# Definitions

1.) Given $x \in \mathbb{R}^n$ and $r > 0$, the open ball of radius $r$ centered at $x$ is defined as follows:

$$\text{Ball}(x, r) = \left\{ y \in \mathbb{R}^n : \sum_{i=1}^{n} (y_i - x_i)^2 < r^2 \right\}$$

2.) A subset $U \subset \mathbb{R}^n$ is open if for all $x \in U$, there exists $r > 0$ where $\text{Ball}(x, r) \subset U$.

3.) Given $A \subset X$, the complement $A'$ of $A$ is the set defined as follows:

$$A' = \{x \in X : x \notin A\}$$

4.) A subset $U \subset \mathbb{R}^n$ is closed if its complement is open.

# Proof

Let $K \subset \mathbb{R}^n$ be closed. For the sake of establishing a contradiction, let $x \in K'$ and $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of points in $K$ where $a_n \to x$. Since $K$ is closed, then by definition $K'$ is open, thus since $x \in K'$, there exists $r > 0$ where $\text{Ball}(x, r) \subset K'$. From this we can see that if $y \in \mathbb{R}^n$ and $d(x, y) < r$, then $y \notin K$, thus there exists $\varepsilon > 0$, where $d(x, a_n) \geq \varepsilon$ for all $n \in \mathbb{N}$, thus $a_n$ does not converge to $x$. $\Rightarrow\Leftarrow$ Thus $x \in K$, thus if $K$ is closed, then any convergent sequence of points in $K$ converges to another point in $K$.

Next, assume that all convergent sequences $\{a_n\}_{n \in \mathbb{N}}$ of points in $K$ converge to other points in $K$. Let $x \in K'$. Since $x \notin K$, there exists no sequence of points in $K$ that converge to $x$, thus for some $\varepsilon > 0$, $d(x, a_n) \geq \varepsilon$ for all $a_n \in K$, thus $\text{Ball}(x, \varepsilon) \subset K'$, thus for all $x \in K'$, there exists $r > 0$ where $\text{Ball}(x, r) \subset K'$, thus $K'$ is open, and thus $K$ is closed. ■