11) The set $\mathcal{S}$ of all $x \in [0,1]$ that can be represented without a 4 in its decimal expansion has measure 0.

Proof: Define $\mathcal{S}_n$ as containing all $x \in [0,1]$ where the first n digits of its decimal representation $\neq 4$, thus $\mathcal{S} = \lim_{n\to\infty} m(\mathcal{S}_n)$. Since $0.4 = 0.3\overline{9}$, we have that $\mathcal{S}_1 = [0, 4/10] \cup [5/10, 1]$, thus $m(\mathcal{S}_1) = 9/10$. At each n, we eliminate a tenth of the numbers, so we have that $m(\mathcal{S}_n) = \frac{9}{10} m(\mathcal{S}_{n-1})$, thus $m(\mathcal{S}_n) = (9/10)^n$, and thus $m(\mathcal{S}) = \lim_{n\to\infty} (9/10)^n = 0$. ■

16) Let $\{E_k\}_{k\in\mathbb{N}}$ be a countable collection of measurable sets where

$$M = \sum_{k\in\mathbb{N}} m(E_k) < \infty$$

and define $E = \left\{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\right\}$, then we have that E is measurable and $m(E) = 0$.

Proof: For all $x \in E$, we have that $x \in \bigcup_{k>N} E_k$ for arbitrary $N \in \mathbb{N}$, thus $E \subseteq \bigcup_{k>N} E_k$. Fix $\varepsilon > 0$, then for each E_k we can choose an open set O_k where $E_k \subseteq O_k$ and $m_*(O_k) < m_*(E_k) + \varepsilon/2^k$. [1] Define the open set $O = \bigcup_{k>N} O_k$, then $E \subseteq O$ and

$$m_*(O) = m_*\left(\bigcup_{k>N} O_k\right) \leq \sum_{k>N} m_*(O_k) < \sum_{k>N} \left(m_*(E_k) + \frac{\varepsilon}{2^k}\right) < \sum_{k>N} m_*(E_k) + \varepsilon,$$

which implies

$$m_*(O \setminus E) = m_*(O) - m_*(E) < \sum_{k>N} m_*(E_k) + \varepsilon.$$

As $N \to \infty$, we have that $\sum_{k>N} m_*(E_k) \to 0$ since $M < \infty$, thus $m_*(O \setminus E) < \varepsilon$ and E is measurable. Since $E \subseteq \bigcup_{k>N} E_k$, we have $m_*(E) \leq \sum_{k>N} m_*(E_k)$, but $\sum_{k>N} m_*(E_k) \to 0$, so $m_*(E) \leq 0$, thus $m_*(E) = m(E) = 0$. ■

25) Fix $\varepsilon > 0$ and let $E \subseteq \mathbb{R}^d$, then the following are equivalent:

(1) There exists an open set O where $E \subseteq O$ and $m_*(O \setminus E) < \varepsilon$

(2) There exists a closed set F where $F \subseteq E$ and $m_*(E \setminus F) < \varepsilon$

Proof: By **theorem 3.4**, we have that (1) $\implies$ (2).

Now assume (2) holds for E and let $F \subseteq E$ be a closed set where $m_*(E \setminus F) < \varepsilon$. We have that $F^\complement \setminus E^\complement \subseteq E \setminus F$, thus $m_*(F^\complement \setminus E^\complement) \leq m_*(E \setminus F) < \varepsilon$. Since $F^\complement$ is open and $E^\complement \subseteq F^\complement$, we have that (1) holds for $E^\complement$, and thus (2) does as well. But then we can choose $E = E^\complement$ and use the same argument to show that (1) holds for E. Thus (2) $\implies$ (1) and we are finished. ■

26) Fix measurable sets A and B with finite measure and let E be a set where $A \subseteq E \subseteq B$. If $m(A) = m(B)$, then E is measurable.

Proof: Fix $\varepsilon > 0$. We can choose an open set O where $B \subseteq O$ and $m_*(O) - m_*(B) < \varepsilon/2$ and a closed set F where $F \subseteq A$ and $m_*(A) - m_*(F) < \varepsilon/2$, thus we have $m_*(O) - m_*(B) + m_*(A) - m_*(F) = m_*(O) - m_*(F) < \varepsilon$. Since $A \subseteq E$, we have $m_*(A) \leq m_*(E)$ and thus $m_*(F) \leq m_*(E)$. From this we have $m_*(O) - m_*(E) \leq m_*(O) - m_*(F) < \varepsilon$, which shows $m_*(O \setminus E) < \varepsilon$ and thus E is measurable ∎

References

[1] https://proofwiki.org/wiki/Measure_of_Set_Difference_with_Subset. Accessed on 9/17/24.