Let $X$ and $Y$ be compact topological spaces, and consider the product space $X \times Y$. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $X \times Y$, then for each $\alpha \in \mathcal{A}$, there exist collections $\{U_\beta\}_{\beta \in \mathcal{B}_\alpha}$ and $\{V_\beta\}_{\beta \in \mathcal{B}_\alpha}$ of open sets in $X$ and $Y$ respectively, where

$$U_\alpha = \bigcup_{\beta \in \mathcal{B}_\alpha} U_\beta \times V_\beta.$$

Now, let $\left\{U'_\beta\right\}_{\beta \in \mathcal{B}}$ be a collection of open sets in $X$ where

$$\mathcal{B} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{B}_\alpha$$

We can show that this collection is an open cover of $X$. Let $x \in X$, then we know that there exists $y \in Y$ where $(x, y) \in X \times Y$. Since $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a cover of $X \times Y$, we know that for all $(x, y) \in X \times Y$, there exists $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}_\alpha$ where $x \in U_\beta$, and thus there exists $\beta \in \mathcal{B}$ where $x \in U'_\beta$, thus we know that $\left\{U'_\beta\right\}_{\beta \in \mathcal{B}}$ is an open cover of $X$. A similar argument admits an open cover $\left\{V'_\beta\right\}_{\beta \in \mathcal{B}}$ of $Y$ that depends on $\{U_\alpha\}_{\alpha \in \mathcal{A}}$. Since $X$ and $Y$ are compact, there exist finite subcovers $\left\{U'_\beta\right\}_{\beta \in \mathcal{B}'}$ and $\left\{V'_\beta\right\}_{\beta \in \mathcal{B}'}$ of $X$ and $Y$ respectively for some $\mathcal{B}' \subseteq \mathcal{B}$. Next, let $\{P_\beta\}_{\beta \in \mathcal{B}'}$ be a finite collection of sets where $P_\beta = U'_\beta \times V'_\beta$. For each $(x, y) \in X \times Y$, we know that there exists $\beta \in \mathcal{B}$ where $x \in U'_\beta$ and $y \in V'_\beta$, and thus there exists $\beta' \in \mathcal{B}'$ where $x \in U'_{\beta'}$ and $y \in V'_{\beta'}$, thus $(x, y) \in P_{\beta'}$, thus $\{P_\beta\}_{\beta \in \mathcal{B}'}$ is a finite cover of $X \times Y$. Lastly, there exists $\alpha$ where $\beta \in \mathcal{B}_\alpha$, thus