u_2 y_2$$

Where $y_1 = \sin t$ and $y_2 = \cos t$. Finding the derivatives of Y:

$$Y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2', \quad \text{assume } u_1' y_1 + u_2' y_2 = 0$$

$$\implies Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Thus

$$Y'' + Y = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + u_1 y_1 + u_2 y_2 = \tan t$$

$$\implies u_1' y_1' + u_2' y_2' + u_1(y_1'' + y_1) + u_2(y_2'' + y_2) = u_1' y_1' + u_2' y_2' + 0 + 0 = \tan t$$

Solving for u_1' and u_2':

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = \tan t$$

$$\implies u_1' = -\frac{u_2' y_2}{y_1}$$

$$\implies u_1' y_1' + u_2' y_2' = u_2' \left(y_2' - \frac{y_2 y_1'}{y_1} \right) = u_2' \left(\frac{y_1 y_2' - y_2 y_1'}{y_1} \right) = \tan t$$

$$\implies u_2' = \frac{y_1 \tan t}{y_1 y_2' - y_2 y_1'}$$

$$\implies u_1' = -\frac{y_1 y_2 \tan t}{y_1 (y_1 y_2' - y_2 y_1')} = -\frac{y_2 \tan t}{y_1 y_2' - y_2 y_1'}$$

Integrating u_1' and u_2':

$$u_1 = -\int \frac{y_2 \tan t}{y_1 y_2' - y_2 y_1'}\, dt = -\int \frac{\cos t \tan t}{W[\sin t, \cos t]}\, dt = \int \cos t \frac{\sin t}{\cos t}\, dt = -\cos t$$

$$u_2 = \int \frac{\sin t \tan t}{W[\sin t, \cos t]}\, dt = -\int \frac{\sin^2 t}{\cos t}\, dt = -\int \frac{1 - \cos^2 t}{\cos t}\, dt = -\int \frac{1}{\cos t} - \cos t\, dt$$

$$= \sin t - \ln|\tan t + \sec t|$$

$$\therefore\ Y = -\sin t \cos t + \sin t \cos t - \cos t \ln|\tan t + \sec t|$$

$$\therefore\ y_h + Y = c_1 \sin t + c_2 \cos t - \cos t \ln|\tan t + \sec t|$$

Which is the general solution.

5.) Find the homogeneous solution:

$$y'' + 9y = 0 \implies r = \pm 3i$$

$$\implies y_h = c_1 \sin(3t) + c_2 \cos(3t)$$

Now find Y:

$$Y = u_1 y_1 + u_2 y_2,\ Y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' = u_1 y_1' + u_2 y_2'$$

$$Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$\therefore\ Y'' + 9Y = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + 9(u_1 y_1 + u_2 y_2)$$

$$= u_1' y_1' + u_2' y_2' + u_1(y_1'' + 9y_1) + u_2(y_2'' + 9y_2) = u_1' y_1' + u_2' y_2' + 0 + 0 = 9\sec^2(3t)$$

Solving for u_1' and u_2':

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = 9\sec^2(3t)$$

$$\implies u_1' = -\frac{9y_2 \sec^2(3t)}{y_1 y_2' - y_2 y_1'} = 9\cos(3t)\sec^2(3t)$$

$$\implies u_2' = \frac{9y_1 \sec^2(3t)}{y_1 y_2' - y_2 y_1'} = -9\sin(3t)\sec^2(3t)$$

Integrating:

$$u_1 = \int 9\cos(3t)\sec^2(3t)\,dt = 9\int \sec(3t)\,dt = 3\ln|\tan(3t) + \sec(3t)|$$

$$u_2 = -9\int \sin(3t)\sec^2(3t)\,dt = -9\int \tan(3t)\sec(3t)\,dt = -3\sec(3t)$$

$$\therefore\ Y = 3\left[\sin(3t)\ln|\tan(3t) + \sec(3t)| - \cos(3t)\sec(3t)\right]$$

$$\therefore\ y_h + Y = c_1 \sin(3t) + c_2 \cos(3t) + 3\left[\sin(3t)\ln|\tan(3t) + \sec(3t)| - 1\right]$$

Which is the general solution.

6.) Find y_h:

$$y'' + 4y' + 4y = 0 \implies r_1 = r_2 = -2$$

$$\implies y_h = c_1 e^{-2t} + c_2 t e^{-2t}$$

Find Y:

$$Y = u_1 y_1 + u_2 y_2,\ Y' = u_1 y_1' + u_2 y_2',\ Y'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$\therefore\ Y'' + 4Y' + 4Y = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + 4(u_1 y_1' + u_2 y_2') + 4(u_1 y_1 + u_2 y_2)$$

$$= u_1' y_1' + u_2' y_2' = t^{-2} e^{-2t}$$

Solving for u_1' and u_2':

$$u_1' = -\frac{y_2 t^{-2} e^{-2t}}{y_1 y_2' - y_2 y_1'} = -\frac{t^{-1} e^{-4t}}{e^{-4t}} = -t^{-1}$$

$$u_2' = \frac{y_1 t^{-2} e^{-2t}}{y_1 y_2' - y_2 y_1'} = \frac{t^{-2} e^{-4t}}{e^{-4t}} = t^{-2}$$

$$u_1 = -\int -t^{-1}\, dt = \ln|t|$$

$$u_2 = \int t^{-2}\, dt = -t^{-1}$$

$$\therefore\ Y = -e^{-2t} \ln|t| - e^{-2t}$$

$$\therefore\ y_h + Y = (c_1 - 1)e^{-2t} + c_2 t e^{-2t} - e^{-2t} \ln|t|$$

Which is the general solution.***

7.) Find y_h:

$$4y'' + y = 0 \implies y'' + \frac{1}{4} y = 0 \implies r = \pm \frac{1}{2} i$$

$$\implies y_h = c_1 \sin(t/2) + c_2 \cos(t/2)$$

Find Y:

$$u_1' y_1' + u_2' y_2' = 2\sec(t/2)$$

$$u_1' = -\frac{2y_2 \sec(t/2)}{-1} = 2\cos(t/2)\sec(t/2) = 2$$

$$u_2' = \frac{2y_1 \sec(t/2)}{-1} = -2\sin(t/2)\sec(t/2) = -2\tan(t/2)$$

$$u_1 = \int 2\, dt = 2t$$

$$u_2 = -2\int \tan(t/2)\, dt = 4\ln|\cos(t/2)|$$

$$\therefore\ Y = 2t\sin(t/2) + 4\cos(t/2)\ln|\cos(t/2)|$$

$$\therefore\ y_h + Y = c_1 \sin(t/2) + c_2 \sin(t/2) + 2\left[t\sin(t/2) + 2\cos(t/2)\ln|\cos(t/2)|\right]$$

Which is the general solution.

8.) Find y_h:

$$y'' - 2y' + y = 0 \implies r_1 = r_2 = 1$$

$$\implies y_h = c_1 e^t + c_2 t e^t$$

Find Y:

$$W[e^t, te^t] = e^{2t}$$

$$u_1 = -\int \frac{te^{2t}}{e^{2t}(1+t^2)}\,dt = -\int \frac{t}{1+t^2}\,dt = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln\left|1+x^2\right|$$

$$u_2 = \int \frac{e^{2t}}{e^{2t}(1+x^2)}\,dt = \int \frac{1}{1+x^2}\,dt = \tan^{-1} t$$

$$\therefore\ Y = -\frac{1}{2}e^t \ln\left|1+x^2\right| + te^t \tan^{-1} t$$

$$\therefore\ y_h + Y = c_1e^t + c_2te^t - \frac{1}{2}e^t \ln\left|1+x^2\right| + te^t \tan^{-1} t$$

Which is the general solution.

9.) Find y_h:

$$y'' - 5y' + 6y = 0 \implies r = 2, 3$$

$$\implies y_h = c_1e^{2t} + c_2e^{3t}$$

Find Y:

$$W[e^{2t}, e^{3t}] = e^{5t}$$

$$u_1 = -\int \frac{e^{3t}g(t)}{e^{5t}}\,dt = -\int e^{-2t}g(t)\,dt$$

$$u_2 = \int \frac{e^{2t}g(t)}{e^{5t}}\,dt = \int e^{-3t}g(t)\,dt$$

$$\therefore\ Y = e^{3t}\int e^{-3t}g(t)\,dt - e^{2t}\int e^{-2t}g(t)\,dt$$

$$\therefore\ y_h + Y = c_1e^{2t} + c_2e^{3t} + e^{3t}\int e^{-3t}g(t)\,dt - e^{2t}\int e^{-2t}g(t)\,dt$$

Which is the general solution.

10.) Check y_1 and y_2:

$$y_1 = t^2,\ y_1' = 2t,\ y_1'' = 2$$

$$t^2y'' - 2y = 2t^2 - 2t^2 = 0$$

$$y_2 = t^{-1},\ y_2' = -t^{-2},\ y_2'' = 2t^{-3}$$

$$2t^2t^{-3} - 2t^{-1} = 2t^{-1} - 2t^{-1} = 0$$

Thus y_1 and y_2 are solutions to the homogeneous equation. Find Y:

$$g = \frac{3t^2 - 1}{t^2} = 3 - t^{-2}$$

$$W[t^2, t^{-1}] = -3$$

$$u_1 = -\int \frac{t^{-1}(3 - t^{-2})}{-3}\, dt = \frac{1}{3}\int 3t^{-1} - t^{-3}\, dt = \ln|t| + \frac{1}{6}t^{-2}$$

$$u_2 = \int \frac{t^2(3 - t^{-2})}{-3}\, dt = -\frac{1}{3}\int 3t^2 - 1\, dt = -\frac{1}{3}t^3 + \frac{1}{3}t$$

$$\therefore\ Y = t^2 \ln|t| + \frac{1}{6} - \frac{1}{3}t^2 + \frac{1}{3} = \frac{1}{2} + t^2\left(\ln|t| - \frac{1}{3}\right)$$

Which is the particular solution.