9) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-negative integrable function. If $\alpha > 0$ and $E_\alpha = \left\{x \in \mathbb{R}^d : f(x) > \alpha\right\}$, then

$$m(E_\alpha) \leq \frac{1}{\alpha} \int_{E_\alpha} f(x)\, dx.$$

Proof: We can see that

$$m(E_\alpha) = \int_{E_\alpha} 1\, dx.$$

Additionally, if $x \in E_\alpha$, then $f(x) > \alpha$, hence $f(x)/\alpha > 1$. Thus, by monotonicity, we have that

$$\int_{E_\alpha} 1\, dx < \int_{E_\alpha} \frac{f(x)}{\alpha}\, dx = \frac{1}{\alpha} \int_{E_\alpha} f(x)\, dx,$$

which proves the inequality. ■

17) Given a positive convergent series $\sum_{n \in \mathbb{N}_0} b_n$, define $a_n = \sum_{0 \leq k \leq n} b_k$, and let f be a function on $\mathbb{R}^2$ defined as follows:

$$f(x, y) = \begin{cases} a_n & \text{if } x \in [n, n+1) \text{ and } y \in [n, n+1) \\ -a_n & \text{if } x \in [n, n+1) \text{ and } y \in [n+1, n+2) \\ 0 & \text{otherwise }, \end{cases}$$

where n is always non-negative. Then we have that f^y and f_x are integrable over $\mathbb{R}$, and that $\int f_x(y)\, dy = 0$ for all $x \in \mathbb{R}$, showing that $\int \left(\int f(x, y)\, dy\right) dx = 0$.

However, we also have that $\int f^y(x)\, dx = a_0$ for $0 \leq y < 1$, and $\int f^y(x)\, dx = a_n - a_{n-1}$ if $n \leq y < n+1$ for some $n \in \mathbb{N}$. As a result, $\int_{(0,\infty)} \int f^y(x)\, dx\, dy = \sum_{k \in \mathbb{N}_0} b_k$.

Finally, we have that $\iint |f(x, y)|\, dx\, dy = \infty$.

Proof: Since n is always non-negative, we have that $f(x, y) = 0$ if either x or y is negative. Fix $x \geq 0$ and let $n = \lfloor x \rfloor$. If $n \leq y < n+1$, then $f_x(y) = a_n$, and if $n+1 \leq y < n+2$, then $f_x(y) = -a_n$. Otherwise, $f_x(y) = 0$. Thus, we have that

$$\int f_x(y)\, dy = \int_{[n,n+2)} f_x(y)\, dy = \int_{[n,n+1)} f_x(y)\, dy + \int_{[n+1,n+2)} f_x(y)\, dy = a_n - a_n = 0,$$

which shows that f_x is integrable and $\int f_x(y)\, dy = 0$. From this we also obtain

$$\iint f(x, y)\, dy\, dx = \iint f_x(y)\, dy\, dx = \int 0\, dx = 0.$$

Fix $y \geq 0$ and redefine $n = \lfloor y \rfloor$. If $y < 1$, then $f^y(x) = a_0$ only when $0 \leq x < 1$, and is otherwise 0, thus we would have

$$\int f^y(x)\,dx = \int_{[0,1)} f^y(x)\,dx = \int_{[0,1)} a_0\,dx = a_0.$$

Now let $y \geq 1$. If $n \leq x < n+1$, then $f^y(x) = a_n$, and if $n-1 \leq x < n$, then $f^y(x) = -a_{n-1}$. This shows that

$$\int f^y(x)\,dx = \int_{[n,n+2)} f^y(x)\,dx = \int_{[n,n+1)} f^y(x)\,dx + \int_{[n+1,n+2)} f^y(x)\,dx$$

$$= a_n - a_{n-1} = b_n,$$

thus f^y is integrable. Additionally, since $[0,\infty)$ is the union of the disjoint intervals $[0,1),[1,2),[2,3),\dots$ we have by additivity that

$$\int_{(0,\infty)} \int f^y(x)\,dx\,dy = \int_{[0,\infty)} \int f^y(x)\,dx\,dy = \sum_{k\in\mathbb{N}_0} \int_{[k,k+1)} \int f^y(x)\,dx\,dy$$

$$= a_0 + \lim_{n\to\infty} \sum_{1\leq k\leq n} \int_{[k,k+1)} a_{\lfloor y\rfloor} - a_{\lfloor y-1\rfloor}\,dy = b_0 + \lim_{n\to\infty} \sum_{1\leq k\leq n} a_k - a_{k-1} = b_0 + \lim_{n\to\infty} \sum_{1\leq k\leq n} b_k$$

$$= b_0 + \sum_{k\in\mathbb{N}} b_k = \sum_{k\in\mathbb{N}_0} b_k,$$

thus proving the first part of the theorem.

Finally, fix $n \leq x < n+1$. If $n \leq y < n+2$, then $|f_x(y)| = a_n$, otherwise $|f_x(y)| = 0$. Additionally, since $\{b_n\}$ is a positive sequence, we have that $n_1 < n_2$ implies $a_{n_1} < a_{n_2}$, thus using **Corollary 10** we find that

$$\iint |f(x,y)|\;dy\,dx = \iint |f_x(y)|\;dy\,dx = \int \sum_{k\in\mathbb{N}_0} \int_{[k,k+2)} a_k\,dy\,dx = \int \sum_{k\in\mathbb{N}_0} 2a_k\,dx$$

$$= \sum_{k\in\mathbb{N}_0} \int 2a_k dx.$$

The integral of a positive constant over $\mathbb{R}$ is ∞, which shows that

$$\iint |f(x,y)|\;dy\,dx = \sum_{k\in\mathbb{N}_0} \int 2a_k dx = \infty,$$

thus completing the proof. ■