Part 1

Theorem 0.1 Let A and B be matrices over a field F, then

1.) $(A^T)^T = A$

Proof: We have $(A^T)^T_{i,j} = A^T_{j,i} = A_{i,j}$. ■

2.) $(A+B)^T = A^T + B^T$

Proof: We have $(A+B)^T_{i,j} = (A+B)_{j,i} = A_{j,i} + B_{j,i} = A^T_{i,j} + B^T_{i,j}$. ■

3.) $(rA)^T = r(A^T)$ for all $r \in F$

Proof: We have $(rA)^T_{i,j} = (rA)_{j,i} = r(A_{j,i}) = r(A^T_{i,j})$. ■

4.) $(AB)^T = B^T A^T$ given AB is defined

Proof: Let $A \in \mathcal{M}_{m,n}$ and $B \in \mathcal{M}_{n,p}$. We know that

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j} \ ,$$

so we have

$$(AB)^T_{i,j} = (AB)_{j,i} = \sum_{k=1}^{n} A_{j,k} B_{k,i} = \sum_{k=1}^{n} A^T_{k,j} B^T_{i,k} = \sum_{k=1}^{n} B^T_{i,k} A^T_{k,j} = \left(B^T A^T\right)_{i,j} \ .$$

■

5.) $\det(A) = \det(A^T)$

Proof: The row-wise determinant of A is equal to the column-wise determinant of A^T. ■

Theorem 0.2 Matrices A and B are *row equivalent* if one can be obtained by a series of elementary row operations on the other.

1.) Row equivalence is an equivalence relation

Proof: Let A, B, and C be matrices of the same size. Since multiplying any row of A by 1 leaves it the same, we know that $A \sim A$, thus row equivalence is reflexive.

Let $A \sim B$, then $A = E_1 \cdots E_n B$ where E_i is an elementary matrix. Since elementary matrices are invertible, we know that $B = E_n^{-1} \cdots E_1^{-1} A$, thus $B \sim A$, thus row equivalence is symmetric.

Finally, let $A \sim B$ and $B \sim C$, thus $A = E_1 \cdots E_m B$ and $B = E'_1 \cdots E'_n C$ where E_i and E'_i are elementary matrices. We can see that $A = E_1 \cdots E_m B = E_1 \cdots E_m E'_1 \cdots E'_n C$, thus $A \sim C$, thus row equivalence is transitive, and thus an equivalence relation. ■

2.) A given matrix A is row equivalent to a unique matrix R that is in reduced row echelon form.

Proof: We know that for all matrices A, there exists some matrix R in reduced row echelon form where $A \sim R$. Suppose R_1 and R_2 are matrices in reduced row echelon form and that $A \sim R_1$ and $A \sim R_2$, then $A = E_1 \cdots E_m R_1$ and $A = E'_1 \cdots E'_n R_2$ where E_i and E'_i are elementary matrices. We can see that

3.) A given matrix A is invertible if and only if its reduced row echelon form is an identity matrix.

Proof: Assume A is invertible, and let R be a matrix in reduced row echelon form where $A \sim R$, thus $A = E_1 \cdots E_n R$ where E_i is an elementary matrix. We know that E_i is invertible, so $(E_1 \cdots E_n)^{-1} A = R$. Since the product of two invertible matrices is invertible, we know that R is invertible, and since it is in reduced row echelon form, it either has a zero row or column, or is the identity matrix, but R is invertible, so it cannot have any zero rows or columns, thus R is the identity matrix. ■