Problems

5.1.8. Let $\varepsilon > 0$ be given and consider the following:

$$\left|\sqrt{x}-\sqrt{x_0}\right| < \varepsilon \implies \left|\sqrt{x}-\sqrt{x_0}\right|\left|\sqrt{x}+\sqrt{x_0}\right| = |x-x_0| < \varepsilon\left|\sqrt{x}+\sqrt{x_0}\right|$$

Since $\sqrt{x}$ is only defined for $x \geq 0$, we know that $x, x_0 \geq 0$, thus $\sqrt{x}, \sqrt{x_0} > 0$, thus

$$|x-x_0| < \varepsilon\left|\sqrt{x}+\sqrt{x_0}\right| < \implies \frac{|x-x_0|}{\left|\sqrt{x}+\sqrt{x_0}\right|} < \frac{|x-x_0|}{\sqrt{x_0}} < \varepsilon \implies |x-x_0| < \varepsilon\sqrt{x_0}$$

Let $\delta = \varepsilon\sqrt{x_0}$, thus

$$0 < |x-x_0| < \delta \implies |x-x_0| < \varepsilon\sqrt{x_0} \implies \frac{|x-x_0|}{\sqrt{x_0}} < \varepsilon \implies \frac{|x-x_0|}{\left|\sqrt{x}+\sqrt{x_0}\right|} < \frac{|x-x_0|}{\sqrt{x_0}} < \varepsilon$$
$$\implies \left|\sqrt{x}-\sqrt{x_0}\right| < \varepsilon$$

Thus $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$. ■

5.1.16 Let $f(x) = |x|/x$. We will evaluate $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

First, consider $\lim_{x\to 0^-} f(x)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence where $x_n \to 0$ and $x_n < 0$ for all $n \in \mathbb{N}$. We can see the following:

$$f(x_n) = \frac{|x_n|}{x_n} = \frac{-x_n}{x_n} = -1$$

Thus $\lim_{x\to 0^-} f(x) = -1$. Next, consider $\lim_{x\to 0^+} f(x)$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence where $y_n \to 0$ and $y_n > 0$ for all $n \in \mathbb{N}$. We can see the following:

$$f(y_n) = \frac{|y_n|}{y_n} = \frac{y_n}{y_n} = 1$$

Thus $\lim_{x\to 0^+} f(x) = 1$, thus $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, thus $\lim_{x\to 0} f(x)$ does not exist. ■

5.6.12 Let f be not uniformly continuous on an interval I, then there exists $\varepsilon > 0$ where for all $\delta > 0$, the following is true:

$$x, y \in I \text{ and } 0 < |x-y| < \delta \implies |f(x)-f(y)| > \varepsilon$$

Consider two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ where $x_n, y_n \in I$ for all $n \in \mathbb{N}$, and where $x_n - y_n \to 0$, thus for some $N \in \mathbb{N}$, the following is true:

$$n > N \implies |x_n - y_n| < \delta$$

Let $n > N$, then we can conclude that

$$x_n, y_n \in I \text{ and } |x_n - y_n| < \delta \implies |f(x_n) - f(y_n)| > \varepsilon$$

Thus there exist seqeuences where $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)| > c$ for $c > 0$. ■