29.) a.) Since $S \neq \varnothing$ and is bounded below, there exists $l \in \mathbb{R}$ where $l \leq x$ for all $x \in S$, thus $l \in \mathscr{L}$, thus $\mathscr{L} \neq \varnothing$. Q.E.D.

b.) Let $w = \sup(\mathscr{L})$, thus $w \geq l$ for all $l \in \mathscr{L}$. For the sake of establishing a contradiction, suppose $w > x$ for some $x \in S$. Let $m = \mathrm{mid}(w, x)$, thus $w > m > x$. Since $m > x$ for some $x \in S$, m cannot be a lower bound of S, thus w cannot be a lower bound of S, thus $w \neq \sup(\mathscr{L}) \Rightarrow\Leftarrow$, thus $w \leq x$ for all $x \in S$, thus $w = \sup(\mathscr{L})$ is a lower bound of S, thus $w \in \mathscr{L}$. Q.E.D.

c.) Since $w = \sup(\mathscr{L})$, $w \geq l$ for all $l \in \mathscr{L}$ and thus all lower bounds l of S, thus $w = \inf(S)$. Q.E.D.

33.) Consider $|x|$:

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$$

Now consider $|y|$:

$$|y| = |y - x + x| \leq |y - x| + |x| \implies |y| - |x| \leq |y - x| = |x - y|$$

$$\implies |x| - |y| \geq -|x - y|$$

Since $-|x - y| \leq |x| - |y| \leq |x - y|$, $\big||x| - |y|\big| \leq |x - y|$, thus the inequality holds. Q.E.D.

37.) Since $b \in \mathbb{B} \implies b \geq a$ for all $a \in A$, b is an upper bound of A for all $b \in B$. Since $A \neq \varnothing$ and is bounded above, there exists $u \in \mathbb{R}$ where $u = \sup(A)$. For the sake of establishing a contradiction, suppose $u > b$ for some $b \in B$. Let $m = \mathrm{mid}(u, b)$, thus $u > m > b$. Since $m > b$ for some $b \in B$, $m > a$ for all $a \in A$, thus m is an upper bound of A, but since $u > m$, $u \neq \sup(A) \Rightarrow\Leftarrow$, thus $u \leq b$ for all $b \in B$, thus u is a lower bound of B, and thus $u = \sup(A) \leq \inf(B)$. Q.E.D.

38.) Suppose S is uniformly discrete and $S \neq \varnothing$. For the sake of establishing a contradiction, suppose S has no maximal element, thus $x \in S \implies (x+\varepsilon) \in S$ for some $\varepsilon > 0$. Let $x \in S$, thus there exists $\varepsilon > 0$ where $(x+\varepsilon) \in S$. Since $x, (x+\varepsilon) \in S$ and $x, (x+\varepsilon) \in (x-2\varepsilon, x+2\varepsilon)$, then $\{x, x+\varepsilon\} \subseteq S \cap (x-2\varepsilon, x+2\varepsilon)$. Let $\varepsilon_0 = 2\varepsilon$, thus for some $\varepsilon_0 > 0$, $S \cap (x-\varepsilon_0, x+\varepsilon_0) \neq \{x\}$, thus S is not uniformly discrete $\Rightarrow\Leftarrow$, thus if S is uniformly discrete, then S has a maximal element $m \in S$, thus $m = \sup(S)$, thus $\sup(S) \in S$. Q.E.D.

45.) Since $y_n \to B$, then for all $\varepsilon_0 > 0$, there exists $k_0 \in \mathbb{N}$ where

$$n \geq k_0 \implies |y_n - B| < \varepsilon_0$$

Let $\lambda = |B|/2$. Since $B \neq 0$, there exists $k_1 \in \mathbb{N}$ where

$$n \geq k_1 \implies |y_n| > \lambda$$

Let $n \geq \max(k_0, k_1)$, then

$$\left|\frac{1}{y_n} - \frac{1}{B}\right| = \frac{|y_n - B|}{|y_n|\,|B|} < \frac{\varepsilon_0}{\lambda\,|B|}$$

$$\frac{\varepsilon_0}{\lambda\,|B|} = \varepsilon \implies \varepsilon_0 = \varepsilon\,|B|\,\lambda$$

Let $\varepsilon_0 = \varepsilon\,|B|\,\lambda$ and $n \geq \max(k_0, k_1)$, then

$$|y_n - B| < \varepsilon_0 \implies |y_n - B| < \varepsilon\,|B|\,\lambda \implies \frac{|y_n - B|}{|y_n|\,|B|} < \frac{|y_n - B|}{\lambda\,|B|} < \varepsilon$$

$$\implies \frac{|y_n - B|}{|y_n|\,|B|} = \left|\frac{1}{y_n} - \frac{1}{B}\right| < \varepsilon$$

Thus if $y_n \to B$ and $B \neq 0$, then $1/y_n \to 1/B$. Q.E.D.

47.) Since $y_n \to 0$, then for all $\varepsilon_0 > 0$, there exists $k \in \mathbb{R}$ where

$$n \geq k \implies |y_n - 0| < \varepsilon_0$$

Since x_n is bounded, there exists $M \in \mathbb{R}$ where $M \geq |x_n|$ for all $n \in \mathbb{N}$. Let $\varepsilon_0 = \varepsilon/M$ and $n \geq k$:

$$|y_n - 0| < \varepsilon_0 \implies |y_n - 0| < \frac{\varepsilon}{M} \leq \frac{\varepsilon}{|x_n|} \implies |y_n - 0| < \frac{\varepsilon}{|x_n|}$$
$$\implies |x_n|\,|y_n - 0| = |x_n y_n - 0| < \varepsilon$$

Thus if x_n is bounded and $y_n \to 0$, then $x_n y_n \to 0$. Q.E.D.

48.) Let x_n and y_n be sequences defined as follows:

$$x_n = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \qquad y_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases}$$

Since $x_n, y_n < 2$ for all $n \in \mathbb{N}$, x_n and y_n are bounded. In addition, x_n and y_n do not converge. However, $x_n y_n = 0$ for all $n \in \mathbb{N}$, thus $x_n y_n \to 0$. Q.E.D.

49.)

$$\lim_{n\to\infty} \frac{1 - 5n^2 + 40x^3 + 2n^{-2}}{4 - 12n - 2n^3} = \lim_{n\to\infty} \frac{n^{-3}}{n^{-3}} \cdot \frac{1 - 5n^2 + 40x^3 + 2n^{-2}}{4 - 12n - 2n^3}$$
$$= \lim_{n\to\infty} \frac{x^{-3} - 5n^{-1} + 40 + 2n^{-5}}{4n^{-3} - 12n^{-2} - 2} = \frac{\lim_{n\to\infty} \frac{1}{n^3} - \frac{5}{n} + 40 + \frac{2}{n^5}}{\lim_{n\to\infty} \frac{4}{n^3} - \frac{12}{n^2} - 2} = \frac{0 - 0 + 40 + 0}{0 - 0 - 2}$$
$$= \frac{40}{-2} = -20$$

50.) Since $x_n \to A$, then for all $\varepsilon_0 > 0$, there exists $k \in \mathbb{N}$ where

$$n \geq k \implies |x_n - A| < \varepsilon_0$$

Let $\varepsilon_0 = \varepsilon/\,|c|$, then

$$|x_n - A| < \varepsilon_0 \implies |x_n - A| < \frac{\varepsilon}{|c|} \implies |c|\,|x_n - A| = |cx_n - cA| < \varepsilon$$

Thus if $x_n \to A$, then $cx_n \to cA$. Q.E.D.

63.) Consider the cases of S:

Case: S has a minimal element: then $\inf(S) \in S$. Let $x_n = \inf(S)$, thus $x_n \in S$ for all $n \in \mathbb{N}$ and $x_n \to \inf(S)$.

Case: S has no minimal element, then for all $x \in S$, there exists $\varepsilon > 0$ where $(x - \varepsilon) \in S$. Let $x \in S$ be given and x_n be a sequence defined as follows:

$$x_1 = x, \quad x_n = x_{n-1} - \varepsilon \ \text{ where } \varepsilon > 0 \text{ and } (x_{n-1} - \varepsilon) \in S$$

Since S has no minimal element, x_n is well defined. Since $x_n \in S$ for all $n \in \mathbb{N}$, then $\inf(S) \leq x_n$ for all $n \in \mathbb{N}$, thus x_n is bounded below. Since $x_n = x_{n-1} - \varepsilon$, $x_{n-1} - x_n = \varepsilon > 0$, thus $x_{n-1} > x_n$, thus x_n is strictly decreasing and thus monotone. Finally, according to the monotone convergence theorem, $x_n \to \inf(S)$.

Thus for all $S \subseteq \mathbb{R}$ where S is bounded below, there exists a sequence x_n where $x_n \in S$ for all $x \in \mathbb{N}$ and $x_n \to \inf(S)$. Q.E.D.

86.) a.) True; Let $n_k = k$, thus $x_k = x_{n_k}$, thus $x \preceq x$.

b.) False; Let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$.

c.) True

d.) False; Let $x_n = 1/2n$, $y_n = 1/n$, $z_n = 1/3n$, and $w_n = 1/n$. $x_n + z_n = 1/2n + 1/3n = 5/6n$ and $y_n + w_n = 2/n$, but $2/n \neq 5/6$ for all $n \in \mathbb{N}$, thus $x_n + z_n \not\preceq y_n + w_n$.

e.) False; Let $x_n = 1$ and $y_n = 2$.

87.) Let x_n and y_n be defined as follows:

$$x_n = \begin{cases} -1 & n = 1 \\ 1 & n = 2 \\ 2 & n > 2 \end{cases} \quad y_n = (-1)^n$$

By definition, $X = \{-1, 1, 2\}$, and $Y = \{-1, 1\}$. In this case, $Y \subset X$, however $y_n \not\preceq x_n$, thus the implication does not hold for all x_n, y_n. Q.E.D.

105.) Suppose $y_n = 1/n$. $y_n \neq 0$ for all $n \in \mathbb{N}$, yet $y_n \to 0$. Since y_n converges, it is also cauchy. Because $y_n \to 0$, x_n/y_n does not converge, thus $z_n = x_n/y_n$ does not converge, thus z_n is not cauchy. Q.E.D.

109.) a.) A cauchy sequence is a sequence whose terms, past a certain point, get arbitrarily close to eachother. A type-C sequence is a sequence whose terms, past a certain point, remain constant.

b.) Since $n \neq m \implies 1/n \neq 1/m$, there exists no N such that $n, m \geq N \implies |x_n - x_m| < \varepsilon$ for all $\varepsilon > 0$, thus $x_n = 1/n$ is not type-C.

c.) Let $n \in \mathbb{N}$ be fixed, and consider y_n, y_{n+1}, and y_{n+2}:

$$|y_n - y_{n+1}| = 2, \ \ |y_n - y_{n+2}| = 0$$

Since the distance between any two terms of y_n is either 0 or 2, it cannot be less than all $\varepsilon > 0$ given $m, n > N$ for some $N \in \mathbb{N}$, thus y_n is not type-C.

d.) Since $\sin(n)$ is an oscillating function, $\sin(n)$ never becomes a constant sequence, thus $2^{-n}\sin(n)$ cannot be type-C.

e.) Since any type-C sequence eventually reaches a point where its terms remain constant, we know that every type-C sequence converges to this constant. Since it converges, it is also cauchy.

f.) $1/n$ is cauchy, but not a type-C sequence, thus not every cauchy sequence is type-C.

121.) $E = \{1\}$

122.) $E = \{x_n\} \cup \{y_n\} \cup \{z_n\}$ where $x_n = 1/n$, $y_n = (n+1)/n$, and $z_n = (2n+1)/n$.

123.) $E = \displaystyle\bigcup_{k \in \mathbb{N}} \left\{ \frac{n}{kn+1} \right\}_{n \in \mathbb{N}}$ where $\left\{ \dfrac{n}{kn+1} \right\}_{n \in \mathbb{N}}$ is a sequence given $k \in \mathbb{N}$.

130.) Let x_n be sequence where $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbb{N}$ and where $x_n \to c$. By theorem 25, since $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $f(x_n) \to L$ and $h(x_n) \to L$ as $n \to \infty$. Consider $g(x)$. Since $f(x) \leq g(x) \leq h(x)$ for all $x \neq c$, $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbb{N}$. Consider $g(x_n)$ as $n \to \infty$. Since $f(x_n) \to L$ and $h(x_n) \to L$, and since $f(x_n) \leq g(x_n) \leq h(x_n)$, then by the squeeze theorem, $g(x_n) \to L$, thus $\lim_{x \to c} g(x) = L$. Q.E.D.

132.) Intuitively, $f(x)$ being continuous at c means the limit of $f(x)$ as $x \to c$ exists, and is equal to the value of the function at that point.

135.) For $|x|$ to be continuous over all $c \in \mathbb{R}$, then for all $\varepsilon > 0$, there must exist $\delta > 0$ where

$$|x - c| < \delta \implies \big||x| - |c|\big| < \varepsilon$$

By the reverse triangle inequality, we know that $\big||x| - |c|\big| \leq |x - c|$, thus

$$\big||x| - |c|\big| \leq |x - c| < \delta$$

Let $\delta = \varepsilon$, then

$$\big||x| - |c|\big| < \varepsilon$$

Thus $|x|$ is continuous for all $c \in \mathbb{R}$. Q.E.D.

136.) For $g(x)$ to be continuous at $c \in \mathbb{R}$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - c| < \delta \implies |g(x) - g(c)| < \varepsilon$$

Consider $|g(x) - g(c)|$:

$$|g(x) - g(c)| = |(x - c)f(x) - (c - c)f(x)| = |(x - c)f(x)| = |x - c|\,|f(x)|$$
$$\leq M\,|x - c| < \varepsilon \implies |x - c| < \frac{\varepsilon}{M}$$

Let $\delta = \varepsilon/M$:

$$|x - c| < \delta \implies |x - c| < \frac{\varepsilon}{M} \implies M\,|x - c| < \varepsilon \implies |f(x)|\,|x - c| < \varepsilon$$
$$\implies |(x - c)f(x) - 0| < \varepsilon \implies |g(x) - g(c)| < \varepsilon$$

Thus if $f(x)$ is bounded and $g(x) = (x - c)f(x)$, then $g(x)$ is continuous at c. Q.E.D.

137.) $g(x) = \lfloor x \rfloor$ is continuous over all $x \in \mathbb{R} - \mathbb{Z}$.

139.) Let $f : \mathbb{R} \to \mathbb{R}$ be 1/2-Hölder. For f to be continuous over all $\mathbb{R}$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Let $c = y$ and $\delta = (\varepsilon/C)^2$:

$$|x - y| < \delta \implies |x - y| < \left(\frac{\varepsilon}{C}\right)^2 \implies |x - y|^{1/2} < \frac{\varepsilon}{C} \implies C\,|x - y|^{1/2} < \varepsilon$$

Since f is 1/2-Hölder, $|f(x) - f(y)| < C\,|x - y|^{1/2}$, thus $|f(x) - f(y)| < \varepsilon$, thus f is continuous over all $\mathbb{R}$. Q.E.D.

145.) Without loss of generality, let $f : (0, 1) \to (0, 1)$ where $f(x) = x^2$. For $f(x)$ to have a fixed point, then $f(x)$ has to intersect the line $y = x$ at some $x \in (0, 1)$. However, $0 < x < 1 \implies 0 < x^2 < x$, thus $f(x) < x$ for all $x \in (0, 1)$, thus $f(x) \neq x$ for all $x \in (0, 1)$, thus $f(x)$ does not have a fixed point, thus $f : (a, b) \to (a, b)$ being continuous does not imply that $f(x)$ has a fixed point. Q.E.D.

150.) a.) For $f(x)$ to be uniformly continuous over $(3, 5)$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$x, y \in (3, 5) \land |x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$$

Let $\delta = 25\varepsilon$:

$$|x - y| < \delta \implies |x - y| < 25\varepsilon \implies \frac{|x - y|}{25} < \varepsilon \implies \frac{|x - y|}{|xy|} < \varepsilon$$

$$\implies \left|\frac{x}{xy} - \frac{y}{xy}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$$

Thus $f(x)$ is uniformly continuous over $(3, 5)$. Q.E.D.

b.) Suppose $f(x)$ is uniformly continuous over $(0, 2)$, then for all $\varepsilon > 0$, there exists $\delta > 0$ where

$$x, y \in (0, 2) \wedge |x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$$

Let $x < \delta$ and $y = x/2$:

$$|x - y| = \left|x - \frac{x}{2}\right| = \left|\frac{x}{2}\right| = \frac{x}{2} < \delta$$

Thus $|x - y| < \delta$. Let $\varepsilon = 1$:

$$|x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{2}{x}\right| = \left|-\frac{1}{x}\right| = \frac{2}{x} < \varepsilon = 1$$

$$\implies 2 < x$$

But since $x \in (0, 2)$, $x < 2 \Rightarrow\!\Leftarrow$, thus $f(x)$ is not uniformly continuous over $(0, 2)$. Q.E.D.